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The space $BV(S^2, S^1)$: minimal connection and optimal lifting

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Abstract

We show that topological singularities of maps in $BV(S^2, S^1)$ can be detected by its distributional Jacobian. As an application, we construct an optimal lifting and we compute its total variation.

Résumé

On montre que le jacobien d'une fonction $u \in BV(S^2, S^1)$ permet de localiser les singularités topologiques de u. On applique ce résultat à la construction d'un relèvement optimal et on calcule sa variation totale.

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1. Introduction

Let $u \in BV(S^2, S^1)$, i.e. $u = (u_1, u_2) \in L^1(S^2, \mathbb{R}^2)$, $|u(x)| = 1$ for a.e. $x \in S^2$ and the derivative of u (in the sense of the distributions) is a finite 2×2-matrix Radon measure

$$\int_{S^2} |Du| = \sup\left\{ \int_{S^2} \sum_{k=1}^{2} u_k \operatorname{div} \zeta_k \, d\mathcal{H}^2\colon\ \zeta_k \in C^1(S^2, \mathbb{R}^2),\ \sum_{k=1}^{2} \left|\zeta_k(x)\right|^2 \leqslant 1,\ \forall x \in S^2 \right\} < \infty,$$

where the norm in $\mathbb{R}^2$ is the Euclidean norm. Observe that the total variation of Du is independent of the choice of the orthonormal frame (x, y) on S^2; a frame (x, y) is always taken such that (x, y, e) is direct, where e is the outward normal to the sphere S^2.

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We begin with the notion of minimal connection between point singularities of u. The concept of a minimal connection associated to a function from $\mathbb{R}^3$ into S^2 was originally introduced by Brezis, Coron and Lieb [3]. Following the ideas in [3] and [6], Brezis, Mironescu and Ponce [4] studied the topological singularities of functions $g \in W^{1,1}(S^2, S^1)$. They show that the distributional Jacobian of g describes the location and the topological charge of the singular set of g. More precisely, let $T(g) \in \mathcal{D}'(S^2, \mathbb{R})$ be defined as

$$T(g) = 2\det(\nabla g) = -(g \wedge g_x)_y + (g \wedge g_y)_x;$$

then there exist two sequences of points (p_k), (n_k) in S^2 such that

$$\sum_k |p_k - n_k| < \infty \quad \text{and} \quad T(g) = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}).$$

Our aim is to extend these notions for functions $u \in BV(S^2, S^1)$. In this case, the difficulty of the analysis of the singular set arises from the existence of more than one type of singularity: besides the point singularities carrying a degree, the jump singularities of u should be taken into account.

We start by introducing some notation. Write the finite Radon 2×2-matrix measure Du as

$$Du = D^a u + D^c u + D^j u,$$

where $D^a u$, $D^c u$ and $D^j u$ are the absolutely continuous part, the Cantor part and the jump part of Du (see e.g. [1]). We recall that $D^j u$ can be written as

$$D^j u = (u^+ - u^-) \otimes \nu_u \mathcal{H}^1 \llcorner S(u),$$

where $S(u)$ denotes the set of jump points of u; $S(u)$ is a countably $\mathcal{H}^1$-rectifiable set on S^2 oriented by the Borel map $\nu_u : S(u) \to S^1$. The Borel functions $u^+, u^- : S(u) \to S^1$ are the traces of u on the jump set $S(u)$ with respect to the orientation ν_u. Throughout the paper we identify u by its precise representative that is defined $\mathcal{H}^1$-a.e. in $S^2 \setminus S(u)$.

We now introduce the distribution $T(u) \in \mathcal{D}'(S^2, \mathbb{R})$ as

$$\langle T(u), \zeta \rangle = \int_{S^2} \nabla^\perp \zeta \cdot \big(u \wedge (D^a u + D^c u)\big) + \int_{S(u)} \rho(u^+, u^-) \nu_u \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1, \quad \forall \zeta \in C^1(S^2, \mathbb{R}). \tag{1}$$

Here, $\nabla^\perp \zeta = (\zeta_y, -\zeta_x)$,

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \wedge \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = (u \wedge a, u \wedge b) = (u_1 a_2 - u_2 a_1, u_1 b_2 - u_2 b_1),$$

where $a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ and $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$. The function $\rho(\cdot,\cdot) : S^1 \times S^1 \to [-\pi, \pi]$ is the signed geodesic distance on S^1 defined as

$$\rho(\omega_1, \omega_2) = \begin{cases} \operatorname{Arg}(\frac{\omega_1}{\omega_2}) & \text{if } \frac{\omega_1}{\omega_2} \neq -1, \\ \operatorname{Arg}(\omega_1) - \operatorname{Arg}(\omega_2) & \text{if } \frac{\omega_1}{\omega_2} = -1, \end{cases} \quad \forall \omega_1, \omega_2 \in S^1,$$

where $\operatorname{Arg}(\omega) \in (-\pi, \pi]$ stands for the argument of the unit complex number $\omega \in S^1$. $T(u)$ represents the distributional determinant of the absolutely continuous part and the Cantor part of Du which is adjusted on $S(u)$ by the tangential derivative of $\rho(u^+, u^-)$. The second term in the RHS of (1) is motivated by the study of $BV(S^1, S^1)$ functions (see [9]): we defined there a similar quantity that represents a pseudo-degree for $BV(S^1, S^1)$ functions.

Remark 1. (i) The integrand in (1) is computed pointwise in any orthonormal frame (x, y) and the corresponding quantity is frame-invariant.

(ii) The 2-vector measure

$$\mu=(\mu_1,\mu_2)=u\wedge(D^a u+D^c u)=\big(u\wedge\big((u_x)^a+(u_x)^c\big),u\wedge\big((u_y)^a+(u_y)^c\big)\big)$$

is well-defined since $D^a u+D^c u$ vanishes on sets which are σ-finite with respect to $\mathcal{H}^1$.

(iii) Notice that the function ρ is antisymmetric, i.e.

$$\rho(\omega_1,\omega_2)=-\rho(\omega_2,\omega_1),\quad \forall\omega_1,\omega_2\in S^1$$

and therefore, $T(u)$ does not depend of the choice of the orientation ν_u on the jump set $S(u)$. By Lemma 5 (see below), we obtain

$$\big|\langle T(u),\zeta\rangle\big|\leqslant|u|_{BV\,S^1},\quad \forall\zeta\in C^1(S^2,\mathbb{R})\text{ with }|\nabla\zeta|\leqslant 1,$$

where $|u|_{BV\,S^1}=\int_{S^2}\big(|D^a u|+|D^c u|\big)+\int_{S(u)}d_{S^1}(u^+,u^-)\,\mathrm{d}\mathcal{H}^1$ and d_{S^1} stands for the geodesic distance on S^1. Therefore, $T(u)$ is indeed a distribution (of order 1) on S^2.

For a compact Riemannian manifold X with the induced distance d, define

$$\mathcal{Z}(X)=\Big\{\Lambda\in\big[C^1(X)\big]^*:\ \exists(p_k),(n_k)\subset X,\sum_k d(p_k,n_k)<\infty\text{ and }\Lambda=2\pi\sum_k(\delta_{p_k}-\delta_{n_k})\Big\}.$$

$\mathcal{Z}(X)$ is the set of distributions that can be written as a countable sum of dipoles.

Remark 2. (i) In general, $\Lambda\in\mathcal{Z}(X)$ is not a measure. In fact, it can be shown that Λ is a measure if and only if Λ is a finite sum of dipoles (see Smets [11] and also Ponce [10]).

(ii) $\Lambda\in\mathcal{Z}(X)$ has always infinitely many representations as a sum of dipoles and these representations need not be equivalent modulo a permutation of points. For example, a dipole $\delta_p-\delta_n$ may be represented as $\delta_p-\delta_{n_1}+\sum_{k\geqslant 1}(\delta_{n_k}-\delta_{n_{k+1}})$ for any sequence (n_k) rapidly converging to n.

For each $\Lambda\in\mathcal{Z}(X)$, the length of a minimal connection between the singularities is defined as

$$\|\Lambda\|=\sup_{\substack{\zeta\in C^1(X)\\|\nabla\zeta|\leqslant 1}}\langle\Lambda,\zeta\rangle.$$

For example, when $\Lambda=2\pi\sum_{k=1}^m(\delta_{p_k}-\delta_{n_k})$ is a finite sum of dipoles, Brezis, Coron and Lieb [3] showed that

$$\|\Lambda\|=2\pi\min_{\sigma\in S_m}\sum_{k=1}^m d(p_k,n_{\sigma(k)}),$$

where S_m denotes the group of permutation of $\{1,2,\ldots,m\}$. In general, for an arbitrary $\Lambda\in\mathcal{Z}(X)$, Bourgain, Brezis and Mironescu [2] proved the following characterization of the length of a minimal connection:

$$\|\Lambda\|=\inf_{(p_k),(n_k)}\Big\{2\pi\sum_k d(p_k,n_k):\ \Lambda=2\pi\sum_k(\delta_{p_k}-\delta_{n_k})\text{ and }\sum_k d(p_k,n_k)<\infty\Big\}.\tag{2}$$

From (2), one can deduce that $\mathcal{Z}(X)$ is a complete metric space with respect to the distance induced by $\|\cdot\|$ (see e.g. [10]).

Our first theorem states that $T(u)$ is a countable sum of dipoles. It is the extension to the BV case of the result in [4] mentioned in the beginning.

Theorem 1. *For every $u \in BV(S^2; S^1)$, we have $T(u) \in \mathcal{Z}(S^2)$, i.e. there exist (p_k), (n_k) in S^2 such that*

$$\sum_k |p_k - n_k| < \infty \quad \textit{and} \quad T(u) = 2\pi \sum_k (\delta_{p_k} - \delta_{n_k}).$$

The proof relies on the fact that the derivative (in the sense of distributions) of the characteristic function of a bounded measurable set in $\mathbb{R}$ can be written as a sum of differences between Dirac masses:

Lemma 1. *Let $I \subset \mathbb{R}$ be a compact interval and $f : I \to 2\pi\mathbb{Z}$ be an integrable function. Define*

$$\left\langle \frac{\mathrm{d}f}{\mathrm{d}t}, \zeta \right\rangle := -\int_I f(t)\zeta'(t)\,\mathrm{d}t, \quad \forall \zeta \in C^1(I).$$

Then

$$\frac{\mathrm{d}f}{\mathrm{d}t} \in \mathcal{Z}(I) \quad \textit{and} \quad \left\| \frac{\mathrm{d}f}{\mathrm{d}t} \right\| = \int_I |f|\,\mathrm{d}t.$$

The same property is valid for the distributional tangential derivative of an integrable function taking values in $2\pi\mathbb{Z}$ and defined on a C^1 1-graph (see Remark 3). Since every countably $\mathcal{H}^1$-rectifiable set $S \subset S^2$ can be covered $\mathcal{H}^1$-a.e. by a sequence of C^1 1-graphs, it makes sense to define for every $\Lambda \in \mathcal{Z}(S^2)$ the set

$$\mathcal{J}(\Lambda) = \Big\{ (f, S, \nu) \colon S \text{ is a countably } \mathcal{H}^1\text{-rectifiable set in } S^2,\ \nu \text{ is an orientation on } S, \\ f \in L^1(S, 2\pi\mathbb{Z}) \text{ is such that } \int_S f\nu \cdot \nabla^{\perp}\zeta\,\mathrm{d}\mathcal{H}^1 = \langle \Lambda, \zeta \rangle,\ \forall \zeta \in C^1(S^2) \Big\}.$$

We have the following reformulation of (2):

Lemma 2. *For every $\Lambda \in \mathcal{Z}(S^2)$, we have*

$$\|\Lambda\| = \min_{(f,S,\nu)\in\mathcal{J}(\Lambda)} \int_S |f|\,\mathrm{d}\mathcal{H}^1.$$

It is known that the infimum in (2) is not achieved in general (see [10]); the advantage of the above formula is that the minimum is always attained. It means that the length of Λ represents the minimal mass that an $\mathcal{H}^1$-integrable function with values into $2\pi\mathbb{Z}$ could carry between the dipoles of Λ.

In the sequel we are concerned with the lifting of $u \in BV(S^2, S^1)$. We call *BV* lifting of u every function $\varphi \in BV(S^2, \mathbb{R})$ such that

$$u = \mathrm{e}^{i\varphi} \quad \text{a.e. in } S^2.$$

The existence of a *BV* lifting for functions $u \in BV(S^2, S^1)$ was initially shown by Giaquinta, Modica and Souček [8]. Later, Dávila and Ignat [5] proved the existence of a lifting $\varphi \in BV \cap L^\infty(S^2, \mathbb{R})$ such that

$$\int_{S^2} |D\varphi| \leqslant 2 \int_{S^2} |Du|; \tag{3}$$

moreover, the constant 2 in (3) is the best constant (see Example 1 and Proposition 3 below).

We give the following characterization for a lifting of u:

Lemma 3. *Let $u \in BV(S^2, S^1)$. For every lifting $\varphi \in BV(S^2, \mathbb{R})$ of u, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ such that*

$$D\varphi = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - f\nu \mathcal{H}^1 \llcorner S. \tag{4}$$

Conversely, for every triple $(f, S, \nu) \in \mathcal{J}(T(u))$ there exists a lifting $\varphi \in BV(S^2, \mathbb{R})$ of u such that (4) *holds.*

In this framework, it is natural to investigate the quantity

$$E(u) = \inf\left\{ \int_{S^2} |D\varphi|\colon\ \varphi \in BV(S^2, \mathbb{R}),\ \mathrm{e}^{\mathrm{i}\varphi} = u \text{ a.e. in } S^2 \right\}. \tag{5}$$

The infimum from above is achieved and it is equal to the relaxed energy

$$E_{\mathrm{rel}}(u) = \inf\left\{ \liminf_{k\to\infty} \int_{S^2} |\nabla u_k|\,\mathrm{d}\mathcal{H}^2\colon\ u_k \in C^\infty(S^2, S^1), u_k \to u \text{ a.e. in } S^2 \right\} \tag{6}$$

(see Remark 4). A lifting $\varphi \in BV(S^2, \mathbb{R})$ of u is called optimal if

$$E(u) = \int_{S^2} |D\varphi|.$$

An optimal lifting need not be unique (see Proposition 3). Remark also that for $u \in BV(S^2, S^1)$, there could be no optimal *BV* lifting of u that belongs to L^∞ (see Example 3).

Our aim is to compute the total variation $E(u)$ of an optimal lifting and to construct an optimal lifting. Theorem 2 establishes the formula for $E(u)$ using the distribution $T(u)$.

Theorem 2. *For every $u \in BV(S^2, S^1)$, we have*

$$E(u) = \int_{S^2} \big(|D^a u| + |D^c u|\big) + \min_{(f,S,\nu)\in\mathcal{J}(T(u))} \int_{S\cup S(u)} \big| f\nu\chi_S - \rho(u^+, u^-)\nu_u \chi_{S(u)} \big|\,\mathrm{d}\mathcal{H}^1. \tag{7}$$

We refer the reader to [8] for related results in terms of Cartesian currents.

As a consequence of Theorem 2, we recover the result of Brezis, Mironescu and Ponce [4] about the total variation of an optimal *BV* lifting for functions $g \in W^{1,1}(S^2, S^1)$: the gap

$$E(g) - \int_{S^2} |\nabla g|\,\mathrm{d}\mathcal{H}^2$$

is equal to the length of a minimal connection connecting the topological singularities of g.

Corollary 1. *For every $g \in W^{1,1}(S^2, S^1)$, we have*

$$E(g) = \int_{S^2} |\nabla g|\,\mathrm{d}\mathcal{H}^2 + \big\|T(g)\big\|.$$

From (7), we deduce an estimate for $E(u)$ (which is a weaker form of inequality (3)):

Corollary 2. *For every $u \in BV(S^2, S^1)$, we have*

$$E(u) \leqslant 2|u|_{BV\,S^1}.$$

In the spirit of [4], we have the following interpretation of $\|T(u)\|$ as a distance:

Theorem 3. *For every $u \in BV(S^2, S^1)$, we have*

$$\|T(u)\| = \min_{\psi \in BV(S^2,\mathbb{R})} \int_{S^2} \left| u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\psi \right|. \tag{8}$$

Moreover, there is at least one minimizer $\psi \in BV(S^2, \mathbb{R})$ of (8) *that is a lifting of u.*

Remark that in general, $\|T(u)\|$ is not the distance of the measure

$$u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u)$$

to the class of gradient maps. In Example 4, we construct a function $u \in BV(S^2, S^1)$ such that

$$\|T(u)\| < \inf_{\psi \in C^\infty(S^2,\mathbb{R})} \int_{S^2} \left| u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\psi \right|.$$

In Section 2, we present the proofs of Lemmas 1, 2 and 3, Theorems 1, 2 and 3 and Corollaries 1 and 2. Some examples and interesting properties of $T(u)$ are given in Section 3. Among other things, we show that $T : BV(S^2, S^1) \to \mathcal{Z}(S^2)$ is discontinuous and we analyze some algebraic properties of $T(u)$. We also discuss the meaning of the point singularities of $T(u)$ and about their location on S^2.

All the results included here can be easily adapted for functions in $BV(\Omega, S^1)$ where Ω is a more general simply connected Riemannian manifold of dimension 2.

2. Remarks and proofs of the main results

We start by proving Lemma 1:

Proof of Lemma 1. Firstly, let us suppose that $f = 2\pi\chi_A$ where $A \subset I$ is an open set. Write $A = \bigcup_{j\in\mathbb{N}} (a_j, b_j)$ as a countable reunion of disjoint intervals. It is clear that

$$\left\langle \frac{\mathrm{d}\chi_A}{\mathrm{d}t}, \zeta \right\rangle = \sum_{j\in\mathbb{N}} \big(\zeta(a_j) - \zeta(b_j)\big), \quad \forall \zeta \in C^1(I)$$

and $\sum_{j\in\mathbb{N}} (b_j - a_j) = \mathcal{H}^1(A)$. Thus $2\pi \frac{\mathrm{d}\chi_A}{\mathrm{d}t} \in \mathcal{Z}(I)$ and

$$\left\| \frac{\mathrm{d}f}{\mathrm{d}t} \right\| = 2\pi \sup_{\substack{\zeta \in C^1(I)\\ |\zeta'| \leqslant 1}} \int_I \chi_A \zeta' \,\mathrm{d}t = 2\pi \sup_{\substack{\psi \in C(I)\\ |\psi| \leqslant 1}} \int_I \chi_A \psi \,\mathrm{d}t = 2\pi \mathcal{H}^1(A).$$

Moreover, let $A \subset I$ be a Lebesgue measurable set and $f = 2\pi\chi_A$. Using the regularity of the Lebesgue measure, there exists a decreasing sequence of open sets $A \subset A_{k+1} \subset A_k \subset I$, $k \in \mathbb{N}$ such that $\lim_{k\to\infty} \mathcal{H}^1(A_k) = \mathcal{H}^1(A)$. Observe that $\frac{\mathrm{d}\chi_{A_k}}{\mathrm{d}t} \to \frac{\mathrm{d}\chi_A}{\mathrm{d}t}$ in $[C^1(I)]^*$. Since $\mathcal{Z}(I)$ is a complete metric space, we conclude that $2\pi \frac{\mathrm{d}\chi_A}{\mathrm{d}t} \in \mathcal{Z}(I)$ and $\left\| 2\pi \frac{\mathrm{d}\chi_A}{\mathrm{d}t} \right\| = 2\pi \mathcal{H}^1(A)$. In the general case of an integrable function $f : I \to 2\pi\mathbb{Z}$, write

$$f = 2\pi \sum_{k\in\mathbb{Z}} k\chi_{E_k} \quad \text{in } L^1, \tag{9}$$

where $E_k = \{x \in I:\ f(x) = 2\pi k\}$. Notice that $2\pi \frac{\mathrm{d}(k\chi_{E_k})}{\mathrm{d}t} \in \mathcal{Z}(I)$ and the series $\sum_{k\in\mathbb{Z}} 2\pi \frac{\mathrm{d}(k\chi_{E_k})}{\mathrm{d}t}$ converges absolutely; indeed, we have

$$\sum_{k\in\mathbb{Z}}\left\|2\pi\frac{\mathrm{d}(k\chi_{E_k})}{\mathrm{d}t}\right\| = 2\pi\sum_{k\in\mathbb{Z}}|k|\mathcal{H}^1(E_k) = \int_I |f|\,\mathrm{d}t < \infty.$$

By (9), we conclude that $\frac{\mathrm{d}f}{\mathrm{d}t} \in \mathcal{Z}(I)$ and

$$\left\|\frac{\mathrm{d}f}{\mathrm{d}t}\right\| = \sup_{\substack{\zeta\in C^1(I)\\ |\zeta'|\leqslant 1}}\int_I f\zeta'\,\mathrm{d}t = \sup_{\substack{\psi\in C(I)\\ |\psi|\leqslant 1}}\int_I f\psi\,\mathrm{d}t = \int_I |f|\,\mathrm{d}t. \qquad \square$$

Remark 3. The conclusion of Lemma 1 is also true for $\mathcal{H}^1$-integrable functions with values in $2\pi\mathbb{Z}$ that are defined on C^1 1-graphs. For simplicity, we restrict to C^1 1-graphs in S^2, i.e. for an orthonormal frame (x, y) on S^2, we consider the set

$$\Gamma = \{(x, y):\ \phi(x) = y\}$$

where ϕ is a C^1 function. Suppose $c:[0,1]\to\Gamma$ is a parameterization of Γ and set $\tau(c(t)) = \frac{c'(t)}{|c'(t)|}$ the tangent unit vector to the curve Γ at $c(t)$, $\forall t\in(0,1)$. Let $f:\Gamma\to 2\pi\mathbb{Z}$ be an $\mathcal{H}^1$-integrable function on Γ. Define

$$\left\langle\frac{\partial f}{\partial\tau},\zeta\right\rangle := -\int_0^1 f\circ c(t)(\zeta\circ c)'(t)\,\mathrm{d}t, \quad \forall\zeta\in C^1(\Gamma).$$

By Lemma 1, we have

$$\frac{\partial f}{\partial\tau}\in\mathcal{Z}(\Gamma) \quad \text{and} \quad \left\|\frac{\partial f}{\partial\tau}\right\| = \int_0^1 |f|(c(t))\left|c'(t)\right|\mathrm{d}t.$$

Before proving Lemma 3, we give the following result:

Lemma 4. *For every $u\in BV(S^2, S^1)$, we have*

$$u\wedge(D^a u + D^c u) = \frac{1}{\mathrm{i}}\bar{u}(D^a u + D^c u)$$
$$\textit{and} \quad \left|u\wedge(D^a u + D^c u)\right| = |D^a u| + |D^c u|.$$

Proof. Write $u = (u_1, u_2) = u_1 + \mathrm{i}u_2$. We can consider the 2×2 matrix of real measures Du as a 2-vector of complex measures, i.e. $Du = Du_1 + \mathrm{i}Du_2$. Since $u_1^2 + u_2^2 = 1$, it results $D(u_1^2 + u_2^2) = 0$. By the chain rule (see e.g. [1]), we obtain

$$u_1(D^a u_1 + D^c u_1) + u_2(D^a u_2 + D^c u_2) = 0,$$

i.e. the real part of the $\mathbb{C}^2$-measure $\bar{u}(D^a u + D^c u)$ vanishes. Therefore,

$$u\wedge(D^a u + D^c u) = \frac{1}{\mathrm{i}}\bar{u}(D^a u + D^c u).$$

Hence, using the fact that the absolutely continuous part and the Cantor part of Du are mutually singular, we conclude that

$$|u\wedge(D^a u + D^c u)| = |u|(|D^a u| + |D^c u|) = |D^a u| + |D^c u|. \qquad \square$$

Proof of Lemma 3. Let $\varphi \in BV(S^2, \mathbb{R})$ be a lifting of u. Write

$$D\varphi = D^a\varphi + D^c\varphi + (\varphi^+ - \varphi^-)\nu_\varphi \mathcal{H}^1 \llcorner S(\varphi).$$

By the chain rule and Lemma 4, we obtain

$$D^a\varphi + D^c\varphi = \frac{1}{\mathrm{i}}\bar{u}(D^a u + D^c u) = u \wedge (D^a u + D^c u).$$

Since $u = \mathrm{e}^{\mathrm{i}\varphi}$ a.e. in S^2, we have that $S(u) \subset S(\varphi)$ and by changing the orientation ν_φ, we may assume

$$\begin{cases} \nu_\varphi = \nu_u \\ \mathrm{e}^{\mathrm{i}\varphi+} = u^+ \quad \mathcal{H}^1\text{-a.e. on } S(u). \\ \mathrm{e}^{\mathrm{i}\varphi-} = u^- \end{cases}$$

Therefore,

$$\varphi^+ - \varphi^- \equiv \rho(u^+, u^-) \pmod{2\pi}\ \mathcal{H}^1\text{-a.e. in } S(u)$$
$$\text{and} \quad \varphi^+ - \varphi^- \equiv 0 \pmod{2\pi}\ \mathcal{H}^1\text{-a.e. in } S(\varphi) \setminus S(u).$$

Hence, there exists $f_\varphi : S(\varphi) \to 2\pi\mathbb{Z}$ a measurable function such that

$$D\varphi = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - f_\varphi \nu_\varphi \mathcal{H}^1 \llcorner S(\varphi). \tag{10}$$

Observe that f_φ is an $\mathcal{H}^1$-integrable function since

$$\left|\rho(u^+, u^-)\right| = d_{S^1}(u^+, u^-) \leqslant \frac{\pi}{2}|u^+ - u^-|.$$

Since $D\varphi$ is a measure, we have

$$\operatorname{curl} D\varphi = 0 \quad \text{in } \mathcal{D}',$$

i.e. for every $\zeta \in C^1(S^2, \mathbb{R})$,

$$\int_{S^2} \nabla^\perp \zeta \, D\varphi = 0.$$

By (10), it yields

$$\left\langle T(u), \zeta \right\rangle = \int_{S(\varphi)} f_\varphi \nabla^\perp \zeta \cdot \nu_\varphi \, \mathrm{d}\mathcal{H}^1, \quad \forall \zeta \in C^1(S^2)$$

and therefore, $(f_\varphi, S(\varphi), \nu_\varphi) \in \mathcal{J}(T(u))$.

Conversely, take $(f, S, \nu) \in \mathcal{J}(T(u))$. Without loss of generality, we may consider $S = \{f \neq 0\}$. Consider the finite Radon $\mathbb{R}^2$-valued measure

$$\mu = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - f\nu \mathcal{H}^1 \llcorner S.$$

We check that $\operatorname{curl} \mu = 0$ in $\mathcal{D}'(S^2)$. Indeed, for every $\zeta \in C^1(S^2, \mathbb{R})$,

$$-\langle \operatorname{curl} \mu, \zeta \rangle = \int_{S^2} \nabla^\perp \zeta \, \mathrm{d}\mu = \left\langle T(u), \zeta \right\rangle - \int_S f \nabla^\perp \zeta \cdot \nu \, \mathrm{d}\mathcal{H}^1 = 0.$$

By the *BV* version of Poincare's lemma, there exists $\varphi \in BV(S^2, \mathbb{R})$ such that $D\varphi = \mu$ in $\mathcal{D}'(S^2, \mathbb{R}^2)$. Here, $S \cup S(u)$ is the jump set of φ. On the set $S \cup S(u)$, we choose an orientation ν_φ such that $\nu_\varphi = \nu_u$ on $S(u)$. We have

$$\begin{cases} D^a\varphi + D^c\varphi = u \wedge (D^a u + D^c u) = \frac{1}{\mathrm{i}}\bar{u}(D^a u + D^c u), \\ \varphi^+ - \varphi^- \equiv \rho(u^+, u^-) \pmod{2\pi}\ \mathcal{H}^1\text{-a.e. in } S(u), \\ \varphi^+ - \varphi^- \equiv 0 \pmod{2\pi}\ \mathcal{H}^1\text{-a.e. in } S \setminus S(u). \end{cases}$$

We now show that

$$D(u\,\mathrm{e}^{-\mathrm{i}\varphi}) = 0.$$

By the chain rule, we get

$$D(\mathrm{e}^{-\mathrm{i}\varphi}) = -\mathrm{i}\mathrm{e}^{-\mathrm{i}\varphi}(D^a\varphi + D^c\varphi) + (\mathrm{e}^{-\mathrm{i}\varphi^+} - \mathrm{e}^{-\mathrm{i}\varphi^-}) \otimes \nu_u \mathcal{H}^1 \llcorner S(u)$$
$$= -\mathrm{e}^{-\mathrm{i}\varphi}\bar{u}(D^a u + D^c u) + (\mathrm{e}^{-\mathrm{i}\varphi^+} - \mathrm{e}^{-\mathrm{i}\varphi^-}) \otimes \nu_u \mathcal{H}^1 \llcorner S(u).$$

Remark that the space $BV(S^2, \mathbb{C}) \cap L^\infty$ is an algebra. Differentiating the product $u\,\mathrm{e}^{-\mathrm{i}\varphi}$, we obtain

$$D(u\,\mathrm{e}^{-\mathrm{i}\varphi}) = \mathrm{e}^{-\mathrm{i}\varphi}(D^a u + D^c u) - u\,\mathrm{e}^{-\mathrm{i}\varphi}\bar{u}(D^a u + D^c u) + (u^+\,\mathrm{e}^{-\mathrm{i}\varphi^+} - u^-\,\mathrm{e}^{-\mathrm{i}\varphi^-}) \otimes \nu_u \mathcal{H}^1 \llcorner S(u) = 0.$$

Thus, up to an additive constant, φ is a BV lifting of u and (4) is fulfilled. □

Proof of Theorem 1. Let $\varphi \in BV(S^2, \mathbb{R})$ be a lifting of u. By Lemma 3, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ such that (4) holds. Denote by $\tau : S \to S^1$ the tangent vector at $\mathcal{H}^1$-a.e. point of S such that (ν, τ, e) is direct. By (4),

$$\langle T(u), \zeta \rangle = \int_S f \nabla^\perp \zeta \cdot \nu \,\mathrm{d}\mathcal{H}^1 = \int_S f \frac{\partial \zeta}{\partial \tau}\,\mathrm{d}\mathcal{H}^1 = \sum_{k\in\mathbb{N}} \int_{I_k} \chi_S f \frac{\partial \zeta}{\partial \tau}\,\mathrm{d}\mathcal{H}^1, \quad \forall \zeta \in C^1(S^2)$$

where $\{I_k\}_{k\in\mathbb{N}}$ is a family of disjoint compact C^1 1-graphs that covers $\mathcal{H}^1$-almost all of the countably rectifiable set S, i.e.

$$\mathcal{H}^1\Big(S \Big\backslash \bigcup_{k\in\mathbb{N}} I_k\Big) = 0.$$

According to Lemma 1 and Remark 3, we conclude $T(u) \in \mathcal{Z}(S^2)$ and $\|T(u)\| \leqslant \int_S |f|\,\mathrm{d}\mathcal{H}^1$. □

Before proving Theorem 2, let us make some remarks about $E(u)$ and $E_{\mathrm{rel}}(u)$ for $u \in BV(S^2, S^1)$ (see also [4]):

Remark 4. (i) $E(u) < \infty$ and $E_{\mathrm{rel}}(u) < \infty$;

(ii) The infimum in (5) is achieved; indeed, let $\varphi_k \in BV(S^2, \mathbb{R})$, $\mathrm{e}^{\mathrm{i}\varphi_k} = u$ a.e. in S^2, be such that

$$\lim_{k\to\infty} \int_{S^2} |D\varphi_k| = E(u) < \infty.$$

By Poincaré's inequality, there exists a universal constant $C > 0$ such that

$$\int_{S^2} \bigg|\varphi_k - \fint_{S^2}\varphi_k\bigg|\,\mathrm{d}\mathcal{H}^2 \leqslant C \int_{S^2} |D\varphi_k|, \quad \forall k \in \mathbb{N}$$

(where $\fint_{S^2}$ stands for the average). Therefore, by subtracting a suitable integer multiple of 2π, we may assume that $(\varphi_k)_{k\in\mathbb{N}}$ is bounded in $BV(S^2, \mathbb{R})$. After passing to a subsequence if necessary, we may assume that $\varphi_k \to \varphi$ a.e. and L^1 for some $\varphi \in BV(S^2, \mathbb{R})$. It follows that φ is a lifting of u on S^2 and

$$E(u) = \lim_{k\to\infty} \int_{S^2} |D\varphi_k| \geqslant \int_{S^2} |D\varphi| \geqslant E(u);$$

(iii) The infimum in (6) is also achieved; take $u_k^m \in C^\infty(S^2, S^1)$ such that for each $k \in \mathbb{N}$,

$$u_k^m \to u \quad \text{a.e. in } S^2 \quad \text{and} \quad \int_{S^2} |\nabla u_k^m|\,\mathrm{d}\mathcal{H}^2 \searrow a_k \in \mathbb{R} \quad \text{as } m \to \infty$$

and $\lim_{k\to\infty} a_k = E_{\mathrm{rel}}(u)$. Subtracting a subsequence, we may assume that for each $k \in \mathbb{N}$,

$$\int_{S^2} |u_k^m - u| \,\mathrm{d}\mathcal{H}^2 < \frac{1}{k} \quad \text{and} \quad \int_{S^2} |\nabla u_k^m| \,\mathrm{d}\mathcal{H}^2 - a_k < \frac{1}{k}, \quad \forall m \geqslant 1.$$

Therefore, $u_k^k \to u$ in L^1 and

$$\lim_{k\to\infty} \int_{S^2} |\nabla u_k^k| \,\mathrm{d}\mathcal{H}^2 = E_{\mathrm{rel}}(u).$$

(iv) $E(u) = E_{\mathrm{rel}}(u)$. For "$\leqslant$", take $u_k \in C^\infty(S^2, S^1)$, $\forall k \in \mathbb{N}$ such that $u_k \to u$ a.e. in S^2 and

$$\sup_{k\in\mathbb{N}} \int_{S^2} |\nabla u_k| \,\mathrm{d}\mathcal{H}^2 < \infty.$$

Since S^2 is simply connected, there exists $\varphi_k \in C^\infty(S^2, \mathbb{R})$ such that $\mathrm{e}^{\mathrm{i}\varphi_k} = u_k$. Moreover,

$$\int_{S^2} |\nabla \varphi_k| \mathrm{d}\mathcal{H}^2 = \int_{S^2} |\nabla u_k| \,\mathrm{d}\mathcal{H}^2.$$

Using the same argument as in ii), we may assume that $\varphi_k \to \varphi$ a.e. and L^1 for some $\varphi \in BV(S^2, \mathbb{R})$. Therefore, $\mathrm{e}^{\mathrm{i}\varphi} = u$ a.e. in S^2 and

$$E(u) \leqslant \int_{S^2} |D\varphi| \leqslant \liminf_{k\to\infty} \int_{S^2} |\nabla \varphi_k| \,\mathrm{d}\mathcal{H}^2 = \liminf_{k\to\infty} \int_{S^2} |\nabla u_k| \,\mathrm{d}\mathcal{H}^2.$$

For "$\geqslant$", consider a BV lifting φ of u and take an approximating sequence $\varphi_k \in C^\infty(S^2, \mathbb{R})$ such that $\varphi_k \to \varphi$ a.e. and $|D\varphi|(S^2) = \lim_{k\to\infty} \int_{S^2} |\nabla \varphi_k| \,\mathrm{d}\mathcal{H}^2$. With $u_k = \mathrm{e}^{\mathrm{i}\varphi_k} \in C^\infty(S^2, S^1)$, we have $u_k \to u$ a.e. in S^2 and

$$E_{\mathrm{rel}}(u) \leqslant \lim_{k\to\infty} \int_{S^2} |\nabla u_k| \,\mathrm{d}\mathcal{H}^2 = \lim_{k\to\infty} \int_{S^2} |\nabla \varphi_k| \,\mathrm{d}\mathcal{H}^2 = \int_{S^2} |D\varphi|.$$

Proof of Theorem 2. For "$\leqslant$", take $(f, S, \nu) \in \mathcal{J}(T(u))$. By Lemma 3, there exists a lifting $\varphi \in BV(S^2, \mathbb{R})$ of u such that (4) holds. It follows that

$$E(u) \leqslant \int_{S^2} |D\varphi| = \int_{S^2} \big(|D^a u| + |D^c u|\big) + \int_{S\cup S(u)} \big| f\nu\chi_S - \rho(u^+, u^-)\nu_u \chi_{S(u)} \big| \,\mathrm{d}\mathcal{H}^1.$$

Let us prove now "$\geqslant$". By Remark 4, there is an optimal BV lifting φ of u, i.e. $E(u) = \int_{S^2} |D\varphi|$. By Lemma 3, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ such that (4) holds. It results that

$$E(u) = \int_{S^2} |D\varphi| = \int_{S^2} \big(|D^a u| + |D^c u|\big) + \int_{S\cup S(u)} \big| f\nu\chi_S - \rho(u^+, u^-)\nu_u \chi_{S(u)} \big| \,\mathrm{d}\mathcal{H}^1.$$

From here, we also deduce that the minimum inside the RHS of (7) is achieved. □

Remark 5 (Construction of an optimal lifting). Take $(f,S,\nu)\in\mathcal{J}(T(u))$ that achieves the minimum

$$\min_{(f,S,\nu)\in\mathcal{J}(T(u))}\int\limits_{S\cup S(u)}\left|f\nu\chi_S-\rho(u^+,u^-)\nu_u\chi_{S(u)}\right|\mathrm{d}\mathcal{H}^1. \tag{11}$$

By Lemma 3, there exists a lifting $\varphi\in BV(S^2,\mathbb{R})$ of u such that (4) holds. Then

$$\int\limits_{S^2}|D\varphi|=\int\limits_{S^2}\left(|D^a u|+|D^c u|\right)+\int\limits_{S\cup S(u)}\left|f\nu\chi_S-\rho(u^+,u^-)\nu_u\chi_{S(u)}\right|\mathrm{d}\mathcal{H}^1=E(u)$$

and therefore, φ is an optimal lifting of u.

Proof of Lemma 2. For "$\leqslant$", it is easy to see that if $(f,S,\nu)\in\mathcal{J}(\Lambda)$ then for every $\zeta\in C^1(S^2)$ with $|\nabla\zeta|\leqslant 1$,

$$\langle\Lambda,\zeta\rangle=\int\limits_S f\nu\cdot\nabla^\perp\zeta\,\mathrm{d}\mathcal{H}^1\leqslant\int\limits_S|f|\,\mathrm{d}\mathcal{H}^1.$$

For "$\geqslant$", we use characterization (2) of the distribution $\Lambda\in\mathcal{Z}(S^2)$. We denote by d_{S^2} the geodesic distance on S^2. Let $\Lambda=2\pi\sum_k(\delta_{p_k}-\delta_{n_k})$ where $(p_k)_{k\in\mathbb{N}}$, $(n_k)_{k\in\mathbb{N}}$ belong to S^2 such that $\sum_k d_{S^2}(p_k,n_k)<\infty$. For every $k\in\mathbb{N}$, consider $\widehat{n_k p_k}$ a geodesic arc on S^2 oriented from n_k to p_k. Take ν_k the normal vector to $\widehat{n_k p_k}$ in the frame (x,y). Set $S=\bigcup_k\widehat{n_k p_k}$. Since $\sum_k d_{S^2}(p_k,n_k)<\infty$, there exist an orientation $\nu:S\to S^1$ on S and an $\mathcal{H}^1$-integrable function $f:S\to 2\pi\mathbb{Z}$ such that

$$f\nu\chi_S=\sum_k 2\pi\nu_k\chi_{\widehat{n_k p_k}}\quad\text{in }L^1(S,\mathbb{R}^2). \tag{12}$$

Then

$$\int\limits_S f\nu\cdot\nabla^\perp\zeta\,\mathrm{d}\mathcal{H}^1=2\pi\sum_k\int\limits_{\widehat{n_k p_k}}\nu_k\cdot\nabla^\perp\zeta\,\mathrm{d}\mathcal{H}^1=2\pi\sum_k\left(\zeta(p_k)-\zeta(n_k)\right)=\langle\Lambda,\zeta\rangle,\quad\forall\zeta\in C^1(S^2).$$

It follows that $(f,S,\nu)\in\mathcal{J}(\Lambda)$ and by (12),

$$\int\limits_S|f|\,\mathrm{d}\mathcal{H}^1\leqslant\sum_k 2\pi d_{S^2}(n_k,p_k).$$

Minimizing after all suitable pairs $(p_k,n_k)_{k\in\mathbb{N}}$, it follows by (2),

$$\|\Lambda\|=\inf_{(f,S,\nu)\in\mathcal{J}(\Lambda)}\int\limits_S|f|\,\mathrm{d}\mathcal{H}^1. \tag{13}$$

We now show that the infimum in (13) is indeed achieved. By a dipole construction (see [2], Lemma 16), there exists $u\in W^{1,1}(S^2,S^1)$ such that $\Lambda=T(u)$. We choose $(f_k,S_k,\nu_k)\in\mathcal{J}(T(u))$ such that

$$\left\|T(u)\right\|=\lim_k\int\limits_{S_k}|f_k|\,\mathrm{d}\mathcal{H}^1.$$

By Lemma 3, we construct a lifting $\varphi_k\in BV(S^2,\mathbb{R})$ of u such that

$$D\varphi_k=u\wedge(D^a u+D^c u)+\rho(u^+,u^-)\nu_u\mathcal{H}^1\llcorner S(u)-f_k\nu_k\mathcal{H}^1\llcorner S_k.$$

Remark that

$$\int_{S^2} |D\varphi_k| \leqslant \int_{S^2} \big(|D^a u| + |D^c u|\big) + \int_{S(u)} \big|\rho(u^+, u^-)\big| \, d\mathcal{H}^1 + \int_{S_k} |f_k| \, d\mathcal{H}^1.$$

Subtracting a suitable number in $2\pi\mathbb{Z}$, we may assume that (φ_k) is a bounded sequence in $BV(S^2, \mathbb{R})$. Up to a subsequence, we find $\varphi \in BV(S^2, \mathbb{R})$ such that

$$\varphi_k \to \varphi \quad \text{a.e. in } S^2 \quad \text{and} \quad D\varphi_k \overset{*}{\rightharpoonup} D\varphi \quad \text{in the measure sense.}$$

Therefore, φ is a BV lifting of u and by Lemma 3, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ such that

$$D\varphi = u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - f\nu \mathcal{H}^1 \llcorner S.$$

We conclude

$$\begin{aligned} \int_S |f| \, d\mathcal{H}^1 &= \int_{S^2} \big|u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\varphi\big| \\ &\leqslant \liminf_k \int_{S^2} \big|u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\varphi_k\big| \\ &= \lim_k \int_{S_k} |f_k| \, d\mathcal{H}^1 = \|T(u)\|. \qquad \square \end{aligned}$$

Proof of Theorem 3. Let $\psi \in BV(S^2, \mathbb{R})$ and $\zeta \in C^1(S^2)$ be such that $|\nabla\zeta| \leqslant 1$. Then

$$\int_{S^2} \big|u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\psi\big| \geqslant \big\langle T(u), \zeta\big\rangle - \int_{S^2} D\psi \cdot \nabla^\perp \zeta = \big\langle T(u), \zeta\big\rangle.$$

By taking the supremum over ζ, we obtain

$$\int_{S^2} \big|u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\psi\big| \geqslant \big\|T(u)\big\|.$$

We now show that there is a lifting $\varphi \in BV(S^2, \mathbb{R})$ of u such that the minimum in (8) is achieved. By Lemma 2, choose $(f, S, \nu) \in \mathcal{J}(T(u))$ such that

$$\big\|T(u)\big\| = \int_S |f| \, d\mathcal{H}^1.$$

Using Lemma 3, we construct a lifting $\varphi \in BV(S^2, \mathbb{R})$ such that (4) holds. Thus,

$$\big\|T(u)\big\| = \int_S |f| d\mathcal{H}^1 = \int_{S^2} \big|u \wedge (D^a u + D^c u) + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - D\varphi\big|. \qquad \square$$

Proof of Corollary 1. The result is a straightforward consequence of Theorem 2 and Lemma 2. $\square$

In order to prove Corollary 2, we need the following estimation of $\|T(u)\|$ in terms of the seminorm $|u|_{BV\,S^1}$:

Lemma 5. *We have* $\|T(u)\| \leqslant |u|_{BV\,S^1}$, $\forall u \in BV(S^2, S^1)$.

Proof. By Lemma 4, it results that for every $\zeta \in C^1(S^2)$ with $|\nabla\zeta| \leqslant 1$,

$$\left|\langle T(u), \zeta\rangle\right| \leqslant \int_{S^2} \left|u \wedge (D^a u + D^c u)\right| + \int_{S(u)} \left|\rho(u^+, u^-)\right| \mathrm{d}\mathcal{H}^1$$
$$= \int_{S^2} \left(|D^a u| + |D^c u|\right) + \int_{S(u)} d_{S^1}(u^+, u^-)\,\mathrm{d}\mathcal{H}^1;$$

therefore

$$\left\|T(u)\right\| \leqslant |u|_{BV\,S^1}. \qquad \square$$

Proof of Corollary 2. By Theorem 2, Lemmas 2 and 5, we conclude that

$$E(u) \leqslant \int_{S^2} \left(|D^a u| + |D^c u|\right) + \int_{S(u)} \left|\rho(u^+, u^-)\right| \mathrm{d}\mathcal{H}^1 + \min_{(f,S,\nu)\in\mathcal{J}(T(u))} \int_S |f|\,\mathrm{d}\mathcal{H}^1$$
$$= |u|_{BV\,S^1} + \left\|T(u)\right\| \leqslant 2|u|_{BV\,S^1}. \qquad \square$$

Let $|u|_{BV} = \int_{S^2} |Du| = \int_{S^2} \left(|D^a u| + |D^c u|\right) + \int_{S(u)} |u^+ - u^-|\,\mathrm{d}\mathcal{H}^1$; we deduce that

$$|u|_{BV} \leqslant |u|_{BV\,S^1} \leqslant \frac{\pi}{2}|u|_{BV}, \quad \forall u \in BV(S^2, S^1).$$

Therefore, Corollary 2 is a weaker estimate of $E(u)$ than inequality (3) obtained in [5].

3. Some other properties of the distribution T

We start by observing that $T : BV(S^2, S^1) \to \mathcal{D}'(S^2, \mathbb{R})$ is not continuous, i.e. there exists a sequence of functions $u_k \in BV(S^2, S^1)$ such that $u_k \to u$ strongly in $BV(S^2, S^1)$ and $T(u_k) \nrightarrow T(u)$ in $\mathcal{D}'(S^2, \mathbb{R})$. The reason for that is the discontinuity of the function ρ that enters in the definition of T.

Proposition 1. *The map $T : BV(S^2, S^1) \to \mathcal{D}'(S^2, \mathbb{R})$ is discontinuous.*

Proof. Write

$$S^2 = \left\{(\cos\theta\sin\alpha, \sin\theta\sin\alpha, \cos\alpha):\ \alpha \in [0, \pi],\ \theta \in (0, 2\pi]\right\}.$$

In the spherical coordinates $(\alpha, \theta) \in [0, \pi] \times [0, 2\pi]$, consider the BV functions φ and u defined as

$$\varphi(\alpha, \theta) = \begin{cases} -2\theta & \text{if } \theta \in (0, \frac{\pi}{2}), \alpha \in (0, \frac{\pi}{2}), \\ -\pi & \text{if } \theta \in (\frac{\pi}{2}, \frac{3\pi}{2}), \alpha \in (0, \frac{\pi}{2}), \\ 2(\theta - 2\pi) & \text{if } \theta \in (\frac{3\pi}{2}, 2\pi), \alpha \in (0, \frac{\pi}{2}), \\ 0 & \text{if } \theta \in (0, 2\pi), \alpha \in (\frac{\pi}{2}, \pi) \end{cases} \quad \text{and} \quad u = \mathrm{e}^{\mathrm{i}\varphi}. \tag{14}$$

We have that the jump set of u and φ is concentrated on the equator $\{\alpha = \frac{\pi}{2}\}$ of the sphere S^2, i.e.

$$S(\varphi) = S(u) = \left\{\alpha = \frac{\pi}{2}\right\}.$$

On the equator we choose the orientation given by the normal vector $\vec{\alpha}$ oriented from the north to the south; so $(\vec{\alpha}, \vec{\theta}, \vec{e})$ is direct. We show that

$$T(u) = 2\pi(\delta_p - \delta_n), \tag{15}$$

where $n = (\frac{\pi}{2}, \frac{3\pi}{2})$ and $p = (\frac{\pi}{2}, \frac{\pi}{2})$ in the frame (α, θ). Indeed, we remark that

$$\varphi^+ - \varphi^- = \rho(u^+, u^-) + 2\pi\chi_{\widehat{np}} \quad \text{in } S(u);$$

by Lemma 3, we obtain

$$D\varphi = u \wedge \nabla u \mathcal{H}^2 + \rho(u^+, u^-)\vec{\alpha}\mathcal{H}^1 \llcorner S(u) + 2\pi\vec{\alpha}\mathcal{H}^1 \llcorner \widehat{np}$$

and it yields

$$\langle T(u), \zeta \rangle = -2\pi \int_{\widehat{np}} \vec{\alpha} \cdot \nabla^\perp \zeta \, d\mathcal{H}^1 = -2\pi \int_p^n \frac{\partial \zeta}{\partial \theta} \, d\mathcal{H}^1 = 2\pi\big(\zeta(p) - \zeta(n)\big), \quad \forall \zeta \in C^1(S^2, \mathbb{R}).$$

Construct the approximation sequence $\varphi_\varepsilon \in BV(S^2, \mathbb{R})$, $\varepsilon \in (0,1)$ defined (in the spherical coordinates) as

$$\varphi_\varepsilon(\alpha, \theta) = \begin{cases} -2\theta & \text{if } \theta \in (0, \frac{\pi-\varepsilon}{2}), \alpha \in (0, \frac{\pi}{2}), \\ -\pi + \varepsilon & \text{if } \theta \in (\frac{\pi-\varepsilon}{2}, \frac{3\pi+\varepsilon}{2}), \alpha \in (0, \frac{\pi}{2}), \\ 2(\theta - 2\pi) & \text{if } \theta \in (\frac{3\pi+\varepsilon}{2}, 2\pi), \alpha \in (0, \frac{\pi}{2}), \\ 0 & \text{if } \theta \in (0, 2\pi), \alpha \in (\frac{\pi}{2}, \pi) \end{cases}$$

and set $u_\varepsilon = e^{i\varphi_\varepsilon}$. An easy computation shows that $\varphi_\varepsilon \to \varphi$ strongly in BV; therefore, $u_\varepsilon \to u$ strongly in BV as $\varepsilon \to 0$. As before, we have

$$S(\varphi_\varepsilon) = S(u_\varepsilon) = \left\{\alpha = \frac{\pi}{2}\right\} \quad \text{and} \quad \varphi_\varepsilon^+ - \varphi_\varepsilon^- = \rho(u_\varepsilon^+, u_\varepsilon^-) \quad \text{in } \left\{\alpha = \frac{\pi}{2}\right\}.$$

It follows that $T(u_\varepsilon) = 0$ and we conclude

$$T(u_\varepsilon) \nrightarrow T(u) \quad \text{in } \mathcal{D}'(S^2, \mathbb{R}). \qquad \square$$

As Brezis, Mironescu and Ponce proved in [4], if we restrict ourselves to $W^{1,1}(S^2, S^1)$, then the map $T|_{W^{1,1}(S^2,S^1)} : W^{1,1}(S^2, S^1) \to \mathcal{Z}(S^2)$ is continuous, i.e. if $g, g_k \in W^{1,1}(S^2, S^1)$ such that $g_k \to g$ in $W^{1,1}$ then $\|T(g_k) - T(g)\| \to 0$ as $k \to \infty$. It is natural to ask if one could change the antisymmetric function ρ in order that the corresponding map T become continuous. The answer is negative:

Proposition 2. *There is no antisymmetric function $\gamma : S^1 \times S^1 \to \mathbb{R}$ such that the map $T_\gamma : BV(S^2, S^1) \to \mathcal{Z}(S^2)$ given for every $u \in BV(S^2, S^1)$ as*

$$\langle T_\gamma(u), \zeta \rangle = \int_{S^2} \nabla^\perp \zeta \cdot \big(u \wedge (D^a u + D^c u)\big) + \int_{S(u)} \gamma(u^+, u^-)\nu_u \cdot \nabla^\perp \zeta \, d\mathcal{H}^1, \quad \forall \zeta \in C^1(S^2, \mathbb{R})$$

is well-defined and continuous.

Proof. By contradiction, suppose that there exists such a function γ. First we show that

$$\gamma(\omega_1, \omega_2) \equiv \operatorname{Arg}(\omega_1) - \operatorname{Arg}(\omega_2) \pmod{2\pi}, \quad \forall \omega_1, \omega_2 \in S^1. \tag{16}$$

Indeed, fix $\omega_1, \omega_2 \in S^1$. Take $f : [0, 2\pi] \to \mathbb{R}$ the linear function satisfying $f(0) = \operatorname{Arg}(\omega_1)$ and $f(2\pi) = \operatorname{Arg}(\omega_2)$; define $u \in BV(S^2, S^1)$ as

$$u(\alpha, \theta) = e^{if(\theta)}, \quad \forall \alpha \in (0, \pi), \ \theta \in (0, 2\pi).$$

Consider the lifting $\varphi \in BV(S^2, \mathbb{R})$ of u given by

$$\varphi(\alpha, \theta) = f(\theta), \quad \forall \alpha \in (0, \pi),\ \theta \in (0, 2\pi).$$

If $\omega_1 \neq \omega_2$, the jump set of u and φ is concentrated on the meridian $\{\theta = 0\}$ orientated counterclockwise by the unit vector $\vec{\theta}$. We have that

$$D\varphi = u \wedge \nabla u \mathcal{H}^2 + \big(\mathrm{Arg}(\omega_1) - \mathrm{Arg}(\omega_2)\big)\vec{\theta}\mathcal{H}^1 \llcorner \{\theta = 0\}.$$

Since curl $D\varphi = 0$ in $\mathcal{D}'$, it yields

$$\begin{aligned}\int_{S^2} u \wedge \nabla u \cdot \nabla^{\perp}\zeta \,\mathrm{d}\mathcal{H}^2 &= -\int_{\{\theta=0\}} \big(\mathrm{Arg}(\omega_1) - \mathrm{Arg}(\omega_2)\big)\vec{\theta} \cdot \nabla^{\perp}\zeta \,\mathrm{d}\mathcal{H}^1 \\ &= \big(\mathrm{Arg}(\omega_1) - \mathrm{Arg}(\omega_2)\big) \int_p^n \frac{\partial \zeta}{\partial \alpha} \,\mathrm{d}\mathcal{H}^1 \\ &= \big(\mathrm{Arg}(\omega_2) - \mathrm{Arg}(\omega_1)\big)\big(\zeta(p) - \zeta(n)\big), \quad \forall \zeta \in C^1(S^2)\end{aligned}$$

where $p = (0, 0)$ and $n = (\pi, 0)$ (in the spherical coordinates) are the north and the south pole of S^2. We obtain that

$$\begin{aligned}\langle T_\gamma(u), \zeta\rangle &= \int_{S^2} \nabla^{\perp}\zeta \cdot (u \wedge \nabla u) \,\mathrm{d}\mathcal{H}^2 + \gamma(\omega_1, \omega_2) \int_{\{\theta=0\}} \vec{\theta} \cdot \nabla^{\perp}\zeta \,\mathrm{d}\mathcal{H}^1 \\ &= \big(\mathrm{Arg}(\omega_2) - \mathrm{Arg}(\omega_1) + \gamma(\omega_1, \omega_2)\big)\big(\zeta(p) - \zeta(n)\big), \quad \forall \zeta \in C^1(S^2, \mathbb{R}).\end{aligned}$$

From the definition we know that $T_\gamma(u) \in \mathcal{Z}(S^2)$ and therefore, (16) holds. If $\omega_1 = \omega_2$, by the antisymmetry of γ, we have $\gamma(\omega_1, \omega_2) = 0$ and so, (16) is obvious.

Second we prove that the continuity of T_γ implies that γ is continuous on $S^1 \times S^1$. Indeed, let (ω_1^ε) and (ω_2^ε) be two sequences in S^1 such that $\omega_1^\varepsilon \to \omega_1$ and $\omega_2^\varepsilon \to \omega_2$. We want that

$$\gamma(\omega_1^\varepsilon, \omega_2^\varepsilon) \to \gamma(\omega_1, \omega_2). \tag{17}$$

Take $\beta \in [0, 2\pi)$ such that $\mathrm{e}^{\mathrm{i}\beta}$ is different from ω_1 and ω_2. For each $\omega \in S^1$ denote by $\arg_\beta(\omega) \in (\beta - 2\pi, \beta]$ the argument of ω, i.e.

$$\mathrm{e}^{\mathrm{i}\arg_\beta(\omega)} = \omega. \tag{18}$$

As above, define $f_\varepsilon : [0, 2\pi] \to \mathbb{R}$ as the linear function satisfying $f_\varepsilon(0) = \arg_\beta(\omega_1^\varepsilon)$ and $f_\varepsilon(2\pi) = \arg_\beta(\omega_2^\varepsilon)$ and consider $u_\varepsilon \in BV(S^2, S^1)$ such that

$$u_\varepsilon(\alpha, \theta) = \mathrm{e}^{\mathrm{i}f_\varepsilon(\theta)}, \quad \forall \alpha \in (0, \pi),\ \theta \in (0, 2\pi).$$

It is easy to check that $u_\varepsilon \to u$ strongly in BV, where $u(\alpha, \theta) = \mathrm{e}^{\mathrm{i}f(\theta)}$ and f is the linear function satisfying $f(0) = \arg_\beta(\omega_1)$ and $f(2\pi) = \arg_\beta(\omega_2)$. As before, we obtain

$$\begin{aligned}&T_\gamma(u_\varepsilon) = \big(\arg_\beta(\omega_2^\varepsilon) - \arg_\beta(\omega_1^\varepsilon) + \gamma(\omega_1^\varepsilon, \omega_2^\varepsilon)\big)(\delta_p - \delta_n) \\ \text{and}\quad &T_\gamma(u) = \big(\arg_\beta(\omega_2) - \arg_\beta(\omega_1) + \gamma(\omega_1, \omega_2)\big)(\delta_p - \delta_n).\end{aligned}$$

Since T_γ and $\arg_\beta$ are continuous on $BV(S^2, S^1)$, respectively on $S^1 \setminus \{\mathrm{e}^{\mathrm{i}\beta}\}$, we deduce that (17) holds.

Observe now that the function

$$(\omega_1, \omega_2) \mapsto \gamma(\omega_1, \omega_2) - \mathrm{Arg}(\omega_1) + \mathrm{Arg}(\omega_2)$$

is continuous on the connected set $S^1 \setminus \{-1\} \times S^1 \setminus \{-1\}$ and takes values in $2\pi\mathbb{Z}$. Therefore, there exists $k \in \mathbb{Z}$ such that

$$\gamma(\omega_1, \omega_2) = \mathrm{Arg}(\omega_1) - \mathrm{Arg}(\omega_2) - 2\pi k \quad \text{in } S^1 \setminus \{-1\} \times S^1 \setminus \{-1\}.$$

In fact, $k = 0$ if one takes $\omega_1 = \omega_2$. But $\mathrm{Arg}(\cdot)$ is not a continuous map on S^1 which is a contradiction with the continuity of γ on $S^1 \times S^1$. □

The algebraic properties of T restricted to $W^{1,1}(S^2, S^1)$ (see [4], Lemma 1) do not hold in general for $BV(S^2, S^1)$ functions.

Remark 6. (a) There exists $u \in BV(S^2, S^1)$ such that $T(\bar{u}) \neq -T(u)$. Indeed, take the function u defined in (14). A similar computation gives us that $T(\bar{u}) = 0 \neq -T(u)$.

(b) The relation $T(gh) = T(g) + T(h)$, $\forall g, h \in W^{1,1}(S^2, S^1)$ need not hold for $BV(S^2, S^1)$ functions. As before, consider the function u in (14). Then $T(-u) = 0$. Since $T(-1) = 0$, we conclude $T(-u) \neq T(u) + T(-1)$.

In the following we discuss the nature of the singularities of the distribution $T(u)$. As it was mentioned in the beginning, we deal with two types of singularity:

(i) topological singularities carrying a degree which are created by the absolutely continuous part and the Cantor part of the distributional determinant of u;
(ii) point singularities coming from the jump part of the derivative Du.

We give some examples in order to point out these two different kind of singularity. In Example 1, $T(u)$ is a dipole made up by two vortices of degree $+1$ and -1; these two vortices are generated by the absolutely continuous part of $\det(\nabla u)$ in (a), respectively by the Cantor part of the distributional Jacobian of u in (b).

Example 1. (a) Let us analyze the function $g \in W^{1,1}(S^2, S^1)$,

$$g(\alpha, \theta) = \mathrm{e}^{\mathrm{i}\theta}, \quad \forall \alpha \in (0, \pi),\ \theta \in [0, 2\pi).$$

Denote p and n the north and respectively the south pole of the unit sphere. We consider the lifting $\varphi \in BV(S^2, \mathbb{R})$ of u given by $\varphi(\alpha, \theta) = \theta$ for every $\alpha \in (0, \pi)$, $\theta \in (0, 2\pi)$. Then the jump set of φ is concentrated on the meridian $\{\theta = 0\}$ oriented counterclockwise by the unit vector $\vec{\theta}$. We have

$$D\varphi = g \wedge \nabla g \mathcal{H}^2 - 2\pi \vec{\theta} \mathcal{H}^1 \llcorner \widehat{np}.$$

Therefore, $T(g) = 2\pi(\delta_p - \delta_n)$. The two poles are the vortices of the function g.

(b) The same situation may occur for some purely Cantor functions. Let us consider the standard Cantor function $f : [0, 1] \to [0, 1]$; f is a continuous, nondecreasing function with $f(0) = 0$, $f(1) = 1$ and $f'(x) = 0$ for a.e. $x \in (0, 1)$. Take $v \in BV(S^2, S^1)$ defined as

$$v(\alpha, \theta) = \mathrm{e}^{2\pi \mathrm{i} f(\theta/2\pi)}, \quad \forall \alpha \in (0, \pi),\ \theta \in [0, 2\pi).$$

The lifting $\varphi \in BV(S^2, \mathbb{R})$ given by $\varphi(\alpha, \theta) = 2\pi f(\theta/2\pi)$ for every $\alpha \in (0, \pi), \theta \in (0, 2\pi)$ has the jump set concentrated on the meridian $\{\theta = 0\}$ and

$$D\varphi = v \wedge D^c v - 2\pi \vec{\theta} \mathcal{H}^1 \llcorner \widehat{np}.$$

As before, we obtain that $T(v) = 2\pi(\delta_p - \delta_n)$ where p and n are the poles of S^2.

Remark also that for the two functions constructed in Example 1, the constant 2 in inequality (3) is optimal and we have a specific structure for an optimal lifting:

Proposition 3. *Let $u \in BV(S^2, S^1)$ be one of the two functions defined in Example 1. Then for every lifting $\varphi \in BV(S^2, \mathbb{R})$ of u we have*

$$\int_{S^2} |D\varphi| \geqslant 2 \int_{S^2} |Du|.$$

Moreover, the set of all optimal liftings of u is given by

$$\{\arg_\beta(u) + 2\pi k\colon\ \beta \in [0, 2\pi),\ k \in \mathbb{Z}\}$$

where $\arg_\beta(\omega) \in (\beta - 2\pi, \beta]$ stands for the argument of $\omega \in S^1$ (as in (18)).

Proof. First we notice that

$$\int_{S^2} |Du| = 2\pi^2 \quad \text{and} \quad \|T(u)\| = 2\pi d_{S^2}(n, p) = 2\pi^2$$

where n and p are the two poles of S^2.

Let $\varphi \in BV(S^2, \mathbb{R})$ be a lifting of u. By Theorem 2 and Lemma 2, we obtain

$$\int_{S^2} |D\varphi| \geqslant E(u) = \int_{S^2} |Du| + \|T(u)\| = 4\pi^2 = 2 \int_{S^2} |Du|.$$

Take now $\varphi \in BV(S^2, \mathbb{R})$ an optimal lifting of u. By Lemma 3, there exists $(f, S, \nu) \in \mathcal{J}(T(u))$ that achieves the minimum in (11) and satisfies

$$D\varphi = u \wedge Du - f\nu \mathcal{H}^1 \llcorner S.$$

That means

$$D^j\varphi = -f\nu \mathcal{H}^1 \llcorner S \quad \text{and} \quad \int_S |f| = 2\pi d_{S^2}(n, p). \tag{19}$$

We may assume here that $S = \{f \neq 0\}$. For every $\alpha \in (0, \pi)$ we denote L_α the latitude on S^2 corresponding to α and $\varphi_\alpha : L_\alpha \to \mathbb{R}$ the restriction of φ to L_α. Using the Characterization Theorem of *BV* functions by sections and Theorem 3.108 in [1], it results that for a.e. $\alpha \in (0, \pi)$, $\varphi_\alpha \in BV(L_\alpha, \mathbb{R})$ and the discontinuity set of φ_α is $S \cap L_\alpha$. Remark that $\deg(u; L_\alpha) = 1$ for every $\alpha \in (0, \pi)$. Thus, for a.e. $\alpha \in (0, \pi)$, φ_α will have at least one jump on L_α and the length of a jump is not less than 2π. It yields $\mathcal{H}^1(S) \geqslant \pi$ and $|f| \geqslant 2\pi \mathcal{H}^1 -$ a.e. in S. By (19), we deduce that

$$|f| = 2\pi \quad \mathcal{H}^1\text{-a.e. in } S \quad \text{and} \quad \mathcal{H}^1(S) = \pi.$$

We know that

$$\int_S \frac{f}{2\pi} \nu \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1 = \zeta(p) - \zeta(n), \quad \forall \zeta \in C^1(S^2).$$

By [7] (Section 4.2.25), it results that S covers $\mathcal{H}^1$-almost all of a Lipschitz univalent path c between the two poles. Since $\mathcal{H}^1(S) = d_{S^2}(n, p)$ we deduce that S is a geodesic arc on S^2 between n and p and $\frac{f}{2\pi}\nu$ is the normal unit vector to the curve c. Take $\beta \in [0, 2\pi)$ such that $S = \{\theta = \beta\}$ in the spherical coordinates. We have that $\varphi - \arg_\beta(u) : S^2 \setminus S \to 2\pi\mathbb{Z}$ is continuous on the connected set $S^2 \setminus S$. Therefore, there exists $k \in \mathbb{Z}$ such that

$$\varphi = \arg_\beta(u) + 2\pi k$$

and the conclusion follows. $\square$

The appearance of non-topological singularities in the writing of $T(u)$ for $u \in BV(S^2, S^1)$ was already seen in the example (14); there the distribution $T(u)$ is a dipole even if the function u does not have any vortex. One should notice that the dipole (15) is created on the jump set of u by the discontinuity of the chosen argument Arg. In Remark 7, we will see that a dipole could disappear if we change the choice of the argument.

Remark 7. Let $\beta \in [0, 2\pi)$. Define the antisymmetric function $\gamma_\beta(\cdot,\cdot): S^1 \times S^1 \to [-\pi, \pi]$ as

$$\gamma_\beta(\omega_1, \omega_2) = \begin{cases} \operatorname{Arg}\left(\frac{\omega_1}{\omega_2}\right) & \text{if } \frac{\omega_1}{\omega_2} \neq -1, \\ \arg_\beta(\omega_1) - \arg_\beta(\omega_2) & \text{if } \frac{\omega_1}{\omega_2} = -1, \end{cases} \quad \forall \omega_1, \omega_2 \in S^1.$$

Consider now the distribution $T_{\gamma_\beta}(u) \in \mathcal{D}'(S^2, \mathbb{R})$ given as in Proposition 2:

$$\left\langle T_{\gamma_\beta}(u), \zeta \right\rangle = \int_{S^2} \nabla^\perp \zeta \cdot \left(u \wedge (D^a u + D^c u)\right) + \int_{S(u)} \gamma_\beta(u^+, u^-) \nu_u \cdot \nabla^\perp \zeta \, \mathrm{d}\mathcal{H}^1, \quad \forall \zeta \in C^1(S^2, \mathbb{R}).$$

Observe that T_{γ_β} inherits the properties of T given in Theorems 1, 2 and 3. However, the structure of the singularities of $T_{\gamma_\beta}(u)$ may be different from $T(u)$. Indeed, consider $u \in BV(S^2, S^1)$ the function constructed in (14). We saw that $T(u) = 2\pi(\delta_p - \delta_n)$ where $n = (\frac{\pi}{2}, \frac{3\pi}{2})$ and $p = (\frac{\pi}{2}, \frac{\pi}{2})$ (in the spherical coordinates). The same computation gives us $T_{\gamma_{\pi/2}}(u) = 0$. The difference between $T(u)$ and $T_{\gamma_{\pi/2}}(u)$ arises from the choice of the argument.

An interesting phenomenon is observed in Example 2 where the two types of singularity are mixed: some topological vortices may be located on the jump set of u.

Example 2. (a) An example that points out the mixture of the two type of singularity is given by functions with pseudo-vortices: define $u \in BV(S^2, S^1)$ as

$$u(\alpha, \theta) = \mathrm{e}^{3\mathrm{i}\theta/2}, \quad \forall \alpha \in (0, \pi),\ \theta \in (0, 2\pi).$$

The jump set of u is the meridian $\{\theta = 0\}$. We have

$$T(u) = 2\pi(\delta_p - \delta_n) \quad \text{and} \quad T_{\gamma_{\pi/2}}(u) = 4\pi(\delta_p - \delta_n).$$

The two poles p and n arise on the jump set of u and behave like some pseudo-vortices, i.e. after a complete turn, the function u rotates $3/2$ times around the poles (with different signs: '$+$' around p and '$-$' around n). According to the choice of the argument in the definition of γ_β, the distribution $T_{\gamma_\beta}(u)$ will count once or twice the dipole.

(b) A piecewise constant function $u \in BV(S^2, S^1)$ may create a dipole for $T(u)$. Indeed, let us define $\varphi \in BV(S^2, \mathbb{R})$ as

$$\varphi(\alpha, \theta) = \begin{cases} 0 & \text{if } \theta \in (0, 2\pi/3),\ \alpha \in (0, \pi), \\ 2\pi/3 & \text{if } \theta \in (2\pi/3, 4\pi/3),\ \alpha \in (0, \pi), \\ 4\pi/3 & \text{if } \theta \in (4\pi/3, 2\pi),\ \alpha \in (0, \pi) \end{cases}$$

and set $u = \mathrm{e}^{\mathrm{i}\varphi}$. The jump set of u and φ is the union of three meridians

$$S(u) = S(\varphi) = \{\theta = 0\} \cup \{\theta = 2\pi/3\} \cup \{\theta = 4\pi/3\}.$$

We have

$$\varphi^+ - \varphi^- = \rho(u^+, u^-) - 2\pi \chi_{\{\theta=0\}} \quad \text{in } S(\varphi).$$

We obtain $T(u) = 2\pi(\delta_p - \delta_n)$ where p and n are the two poles of the unit sphere. For every $\beta \in [0, 2\pi)$, T_{γ_β} has the same behavior, i.e. $T_{\gamma_\beta}(u) = 2\pi(\delta_p - \delta_n)$.

(c) Let $u \in BV(S^2, S^1)$ be the function defined above in (b) and take g the function constructed in Example 1(a). Set $w = gu \in BV(S^2, S^1)$. We have $S(w) = \{\theta = 0\} \cup \{\theta = 2\pi/3\} \cup \{\theta = 4\pi/3\}$. We show that $T(w) = 4\pi(\delta_p - \delta_n)$. Indeed, construct the lifting $\psi \in BV(S^2, \mathbb{R})$ of w as

$$\psi(\alpha,\theta) = \begin{cases} \theta & \text{if } \theta \in (0, 2\pi/3), \alpha \in (0,\pi), \\ \theta + 2\pi/3 & \text{if } \theta \in (2\pi/3, 4\pi/3), \alpha \in (0,\pi), \\ \theta - 2\pi/3 & \text{if } \theta \in (4\pi/3, 2\pi), \alpha \in (0,\pi). \end{cases}$$

Observe that

$$\psi^+ - \psi^- = \rho(w^+, w^-) - 2\pi\chi_{\{\theta=0\}} - 2\pi\chi_{\{\theta=4\pi/3\}} \quad \text{in } S(w)$$

and conclude that $T(w) = 4\pi(\delta_p - \delta_n)$. So, the north pole p and the south pole n which are the vortices of g remain singularities for the function w; they appear now on the jump part of w. The same behavior happens to T_{γ_β} for every $\beta \in [0, 2\pi)$, i.e. $T_{\gamma_\beta}(w) = 4\pi(\delta_p - \delta_n)$.

As we mentioned before, for every $u \in BV(S^2, S^1)$ there exists a bounded lifting $\varphi \in BV \cap L^\infty(S^2, \mathbb{R})$ (see [5]). The striking fact is that we can construct functions $u \in BV(S^2, S^1)$ such that no optimal lifting belongs to L^∞. We give such an example in the following:

Example 3. On the interval $(0, 2\pi)$ we consider

$$p_1 = 1, \quad n_k = p_k + \frac{1}{4^k} \quad \text{and} \quad p_{k+1} = n_k + \frac{1}{2^k}, \quad \forall k \geqslant 1.$$

Suppose that this configuration of points lies on the equator $\{\frac{\pi}{2}\} \times [0, 2\pi]$ (in the spherical coordinates) of S^2 and we consider that each dipole (p_k, n_k) appears k times. Since $\sum_{k\geqslant 1} k d_{S^2}(p_k, n_k) < \infty$, set

$$\Lambda = 2\pi \sum_{k\geqslant 1} k(\delta_{p_k} - \delta_{n_k}) \in \mathcal{Z}(S^2).$$

By [2] (Lemma 16),

$$T\big(W^{1,1}(S^2, S^1)\big) = \mathcal{Z}(S^2).$$

Thus, take $g \in W^{1,1}(S^2, S^1)$ such that $T(g) = \Lambda$. Using (2), it follows that

$$\big\|T(g)\big\| = 2\pi \sum_{k\geqslant 1} k d_{S^2}(p_k, n_k).$$

Let $\varphi \in BV(S^2, \mathbb{R})$ be an optimal lifting of g. Then there is a triple $(f, S, \nu) \in \mathcal{J}(T(g))$ such that

$$D\varphi = g \wedge \nabla g \mathcal{H}^2 - f\nu \mathcal{H}^1 \llcorner S \quad \text{and} \quad \int_S |f| \, d\mathcal{H}^1 = \|T(g)\|. \tag{20}$$

We may assume that $S = \{f \neq 0\}$.

We know that $\int_S f\nu \cdot \nabla^\perp \zeta \, d\mathcal{H}^1 = 2\pi \sum_{k\geqslant 1} k\big(\zeta(p_k) - \zeta(n_k)\big)$, $\forall \zeta \in C^1(S^2)$. For each $k \geqslant 1$, we denote in the spherical coordinates $V_k = (0,\pi) \times \left(p_k - \frac{1}{8^k}, n_k + \frac{1}{8^k}\right)$. Then

$$\int_S f\nu \cdot \nabla^\perp \zeta \, d\mathcal{H}^1 = 2\pi k\big(\zeta(p_k) - \zeta(n_k)\big), \quad \forall \zeta \in C^1(S^2) \text{ with } \operatorname{supp} \zeta \subset V_k.$$

By (20), it follows that

$$\int_{S\cap V_k} |f|\, \mathrm{d}\mathcal{H}^1 = 2\pi k d_{S^2}(p_k, n_k).$$

Using the same argument as in the proof of Proposition 3, we deduce that for each $k \in \mathbb{N}$,

$$S(\varphi) \cap V_k = S \cap V_k = \widehat{n_k p_k} \quad \text{and} \quad |\varphi^+ - \varphi^-| = |f| = 2k\pi \quad \mathcal{H}^1\text{-a.e. on } \widehat{n_k p_k}$$

where $\widehat{n_k p_k}$ is the geodesic arc connecting n_k and p_k. It yields that $\varphi \notin L^\infty$. So, every optimal BV lifting of g does not belong to L^∞.

In the next example, we show that Theorem 3 fails if we minimize the energy in (8) just over the class of gradient maps:

Example 4. Let $u \in BV(S^2, S^1)$ be defined as

$$u(\alpha, \theta) = \mathrm{e}^{\mathrm{i}\theta/3}, \quad \forall \alpha \in (0, \pi),\ \theta \in (0, 2\pi).$$

The jump set of u is the meridian $\{\theta = 0\}$ oriented counterclockwise and $\rho(u^+, u^-) = -2\pi/3$ on $S(u)$. We have that $T(u) = 0$. On the other hand, for every $\psi \in C^\infty(S^2, \mathbb{R})$, we have

$$\int_{S^2} \left|u \wedge \nabla u \mathcal{H}^2 + \rho(u^+, u^-)\nu_u \mathcal{H}^1 \llcorner S(u) - \nabla\psi \mathcal{H}^2\right| = \int_{S^2} |u \wedge \nabla u - \nabla\psi|\, \mathrm{d}\mathcal{H}^2 + \int_{S(u)} \left|\rho(u^+, u^-)\right| \mathrm{d}\mathcal{H}^1$$

$$\geqslant \int_{S(u)} \frac{2\pi}{3}\, \mathrm{d}\mathcal{H}^1 = \frac{2\pi^2}{3} > \left\|T(u)\right\|.$$

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