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Existence of a minimal non-scattering solution to the mass-subcritical generalized Korteweg–de Vries equation

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Abstract

In this article, we prove the existence of a non-scattering solution, which is minimal in some sense, to the mass-subcritical generalized Korteweg–de Vries (gKdV) equation in the scale critical $\hat{L}^r$ space where $\hat{L}^r = \{f \in \mathcal{S}'(\mathbb{R}) | \, \|f\|_{\hat{L}^r} = \|\hat{f}\|_{L^{r'}} < \infty\}$. We construct this solution by a concentration compactness argument. Then, key ingredients are a linear profile decomposition result adopted to $\hat{L}^r$-framework and approximation of solutions to the gKdV equation which involves rapid linear oscillation by means of solutions to the nonlinear Schrödinger equation.

MSC: primary 35Q53, 35B40; secondary 35B30

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1. Introduction

In this article, we consider the generalized Korteweg–de Vries (gKdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u = \mu \partial_x (|u|^{2\alpha} u), & t, x \in \mathbb{R}, \\ u(0, x) = u_0(x) \in \hat{L}^\alpha(\mathbb{R}), & x \in \mathbb{R} \end{cases} \tag{gKdV}$$

where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is an unknown function, $u_0 : \mathbb{R} \to \mathbb{R}$ is a given data, and $\mu = \pm 1$ and $\alpha > 0$ are constants. The space $\hat{L}^r$ is defined for $1 \leqslant r \leqslant \infty$ by

$$\hat{L}^r = \hat{L}^r(\mathbb{R}) := \{f \in \mathcal{S}'(\mathbb{R}) | \, \|f\|_{\hat{L}^r} = \|\hat{f}\|_{L^{r'}} < \infty\},$$

where $\hat{f} = \mathcal{F} f$ stands for Fourier transform of f with respect to space variable and $r' = (1 - 1/r)^{-1}$ denotes the Hölder conjugate of r with conventions $1' = \infty$ and $\infty' = 1$. We say that (gKdV) is defocusing if $\mu = +1$ and

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focusing if $\mu = -1$. Our aim here is to study time global behavior of solutions to (gKdV) with focusing nonlinearities in the *mass-subcritical* range $\alpha < 2$. More specifically, we investigate the existence of a threshold solution which lies on the boundary of small scattering solutions around zero and other solutions.

The class of equations (gKdV) arises in several fields of physics. Equation (gKdV) is a generalization of the Korteweg–de Vries equation which models long waves propagating in a channel [35]. Equation (gKdV) with $\alpha = 1$ is also known as the modified Korteweg–de Vries equation which describes a time evolution for the curvature of certain types of helical space curves [36].

Equation (gKdV) has the following scaling property: if $u(t,x)$ is a solution to (gKdV), then $u_\lambda(t,x) := \lambda^{1/\alpha} u(\lambda^3 t, \lambda x)$ is also a solution to (gKdV) with initial data $u_\lambda(0,x) = \lambda^{1/\alpha} u_0(\lambda x)$ for any $\lambda > 0$. When $\alpha = 2$, (gKdV) is called mass-critical because the above scaling leaves the mass invariant.

The small data global existence results of (gKdV) have been studied by several authors in the framework of scale subcritical and critical spaces. Here we mention the known results in the scale critical spaces only. For the small data global existence in the scale subcritical spaces, see [11,12,19–22,24,51,52,56,57]. For the scaling critical case, Kenig–Ponce–Vega [28] proved the small data global well-posedness and scattering of (gKdV) in the scale critical space $\dot{H}^{s_\alpha}$ for $\alpha \geqslant 2$, where $s_\alpha := 1/2 - 1/\alpha$ is a scale critical exponent. Since the scale critical exponent s_α is negative in the mass-subcritical case $\alpha < 2$, the well-posedness of (gKdV) in $\dot{H}^{s_\alpha}$ becomes rather a difficult problem. Tao [58] proved global well-posedness for small data for (gKdV) with the quartic nonlinearity $\mu \partial_x (u^4)$ in $\dot{H}^{s_{3/2}}$. Later on, Koch–Marzuola [34] simplified Tao's proof and extended his result to a Besov space $\dot{B}^{s_{3/2}}_{\infty,2}$. As for the $\hat{L}^r$-framework, Grünrock and his collaborator proved well-posedness for various nonlinear dispersive equations, see [16–18].

The well-posedness of (gKdV) and small data scattering in $\hat{L}^\alpha$ is established by the authors as long as $8/5 < \alpha < 10/3$ by introducing a generalized version of Stichartz' estimates adopted to the $\hat{L}^r$-framework, see [46]. The mass

$$M[u] = \frac{1}{2} \|u\|_{L^2}^2$$

and the energy

$$E[u] = \frac{1}{2} \|\partial_x u\|_{L^2}^2 + \frac{\mu}{2\alpha + 2} \|u\|_{L^{2\alpha+2}}^{2\alpha+2}$$

are well-known conserved quantities for (gKdV). However, neither makes sense in general for $\hat{L}^\alpha$-solutions. Thus, global existence is nontrivial for large data even in the mass-subcritical range $\alpha < 2$.

As a step next to small data scattering, in this article, we consider existence of a threshold solution which lies on the boundary of small scattering solutions around zero and other solutions, via concentration compactness argument. Let us make our setup more precise. We say an $\hat{L}^\alpha$-solution $u(t)$ scatters forward in time (resp. backward in time) if maximal existence interval of $u(t)$ is not bounded from above (resp. from below) and if $e^{t\partial_x^3} u(t)$ converges in $\hat{L}^\alpha$ as $t \to \infty$ (resp. $t \to -\infty$). We define a *forward scattering set* $\mathcal{S}_+$ as follows

$$\mathcal{S}_+ := \left\{ u_0 \in \hat{L}^\alpha \;\middle|\; \begin{array}{l} \text{a solution } u(t) \text{ to (gKdV) with } u_{|t=0} = u_0 \\ \text{scatters forward in time} \end{array} \right\}.$$

A *backward scattering set* $\mathcal{S}_-$ is defined in a similar way. We now introduce a quantity

$$\widetilde{d} := \inf_{u_0 \in \hat{L}^\alpha \setminus \mathcal{S}_+} \|u_0\|_{\hat{L}^\alpha}. \tag{1}$$

The question we address in this article is that existence of a special solution which belongs to $\hat{L}^\alpha \setminus \mathcal{S}_+$ at each time and attains $\widetilde{d}$ in a suitable sense. By small data scattering result in [46], we know that $\widetilde{d}$ is bounded by a positive constant from below. Remark that there are several choices on notion of minimality of non-scattering solutions since $\|u(t)\|_{\hat{L}^\alpha}$ is not a conserved quantity. The above $\widetilde{d}$ is a number that gives a sharp scattering criterion; if $\|u_0\|_{\hat{L}^\alpha} < \widetilde{d}$ then a corresponding solution scatters for positive time direction. However, we actually work with a weaker formulation by some technical reason (see (5), below).

The above problem has a connection with stability of solitons. In the focusing case (i.e., $\mu = -1$), (gKdV) admits a soliton solution $Q_c(t,x) = c^{1/\alpha} Q(c(x - c^2 t))$, where $Q(x)$ is a (unique) positive even solution of $-Q'' + Q = Q^{2\alpha+1}$ and $c > 0$ is a parameter describing amplitude and propagating speed of soliton. Let us remind ourselves that we consider the mass-subcritical problem. It is well known that Q is orbitally stable if $\alpha < 2$ [2,63] and unstable if $\alpha \geqslant 2$ (see [3] for $\alpha > 2$ and [37] for $\alpha = 2$). When the soliton solutions are unstable, for example in the mass-critical case $\alpha = 2$, it is conjectured that the above $\widetilde{d}$ coincide with L^2-norm (since $\alpha = 2$) of Q_c. So far, it is known that if $\alpha = 2$ then Q lies on the boundary of sets of *global* solutions and non-global solutions in H^1, see Weinstein [62] for the sharp global existence result and Martel–Merle [38] for the existence of a finite time blow up solution.

On the other hand, in mass-subcritical case, solitons are stable (in H^1) and so they are not thresholds any longer. Indeed, it follows from [46, Theorem 1.10] that $\widetilde{d} \leqslant c_\alpha \|Q\|_{\hat{L}^\alpha}$, where

$$c_\alpha = \left(\frac{(\alpha+1)\,\|Q'\|_{L^2}^2}{\|Q\|_{L^{2\alpha+2}}^{2\alpha+2}} \right)^{1/(2\alpha)} < 1 \tag{2}$$

is a constant such that $E[c_\alpha Q] = 0$.

Recently, there are much progress on analysis of global behavior of dispersive equations by so-called concentration compactness/rigidity argument, after a pioneering work by Kenig–Merle [26]. The existence of a critical element is one of the main step of the argument. As for generalized KdV equation (gKdV), the mass-critical case is most extensively studied in this direction. Killip–Kwon–Shao–Visan [30] constructed a minimal blow-up solution to the mass critical KdV equation in L^2 under the assumption on the space time bounds for the one dimensional mass-critical Schrödinger (NLS) equation. Subsequently, Dodson [14] proved the global well-posedness for the one dimensional, defocusing, mass-critical NLS in L^2. As by product of his result, the assumption imposed in [30] was removed for the defocusing case. Furthermore, Dodson [15] has shown the global well-posedness for the defocusing mass-critical KdV equation for any initial data in L^2. For the focusing mass-critical KdV equation, Martel–Merle–Raphaël [40–42] and Martel–Merle–Nakanishi–Raphaël [39] classified the dynamics of solution into three cases (blow-up, soliton, away from soliton) in the small neighborhood of Q. As for the mass-subcritical nonlinear Schrödinger equation, the first author treated a minimization problem similar to (1) in a framework of weighted spaces and showed existence of a threshold solution which is smaller than ground state solutions (see [43,44]).

A main contribution of this article is to extend the concentration compactness argument to $\hat{L}^\alpha$-framework. We then come across two difficulties because of the fact that the $\hat{L}^\alpha$-norm is invariant under the following four group actions;

(i) Translation in physical space: $(T(a)f)(x) = f(x-a)$, $a \in \mathbb{R}$,
(ii) Translation in Fourier space: $(P(\xi)f)(x) = e^{-ix\xi} f(x)$, $\xi \in \mathbb{R}$,
(iii) Dilation: $(D(h)f)(x) = (D_\alpha(h)f)(x) = h^{1/\alpha} f(hx)$, $h \in 2^{\mathbb{Z}}$,
(iv) Airy flow: $(A(t)f)(x) = e^{-t\partial_x^3} f(x)$, $t \in \mathbb{R}$.

They are one parameter groups of linear isometries in $\hat{L}^\alpha$. In this article, we call a bijective linear isometry from a Banach space X to X itself a *deformation on* X. Further, we refer to a deformation of the form $\times\phi(x)$ as a *phase-like deformation*, and a deformation of the form $\phi((1/i)\partial_x) = \mathcal{F}^{-1}\phi(\xi)\mathcal{F}$ as a *multiplier-like deformation*, where $\phi(x) : \mathbb{R} \to \mathbb{C}$ is some function with $|\phi| = 1$. With these terminologies, $T(a) = e^{-a\partial_x}$ and $A(t)$ are multiplier-like deformations on $\hat{L}^\alpha$ and $P(\xi)$ is a phase-like deformation on $\hat{L}^\alpha$.

The first difficulty lies in a linear profile decomposition, which is roughly speaking a decomposition of a bounded sequence of functions into a sum of characteristic profiles and a remainder by finding weak limit(s) of the sequence modulo deformations. Intuitively, this decomposition is done by a recursive use of a suitable concentration compactness result. Then, to ensure smallness of remainder as the number of detected profiles increases, a decoupling equality, so-called the Pythagorean decomposition, plays a crucial role.

Let us now be more precise on the Pythagorean decomposition. Let $\{f_n\}$ be a bounded sequence of $\hat{L}^\alpha$. Since $\hat{L}^\alpha$ is reflexive as long as $1 < \alpha < \infty$, by extracting subsequence, f_n converges to some function $f \in \hat{L}^\alpha$ in weak $\hat{L}^\alpha$ sense. Now we suppose that $f \neq 0$. Then, the Pythagorean decomposition is a decoupling equality of the form

$$\|\hat{f}_n\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} = \|\hat{f}\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} + \|\hat{f} - \hat{f}_n\|_{L^{\alpha'}(\mathbb{R})}^{\alpha'} + o(1) \tag{3}$$

as $n \to \infty$. It is well-known that the above decoupling holds for $\alpha = 2$ and may fail for $\alpha \neq 2$. Remark that the Brezis–Lieb lemma tells us that a sufficient condition for the decoupling (for $\alpha \neq 2$) is that $\hat{f}_n$ converges to $\hat{f}$ almost everywhere. However, in our case, due to multiplier-like deformations T and A, which are phase-like in the Fourier side, Fourier transform of considering sequence does not necessarily converge almost everywhere. Thus, we may not expect that (3) holds for $\hat{L}^\alpha$-norm.[1] This respect is rather a serious problem for linear profile decomposition, because a decoupling like (3) is a key for obtaining smallness of remainder term as the number of detected profiles increases, as mentioned above.

To overcome this difficulty, we shall show a decoupling *inequality* with respect to a weaker norm, a *generalized Morrey* norm, defined as follows:

Definition 1.1. For $4/3 < \alpha < 2$ and for $\sigma \in (\alpha', \frac{6\alpha}{3\alpha-2})$, we introduce a *generalized Morrey norm* $\|\cdot\|_{\hat{M}^\alpha_{2,\sigma}}$ by

$$\|f\|_{\hat{M}^\alpha_{2,\sigma}} = \left\| 2^{j(\frac{1}{\alpha}-\frac{1}{2})} \|\hat{f}\|_{L^2(\tau^j_k)} \right\|_{\ell^\sigma_{j,k}} = \left\| |\tau^j_k|^{\frac{1}{2}-\frac{1}{\alpha}} \|\hat{f}\|_{L^2(\tau^j_k)} \right\|_{\ell^\sigma_{j,k}},$$

where $\tau^j_k = [k2^{-j}, (k+1)2^{-j})$. Further, we introduce

$$\ell(u) = \ell_\sigma(u) := \inf_{\xi \in \mathbb{R}} \|P(\xi)u\|_{\hat{M}^\alpha_{2,\sigma}} \qquad \text{for } u \in \hat{L}^\alpha. \tag{4}$$

Details on generalized Morrey space are summarized in Section 2. Here, we only note that the embedding $\hat{L}^\alpha \hookrightarrow \hat{M}^\alpha_{2,\sigma}$ holds, that $\ell(f) \sim \|f\|_{\hat{M}^\alpha_{2,\sigma}}$, and so that $\ell(f)$ is a quasi-norm and makes sense for all $f \in \hat{L}^\alpha$. It is obvious by definition that $T(a)$ and $A(t)$ are deformations on $\hat{M}^\alpha_{2,\sigma}$ for any $a, t \in \mathbb{R}$. Similarly, $D(h)$ is a deformation on $\hat{M}^\alpha_{2,\sigma}$ if h is a dyadic number. We introduce $\ell(\cdot)$ because $\hat{M}^\alpha_{2,\sigma}$ norm is not invariant (but bounded from above and below) under $P(\xi)$ action. The heart of matter is that local (in the Fourier side) L^2 norm decouples even under presence of multiplier-like deformations T and A. Hence, summing up the local L^2 decoupling with respect to intervals, we recover a decoupling *inequality* for $\ell(\cdot)$. Thus, we obtain a linear profile decomposition in the $\hat{M}^\alpha_{2,\sigma}$-framework. This is one of the main ideas of this article.

Because our decoupling inequality is established only for $\ell(\cdot)$, a natural choice of the meaning of *minimality* of the solution is not with respect to $\|\cdot\|_{\hat{L}^\alpha}$ any longer but to $\ell(\cdot)$. Thus, we consider the minimization problem for

$$d_+ = d_+(\sigma, M) := \inf\{\ell(u_0) \mid u_0 \in B_M \setminus \mathcal{S}_+\}, \tag{5}$$

where $M > 0$ is a parameter and $B_M := \{f \in \hat{L}^\alpha \mid \|f\|_{\hat{L}^\alpha} \leqslant M\}$ is a ball. We consider minimization problem in a ball in $\hat{L}^\alpha$ because well-posedness of (gKdV) is not known in the generalized Morrey space $M^\alpha_{2,\sigma}$. As a result, our threshold solution may depend on M.

Here, it is worth mentioning that the generalized Morrey space naturally appears in the context of refinement of Stein–Tomas inequality, which corresponds to the diagonal version of Strichartz' estimate in $\hat{L}^\alpha$. The refinement goes back to Bourgain [4] (see also [5,6,29,49,50]), and has been used for the linear profile decomposition in L^2-framework. See [1,8,47] for decomposition associated with Schrödinger equation and see [55] for that with Airy equation. We show a version of the refined Stein–Tomas inequality

$$\|e^{-t\partial_x^3} f\|_{L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant C \left(\sup_{N \in 2^{\mathbb{Z}}} \|e^{-t\partial_x^3} P_N f\|_{L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R})} \right)^{1-\frac{\sigma}{3\alpha}} \|f\|^{\frac{\sigma}{3\alpha}}_{\hat{M}^\alpha_{2,\sigma}},$$

where P_N is the standard frequency cut-off operator to $|\xi| \sim N \in 2^{\mathbb{Z}}$. For the details on this estimate, see Lemma 5.9 in Section 5 (see also Theorem B.1 in Appendix B). The refinement is crucial for the linear profile decomposition in $\hat{M}^\alpha_{2,\sigma}$.

The second difficulty comes from a linking between generalized KdV equation and nonlinear Schrödinger equation caused by the presence of P-deformation. More precisely, if an initial data is of the form $u_0(x) = \mathrm{Re}[P(\xi)\phi(x)]$ then a corresponding solution to (gKdV) can be approximated in terms of a solution to nonlinear Schrödinger equation

$$\begin{cases} i\partial_t v - \partial_x^2 v = -\mu|v|^{2\alpha} v, & t, x \in \mathbb{R}, \\ v(0,x) = v_0(x), & x \in \mathbb{R}, \end{cases} \tag{NLS}$$

in the limit $|\xi| \to \infty$. This interesting phenomena is known in [30,59] (see also [7,10,54]).

[1] Actually, when $\alpha' = 4$, $f_n = f + T(n)g$ with $f, g \in \hat{L}^{4/3}$ is a counter example to the above decoupling.

As for linear Airly equation, the linking with linear Schrödinger equation can be explained by an elemental identity

$$A(t)P(\xi) = e^{-it\xi^3} P(\xi)T(-3\xi^2 t)e^{3i\xi t\partial_x^2} A(t). \tag{6}$$

The identity infers that the presence of P on the initial data produces Schrödinger group $e^{3i\xi t\partial_x^2}$. Furthermore, in fact, the Schrödinger evolution takes a main part in the limit $|\xi| \to \infty$ because the speed of Schrödinger evolution becomes much faster than that of Airy evolution. The above identity is a kind of Galilean transform, and can be compared with the one for Schrödinger equations;

$$e^{it\partial_x^2} P(\xi) = e^{-it|\xi|^2} P(\xi)T(-2t\xi)e^{it\partial_x^2}. \tag{7}$$

Roughly speaking, as a nonlinear evolution generated by a class of nonlinear Schrödinger equation, such as (NLS), inherits the Galilean transform (7), the effect on the nonlinear problem (gKdV) which is caused by the presence of P in initial data is similar to that on the Airy equation described as in (6).

Because of the above linking, existence of a threshold solution is shown under the assumption

$$d_+ < 2^{\frac{1}{\sigma}-1} \left(\frac{3\sqrt{\pi}\Gamma(\alpha+2)}{2\Gamma(\alpha+\frac{3}{2})} \right)^{\frac{1}{2\alpha}} d_{\mathrm{NLS}}, \tag{8}$$

where d_+ is the number given in (5), σ is a parameter chosen to define $\ell(\cdot)$, $\Gamma(x)$ is the Gamma function, and

$$d_{\mathrm{NLS}} = d_{\mathrm{NLS}}(\sigma, M) := \inf\{\ell(v_0) \mid v_0 \in B_M \setminus \mathcal{S}_{\mathrm{NLS}}\} \tag{9}$$

with

$$\mathcal{S}_{\mathrm{NLS}} := \left\{ v_0 \in \hat{L}^\alpha \;\middle|\; \begin{array}{l} \text{a solution } v(t) \text{ to (NLS) with } v_{|t=0} = v_0 \\ \text{scatters forward and backward in time} \end{array} \right\}.$$

Here, the notion of scattering of $\hat{L}^\alpha$-solution $v(t)$ to (NLS) forward in time (resp. backward in time) is defined as validity of the following two; (i) maximal existence interval of $v(t)$ is not bounded from above (resp. from below); (ii) $e^{it\partial_x^2}v(t)$ converges in $\hat{L}^\alpha$ as $t \to \infty$ (resp. $t \to -\infty$). It was pointed out in [30,59] that, in the mass-critical case $\alpha = 2$, the problem of a threshold solution for (gKdV) relates to the same problem for (NLS). Although we are working in the mass-subcritical case, the same linking appears because it is due to the presence of the P-deformation. When $\alpha = 2$, the assumption (8) essentially coincides with those in [30,59].

The justification of the Schrödinger approximation is done essentially in the same way as in [30] for the mass critical KdV equation. One of the key point to justify the approximation is how to pick up the *resonance* term from the nonlinear term. Notice that for the mass critical case, the nonlinear term is polynomial in u and $\overline{u}$, so it is easy to pick up the resonance term from the nonlinear term. On the other hand, in our setting it is non-trivial to pick up the resonance term from the nonlinear term because the power of the nonlinear term is fractional. A main idea for dealing with nonlinearity is to use a Fourier series expansion

$$|\cos\theta|^{2\alpha} \sin\theta = \sum_{k=1}^{\infty} C_k \sin(k\theta).$$

The constant in assumption (8) given in terms of the first coefficient C_1 of the expansion. By using the Fourier series expansion, we are able to pick up the resonance term from the nonlinear term. Consequently, we have to take care of convergence for the above Fourier series. Fortunately, we can show its convergence thanks to the enough smoothness of the nonlinearity. Further, to justify approximation, we also establish local well-posedness of (NLS) in a scale critical $\hat{L}^\alpha$ space, which seems already a new result.

1.1. Main results

In what follows, we consider the focusing case $\mu = -1$ only. However, the focusing assumption is used only for $d_+(M) < \infty$. Our analysis work also in the defocusing case $\mu = +1$ if we assume $d_+(M) < \infty$.

Theorem 1.2. *Let $3/2+\sqrt{7/60}<\alpha<2$ and $\sigma\in(\alpha',\frac{3\alpha(5\alpha-8)}{2(3\alpha-4)})$. Let $M>0$ so that $B_M\cap S_+^c\neq\emptyset$. If the assumption (8) is true then there exists a special solution $u_c(t)$ to (gKdV) with maximal interval $I_{\max}(u_c)\ni 0$ such that*

(i) $u_c(0)\notin\mathcal{S}_+$;
(ii) u_c attains d_+ in such a sense that one of the following two properties holds;
  (a) $u_c(0)\in B_M$ and $\ell(u_c(0))=d_+$;
  (b) $u_c(0)\in\mathcal{S}_-$ and scatters backward in time to $u_{c,-}$ satisfying $u_{c,-}\in B_M$ and $\ell(u_{c,-})=d_+$.

In this article we call u_c constructed in Theorem 1.2 by a *minimal non-scattering solution*.

Remark 1.3. As mentioned above, $d_+(M)$ gives a scattering criterion; if $u_0\in\hat{L}^\alpha$ satisfies $\|u_0\|_{\hat{L}^\alpha}\leqslant M$ and $\ell(u_0)<d_+$ then $u_0\in\mathcal{S}_+$. By definition of d_+, this is sharp in such a sense that d_+ cannot be replaced by a larger number. It is not clear whether we can replace $\ell(u_0)<d_+$ by $\ell(u_0)\leqslant d_+$.

Remark 1.4. So far, we do not have any additional property, such as precompactness of the flow, of the critical solution u_c constructed in Theorem 1.2. It is not necessarily by a technical reason. Indeed, a similar minimization problem is considered for energy critical nonlinear Schrödinger equation in [45], and a minimizer satisfying properties (i) and (ii)-(b) is given. (Furthermore, in this case there is no minimizer which attains minimum value at finite time as in (ii)-(a).) Remark that the minimizer satisfying (ii)-(b) does not possess precompactness of the flow for negative time direction.

The assumption $B_M\cap S_+^c\neq\emptyset$ is fulfilled for $M\geqslant c_\alpha\|Q\|_{\hat{L}^\alpha}$ because $c_\alpha Q\notin S_+$ by means of [46, Theorem 1.10]. By the same reason, we have the following:

Theorem 1.5. *Let $3/2+\sqrt{7/60}<\alpha<2$ and $\sigma\in(\alpha',\frac{3\alpha(5\alpha-8)}{2(3\alpha-4)})$. Let $M>0$ so that $B_M\cap S_+^c\neq\emptyset$. Then, $d_+\leqslant c_\alpha\ell(Q)$, where c_α is the constant in (2). In particular, the minimal non-scattering solution u_c given in Theorem 1.2 is not a soliton.*

Let us discuss the assumption (8). We do not know whether the assumption is true. However, it would be reasonable to expect that the assumption (8) is true for focusing case at least when α and σ are close to 2 and α', respectively. One positive reason is that it is true in the limiting mass-critical case $\alpha=2$ with a modification $\ell(\cdot)=\|\cdot\|_{L^2}$ and $\sigma=2$. Killip–Kwon–Shao–Vişan [30] proved a similar theorem[2] under the assumption $d_+<(6/5)^{1/4}d_{\mathrm{NLS}}=2^{\frac12-1}(3\sqrt{\pi}\Gamma(4)/2\Gamma(7/2))^{1/4}d_{\mathrm{NLS}}$. We see that this assumption is true by combining the fact by Dodson [13] that $d_{\mathrm{NLS}}=\|Q\|_{L^2}$ (see also [31,33]) and trivial bound $d_+\leqslant\|Q\|_{L^2}$. Nevertheless, it does not imply the assumption is true because continuity property for d_+ and d_{NLS} in α and σ is not known. Hence, we show existence of a minimal non-scattering solution without the assumption (8) by modifying the minimization problem, which is our second result.

For fixed $8/5<\widetilde{\alpha}<\alpha$ and $0<\widetilde{s}<2\alpha+1$, define $\widetilde{B}_M=\{f\in\hat{L}^\alpha\mid\|f\|_{\hat{L}^{\widetilde{\alpha}}}+\|f\|_{\dot{H}^{\widetilde{s}}}\leqslant M\}$. As for a minimizing problem for

$$d'_+=d'_+(\sigma,M):=\inf\{\ell(u_0)\mid u_0\in\widetilde{B}_M\cap\mathcal{S}_+^c\},$$

a minimizer exists *without* the assumption (8).

Theorem 1.6. *Let $3/2+\sqrt{7/60}<\alpha<2$ and $\sigma\in(\alpha',\frac{3\alpha(5\alpha-8)}{2(3\alpha-4)})$. Let $M>0$ so that $\widetilde{B}_M\cap S_+^c\neq\emptyset$. Then, there exists a special solution $\widetilde{u}_c(t)$ to (gKdV) which attains $\widetilde{d}_+$ in a similar way to Theorem 1.2.*

Now let us introduce several consequential results which follow from the arguments which we establish to prove our main results. We begin with two scattering results. The first one is as follows;

[2] We can construct a minimizer to d_+ which possesses the properties described in [30, Theorem 1.13]. Although their result is based on the assumption $d_+<\|Q\|_{L^2}$, which is conjectured to be false, their argument works under a weaker assumption $d_+\leqslant\|Q\|_{L^2}$, which is true.

Theorem 1.7. *Let $5/3 \leqslant \alpha < 20/9$. For any $M > 0$ there exists $\delta = \delta(M) > 0$ such that if $u_0 \in \hat{L}^\alpha$ satisfies $\|u_0\|_{\hat{L}^\alpha} \leqslant M$ and*

$$\||\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} u_0\|_{L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant \delta$$

then a corresponding solution $u(t)$ to (gKdV) exists globally and scatters for both time directions.

The above theorem is a variant of small data scattering, and a consequence of a stability type estimate which is so-called long time stability. Notice that it contains the case that the data is not small in the $\hat{L}^\alpha$ topology.

Remark 1.8. The proof of [46, Theorem 1.7] shows that there exists a constant δ' independent of $\|u_0\|_{\hat{L}^\alpha}$ such that if

$$\||\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} u_0\|_{L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R})} + \|e^{-t\partial_x^3} u_0\|_{L_x^{\frac{5\alpha}{2}}(\mathbb{R}, L_t^{5\alpha}(\mathbb{R}))} \leqslant \delta'$$

then the solution scatters for both time directions. In Theorem 1.7, smallness assumption on the second term of the left hand side is removed, however the constant δ may depends on $\|u_0\|_{\hat{L}^\alpha}$.

The second scattering result is the following.

Theorem 1.9 *(Scattering due to irrelevant deformations). Let $5/3 \leqslant \alpha < 2$. Let $\{u_{0,n}\}_n \subset \hat{L}^\alpha$ be a bounded sequence. Let $u_n(t)$ be a solution to (gKdV) with $u_n(0) = u_{0,n}$. If a set*

$$\left\{ \phi \in \hat{L}^\alpha \;\middle|\; \begin{array}{l} \phi = \lim_{k\to\infty} (D(h_k)A(s_k)T(y_k)P(\xi_k))^{-1} u_{0,n_k} \text{ weakly in } \hat{L}^\alpha, \\ \exists (h_k, \xi_k, s_k, y_k) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R},\ \exists \text{subsequence } n_k \end{array} \right\}$$

is equal to $\{0\}$ then there exists N_0 such that $u_n(t)$ is global and scatters for both time directions as long as $n \geqslant N_0$.

This theorem is a consequence of Theorem 1.7 and a concentration compactness argument. An example of sequence $\{u_{0,n}\}_n$ that satisfies the assumption of Theorem 1.9 is $u_{0,n} = e^{in\partial_x^4} f$, $f \in \hat{L}^\alpha$. As a corollary, we also see that $\mathcal{S}_+ \cap \mathcal{S}_-$ is unbounded in $\hat{L}^\alpha$ topology.

Corollary 1.10. *For any $f \in \hat{L}^\alpha$, there exists $T > 0$ such that $e^{it\partial_x^4} f \in \mathcal{S}_+ \cap \mathcal{S}_-$ for $|t| \geqslant T$. In particular, $\mathcal{S}_+ \cap \mathcal{S}_-$ is an unbounded subset of $\hat{L}^\alpha$.*

Unboundedness of each $\mathcal{S}_+$ and $\mathcal{S}_-$ are seen by considering an orbit $\{A(t)f \mid t \in \mathbb{R}\}$ of $f \in \hat{L}^\alpha$. However, this argument does not yield that of the intersection of the both.

Finally, we state the results on the well-posedness of nonlinear Schrödinger equation (NLS) in $\hat{L}^\alpha$ and $\hat{M}^\alpha_{2,\sigma}$. Although the analysis of (NLS) is not an original purpose of the article, this is necessary for our analysis because there is a linking between (gKdV) and (NLS) due to the presence of P-deformation.

Theorem 1.11 *(Local well-posedness of (NLS) in $\hat{L}^\alpha$). The equation (NLS) is locally well-posed in $\hat{L}^\alpha$ if $4/3 < \alpha < 4$.*

Theorem 1.12 *(Local well-posedness of (NLS) in $\hat{M}^\alpha_{2,\sigma}$). The equation (NLS) is locally well-posed in $\hat{M}^\alpha_{2,\sigma}$ if $4/3 < \alpha < 2$ and $\alpha' < \sigma \leqslant \frac{6\alpha}{3\alpha-2}$.*

The rest of the article is organized as follows. The main theorems are proven in Section 4 after preliminaries on notations and basic facts (Section 2) and stability estimate (Section 3). For the proof, we rely on two important ingredient, linear profile decomposition (Theorem 4.3) and NLS approximation (Theorem 4.4). We prove Theorem 4.3 in Sections 5 and 6. Finally, we turn to the proof of Theorem 4.4 in Sections 7 and 8. On the other hand, consequential results are shown when we are ready; Theorem 1.7 is proven in Section 3, Theorem 1.9 is in Section 6, and Theorems 1.11 and 1.12 are in Section 7.

The following notation will be used throughout this paper: We use the notation $A \sim B$ to represent $C_1 A \leqslant B \leqslant C_2 A$ for some constants C_1 and C_2. We also use the notation $A \lesssim B$ to denote $A \leqslant CB$ for some constant C. $|\partial_x|^s = (-\partial_x^2)^{s/2}$ denotes the Riesz potential of order $-s$. For $1 \leqslant p, q \leqslant \infty$ and $I \subset \mathbb{R}$, let us define a space–time norm $\|f\|_{L_x^p L_t^q(I)} = \| \|f(\cdot, x)\|_{L_t^q(I)} \|_{L_x^p(\mathbb{R})}$.

2. Notations and basic facts

In this section, we introduce several notations and give lemmas which are needed to prove main results.

2.1. Deformations

Let us first collect elementary facts on the deformations which is used throughout the article. As in the introduction, we set

- $(T(y)f)(x) = f(x-y)$, $y \in \mathbb{R}$,
- $(P(\xi)f)(x) = e^{-ix\xi} f(x)$, $xi \in \mathbb{R}$,
- $(D_p(h)f)(x) = h^{1/p} f(hx)$, $h \in 2^{\mathbb{Z}}$,
- $(A(t)f)(x) = e^{-t\partial_x^3} f(x)$, $t \in \mathbb{R}$.

They are deformations on $\hat{L}^p$ for any $1 \leqslant p \leqslant \infty$. Denote $D(h) = D_\alpha(h)$, where α is the number in (gKdV). Let $S(t) = e^{it\partial_x^2}$ be a Schrödinger group. Notice that $S(t)$ is also a deformation on $\hat{L}^p$, $1 \leqslant p \leqslant \infty$. The inverses of $A(t)$, $S(t)$, $T(y)$, and $P(\xi)$ are $A(-t)$, $S(-t)$, $T(-y)$, and $P(-\xi)$, respectively. Further, $D_p(h)^{-1} = D_p(h^{-1})$.

We use a notation $\hat{X} := \mathcal{F} X \mathcal{F}^{-1}$, or equivalently, $\mathcal{F}X = \hat{X}\mathcal{F}$, for $X = A, S, T, P, D$. More specifically, $\hat{A}(t) = e^{it\xi^3}$, $\hat{S}(t) = e^{-it\xi^2}$, $\hat{T}(y) = P(y)$, $\hat{P}(\xi) = T(-\xi)$, and $\hat{D_\alpha}(h) = D_{\alpha'}(h^{-1})$. With this notation, the identity (6) is easily obtained as follows.

$$\hat{P}(\xi_0)^{-1} \hat{A}(t) \hat{P}(\xi_0) = e^{it(\xi-\xi_0)^3} = e^{-i\xi_0^3 t} \hat{T}(-3\xi_0^2 t) \hat{S}(3\xi_0 t) \hat{A}(t).$$

Next, we collect commutations of the above deformations. We have

$$[A(t), S(t)] = [A(t), T(y)] = [S(t), T(y)] = 0, \quad T(y)P(\xi) = e^{iy\xi} P(\xi) T(y).$$

Commutation property for $D(h)$ is as follows:

$$A(t)D(h) = D(h)A(h^3 t), \qquad S(t)D(h) = D(h)S(h^2 t),$$
$$T(y)D(h) = D(h)T(hy), \qquad P(\xi)D(h) = D(h)P(h^{-1}\xi).$$

Combining above relations, we have the following identity

$$\begin{aligned} &(D(\widetilde{h})T(\widetilde{y})A(\widetilde{s})P(\widetilde{\xi}))^{-1}(D(h)T(y)A(s)P(\xi)) \\ &\qquad = e^{i\gamma} D\left(\frac{h}{\widetilde{h}}\right) P\left(\xi - \frac{\widetilde{h}}{h}\widetilde{\xi}\right) A\left(s - \left(\frac{h}{\widetilde{h}}\right)^3 \widetilde{s}\right) \\ &\quad S\left(3\left(s - \left(\frac{h}{\widetilde{h}}\right)^3 \widetilde{s}\right)\xi\right) T\left(y - \frac{h}{\widetilde{h}}\widetilde{y} - 3\left(s - \left(\frac{h}{\widetilde{h}}\right)^3 \widetilde{s}\right)\xi^2\right), \end{aligned} \tag{10}$$

where γ is a real number given by $h, y, s, \xi, \widetilde{h}, \widetilde{y}, \widetilde{s}, \widetilde{\xi}$. This identity is useful for linear profile decomposition (see Remark 4.2).

2.2. Generalized Morrey space

For $j \in \mathbb{Z}$, we set $\mathcal{D}_j := \{[k2^{-j}, (k+1)2^{-j}) \mid k \in \mathbb{Z}\}$. Let $\mathcal{D} := \cup_{j\in\mathbb{Z}} \mathcal{D}_j$. For a function $a : \mathcal{D} \to \mathbb{C}$, we denote $\|a\|_{\ell^r_{\mathcal{D}}} := (\sum_{I\in\mathcal{D}} |a(I)|^r)^{1/r}$ if $0 < r < \infty$ and $\|a\|_{\ell^\infty_{\mathcal{D}}} := \sup_{I\in\mathcal{D}} |a(I)|$.

Definition 2.1. For $1 \leqslant q \leqslant p < \infty$ and for $r \in (p, \infty]$, we introduce a *generalized Morrey norm* $\|\cdot\|_{M^p_{q,r}}$ by

$$\|f\|_{M^p_{q,r}} = \left\| |I|^{\frac{1}{p}-\frac{1}{q}} \|f\|_{L^q(I)} \right\|_{\ell^r_{\mathcal{D}}}.$$

Here, the case $p = q$ and $r < \infty$ is excluded. For $1 < p \leqslant q \leqslant \infty$ and for $r \in (p', \infty]$, we also introduce $\|f\|_{\hat{M}^p_{q,r}} := \|\hat{f}\|_{M^{p'}_{q',r}}$, i.e.,

$$\|f\|_{\hat{M}^p_{q,r}} = \left\| |I|^{\frac{1}{q}-\frac{1}{p}} \|\hat{f}\|_{L^{q'}(I)} \right\|_{\ell^r_{\mathcal{D}}}.$$

Banach spaces $M^p_{q,r}$ and $\hat{M}^p_{q,r}$ are defined as sets of tempered distributions of which above norms are finite, respectively.

Remark 2.2. (i) $M^p_{q,\infty}$ is a usual Morrey space. $M^p_{p,\infty} = L^p$ with equal norm.

(ii) For any $1 \leqslant q_2 \leqslant q_1 \leqslant p < \infty$ and $p < r_1 \leqslant r_2 \leqslant \infty$, it holds that $M^p_{q_1,r_1} \hookrightarrow M^p_{q_2,r_2}$.

(iii) For any $1 < p \leqslant q_1 \leqslant q_2 \leqslant \infty$ and $p' < r_1 \leqslant r_2 \leqslant \infty$, it holds that $\hat{M}^p_{q_1,r_1} \hookrightarrow \hat{M}^p_{q_2,r_2}$.

(iv) $L^p \hookrightarrow M^p_{q,r}$ holds as long as $1 \leqslant q < p < r \leqslant \infty$.

(v) $\hat{L}^p \hookrightarrow \hat{M}^p_{q,r}$ holds as long as $1 \leqslant q' < p' < r \leqslant \infty$.

For the last two assertions, see Proposition A.1.

Lemma 2.3. *Let $1 < p \leqslant q \leqslant \infty$ and let $r \in (p', \infty]$. There exists $C \geqslant 1$ such that $C^{-1}\|f\|_{\hat{M}^p_{q,r}} \leqslant \|D_p(h)A(s)T(y)P(\xi)f\|_{\hat{M}^p_{q,r}} \leqslant C\|f\|_{\hat{M}^p_{q,r}}$ for any $f \in \hat{M}^p_{q,r}$ and any $(h, \xi, s, y) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$. Further, if $\xi = 0$ then the above inequality holds with $C = 1$.*

Proof of Lemma 2.3. We only consider $q > 1$. Notice that

$$|\mathcal{F}D_p(h)A(s)T(y)P(\xi)f|(x) = h^{-\frac{1}{p'}} |\mathcal{F}f| \left(\frac{x}{h} + \xi\right).$$

Therefore, for any $\tau^j_k = [k/2^j, (k+1)/2^j) \in \mathcal{D}_j$ we have

$$|\tau^j_k|^{\frac{1}{p'}-\frac{1}{q'}} \left\|\mathcal{F}D_p(h)A(s)T(y)P(\xi)f\right\|_{L^{q'}(\tau^j_k)} = |\widetilde{\tau}^j_k|^{\frac{1}{p'}-\frac{1}{q'}} \|\mathcal{F}f\|_{L^{q'}(\widetilde{\tau}^j_k)},$$

where $\widetilde{\tau}^j_k = \left[\frac{k}{h2^j} + \xi, \frac{k+1}{h2^j} + \xi\right)$. Denote $h = 2^{j_0}$. We choose $k_0 = k_0(j)$ so that $k_0 \leqslant 2^{j+j_0}\xi < k_0 + 1$. Then, $\widetilde{\tau}^j_k \subset \tau^{j+j_0}_{k+k_0} \cup \tau^{j+j_0}_{k+k_0+1}$ and $|\widetilde{\tau}^j_k| = |\tau^{j+j_0}_{k+k_0}| = |\tau^{j+j_0}_{k+k_0+1}|$. Thus,

$$\begin{aligned} &|\widetilde{\tau}^j_k|^{\frac{1}{p'}-\frac{1}{q'}} \|\mathcal{F}f\|_{L^{q'}(\widetilde{\tau}^j_k)} \\ &\leqslant |\tau^{j+j_0}_{k+k_0}|^{\frac{1}{p'}-\frac{1}{q'}} \|\mathcal{F}f\|_{L^{q'}(\tau^{j+j_0}_{k+k_0})} + |\tau^{j+j_0}_{k+k_0+1}|^{\frac{1}{p'}-\frac{1}{q'}} \|\mathcal{F}f\|_{L^{q'}(\tau^{j+j_0}_{k+k_0+1})}. \end{aligned}$$

We take ℓ^r_k norm and then ℓ^r_j norm to obtain the second inequality with $C = 2$. It is obvious that if $\xi = 0$ then $\widetilde{\tau}^j_k = \tau^{j+j_0}_k$ and $|\tau^j_k|/h = |\tau^{j+j_0}_k|$ hold and so we can take $C = 1$. The first inequality follows in a similar way. We repeat the same argument from $|\mathcal{F}f|(y) = |\mathcal{F}D(h)A(s)T(y)P(\xi)f|(hy - h\xi)$. □

2.3. Generalized Strichartz' estimates

In this subsection we give a generalized Strichartz' estimates for the Airy equation. To this end, we introduce several notations.

Definition 2.4. (i) A pair $(s, r) \in \mathbb{R} \times [1, \infty]$ is said to be *acceptable* if $1/r \in [0, 3/4)$ and

$$s \in \begin{cases} [-\frac{1}{2r}, \frac{2}{r}] & 0 \leqslant \frac{1}{r} \leqslant \frac{1}{2}, \\ (\frac{2}{r} - \frac{5}{4}, \frac{5}{2} - \frac{3}{r}) & \frac{1}{2} < \frac{1}{r} < \frac{3}{4}. \end{cases}$$

(ii) A pair $(s, r) \in \mathbb{R} \times [1, \infty]$ is said to be *conjugate-acceptable* if $(1 - s, r')$ is acceptable, where $\frac{1}{r'} = 1 - \frac{1}{r} \in [0, 1]$.

For an interval $I \subset \mathbb{R}$ and an acceptable pair (s, r), we define a function space $X(I; s, r)$ of space–time functions with the following norm

$$\|f\|_{X(I;s,r)} = \left\| |\partial_x|^s f \right\|_{L_x^{p(s,r)}(\mathbb{R}; L_t^{q(s,r)}(I))},$$

where the exponents $p(s, r)$ and $q(s, r)$ are given by

$$\frac{2}{p(s,r)} + \frac{1}{q(s,r)} = \frac{1}{r}, \quad -\frac{1}{p(s,r)} + \frac{2}{q(s,r)} = s. \tag{11}$$

We refer $X(I; s, r)$ to as an $\hat{L}^r$*-admissible space*.

For an interval $I \subset \mathbb{R}$ and a conjugate-acceptable pair (s, r), we define a function space $Y(I; s, r)$ by the norm

$$\|f\|_{Y(I;s,r)} = \left\| |\partial_x|^s f \right\|_{L_x^{\tilde{p}(s,r)}(\mathbb{R}; L_t^{\tilde{q}(s,r)}(I))},$$

where the exponents $\tilde{p}(s, r)$ and $\tilde{q}(s, r)$ are given by

$$\frac{2}{\tilde{p}(s,r)} + \frac{1}{\tilde{q}(s,r)} = 2 + \frac{1}{r}, \quad -\frac{1}{\tilde{p}(s,r)} + \frac{2}{\tilde{q}(s,r)} = s. \tag{12}$$

Let us define some specific $X(I; s, r)$ and $Y(I; s, r)$ type spaces by choosing specific degrees $s = s(r)$.

Definition 2.5. Set $s(L) = s(L, \alpha) := 1/(3\alpha)$, $s(S) = s(S, \alpha) := 0$, $s(K) = s(K, \alpha) := \frac{5}{2} - \frac{3}{\alpha} - \varepsilon$, and $s(Z) = s(Z, r) := \frac{2}{\alpha} - \frac{5}{4} + \varepsilon$, where $\varepsilon > 0$ is a sufficiently small number. For $W = L, S, K, Z$, we define $W(I) := X(I; s(W), \alpha)$. Also define $N(I) := Y(I; s(L), \alpha)$. We use the notation $(p(W), q(W)) := (p(s(W), \alpha), q(s(W), \alpha))$ for $W = L, S, K, Z$ and $(\widetilde{p}(N), \widetilde{q}(N)) := (\widetilde{p}(s(L), \alpha), \widetilde{q}(s(L), \alpha))$.

From the definition, we have $(p(S), q(S)) = (\frac{5}{2}\alpha, 5\alpha)$ and $(p(L), q(L)) = (3\alpha, 3\alpha)$. For details of choice of $s(Z)$ and $s(K)$, see Remark 4.12 below.

Remark 2.6. The $S(I)$ norm is so-called *scattering norm*. This norm plays an important role on well-posedness theory. For example, criterions for blowup and scattering are given in terms of the scattering norm (see [46, Theorems 1.8 and 1.9]). Notice that the pair $(0, \alpha)$ is admissible only if $\alpha > 8/5$. The $L(I)$ norm is a non-mixed space. This norm appears in refinement of Stein–Tomas type inequality, see Lemma 5.9, below. A pair $(s_L(\alpha), \alpha)$ is acceptable and conjugate-acceptable if $5/3 \leqslant \alpha < 20/9$. Remark that there exists an acceptable and conjugate-acceptable pair under a weaker assumption $10/7 < \alpha < 10/3$ (see [46, Remark 4.1]).

We have the following generalized version of Strichartz' estimate.

Proposition 2.7 *(Generalized Strichartz' estimates).*

(i) (homogeneous estimate) It holds for any acceptable pair (s, r) *and interval* I *that*

$$\|e^{-t\partial_x^3} f\|_{X(I;s,r)} \leqslant C \|f\|_{\hat{L}^r}, \tag{13}$$

where the constant C *depends only on* s *and* r.

(ii) (inhomogeneous estimate) Let $4/3 < r < 4$. *Let* (s_1, r) *be an acceptable pair and let* (s_2, r) *be a conjugate-acceptable pair. Then, it holds for any* $t_0 \in I \subset \mathbb{R}$ *that*

$$\left\| \int_{t_0}^{t} e^{-(t-t')\partial_x^3} \partial_x F(t') dt' \right\|_{L_t^\infty(I;\hat{L}_x^r) \cap X(I,s_1,r)} \leqslant C \|F\|_{Y(I,s_2,r)}, \tag{14}$$

where the constant C depends on s_1, s_2 and r.

Proof of Proposition 2.7. The inequality (13) is obtained by interpolating the notable Kato's smoothing effect, the Kenig–Ruiz estimate and the Stein–Tomas inequality. See [46, Proposition 2.1] for the detail. Moreover, the inhomogeneous estimate (14) follows from the combination of the homogeneous inequality (13) and the Christ–Kiselev lemma. See [46, Proposition 2.5] for the detail. □

To handle $X(I; s, r)$ and $Y(I; s, r)$ spaces, the following lemma is useful.

Lemma 2.8. *Let $1 < p_i, q_i < \infty$ and $s_i \in \mathbb{R}$ for $i = 1, 2$. Let p, q, s satisfy*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}, \quad s = \theta s_1 + (1-\theta) s_2$$

for some $\theta \in (0, 1)$. Then, there exists a positive constant C such that the inequality $\||\partial_x|^s f\|_{L_x^p L_t^q} \leqslant C \||\partial_x|^{s_1} f\|_{L_x^{p_1} L_t^{q_1}}^{\theta} \||\partial_x|^{s_2} f\|_{L_x^{p_2} L_t^{q_2}}^{1-\theta}$ holds for any f such that $|\partial_x|^{s_1} f \in L_x^{p_1} L_t^{q_1}$ and $|\partial_x|^{s_2} f \in L_x^{p_2} L_t^{q_2}$.

Proof of Lemma 2.8. See [46, Lemma 3.3]. □

To evaluate the nonlinear term, we need the following lemma.

Lemma 2.9. *Suppose that $8/5 < \alpha < 10/3$. Let (s, r) be a pair which is acceptable and conjugate-acceptable. Then, the following two assertions hold:*

(i) If $u \in S(I) \cap X(I; s, r)$ then $|u|^{2\alpha} u \in Y(I; s, r)$. Moreover, there exists a positive constant C such that the inequality

$$\||u|^{2\alpha} u\|_{Y(I;s,r)} \leqslant C \|u\|_{S(I)}^{2\alpha} \|u\|_{X(I;s,r)}$$

holds for any $u \in S(I) \cap X(I; s, r)$.

(ii) There exists a positive constant C such that the inequality

$$\begin{aligned} &\||u|^{2\alpha} u - |v|^{2\alpha} v\|_{Y(I;s,r)} \\ &\leqslant C(\|u\|_{X(I;s,r)} + \|v\|_{X(I;s,r)})(\|u\|_{S(I)} + \|v\|_{S(I)})^{2\alpha-1} \|u - v\|_{S(I)} \\ &\quad + C(\|u\|_{S(I)} + \|v\|_{S(I)})^{2\alpha} \|u - v\|_{X(I;s,r)} \end{aligned}$$

holds for any $u, v \in S(I) \cap X(I; s, r)$.

Proof of Lemma 2.9. See [46, Proposition 3.4]. □

3. Stability estimates

3.1. Stability for gKdV

We consider the generalized KdV equation with the perturbation:

$$\begin{cases} \partial_t \tilde{u} + \partial_x^3 \tilde{u} = \mu \partial_x (|\tilde{u}|^{2\alpha} \tilde{u}) + \partial_x e, & t, x \in \mathbb{R}, \\ \tilde{u}(\hat{t}, x) = \tilde{u}_0(x), & x \in \mathbb{R}, \end{cases} \tag{15}$$

where the perturbation e is small in a suitable sense and the initial data $\tilde{u}_0$ is close to u_0.

Although the estimates in this section are restricted to $5/3 \leqslant \alpha < 20/9$, one can easily extend the results for $8/5 < \alpha < 10/3$ by modifying the definitions of $L(I)$ and $N(I)$ spaces. See Remark 2.6 for the meaning of the above restriction on α.

Lemma 3.1 *(Short time stability for gKdV). Assume $5/3 \leqslant \alpha < 20/9$ and $\hat{t} \in \mathbb{R}$. Let I be a time interval containing $\hat{t}$ and let $\tilde{u}$ be a solution to (15) on $I \times \mathbb{R}$ for some function e. Then, there exists $\varepsilon_0 > 0$ such that if $\tilde{u}$ and e satisfy $\|\tilde{u}\|_{S(I)} + \|\tilde{u}\|_{L(I)} \leqslant \varepsilon_0$, and*

$$\|e^{-(t-\hat{t})\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{S(I)} + \|e^{-(t-\hat{t})\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{L(I)} + \|e\|_{N(I)} \leqslant \varepsilon,$$

and if $0 < \varepsilon < \varepsilon_0$ hold, then there exists a unique solution $u \in S(I) \cap L(I)$ to (gKdV) satisfying

$$\|u - \tilde{u}\|_{S(I)} + \|u - \tilde{u}\|_{L(I)} \leqslant C\varepsilon, \tag{16}$$

$$\||u|^{2\alpha}u - |\tilde{u}|^{2\alpha}\tilde{u}\|_{N(I)} \leqslant C\varepsilon. \tag{17}$$

If further $u(\hat{t}) - \tilde{u}(\hat{t}) \in \hat{L}^\alpha$ holds then

$$\|u - \tilde{u}\|_{L_t^\infty(I;\hat{L}_x^\alpha)} \leqslant \|u(\hat{t}) - \tilde{u}(\hat{t})\|_{\hat{L}_x^\alpha} + C\varepsilon. \tag{18}$$

Proof of Lemma 3.1. By the local well-posedness theory, it suffices to show (16), (17), and (18) as a priori estimates. Let $w := u - \tilde{u}$. Then w satisfies

$$w(t) = e^{-(t-\hat{t})\partial_x^3}w(\hat{t}) + \mu \int_{\hat{t}}^{t} e^{-(t-t')\partial_x^3}\partial_x\{|\tilde{u} + w|^{2\alpha}(\tilde{u} + w) - |\tilde{u}|^{2\alpha}\tilde{u}\}dt'$$
$$- \int_{\hat{t}}^{t} e^{-(t-t')\partial_x^3}\partial_x e(t')dt'.$$

For $t \in I$, set

$$F(t) := \|w\|_{S(0,t)} + \|w\|_{L(0,t)},$$
$$G(t) := \||\tilde{u} + w|^{2\alpha}(\tilde{u} + w) - |\tilde{u}|^{2\alpha}\tilde{u}\|_{N(0,t)},$$

where we use abbreviation such as $S(0,t) = S([0,t))$ to simplify notation. Then the assumptions on $u(\hat{t})$, $\tilde{u}(\hat{t})$ and e, and Proposition 2.7 (14) lead us to

$$\begin{aligned} F(t) &\leqslant \|e^{-(t-\hat{t})\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{S(0,t)} + \|e^{-(t-\hat{t})\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{L(0,t)} \\ &\quad + CG(t) + C\|e\|_{N(0,t)} \\ &\leqslant C\varepsilon + CG(t). \end{aligned}$$

Lemma 2.9 (ii) yields

$$\begin{aligned} G(t) &\leqslant C(\|\tilde{u} + w\|_{L(0,t)} + \|\tilde{u}\|_{L(0,t)}) \\ &\qquad \times (\|\tilde{u} + w\|_{S(0,t)} + \|\tilde{u}\|_{S(0,t)})^{2\alpha-1}\|w\|_{S(0,t)} \\ &\quad + C(\|\tilde{u} + w\|_{S(0,t)} + \|\tilde{u}\|_{S(0,t)})^{2\alpha}\|w\|_{L(0,t)} \\ &\leqslant C(\varepsilon_0 + F(t))^{2\alpha}F(t). \end{aligned} \tag{19}$$

Hence, $F(t) \leqslant C\varepsilon + C\varepsilon_0^{2\alpha}F(t) + CF(t)^{2\alpha+1}$. Since $F(0) = 0$, by the continuity argument, we have that if $C\varepsilon_0^{2\alpha} < 1$, then $F(t) \leqslant C\varepsilon$ for any $t \in I$. Hence we have (16). Combining (16) and (19), we have (17).

Now we suppose that $w(\hat{t}) = u(\hat{t}) - \tilde{u}(\hat{t}) \in \hat{L}^\alpha$. Then, Proposition 2.7 (13) and (14) yield

$$\begin{aligned} &\|w\|_{L_t^\infty(I)\hat{L}_x^\alpha} \\ &\quad \leqslant \|w(\hat{t})\|_{\hat{L}_x^\alpha} + C\||\tilde{u} + w|^{2\alpha}(\tilde{u} + w) - |\tilde{u}|^{2\alpha}\tilde{u}\|_{N(I)} + C\|e\|_{N(I)} \\ &\quad \leqslant \|w(\hat{t})\|_{\hat{L}_x^\alpha} + C\varepsilon. \end{aligned}$$

Hence we have (18). This competes the proof of Lemma 3.1. □

Proposition 3.2 *(Long time stability for gKdV). Assume $5/3 \leqslant \alpha < 20/9$ and $\hat{t} \in \mathbb{R}$. Let $I \subset \mathbb{R}$ be an interval containing $\hat{t}$. Let $\tilde{u}$ be a solution to (15) on $I \times \mathbb{R}$ for some function e. Assume that $\tilde{u}$ satisfies*

$$\|\tilde{u}\|_{S(I)} + \|\tilde{u}\|_{L(I)} \leqslant M,$$

for some $M > 0$. Then there exists $\varepsilon_1 = \varepsilon_1(M) > 0$ such that if

$$\|e^{-t\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{S(I)} + \|e^{-t\partial_x^3}(u(\hat{t}) - \tilde{u}(\hat{t}))\|_{L(I)} + \|e\|_{N(I)} \leqslant \varepsilon$$

and $0 < \varepsilon < \varepsilon_1$, then there exists a solution u to (gKdV) on $I \times \mathbb{R}$ satisfies

$$\|u - \tilde{u}\|_{S(I)} + \|u - \tilde{u}\|_{L(I)} \leqslant C\varepsilon, \tag{20}$$

$$\||u|^{2\alpha}u - |\tilde{u}|^{2\alpha}\tilde{u}\|_{N(I)} \leqslant C\varepsilon, \tag{21}$$

where the constant C depends only on M. Further, if $u(\hat{t}) - \tilde{u}(\hat{t}) \in \hat{L}^\alpha$ for some $\hat{t} \in I$ then, it also holds that

$$\|u - \tilde{u}\|_{L_t^\infty(I;\hat{L}_x^\alpha)} \leqslant \|u(\hat{t}) - \tilde{u}(\hat{t})\|_{\hat{L}^\alpha} + C\varepsilon. \tag{22}$$

Proof of Proposition 3.2. The proof is the combination of Lemma 3.1 and an iterative procedure. Without loss of generality, we may assume that $\hat{t} = 0$ and $\inf I = 0$. Now let ε_0 be the constant given in Lemma 3.1. We first show the following claim: There exists a positive integer $N \leqslant 1 + (2M/\varepsilon_0)^{q(S)}$ such that $I = \bigcup_{j=1}^N I_j$, $I_j = [t_{j-1}, t_j]$ with the property $\|\tilde{u}\|_{S(I_j)} + \|\tilde{u}\|_{L(I_j)} \leqslant \varepsilon_0$ for any $1 \leqslant j \leqslant N$. Suppose $M > \varepsilon_0$, otherwise there is nothing to prove. Take $t_1 \in I$ so that $t_0 < t_1$ and $\|\tilde{u}\|_{S(I_1)} + \|\tilde{u}\|_{L(I_1)} = \varepsilon_0$. Similarly, as long as $\|\tilde{u}\|_{S((t_{j-1}, \sup I))} + \|\tilde{u}\|_{L((t_{j-1}, \sup I))} > \varepsilon_0$ we define $t_j \in I$ so that $t_{j-1} < t_j$ and $\|\tilde{u}\|_{S(I_j)} + \|\tilde{u}\|_{L(I_j)} = \varepsilon_0$. Now we show that $N \leqslant 1 + (2M/\varepsilon_0)^{q(S)}$ by the contradiction argument. Suppose that $1 + (2M/\varepsilon_0)^{q(S)} < N \leqslant \infty$. Let N' be an integer defined by $N' = N$ if N is finite and N' any integer satisfying $1 + (2M/\varepsilon_0)^{q(S)} < N'$ if N is infinite.

For $W = S, L$ and $1 \leqslant j \leqslant N'$, let us introduce two functions $f_{W,j}(x) := \||\partial_x|^{s(W)}\tilde{u}(\cdot, x)\|_{L_t^{q(W)}(I_j)}$. Then

$$M \geqslant \left\| \left(\|\tilde{u}(\cdot, x)\|^{q(W)}_{L_t^{q(W)}((0,t_{N'}))} \right)^{\frac{1}{q(W)}} \right\|_{L_x^{p(W)}} = \left\| \left(\sum_{j=1}^{N'} |f_{W,j}(x)|^{q(W)} \right)^{\frac{1}{q(W)}} \right\|_{L_x^{p(W)}}.$$

Noting $p(S) < q(S)$ and $p(L) = q(L)$, by the above inequality and the Hölder inequality, we obtain

$$\begin{aligned}
\varepsilon_0 N' &= \sum_{W=S,L} \sum_{j=1}^{N'} \|\tilde{u}\|_{W(I_j)} \leqslant \sum_{W=S,L} (N')^{1-\frac{1}{p(W)}} \left(\sum_{j=1}^{N'} \|\tilde{u}\|^{p(W)}_{W(I_j)} \right)^{\frac{1}{p(W)}} \\
&= \sum_{W=S,L} (N')^{1-\frac{1}{p(W)}} \left\| \left(\sum_{j=1}^{N'} |f_{W,j}(x)|^{p(W)} \right)^{\frac{1}{p(W)}} \right\|_{L_x^{p(W)}} \\
&\leqslant \sum_{W=S,L} (N')^{1-\frac{1}{q(W)}} \left\| \left(\sum_{j=1}^{N'} |f_{W,j}(x)|^{q(W)} \right)^{\frac{1}{q(W)}} \right\|_{L_x^{p(W)}} \\
&\leqslant \sum_{W=S,L} (N')^{1-\frac{1}{q(W)}} M.
\end{aligned}$$

Since $q(L) > q(S)$, we obtain $N' \leqslant 1 + (2M/\varepsilon_0)^{q(L)}$. This contradicts the definition of N', which proves the claim.

Combining Lemma 3.1 and the argument by [44, Theorem 3.8], we have that if we choose $\varepsilon_1 = \varepsilon_1(M)$ sufficiently small, then we have (20) and (21). Finally if $w(0) \in \hat{L}^\alpha$, we use (21) to obtain

$$\begin{aligned}
\|w\|_{L^\infty(I,\hat{L}^\alpha)} &\leqslant \|w(0)\|_{\hat{L}^\alpha} + C \left\| |u|^{2\alpha}u - |\tilde{u}|^{2\alpha}\tilde{u} \right\|_{N(I)} + C\|e\|_{N(I)} \\
&\leqslant \|w(0)\|_{\hat{L}^\alpha} + C\varepsilon
\end{aligned}$$

This completes the proof of Proposition 3.2. □

3.2. A version of small data scattering

As a simple consequence of Proposition 3.2, we have the following result, which is Theorem 1.7.

Corollary 3.3. *Let $5/3 \leqslant \alpha < 20/9$. For any $M > 0$ there exists $\delta = \delta(M) > 0$ such that if $u_0 \in \hat{L}^\alpha$ satisfies $\|u_0\|_{\hat{L}^\alpha} \leqslant M$ and $\varepsilon := \|e^{-t\partial_x^3} u_0\|_{L(\mathbb{R})} \leqslant \delta$ then a corresponding solution $u(t)$ to (gKdV) exists globally and scatters for both time directions. Further, it holds that*

$$\|u\|_{S(\mathbb{R})} + \|u\|_{L(\mathbb{R})} \leqslant M + CM^{2\alpha}\varepsilon$$

for some constant C.

Proof of Corollary 3.3. We just apply Proposition 3.2 with $\widetilde{u}(t,x) = e^{-t\partial_x^3} u_0$, $I = \mathbb{R}$, and $\hat{t} = 0$. Remark that $\|\widetilde{u}\|_{S(\mathbb{R})} + \|\widetilde{u}\|_{L(\mathbb{R})} \leqslant C\|u_0\|_{\hat{L}^\alpha} \leqslant CM$ follows from (13) and by assumption. Further, $u(0) - \widetilde{u}(0) \equiv 0$ and

$$\|e\|_{N(\mathbb{R})} = \left\| |\widetilde{u}|^{2\alpha}\widetilde{u} \right\|_{N(\mathbb{R})} \leqslant C\|\widetilde{u}\|_{S(\mathbb{R})}^{2\alpha} \|\widetilde{u}\|_{L(\mathbb{R})} \leqslant CM^{2\alpha}\varepsilon \leqslant \varepsilon_1$$

for sufficiently small $\delta = \delta(M)$, where ε_1 is the constant given in Proposition 3.2. Therefore, the assumption of Proposition 3.2 is satisfied. □

4. Proof of main theorems

4.1. Two tools

For the proof of Theorem 1.2, we introduce the following two tools.

The first one is a linear profile decomposition for $\hat{L}^\alpha$-bounded sequences. Let us define a set of deformations as follows

$$G := \{D(h)A(s)T(y)P(\xi) \mid \Gamma = (h, \xi, s, y) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\}. \tag{23}$$

We often identify $\mathcal{G} \in G$ with a corresponding parameter $\Gamma \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ if there is no fear of confusion. Let us now introduce a notion of orthogonality between two families of deformations.

Definition 4.1. We say two families of deformations $\{\mathcal{G}_n\} \subset G$ and $\{\widetilde{\mathcal{G}}_n\} \subset G$ are *orthogonal* if corresponding parameters $\Gamma_n, \widetilde{\Gamma}_n \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ satisfies

$$\lim_{n\to\infty} \left(\left|\log \frac{h_n}{\widetilde{h}_n}\right| + \left|\xi_n - \frac{\widetilde{h}_n}{h_n}\widetilde{\xi}_n\right| + \left|s_n - \left(\frac{h_n}{\widetilde{h}_n}\right)^3 \widetilde{s}_n\right|(1+|\xi_n|) \right.$$
$$\left. + \left| y_n - \frac{h_n}{\widetilde{h}_n}\widetilde{y}_n - 3\left(s_n - \left(\frac{h_n}{\widetilde{h}_n}\right)^3 \widetilde{s}_n\right)(\xi_n)^2 \right| \right) = +\infty. \tag{24}$$

Remark 4.2. It follows from (10) that

$$(\widetilde{\mathcal{G}}_n)^{-1}\mathcal{G}_n = e^{i\gamma_n} D\left(\frac{h_n}{\widetilde{h}_n}\right) P\left(\xi_n - \frac{\widetilde{h}_n}{h_n}\widetilde{\xi}_n\right) A\left(s_n - \left(\frac{h_n}{\widetilde{h}_n}\right)^3 \widetilde{s}_n\right)$$
$$S\left(3\left(s_n - \left(\frac{h_n}{\widetilde{h}_n}\right)^3 \widetilde{s}_n\right)\xi_n\right) T\left(y_n - \frac{h_n}{\widetilde{h}_n}\widetilde{y}_n - 3\left(s_n - \left(\frac{h_n}{\widetilde{h}_n}\right)^3 \widetilde{s}_n\right)\xi_n^2\right),$$

where Γ_n and $\widetilde{\Gamma}_n$ are parameters associated with $\mathcal{G}_n$ and $\widetilde{\mathcal{G}}_n$, respectively, and γ_n is a real constant given by Γ_n and $\widetilde{\Gamma}_n$. Intuitively, the orthogonality given in Definition 4.1 implies at least one of the deformations in the right hand side produces bad behavior.

Theorem 4.3 *(Linear profile decomposition for "real valued" functions). Let $4/3 < \alpha < 2$ and $\alpha' < \sigma < \frac{6\alpha}{3\alpha-2}$. Let $u = \{u_n\}_n$ be a sequence of real-valued functions in B_M. Then, there exist $\psi^j \in B_M$, $r_n^j \in B_{(2j+1)M}$ and pairwise orthogonal families of deformations $\{\mathcal{G}_n^j\}_n \subset G$ $(j = 1, 2, \dots)$ parametrized by $\{\Gamma_n^j = (h_n^j, \xi_n^j, s_n^j, y_n^j)\}_n$ such that, extracting a subsequence in n,*

$$u_n = \sum_{j=1}^{l} \mathrm{Re}(\mathcal{G}_n^j \psi^j) + r_n^l \tag{25}$$

for all $l \geqslant 1$ and

$$\limsup_{n\to\infty} \left\| e^{-t\partial_x^3} r_n^l \right\|_{L(\mathbb{R})} \to 0 \tag{26}$$

as $l \to \infty$. For all $j \geqslant 1$,

$$\text{either} \quad \xi_n^j = 0, \ \forall n \geqslant 0 \quad \text{or} \quad \xi_n^j \to \infty \text{ as } n \to \infty.$$

Moreover, a decoupling inequality

$$\limsup_{n\to\infty} \ell(u_n) \geqslant \left(\sum_{j=1}^{J} c_j^{1-\sigma} \ell(\psi^j)^\sigma \right)^{1/\sigma} + \limsup_{n\to\infty} \ell(r_n^J) \tag{27}$$

holds for all $J \geqslant 1$, where

$$c_j = \begin{cases} 1 & \text{if } \xi_n^j \equiv 0, \\ 2 & \text{if } \xi_n^j \to \infty \text{ as } n \to \infty. \end{cases}$$

Furthermore, it holds for any j that

$$c_j \left\| \psi^j \right\|_{\hat{L}^\alpha} \leqslant \limsup_{n\to\infty} \|u_n\|_{\hat{L}^\alpha}. \tag{28}$$

The second tool to prove Theorem 1.2 is uniform boundedness of solutions with highly oscillating initial data. The assumption (8) is necessary for this boundedness.

Theorem 4.4. *Let $12/7 < \alpha < 2$. Assume (8). Let $\phi \in \hat{L}_x^\alpha(\mathbb{R})$ be a complex valued function such that $\ell(\phi) < 2^{1-\frac{1}{\sigma}} d_+$. Let $\{\xi_n\}_{n\geqslant 1} \subset (0, \infty)$ with $\xi_n \to \infty$ and let $\{t_n\}_{n\geqslant 1} \subset \mathbb{R}$ be such that $-3t_n\xi_n$ converges to some $T_0 \subset [-\infty, \infty]$. Then for n sufficiently large, a corresponding $\hat{L}^\alpha$-solution u_n to (gKdV) with the initial condition*

$$u_n(t_n, x) = A(t_n) Re[P(\xi_n)\phi(x)] \tag{29}$$

exists globally in time. Moreover, the solution u_n satisfies a uniform space–time bound

$$\|u_n\|_{S(\mathbb{R})} + \|u_n\|_{L(\mathbb{R})} \leqslant C, \tag{30}$$

where C is a positive constant depending only on ϕ, and for any $\varepsilon > 0$, there exist $N_\varepsilon \in \mathbb{N}$ and $\psi_\varepsilon \in C_c^\infty(\mathbb{R} \times \mathbb{R})$ such that

$$\|u_n(t, x) - Re[e^{-ix\xi_n - it\xi_n^3} \psi_\varepsilon(-3\xi_n t, x + 3\xi_n^2 t)]\|_{S(\mathbb{R})} < \varepsilon. \tag{31}$$

We postpone the proof of Theorems 4.3 and 4.4 to Sections 5 and 8, respectively.

4.2. Proof of Theorem 1.2

Step 1

Take a minimizing sequence $\{u_n\}_n$ as follows;

$$u_n \in B_M \setminus \mathcal{S}_+, \quad \ell(u_n) \leqslant d_+ + \frac{1}{n}. \tag{32}$$

We apply the linear profile decomposition theorem (Theorem 4.3) to the sequence $\{u_n\}_n$. Then, up to subsequence, we obtain a decomposition

$$u_n = \sum_{j=1}^{l} \mathrm{Re}(\mathcal{G}_n^j \psi^j) + r_n^l \tag{33}$$

for $n, l \geqslant 1$. By extracting subsequence and changing notations if necessary, we may assume that for each j and $\{x_n^j\}_{n,j} = \{\log h_n^j\}_{n,j}, \{t_n^j\}_{n,j}, \{y_n^j\}_{n,j}, \{3\xi_n^j t_n^j\}$, either $x_n^j \equiv 0$, $x_n^j \to \infty$ as $n \to \infty$, or $x_n^j \to -\infty$ as $n \to \infty$ holds.

Step 2

In this step and the next step, we shall show that $\psi^j \equiv 0$ except for at most one j.

Suppose not. Then, by means of (27), we have $c_j^{\frac{1}{\sigma}-1} \ell(\psi^j) < d_+$ for all j. Let us define $V_n^j(t,x)$ as follows:

- When $\xi_n \equiv 0$, we let $V_n^j(t) = D(h_n^j)T(y_n^j)\Psi^j((h_n^j)^3 t + t_n^j)$, where $\Psi^j(t)$ is a nonlinear profile associated with $(\mathrm{Re}\,\psi^j, t_n^j)$, that is,
  - if $t_n^j \equiv 0$ then $\Psi^j(t)$ is a solution to (gKdV) with $\Psi^j(0) = \mathrm{Re}\,\psi^j$;
  - if $t_n^j \to \infty$ as $n \to \infty$ then $\Psi^j(t)$ is a solution to (gKdV) that scatters forward in time to $e^{-t\partial_x^3} \mathrm{Re}\,\psi^j$;
  - if $t_n^j \to -\infty$ as $n \to \infty$ then $\Psi^j(t)$ is a solution to (gKdV) that scatters backward in time to $e^{-t\partial_x^3} \mathrm{Re}\,\psi^j$;
- When $\xi_n \to \infty$, we let $V_n^j(t) = D(h_n^j)T(y_n^j)\Psi_n^j((h_n^j)^3 t + t_n^j)$, where Ψ_n^j is a solution to (gKdV) with $\Psi_n^j(t_n^j) = A(t_n^j)\,\mathrm{Re}(P(\xi_n^j)\psi^j)$.

Let us show the following two lemmas.

Lemma 4.5 *(Uniform bound on the approximate solution). There exists $M > 0$ such that $\|V_n^j\|_{K(\mathbb{R}_+)} + \|V_n^j\|_{Z(\mathbb{R}_+)} \leqslant M$ holds for any $j, n \geqslant 1$.*

Proof of Lemma 4.5. The case $\xi_n^j \to \infty$ follows from Theorem 4.4. Hence, here we assume that $\xi_n^j \equiv 0$. Note that $c_j = 1$. Since the deformations $D(h_n^j)$ and $T(y_n^j)$ leave the left hand side invariant, it suffices to show that $\|\Psi^j\|_{K((t_n^j,\infty))} + \|\Psi^j\|_{Z((t_n^j,\infty))}$ is bounded uniformly in n. Since $\ell(\psi^j) < d_+$ by assumption, Ψ^j scatters forward in time. Hence, if $t_n^j \equiv 0$ or if $t_n^j \to \infty$ as $n \to \infty$ then

$$\left\|\Psi^j\right\|_{K((t_n^j,\infty))} + \left\|\Psi^j\right\|_{Z((t_n^j,\infty))} \leqslant \left\|\Psi^j\right\|_{K((0,\infty))} + \left\|\Psi^j\right\|_{Z((0,\infty))} < \infty$$

by scattering criterion. If $t_n \to -\infty$ as $n \to \infty$ then Ψ^j scatters for both time directions and so

$$\left\|\Psi^j\right\|_{K((t_n^j,\infty))} + \left\|\Psi^j\right\|_{Z((t_n^j,\infty))} \leqslant \left\|\Psi^j\right\|_{K(\mathbb{R})} + \left\|\Psi^j\right\|_{Z(\mathbb{R})} < \infty.$$

Hence, we obtain Lemma 4.5. □

Next lemma is concerned with the decoupling of the nonlinear profiles.

Lemma 4.6. *For any $j \neq k$, we have $\lim_{n\to\infty} \|V_n^j V_n^k\|_{L_x^{p(S)/2} L_t^{q(S)/2}(\mathbb{R})} = 0$, where $(p(S), q(S)) = (\frac{5}{2}\alpha, 5\alpha)$.*

By Theorem 4.4 (31), to prove Lemma 4.6, it suffices to show the following lemma.

Lemma 4.7. *Set*

$$\tilde{V}_n^j := \begin{cases} D(h_n^j)T(y_n^j)\Psi^j((h_n^j)^3 t + t_n^j, x) \qquad (\text{if } \xi_n^j \equiv 0), \\ D(h_n^j)T(y_n^j)\Psi^j(-3(\xi_n^j)[(h_n^j)^3 t + t_n^j], x + 3(\xi_n^j)^2[(h_n^j)^3 t + t_n^j]) \\ \qquad\qquad\qquad\qquad\qquad (\text{if } \xi_n^j \to \infty \text{ as } n \to \infty). \end{cases}$$

Then for any $j \neq k$, we have

$$\lim_{n\to\infty} \left\| \tilde{V}_n^j \tilde{V}_n^k \right\|_{L_x^{\frac{p(S)}{2}} L_t^{\frac{q(S)}{2}}(\mathbb{R})} = 0. \tag{34}$$

Proof of Lemma 4.7. The proof is now standard (see [30, Lemma 2.6] for instance), so we omit the detail. □

Lemma 4.8. *Let $F(z) = |z|^{2\alpha} z$. For any $\mathcal{J} \subset \mathbb{Z}_+$,*

$$\left\| F\left(\sum_{j\in\mathcal{J}} V_n^j\right) - \sum_{j\in\mathcal{J}} F(V_n^j) \right\|_{L_x^{\frac{p(S)}{2\alpha+1}} L_t^{\frac{q(S)}{2\alpha+1}}(\mathbb{R}_+)} = o(1)$$

as $n \to \infty$. Similarly,

$$\left\| F\left(\sum_{j\in\mathcal{J}} V_n^j\right) - \sum_{j\in\mathcal{J}} F(V_n^j) \right\|_{N(\mathbb{R}_+)} = o(1)$$

as $n \to \infty$.

Proof of Lemma 4.8. The former estimate is a consequence of Lemma 4.6. Indeed, we see that

$$\left| F\left(\sum_{j\in\mathcal{J}} V_n^j\right) - \sum_{j\in\mathcal{J}} F(V_n^j) \right| \leqslant C \left| \sum_{j_1,j_2\in\mathcal{J}, j_1\neq j_2} V_n^{j_1} V_n^{j_2} \right| \left| \sum_{j_3,j_4\in\mathcal{J}, j_3\neq j_4} V_n^{j_3} V_n^{j_4} \right| \left| \sum_{j_5,j_6\in\mathcal{J}, j_5\neq j_6} V_n^{j_5} V_n^{j_6} \right|^{\frac{2\alpha-3}{2}}.$$

Therefore, the Hölder inequality and Lemma 4.6 give us the desired estimate. Take a conjugate-acceptable pair $(s(N'), \alpha)$ so that $0 < s(N') - s(N) \ll 1$, and set $N'(I) := Y(I; s(N'), \alpha)$. By means of the interpolation estimate (Lemma 2.8), the latter estimate follows if we show that

$$\left\| \left(F\left(\sum_{j\in\mathcal{J}} V_n^j\right) - \sum_{j\in\mathcal{J}} F(V_n^j) \right) \right\|_{N'(\mathbb{R}_+)}$$

is bounded uniformly in n. When $s(N')$ is chosen sufficiently close to $s(N)$, we have $\|F(u)\|_{N'(\mathbb{R}_+)} \leqslant C \|u\|_{L(\mathbb{R}_+)\cap S(\mathbb{R}_+)}^{2\alpha+1}$ just as in the proof of Lemma 2.9. Therefore,

$$\begin{aligned} &\left\| \left(F\left(\sum_{j\in\mathcal{J}} V_n^j\right) - \sum_{j\in\mathcal{J}} F(V_n^j) \right) \right\|_{N'(\mathbb{R}_+)} \\ &\leqslant \left\| F\left(\sum_{j\in\mathcal{J}} V_n^j\right) \right\|_{N'(\mathbb{R}_+)} + \sum_{j\in\mathcal{J}} \left\| F(V_n^j) \right\|_{N'(\mathbb{R}_+)} \\ &\leqslant C \left\| \sum_{j\in\mathcal{J}} V_n^j \right\|_{L(\mathbb{R}_+)\cap S(\mathbb{R}_+)}^{2\alpha+1} + C \sum_{j\in\mathcal{J}} \left\| V_n^j \right\|_{S(\mathbb{R}_+)\cap S(\mathbb{R}_+)}^{2\alpha+1} \end{aligned}$$

$$\leqslant C \sum_{j\in\mathcal{J}} \left\| V_n^j \right\|_{L(\mathbb{R}_+)\cap S(\mathbb{R}_+)}^{2\alpha+1}.$$

The right hand side is bounded uniformly in n, thanks to Lemma 4.8, which completes the proof. □

Step 3

Here, we define an approximate solution

$$\tilde{u}_n^J(t,x) = \sum_{j=1}^{J} V_n^j(t,x) + e^{-t\partial_x^3} r_n^J, \tag{35}$$

where V_n^j is given in Step 2. To apply long time stability, we now check that $\tilde{u}_n^J$ satisfies the assumption.

Proposition 4.9 *(Asymptotic agreement at the initial time). Let $\tilde{u}_n^J$ and u_n be given by (35) and (32), respectively. Then it holds for any $J \geqslant 1$ that $\left\|\tilde{u}_n^J(0) - u_n\right\|_{\hat{L}^\alpha} \to 0$ as $n \to \infty$.*

Proof of Proposition 4.9. This follows from $V_n^j(0) - \mathcal{G}_n^j \psi^j \to 0$ in $\hat{L}^\alpha$ for each j, which is an immediate consequence of the way V_n^j are constructed. □

Proposition 4.10 *(Uniform bound on the approximate solution). There exists $M > 0$ such that $\left\|\tilde{u}_n^J\right\|_{K(\mathbb{R}_+)} + \left\|\tilde{u}_n^J\right\|_{Z(\mathbb{R}_+)} \leqslant M$ holds for any $J \geqslant 1$ and $n \geqslant N(J)$.*

Recall that each V_n^j ($j \geqslant 1$) is bounded in $K(0,\infty) \cap Z(0,\infty)$ uniformly in n (Lemma 4.5). Further, $e^{-t\partial_x^3} r_n^J$ is also bounded uniformly in $J, n \geqslant 1$. Hence, we shall show that there exists J_0 such that

$$\left\| \sum_{j=J_0+1}^{J_0+k} V_n^j \right\|_{K(\mathbb{R}_+)} + \left\| \sum_{j=J_0+1}^{J_0+k} V_n^j \right\|_{Z(\mathbb{R}_+)} \leqslant C$$

for any $k \geqslant 1$ and $n \geqslant N(k)$. To this end, we need the following.

Lemma 4.11. *For any $\varepsilon > 0$, there exists $J_0 = J_0(\varepsilon)$ such that*

$$\left\| \sum_{j=J_0+1}^{J_0+k} e^{-t\partial_x^3} V_n^j(0) \right\|_{L(\mathbb{R}_+)} + \left\| \sum_{j=J_0+1}^{J_0+k} e^{-t\partial_x^3} V_n^j(0) \right\|_{S(\mathbb{R}_+)} \leqslant \varepsilon$$

for any $k \geqslant 1$ and $n \geqslant N(k)$.

Proof of Lemma 4.11. By Proposition 4.9, it suffices to prove the estimate for $e^{-t\partial_x^3}\mathcal{G}_n^j \psi^j$ instead of $e^{-t\partial_x^3} V_n^j(0)$. By Theorem B.1 in Appendix B, we see that

$$\begin{aligned}\left\| \sum_{j=J_0+1}^{J_0+k} e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{L(\mathbb{R}_+)} &\leqslant C \left\| \sum_{j=J_0+1}^{J_0+k} \mathcal{G}_n^j \psi^j \right\|_{\hat{M}_{2,\sigma}^\alpha} \\ &\leqslant C \left(\sum_{j=J_0+1}^{J_0+k} \left\| \psi^j \right\|_{\hat{M}_{2,\sigma}^\alpha}^\sigma \right)^{1/\sigma} + o(1).\end{aligned}$$

Therefore, for any $\varepsilon > 0$, we can choose $J_0(\varepsilon)$ so that

$$\left\| \sum_{j=J_0+1}^{J_0+k} e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{L(\mathbb{R}_+)} \leqslant \frac{\varepsilon}{2}$$

for any $k \geqslant 1$ and $n \geqslant n(k)$.

On the other hand,

$$\left\| \sum_{j=J_0+1}^{J_0+k} e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{S(\mathbb{R}_+)}^{p(S)} \leqslant \sum_{j=J_0+1}^{J_0+k} \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{S(\mathbb{R}_+)}^{p(S)} + o(1)$$

as $n \to \infty$. Since

$$\begin{aligned} \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{S(\mathbb{R}_+)} &\leqslant C \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{L(\mathbb{R}_+)}^{\theta} \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{Z(\mathbb{R}_+)}^{1-\theta} \\ &\leqslant C \left\| \mathcal{G}_n^j \psi^j \right\|_{\hat{M}_{2,\sigma}^{\alpha}}^{\theta} \left\| \psi^i \right\|_{\hat{L}^{\alpha}}^{1-\theta} \leqslant C \left\| \mathcal{G}_n^j \psi^j \right\|_{\hat{M}_{2,\sigma}^{\alpha}}^{\theta}, \end{aligned}$$

where $\theta = -s(Z)/(s(L) - s(Z))$, one verifies that

$$\sum_{j=J_0+1}^{J_0+k} \left\| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \right\|_{S(\mathbb{R}_+)}^{p(S)} \leqslant C \sum_{j=J_0+1}^{\infty} \left\| \mathcal{G}_n^j \psi^j \right\|_{\hat{M}_{2,\sigma}^{\alpha}}^{\theta p(S)}.$$

The right hand side is bounded since $\theta p(S) > \sigma$. Hence, we can choose $J_1(\varepsilon)$ so that $\sum_{j=J_1+1}^{J_1+k} \| e^{-t\partial_x^3} \mathcal{G}_n^j \psi^j \|_{S(\mathbb{R}_+)} \leqslant \frac{\varepsilon}{2}$ for any $k \geqslant 1$ and $n \geqslant n(k)$. □

Remark 4.12. Our assumption $\alpha > 3/2 + \sqrt{7/60}$ comes from the condition $\theta p(S) > \sigma$ in this lemma. By letting $s(Z) \downarrow \frac{2}{\alpha} - \frac{5}{4}$, we have $\theta p(S) \to \frac{3\alpha(5\alpha-8)}{2(3\alpha-4)}$. This upper bound of σ, which restricts us to the above range of α with lower bound $\sigma > \alpha'$, is used only in this lemma, and all other arguments work with a weaker assumption $\sigma < 6\alpha/(3\alpha - 2)$. Here, we also remark on the choice of the space $Z(I)$. For any fixed $\alpha > 3/2 + \sqrt{7/60}$, we are able to choose σ so that $\alpha' < \sigma < \frac{3\alpha(5\alpha-8)}{2(3\alpha-4)}$. Then, we fix the space $Z(I)$ so that $\theta = -s(Z)/(s(L) - s(Z))$ satisfies $\theta p(S) > \sigma$.

We now prove Proposition 4.10.

Proof of Proposition 4.10. The integral equation that $W_n^k := \sum_{j=J_0+1}^{J_0+k} V_n^j$ satisfies is

$$W_n^k = e^{-t\partial_x^3} W_n^k(0) + \mu \int_0^t e^{-(t-s)\partial_x^3} \partial_x (|W_n^k|^{2\alpha} W_n^k + E_n^k) ds,$$

where $-E_n^k = |W_n^k|^{2\alpha} W_n^k - \sum_{j=J_0+1}^{J_0+k} |V_n^j|^{2\alpha} V_n^j$. Therefore,

$$\begin{aligned} \left\| W_n^k \right\|_{L(\mathbb{R}_+)\cap S(\mathbb{R}_+)} \leqslant{}& \left\| e^{-t\partial_x^3} W_n^k(0) \right\|_{L(\mathbb{R}_+)\cap S(\mathbb{R}_+)} + C \left\| W_n^k \right\|_{L(\mathbb{R}_+)\cap S(\mathbb{R}_+)}^{2\alpha+1} \\ &+ C \left\| E_n \right\|_{N(\mathbb{R}_+)}. \end{aligned}$$

Fix $\varepsilon > 0$. Thanks to Lemma 4.11, one can choose J_0 so that

$$\| e^{-t\partial_x^3} W_n^k(0) \|_{L(\mathbb{R}_+)\cap S(\mathbb{R}_+)} \leqslant \varepsilon$$

for any $k \geqslant 1$ and $n \geqslant N(k)$. Further, for this J_0, we have $\| E_n \|_{N(\mathbb{R}_+)} \leqslant \varepsilon$ for any $k \geqslant 1$ and $n \geqslant N(k)$. □

Proposition 4.13 *(Approximate solution to the equation). Let $\tilde{u}_n^J$ be defined by (35). Then $\tilde{u}_n^J$ is an approximate solution to (gKdV) in such a sense that*

$$\lim_{J\to\infty} \limsup_{n\to\infty} \left\| |\partial_x|^{-1} [(\partial_t + \partial_{xxx}) \tilde{u}_n^J - \mu \partial_x (|\tilde{u}_n^J|^{2\alpha} \tilde{u}_n^J)] \right\|_{N(\mathbb{R}_+)} = 0.$$

Proof of Proposition 4.13. First note that the identity

$$\begin{aligned}
&(\partial_t + \partial_{xxx})\tilde{u}_n^J - \mu\partial_x(|\tilde{u}_n^J|^{2\alpha}\tilde{u}_n^J) \\
&= \mu\sum_{j=1}^{J}\partial_x(|V_n^j|^{2\alpha}V_n^j) - \mu\partial_x(|\tilde{u}_n^J|^{2\alpha}\tilde{u}_n^J) \\
&= \mu\partial_x\left\{\sum_{j=1}^{J}(|V_n^j|^{2\alpha}V_n^j) - \left(\left|\sum_{j=1}^{J}V_n^j\right|^{2\alpha}\sum_{j=1}^{J}V_n^j\right)\right\} \\
&\quad + \mu\partial_x\left\{\left(\left|\sum_{j=1}^{J}V_n^j\right|^{2\alpha}\sum_{j=1}^{J}V_n^j\right) - (|\tilde{u}_n^J|^{2\alpha}\tilde{u}_n^J)\right\} =: \partial_x I_1 + \partial_x I_2.
\end{aligned}$$

Lemma 4.8 implies $\lim_{n\to\infty}\|I_1\|_{N(\mathbb{R}_+)} = 0$. Therefore, we only have to handle I_2. From Lemma 2.9 (ii) and Proposition 4.10, we have

$$\begin{aligned}
&\|I_2\|_{N(\mathbb{R}_+)} \\
&\quad \leqslant C(\|\tilde{u}_n^J\|_{L(\mathbb{R}_+)\cap S(\mathbb{R}_+)} + \|e^{-t\partial_x^3}r_n^J\|_{L(\mathbb{R}_+)\cap S(\mathbb{R}_+)})^{2\alpha}\left\|e^{-t\partial_x^3}r_n^J\right\|_{L(\mathbb{R}_+)\cap S(\mathbb{R}_+)} \\
&\quad \leqslant C\left\|e^{-t\partial_x^3}r_n^J\right\|_{L(\mathbb{R}_+)\cap S(\mathbb{R}_+)}
\end{aligned}$$

for any $n \geqslant N(J)$. By (26), $\lim_{J\to\infty}\limsup_{n\to\infty}\|e^{-t\partial_x^3}r_n^J\|_L = 0$ and

$$\begin{aligned}
\limsup_{n\to\infty}\left\|e^{-t\partial_x^3}r_n^J\right\|_{S(\mathbb{R}_+)} &\leqslant C\limsup_{n\to\infty}\left\|e^{-t\partial_x^3}r_n^J\right\|_{L(\mathbb{R}_+)}^{\theta}\left\|e^{-t\partial_x^3}r_n^j\right\|_{Z(\mathbb{R}_+)}^{1-\theta} \\
&\leqslant CM^{1-\theta}\limsup_{n\to\infty}\left\|e^{-t\partial_x^3}r_n^J\right\|_{L(\mathbb{R}_+)}^{\theta} \to 0
\end{aligned}$$

as $J\to\infty$. This yields $\lim_{n\to\infty}\|I_2\|_{N(\mathbb{R}_+)} = 0$. Hence we have the desired estimate. □

Now, we apply long time stability to see that $\|u_n\|_{S(\mathbb{R}_+)} < \infty$ for sufficiently large n. This implies that $u_n \in \mathcal{S}_+$, which contradicts with the definition of $\{u_n\}_n$.

Step 4

We now see that there exists j_0 such that $c_{j_0}^{\frac{1}{\sigma}-1}\ell(\psi^{j_0}) = d_+$. Then, one sees from the definition of $\{u_n\}_n$ and (27) that $\psi^j \equiv 0$ for $j \neq j_0$. For simplicity, we drop index j_0 and write $u_n = \mathcal{G}_n\psi + r_n$, $\tilde{u}_n(t) = V_n(t) + e^{-t\partial_x^3}r_n$ in what follows. Further, we have $\lim_{n\to\infty}\|r_n\|_{\hat{M}_{2,\sigma}^{\alpha}} = 0$ and so $\lim_{n\to\infty}\|e^{-t\partial_x^3}r_n\|_{K\cap Z} = 0$. When $|\xi_n| \to \infty$, as in the previous step, we see from assumption (8) and Theorem 4.4 that $u_n \in S_+$ for large n, a contradiction. Hence, $\xi_n \equiv 0$. Recall that $V_n = D(h_n)T(y_n)\Psi((h_n)^3 t + t_n)$, where $\Psi(t)$ is a nonlinear profile associated with (ψ, t_n). Let us now show that $u_c := \Psi$ is the solution which has the desired property. We have $\Psi(t_n) \notin \mathcal{S}_+$, otherwise $u_n \in \mathcal{S}_+$ for large n by long time stability.

The case $t_n \to \infty$ $(n\to\infty)$ is excluded since this implies $\Psi(t_n) \in \mathcal{S}_+$. If $t_n \equiv 0$ then $\Psi(0) = \psi$ and so $\ell(\Psi(0)) = d_+$. Finally, if $t_n \to -\infty$ as $n \to \infty$ then $\lim_{t\to-\infty} e^{t\partial_x^3}\Psi(t) = \psi$ and putting $u_{c,-} := \lim_{t\to-\infty} e^{t\partial_x^3}\Psi(t)$, we have $\ell(u_{c,-}) = d_+$. This completes the proof of Theorem 1.2. □

4.3. *Proof of Theorem 1.6*

The proof is essentially the same as for Theorem 1.2. We first take a minimizing sequence associated with $\tilde{d}_+$. Then, we apply Theorem 4.3. The difference is that uniform boundedness in $\hat{L}^{\tilde{\alpha}} \cap H^s$ gives us $h_n^j \equiv 1$ and $\xi_n^j \equiv 0$ (see Proposition 6.1). Thus, the assumption (8) is not necessary any longer since it is necessary just to exclude the case $\xi_n^j \to \infty$ via Theorem 4.4. Recall that $\tilde{B}_M \subset B_M$. Hence, the rest of the proof is the same. This completes the proof of Theorem 1.6.

5. Linear profile decomposition

In this section and the next section, we prove the linear profile decomposition (Theorem 4.3). In this section, we first prove a decomposition of sequence of *complex-valued* functions. We derive the desired decomposition for real-valued functions as a corollary in the next section.

To state the main result of this section. Recall the set of the deformations

$$G := \{D(h)A(s)T(y)P(\xi) \mid \Gamma = (h, \xi, s, y) \in 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}\}.$$

Remark 5.1. The set G plays a role of *a group of dislocations* in the sense of [53]. Remark that, however, G is not a group.

Theorem 5.2 *(Decomposition of "$\mathbb{C}$-valued" functions). Let $4/3 < \alpha < 2$ and $\sigma \in (\alpha', \frac{6\alpha}{3\alpha-2})$. Let $u = \{u_n\}_n$ be a bounded sequence of $\mathbb{C}$-valued functions in $\hat{L}^\alpha$. Then, there exist $\{\psi^j\}_j, \{r_n^j\}_{n,j} \subset \hat{L}^\alpha$ and pairwise orthogonal families $\{\mathcal{G}_n^j\}_n \subset G$ $(j = 1, 2, \dots)$ such that, up to subsequence,*

$$u_n = \sum_{j=1}^{l} \mathcal{G}_n^j \psi^j + r_n^l$$

for all $l \geqslant 1$ with

$$\limsup_{n\to\infty} \|e^{-t\partial_x^3} r_n^l\|_{L(\mathbb{R})} \to 0 \tag{36}$$

as $l \to \infty$. Further, the decouple inequality

$$\limsup_{n\to\infty} \ell(u_n)^\sigma \geqslant \sum_{j=1}^{\infty} \ell(\psi^j)^\sigma + \limsup_{n\to\infty} \ell(r_n^J)^\sigma$$

holds for any $J \geqslant 1$. Moreover, it holds that $\|\psi^j\|_{\hat{L}^\alpha} \leqslant \limsup_{n\to\infty} \|u_n\|_{\hat{L}^\alpha}$ for any j. Furthermore, each $x_n = \log h_n^j$, ξ_n^j, s_n^j, and y_n^j satisfies either $x_n = 0$ for all n, $x_n \to \infty$ as $n \to \infty$, or $x_n \to -\infty$ as $n \to \infty$.

As in [8], the proof splits into two parts. The first part, treated in Section 5.3 as Theorem 5.5, is the procedure of finding profiles and obtaining pairwise orthogonality between profiles. The smallness of the remainder term is given in an abstract form. As mentioned in the introduction, a decoupling equality (3) fails by the presence of multiplier-like deformations A and T, and so the main point of our decomposition is to establish a decoupling *inequality* with respect to $\ell(\cdot)$ (Lemma 5.4). The second part is concentration compactness (Theorem 5.7), which shows the abstract smallness of the remainder term obtained in the first step is sufficient for our use. The step also determines the suitable choice of the set of deformations.

5.1. A characterization of orthogonality

To begin with, we give a characterization of orthogonality of two families of deformations given in Definition 4.1.

Lemma 5.3 *(Characterization of orthogonality). Let $\mathcal{G}_n, \widetilde{\mathcal{G}}_n \in G$ be two families of deformations. The following three statements are equivalent:*

(i) $\mathcal{G}_n$ and $\widetilde{\mathcal{G}}_n$ are orthogonal.

(ii) It holds that $(\widetilde{\mathcal{G}}_n)^{-1}\mathcal{G}_n\psi \rightharpoonup 0$ weakly in $\hat{L}^\alpha$ as $n \to \infty$ for any $\psi \in \hat{L}^\alpha$.

(iii) For any subsequence of n_k there exists a sequence $u_k \in \hat{L}^\alpha$ such that, up to subsequence (of k), $(\mathcal{G}_{n_k})^{-1}u_k \rightharpoonup \psi \neq 0$ and $(\widetilde{\mathcal{G}}_{n_k})^{-1}u_k \rightharpoonup 0$ weakly in $\hat{L}^\alpha$ as $k \to \infty$.

Proof of Lemma 5.3. "(ii)⇒(iii)" is immediate by taking $u_k = (\mathcal{G}_{n_k})\psi$ for some $\psi \neq 0$.

We prove "(i)⇒(ii)". Remark that the stated weak convergence is equivalent to $\mathcal{F}(\widetilde{\mathcal{G}}_n)^{-1}\mathcal{G}_n\psi = (\hat{\widetilde{\mathcal{G}}}_n)^{-1}\hat{\mathcal{G}}_n\mathcal{F}\psi \rightharpoonup 0$ weakly in $L^{\alpha'}$ as $n \to \infty$. Set $h'_n = h_n/\widetilde{h}_n$, $\xi'_n = \xi_n - \widetilde{\xi}_n/h'_n$, $s'_n = s_n - (h'_n)^3\widetilde{s}_n$, and $y'_n = y_n - h'_n\widetilde{y}_n - 3s'_n\xi_n^2$. By density argument and (10), it suffices to show that

$$\int \mathbf{1}_K(\xi)[\hat{D}(h'_n)\hat{P}(\xi'_n)\hat{A}(s'_n)\hat{T}(y'_n)\hat{S}(3\xi_n s'_n)\mathbf{1}_L](\xi)d\xi \to 0$$

as $n \to \infty$ for any compact intervals K, L. The Hölder inequality shows the right hand side is bounded by $\min((h'_n)^{-1+\frac{1}{r}}|K|, (h'_n)^{\frac{1}{r}}|L|)$. Hence we have the conclusion when $|\log h'_n| \to \infty$ as $n \to \infty$. On the other hand, if $\limsup_{n\to\infty} |\log h'_n| < \infty$ and if $|\xi'_n| \to \infty$ as $n \to \infty$ then K and the support of $\hat{D}(h'_n)\hat{P}(\xi'_n)\mathbf{1}_L$ are disjoint for large n. These cases are acceptable. We hence assume that $\limsup_{n\to\infty} \left(|\log h'_n| + |\xi'_n|\right) < \infty$. Taking subsequence, we may suppose that $h'_n \to h' \in (0, \infty)$ and $\xi'_n \to \xi' \in \mathbb{R}$ as $n \to \infty$. Then, for any $f \in \hat{L}^\alpha$, $D(h'_n)P(\xi'_n)f$ converges to $D(h')P(\xi')f$ strongly in $\hat{L}^\alpha$. Since $D(h')P(\xi')$ is invertible, we need to show $A(s'_n)T(y'_n)S(3\xi_n\tau'_n)\psi \rightharpoonup 0$ weakly in $\hat{L}^\alpha$ as $n \to \infty$. To do so, it suffices to show $\int_L e^{i\Phi_n(\xi)}d\xi \to 0$ as $n \to \infty$ for any compact interval L, where $\Phi_n(\xi) := s'_n\xi^3 - 3s'_n\xi_n\xi^2 - y'_n\xi$. It is easily shown.

Let us proceed to the proof of "(iii)⇒(i)". Assume for contradiction that h'_n, ξ'_n, s'_n, $3s'_n\xi_n$, and y'_n are uniformly bounded. Then, there exists a subsequence n_k such that these parameters converge as $k \to \infty$. Denote the limits by h', ξ', s', τ', and y', respectively. By refining subsequence if necessary, we may suppose that $e^{i\gamma_{n_k}}$ also converges. In this case, for any $f \in \hat{L}^\alpha$ we have

$$(\widetilde{\mathcal{G}}_{n_k})^{-1}\mathcal{G}_{n_k}f \to e^{i\gamma}D(h')P(\xi')A(s')S(\tau')T(y')f \tag{37}$$

as $k \to \infty$ strongly in $\hat{L}^\alpha$. Now, suppose that there exists a subsequence of k, which we denote again by k, such that $(\mathcal{G}_{n_k})^{-1}u_k$ and $(G_{n_k})^{-1}u_k$ converge weakly in $\hat{L}^\alpha$ to ψ and 0, respectively, as $k \to \infty$. Since $(\mathcal{G}_{n_k})^{-1}u_k$ converges to ψ weakly in $\hat{L}^\alpha$, we see from (37) that

$$(\widetilde{\mathcal{G}}_{n_k})^{-1}u_k = ((\widetilde{\mathcal{G}}_{n_k})^{-1}\mathcal{G}_n)(\mathcal{G}_n)^{-1}u_n \rightharpoonup e^{i\gamma}D(h')P(\xi')A(s')S(\tau')T(y')\psi$$

weakly in $\hat{L}^\alpha$. On the other hand, $(\widetilde{\mathcal{G}}_{n_k})^{-1}u_k \rightharpoonup 0$ weakly in $\hat{L}^\alpha$ by assumption. Thanks to uniqueness of weak limit, we see that $\psi \equiv 0$, a contradiction. □

5.2. Decoupling inequality

We next prove a decoupling inequality for ℓ. The idea of the proof is to sum up the local (in the Fourier side) L^2 decoupling with respect to intervals.

Lemma 5.4 *(Decoupling inequality). Let $4/3 < \alpha < 2$ and $\alpha' < \sigma \leqslant \frac{6\alpha}{3\alpha-2}$. Let $\{u_n\}_n$ be a bounded sequence in $\hat{L}^\alpha$. Suppose that $\mathcal{G}_n^{-1}u_n$ converges to ψ weakly in $\hat{L}^\alpha$ as $n \to \infty$ with some $\{\mathcal{G}_n\}_n \subset G$. Set $r_n := u_n - \mathcal{G}_n\psi$. Then, for any $\gamma > 1$ and $\xi_0 \in \mathbb{R}$, it holds that*

$$\gamma \|P(\xi_0)u_n\|^\sigma_{\hat{M}^\alpha_{2,\sigma}} \geqslant \|P(\xi_0)\mathcal{G}_n\psi\|^\sigma_{\hat{M}^\alpha_{2,\sigma}} + \|P(\xi_0)r_n\|^\sigma_{\hat{M}^\alpha_{2,\sigma}} + o_\gamma(1) \tag{38}$$

as $n \to \infty$.

Proof of Lemma 5.4. We only consider the case $\xi_0 = 0$. The other cases handled in the same way because the presence of $P(\xi_0)$ causes merely a universal translation in the Fourier side. It is also clear from the proof that the small error term can be taken independently of ξ_0.

Denote $\mathcal{D} := \{\tau_k^l := [k/2^l, (k+1)/2^l) \mid k, l \in \mathbb{Z}\}$. For each $\tau_k^l \in \mathcal{D}$, we have the decoupling in L^2;

$$\|\mathcal{F}u_n\|^2_{L^2(\tau_k^l)} = \|\mathcal{F}\mathcal{G}_n\psi\|^2_{L^2(\tau_k^l)} + \|\mathcal{F}r_n\|^2_{L^2(\tau_k^l)} + 2\,\mathrm{Re}\,\langle\mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n\rangle_{\tau_k^l}.$$

Let $\gamma = \frac{m+1}{m}$, $m > 0$. By an elementary inequality $(a-b)^{\frac{\sigma}{2}} \geqslant (\frac{m}{m+1})^{\frac{\sigma-2}{2}}a^{\frac{\sigma}{2}} - m^{\frac{\sigma-2}{2}}b^{\frac{\sigma}{2}}$ for any $a \geqslant b \geqslant 0$ and $m > 0$ and by embedding $\ell^2_{\mathcal{D}} \hookrightarrow \ell^\sigma_{\mathcal{D}}$, it follows that

$$
\begin{aligned}
&\sum_{k,l\in\mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}u_n\|_{L^2(\tau_k^l)}^{\sigma} \\
&\geqslant \sum_{k,l\in\mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left(\|\mathcal{F}\mathcal{G}_n\psi\|_{L^2(\tau_k^l)}^2 + \|\mathcal{F}r_n\|_{L^2(\tau_k^l)}^2 - 2\left|\langle \mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n\rangle_{\tau_k^l}\right|\right)^{\frac{\sigma}{2}} \\
&\geqslant \left(\frac{m}{m+1}\right)^{\frac{\sigma-2}{2}} \left(\sum_{k,l\in\mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}\mathcal{G}_n\psi\|_{L^2(\tau_k^l)}^{\sigma} + \sum_{k,l\in\mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}r_n\|_{L^2(\tau_k^l)}^{\sigma}\right) \\
&\quad - 2^{\frac{\sigma}{2}} m^{\frac{\sigma-2}{2}} \sum_{k,l\in\mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left|\langle \mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n\rangle_{\tau_k^l}\right|^{\frac{\sigma}{2}}
\end{aligned}
$$

To obtain (38), it therefore suffices to show that

$$
R_n := \sum_{k,l\in\mathbb{Z}} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left|\langle \mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n\rangle_{\tau_k^l}\right|^{\frac{\sigma}{2}} \to 0 \tag{39}
$$

as $n \to \infty$. A computation shows that

$$
|I|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left|\langle \mathcal{F}\mathcal{G}_n\psi, \mathcal{F}r_n\rangle_I\right|^{\frac{\sigma}{2}} = |J_n|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left|\left\langle \mathcal{F}\psi, \mathcal{F}(\mathcal{G}_n)^{-1}r_n\right\rangle_{J_n}\right|^{\frac{\sigma}{2}}
$$

for any $I \subset \mathbb{R}$, where $J_n = I/h_n + \xi_n$ with the parameters h_n, ξ_n associated with $\mathcal{G}_n$. By changing notation if necessary, one sees that

$$
R_n = \sum_{k,l\in\mathbb{Z}} |\widetilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left|\left\langle \mathcal{F}\psi^1, \mathcal{F}(\mathcal{G}_n^1)^{-1}r_n^1\right\rangle_{\widetilde{\tau}_k^l}\right|^{\frac{\sigma}{2}},
$$

where $\widetilde{\tau}_k^l = \tau_k^l + 2^{-l}\sigma_n$ with some $0 \leqslant \sigma_n < 1$. Fix $\varepsilon > 0$. Since $\|\psi\|_{\hat{M}_{2,\sigma}^{\alpha}} \leqslant C\|\psi\|_{\hat{L}^{\alpha}} < \infty$, there exist $k_0(\varepsilon)$ and $l_0(\varepsilon)$ such that $D := \{|l| \leqslant l_0,\ |k| \leqslant k_0\} \subset \mathbb{Z}^2$ satisfies $\sum_{(k,l)\in\mathbb{Z}^2\setminus D} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \|\mathcal{F}\psi\|_{L^2(\tau_k^l)}^{\sigma} \leqslant \varepsilon$. It is obvious that $\widetilde{\tau}_k^l \subset \tau_k^l \cup \tau_{k+1}^l$ and $|\widetilde{\tau}_k^l| = |\tau_k^l| = |\tau_{k+1}^l|$ for each l, k. Hence, denoting $D' := \{|l| \leqslant l_0,\ |k| \leqslant k_0+1\} \subset \mathbb{Z}^2$, we have

$$
\begin{aligned}
&\sum_{(k,l)\in\mathbb{Z}^2\setminus D'} |\widetilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left\|\mathcal{F}\psi^1\right\|_{L^2(\widetilde{\tau}_k^l)}^{\sigma} \\
&\quad \leqslant \sum_{(k,l)\in\mathbb{Z}^2\setminus D'} |\widetilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left(\left\|\mathcal{F}\psi^1\right\|_{L^2(\tau_k^l)}^2 + \left\|\mathcal{F}\psi^1\right\|_{L^2(\tau_{k+1}^l)}^2\right)^{\sigma/2} \\
&\quad \leqslant 2^{\frac{\sigma}{2}} \sum_{(k,l)\in\mathbb{Z}^2\setminus D} |\tau_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left\|\mathcal{F}\psi^1\right\|_{L^2(\tau_k^l)}^{\sigma} \leqslant C\varepsilon.
\end{aligned}
$$

Then, by Schwartz' inequality,

$$
\begin{aligned}
&\sum_{(k,l)\in\mathbb{Z}^2\setminus D'} |\widetilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left|\left\langle \mathcal{F}\psi^1, \mathcal{F}(\mathcal{G}_n)^{-1}r_n^1\right\rangle_{\widetilde{\tau}_k^l}\right|^{\frac{\sigma}{2}} \\
&\quad \leqslant \left(\sum_{(k,l)\in\mathbb{Z}^2\setminus D'} |\widetilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left\|\mathcal{F}\psi^1\right\|_{L^2(\widetilde{\tau}_k^l)}^{\sigma}\right)^{\frac{1}{2}} \\
&\qquad \times \left(\sum_{(k,l)\in\mathbb{Z}^2\setminus D'} |\widetilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left\|\mathcal{F}(\mathcal{G}_n)^{-1}r_n^1\right\|_{L^2(\widetilde{\tau}_k^l)}^{\sigma}\right)^{\frac{1}{2}} \\
&\quad \leqslant C\varepsilon^{\frac{1}{2}} \left\|(\mathcal{G}_n)^{-1}r_n^1\right\|_{\hat{M}_{2,\sigma}^{\alpha}}^{\frac{\sigma}{2}} \leqslant C\varepsilon^{\frac{1}{2}} \left\|(\mathcal{G}_n)^{-1}r_n^1\right\|_{\hat{L}^{\alpha}}^{\frac{\sigma}{2}} = C\varepsilon^{\frac{1}{2}} \left\|r_n^1\right\|_{\hat{L}^{\alpha}}^{\frac{\sigma}{2}}.
\end{aligned}
$$

Remark that

$$\limsup_{n\to\infty} \|r_n\|_{\hat{L}^\alpha} \leqslant \limsup_{n\to\infty} \|u_n\|_{\hat{L}^\alpha} + \|\psi\|_{\hat{L}^\alpha} \leqslant 2\limsup_{n\to\infty} \|u_n\|_{\hat{L}^\alpha} \leqslant C.$$

Hence, the proof of (39) is reduced to showing

$$\sum_{(k,l)\in D'} |\widetilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left| \left\langle \mathcal{F}\psi^1, \mathcal{F}(\mathcal{G}_n)^{-1} r_n^1 \right\rangle_{\widetilde{\tau}_k^l} \right|^{\frac{\sigma}{2}} \to 0 \tag{40}$$

as $n \to \infty$. For $l \in [-l_0, l_0]$, set $f_n^l(x) := \left| \langle \mathcal{F}\psi^1, \mathcal{F}(\mathcal{G}_n)^{-1} r_n^1 \rangle_{[x,x+2^{-l}]} \right|$ with domain $x \in [-k_0/2^l, (k_0+1)/2^l]$. Then, there exists a constant $C = C(k_0, l_0) = C(\varepsilon)$ such that

$$\sum_{(k,l)\in D'} |\widetilde{\tau}_k^l|^{\sigma(\frac{1}{2}-\frac{1}{\alpha})} \left| \left\langle \mathcal{F}\psi^1, \mathcal{F}(\mathcal{G}_n)^{-1} r_n^1 \right\rangle_{\widetilde{\tau}_k^l} \right|^{\frac{\sigma}{2}}$$
$$\leqslant C(\varepsilon) \max_{l\in[-l_0,l_0]} \left(\sup_{x\in[-k_0/2^l,(k_0+1)/2^l]} f_n^l(x) \right).$$

Therefore, we obtain (40) if we show the uniform convergence

$$\sup_{x\in[-k_0/2^l,(k_0+1)/2^l]} f_n^l(x) \to 0 \tag{41}$$

as $n \to \infty$. Since $(\mathcal{G}_n)^{-1} r_n$ converges to zero weakly in $\hat{L}^\alpha$ as $n \to \infty$ by definition, $\lim_{n\to\infty} f_n(x) = 0$ follows for each x. Further, by the Hölder inequality,

$$|f_n^l(x+\delta) - f_n^l(x)|$$
$$\leqslant C \left(\sup_n \left\| (\mathcal{G}_n)^{-1} r_n^1 \right\|_{\hat{L}^\alpha} \right) \left\| \mathcal{F}\psi^1 \right\|_{L^\alpha([x,x+\delta]\cup[x+2^{-l},x+2^{-l}+\delta])}$$

for small $\delta > 0$. The right hand side is independent of n and tends to zero as $\delta \downarrow 0$. Therefore, $\{f_n^l\}_n$ is equicontinuous. By a similar argument, $\sup_{x\in[-k_0/2^l,(k_0+1)/2^l]} f_n^l(x)$ is bounded uniformly in n. Therefore, the Ascoli–Arzela theorem gives us the desired convergence (41). This completes the proof of Lemma 5.4. □

5.3. Decomposition procedure

For a bounded sequence $P = \{P_n\}_n \subset \hat{L}^\alpha$, we introduce a *set of weak limits modulo deformations*

$$\mathcal{V}(P) := \left\{ \phi \in \hat{L}^\alpha \;\middle|\; \begin{array}{l} \phi = \lim_{k\to\infty} \mathcal{G}_{n_k}^{-1} P_{n_k} \text{ weakly in } \hat{L}^\alpha, \\ \exists \mathcal{G}_n \in G, \exists \text{subsequence } n_k \end{array} \right\}.$$

and define $\eta(P) := \sup_{\phi\in\mathcal{V}(P)} \ell(\phi)$. By definition, $\eta(P) = 0$ implies that we may not find any weak limit from a sequence $\{P_n\}_n$ even modulo the orbit by deformations G. Conversely, if $\eta(P) > 0$ we can find a non-zero weak limit modulo G. The main result of this section is decomposition with a smallness of remainder with respect to η.

Theorem 5.5. *Let $4/3 < \alpha < 2$ and $\alpha' < \sigma \leqslant \frac{6\alpha}{3\alpha-2}$. Let $u = \{u_n\}_n$ be a bounded sequence of $\mathbb{C}$-valued functions in $\hat{L}^\alpha$. Then, there exist $\psi^j \in \mathcal{V}(u)$ and pairwise orthogonal families $\{\mathcal{G}_n^j\}_n \subset G$ ($j = 1, 2, \dots$) such that*

$$u_n = \sum_{j=1}^{l} \mathcal{G}_n^j \psi^j + r_n^l \tag{42}$$

for all $l \geqslant 1$ with $\eta(r^l) \to 0$ as $l \to \infty$. Further, a decoupling inequality

$$\limsup_{n\to\infty} \ell(u_n)^\sigma \geqslant \sum_{j=1}^{J} \ell(\psi^j)^\sigma + \limsup_{n\to\infty} \ell(r_n^J)^\sigma \tag{43}$$

holds for all $J \geqslant 1$. Further, it holds that

$$\left\| \psi^j \right\|_{\hat{L}^\alpha} \leqslant \limsup_{n\to\infty} \|u_n\|_{\hat{L}^\alpha} \tag{44}$$

and

$$\limsup_{n\to\infty} \left\| r_n^j \right\|_{\hat{L}^\alpha} \leqslant (j+1) \limsup_{n\to\infty} \|u_n\|_{\hat{L}^\alpha} \tag{45}$$

for any j. Furthermore, each $x_n = \log h_n^j$, ξ_n^j, s_n^j, and y_n^j satisfies either $x_n = 0$ for all n, $x_n \to \infty$ as $n \to \infty$, or $x_n \to -\infty$ as $n \to \infty$.

Remark 5.6. The important thing in decoupling inequality (43) is that the coefficient of each term in the right hand is equal to one. This is the reason why we work not with $\|\cdot\|_{\hat{M}^\alpha_{2,\sigma}}$ but with $\ell(\cdot)$.

The main technical issue of Theorem 5.5 is essentially settled with the above preliminaries and so now the theorem follows by a standard argument (see [8] and references therein). We give a proof in order to give details of the decoupling inequality (43).

Proof of Theorem 5.5. We may suppose $\eta(u) > 0$, otherwise the result holds with $\phi^j \equiv 0$ and $r_n^j = u_n$ for all $j \geqslant 1$. Then, we can choose $\psi^1 \in \mathcal{V}(u)$ so that $\ell(\psi^1) \geqslant \frac{1}{2}\eta(u)$ by definition of η. Then, by definition of $\mathcal{V}(u)$, one finds $\mathcal{G}_n^1 \in G$ such that $(\mathcal{G}_n^1)^{-1} u_n \rightharpoonup \psi^1$ weakly in $\hat{L}^\alpha$ as $n \to \infty$ up to subsequence. By extracting subsequence and changing notation if necessary, one may suppose that each $x_n = \log h_n^1$, ξ_n^1, s_n^1, and y_n^1 satisfies either $x_n = 0$ for all n, $x_n \to \infty$ as $n \to \infty$, or $x_n \to -\infty$ as $n \to \infty$. By lower semicontinuity of weak limit, we obtain (44) for $j = 1$. Define $r_n^1 := u_n - \mathcal{G}_n^1 \psi^1$. Then, it is obvious that $(\mathcal{G}_n^1)^{-1} r_n^1 \rightharpoonup \psi^1 - \psi^1 = 0$ weakly in $\hat{L}^\alpha$ as $n \to \infty$. The boundedness (45) for $j = 1$ is also obvious by (44). By Lemma 5.4,

$$\gamma \left\| P(\xi_0) u_n \right\|^\sigma_{\hat{M}^\alpha_{2,\sigma}} \geqslant \left\| P(\xi_0) \mathcal{G}_n^1 \psi^1 \right\|^\sigma_{\hat{M}^\alpha_{2,\sigma}} + \left\| P(\xi_0) r_n^1 \right\|^\sigma_{\hat{M}^\alpha_{2,\sigma}} + o_\gamma(1) \tag{46}$$

as $n \to \infty$ for any constant $\gamma > 1$ and $\xi_0 \in \mathbb{R}$. Since $\gamma > 1$ and ξ_0 are arbitrary, the decoupling inequality (43) holds for $J = 1$.

If $\eta(r^1) = 0$ then the proof is completed by taking $\psi^j \equiv 0$ for $j \geqslant 2$. Otherwise, we can choose $\psi^2 \in \mathcal{V}(r^1)$ so that $\ell(\psi^2) \geqslant \frac{1}{2}\eta(r^1)$. Then, as in the previous step, one can take $\mathcal{G}_n^2 \in G$ so that $(\mathcal{G}_n^2)^{-1} r_n^1 \rightharpoonup \psi^2$ weakly in $\hat{L}^\alpha$ as $n \to \infty$, up to subsequence. By extracting subsequence and changing notation if necessary, one may suppose that each $x_n = \log h_n^2$, ξ_n^2, s_n^2, and y_n^2 satisfies either $x_n = 0$ for all n, $x_n \to \infty$ as $n \to \infty$, or $x_n \to -\infty$ as $n \to \infty$. In particular, $\psi^2 \not\equiv 0$. Together with $\text{w-lim}_{n\to\infty} (\mathcal{G}_n^1)^{-1} r_n^1 = 0$ in $\hat{L}^\alpha$, Lemma 5.3 gives us that two families $\mathcal{G}_n^1$ and $\mathcal{G}_n^2$ are orthogonal. Then, $(\mathcal{G}_n^2)^{-1} u_n = (\mathcal{G}_n^2)^{-1} \mathcal{G}_n^1 \psi^1 + (\mathcal{G}_n^2)^{-1} r_n^1 \rightharpoonup 0 + \psi^2$ weakly in $\hat{L}^\alpha$ as $n \to \infty$. Hence, we obtain $\psi^2 \in \mathcal{V}(u)$ and so (44) for $j = 2$. Set $r_n^2 := r_n^1 - \mathcal{G}_n^2 \psi^2$. Then, (45) for $j = 2$ follows from

$$\limsup_{n\to\infty} \left\| r_n^2 \right\|_{\hat{L}^\alpha} \leqslant \limsup_{n\to\infty} \left\| r_n^1 \right\|_{\hat{L}^\alpha} + \left\| \psi^2 \right\|_{\hat{L}^\alpha} \leqslant 3 \limsup_{n\to\infty} \|u_n\|_{\hat{L}^\alpha}.$$

Further, one deduces from Lemma 5.4 that

$$\gamma \left\| P(\xi_0) r_n^1 \right\|^\sigma_{\hat{M}^\alpha_{2,\sigma}} \geqslant \left\| P(\xi_0) \mathcal{G}_n^2 \psi^2 \right\|^\sigma_{\hat{M}^\alpha_{2,\sigma}} + \left\| P(\xi_0) r_n^2 \right\|^\sigma_{\hat{M}^\alpha_{2,\sigma}} + o_\gamma(1)$$

as $n \to \infty$ for any $\gamma > 1$ and $\xi_0 \in \mathbb{R}$. This implies (43) for $J = 2$ with the help of (46).

Repeat this argument and construct $\psi^j \in \mathcal{V}(r^{j-1})$ and $\mathcal{G}_n^j \in G$, inductively. If we have $\eta(r^{j_0}) = 0$ for some j_0, then we define $\psi^j \equiv 0$ for $j \geqslant j_0 + 1$. In what follows, we may suppose that $\eta(r^j) > 0$ for all $j \geqslant 1$. In each step, r_n^j is defined by the formula $r_n^j = r_n^{j-1} - \mathcal{G}_n^j \psi^j$. The property (42) is obvious by construction.

Let us now prove that pairwise orthogonality. To this end, we demonstrate that $\mathcal{G}_n^j$ is orthogonal to $\mathcal{G}_n^k$ for $1 \leqslant k \leqslant j-1$. Since $(\mathcal{G}_n^j)^{-1} r_n^j \rightharpoonup \psi^j$ and $(\mathcal{G}_n^{j-1})^{-1} r_n^j \rightharpoonup 0$ in $\hat{L}^\alpha$ as $n \to \infty$, Lemma 5.3 implies that $\mathcal{G}_n^j$ and $\mathcal{G}_n^{j-1}$ are orthogonal. If $\mathcal{G}_n^j$ is orthogonal to $\mathcal{G}_n^k$ for $k_0 \leqslant k \leqslant j-1$ then Lemma 5.3 yields $(\mathcal{G}_n^j)^{-1} r_n^{k_0-1} = \sum_{k=k_0}^{j-1} (\mathcal{G}_n^j)^{-1} \mathcal{G}_n^k \psi^k +$

$(\mathcal{G}_n^j)^{-1} r_n^{j-1} \rightharpoonup \psi^j$ as $n \to \infty$. On the other hand, $(\mathcal{G}_n^{k_0-1})^{-1} r_n^{k_0-1} \rightharpoonup 0$ as $n \to \infty$. We therefore see from Lemma 5.3 that $\mathcal{G}_n^j$ and $\mathcal{G}_n^{k_0-1}$ are orthogonal. Hence, $\mathcal{G}_n^j$ is orthogonal to $\mathcal{G}_n^k$ for $1 \leqslant k \leqslant j-1$. Then, by (42) and by Lemma 5.3, we have $\psi^j \in \mathcal{V}(u)$, from which boundedness (44) and (45) follow.

To conclude the proof, we shall show $\lim_{j\to\infty} \eta(r^j) = 0$ and (43). Notice that the inductive construction gives us $\ell(\psi^{j+1}) \geqslant \frac{1}{2}\eta(r^j)$ for all $j \geqslant 1$ and

$$\begin{aligned} &\gamma \left\| P(\xi_0) r_n^j \right\|_{\hat{M}^\alpha_{2,\sigma}}^\sigma \\ &\quad \geqslant \left\| P(\xi_0) \mathcal{G}_n^{j+1} \psi^{j+1} \right\|_{\hat{M}^\alpha_{2,\sigma}}^\sigma + \left\| P(\xi_0) r_n^{j+1} \right\|_{\hat{M}^\alpha_{2,\sigma}}^\sigma + o_{\gamma,j}(1), \end{aligned} \tag{47}$$

as $n \to \infty$ for (fixed) $j \geqslant 1$ and any $\gamma > 1$ and $\xi_0 \in \mathbb{R}$. Combining (46) and (47) for $1 \leqslant j \leqslant J$, we have

$$\begin{aligned} \gamma^J \left\| P(\xi_0) u_n \right\|_{\hat{M}^\alpha_{2,\sigma}}^\sigma &\geqslant \sum_{j=1}^J \gamma^{J-j} \left\| P(\xi_0) \mathcal{G}_n^j \psi^j \right\|_{\hat{M}^\alpha_{2,\sigma}}^\sigma + \left\| P(\xi_0) r_n^J \right\|_{\hat{M}^\alpha_{2,\sigma}}^\sigma + o_{\gamma,J}(1) \\ &\geqslant \sum_{j=1}^J \gamma^{J-j} \ell(\psi^j)^\sigma + \ell(r_n^J)^\sigma + o_{\gamma,J}(1). \end{aligned}$$

Take first infimum with respect to ξ_0 and then limit supremum in n to obtain

$$\limsup_{n\to\infty} \ell(u_n)^\sigma \geqslant \sum_{j=1}^J \gamma^{-j} \ell(\psi^j)^\sigma + \gamma^{-J} \limsup_{n\to\infty} \ell(r_n^J)^\sigma.$$

Since $\gamma > 1$ is arbitrary, we obtain (43). This also implies $\lim_{j\to\infty} \eta(r^j) \leqslant 2 \lim_{j\to\infty} \ell(\psi^{j+1}) = 0$, which completes the proof of Theorem 5.5.

5.4. Concentration compactness

Let us proceed to the concentration compactness. Intuitively, the meaning of the concentration compactness here is as follows. Let us consider a bonded sequence $\{u_n\}_n \subset X$. Here, X is a Banach space. In addition to the boundedness with respect to X, we make an *additional assumption* on the sequence. If the additional assumption is so strong that it removes almost all possible deformations for $\{u_n\}_n$ with few exceptions, say G, then we can find a *non-zero* weak limit modulo G. In our case, $X = \hat{M}^\alpha_{2,\sigma}$ and we use (49) below as the additional assumption. It will turn out that this assumption removes almost all deformations. The exception is G given in (23). This is the reason why we use the set G of deformations in Theorems 4.3 or 5.5. The precise statement is as follows.

Theorem 5.7 *(Concentration compactness). Let $4/3 < \alpha < 2$ and $\alpha' < \sigma < \frac{6\alpha}{3\alpha-2}$. Let a bounded sequence $\{u_n\} \subset \hat{L}^\alpha$ satisfy*

$$\|u_n\|_{\hat{M}^\alpha_{2,\sigma}} \leqslant M \tag{48}$$

and

$$\|e^{-t\partial_x^3} u_n\|_{L(\mathbb{R})} \geqslant m \tag{49}$$

for some positive constants m, M. Then, there exist $\mathcal{G}_n \in G$ and $\psi \in \hat{L}^\alpha$ such that, up to subsequence, $\mathcal{G}_n^{-1} u_n \rightharpoonup \psi$ weakly in $\hat{L}^\alpha$ as $n \to \infty$ and $\|\psi\|_{\hat{M}^\alpha_{2,\sigma}} \geqslant \beta(m, M)$, where $\beta(m, M)$ is a positive constant depending only on m, M. In particular, $\eta(u) \geqslant C\beta(m, M)$ holds for some constant C.

Remark 5.8. We would emphasize that $\{u_n\}_n$ should be a bounded sequence of $\hat{L}^\alpha$ functions but the constant β is chosen independently of the value of $\limsup_{n\to\infty} \|u_n\|_{\hat{L}^\alpha}$. This respect is crucial to obtain Theorem 5.2 because an $\hat{L}^\alpha$-bound on r_n^J given in Theorem 5.5 is no more than (45).

We use an argument similar to [32]. See [47,8,55] for the decomposition in the L^2 case $\alpha = 2$. For the proof Theorem 5.7, we introduce a refinement of Strichartz estimate (13). Let P_N be the standard frequency cut-off operator to $|\xi| \sim N \in 2^{\mathbb{Z}}$.

Lemma 5.9 *(Refined Strichartz). Let* $4/3 < \alpha < 2$ *and* $\sigma \in (\alpha', 6\alpha/(3\alpha - 2))$. *Then,*

$$\|e^{-t\partial_x^3} f\|_{L(\mathbb{R})} \leqslant C \left(\sup_{N\in 2^{\mathbb{Z}}} \|e^{-t\partial_x^3} P_N f\|_{L(\mathbb{R})} \right)^{1-\frac{\sigma}{3\alpha}} \|f\|_{\hat{M}^\alpha_{2,\sigma}}^{\frac{\sigma}{3\alpha}}. \tag{50}$$

Proof of Lemma 5.9. By the square function estimate, we have

$$\left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} f \right\|_{L^{3\alpha}_{t,x}} \sim \left\| \sqrt{\sum_{N\in 2^{\mathbb{Z}}} |P_N |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} f|^2} \right\|_{L^{3\alpha}_{t,x}}. \tag{51}$$

As $4 < 3\alpha < 6$,

$$\begin{aligned}(\text{R.H.S. of (51)})^{3\alpha} &= \iint \prod_{k=1}^{3} \left(\sum_{N_k\in 2^{\mathbb{Z}}} |P_{N_k} |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} f|^2 \right)^{\frac{\alpha}{2}} dxdt \\ &\leqslant C \sum_{N_1\leqslant N_2\leqslant N_3} \iint \prod_{k=1}^{3} |P_{N_k} |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} f|^\alpha dxdt\end{aligned}$$

Since the summation over the case $N_1 = N_2 = N_3$ is handled easily, we may exclude the case and suppose $N_1 < N_3$ in what follows. By the Hölder inequality, the summand is bounded by

$$\begin{aligned}&\left(\sup_{N\in 2^{\mathbb{Z}}} \left\| P_N |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} f \right\|_{L^{3\alpha}_{t,x}} \right)^{3\alpha-\sigma} \left\| |\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} P_{N_2} f \right\|_{L^{3\alpha}_{t,x}}^{\sigma-\zeta} \\ &\qquad \times \left\| (|\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} P_{N_1} f)(|\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} P_{N_3} f) \right\|_{L^{\frac{3\alpha}{2}}_{t,x}}^{\frac{\zeta}{2}},\end{aligned}$$

where $\zeta := \min(\sigma, 2\alpha)$. By the refined Strichartz estimates (Proposition B.3 (B.2)), we see that the middle term is bounded by $C \| P_{N_2} f \|_{\hat{M}^\alpha_{2,\sigma}}^{\sigma-\zeta}$. By the bilinear Strichartz estimates (Proposition B.3), we have

$$\begin{aligned}&\left\| (|\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} P_{N_1} f)(|\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} P_{N_3} f) \right\|_{L^{\frac{3\alpha}{2}}_{t,x}} \\ &\quad \lesssim \left(\frac{N_1}{N_3} \right)^{\frac{1}{3\alpha}} (N_1^{-\frac{1}{3\alpha}} \|\hat{f}\|_{L^{(\frac{3\alpha}{2})'}(|\xi|\sim N_1)})(N_3^{-\frac{1}{3\alpha}} \|\hat{f}\|_{L^{(\frac{3\alpha}{2})'}(|\xi|\sim N_3)}).\end{aligned}$$

Put $a_N := N^{-1/3\alpha} \|\mathcal{F}(P_N f)\|_{L^{(3\alpha/2)'}(|\xi|\sim N)}$ and $b_N := \|P_N f\|_{\hat{M}^\alpha_{2,\sigma}}$ for $N \in 2^{\mathbb{Z}}$. It is easy to see that $\|a_N\|_{\ell^\sigma_N} + \|b_N\|_{\ell^\sigma_N} \leqslant C\|f\|_{\hat{M}^\alpha_{2,\sigma}}$. Then, we have

$$\begin{aligned}\sum_{N_1\leqslant N_2\leqslant N_3} a_{N_1}^{\frac{\zeta}{2}} b_{N_2}^{\sigma-\zeta} a_{N_3}^{\frac{\zeta}{2}} \left(\frac{N_1}{N_3} \right)^{\frac{\zeta}{6\alpha}} &\leqslant \sum_{1\leqslant L\leqslant M} M^{-\frac{\zeta}{6\alpha}} \sum_{N_1} a_{N_1}^{\frac{\zeta}{2}} b_{(LN_1)}^{\sigma-\zeta} a_{(MN_1)}^{\frac{\zeta}{2}} \\ &\leqslant C\, \|a_N\|_{\ell^\sigma_N}^{\zeta} \|b_N\|_{\ell^\sigma_N}^{\sigma-\zeta} \sum_{M\geqslant 1} M^{-\frac{\zeta}{6\alpha}} (1 + \log M) \leqslant C\, \|f\|_{\hat{M}^\alpha_{2,\sigma}}^{\sigma},\end{aligned}$$

which completes the proof. □

Proof of Theorem 5.7. By Lemma 5.9, the assumption of the theorem implies that there exists a sequence $\{N_n\} \subset 2^{\mathbb{Z}}$ such that

$$|N_n|^{\frac{1}{3\alpha}} \left\| P_{N_n} e^{-t\partial_x^3} u_n \right\|_{L^{3\alpha}_{t,x}} \geqslant C(M,m).$$

Take $\theta \in (0,1)$ sufficiently close to one. By Theorem B.1 and the embedding $\hat{M}^{\theta\alpha}_{2,\sigma} \hookrightarrow \hat{M}^{\theta\alpha}_{\frac{3}{2}\theta\alpha, 2(\frac{3}{2}\theta\alpha)'}$, we see

$$\begin{aligned}\left\| P_{N_n} e^{-t\partial_x^3} u_n \right\|_{L^{3\alpha}_{t,x}} &\leqslant \left\| P_{N_n} e^{-t\partial_x^3} u_n \right\|^{1-\theta}_{L^{\infty}_{t,x}} \left\| P_{N_n} e^{-t\partial_x^3} u_n \right\|^{\theta}_{L^{3\theta\alpha}_{t,x}} \\ &\leqslant C \left\| P_{N_n} e^{-t\partial_x^3} u_n \right\|^{1-\theta}_{L^{\infty}_{t,x}} \left(M N_n^{-\frac{4}{3\alpha\theta}+\frac{1}{\alpha}} \right)^{\theta}.\end{aligned}$$

Hence, we obtain $(N_n)^{-\frac{1}{\alpha}} \|e^{-t\partial_x^3} P_{N_n} u_n\|_{L^\infty_{t,x}} \geqslant C(M,m)$. With N_n given above, we set $v_n(x) := (N_n)^{1/\alpha} u_n(N_n x)$ to get $\|P_1 e^{-t\partial_x^3} v_n\|_{L^\infty_{t,x}} \geqslant C(M,m)$. Thus, there exists $(s_n, y_s) \in \mathbb{R}^2$ such that

$$|P_1 e^{s_n\partial_x^3} v_n|(-y_n) \geqslant C(M,m). \tag{52}$$

Let $\psi \in \hat{L}^\alpha$ be a weak limit of $T(-y_n)e^{s_n\partial_x^3} v_n$ along a subsequence. Then, by a standard argument, we conclude from (52) that $\|\psi\|_{\hat{M}^\alpha_{2,\sigma}} \geqslant \beta(M,m)$. □

5.5. *Proof of Theorem 5.2*

Plugging Theorem 5.7 to Theorem 5.5, we obtain a decomposition result.

Proof of Theorem 5.2. By means of Theorem 5.5, it suffices to show (36) as $l \to \infty$. Assume for contradiction that a sequence r_n^l given in Theorem 5.5 satisfies $\limsup_{l\to\infty} \limsup_{n\to\infty} \|e^{-t\partial_x^3} r_n^l\|_{L(\mathbb{R})} > 0$. Then, we can choose $m > 0$ and a subsequence l_k with $l_k \to \infty$ as $k \to \infty$ such that the assumption of Theorem 5.7 is fulfilled for each k. Then, Theorem 5.7 implies $\eta(r^{l_k}) \geqslant C\beta > 0$, which contradicts to $\lim_{l\to\infty} \eta(r^l) = 0$. □

A similar argument yields Theorem 1.9. We restate it in terms of η.

Theorem 5.10 *(Scattering due to irrelevant deformations). Let $\{u_{0,n}\}_n \subset \hat{L}^\alpha$ be a bounded sequence. Let $u_n(t)$ be a solution to (gKdV) with $u_n(0) = u_{0,n}$. If $\eta(\{u_{0,n}\}_n) = 0$ then there exists N_0 such that $u_n(t)$ is global and scatters for both time direction as long as $n \geqslant N_0$. Furthermore,*

$$\|u_n\|_{S(\mathbb{R})} + \|u_n\|_{L(\mathbb{R})} \leqslant 2 \limsup_{n\to\infty} \|u_{0,n}\|_{\hat{L}^\alpha}$$

for $n \geqslant N_0$.

Proof of Theorem 5.10. Just as in the proof of Theorem 5.2, we deduce from $\eta(\{u_{0,n}\}_n) = 0$ that $\lim_{n\to\infty} \||\partial_x|^{\frac{1}{3\alpha}} e^{-t\partial_x^3} u_{0,n}\|_{L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R})} = 0$ thanks to Theorem 5.7. Then, the result follows from Corollary 3.3. □

6. Two refinements of the profile decomposition

We consider two improvements of Theorem 5.2, under some additional assumptions.

6.1. *Decomposition of a sequence of real-valued functions*

The first one is the case when functions in a sequence are real-valued. This is nothing but the case of Theorem 4.3.

Proof of Theorem 4.3. In addition to the assumption of Theorem 5.2, we assume that u_n is real valued. We already have a decomposition

$$u_n = \sum_{j=1}^{J} \mathcal{G}_n^j \psi^j + r_n^J$$

by Theorem 5.2. We now show that this is rewritten as in (25). Fix j. If $\xi_n^j = 0$ for all n then $(\mathcal{G}_n^j)^{-1} u_n \rightharpoonup \psi^j$ in $\hat{L}^\alpha$ implies

$$A(s_n^j)^{-1} T(y_n^j)^{-1} D(h_n^j)^{-1} u_n \rightharpoonup \psi^j \quad \text{in } \hat{L}^\alpha.$$

Since the left hand side is real-valued, so is ψ^j. Hence, $\mathcal{G}_n^j \psi^j = \mathrm{Re}(\mathcal{G}_n^j \psi^j)$.

Next consider the case $\xi_n^j \to \infty$ as $n \to \infty$. Then, the convergence $\overline{(\mathcal{G}_n^j)^{-1} u_n} \rightharpoonup \overline{\psi^j}$ in $\hat{L}^\alpha$ implies

$$P(-\xi_n^j)^{-1} A(s_n^j)^{-1} T(y_n^j)^{-1} D(h_n^j)^{-1} u_n \rightharpoonup \overline{\psi^j} \quad \text{in } \hat{L}^\alpha.$$

Therefore, there exists k such that $\{\mathcal{G}_n^k\}_n$ is not orthogonal to the family $\{\overline{\mathcal{G}_n^j}\}_n := \{D(h_n^j)T(y_n^j)A(s_n^j)P(-\xi_n^j)\}_n$. Indeed, if not then the above convergence implies $\eta(r_n^J) \geqslant \|\overline{\psi^j}\|_{\hat{M}^\alpha_{2,\sigma}}$ for all $J \geqslant 1$, a contradiction. Then, one can replace $\{\mathcal{G}_n^k\}_n$ and ψ^k by $\{\overline{\mathcal{G}_n^j}\}_n$ and $\overline{\psi^j}$, respectively. Denoting $\psi^j/2$ again by ψ^j, we obtain the result. This is the reason why $c_j = 2$ when $|\xi_n^j| \to \infty$ as $n \to \infty$. This completes the proof of Theorem 4.3. □

6.2. Decomposition of a sequence with stronger boundedness

The second one is exclusion of deformations $D(h)$ and $P(\xi)$ under uniform boundedness in a stronger topologies. This is the key for Theorem 1.6.

Proposition 6.1. *(i) Under the assumptions in Theorem 5.2, assume in addition that $\{u_n\}_n$ is uniformly bounded in $\hat{L}^{\alpha_1} \cap \hat{L}^{\alpha_2}$ for some $1 < \alpha_1 < \alpha < \alpha_2 < \infty$. Then, the assertions of Theorem 5.2 hold with $h_n^j \equiv 1$. Furthermore, we have $\|\psi^j\|_{\hat{L}^\rho} \leqslant \limsup_{n\to\infty} \|u_n\|_{\hat{L}^\rho}$ for all $j \geqslant 1$ and $\alpha_1 \leqslant \rho \leqslant \alpha_2$.*

(ii) In addition to the assumption of Theorem 5.2, if $\{u_n\}_n$ is uniformly bounded in $\hat{L}^{\alpha_1} \cap \dot{H}^s$ for some $1 < \alpha_1 < \alpha$ and $s > 0$ then, the assertions of Theorem 5.2 hold with $h_n^j \equiv 1$, $\xi_n^j \equiv 0$. Furthermore, we have $\|\psi^j\|_{\dot{H}^s} \leqslant \limsup_{n\to\infty} \|u_n\|_{\dot{H}^s}$ for all $j \geqslant 1$.

Proof of Proposition 6.1. Suppose that u_n is uniformly bounded in $\hat{L}^{\alpha_1} \cap \hat{L}^{\alpha_2}$, $\alpha_1 < \alpha < \alpha_2$, and that $P(\xi_n)^{-1} A(s_n)^{-1} T(y_n)^{-1} D_\alpha(h_n)^{-1} u_n \rightharpoonup \psi$ in $\hat{L}^\alpha$ as $n \to \infty$ for some $\psi \not\equiv 0$. Then, for $g \in C^\infty$ such that $\hat{g}$ has a compact support,

$$\begin{aligned}
|(\psi, g)| &\leqslant 2 \left| \int (P(\xi_n)^{-1} A(s_n)^{-1} T(y_n)^{-1} D_\alpha(h_n)^{-1} u_n)(x) \overline{g(x)} dx \right| \\
&= 2 \left| \int u_n(x) \overline{(D_{\alpha'}(h_n) T(y_n) A(s_n) P(\xi_n) g)(x)} dx \right| \\
&\leqslant 2 \|u_n\|_{\hat{L}^{\alpha_1} \cap \hat{L}^{\alpha_2}} \|D_{\alpha'}(h_n) T(y_n) A(s_n) P(\xi_n) g\|_{\hat{L}^{\alpha_1'} + \hat{L}^{\alpha_2'}} \\
&\leqslant C \|D_{\alpha'}(h_n) T(y_n) A(s_n) P(\xi_n) g\|_{\hat{L}^{\alpha_1'} + \hat{L}^{\alpha_2'}}
\end{aligned}$$

for large n. If $h_n \to 0$ as $n \to \infty$ then

$$\left\| D_{p'}(h_n) T(y_n) A(s_n) P(\xi_n) g \right\|_{\hat{L}^{\alpha_1'}} = (h_n)^{\frac{1}{\alpha_1} - \frac{1}{\alpha}} \|g\|_{\hat{L}^{\alpha_1'}} \to 0$$

as $n \to \infty$. Similarly, if $h_n \to \infty$ as $n \to \infty$ then

$$\|D_{\alpha'}(h_n) T(y_n) A(s_n) P(\xi_n) g\|_{\hat{L}^{\alpha_2'}} = (h_n)^{\frac{1}{\alpha_2} - \frac{1}{\alpha}} \|g\|_{\hat{L}^{\alpha_2'}} \to 0$$

as $n \to \infty$. In the both cases, we have $\psi \equiv 0$, a contradiction. Thus, we conclude that $|\log h_n|$ is bounded. Extracting subsequence, we have $h_n \to h_0 > 0$ as $n \to \infty$. Then,

$$\begin{aligned}
&P(h_n \xi_n)^{-1} A(s_n/(h_n)^3)^{-1} T(y_n/h_n)^{-1} u_n \\
&= D_\alpha(h_n) P(\xi_n)^{-1} A(s_n)^{-1} T(y_n)^{-1} D_\alpha(h_n)^{-1} u_n \rightharpoonup D_\alpha(h_0)\psi \quad \text{in } \hat{L}^p.
\end{aligned}$$

Hence, denoting $(h_n\xi_n, s_n/(h_n)^3, y_n/h_n)$ and $D_\alpha(h_0)\psi$ again by (ξ_n, s_n, y_n) and ψ, respectively, we may let $h_n \equiv 1$. Under the new notation, we have

$$\|\psi\|_{\hat{L}^\rho} \leqslant \limsup_{n\to\infty} \left\| P(\xi_n)^{-1} A(s_n)^{-1} T(y_n)^{-1} u_n \right\|_{\hat{L}^\rho} = \limsup_{n\to\infty} \|u_n\|_{\hat{L}^\rho}.$$

for all $\alpha_1 \leqslant \rho \leqslant \alpha_2$.

Next, let us suppose that u_n is bounded in $\hat{L}^{\alpha_1} \cap \dot{H}^s$ ($\alpha_1 < \alpha$, $s > 0$). Note that this implies u_n is bounded in L^2. Hence, the above argument gives us $h_n^j \equiv 1$ for all $j \geqslant 1$. Let us show $\xi_n^j \equiv 0$ for all $j \geqslant 1$. For $g \in C^\infty$ such that $\hat{g}$ has a compact support, we have

$$\begin{aligned}
|(\psi, g)| &\leqslant 2 \left| \int (P(\xi_n)^{-1} A(s_n)^{-1} T(y_n)^{-1} u_n)(x) \overline{g(x)} dx \right| \\
&= 2 \left| \int u_n(x) \overline{(T(y_n) A(s_n) P(\xi_n) g)(x)} dx \right| \\
&\leqslant 2 \|u_n\|_{\dot{H}^s} \|T(y_n) A(s_n) P(\xi_n) g\|_{\dot{H}^{-s}} \leqslant C \|P(\xi_n) g\|_{\dot{H}^{-s}}
\end{aligned}$$

for large n. If $|\xi_n| \to \infty$ as $n \to \infty$ then $\|P(\xi_n)g\|_{\dot{H}^{-s}} \to 0$ as $n \to \infty$. This gives us $\psi \equiv 0$, a contradiction. Hence, ξ_n is bounded. By extracting subsequence, $\xi_n \to \xi_0 \in \mathbb{R}$ as $n \to \infty$. Then,

$$A(s_n)^{-1} T(y_n)^{-1} u_n = P(\xi_n) P(\xi_n)^{-1} A(s_n)^{-1} T(y_n)^{-1} u_n \rightharpoonup P(\xi_0)\psi \quad \text{in } \hat{L}^p.$$

Thus, denoting $P(\xi_0)\psi$ again by ψ, we may let $\xi_n \equiv 0$ and we have the bound $\|\psi\|_{\dot{H}^s} \leqslant \limsup_{n\to\infty} \|u_n\|_{\dot{H}^s}$. □

7. Quick review on well-posedness of (NLS)

In this section, we briefly summarize well-posedness and stability results for (NLS) which are need to prove Theorem 4.4.

7.1. Well-posedness for NLS

We first consider well-posedness for (NLS) in $\hat{L}^\alpha$-space and $\hat{M}^\alpha_{\rho,\sigma}$-space. The initial value problem (NLS) is formulated as

$$v(t) = e^{-it\partial_x^2} v_0 + i\mu \int_0^t e^{-i(t-t')\partial_x^2} (|v|^{2\alpha} v)(t') dt'. \tag{53}$$

The following well-posedness result plays an important role in this subsection. This kind of result is well known (see [9,25,61], for example).

Proposition 7.1. *Let $4/3 < \alpha < 4$. Then there exists a number $\delta > 0$ such that if a data $v_0 \in \mathcal{S}'$ and an interval $I \ni 0$ satisfies*

$$\|e^{-it\partial_x^2} v_0\|_{L^{3\alpha}_{t,x}(I\times\mathbb{R})} \leqslant \delta$$

then there exists a unique solution $v(t)$ to (53) which satisfies

$$\|v\|_{L^{3\alpha}_{t,x}(I\times\mathbb{R})} \leqslant 2\|e^{-it\partial_x^2} v_0\|_{L^{3\alpha}_{t,x}(I\times\mathbb{R})}.$$

Further, the solution belongs to $L^p_t(I, L^q_x)$ for any $p, q \in (2, \infty)$ with $2/p + 1/q = 1/\alpha$.

Proof of Proposition 7.1. Proposition 7.1 is an immediate consequence of the estimate

$$\|\Phi[v]\|_{L^{3\alpha}_{t,x}(I\times\mathbb{R})} \leqslant \|e^{-it\partial_x^2} v_0\|_{L^{3\alpha}_{t,x}(I\times\mathbb{R})} + C\|v\|^{2\alpha+1}_{L^{3\alpha}_{t,x}(I\times\mathbb{R})}$$

for $4/3 < \alpha < 4$, where $\Phi[v]$ is the right hand side of (53). This inequality follows from Strichartz' estimate for non-admissible pairs (see Kato [25] or Lemma 7.3 (ii), below), and Hölder inequality. □

Remark 7.2. Well-posedness of (53) in a space like $L_t^p(I; L_x^q)$ also holds for $\frac{1+\sqrt{17}}{4} < \alpha \leqslant 4/3$ if we allow the case $p \neq q$.

To prove well-posedness in $\hat{L}^\alpha$-space, we show the following generalized Strichartz estimate for the Schrödinger equation.

Lemma 7.3. *(i) (homogeneous estimates) Let I be an interval. Let (p, q) satisfy*

$$0 \leqslant \frac{1}{p} < \frac{1}{4}, \qquad 0 \leqslant \frac{1}{q} < \frac{1}{2} - \frac{1}{p}.$$

Then, for any $f \in \hat{L}^r$,

$$\| |\partial_x|^\tau e^{-it\partial_x^2} f \|_{L_x^p L_t^q(I)} \leqslant C \|f\|_{\hat{L}^r}, \tag{54}$$

where

$$\frac{1}{r} = \frac{2}{p} + \frac{1}{q}, \quad \tau = -\frac{1}{p} + \frac{1}{q}.$$

and positive constant C depends only on r and s.

(ii) (inhomogeneous estimates) Let $4/3 < r < 4$ and let (p_j, q_j) $(j = 1, 2)$ satisfy

$$0 \leqslant \frac{1}{p_j} < \frac{1}{4}, \qquad 0 \leqslant \frac{1}{q_j} < \frac{1}{2} - \frac{1}{p_j}.$$

Then, the inequalities

$$\left\| \int_0^t e^{-i(t-t')\partial_x^2} F(t') dt' \right\|_{L_t^\infty(I; \hat{L}_x^r)} \leqslant C_1 \| |\partial_x|^{-\tau_2} F \|_{L_x^{p_2'} L_t^{q_2'}(I)}, \tag{55}$$

$$\left\| |\partial_x|^{\tau_1} \int_0^t e^{-i(t-t')\partial_x^2} F(t') dt' \right\|_{L_x^{p_1} L_t^{q_1}(I)} \leqslant C_2 \| |\partial_x|^{-\tau_2} F \|_{L_x^{p_2'} L_t^{q_2'}(I)} \tag{56}$$

hold for any F satisfying $|\partial_x|^{-\tau_2} F \in L_x^{p_2'} L_t^{q_2'}$, where

$$\frac{1}{r} = \frac{2}{p_1} + \frac{1}{q_1}, \quad \tau_1 = -\frac{1}{p_1} + \frac{1}{q_1}, \quad \frac{1}{r'} = \frac{2}{p_2} + \frac{1}{q_2}, \quad \text{and} \quad \tau_2 = -\frac{1}{p_2} + \frac{1}{q_2},$$

where the constant C_1 depends on r, τ_1 and I, and the constant C_2 depends on r, τ_1, τ_2 and I.

Remark 7.4. Remark that we take a space–time norm of the form $L_x^p L_t^q$ in (54). This is why we gain derivative by $|\partial_x|^\tau$. Also remark that a similar estimate for a space–time norm of the form $L_t^p L_x^q$ is known in [23].

Proof of Lemma 7.3. The homogeneous estimate (54) is obtained by interpolating the Kato smoothing effect [27, Theorem 4.1], the Kenig–Ruiz estimate [27, Theorem 2.5] and the Stein–Tomas estimate for the Schrödinger equation [60]. The inhomogeneous estimates (55) and (56) follows from the homogeneous estimate (54) and the Christ–Kiselev lemma by [48, Lemma 2]. □

Inequality (54) and the following inequality yields the local well-posedness in $\hat{L}^\alpha$ and $\hat{M}^\alpha_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}$, respectively;

Proposition 7.5. *Assume that $\alpha > 4/3$. Then,*

$$\|e^{-it\partial_x^2} f\|_{L_{t,x}^{3\alpha}(\mathbb{R}\times\mathbb{R})} \leqslant C \|f\|_{\hat{M}^\alpha_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}}$$

holds for all $f \in \hat{M}^\alpha_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}$. Further, the embedding $\hat{L}^\alpha \hookrightarrow \hat{M}^\alpha_{\frac{3\alpha}{2}, 2(\frac{3\alpha}{2})'}$ holds if $\alpha > 4/3$.

The inequality is shown as in Theorem B.1 in Appendix B. The $\alpha = 2$ case is given in [1,8]. Now, let us see how the well-posedness results are deduced. If either $v_0 \in \hat{M}^{\alpha}_{\frac{3\alpha}{2},2(\frac{3\alpha}{2})'}$ or $v_0 \in \hat{L}^{\alpha}$ then the above inequalities imply that $\|e^{-it\partial_x^2} v_0\|_{L^{3\alpha}_{t,x}(I\times\mathbb{R})} \leqslant \delta$ holds at least for small interval $I = I(v_0)$. Then, we obtain a solution $u(t)$ on I belonging to $L^{3\alpha}_{t,x}(I \times \mathbb{R})$ thanks to Proposition 7.1. Further, by applying (55), we see that

$$\|\Phi[v] - e^{-it\partial_x^2} v_0\|_{L^\infty_t(I,\hat{L}^\alpha_x)} \leqslant C \|v\|^{2\alpha+1}_{L^{3\alpha}_{t,x}(I\times\mathbb{R})}.$$

Finally, the linear part $e^{-it\partial_x^2} v_0$ belongs to $C(\mathbb{R}; \hat{L}^\alpha)$ (resp. $C(\mathbb{R}; \hat{M}^{\alpha}_{\frac{3\alpha}{2},2(\frac{3\alpha}{2})'})$) if $v_0 \in \hat{L}^\alpha$ (resp. $v_0 \in \hat{M}^{\alpha}_{\frac{3\alpha}{2},2(\frac{3\alpha}{2})'}$). Thus, we obtain the following.

Proposition 7.6 *(Local well-posedness in $\hat{L}^\alpha$ and $\hat{M}^{\alpha}_{\frac{3\alpha}{2},2(\frac{3\alpha}{2})'}$). Let $4/3 < \alpha < 4$.*

(i) For any $u_0 \in \hat{L}^\alpha_x$, there exists a unique solution $u(t) \in C(I; \hat{L}^\alpha)$.

(ii) For any $u_0 \in \hat{M}^{\alpha}_{\frac{3\alpha}{2},2(\frac{3\alpha}{2})'}$, there exists a unique solution $u(t) \in C(I; \hat{M}^{\alpha}_{\frac{3\alpha}{2},2(\frac{3\alpha}{2})'})$. Furthermore, $u(t) - e^{-it\partial_x^2} u_0 \in C(I, \hat{L}^\alpha)$ holds.

Remark 7.7. It is obvious from the proof that a similar well-posedness result holds in all $\hat{M}^{\alpha}_{\rho,\sigma}$ space satisfying $\hat{L}^\alpha \hookrightarrow \hat{M}^{\alpha}_{\rho,\sigma} \hookrightarrow \hat{M}^{\alpha}_{\frac{3\alpha}{2},2(\frac{3\alpha}{2})'}$. Notice that the $\hat{M}^{\alpha}_{2,\sigma}$ space satisfies the above relation if $4/3 < \alpha < 2$ and $\alpha' < \sigma \leqslant 2(\frac{3\alpha}{2})' = 6\alpha/(3\alpha - 2)$. This is nothing but Theorem 1.12. On the other hand, the first assertion of the above proposition is Theorem 1.11.

As a corollary of this proposition, we obtain small data scattering in $\hat{M}^{\alpha}_{\frac{3\alpha}{2},2(\frac{3\alpha}{2})'}$.

Corollary 7.8. *Let $4/3 < \alpha < 4$. Assume that $v_0 \in \hat{L}^\alpha$ or $v_0 \in \hat{M}^{\alpha}_{\frac{3\alpha}{2},2(\frac{3\alpha}{2})'}$. There exists $\varepsilon > 0$ such that if $\hat{M}^{\alpha}_{\frac{3\alpha}{2},2(\frac{3\alpha}{2})'} < \varepsilon$ then $v_0 \in \mathcal{S}_{\mathrm{NLS}}$.*

7.2. Persistence of regularity for NLS

Next we show the persistent property of solution to (NLS).

Lemma 7.9 *(Persistence of $L^p_x L^q_t$- and $L^p_t L^q_x$-regularities). Let $4/3 < \alpha < 4$ and $s \geqslant 0$. Let $\hat{t} \in \mathbb{R}$ and let I be a time interval containing $\hat{t}$. Assume that $v \in C(I; \hat{L}^\alpha_x(\mathbb{R}))$ is a solution to (NLS) satisfying $\|v\|_{L^{3\alpha}_{t,x}(I\times\mathbb{R})} \leqslant M$ for some M. Then, the following two assertions hold:*

(i) If $|\partial_x|^s v(\hat{t}) \in \hat{L}^\alpha(\mathbb{R})$ then, for any

$$\tau \in \begin{cases} (\frac{1}{\alpha} - \frac{3}{4}, \frac{3}{2} - \frac{2}{\alpha}) & \text{if } \alpha < 2, \\ [-\frac{1}{2\alpha}, \frac{1}{\alpha}] & \text{if } \alpha \geqslant 2, \end{cases}$$

there exists a constant $C = C(\alpha, s, \tau, M)$ such that

$$\||\partial_x|^s v\|_{L^\infty_t \hat{L}^\alpha_x(I\times\mathbb{R})} + \||\partial_x|^{s+\tau} v\|_{L^p_x L^q_t(I)} \leqslant C \||\partial_x|^s v(\hat{t})\|_{\hat{L}^\alpha}, \tag{57}$$

holds, where (p, q) satisfies

$$\frac{1}{\alpha} = \frac{2}{p} + \frac{1}{q}, \qquad \tau = -\frac{1}{p} + \frac{1}{q}. \tag{58}$$

(ii) If $v(\hat{t}) \in \dot{H}^s(\mathbb{R})$ then, there exists $C = C(M)$ such that

$$\|v\|_{L^\infty_t(I;\dot{H}^s(\mathbb{R}))} + \||\partial_x|^s v\|_{L^p_t(I;L^q_x(\mathbb{R}))} \leqslant C\|v(\hat{t})\|_{\dot{H}^s}. \tag{59}$$

holds, where (p, q) satisfies

$$0 \leqslant \frac{1}{p} \leqslant \frac{1}{4}, \qquad \frac{1}{2} = \frac{2}{p} + \frac{1}{q}. \tag{60}$$

Proof of Lemma 7.9. Without loss of generality, we may assume that $\hat{t} = 0$ and $\inf I = 0$. We divide the time interval I into N subintervals such that

$$N \leqslant 1 + \left(\frac{M}{\eta}\right)^{3\alpha}, \qquad I = \bigcup_{j=1}^{N} I_j, \qquad I_j = [t_{j-1}, t_j]$$

with $\|v\|_{L^{3\alpha}_{t,x}(I_j \times \mathbb{R})} \leqslant \eta$ for any $1 \leqslant j \leqslant N$, where η is fixed later. Notice that such subdivision exists by the argument similar to the proof of Proposition 3.2.

We shall prove (57). To this end, we show

$$\||\partial_x|^s v\|_{L^\infty_t \hat{L}^\alpha_x(I_j \times \mathbb{R})} + \||\partial_x|^{s+\tau} v\|_{L^p_x L^q_t(I_j)} \leqslant C\||\partial_x|^s v(t_j)\|_{\hat{L}^\alpha} \tag{61}$$

for any $1 \leqslant j \leqslant N$, where p, q satisfy (58). We first consider the case $j = 1$. By Lemma 7.3, we have

$$\begin{aligned}
&\||\partial_x|^s v\|_{L^\infty_t \hat{L}^\alpha_x(I_j \times \mathbb{R})} + \||\partial_x|^{s+\tau} v\|_{L^p_x L^q_t(I_j)} + \||\partial_x|^s v\|_{L^{3\alpha}_{t,x}(I_j \times \mathbb{R})} \\
&\quad \leqslant C\||\partial_x|^s v(0)\|_{\hat{L}^\alpha} + C\||\partial_x|^s (|v|^{2\alpha} v)\|_{L^{\frac{3\alpha}{2\alpha+1}}_{t,x}(I_j)} \\
&\quad \leqslant C\||\partial_x|^s v(0)\|_{\hat{L}^\alpha} + C\|v\|^{2\alpha}_{L^{3\alpha}_{t,x}(I_j \times \mathbb{R})} \||\partial_x|^s v\|_{L^{3\alpha}_{t,x}(I_j \times \mathbb{R})} \\
&\quad \leqslant C\||\partial_x|^s v(0)\|_{\hat{L}^\alpha} + C\eta^{2\alpha} \||\partial_x|^s v\|_{L^{3\alpha}_{t,x}(I_j \times \mathbb{R})}.
\end{aligned}$$

Choosing η sufficiently small so that $C\eta^{2\alpha} < 1$, we have (61) for $j = 1$. In particular, we obtain $\||\partial_x|^s v(t_1)\|_{\hat{L}^\alpha} \leqslant C$. Hence a similar argument as above we have (61) for $j = 2$. Repeating this argument, we obtain (61) for any $1 \leqslant j \leqslant N$. Summing the inequalities (61) over all subintervals, we have (57).

The proof of (59) is done in a similar way. We use (usual) Strichartz' estimates instead. □

7.3. Stability for NLS

In this section we consider the nonlinear Schrödinger equation with the perturbation:

$$\begin{cases} i\partial_t \tilde{v} - \partial_x^2 \tilde{v} = -\mu|\tilde{v}|^{2\alpha}\tilde{v} + e, & t, x \in \mathbb{R}, \\ \tilde{v}(\hat{t}, x) = \tilde{v}_0(x), & x \in \mathbb{R} \end{cases} \tag{62}$$

with the perturbation e small in a suitable sense and the initial data $\tilde{v}_0$ close to v_0.

Proposition 7.10 *(Long time stability for NLS). Assume $4/3 < \alpha < 4$ and $\hat{t} \in \mathbb{R}$. Let I be a time interval containing $\hat{t}$ and let $\tilde{v}$ be a solution to (62) on $I \times \mathbb{R}$ for some function e. Assume that $\tilde{v}$ satisfies*

$$\|\tilde{v}\|_{L^{3\alpha}_{t,x}(I \times \mathbb{R})} \leqslant M,$$

for some $M > 0$. Then there exists $\varepsilon_1 = \varepsilon_1(M) > 0$ such that if $v(\hat{t})$ and $\tilde{v}(\hat{t})$ satisfy

$$\|e^{-i(t-\hat{t})\partial_x^2}(v(\hat{t}) - \tilde{v}(\hat{t}))\|_{L^{3\alpha}_{t,x}(I \times \mathbb{R})} + \|e\|_{L^{\frac{3\alpha}{2\alpha+1}}_{t,x}(I \times \mathbb{R})} \leqslant \varepsilon$$

and $0 < \varepsilon < \varepsilon_1$, then there exists a solution $v \in L^{3\alpha}_{t,x}(I \times \mathbb{R})$ to (NLS) on $I \times \mathbb{R}$ satisfies

$$\|v - \tilde{v}\|_{L^{3\alpha}_{t,x}(I \times \mathbb{R})} \leqslant C\varepsilon, \tag{63}$$

$$\||v|^{2\alpha} v - |\tilde{v}|^{2\alpha}\tilde{v}\|_{L^{\frac{3\alpha}{2\alpha+1}}_{t,x}(I \times \mathbb{R})} \leqslant C\varepsilon, \tag{64}$$

where the constant C depends on M. If, further, if $v(\hat{t}) - \tilde{v}(\hat{t}) \in \hat{L}^\alpha$ then

$$\|v - \tilde{v}\|_{L^\infty(I; \hat{L}^\alpha_x)} \leqslant \|v(\hat{t}) - \tilde{v}(\hat{t})\|_{\hat{L}^\alpha_x} + C\varepsilon. \tag{65}$$

Proof of Proposition 7.10. The proof follows from the argument similar to the proof of Proposition 3.2 or as in [43]. We omit the detail. □

8. Embedding NLS into gKdV

In this section, we prove Theorem 4.4. As we mentioned in Introduction, we prove existence of a global solution u_n to (gKdV) by constructing approximating solution via the solution to the one dimensional nonlinear Schrödinger equation

$$i\partial_t v - \partial_x^2 v = -\mu C_0 |v|^{2\alpha} v, \tag{66}$$

where

$$C_0 = \frac{2\Gamma(\alpha + \frac{3}{2})}{3\sqrt{\pi}\Gamma(\alpha+2)}.$$

With this constant, assumption (8) is written as $d_+ < 2^{\frac{1}{\sigma}-1}(C_0)^{-\frac{1}{2\alpha}} d_{\mathrm{NLS}}$. Let v be a solution to (66) with the following conditions;

$$\begin{cases} v(T_0) = e^{-iT_0\partial_x^2}\phi & \text{if } |T_0| < \infty, \\ \lim_{t\to T_0} \|v(t) - e^{-it\partial_x^2}\phi\|_{\hat{L}_x^\alpha} = 0 & \text{if } T_0 = \pm\infty. \end{cases} \tag{67}$$

We now claim that v global and scatters for both time direction. Let us begin with the case $T_0 \in \mathbb{R}$. Remark that if v solves (66) then $(C_0)^{\frac{1}{2\alpha}} v$ solves (NLS). Hence, assumption of the theorem yields

$$\|(C_0)^{\frac{1}{2\alpha}}\phi\|_{\hat{M}^\alpha_{2,\sigma}} < 2^{1-\frac{1}{\sigma}}(C_0)^{\frac{1}{2\alpha}} d_+ < d_{\mathrm{NLS}}.$$

Since $e^{-t\partial_x^3}$ is isometry in $\hat{M}^\alpha_{2,\sigma}$, $(C_0)^{\frac{1}{2\alpha}} e^{-T_0\partial_x^3}\phi \in \mathcal{S}_{+,\mathrm{NLS}} \cap \mathcal{S}_{-,\mathrm{NLS}}$ and so v scatters for both time directions. Next, if $T_0 = \infty$ then by definition v scatters for positive time direction and $\|v(t)\|_{\hat{M}^\alpha_{2,\sigma}} \to \|\phi\|_{\hat{M}^\alpha_{2,\sigma}}$ as $t \to \infty$. Therefore, we can take $T \in \mathbb{R}$ from maximal existence time of v so that $\left\|(C_0)^{1/2\alpha} v(T)\right\|_{\hat{M}^\alpha_{2,\sigma}} < d_{\mathrm{NLS}}$. This implies that v scatters also for negative time. The case $T_0 = -\infty$ is handled in the same way. Thus, $v \in C(\mathbb{R}; \hat{L}^\alpha_x(\mathbb{R})) \cap L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R})$. We let $v_\pm \in \hat{L}^\alpha_x$ be scattering states such that

$$\lim_{T\to\infty} \|v(\pm T) - e^{\mp iT\partial_x^2} v_\pm\|_{\hat{L}^\alpha_x} = 0. \tag{68}$$

We further introduce v_n as a solution of (66) with

$$\begin{cases} v_n(T_0) = P_{|\xi|\leqslant \xi_n^{1/4}} e^{-iT_0\partial_x^2}\phi & \text{if } |T_0| < \infty, \\ \lim_{t\to T_0} \|v_n(t) - P_{|\xi|\leqslant\xi_n^{1/4}} e^{-it\partial_x^2}\phi\|_{\hat{L}^\alpha_x} = 0 & \text{if } T_0 = \pm\infty, \end{cases} \tag{69}$$

where $P_{|\xi|\leqslant a} = \mathcal{F}^{-1}\varphi(\xi)\mathcal{F}$ with even bump function φ satisfying $supp\varphi \subset [-a, a]$. The long time stability for NLS (Proposition 7.10) yields

$$v_n \to v \text{ in } C(\mathbb{R}; \hat{L}^\alpha_x(\mathbb{R})) \cap L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R}). \tag{70}$$

In particular, v_n satisfies the uniform (in n) space–time bound

$$\|v_n\|_{L^{3\alpha}_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant C(\phi).$$

By the persistence of regularity for (NLS) (Lemma 7.9), we obtain

$$\||\partial_x|^s v_n\|_{L^\infty_t \hat{L}^\alpha_x} + \||\partial_x|^{s+\tau} v\|_{L^p_x L^q_t} \leqslant C\xi_n^{s/4} \tag{71}$$

for any $s \geqslant 0$, where $1/\alpha - 3/4 < \tau < 3/2 - 2/\alpha$ and (p, q) satisfies (58). Further, since $\|P_{|\xi|\leqslant\xi_n^{1/4}} e^{-iT_0\partial_x^2}\phi\|_{H^s_x} = O(\xi_n^{\frac{s}{4}-\frac{1}{8}+\frac{1}{4\alpha}})$ for any $s \geqslant 0$, it follows that

$$\||\partial_x|^s v_n\|_{L_t^p(\mathbb{R},L_x^q)} = O(\xi_n^{\frac{s}{4}-\frac{1}{8}+\frac{1}{4\alpha}}), \tag{72}$$
$$\||\partial_x|^s \partial_t v_n\|_{L_t^p(\mathbb{R},L_x^q)} = O(\xi_n^{\frac{s+2}{4}-\frac{1}{8}+\frac{1}{4\alpha}})$$

for any Schrödinger admissible pair (p,q) (i.e., (p,q) satisfies (60)) and $0 \leqslant s < 2\alpha$.

The convergence (70) gives us

$$\lim_{T\to\infty} \sup_n \|v_n\|_{L(|t|>T)} = 0. \tag{73}$$

Similarly, by (68) and (70),

$$\lim_{T\to\infty} \sup_n \|v_n(\pm T) - e^{\mp iT\partial_x^2} v_\pm\|_{\hat{L}_x^\alpha} = 0. \tag{74}$$

Next, we construct a global solution u_n to (gKdV). As in [30], we introduce an approximate solution $\tilde{u}$ to (gKdV):

$$\tilde{u}_n(t,x) := \begin{cases} Re[e^{-ix\xi_n - it\xi_n^3} v_n(-3\xi_n t, x + 3\xi_n^2 t)], & \text{if } |t| \leqslant \frac{T}{3\xi_n}, \\ e^{-(t-\frac{T}{3\xi_n})\partial_x^3} Re[e^{-ix\xi_n - \frac{i}{3}T\xi_n^2} v_n(-T, x + \xi_n T)], & \text{if } t > \frac{T}{3\xi_n}, \\ e^{-(t+\frac{T}{3\xi_n})\partial_x^3} Re[e^{-ix\xi_n + \frac{i}{3}T\xi_n^2} v_n(T, x - \xi_n T)], & \text{if } t < -\frac{T}{3\xi_n}, \end{cases} \tag{75}$$

where T is a large parameter independent of n which will be chosen later.

Lemma 8.1 *(Space–time bound for $\tilde{u}_n$). Assume $5/3 < \alpha < 2$. We have*

$$\|\tilde{u}_n\|_{L_t^\infty(\mathbb{R};\hat{L}_x^\alpha)} + \|\tilde{u}_n\|_{L(\mathbb{R})} + \|\tilde{u}_n\|_{S(\mathbb{R})} \leqslant C, \tag{76}$$

where C is a positive constant independent of T and n.

Proof of Lemma 8.1. We split the interval of integrals into $|t| > T/(3\xi_n)$ and $|t| \leqslant T/(3\xi_n)$. In the interval $|t| > T/(3\xi_n)$, each norms appearing in the left hand side of (76) are uniformly bounded in n by the homogenous estimate for Airy equation (Proposition 13) and the uniform space–time bound for v_n (71). In the interval $|t| \leqslant T/(3\xi_n)$, the space–time bound for v_n (71) and the interpolation inequality yield (76). □

Lemma 8.2 *(Approximation of gKdV for large time). Assume $5/3 < \alpha < 2$. Let $\tilde{u}_n$ be given by (75). Then we have*

$$\lim_{T\to\infty} \limsup_{n\to\infty} \||\partial_x|^{-1}\{(\partial_t + \partial_x^3)\tilde{u}_n - \mu\partial_x(|\tilde{u}_n|^{2\alpha}\tilde{u}_n)\}\|_{N(|t|>\frac{T}{3\xi_n})} = 0. \tag{77}$$

Proof of Lemma 8.2. Since the proof follows from the argument similar to [30, Lemmas 4.2 and 4.3], we omit the detail. □

Next, we consider the approximation of gKdV in the middle interval $|t| \leqslant T/(3\xi_n)$ which is a crucial part of the proof of Theorem 4.4. Let us introduce $\Theta = \Theta(n,t,x) = -x\xi_n - t\xi_n^3$, $\tau = \tau(n,t) = -3\xi_n t$, and $y = y(n,t,x) = x + 3\xi_n^2 t$. A direct calculation yields

$$(\partial_t + \partial_x^3)\tilde{u}_n = 3\mu C_0 \xi_n \operatorname{Im}[e^{i\Theta}(|v_n|^{2\alpha} v_n)(\tau,y)] + \operatorname{Re}[e^{i\Theta}(\partial_x^3 v_n)(\tau,y)] \tag{78}$$

and

$$\mu\partial_x(|\tilde{u}_n|^{2\alpha}\tilde{u}_n) = (2\alpha+1)\mu\xi_n|\operatorname{Re}[e^{i\Theta}v_n(\tau,y)]|^{2\alpha}\operatorname{Im}[e^{i\Theta}v_n(\tau,y)]$$
$$+ (2\alpha+1)\mu|\operatorname{Re}[e^{i\Theta}v_n(\tau,y)]|^{2\alpha}\operatorname{Re}[e^{i\Theta}(\partial_x v_n)(\tau,y)].$$

To extract a main contribution from the first term, we use the Fourier expansion

$$f_\alpha(\theta) := |\cos\theta|^{2\alpha}\sin\theta = \sum_{k=1}^{\infty} C_k \sin(k\theta).$$

Here C_k is a k-th Fourier-sin coefficient $C_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f_\alpha(\theta)\sin(k\theta)d\theta$. The expansion gives us

$$\begin{aligned}
&|\operatorname{Re}[e^{i\Theta}v_n(\tau,y)]|^{2\alpha}\operatorname{Im}[e^{i\Theta}v_n(\tau,y)]\\
&\quad=(|v_n|^{2\alpha+1})(\tau,y)f_\alpha(\Theta+\arg v_n)\\
&\quad=(|v_n|^{2\alpha+1})(\tau,y)\sum_{k=1}^{\infty}C_k\sin k(\Theta+\arg v_n)\\
&\quad=C_1\operatorname{Im}[e^{i\Theta}(|v_n|^{2\alpha}v_n)(\tau,y)]+\sum_{k=2}^{\infty}C_k\operatorname{Im}[e^{ik\Theta}(|v_n|^{2\alpha+1-k}v_n^k)(\tau,y)],
\end{aligned}$$

where $\arg v_n = \arg v_n(\tau, y)$. An elementary computation shows that

$$C_1=\frac{2\Gamma(\alpha+\frac12)\Gamma(\frac32)}{\pi\Gamma(\alpha+2)}=\frac{2\Gamma(\alpha+\frac32)}{\sqrt{\pi}(2\alpha+1)\Gamma(\alpha+2)}=\frac{3}{2\alpha+1}C_0.$$

Then we have

$$\begin{aligned}
&(\partial_t+\partial_x^3)\tilde u_n-\mu\partial_x(|\tilde u_n|^{2\alpha}\tilde u_n)\\
&=\operatorname{Re}[e^{i\Theta}(\partial_x^3v_n)(\tau,y)]\\
&\quad-(2\alpha+1)\mu|\operatorname{Re}[e^{i\Theta}v_n(\tau,y)]|^{2\alpha}\operatorname{Re}[e^{i\Theta}(\partial_xv_n)(\tau,y)]\\
&\quad-(2\alpha+1)\mu\xi_n\sum_{k=2}^{\infty}C_k\operatorname{Im}[e^{ik\Theta}(|v_n|^{2\alpha+1-k}v_n^k)(\tau,y)]\\
&=:R_n^1+R_n^2+R_n^3.
\end{aligned}\tag{79}$$

To evaluate the right hand side of (79), we introduce a function e_n by

$$\begin{cases}(\partial_t+\partial_x^3)e_n=R_n^1+R_n^2+R_n^3,\\ e_n(0,x)=0.\end{cases}\tag{80}$$

Set $e_n =: e_{n,1} + e_{n,2}$, where

$$\begin{aligned}
e_{n,1}&=(2\alpha+1)\mu\xi_n^{-2}\\
&\quad\times\sum_{k=2}^{\infty}C_k\operatorname{Im}\left[e^{-ikx\xi_n}\frac{e^{-ikt\xi_n^3}-e^{-ik^3t\xi_n^3}}{i(k-k^3)}(|v_n|^{2\alpha+1-k}v_n^k)(\tau,y)\right].
\end{aligned}$$

A direct calculation yields

$$\begin{aligned}
&(\partial_t+\partial_x^3)e_{n,1}=R_n^3+R_n^4, &&e_{n,1}(0,x)=0,\\
&(\partial_t+\partial_x^3)e_{n,2}=R_n^1+R_n^2-R_n^4, &&e_{n,2}(0,x)=0,
\end{aligned}$$

where R_n^4 is given by

$$R_n^4=\sum_{\ell=1}^{4}\sum_{k=2}^{\infty}\operatorname{Im}\left[G_n^{\ell,k}(\tau,y)(e^{-ikt\xi_n^3}-e^{-ik^3t\xi_n^3})e^{-ikx\xi_n}\right]$$

with

$$\begin{aligned}
G_n^{1,k}(t,x)&=3(2\alpha+1)\mu\frac{C_k}{ik}\partial_x(|v_n|^{2\alpha+1-k}v_n^k)(t,x),\\
G_n^{2,k}(t,x)&=-3(2\alpha+1)\mu\frac{C_k}{1-k^2}\xi_n^{-1}\partial_x^2(|v_n|^{2\alpha+1-k}v_n^k)(t,x),\\
G_n^{3,k}(t,x)&=-3(2\alpha+1)\mu\frac{C_k}{i(k-k^3)}\xi_n^{-1}\partial_t(|v_n|^{2\alpha+1-k}v_n^k)(t,x),\\
G_n^{4,k}(t,x)&=(2\alpha+1)\mu\frac{C_k}{i(k-k^3)}\xi_n^{-2}\partial_x^3(|v_n|^{2\alpha+1-k}v_n^k)(t,x).
\end{aligned}$$

Lemma 8.3 *(Error control). Fix $T > 0$. Let e_n be a solution to (80). Then,*

$$\lim_{n\to\infty}\left(\|e_n\|_{L_t^\infty \hat{L}_x^\alpha([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])} + \|e_n\|_{L([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])} + \|e_n\|_{S([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])}\right) = 0. \tag{81}$$

Proof of Lemma 8.3. We keep the notation $\tau = \tau(n,t,x) = -3\xi_n t$ and $y = y(n,t,x) = x + 3\xi_n^2 t$. We first evaluate $L_t^\infty \hat{L}_x^\alpha$-norm of $e_{n,1}$. By the definition of $e_{n,1}$, we have

$$\begin{aligned}
&\|e_{n,1}\|_{L_t^\infty \hat{L}_x^\alpha([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])} \\
&\leqslant C\xi_n^{-2}\sum_{k\geqslant 2}\frac{|C_k|}{k^3}\|(|v_n|^{2\alpha+1-k}v_n^k)(\tau,y)\|_{L_t^\infty \hat{L}_x^\alpha([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])}.
\end{aligned}$$

Since $L^\alpha \hookrightarrow \hat{L}^\alpha$ and $\dot{H}^{\frac12-\frac{1}{\alpha(2\alpha+1)}} \hookrightarrow L^{\alpha(2\alpha+1)}$ for $1 < \alpha \leqslant 2$, we see from (72) that

$$\begin{aligned}
\|(|v_n|^{2\alpha+1-k}v_n^k)(\tau,y)\|_{L_t^\infty \hat{L}_x^\alpha([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])} &= C\||v_n|^{2\alpha+1-k}v_n^k\|_{L_t^\infty \hat{L}_x^\alpha([-T,T])} \\
&\leqslant C\||v_n|^{2\alpha+1-k}v_n^k\|_{L_t^\infty L_x^\alpha([-T,T])} \leqslant C\|v_n\|^{2\alpha+1}_{L_t^\infty H_x^{\frac12-\frac{1}{\alpha(2\alpha+1)}}} \leqslant C\xi_n^{\frac12},
\end{aligned}$$

which implies

$$\|e_{n,1}\|_{L_t^\infty \hat{L}_x^\alpha([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])} \leqslant C\xi_n^{-\frac32}\sum_{k=2}^\infty \frac{|C_k|}{k^3} \leqslant C\xi_n^{-\frac32} \to 0 \tag{82}$$

as $n\to\infty$. Next we evaluate the L-norm of $e_{n,1}$. An interpolation shows

$$\|e_{n,1}\|_{L([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])} \leqslant \|e_{n,1}\|^{1-\frac{1}{3\alpha}}_{L_{t,x}^{3\alpha}([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])}\|\partial_x e_{n,1}\|^{\frac{1}{3\alpha}}_{L_{t,x}^{3\alpha}([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])}.$$

In the same manner as in the estimate for $L_t^\infty \hat{L}_x^\alpha$-norm of $e_{n,1}$, we have $\|\partial_x^j e_{n,1}\|_{L_{t,x}^{3\alpha}([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])} \leqslant C\xi_n^{-\frac32-\frac{1}{3\alpha}+j}$ for $j = 0, 1$. Hence,

$$\|e_{n,1}\|_{L([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])} \leqslant C\xi_n^{-\frac32}. \tag{83}$$

Let us proceed to the evaluation of $S([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])$-norm of $e_{n,1}$. We put $\rho = 5\alpha(2\alpha+1)$. Then we easily see

$$\left\|(|v_n|^{2\alpha+1-k}v_n^k)(\tau,y)\right\|_{L_x^{\frac{5\alpha}{2}}L_t^{5\alpha}([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])} = \|v_n(\tau,y)\|^{2\alpha+1}_{L_x^{\frac{\rho}{2}}L_t^\rho([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])}.$$

Change of variables and the Gagliardo–Nirenberg inequality yield

$$\begin{aligned}
\|v_n(\tau,y)\|_{L_t^\rho([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])} &\leqslant C\xi_n^{-\frac{1}{\rho}}\|v_n(t,x-\xi_n t)\|_{L_t^\rho(\mathbb{R})} \\
&\leqslant C\xi_n^{-\frac{1}{\rho}}\|v_n(t,x-\xi_n t)\|^{1-\frac{1}{\rho}}_{L_t^{\frac{\rho}{2}}(\mathbb{R})}\|\partial_t(v_n(t,x-\xi_n t))\|^{\frac{1}{\rho}}_{L_t^{\frac{\rho}{2}}(\mathbb{R})}.
\end{aligned}$$

Hence,

$$\|v_n(\tau,y)\|_{L_x^{\frac{\rho}{2}}L_t^\rho([-\frac{T}{3\xi_n},\frac{T}{3\xi_n}])} \leqslant C\xi_n^{-\frac{1}{\rho}}\|v_n\|^{1-\frac{1}{\rho}}_{L_{t,x}^{\frac{\rho}{2}}(\mathbb{R}^2)}\|\partial_t v_n - \xi_n\partial_x v_n\|^{\frac{1}{\rho}}_{L_{t,x}^{\frac{\rho}{2}}(\mathbb{R}^2)}.$$

Since $(\frac{\rho}{2},\frac{2\rho}{\rho-8})$ is a Schrödinger admissible pair, it follows from (72) that

$$\|v_n\|_{L_{t,x}^{\frac{\rho}{2}}} \leqslant \||\partial_x|^{\frac12-\frac{6}{\rho}}v_n\|_{L_t^{\frac{\rho}{2}}L_x^{\frac{2\rho}{\rho-8}}} = O(\xi_n^{-\frac{3}{2\rho}+\frac{1}{4\alpha}}).$$

Similar estimates hold for $\partial_t v_n$ and $\partial_x v_n$. Combining above estimates, we conclude that $\|v_n(\tau, y)\|_{L_x^{\rho/2} L_t^{\rho}([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} = O(\xi_n^{1/2(2\alpha+1)})$. Thus, $\|e_{n,1}\|_{S([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} = O(\xi_n^{-3/2})$.

To evaluate $e_{n,2}$, we employ the inhomogeneous estimate for Airy equation (14). Since $(1, \alpha)$ is a conjugate-acceptable pair,

$$\begin{aligned}
&\|e_{n,2}\|_{L_t^\infty \hat{L}_x^\alpha([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} + \|e_{n,2}\|_{L([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} + \|e_{n,2}\|_{S([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\
&\quad \leqslant \|R_n^1\|_{L_t^1 \hat{L}_x^\alpha([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} + \|R_n^2\|_{L_x^{\tilde{p}(1,\alpha)} L_t^{\tilde{q}(1,\alpha)}([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\
&\qquad + \|R_n^4\|_{L_x^{\tilde{p}(1,\alpha)} L_t^{\tilde{q}(1,\alpha)}([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])}.
\end{aligned} \tag{84}$$

By (71), we have

$$\|R_n^1\|_{L_t^1 \hat{L}_x^\alpha([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \leqslant C\xi_n^{-\frac{1}{4}} T \|v_n\|_{L_t^\infty \hat{L}_x^\alpha([-T,T])} \to 0 \tag{85}$$

as $n \to \infty$ and

$$\begin{aligned}
&\|R_n^2\|_{L_x^{\tilde{p}(1,\alpha)} L_t^{\tilde{q}(1,\alpha)}([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \\
&\quad \leqslant C\xi_n^{-\frac{1}{\tilde{q}(1,\alpha)}} \|v_n\|^{2\alpha}_{L_x^{p(0,\alpha)} L_t^{q(0,\alpha)}([-T,T])} \|\partial_x v_n\|_{L_x^{p(1,\alpha)} L_t^{q(1,\alpha)}([-T,T])} \\
&\quad \leqslant C\xi_n^{-\frac{1}{\tilde{q}(1,\alpha)}} \|v_n\|^{2\alpha}_{L_x^{p(0,\alpha)} L_t^{q(0,\alpha)}([-T,T])} T^{\frac{2-\alpha}{2\alpha}} \|\partial_x v_n\|_{L_x^{\frac{5\alpha}{2-\alpha}} L_t^{\frac{10\alpha}{9\alpha-8}}([-T,T])} \\
&\quad \leqslant C\xi_n^{-\frac{17\alpha-7}{20\alpha}} T^{\frac{2-\alpha}{2\alpha}} \to 0
\end{aligned} \tag{86}$$

as $n \to \infty$. In a similar way

$$\|R_n^4\|_{L_x^{\tilde{p}(1,\alpha)} L_t^{\tilde{q}(1,\alpha)}([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \leqslant C\xi_n^{-\frac{15}{16\tilde{q}(1,\alpha)}} \to 0 \tag{87}$$

as $n \to \infty$. By (84), (85), (86) and (87), we see $\|e_n\|_{S([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} \to 0$ as $n \to \infty$. Together with (82) and (83), it shows (81). □

Lemma 8.4 *(Approximation of gKdV for middle interval). Fix $T \in \mathbb{R}$. Let $\tilde{u}_n$ and e_n be given by (75) and (80). Then we have*

$$\begin{aligned}
&\lim_{n\to\infty} \||\partial_x|^{-1}[(\partial_t + \partial_x^3)(\tilde{u}_n - e_n) \\
&\qquad - \mu\partial_x\{|\tilde{u}_n - e_n|^{2\alpha}(\tilde{u}_n - e_n)\}]\|_{N([-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}])} = 0.
\end{aligned} \tag{88}$$

Proof of Lemma 8.4. (88) easily follows from Lemmas 2.9 and 8.3. □

By the argument similar to [30, Lemma 4.7], we obtain

Lemma 8.5 *(Initial condition). Take a parameter T so that $T > T_0$ if $|T_0| < \infty$ and arbitrarily positive if $T_0 = \pm\infty$. Let $u_n(t_n)$ and $\tilde{u}_n(t)$ be given by (29) and (75), respectively. Then we have*

$$\lim_{n\to\infty} \|u_n(t_n) - \tilde{u}_n(t_n)\|_{\hat{L}_x^\alpha} = 0. \tag{89}$$

We now prove Theorem 4.4.

Proof of Theorem 4.4. By Lemma 8.1, there exist two positive constants A and M which are independent of T and n such that

$$\|\tilde{u}_n\|_{L_t^\infty(\mathbb{R};\hat{L}_x^\alpha)} \leqslant A, \qquad \|\tilde{u}_n\|_{S(\mathbb{R})} + \|\tilde{u}_n\|_{L(\mathbb{R})} \leqslant M.$$

For the above M, let $\varepsilon_1 = \varepsilon_1(M)$ be given by Lemma 3.2 and let C be a constant appearing in Lemma 3.2. Then. Lemma 8.2 yields that for any ε satisfying $0 < \varepsilon < C\varepsilon_1$, there exists a positive constant T_ε such that if $T \geqslant T_\varepsilon$, then

$$\lim_{n\to\infty} \||\partial_x|^{-1}\{(\partial_t + \partial_x^3)\tilde{u}_n - \mu\partial_x(|\tilde{u}_n|^{2\alpha}\tilde{u}_n)\}\|_{N(|t|>\frac{T}{3\xi_n})} < \frac{\varepsilon}{2}. \tag{90}$$

We now choose

$$T := \begin{cases} \max\{T_\varepsilon, 2|T_0|\} & \text{if } |T_0| < \infty, \\ T_\varepsilon & \text{if } T_0 = \pm\infty. \end{cases}$$

We first apply the long time stability for gKdV in the time interval $\{|t| \leqslant T/(3\xi_n)\}$. Lemmas 8.4 and 8.5 lead that there exits a nonnegative integer $N_1 = N_1(\varepsilon, T_\varepsilon)$ such that if $n \geqslant N_1$, then $|t_n| \leqslant T/(3\xi_n)$ and

$$\|u_n(t_n) - \tilde{u}_n(t_n)\|_{\hat{L}_x^\alpha} + \||\partial_x|^{-1}\{(\partial_t + \partial_x^3)\tilde{u}_n - \mu\partial_x(|\tilde{u}_n|^{2\alpha}\tilde{u}_n)\}\|_{N(|t|\leqslant\frac{T}{3\xi_n})} \leqslant \frac{\varepsilon}{C}.$$

Hence, by Proposition 3.2, there exists a unique solution $u \in C(I; \hat{L}_x^\alpha)$ to (gKdV) satisfying

$$\|u_n - \tilde{u}_n\|_{L_t^\infty(I;\hat{L}_x^\alpha)} + \|u_n - \tilde{u}_n\|_{S(I)} + \|u_n - \tilde{u}_n\|_{L(I)} \leqslant \frac{\varepsilon}{2}, \tag{91}$$

where $I = [-\frac{T}{3\xi_n}, \frac{T}{3\xi_n}]$. Especially, we have

$$\left\| u_n\left(\pm\frac{T}{3\xi_n}\right) - \tilde{u}_n\left(\pm\frac{T}{3\xi_n}\right)\right\|_{\hat{L}_x^\alpha} \leqslant \frac{\varepsilon}{2}. \tag{92}$$

Next we apply the long time stability for gKdV in the time intervals $t \geqslant T/(3\xi_n)$ and $t \leqslant -T/(3\xi_n)$, respectively. Combining (90), (91), (92) and Lemma 3.2, we find that there exists a unique global solution $u \in C(\mathbb{R}; \hat{L}_x^\alpha)$ to (gKdV) satisfying

$$\|u_n - \tilde{u}_n\|_{L_t^\infty(\mathbb{R};\hat{L}_x^\alpha)} + \|u_n - \tilde{u}_n\|_S + \|u_n - \tilde{u}_n\|_L \leqslant C\varepsilon. \tag{93}$$

Combining the above inequality and Lemma 8.1 we have (30).

Finally, the inequality (31) follows from the argument by [30, Theorem 4.1]. This completes the proof of Theorem 4.4. □

Conflict of interest statement

There is no conflict of interest.

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Appendix A. On generalized Morrey spaces

In this appendix, we give the following interpolation type inequality for the generalized Morrey spaces.

Proposition A.1. *Suppose that* $0 < q < p < r < \infty$. *If* s *satisfies*

$$\frac{1}{s} \times \left(1 - \frac{p}{r}\right) + \frac{1}{p} \times \frac{p}{r} < \frac{1}{q}$$

then, for any $f \in L^q(\mathbb{R})$, *we have* $\|f\|_{M_{q,r}^p} \leqslant C\,\|f\|_{M_{s,\infty}^p}^{1-\frac{p}{r}}\,\|f\|_{M_{p,\infty}^p}^{\frac{p}{r}}$. *In particular,* $L^p \hookrightarrow M_{q,r}^p$.

Proof of Proposition A.1. Set $f_{n,I}(x) := f(x)\mathbf{1}_{I\cap\{2^n \leqslant |I|^{1/p}|f(x)|\leqslant 2^{n+1}\}}(x)$ for $I \in \mathcal{D}$ and $n \in \mathbb{Z}$. Let $\theta = 1 - p/r$. By the Hölder inequality in x,

$$\int |f_{n,I}|^q\, dx \leqslant \left(\int |f_{n,I}|^{\frac{\theta qp}{p-(1-\theta)q}}\, dx\right)^{1-\frac{(1-\theta)q}{p}} \left(\int |f_{n,I}|^p\, dx\right)^{\frac{(1-\theta)q}{p}}. \tag{A.1}$$

By definition of $f_{n,I}$, we have

$$\left(\int |f_{n,I}|^{\frac{\theta qp}{p-(1-\theta)q}}\, dx\right)^{1-\frac{(1-\theta)q}{p}} \leqslant C2^{\theta qn}|I|^{-\frac{\theta q}{p}} \left(\int\limits_{I\cap\{|f|\geqslant 2^n|I|^{-\frac{1}{p}}\}} dx\right)^{1-\frac{(1-\theta)q}{p}}.$$

One sees from Chebyshev's inequality that

$$\int\limits_{I\cap\{|f|\geqslant 2^n|I|^{-\frac{1}{p}}\}} dx \leqslant \frac{\int_I |f|^s\, dx}{2^{sn}|I|^{-\frac{s}{p}}} \leqslant 2^{-sn}|I|\left(\sup_{I\in\mathcal{D}} |I|^{\frac{1}{p}-\frac{1}{s}}\|f\|_{L^s(I)}\right)^s.$$

Together with the trivial estimate $\int_{I\cap\{|f|\geqslant 2^n|I|^{-1/p}\}} dx \leqslant |I|$,

$$\begin{aligned}
&\left(\int |f_{n,I}|^{\frac{\theta pq}{p-(1-\theta)q}}\, dx\right)^{1-\frac{(1-\theta)q}{p}} \\
&\quad\leqslant C|I|^{1-\frac{q}{p}} \min\left(2^{\theta qn}, 2^{\theta qn-(1-\frac{(1-\theta)q}{p})sn}\|f\|_{M^p_{s,\infty}}^{(1-\frac{(1-\theta)q}{p})s}\right) \\
&\quad= C|I|^{1-\frac{q}{p}}\|f\|_{M^p_{s,\infty}}^{\theta q} \min\left(2^{\theta q(n-n_0)}, 2^{(\theta-\frac{s}{q}+\frac{(1-\theta)s}{p})q(n-n_0)}\right),
\end{aligned}$$

where, we chose $n_0 \in \mathbb{R}$ by $2^{n_0} = \|f\|_{M^q_{s,\infty}}$. Since $\theta - \frac{s}{q} + \frac{(1-\theta)s}{p} < 0 < \theta$ by assumption, there exists $\delta = \delta(p,q,s,\theta) > 0$ such that

$$\left(\int |f_{n,I}|^{\frac{\theta pq}{q-(1-\theta)p}}\, dx\right)^{1-\frac{(1-\theta)p}{q}} \leqslant C2^{-\delta|n-n_0|}|I|^{1-\frac{q}{p}}\|f\|_{M^p_{s,\infty}}^{\theta q} \tag{A.2}$$

for all $n \in \mathbb{Z}$ and $I \in \mathcal{D}$.

Note that $\int_I |f|^q dx = \sum_{n\in\mathbb{Z}}\int_{\mathbb{R}} |f_{n,I}|^q dx$ for any $I \in \mathcal{D}$ since $q < \infty$ by assumption. The inequalities (A.1) and (A.2) yield

$$\begin{aligned}
\|f\|_{M^p_{q,r}}^r &= \sum_{I\in\mathcal{D}}\left(\sum_{n\in\mathbb{Z}} |I|^{\frac{q}{p}-1}\|f_{n,I}\|_{L^q(\mathbb{R})}^q\right)^{r/q} \\
&\leqslant C\sum_{I\in\mathcal{D}}\left(\sum_{n\in\mathbb{Z}} 2^{-\delta|n-n_0|}\|f\|_{M^p_{s,\infty}}^{\theta q}\left(\int |f_{n,I}|^p\, dx\right)^{\frac{(1-\theta)q}{p}}\right)^{r/q} \\
&= C\|f\|_{M^p_{s,\infty}}^{\theta r}\sum_{I\in\mathcal{D}}\left(\sum_{n\in\mathbb{Z}}\left(2^{-\delta'|n-n_0|}\int |f_{n,I}|^p\, dx\right)^{q/r}\right)^{r/q} \\
&\leqslant C_{\delta'}\|f\|_{M^p_{s,\infty}}^{\theta r}\sum_{I\in\mathcal{D}}\sum_{n\in\mathbb{Z}} 2^{-\frac{\delta'}{2}|n-n_0|}\int |f_{n,I}|^p\, dx,
\end{aligned}$$

where we have used the Hölder inequality in n to yield the last line. Thus,

$$\|f\|_{M^p_{q,r}}^r \leqslant C\|f\|_{M^p_{s,\infty}}^{\theta r}\sup_{n\in\mathbb{Z}}\sum_{I\in\mathcal{D}}\int |f_{n,I}|^p\, dx.$$

Finally, for any fixed n, we have

$$\sum_{I\in\mathcal{D}}\int |f_{n,I}|^p\,dx=\sum_{j\in\mathbb{Z}}\sum_{I\in\mathcal{D}_j}\int |f_{n,I}|^p\,dx=\sum_{j\in\mathbb{Z}}\int_{\{2^n\leqslant 2^{-\frac{j}{p}}|f|\leqslant 2^{n+1}\}}|f|^p\,dx,$$

where we have used the fact that elements of $\mathcal{D}_j$ are mutually disjoint and $\cup_{I\in\mathcal{D}_j}I=\mathbb{R}$. Since $\{2^n\leqslant 2^{-j/p}|f|\leqslant 2^{n+1}\}$ and $\{2^n\leqslant 2^{-j'/p}|f|\leqslant 2^{n+1}\}$ are disjoint as long as $|j-j'|>p$, we have

$$\sum_{j\in\mathbb{Z}}\int_{\{2^n\leqslant 2^{-\frac{j}{p}}|f|\leqslant 2^{n+1}\}}|f|^p\,dx\leqslant (p+1)\,\|f\|_{L^p(\mathbb{R})}^p=(p+1)\,\|f\|_{L^p(\mathbb{R})}^{(1-\theta)r},$$

which completes the proof. □

Appendix B. Refinement of the Stein–Tomas inequality

In this subsection, we prove the first inequality of the refined Stein–Tomas estimate. We state it in the bilinear form.

Theorem B.1 *(Bilinear refined Stein–Tomas inequality). Let $4/3\leqslant p<\infty$ and $\sigma\in[0,\min(1/2-2/(3p),1/(3p)]$. Then, there exist a constant $C=C(p,\sigma)$ such that*

$$\left\|(|\partial_x|^{\frac{1}{3p}-\frac{\sigma}{2}}e^{-t\partial_x^3}f)(|\partial_x|^{\frac{1}{3p}-\frac{\sigma}{2}}e^{-t\partial_x^3}g)\right\|_{L^{\frac{3p}{2}}_{t,x}}\leqslant C\,\|f\|_{\hat{M}^{p_\sigma}_{q,2q'}}\|g\|_{\hat{M}^{p_\sigma}_{q,2q'}} \tag{B.1}$$

for any $f,g\in\hat{M}^{p_\sigma}_{q,2q'}$, where $p_\sigma=(\frac{1}{p}+\frac{\sigma}{2})^{-1}$ and $q=(\frac{2}{3p}+\sigma)^{-1}$. Especially, for $4/3\leqslant p<\infty$ we have

$$\left\||\partial_x|^{\frac{1}{3p}}e^{-t\partial_x^3}f\right\|_{L^{3p}_{t,x}}\leqslant C\,\|f\|_{\hat{M}^{p}_{\frac{3}{2}p,2(\frac{3}{2}p)'}}. \tag{B.2}$$

This kind of refinement for the Airy equation was known in the case $p=2$ and $\sigma=0$ (see [29,55]).

Proof of Proposition B.1. We argue as in Shao [55]. It suffices to show under the assumption that supp $\hat{f}$, supp $\hat{g}\subset[0,\infty)$. We further denote $\overline{g}$ by g. Then, the left hand side of (B.1) is equal to

$$\left\|\iint e^{ixa+itb}\frac{|\xi\eta|^{\frac{1}{3p}-\frac{\sigma}{2}}}{|\xi^2-\eta^2|}\hat{f}(\xi)\overline{\hat{g}(\eta)}\,dadb\right\|_{L^{3p/2}_{t,x}}$$

up to constant, where we have introduced $a=\xi-\eta$ and $b=\xi^3-\eta^3$. By applying the Sobolev embedding and the Hausdorff–Young inequality, it is bounded by

$$C\left\|\frac{|ab|^{\sigma}|\xi\eta|^{\frac{1}{3p}-\frac{\sigma}{2}}}{|\xi^2-\eta^2|}\hat{f}(\xi)\overline{\hat{g}(\eta)}\right\|_{L^{q'}_{a,b}}=C\left(\iint_{\mathbb{R}^2}\Lambda(\xi,\eta)\frac{|\hat{f}(\xi)|^{q'}|\hat{g}(\eta)|^{q'}}{|\xi-\eta|^{(\frac{2}{3p}-\sigma)q'}}\,d\xi d\eta\right)^{\frac{1}{q'}},$$

where $\sigma\in[0,\frac{1}{2}-\frac{2}{3p}]$, $\frac{1}{q}=\frac{2}{3p}+\sigma\in(0,1/2]$, and

$$\Lambda(\xi,\eta)=\left(\frac{|\xi^2+\xi\eta+\eta^2|^{3p\sigma}|\xi\eta|^{1-\frac{3p\sigma}{2}}}{|\xi+\eta|^{3p\sigma+2}}\right)^{\frac{1}{3p(1-\sigma)-2}}$$

We now introduce a Whitney-type decomposition. For an interval $I\in\mathcal{D}_j$, there exists a unique interval $J\in\mathcal{D}_{j-1}$ such that $I\subset J$. We call J as a *parent* of I. For two intervals $I,I'\in\mathcal{D}$, we introduce a binary relation $\sim_{\mathcal{W}}$ so that $I\sim_{\mathcal{W}}I'$ holds if the following three conditions are satisfied; (i) I and I' belong to same $\mathcal{D}_j$, that is, $|I|=|I'|$; (ii) I is not neighboring I'; and (iii) a parent of I is neighboring a parent of I'. Set $\mathcal{W}:=\{(I,I')\in\mathcal{D}\times\mathcal{D}\mid I\sim_{\mathcal{W}}I'\}$.

Notice that if $I \sim_{\mathcal{W}} I'$ then $|I| \leqslant \mathrm{dist}(I, I') \leqslant 2|I|$ and that for any $I \in \mathcal{D}$, $\#\{I' \in \mathcal{D} \mid I \sim_{\mathcal{W}} I'\} = 3$. Then, we have the following Whitney-type decomposition of $\mathbb{R} \times \mathbb{R}$;

$$\sum_{(I,I')\in\mathcal{W}} \mathbf{1}_I(\xi)\mathbf{1}_{I'}(\eta) = 1, \qquad (\xi,\eta) \in \mathbb{R}^2 \setminus \{(\xi,\xi) \mid \xi \in \mathbb{R}\}.$$

Let $\mathcal{W}$ be as above. Remark that there exists $C > 0$ such that $\Lambda(\xi,\eta) \leqslant C$ for any $(\xi,\eta) \in \mathbb{R}^2_+$ and that $\frac{1}{|\xi-\eta|} \leqslant |I|^{-1} = |I'|^{-1}$ for any $(\xi,\eta) \in I \times I'$ with $(I, I') \in \mathcal{W}$. We hence obtain

$$\iint_{\mathbb{R}^2} \Lambda(\xi,\eta) \frac{|\hat{f}(\xi)|^{q'} |\hat{g}(\eta)|^{q'}}{|\xi-\eta|^{(\frac{2}{3p}-\sigma)q'}} d\xi d\eta$$
$$\leqslant C \sum_{I\in\mathcal{D}} \sum_{I';\, I\sim_{\mathcal{W}} I'} \left(|I|^{\frac{1}{(p_\sigma)'}-\frac{1}{q'}} \|\hat{f}\|_{L^{q'}(I)}\right)^{q'} \left(|I'|^{\frac{1}{(p_\sigma)'}-\frac{1}{q'}} \|\hat{g}\|_{L^{q'}(I')}\right)^{q'},$$

where $\frac{1}{p_\sigma} = \frac{1}{p} + \frac{\sigma}{2}$. By using the fact that, for $I \in \mathcal{D}$, $\{I' \in \mathcal{D} \mid I \sim_{\mathcal{W}} I'\} \subset \{I + k|I| \in \mathcal{D} \mid k \in \{-3,-2,2,3\}\}$, we obtain

$$\sum_{I\in\mathcal{D}} \sum_{I';\, I\sim_{\mathcal{W}} I'} \left(|I|^{\frac{1}{(p_\sigma)'}-\frac{1}{q'}} \|\hat{f}\|_{L^{q'}(I)}\right)^{q'} \left(|I'|^{\frac{1}{(p_\sigma)'}-\frac{1}{q'}} \|\hat{g}\|_{L^{q'}(I')}\right)^{q'}$$
$$\leqslant 4 \|f\|^{q'}_{\hat{M}^{p_\sigma}_{q,2q'}} \|g\|^{q'}_{\hat{M}^{p_\sigma}_{q,2q'}},$$

which completes the proof. □

Remark B.2. Proposition 7.5 can be shown in the same way (see also [1]).

By modifying the proof, we obtain another version of the bilinear estimate.

Proposition B.3. *Let $4/3 \leqslant p \leqslant \infty$. Let $N_1, N_2 \in 2^{\mathbb{Z}}$ be a dyadic numbers such that $N_1 < N_2$. Let $f_j(x)$ $(j = 1, 2)$ be two functions such that* $\mathrm{supp}\, \widehat{f_j} \subset \{N_j \leqslant |\xi| \leqslant 2N_j\}$ *for $j = 1, 2$. Then,*

$$\left\| (|\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f_1)(|\partial_x|^{\frac{1}{3p}} e^{-t\partial_x^3} f_2) \right\|_{L^{\frac{3p}{2}}_{t,x}(\mathbb{R}\times\mathbb{R})} \leqslant C N_2^{-\frac{2}{3p}} \|f_1\|_{\hat{L}^{\frac{3p}{2}}} \|f_2\|_{\hat{L}^{\frac{3p}{2}}}. \tag{B.3}$$

Proof of Proposition B.3. As in the proof of Theorem B.1, we have

$$(\text{L.H.S of } (B.3)) \leqslant C \left(\iint_{\mathbb{R}^2} \frac{|\xi\eta|^{\frac{1}{3p-2}} |\hat{f}_1(\xi)|^{(\frac{3p}{2})'} |\hat{f}_2(\eta)|^{(\frac{3p}{2})'}}{|\xi^2-\eta^2|^{2/(3p-2)}} d\xi d\eta \right)^{1-\frac{2}{3p}}.$$

By the support condition, we have $|\xi\eta|/|\xi^2-\eta^2| \leqslant C N_2^{-2}$, which yields the desired estimate. □

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