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On the scattering problem for infinitely many fermions in dimensions $d \geq 3$ at positive temperature

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Abstract

In this paper, we study the dynamics of a system of infinitely many fermions in dimensions $d \geq 3$ near thermal equilibrium and prove scattering in the case of small perturbation around equilibrium in a certain generalized Sobolev space of density operators. This work is a continuation of our previous paper [11], and extends the important recent result of M. Lewin and J. Sabin in [19] of a similar type for dimension $d = 2$. In the work at hand, we establish new, improved Strichartz estimates that allow us to control the case $d \geq 3$.

Keywords: Hartree equation; Infinitely many particles; Scattering

1. Introduction

In this paper, we study the dynamics of a system of infinitely many fermions in dimensions $d \geq 3$ near thermal equilibrium. In particular, we prove scattering in the case when the perturbation around equilibrium is small in a certain generalized Sobolev space of density operators. This work is a continuation of our previous paper [11], and extends some important recent result of M. Lewin and J. Sabin in [19] of a similar type for two dimensions ($d = 2$). In the work at hand, we are employing new, improved Strichartz estimates that allow us to access higher dimensions.

To set up the problem, we start with a finite system of N fermions interacting via a pair potential w in mean-field description. The dynamics is described by N coupled Hartree equations

$$\begin{cases} i\partial_t u_1 = (-\Delta + w * \rho)u_1 \quad , & u_1(t=0) = u_{1,0} \\ \qquad \cdots & \qquad \cdots \\ i\partial_t u_N = (-\Delta + w * \rho)u_N \quad , & u_N(t=0) = u_{N,0} \end{cases} \tag{1.1}$$

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where ρ is the total density of particles

$$\rho(t,x) = \sum_{j=1}^{N} |u_j(t,x)|^2 . \tag{1.2}$$

In order to be in agreement with the Pauli principle, we require that the initial data $\{u_{j,0}\}_{j=1}^N$ is an orthonormal family. Given that the Cauchy problem is well-posed in a suitable solution space, the solution $\{u_{j,t}\}_{j=1}^N$ continues to be an orthonormal family for $t > 0$.

We introduce the one-particle density matrix corresponding to (1.1),

$$\gamma_N(t) = \sum_{j=1}^{N} |u_j(t)\rangle\langle u_j(t)|. \tag{1.3}$$

It corresponds to the rank-N orthogonal projection onto the span of the orthonormal family $\{u_j(t)\}_{j=1}^N$. The system (1.1) is then equivalent to a single operator-valued equation

$$i\partial_t \gamma_N = [-\Delta + w * \rho_{\gamma_N}, \gamma_N] \tag{1.4}$$

with initial data

$$\gamma_N(t=0) = \sum_{j=1}^{N} |u_{j,0}\rangle\langle u_{j,0}|, \tag{1.5}$$

where the density function is given by

$$\rho_{\gamma_N}(t,x) = \gamma_N(t,x,x) . \tag{1.6}$$

Orthonormality of the family $\{u_j\}_{j=1}^N$ implies that $0 \le \gamma \le 1$.

The expected particle number $\int \rho_N dx$ diverges as $N \to \infty$ for the system (1.1)–(1.2), respectively (1.4)–(1.6). Therefore, the one-particle density matrix $\gamma = \sum_{j=1}^{\infty} |u_j\rangle\langle u_j|$ is *not* of trace class; on the other hand, it has a bounded operator norm $L^2 \to L^2$.

For a dilute gas with a finite density (for instance, with $\rho(t,x) = \frac{1}{N}\sum_{j=1}^N |u_j(t,x)|^2$ as $N \to \infty$, or $\rho(t,x) = \sum_{j=1}^\infty \lambda_j |u_j(t,x)|^2$ with $\lambda_j > 0$ and $\sum \lambda_j = 1$), the system (1.1) has been extensively analyzed in the literature, see for instance [1,7–10,22]. In this setting, $\gamma = \lim_{N\to\infty} \gamma_N$ is trace class. See also for instance [3,2,12,15,4,20] and the references therein for its derivation from a quantum system of interacting fermions; we remark that the fermionic exchange term is negligible in this limit.

The Cauchy problem, obtained from (1.6) as $N \to \infty$ but with $\rho_\gamma \notin L^1$, is much more difficult than in the earlier works noted above. The main problem is to understand in which framework the Cauchy problem

$$i\partial_t \gamma = [-\Delta + w * \rho_\gamma, \gamma] \tag{1.7}$$

with initial data

$$\gamma(0) = \gamma_0 , \tag{1.8}$$

and density

$$\rho_\gamma(t,x) = \gamma(t,x,x), \tag{1.9}$$

can be meaningfully posed.[1] Lewin and Sabin were the first authors who introduced a framework for this problem [18,19], which can be described as follows. First, we observe that given a non-negative function $f : \mathbb{R} \to \mathbb{R}_{\ge 0}$, the operator $\gamma_f = f(-\Delta)$ is a stationary solution to (1.7) having infinite particle number, i.e., $\rho_{\gamma_f} \notin L^1$, since the density function ρ_{γ_f} is a constant function. Examples of γ_f include the Fermi sea of the non-interacting system. For inverse temperature $\beta > 0$ and chemical potential $\mu > 0$, the Fermi sea γ_f is given by the Fermi–Dirac distribution

[1] Again, it is required that $0 \le \gamma_0 \le 1$, to be in agreement with the Pauli principle; hence, γ has a bounded operator norm.

$$\gamma_f(x,y) = \int_{\mathbb{R}^d} \frac{e^{ip(x-y)}}{e^{\beta(p^2-\mu)}+1} dp = \Big(\frac{1}{e^{\beta(-\Delta-\mu)}+1}\Big)(x,y), \tag{1.10}$$

while in the zero temperature limit,

$$\gamma_f = \Pi_\mu^- = \mathbf{1}_{(-\Delta\le\mu)}. \tag{1.11}$$

Then, the main idea is to consider a perturbation

$$Q := \gamma - \gamma_f \tag{1.12}$$

from the reference state γ_f, which evolves according to the following Cauchy problem:

$$\begin{cases} i\partial_t Q = [-\Delta + w * \rho_Q, Q + \gamma_f], \\ Q(0) = Q_0. \end{cases} \tag{1.13}$$

In [18], Lewin and Sabin proved that the Cauchy problem (1.13) for Q is globally well-posed for $d \ge 2$ in a suitable subspace of the space of compact operators, provided that the pair interaction w is sufficiently regular. An important tool used in [18] was a Strichartz estimate for density functions originally established in [13], which is extended to the optimal range [14]. The case of a more singular interaction potential, with $w = \delta$ given by the Dirac delta, was analyzed by authors of the paper at hand; in [11], we proved global well-posedness of the perturbative system (1.13), at zero temperature $\gamma_f = \Pi_\mu^-$, by employing new Strichartz estimates for *regular* density functions and those for operator kernels, which were established in the same paper [11].

In the case of a sufficiently regular potential w, Lewin–Sabin in [19] proved scattering for Q in $d = 2$ via Strichartz estimates from [13]. The case of higher dimensions was left open, and the purpose of the paper at hand is to address it.

Before we state the main result of this paper in Theorem 1.1, we present a brief review of the notation. For $p \ge 1$, the Schatten class $\mathfrak{S}^p$ is defined via

$$\|A\|_{\mathfrak{S}^p} = (\mathrm{Tr}(|A|^p))^{1/p},$$

while for $\alpha \ge 0$ a Hilbert–Schmidt Sobolev space $\mathcal{H}^\alpha$ is equipped with the norm[2]

$$\|Q\|_{\mathcal{H}^\alpha} = \|\langle\nabla\rangle^\alpha Q \langle\nabla\rangle^\alpha\|_{\mathfrak{S}^2}.$$

Also, we use the standard notation

$$\check{g}(x) = \frac{1}{(2\pi)^d}\int e^{ix\cdot\xi} g(\xi)\, d\xi$$

to denote the inverse Fourier transform of a function g.

Theorem 1.1. *Let $d \ge 3$, $\alpha > \frac{d-2}{2}$ and α_0 be given by*

$$\begin{cases} \alpha_0 = 2\alpha - \frac{d-1}{2} & \text{if } \alpha < \frac{d-1}{2}, \\ \alpha_0 < \alpha & \text{if } \alpha = \frac{d-1}{2}, \\ \alpha_0 = \alpha & \text{if } \alpha > \frac{d-1}{2} \end{cases} \tag{1.14}$$

and $\beta > \frac{d+2}{2}$.

We assume that

(i) (assumptions on f) f is real-valued, $\langle\cdot\rangle^\beta f \in L^\infty_{r\ge0}$, $f'(r) < 0$ for $r > 0$,

$$\int_0^\infty (r^{d/2-1}|f(r)| + |f'(r)|)dr < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{\check{g}(x)}{|x|^{d-2}}dx < \infty, \tag{1.15}$$

where $g(\xi) = f(|\xi|^2)$.

[2] For details, see (3.2).

(ii) (assumption on w) The interaction potential $w = w_1 * w_2 \in L^1$ is even,

$$\|\hat{w}_1\|_{L^{\frac{2d}{d-2}}}, \ \|\hat{w}_2\|_{L^{\frac{2d}{d-2}}}, \ \|\langle\cdot\rangle^{\alpha_0+\frac{1}{2}}\hat{w}_1\|_{L^\infty}, \|\hat{w}_2\|_{L^\infty}, \ \||\cdot|^{-1/2}\langle\cdot\rangle^{-\alpha_0}\hat{w}_2\|_{L^\infty} < \infty, \tag{1.16}$$

and

$$\|\hat{w}_-\|_{L^\infty} < 2|\mathbb{S}^{d-1}|\Big(\int_{\mathbb{R}^d} \frac{|\check{g}(x)|}{|x|^{d-2}}dx\Big)^{-1} \quad \text{and} \quad \hat{w}_+(0) < \frac{2}{\epsilon_g}|\mathbb{S}^{d-1}|, \tag{1.17}$$

where $A_\pm = \max\{\pm A, 0\}$ and ϵ_g is given by (4.4).

Then, there exists small $\epsilon > 0$ such that if $\|Q_0\|_{\mathcal{H}^\alpha} \le \epsilon$, there exists a unique global solution $Q(t) \in C_t(\mathbb{R}; \mathfrak{S}^{2d})$ to the equation (1.13) with initial data Q_0. Moreover, the associated density function ρ_Q obeys the global space–time bound,

$$\|w * \rho_Q\|_{L_t^2(\mathbb{R}; L_x^d)} < \infty, \tag{1.18}$$

and $Q(t)$ scatters in $\mathfrak{S}^{2d}$ as $t \to \pm\infty$; in other words, there exist $Q_\pm \in \mathfrak{S}^{2d}$ such that $e^{-it\Delta}Q(t)e^{it\Delta} \to Q_\pm$ converges strongly in $\mathfrak{S}^{2d}$ as $t \to \pm\infty$.

Remark 1.2. (i) In Theorem 1.1, various conditions are imposed on the reference state γ_f, the interaction potential w, and the initial data Q_0. Our main goal is to prove scattering in high dimensions. We do not pursue any optimality on the hypotheses. Some physically important examples, such as the Fermi–Dirac distribution (1.10), satisfy these assumptions. The assumptions on f and the assumptions in (1.17) are used for the linear response theory (see Proposition 4.1). The assumptions in (1.16) are used for the proof of the global space–time bound (1.18) (see Section 7).
(ii) The method in our paper might be applied to the two-dimensional case with different conditions on the interaction potential w and initial data Q_0 from Lewin and Sabin [19]. However, we omit the case $d = 2$, as it was already proved in [19]; moreover, some exponents would have to be modified in the proof. For instance, we are using the endpoint Strichartz estimate for convenience, but the endpoint estimate is known to be false in $\mathbb{R}^2$ [21].
(iii) As a crucial new ingredient that allow us to extend the work of Lewin–Sabin [19] to dimensions higher than 2, we establish new Strichartz estimates for density functions and density matrices in Section 3. Compared to the Strichartz estimates derived in [13], and used in [19], our Strichartz estimates exhibit an improved summability gain by imposing more regularity on the initial data.

2. Outline of the proof of Theorem 1.1

In this part of our analysis, we explain the strategy to prove the main result of this article, Theorem 1.1. First, in Section 2.1, we show that if the density function ρ_Q of the solution to (1.13) satisfies the global space–time bound (see (2.10)), then the solution $Q(t)$ scatters. Next, in §2.2, we set up a suitable contraction map Γ (see (2.19)) to construct a solution obeying the desired global space–time bound.

2.1. A global space–time bound for a density function implies scattering

We follow the strategy in Lewin and Sabin [19]. For simplicity, we present the argument only for the forward-in-time direction, as it can be easily modified to prove scattering backward in time.

Given a time-dependent potential $V = V(t, x)$, we denote by $\mathcal{U}_V(t)$ the linear propagator for the linear Schrödinger equation

$$i\partial_t u + \Delta u - Vu = 0, \tag{2.1}$$

i.e., $\mathcal{U}_V(t)\phi$ is the solution to (2.1) with initial data ϕ. We define the *"finite-time" wave operator* $\mathcal{W}_V(t)$ by

$$\mathcal{W}_V(t) := e^{-it\Delta}\mathcal{U}_V(t). \tag{2.2}$$

Iterating the Duhamel formula

$$\mathcal{U}_V(t) = e^{it\Delta} - i\int_0^t e^{i(t-t_1)\Delta}V(t_1)\mathcal{U}_V(t_1)dt_1 \tag{2.3}$$

infinitely many times, the wave operator can be written as an infinite sum,

$$\mathcal{W}_V(t) := \sum_{n=0}^{\infty}\mathcal{W}_V^{(n)}(t), \tag{2.4}$$

where $\mathcal{W}_V^{(0)}(t) := \mathrm{Id}$, and for $n \geq 1$,

$$\begin{aligned}\mathcal{W}_V^{(n)}(t) &:= (-i)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1}\cdots\int_0^{t_2} dt_1 e^{-it_n\Delta}V(t_n)e^{it_n\Delta}\\ &\qquad\qquad\cdot e^{-it_{n-1}\Delta}V(t_{n-1})e^{it_{n-1}\Delta}\cdots e^{-it_1\Delta}V(t_1)e^{it_1\Delta}\\ &= (-i)\int_0^t dt_n e^{-it_n\Delta}V(t_n)e^{it_n\Delta}\mathcal{W}_V^{(n-1)}(t_n).\end{aligned} \tag{2.5}$$

By the definition of the finite-time wave operator, the equation (1.13) is equivalent to

$$Q(t) = e^{it\Delta}\mathcal{W}_{w*\rho_Q}(t)(\gamma_f + Q_0)\mathcal{W}_{w*\rho_Q}(t)^* e^{-it\Delta} - \gamma_f, \tag{2.6}$$

because $Q(t) = \gamma(t) - \gamma_f$ and

$$\gamma(t) = \mathcal{U}_V(t)\gamma_0\mathcal{U}_V(t)^*. \tag{2.7}$$

Inserting the sum (2.4) into the equation (2.6), it becomes

$$\begin{aligned}Q(t) &= e^{it\Delta}\Big(\sum_{m=0}^{\infty}\mathcal{W}_{w*\rho_Q}^{(m)}(t)\Big)\gamma_f\Big(\sum_{n=0}^{\infty}\mathcal{W}_{w*\rho_Q}^{(n)}(t)\Big)^* e^{-it\Delta} - \gamma_f\\ &\quad + e^{it\Delta}\Big(\sum_{m=0}^{\infty}\mathcal{W}_{w*\rho_Q}^{(m)}(t)\Big)Q_0\Big(\sum_{n=0}^{\infty}\mathcal{W}_{w*\rho_Q}^{(n)}(t)\Big)^* e^{-it\Delta}\\ &= e^{it\Delta}Q_0e^{-it\Delta} + \sum_{(m,n)\neq(0,0)} e^{it\Delta}\mathcal{W}_{w*\rho_Q}^{(m)}(t)\gamma_f\mathcal{W}_{w*\rho_Q}^{(n)}(t)^* e^{-it\Delta}\\ &\quad + \sum_{(m,n)\neq(0,0)} e^{it\Delta}\mathcal{W}_{w*\rho_Q}^{(m)}(t)Q_0\mathcal{W}_{w*\rho_Q}^{(n)}(t)e^{-it\Delta}.\end{aligned} \tag{2.8}$$

In [13], Frank, Lewin, Lieb and Seiringer prove that if $d \geq 2$, then

$$\|\mathcal{W}_V^{(n)}(t_0)\|_{\mathfrak{S}^{2d}} \leq \frac{1}{(n!)^{\frac{1}{2}-\epsilon}}\big(C\|V\|_{L^2_{[0,+\infty)}L^d_x}\big)^n, \quad \forall n \geq 1 \tag{2.9}$$

for any small $\epsilon > 0$ (see Theorem 2 for $n = 1$ and Theorem 3 for $n \geq 2$ in [13]). Therefore, if the density function obeys the space–time norm bound

$$\|w * \rho_Q\|_{L^2_t([0,+\infty);L^d_x)} < \infty, \tag{2.10}$$

by (2.9) with $V = w * \rho_Q$, the series is absolutely convergent in $C_t([0,+\infty);\mathfrak{S}^{2d})$, so it is well-defined.

Using this series expansion, we prove that the global space–time bound (2.10) implies scattering.

Lemma 2.1 *(A global space–time bound for a density function implies scattering). Let $d \geq 3$. Suppose that $Q(t) \in C_t([0,+\infty);\mathfrak{S}^{2d})$ is a solution to the equation (2.6) and its density satisfies (2.10). Then, $Q(t)$ scatters in $\mathfrak{S}^{2d}$ as $t \to +\infty$.*

Proof. As in the proof of the absolute convergence of the series, applying the inequality (2.9) to the series expansion of the difference between $e^{-it_1\Delta}Q(t)e^{it_1\Delta}$ and $e^{-it_2\Delta}Q(t)e^{it_2\Delta}$, one can show that $e^{-it_1\Delta}Q(t)e^{it_1\Delta} - e^{-it_2\Delta}Q(t)e^{it_2\Delta} \to 0$ in $\mathfrak{S}^{2d}$ as $t_1, t_2 \to +\infty$. Therefore, $e^{-it\Delta}Q(t)e^{it\Delta}$ has a strong limit Q_+ in $\mathfrak{S}^{2d}$ as $t \to +\infty$. That is, $Q(t)$ scatters in $\mathfrak{S}^{2d}$ as $t \to +\infty$. □

2.2. Set-up for the contraction mapping argument

By Lemma 2.1, the goal is now to prove that the equation (2.6) has a unique solution $Q(t)$ in a suitable space obeying the space–time bound (2.10). To this end, as in Lewin–Sabin [19], we write the equation (2.6) as an equation for density functions,

$$\rho_{Q(t)} = \rho\Big[e^{it\Delta}\mathcal{W}_{w*\rho_Q}(t)(\gamma_f + Q_0)\mathcal{W}_{w*\rho_Q}(t)^* e^{-it\Delta}\Big] - \rho_{\gamma_f}. \tag{2.11}$$

One of the advantages of this wave operator formulation in density is that the unknown is given only by the density function, and there is no unknown operator.

We further simplify the equation by splitting the interaction potential w into $w = w_1 * w_2$, and subsequently convolving the density function ρ_Q with w_2,

$$w_2 * \rho_{Q(t)} = w_2 * \rho\Big[e^{it\Delta}\mathcal{W}_{w_1*(w_2*\rho_Q)}(t)(\gamma_f + Q_0)\mathcal{W}_{w_1*(w_2*\rho_Q)}(t)^* e^{-it\Delta}\Big] - w_2 * \rho_{\gamma_f}. \tag{2.12}$$

Now we consider the equation for $w_2 * \rho_Q$. The motivation for this formulation is that the solution $w_2 * \rho_Q$ is expected to be contained in a larger function space (or bounded in a weaker norm) than the one for ρ_Q, provided that w_2 is sufficiently nice; our constructions will exploit this fact.

Next, inserting the sum (2.4) for the finite time wave operators acting on γ_f, we write

$$\begin{aligned} w_2 * \rho_{Q(t)} &= w_2 * \rho\Big[e^{it\Delta}\Big(\sum_{m=0}^{\infty}\mathcal{W}^{(m)}_{w_1*(w_2*\rho_Q)}(t)\Big)\gamma_f\Big(\sum_{n=0}^{\infty}\mathcal{W}^{(n)}_{w_1*(w_2*\rho_Q)}(t)\Big)^* e^{-it\Delta}\Big] \\ &\quad + w_2 * \rho\Big[e^{it\Delta}\mathcal{W}_{w_1*(w_2*\rho_Q)}(t)Q_0\mathcal{W}_{w_1*(w_2*\rho_Q)}(t)^* e^{-it\Delta}\Big] - w_2 * \rho_{\gamma_f} \\ &= w_2 * \rho\Big[e^{it\Delta}\Big(\mathcal{W}^{(1)}_{w_1*(w_2*\rho_Q)}(t)\gamma_f + \gamma_f\mathcal{W}^{(1)}_{w_1*(w_2*\rho_Q)}(t)^*\Big)e^{-it\Delta}\Big] \\ &\quad + \sum_{m,n=1}^{\infty} w_2 * \rho\Big[e^{it\Delta}\mathcal{W}^{(m)}_{w_1*(w_2*\rho_Q)}(t)\gamma_f\mathcal{W}^{(n)}_{w_1*(w_2*\rho_Q)}(t)^* e^{-it\Delta}\Big] \\ &\quad + w_2 * \rho\Big[e^{it\Delta}\mathcal{W}_{w_1*(w_2*\rho_Q)}(t)Q_0\mathcal{W}_{w_1*(w_2*\rho_Q)}(t)^* e^{-it\Delta}\Big]. \end{aligned} \tag{2.13}$$

Then, introducing the operators,

$$\mathcal{L}(\phi)(t) := -w_2 * \rho\Big[e^{it\Delta}\Big(\mathcal{W}^{(1)}_{w_1*\phi}(t)\gamma_f + \gamma_f\mathcal{W}^{(1)}_{w_1*\phi}(t)^*\Big)e^{-it\Delta}\Big], \tag{2.14}$$

$$\mathcal{A}_{m,n}(\phi)(t) := w_2 * \rho\Big[e^{it\Delta}\mathcal{W}^{(m)}_{w_1*\phi}(t)\gamma_f\mathcal{W}^{(n)}_{w_1*\phi}(t)^* e^{-it\Delta}\Big], \tag{2.15}$$

$$\mathcal{B}(\phi)(t) := w_2 * \rho\Big[e^{it\Delta}\mathcal{W}_{w_1*\phi}(t)Q_0\mathcal{W}_{w_1*\phi}(t)^* e^{-it\Delta}\Big], \tag{2.16}$$

we write

$$w_2 * \rho_Q = -\mathcal{L}(w_2 * \rho_Q) + \Big\{\sum_{m,n=1}^{\infty}\mathcal{A}_{m,n}(w_2 * \rho_Q) + \mathcal{B}(w_2 * \rho_Q)\Big\}. \tag{2.17}$$

We note that compared to the formulation in [19], the equation (2.17) is slightly simpler in that $\mathcal{B}(w_2 * \rho_Q)$ is not expanded as an infinite sum. However, due to the linear nature of the operator $\mathcal{L}$, which is not perturbative even for small functions, the series expansion $\sum_{m,n=1}^{\infty}\mathcal{A}_{m,n}(w_2 * \rho_Q)$ does not seem to be avoidable.

Later in Section 4, it will be shown that $(1 + \mathcal{L})$ is invertible on $L^2_{t\geq 0}L^2_x$. As a result, the equation can be reformulated as

$$w_2 * \rho_Q = (1+\mathcal{L})^{-1}\Big\{\sum_{m,n=1}^{\infty}\mathcal{A}_{m,n}(w_2 * \rho_Q) + \mathcal{B}(w_2 * \rho_Q)\Big\}. \tag{2.18}$$

Our goal is now to show that the map Γ, defined by

$$\Gamma(\phi) = (1+\mathcal{L})^{-1}\Big\{\sum_{m,n=1}^{\infty}\mathcal{A}_{m,n}(\phi) + \mathcal{B}(\phi)\Big\}, \tag{2.19}$$

is contractive in a suitable function space, and its solution satisfies the space–time bound

$$\|\phi\|_{L^2_{t\geq 0}L^2_x} < \infty. \tag{2.20}$$

Then, the main theorem follows (see Section 7).

3. Strichartz estimates for density functions

In this section we present the Strichartz estimates that will be used in our analysis. First, we give an overview of the notation.

3.1. Notation

As already mentioned in Section 1, we denote by $\mathfrak{S}^p$ the Schatten spaces, equipped with the norms

$$\|Q\|_{\mathfrak{S}^p} := \Big(\mathrm{Tr}|Q|^p\Big)^{1/p}, \tag{3.1}$$

for $p \geq 1$.

For $\alpha \geq 0$, we define the *Hilbert–Schmidt Sobolev space* $\mathcal{H}^\alpha$ as the collection of Hilbert–Schmidt operators (which are not necessarily self-adjoint) with a finite norm

$$\|\gamma_0\|_{\mathcal{H}^\alpha} := \|\langle\nabla\rangle^\alpha\gamma_0\langle\nabla\rangle^\alpha\|_{\mathfrak{S}^2} = \|\langle\nabla_x\rangle^\alpha\langle\nabla_{x'}\rangle^\alpha\gamma_0(x,x')\|_{L^2_xL^2_{x'}}. \tag{3.2}$$

Here, $\gamma_0(x,x')$ is the integral kernel of γ_0, i.e.,

$$(\gamma_0 g)(x) = \int_{\mathbb{R}^d}\gamma_0(x,x')g(x')dx'. \tag{3.3}$$

In order to review Strichartz estimates for operator kernels in Subsection 3.3, we need to recall notation from [11] related to Strichartz norms. An exponent pair (q,r) is *(Strichartz) admissible* if $2 \leq q, r \leq \infty$, $(q,r,d) \neq (2,\infty,2)$ and

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}. \tag{3.4}$$

Assume that $\gamma(t)$ is a time-dependent operator on an interval $I \subset \mathbb{R}$. Then, its Strichartz norm is defined by

$$\|\gamma(t)\|_{\mathcal{S}^\alpha(I)} := \sup_{(q,r):\text{ admissible}}\Big\{\|\langle\nabla_x\rangle^\alpha\langle\nabla_{x'}\rangle^\alpha\gamma(t,x,x')\|_{L^q_t(I;L^r_xL^2_{x'})} + \|\langle\nabla_x\rangle^\alpha\langle\nabla_{x'}\rangle^\alpha\gamma(t,x,x')\|_{L^q_t(I;L^r_{x'}L^2_x)}\Big\}. \tag{3.5}$$

It is clear that $\mathcal{S}^\alpha(I) \hookrightarrow L^\infty_t(I;\mathcal{H}^\alpha)$.

We identify the operator $e^{it\Delta}\gamma_0 e^{-it\Delta}$ with its integral kernel

$$(e^{it\Delta}\gamma_0 e^{-it\Delta})(x,x') = (e^{it(\Delta_x - \Delta_{x'})}\gamma_0)(x,x'). \tag{3.6}$$

3.2. Strichartz estimates for density functions

In this section, we prove new Strichartz estimates for density functions, which extend Strichartz estimates proved in the authors' previous work [11] by allowing asymmetric derivatives (α_1 not necessarily equal to α_2). Those are presented in Theorem 3.1, and as a main application, we obtain Corollary 3.2, which we use to control the operators $\mathcal{A}_{m,n}$.

Theorem 3.1 *(Strichartz estimates for density functions). Suppose that $\alpha_0, \alpha_1, \alpha_2 \geq 0$. When $d = 1$, we assume that $\alpha = \min\{\alpha_1, \alpha_2\}$. When $d \geq 2$, we assume that*

$$\alpha_1 + \alpha_2 > \tfrac{d-1}{2} \tag{3.7}$$

and

$$\begin{cases} \alpha_0 = \alpha_1 + \alpha_2 - \frac{d-1}{2} & \text{if } \max\{\alpha_1, \alpha_2\} < \frac{d-1}{2}, \\ \alpha_0 < \min\{\alpha_1, \alpha_2\} & \text{if } \max\{\alpha_1, \alpha_2\} = \frac{d-1}{2}, \\ \alpha_0 = \min\{\alpha_1, \alpha_2\} & \text{if } \max\{\alpha_1, \alpha_2\} > \frac{d-1}{2}. \end{cases} \tag{3.8}$$

Then,

$$\left\| |\nabla|^{1/2} \rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}} \right\|_{L^2_{t\in\mathbb{R}} H^{\alpha_0}_x} \lesssim \| \langle\nabla\rangle^{\alpha_1} \gamma_0 \langle\nabla\rangle^{\alpha_2} \|_{\mathfrak{S}^2}. \tag{3.9}$$

Corollary 3.2. *Suppose that α_0, α_1 and α_2 satisfy the assumptions in Theorem 3.1. Then,*

$$\left\| \langle\nabla\rangle^{-\alpha_1} \int_{\mathbb{R}} e^{-it\Delta} V(t) e^{it\Delta} dt \langle\nabla\rangle^{-\alpha_2} \right\|_{\mathfrak{S}^2} \leq c \|V(t)\|_{L^2_t L_x^{\frac{2d}{d+1}}}. \tag{3.10}$$

Proof of Corollary 3.2, assuming Theorem 3.1. For a compactly supported smooth function $V(t,x)$ and a finite rank smooth operator γ_0, we write

$$\begin{aligned} &\operatorname{Tr}\Big(\langle\nabla\rangle^{-\alpha_1} \int_{\mathbb{R}} e^{-it\Delta} V(t) e^{it\Delta} dt \langle\nabla\rangle^{-\alpha_2} \Big) \gamma_0 \\ &= \int_{\mathbb{R}} \operatorname{Tr}\Big(e^{it\Delta} \langle\nabla\rangle^{-\alpha_2} \gamma_0 \langle\nabla\rangle^{-\alpha_1} e^{-it\Delta} V(t) \Big) dt \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}^d} \rho_{e^{it\Delta}\langle\nabla\rangle^{-\alpha_2}\gamma_0\langle\nabla\rangle^{-\alpha_1} e^{-it\Delta}}(x) V(t,x) dx dt, \end{aligned} \tag{3.11}$$

where the first identity is from cyclicity of trace. Therefore, (3.10) is dual to

$$\| \rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}} \|_{L^2_t L_x^{\frac{2d}{d-1}}} \leq c \| \langle\nabla\rangle^{\alpha_2} \gamma_0 \langle\nabla\rangle^{\alpha_1} \|_{\mathfrak{S}^2}, \tag{3.12}$$

which follows from (3.9) and the Sobolev inequality. □

The main strategy to prove the Strichartz estimate for density functions is to reformulate it as an integral estimate through the space–time Fourier transformation. This approach, via bilinear estimates based on the space–time L^2-norm, has been introduced by Bourgain [5,6] and Klainerman and Machedon [16,17], and subsequently developed by many authors.

Lemma 3.3 *(Reduction to an integral estimate). Let $\tilde{\alpha}$ be any real number. Then if the integral*

$$I_{\tau,\xi} := \int_{|\eta| \leq |\xi-\eta|} \frac{|\xi|^{2\tilde{\alpha}} \langle\xi\rangle^{2\alpha_0}}{\langle\eta\rangle^{2\max\{\alpha_1,\alpha_2\}} \langle\xi-\eta\rangle^{2\min\{\alpha_1,\alpha_2\}}} \delta(\tau + |\eta|^2 - |\xi-\eta|^2) d\eta \tag{3.13}$$

is bounded uniformly in τ and ξ, the Strichartz estimate

$$\||\nabla|^{\tilde{\alpha}}\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}}\|_{L^2_{t\in\mathbb{R}}H^{\alpha_0}_x} \lesssim \|\langle\nabla\rangle^{\alpha_1}\gamma_0\langle\nabla\rangle^{\alpha_2}\|_{\mathfrak{S}^2} \tag{3.14}$$

holds.

Proof. The Fourier transform of the density function of γ is given by

$$\begin{aligned}
\widehat{\rho_\gamma}(\xi) &= \mathcal{F}_x\Big\{\frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\hat{\gamma}(\eta,\zeta)e^{ix\cdot(\eta+\zeta)}dxd\zeta\Big\}(\xi)\\
&= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\hat{\gamma}(\eta,\zeta)\mathcal{F}_x\Big\{e^{ix\cdot(\eta+\zeta)}\Big\}(\xi)d\eta d\zeta\\
&= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\hat{\gamma}(\eta,\zeta)\cdot(2\pi)^d\delta(\xi-\eta-\zeta)d\eta d\zeta\\
&= \frac{1}{(2\pi)^{d}}\int_{\mathbb{R}^d}\hat{\gamma}(\eta,\xi-\eta)d\eta.
\end{aligned} \tag{3.15}$$

Hence, the space–time Fourier transform of the density function $\rho_{e^{it\Delta}\gamma e^{-it\Delta}}$ is

$$\begin{aligned}
(\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}})^\sim(\tau,\xi) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\mathcal{F}_t\Big\{e^{-it(|\eta|^2-|\xi-\eta|^2)}\Big\}\hat{\gamma}_0(\eta,\xi-\eta)d\eta\\
&= \frac{1}{(2\pi)^{d-1}}\int_{\mathbb{R}^d}\delta(\tau+|\eta|^2-|\xi-\eta|^2)\hat{\gamma}_0(\eta,\xi-\eta)d\eta.
\end{aligned} \tag{3.16}$$

Thus, by the Plancherel theorem and Cauchy–Schwarz, we get

$$\begin{aligned}
&\||\nabla|^{\tilde{\alpha}}\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}}\|^2_{L^2_{t\in\mathbb{R}}H^{\alpha_0}_x}\\
&= \frac{1}{(2\pi)^{2(d+1)}}\Big\||\xi|^{\tilde{\alpha}}\langle\xi\rangle^{\alpha_0}(\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}})^\sim(\tau,\xi)\Big\|^2_{L^2_{\tau\in\mathbb{R}}L^2_\xi}\\
&= \frac{1}{(2\pi)^{2(d+1)}}\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\xi|^{2\tilde{\alpha}}\langle\xi\rangle^{2\alpha_0}\\
&\qquad\cdot\Big|\frac{1}{(2\pi)^{d-1}}\int_{\mathbb{R}^d}\delta(\tau+|\eta|^2-|\xi-\eta|^2)\hat{\gamma}_0(\eta,\xi-\eta)d\eta\Big|^2 d\xi d\tau\\
&\le \frac{1}{(2\pi)^{4d}}\int_{\mathbb{R}}\int_{\mathbb{R}^d}|\xi|^{2\tilde{\alpha}}\langle\xi\rangle^{2\alpha_0}\Big\{\int_{\mathbb{R}^d}\frac{\delta(\tau+|\eta|^2-|\xi-\eta|^2)}{\langle\eta\rangle^{2\alpha_1}\langle\xi-\eta\rangle^{2\alpha_2}}d\eta\Big\}\\
&\qquad\cdot\Big\{\int_{\mathbb{R}^d}\delta(\tau+|\eta|^2-|\xi-\eta|^2)|(\langle\nabla\rangle^{\alpha_1}\gamma_0\langle\nabla\rangle^{\alpha_2})^\wedge(\eta,\xi-\eta)|^2 d\eta\Big\}d\xi d\tau\\
&\le \sup_{\tau,\xi}\frac{1}{(2\pi)^{4d}}\Big\{\int_{\mathbb{R}^d}\frac{|\xi|^{2\tilde{\alpha}}\langle\xi\rangle^{2\alpha_0}\delta(\tau+|\eta|^2-|\xi-\eta|^2)}{\langle\eta\rangle^{2\alpha_1}\langle\xi-\eta\rangle^{2\alpha_2}}d\eta\Big\}\\
&\qquad\cdot\int_{\mathbb{R}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\delta(\tau+|\eta|^2-|\xi-\eta|^2)|(\langle\nabla\rangle^{\alpha_1}\gamma_0\langle\nabla\rangle^{\alpha_2})^\wedge(\eta,\xi-\eta)|^2 d\eta d\xi d\tau.
\end{aligned} \tag{3.17}$$

Then, integrating out the delta function with respect to τ and using the Plancherel theorem again,

$$\begin{aligned}&\| |\nabla|^{\tilde{\alpha}} \rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}} \|^2_{L^2_{t\in\mathbb{R}} H^{\alpha_0}_x} \\ &\le \sup_{\tau,\xi} \left\{ \int_{\mathbb{R}^d} \frac{|\xi|^{2\tilde{\alpha}} \langle\xi\rangle^{2\alpha_0} \delta(\tau + |\eta|^2 - |\xi-\eta|^2)}{\langle\eta\rangle^{2\alpha_1} \langle\xi-\eta\rangle^{2\alpha_2}} d\eta \right\} \frac{1}{(2\pi)^{2d}} \| \langle\nabla\rangle^{\alpha_1} \gamma_0 \langle\nabla\rangle^{\alpha_2} \|^2_{\mathfrak{S}^2}.\end{aligned} \tag{3.18}$$

Therefore, it suffices to show that $\sup_{\tau,\xi}\{\cdots\}$ is bounded.

We decompose

$$\begin{aligned}&\int_{\mathbb{R}^d} \frac{|\xi|^{2\tilde{\alpha}} \langle\xi\rangle^{2\alpha_0} \delta(\tau + |\eta|^2 - |\xi-\eta|^2)}{\langle\eta\rangle^{2\alpha_1} \langle\xi-\eta\rangle^{2\alpha_2}} d\eta \\ &= \int_{|\eta|\le|\xi-\eta|} + \int_{|\eta|\ge|\xi-\eta|} \frac{|\xi|^{2\tilde{\alpha}} \langle\xi\rangle^{2\alpha_0} \delta(\tau + |\eta|^2 - |\xi-\eta|^2)}{\langle\eta\rangle^{2\alpha_1} \langle\xi-\eta\rangle^{2\alpha_2}} d\eta.\end{aligned} \tag{3.19}$$

By change of the variable $(\xi-\eta) \mapsto \eta$, the second integral becomes

$$\begin{aligned}&\int_{|\eta|\ge|\xi-\eta|} \frac{|\xi|^{2\tilde{\alpha}} \langle\xi\rangle^{2\alpha_0} \delta(\tau + |\eta|^2 - |\xi-\eta|^2)}{\langle\eta\rangle^{2\alpha_1} \langle\xi-\eta\rangle^{2\alpha_2}} d\eta \\ &= \int_{|\eta|\le|\xi-\eta|} \frac{|\xi|^{2\tilde{\alpha}} \langle\xi\rangle^{2\alpha_0} \delta(\tau + |\xi-\eta|^2 - |\eta|^2)}{\langle\eta\rangle^{2\alpha_2} \langle\xi-\eta\rangle^{2\alpha_1}} d\eta \\ &= \int_{|\eta|\le|\xi-\eta|} \frac{|\xi|^{2\tilde{\alpha}} \langle\xi\rangle^{2\alpha_0} \delta(-\tau + |\eta|^2 - |\xi-\eta|^2)}{\langle\eta\rangle^{2\alpha_2} \langle\xi-\eta\rangle^{2\alpha_1}} d\eta.\end{aligned} \tag{3.20}$$

Thus, by the assumption (3.13), we prove the desired uniform bound,

$$\int_{\mathbb{R}^d} \frac{|\xi|^{2\tilde{\alpha}} \langle\xi\rangle^{2\alpha_0} \delta(\tau + |\eta|^2 - |\xi-\eta|^2)}{\langle\eta\rangle^{2\alpha_1} \langle\xi-\eta\rangle^{2\alpha_2}} d\eta \le I_{\tau,\xi} + I_{-\tau,\xi} \le 2\sup_{\tau,\xi} I_{\tau,\xi} < \infty. \quad \square \tag{3.21}$$

Proof of Theorem 3.1. By Lemma 3.3, the proof of Theorem 3.1 can be reduced to the proof of a uniform bound on the integral

$$\begin{aligned}I_{\tau,\xi} &= \int_{\{|\eta|\le|\xi-\eta|\}} \frac{|\xi| \langle\xi\rangle^{2\alpha_0} \delta(\tau + |\eta|^2 - |\xi-\eta|^2)}{\langle\eta\rangle^{2\max\{\alpha_1,\alpha_2\}} \langle\xi-\eta\rangle^{2\min\{\alpha_1,\alpha_2\}}} d\eta \\ &= \int_{\{|\eta|\le|\xi-\eta|\}} \frac{|\xi| \langle\xi\rangle^{2\alpha_0} \delta(\tau - |\xi|^2 + 2\xi\cdot\eta)}{\langle\eta\rangle^{2\max\{\alpha_1,\alpha_2\}} \langle\xi-\eta\rangle^{2\min\{\alpha_1,\alpha_2\}}} d\eta.\end{aligned} \tag{3.22}$$

Here, we may assume that $\tau \ge 0$, since if $\tau < 0$, then $\tau + |\eta|^2 - |\xi-\eta|^2 < 0$ in the integral domain, so the delta function in (3.22) is zero.

When $d = 1$, using the trivial inequality

$$|\xi| \le |\eta| + |\xi-\eta| \le 2|\xi-\eta| \tag{3.23}$$

in the integral domain, we obtain

$$I_{\tau,\xi} \lesssim \frac{\langle\xi\rangle^{2\alpha_0}}{\langle\xi\rangle^{\min\{\alpha_1,\alpha_2\}}} \int_{|\eta|\le|\xi-\eta|} |\xi| \delta(\tau - \xi^2 + 2\xi\eta) d\eta \sim 1.$$

Suppose that $d \ge 2$. Given $\xi \in \mathbb{R}^d$, changing the variable η by a rotation making $(1,0,\cdots,0) \in \mathbb{R}^d_\eta$ parallel to ξ and then integrating out the delta function, we write the integral as

$$\begin{aligned} I_{\tau,\xi} &= \int_{\mathbb{R}^{d-1}} \int_{|\eta_1|\le|\eta_1-|\xi||} \frac{|\xi|\langle\xi\rangle^{2\alpha_0}\delta(\tau-|\xi|^2+2|\xi|\eta_1)}{\langle(\eta_1,\eta')\rangle^{2\max\{\alpha_1,\alpha_2\}}\langle(\eta_1-|\xi|,\eta')\rangle^{2\min\{\alpha_1,\alpha_2\}}} d\eta_1 d\eta' \\ &= \frac{1}{2}\int_{\mathbb{R}^{d-1}} \frac{\langle\xi\rangle^{2\alpha_0} d\eta'}{\langle(\eta_1^*,\eta')\rangle^{\max\{\alpha_1,\alpha_2\}}\langle(\eta_1^*-|\xi|,\eta')\rangle^{2\min\{\alpha_1,\alpha_2\}}}, \end{aligned} \tag{3.24}$$

where $\eta=(\eta_1,\eta')\in\mathbb{R}\times\mathbb{R}^{d-1}$ and $\eta_1^*=\frac{|\xi|^2-\tau}{2|\xi|}$ with $|\eta_1^*|\le|\eta_1^*-|\xi||$. Note that by the trivial inequality as in (3.23), we have $|\eta_1^*-|\xi||\ge\frac{|\xi|}{2}$. Thus, Theorem 3.1 follows from the uniform bound on

$$\tilde{I}_{\tau,\xi} := \int_{\mathbb{R}^{d-1}} \frac{\langle\xi\rangle^{2\alpha_0} d\eta'}{\langle\eta'\rangle^{2\max\{\alpha_1,\alpha_2\}}\langle(\frac{|\xi|}{2},\eta')\rangle^{2\min\{\alpha_1,\alpha_2\}}}.$$

We decompose

$$\tilde{I}_{\tau,\xi} = \int_{|\eta'|\le|\xi|} + \int_{|\eta'|\ge|\xi|} \frac{\langle\xi\rangle^{2\alpha_0} d\eta'}{\langle\eta'\rangle^{2\max\{\alpha_1,\alpha_2\}}\langle(\frac{|\xi|}{2},\eta')\rangle^{2\min\{\alpha_1,\alpha_2\}}} =: \tilde{I}^{(1)}_{\tau,\xi}+\tilde{I}^{(2)}_{\tau,\xi}. \tag{3.25}$$

For the first integral, using that $\frac{|\xi|}{2}\le|(\frac{|\xi|}{2},\eta')|\le\frac{\sqrt{5}}{2}|\xi|$ in the integral domain, we get

$$\begin{aligned} \tilde{I}^{(1)}_{\tau,\xi} &\sim \int_{|\eta'|\le|\xi|} \frac{\langle\xi\rangle^{2\alpha-2\min\{\alpha_1,\alpha_2\}} d\eta'}{\langle\eta'\rangle^{2\max\{\alpha_1,\alpha_2\}}} \\ &\sim \begin{cases} \langle\xi\rangle^{2\alpha_0-2(\alpha_1+\alpha_2)+d-1} & \text{if } 0\le\max\{\alpha_1,\alpha_2\}<\frac{d-1}{2}, \\ \langle\xi\rangle^{2\alpha_0-2\min\{\alpha_1,\alpha_2\}}\ln\langle\xi\rangle & \text{if } \max\{\alpha_1,\alpha_2\}=\frac{d-1}{2}, \\ \langle\xi\rangle^{2\alpha_0-2\min\{\alpha_1,\alpha_2\}} & \text{if } \max\{\alpha_1,\alpha_2\}>\frac{d-1}{2}. \end{cases} \end{aligned} \tag{3.26}$$

The second integral $\tilde{I}^{(2)}_{\tau,\xi}$ is bounded by

$$\int_{|\eta'|\ge|\xi|} \frac{\langle\xi\rangle^{2\alpha_0} d\eta'}{\langle\eta'\rangle^{2\max\{\alpha_1,\alpha_2\}+2\min\{\alpha_1,\alpha_2\}}} = \int_{|\eta'|\ge|\xi|} \frac{\langle\xi\rangle^{2\alpha_0} d\eta'}{\langle\eta'\rangle^{2(\alpha_1+\alpha_2)}} \lesssim \langle\xi\rangle^{2\alpha_0-2(\alpha_1+\alpha_2)+(d-1)}, \tag{3.27}$$

since $2(\alpha_1+\alpha_2)>d-1$. Both $\tilde{I}^{(1)}_{\tau,\xi}$ and $\tilde{I}^{(2)}_{\tau,\xi}$ are uniformly bounded due to the assumption (3.8). □

Next, we prove optimality of the Strichartz estimate (3.9).

Theorem 3.4 *(Optimality of Theorem 3.1). The assumptions in Theorem 3.1 are necessary.*

The following dual formulation is useful to find the necessary conditions on the Strichartz estimate (3.9).

Lemma 3.5 *(Dual inequality). The Strichartz estimate (3.14) holds if and only if*

$$\left\| \frac{|\xi|^{\tilde{\alpha}}\langle\xi\rangle^{\alpha_0}\tilde{V}(-|\eta|^2+|\xi-\eta|^2,\xi)}{\langle\eta\rangle^{\alpha_1}\langle\xi-\eta\rangle^{\alpha_2}} \right\|_{L^2_\xi L^2_\eta} \le \|\tilde{V}(\tau,\xi)\|_{L^2_{\tau\in\mathbb{R}}L^2_\xi}. \tag{3.28}$$

Proof. Using the Plancherel theorem and (3.16) and then integrating out the delta function, we write

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^d} (|\nabla|^{\tilde{\alpha}}\langle\nabla\rangle^{\alpha_0}\rho_{e^{it\Delta}\gamma_0 e^{-it\Delta}})(x)\overline{V(t,x)}dxdt \\
&= \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\Big\{\frac{1}{(2\pi)^{d-1}}\int_{\mathbb{R}^d}\delta(\tau+|\eta|^2-|\xi-\eta|^2)\hat{\gamma}_0(\eta,\xi-\eta)d\eta\Big\} \\
&\qquad\cdot\overline{|\xi|^{\tilde{\alpha}}\langle\xi\rangle^{\alpha_0}\tilde{V}(\tau,\xi)}d\xi d\tau \\
&= \frac{1}{(2\pi)^{2d}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\hat{\gamma}_0(\eta,\xi-\eta)\overline{|\xi|^{\tilde{\alpha}}\langle\xi\rangle^{\alpha_0}\tilde{V}(-|\eta|^2+|\xi-\eta|^2,\xi)}d\eta d\xi.
\end{aligned} \tag{3.29}$$

By Hölder inequality and the Plancherel theorem, it is bounded by

$$\begin{aligned}
&\frac{1}{(2\pi)^{2d}}\|(\langle\nabla\rangle^{\alpha_1}\gamma_0\langle\nabla\rangle^{\alpha_2})^\wedge\|_{L^2_{\xi,\eta}}\Big\|\frac{|\xi|^{\tilde{\alpha}}\langle\xi\rangle^{\alpha_0}\tilde{V}(-|\eta|^2+|\xi-\eta|^2,\xi)}{\langle\eta\rangle^{\alpha_1}\langle\xi-\eta\rangle^{\alpha_2}}\Big\|_{L^2_{\xi,\eta}} \\
&= \frac{1}{(2\pi)^{d}}\|\langle\nabla\rangle^{\alpha_1}\gamma_0\langle\nabla\rangle^{\alpha_2}\|_{\mathfrak{S}^2}\Big\|\frac{|\xi|^{\tilde{\alpha}}\langle\xi\rangle^{\alpha_0}\tilde{V}(-|\eta|^2+|\xi-\eta|^2,\xi)}{\langle\eta\rangle^{\alpha_1}\langle\xi-\eta\rangle^{\alpha_2}}\Big\|_{L^2_{\xi,\eta}}.
\end{aligned} \tag{3.30}$$

Therefore, by duality, (3.14) is equivalent to (3.28). □

Proof of Theorem 3.4. By the duality lemma (Lemma 3.5), the inequality (3.9) holds if and only if

$$\Big\|\frac{|\xi|^{\tilde{\alpha}}\langle\xi\rangle^{\alpha_0}\tilde{V}(-|\eta|^2+|\xi-\eta|^2,\xi)}{\langle\eta\rangle^{\alpha_1}\langle\xi-\eta\rangle^{\alpha_2}}\Big\|_{L^2_\xi L^2_\eta}\lesssim\|\tilde{V}(\tau,\xi)\|_{L^2_{\tau\in\mathbb{R}}L^2_\xi}. \tag{3.31}$$

The square of the left hand side is

$$\begin{aligned}
&\Big\|\frac{|\xi|^{\tilde{\alpha}}\langle\xi\rangle^{\alpha_0}\tilde{V}(-|\eta|^2+|\xi-\eta|^2,\xi)}{\langle\eta\rangle^{\alpha_1}\langle\xi-\eta\rangle^{\alpha_2}}\Big\|^2_{L^2_\xi L^2_\eta} \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|\xi|^{2\tilde{\alpha}}\langle\xi\rangle^{2\alpha_0}}{\langle\eta\rangle^{2\alpha_1}\langle\xi-\eta\rangle^{2\alpha_2}}|\tilde{V}(|\xi|^2-2\xi\cdot\eta,\xi)|^2d\eta d\xi.
\end{aligned} \tag{3.32}$$

Changing the variable η by a rotation making $(1,0,\cdots,0)\in\mathbb{R}^d_\eta$ parallel to ξ and then changing the variable $\tau=|\xi|^2-2|\xi|\eta_1$ as in the proof of Theorem 3.1, we write

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\frac{|\xi|^{2\tilde{\alpha}}\langle\xi\rangle^{2\alpha_0}}{\langle(\eta_1,\eta')\rangle^{2\alpha_1}\langle(\eta_1-|\xi|,\eta')\rangle^{2\alpha_2}}|\tilde{V}(|\xi|^2-2|\xi|\eta_1,\xi)|^2d\eta_1 d\eta' d\xi \\
&= \frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^{d-1}}\int_{\mathbb{R}}\frac{|\xi|^{2\tilde{\alpha}-1}\langle\xi\rangle^{2\alpha_0}}{\langle(\frac{|\xi|^2-\tau}{2|\xi|},\eta')\rangle^{2\alpha_1}\langle(-\frac{|\xi|^2+\tau}{2|\xi|},\eta')\rangle^{2\alpha_2}}|\tilde{V}(\tau,\xi)|^2d\tau d\eta' d\xi \\
&= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}^d}\Big\{\int_{\mathbb{R}^{d-1}}\frac{|\xi|^{2\tilde{\alpha}-1}\langle\xi\rangle^{2\alpha_0}}{\langle(\frac{|\xi|^2-\tau}{2|\xi|},\eta')\rangle^{2\alpha_1}\langle(-\frac{|\xi|^2+\tau}{2|\xi|},\eta')\rangle^{2\alpha_2}}d\eta'\Big\}|\tilde{V}(\tau,\xi)|^2d\xi d\tau,
\end{aligned} \tag{3.33}$$

where $\eta=(\eta_1,\eta')\in\mathbb{R}\times\mathbb{R}^d$.

1. Necessity of the condition (3.7): From the inner integral $\{\cdots\}$ over $\mathbb{R}^{d-1}$ in (3.33), we see that it is necessary to assume that $\alpha_1+\alpha_2>\frac{d-1}{2}$ in (3.7), because if $\alpha_1+\alpha_2\le\frac{d-1}{2}$, the inequality (3.9) fails.

2. Necessity of the homogeneous half derivative on the left hand side of (3.9): Suppose that $\alpha_1+\alpha_2>\frac{d-1}{2}$. Let

$$\tilde{V}_n(\tau,\xi)=n^{\frac{d+2}{2}}\mathbf{1}_{[-\frac{1}{n^2},\frac{1}{n^2}]}(\tau)\mathbf{1}_{B_{0,\frac{1}{n}}}(\xi),$$

where $B_{0,r}$ is the ball of radius r centered at 0 in $\mathbb{R}^d$. Note that for large n, $\tilde{V}_n$ is localized in low frequencies. We observe that by (3.33), if $\tilde{\alpha}<\frac{1}{2}$, then

$$\begin{aligned}&\left\|\frac{|\xi|^{\tilde\alpha}\langle\xi\rangle^{\alpha_0}\tilde V(-|\eta|^2+|\xi-\eta|^2,\xi)}{\langle\eta\rangle^{\alpha_1}\langle\xi-\eta\rangle^{\alpha_2}}\right\|^2_{L^2_\xi L^2_\eta}\\&\sim\int_{\mathbb{R}}\int_{\mathbb{R}^d}\Big\{\int_{\mathbb{R}^{d-1}}\frac{n^{1-2\tilde\alpha}}{\langle\eta'\rangle^{2(\alpha_1+\alpha_2)}}d\eta'\Big\}|\tilde V(\tau,\xi)|^2d\xi d\tau\sim n^{1-2\tilde\alpha}\underset{n\to\infty}{\longrightarrow}\infty,\end{aligned}\tag{3.34}$$

while $\|\tilde V_n\|_{L^2_{\tau\in\mathbb{R}}L^2_\xi}\sim 1$. Thus, the inequality (3.9) fails when $\tilde\alpha<\frac{1}{2}$.

3. Necessity of the condition (3.8): Suppose that $\alpha_1+\alpha_2>\frac{d-1}{2}$ and $\tilde\alpha=\frac{1}{2}$. We further assume that $\alpha_1\geq\alpha_2$. Now we define the sequence $\{V_n\}_{n=1}^\infty$ by

$$\tilde V_n(\tau,\xi)=\mathbf{1}_{[n^2-\frac12,n^2+\frac12]}(\tau)\mathbf{1}_{[n-\frac12,n+\frac12]\times[\frac12,\frac12]^{d-1}}(\xi),\tag{3.35}$$

where $\xi=(\xi_1,\xi')\in\mathbb{R}\times\mathbb{R}^{d-1}$, so that $\|\tilde V_n\|_{L^2_\tau L^2_\xi}=1$. Then, $|\frac{|\xi|^2-\tau}{2|\xi|}|\leq\frac12+o_n(1)$ and $-\frac{|\xi|^2+\tau}{2|\xi|}=-n+o_n(1)$ in the support of $\tilde V(\tau,\xi)$, where $o_n(1)\to 0$ as $n\to\infty$. Hence, by (3.33),

$$\begin{aligned}&\left\|\frac{|\xi|^{1/2}\langle\xi\rangle^{\alpha_0}\tilde V_n(-|\eta|^2+|\xi-\eta|^2,\xi)}{\langle\eta\rangle^{\alpha_1}\langle\xi-\eta\rangle^{\alpha_2}}\right\|^2_{L^2_\xi L^2_\eta}\\&\gtrsim\int_{|\eta'|\leq\frac n2}\frac{n^{2\alpha_0}}{\langle\eta'\rangle^{2\alpha_1}\langle(-n,\eta')\rangle^{2\alpha_2}}d\eta'\sim n^{2\alpha_0-2\alpha_2}\int_{|\eta'|\leq\frac n2}\frac{d\eta'}{\langle\eta'\rangle^{2\alpha_1}}\\&\sim\begin{cases}n^{2\alpha_0-2(\alpha_1+\alpha_2)+d-1} & \text{if } 0\leq\alpha_1<\frac{d-1}{2},\\ n^{2\alpha_0-2\alpha_2}\ln n & \text{if } \alpha_1=\frac{d-1}{2},\\ n^{2\alpha_0-2\alpha_2} & \text{if } \alpha_1>\frac{d-1}{2}\end{cases}\end{aligned}\tag{3.36}$$

for sufficiently large n. Thus, (3.31) fails unless (3.8) is not satisfied.

When $\alpha_1\leq\alpha_2$, we use the sequence $\{V_n\}_{n=1}^\infty$ given by

$$\tilde V(\tau,\xi)=\mathbf{1}_{[-n^2-\frac12,-n^2+\frac12]}(\tau)\mathbf{1}_{[n-\frac12,n+\frac12]\times[\frac12,\frac12]^{d-1}}(\xi)\tag{3.37}$$

to prove that the condition (3.8) is necessary. □

3.3. *Strichartz estimates for operator kernels*

We finish this section by recalling the statement of the Strichartz estimates for operator kernels, that we established in [11].

Theorem 3.6 *(Strichartz estimates for operator kernels). Let $I\subset\mathbb{R}$. Then, we have*

$$\begin{aligned}\|e^{it\Delta}\gamma_0e^{-it\Delta}\|_{\mathcal{S}^\alpha(\mathbb{R})}&\lesssim\|\gamma_0\|_{\mathcal{H}^\alpha},\\ \Big\|\int_0^t e^{i(t-s)\Delta}R(s)e^{-i(t-s)\Delta}ds\Big\|_{\mathcal{S}^\alpha(\mathbb{R})}&\lesssim\|R(t)\|_{L^1_t(\mathbb{R};\mathcal{H}^\alpha)}.\end{aligned}\tag{3.38}$$

4. Linear response theory: invertibility of $(1+\mathcal{L})$

We review the linear response theory from Section 3 of Lewin and Sabin [19], which addresses the invertibility of the operator $(1+\mathcal{L})$, with $\mathcal{L}$ defined by

$$\begin{aligned}\mathcal{L}(\phi)&=-w_2*\rho\Big[e^{it\Delta}\Big(\mathcal{W}^{(1)}_{w_1*\phi}(t)\gamma_f+\gamma_f\mathcal{W}^{(1)}_{w_1*\phi}(t)^*\Big)e^{-it\Delta}\Big]\\&=iw_2*\rho\Big[\int_0^t e^{i(t-t_1)\Delta}\big[(w_1*\phi)(t_1),\gamma_f\big]e^{-i(t-t_1)\Delta}dt_1\Big],\end{aligned}\tag{4.1}$$

where $w = w_1 * w_2$. Roughly speaking, it asserts that $(1+\mathcal{L})$ is invertible on $L^2_{t\geq 0}L^2_x$, provided that f is strictly decreasing, and that $\hat{w}_+(0)$ and $\hat{w}_-$ are not too large, where $A_\pm = \max\{\pm A, 0\}$ so that $A = A_+ - A_-$.

Proposition 4.1 *(Invertibility of $(1+\mathcal{L})$). Let $d \geq 3$. We assume that $f \in L^\infty_{r\geq 0}$ is real-valued, $f'(r) < 0$ for $r > 0$,*

$$\int_0^\infty (r^{d/2-1}|f(r)| + |f'(r)|)dr < \infty \quad \text{and} \quad \int_{\mathbb{R}^d} \frac{\check{g}(x)}{|x|^{d-2}}dx < \infty, \tag{4.2}$$

where $g(\xi) = f(|\xi|^2)$. Moreover, we assume that the interaction potential $w \in L^1$ is even,

$$\|\hat{w}_-\|_{L^\infty} < 2|\mathbb{S}^{d-1}|\Big(\int_{\mathbb{R}^d} \frac{|\check{g}(x)|}{|x|^{d-2}}dx\Big)^{-1} \quad \text{and} \quad \hat{w}_+(0) < \frac{2}{\epsilon_g}|\mathbb{S}^{d-1}|, \tag{4.3}$$

where

$$\epsilon_g := -\liminf_{(\tau,\xi)\to(0,0)} \frac{\mathrm{Re}(m_f(\tau,\xi))}{2|\mathbb{S}^{d-1}|} \tag{4.4}$$

and

$$(\mathcal{F}_t^{-1} m_f)(t,\xi) = 2\mathbf{1}_{t\geq 0}\sqrt{2\pi}\sin(t|\xi|^2)\check{g}(2t\xi). \tag{4.5}$$

Then, $1+\mathcal{L}$ is invertible on $L^2_{t\geq 0}L^2_x$.

Sketch of the proof. We sketch the proof for the sake of completeness of the article and for the convenience of the reader. For details, we refer the reader to [19, Proposition 1, Proposition 2 and Corollary 1]. We assume $d \geq 3$ for brevity, however, the invertibility of $(1+\mathcal{L})$ was proved in [19] for any dimension $d \geq 1$.

The space–time Fourier transformation of $\mathcal{L}(\phi)$ is directly computed as

$$(\mathcal{L}\phi)^\sim(\tau,\xi) = \hat{w}(\xi)m_f(\tau,\xi)\tilde{\phi}(\tau,\xi), \quad \forall \phi \in \mathcal{D}([0,+\infty)\times\mathbb{R}^d), \tag{4.6}$$

with (4.5), in other words,

$$\widehat{\mathcal{L}\phi}(t,\xi) = 2\sqrt{2\pi}\,\hat{w}(\xi)\int_0^\infty \sin(s|\xi|^2)\check{g}(2s\xi)\hat{\phi}(t-s,\xi)ds. \tag{4.7}$$

Note that the operator $\mathcal{L}$ maps $L^2_{t\geq 0}L^2_x$ to itself, because $\widehat{\mathcal{L}\phi}(t,\xi) = 0$ for $t < 0$. Moreover, we have

$$\|m_f\|_{L^\infty_{\tau,\xi}} \leq \frac{1}{2|\mathbb{S}^{d-1}|}\Big(\int_{\mathbb{R}^d} \frac{|\check{g}(x)|}{|x|^{d-2}}dx\Big) \tag{4.8}$$

and

$$\|\mathcal{L}\|_{L^2_{t\geq 0}L^2_x \to L^2_{t\geq 0}L^2_x} \leq \frac{\|\hat{w}\|_{L^\infty}}{2|\mathbb{S}^{d-1}|}\Big(\int_{\mathbb{R}^d} \frac{|\check{g}(x)|}{|x|^{d-2}}dx\Big) \tag{4.9}$$

(see [19, Proposition 1]). We remark that the operator $\mathcal{L}$ looks different from the corresponding linear operator $\mathcal{L}_1$ in Lewin–Sabin [19] at first glance, however they are indeed the same, since

$$\begin{aligned}(\mathcal{L}\phi)^\sim(\tau,\xi) &= \hat{w}_2(\xi)\Big(\rho\Big[i\int_0^t e^{i(t-t_1)\Delta}\big[(w_1*\phi)(t_1),\gamma_f\big]e^{-i(t-t_1)\Delta}dt_1\Big]\Big)^\sim(\tau,\xi)\\ &= \hat{w}_2(\xi)\hat{w}_1(\xi)m_f(\tau,\xi)\tilde{\phi}(\tau,\xi) \quad (\text{by [19, Proposition 1]})\\ &= \hat{w}(\xi)m_f(\tau,\xi)\tilde{\phi}(\tau,\xi).\end{aligned} \tag{4.10}$$

When $\gamma_f = \mathbf{1}_{(-\Delta\leq\mu)}$, one can compute the multiplier $m_d^F(\mu,\tau,\xi) := m_f(\tau,\xi)$ as

$$m_d^F(\mu,\tau,\xi) = \frac{|\mathbb{S}^{d-2}|\mu^{\frac{d-1}{2}}}{(2\pi)^{\frac{d-1}{2}}} \int_0^1 m_1^F(\mu(1-r^2),\tau,\xi) r^{d-2} dr, \tag{4.11}$$

where

$$\begin{aligned} m_1^F(\mu,\tau,\xi) &= \frac{1}{2\sqrt{2\pi}|\xi|} \log\left|\frac{(|\xi|^2+2|\xi|\sqrt{\mu})^2-\tau^2}{(|\xi|^2-2|\xi|\sqrt{\mu})^2-\tau^2}\right| \\ &\quad + i\frac{\sqrt{\pi}}{2\sqrt{2}|\xi|}\left\{\mathbf{1}_{(|\tau+|\xi|^2|\le 2\sqrt{\mu}|\xi|)} - \mathbf{1}_{(|\tau-|\xi|^2|\le 2\sqrt{\mu}|\xi|)}\right\} \end{aligned} \tag{4.12}$$

(see [19, Proposition 2]). By the relation $\gamma_f = f(-\Delta) = -\int_0^\infty \mathbf{1}_{(-\Delta\le s)} f'(s) ds$, m_f can be written in terms of m_d^F as

$$m_f(\tau,\xi) = -\int_0^\infty m_d^F(s,\tau,\xi) f'(s) ds. \tag{4.13}$$

For $\phi \in L^2_{t\ge 0}L^2_x$, the space–time Fourier transformation of $(1+\mathcal{L})\phi$ is given by $(1+\hat{w}(\xi)m_f(\tau,\xi))\tilde{\phi}(\tau,\xi)$. Thus, the invertibility of $(1+\mathcal{L})$ follows from a uniform lower bound on $|1+\hat{w}m_f|$. Let

$$A := \left\{\xi\in\mathbb{R}^d : |\hat{w}(\xi)| \ge \frac{1}{4|\mathbb{S}^{d-1}|}\int_{\mathbb{R}^d} \frac{|\check{g}(x)|}{|x|^{d-2}} dx\right\}. \tag{4.14}$$

Then, by the bound (4.8), $|(1+\hat{w}m_f)| \ge \frac{1}{2}$ on A. Note that A^c is a compact subset in $\mathbb{R}^d$, because $\hat{w}(\xi)\to 0$ as $\xi\to\infty$. Moreover, by (4.13), m_f is continuous on $\mathbb{R}\times(\mathbb{R}^d\setminus\{0\})$ (so as $(1+\hat{w}m_f)$ by the Riemann–Lebesgue lemma), since m_d^F is continuous on $\mathbb{R}\times(\mathbb{R}^d\setminus\{0\})$. Therefore, it suffices to show that $(1+\hat{w}m_f)$ is non-zero for all ξ.

We consider the four cases separately.

Case 1 ($(\tau,\xi)=(0,\xi)$ **with** $\xi\neq 0$) We observe that $m_f(0,\xi)\ge 0$ for $\xi\ne 0$, since $f'(s)<0$ and $m_1^F(s,0,\xi)\ge 0$ in the integral (4.13) (see (4.12)). Hence, it follows that

$$\begin{aligned} m_f(0,\xi)\hat{w}(\xi)+1 &\ge 1-\hat{w}_-(\xi)m_f(0,\xi) \\ &\ge 1-\|\hat{w}_-\|_{L^\infty}\frac{1}{2|\mathbb{S}^{d-1}|}\left(\int_{\mathbb{R}^d}\frac{|\check{g}(x)|}{|x|^{d-2}}dx\right) \quad \text{(by (4.8))} \\ &> 0 \quad \text{(by the assumption (4.3) on } \hat{w}_-\text{)}. \end{aligned} \tag{4.15}$$

Case 2 ($(\tau,\xi)=(\tau,0)$ **with** $\tau\neq 0$) In this case, $m_1^F(\tau,0)=0$, so $(m_f(\tau,0)\hat{w}(0)+1)=1$.
Case 3 ((τ,ξ) **with** $\tau\neq 0$ **and** $\xi\neq 0$) It suffices to show that $\mathrm{Im}(m_f(\tau,\xi))\neq 0$. By the relation $\mathrm{Im}(m_f(-\tau,\xi)) = -\mathrm{Im}(m_f(\tau,\xi))$, we may assume that $\tau>0$. By (4.13) and (4.11), one can write the imaginary part of $m_f(\tau,\xi)$ explicitly as

$$\mathrm{Im}(m_f(\tau,\xi)) = \frac{|\mathbb{S}^{d-2}|}{4(2\pi)^{\frac{d-2}{2}}}\int_0^1 r^{d-2}\left\{\int_{\frac{(\tau-|\xi|^2)^2}{4|\xi|^2(1-r^2)}}^{\frac{(\tau+|\xi|^2)^2}{4|\xi|^2(1-r^2)}} s^{\frac{d-1}{2}} f'(s) ds\right\} dr. \tag{4.16}$$

Since by the assumption $f'(s)<0$, we conclude from (4.16) that $\mathrm{Im}(m_f(\tau,\xi))\neq 0$.
Case 4 ((τ,ξ) **in the neighborhood of** $(0,0)$) By the definition of m_f and (4.13), one can show that

$$-\epsilon_g 2|\mathbb{S}^{d-1}| \le \mathrm{Re}(m_f(\tau,\xi)) \le \frac{1}{2|\mathbb{S}^{d-1}|}\left(\int_{\mathbb{R}^d}\frac{|\check{g}(x)|}{|x|^{d-2}}dx\right) \tag{4.17}$$

near $(0,0)$ (see [19] for details). Thus, by the assumptions on $\hat{w}_\pm$, $\mathrm{Re}(\hat{w}(\xi)m_f(\tau,\xi)+1)>0$. □

5. Bound on $\mathcal{A}_{m,n}(\phi)$

In this section, we estimate the operator $\mathcal{A}_{m,n}$.

Proposition 5.1 *(Bounds on $\mathcal{A}_{m,n}$). Let $d \geq 3$, $\beta > \frac{d+2}{2}$ and $\beta_0 > \frac{1}{4}$. Then, there exists $C_{\mathcal{A}} > 0$ such that for any (m,n) with $m,n \geq 1$,*

$$\|\mathcal{A}_{m,n}(\phi)\|_{L^2_{t,x}} \leq C_{\mathcal{A}}^{m+n+1} \|\langle\cdot\rangle^\beta f\|_{L^\infty} \|\phi\|^{m+n}_{L^2_{t,x}} \tag{5.1}$$

and

$$\begin{aligned} &\|\mathcal{A}_{m,n}(\phi) - \mathcal{A}_{m,n}(\psi)\|_{L^2_{t,x}} \\ &\leq (m+n) C_{\mathcal{A}}^{m+n+1} \|\langle\cdot\rangle^\beta f\|_{L^\infty} \Big\{\|\phi\|_{L^2_{t,x}} + \|\psi\|_{L^2_{t,x}}\Big\}^{m+n-1} \|\phi-\psi\|_{L^2_{t,x}}, \end{aligned} \tag{5.2}$$

where the constant $C_{\mathcal{A}}$ depends only on d, $\|\langle\cdot\rangle^{\beta_0}\hat{w}_1\|_{L^\infty}$, $\|\hat{w}_2\|_{L^\infty}$, $\|\hat{w}_1\|_{L^{\frac{2d}{d-2}}}$, $\|\hat{w}_2\|_{L^{\frac{2d}{d-2}}}$ and $\|\hat{w}_2\|_{L^{\frac{2d}{d-3}}}$.

Proof. We will prove the proposition by the standard duality argument. For notational convenience, we denote $W = w_1 * \phi$. By the definition of $\mathcal{A}_{m,n}$, we write

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \mathcal{A}_{m,n}(\phi)(t) U(t,x)\, dx dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} w_2 * \rho\Big[e^{it\Delta}\mathcal{W}_W^{(m)}(t)\gamma_f \mathcal{W}_W^{(n)}(t)^* e^{-it\Delta}\Big] U(t,x)\, dx dt \\ &= \int_0^\infty \int_{\mathbb{R}^d} \rho\Big[e^{it\Delta}\mathcal{W}_W^{(m)}(t)\gamma_f \mathcal{W}_W^{(n)}(t)^* e^{-it\Delta}\Big] (w_2 * U)(t,x)\, dx dt. \end{aligned} \tag{5.3}$$

Then, by the formal identity

$$\int_{\mathbb{R}^d} \rho_{\gamma_0} V\, dx = \mathrm{Tr}(\gamma_0 V) \tag{5.4}$$

and the cyclicity of the trace, it becomes

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^d} \mathcal{A}_{m,n}(\phi)(t) U(t,x)\, dx dt \\ &= \mathrm{Tr}\Big(\int_0^\infty e^{it\Delta}\mathcal{W}_W^{(m)}(t)\gamma_f \mathcal{W}_W^{(n)}(t)^* e^{-it\Delta}(w_2 * U)(t)\, dt\Big) \\ &= \mathrm{Tr}\Big(\int_0^\infty \mathcal{W}_W^{(m)}(t)\gamma_f \mathcal{W}_W^{(n)}(t)^* e^{-it\Delta}(w_2 * U)(t) e^{it\Delta}\, dt\Big). \end{aligned} \tag{5.5}$$

Note that the application of the formal identity (5.5) in (5.9) will be justified by the estimates below.

First, we consider the higher order terms with $m+n \geq 3$. In this case, we employ the following two inequalities,

$$\Big\| \int_0^\infty e^{-it\Delta} V(t) e^{it\Delta} dt \Big\|_{\mathfrak{S}^{2d}} \le c \|V\|_{L_t^2 L_x^d}, \tag{5.6}$$

$$\Big\| \int_0^\infty e^{-it\Delta} V(t) e^{it\Delta} dt \langle \nabla \rangle^{-\tilde{\beta}} \Big\|_{\mathfrak{S}^{\frac{2d}{d-1}}} \le c \|V\|_{L_t^2 L_x^2}, \tag{5.7}$$

where $\tilde{\beta} = \beta - 2 > \frac{d-2}{2}$. Here, (5.6) is from Theorem 2 in [13] and (5.7) can be obtained from the complex interpolation between (5.6) and (3.10) with $\alpha_1 = 0$ and $\alpha_2 > \frac{d-1}{2}$. Expanding $\mathcal{W}_W^{(m)}(t)$ and $\mathcal{W}_W^{(n)}(t)$ in the expression (5.5) (see (2.5)) and applying the inequality $|\mathrm{Tr}(AB)| \le \mathrm{Tr}(|A||B|)$, we write

$$\begin{aligned}
&\Big| \int_0^\infty \int_{\mathbb{R}^d} \mathcal{A}_{m,n}(\phi)(t) U(t,x) dx dt \Big| \\
&\le \mathrm{Tr}\Big\{ \int_0^\infty \Big(\int_0^\infty \cdots \int_0^\infty e^{-it_m\Delta} |W(t_m)| e^{it_m\Delta} \cdots e^{-it_1\Delta} |W(t_1)| e^{it_1\Delta} dt_1 \cdots dt_m \Big) \gamma_f \\
&\qquad \Big(\int_0^\infty \cdots \int_0^\infty e^{-it_1'\Delta} |W(t_1')| e^{it_1'\Delta} \cdots e^{-it_n'\Delta} |W(t_n')| e^{it_n'\Delta} dt_1' \cdots dt_n' \Big) \\
&\qquad e^{-it\Delta} |w_1 * U(t)| e^{it\Delta} dt \Big\} \\
&= \mathrm{Tr}\Big\{ \Big(\int_0^\infty e^{-it_1\Delta} |W(t_1)| e^{it_1\Delta} dt_1 \Big) \cdots \Big(\int_0^\infty e^{-it_m\Delta} |W(t_m)| e^{it_m\Delta} dt_m \Big) \gamma_f \\
&\quad \cdot \Big(\int_0^\infty e^{-it_n'\Delta} |W(t_n')| e^{it_n'\Delta} dt_n' \Big) \cdots \Big(\int_0^\infty e^{-it_1'\Delta} |W(t_1')| e^{it_1'\Delta} dt_1' \Big) \\
&\quad \cdot \Big(\int_0^\infty e^{-it\Delta} |w_1 * U(t)| e^{it\Delta} dt \Big) \Big\}.
\end{aligned} \tag{5.8}$$

When $m, n \ge 1$, by the Hölder inequality in the Schatten spaces, (5.6) and (5.7), we obtain

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^d} \mathcal{A}_{m,n}(\phi)(t) U(t,x) dx dt \\
&\le \Big\| \int_0^\infty e^{-it\Delta} |W(t)| e^{it\Delta} dt \Big\|_{\mathfrak{S}^{2d}}^{m-1} \Big\| \int_0^\infty e^{-it\Delta} |W(t)| e^{it\Delta} dt \langle \nabla \rangle^{-\tilde{\beta}} \Big\|_{\mathfrak{S}^{\frac{2d}{d-1}}} \\
&\quad \cdot \|(1-\Delta)^{\tilde{\beta}} \gamma_f\|_{\mathcal{B}(L^2)} \Big\| \langle \nabla \rangle^{-\tilde{\beta}} \int_0^\infty e^{-it\Delta} |W(t)| e^{it\Delta} dt \Big\|_{\mathfrak{S}^{\frac{2d}{d-1}}} \\
&\quad \cdot \Big\| \int_0^\infty e^{-it\Delta} |W(t)| e^{it\Delta} dt \Big\|_{\mathfrak{S}^{2d}}^{n-1} \Big\| \int_0^\infty e^{-it\Delta} |w_2 * U(t)| e^{it\Delta} dt \Big\|_{\mathfrak{S}^{2d}} \\
&\le (c\|W\|_{L_t^2 L_x^d})^{m+n-2} (c\|W\|_{L_t^2 L_x^2})^2 \cdot \|(1+|\cdot|)^{\tilde{\beta}} f\|_{L^\infty} \cdot c\|w_2 * U\|_{L_t^2 L_x^d} \quad (W = w_1 * \phi) \\
&\le c^{m+n+1} \|\hat{w}_1\|_{L_x^{\frac{2d}{d-2}}}^{m+n-2} \|\hat{w}_1\|_{L^\infty}^2 \|\hat{w}_2\|_{L^{\frac{2d}{d-2}}} \|(1+|\cdot|)^{\beta} f\|_{L^\infty} \|\phi\|_{L_t^2 L_x^2}^{m+n} \|U\|_{L_t^2 L_x^2},
\end{aligned} \tag{5.9}$$

where $\mathcal{B}(L^2)$ is the operator norm and in the last step, we used that if $r \geq 2$,

$$\begin{aligned}\|w * \phi\|_{L^r} &\leq \|\widehat{w * \phi}\|_{L^{r'}} = \|\hat{w}\hat{\phi}\|_{L^{r'}} \quad \text{(by Hausdorff-Young)}\\ &\leq \|\hat{w}\|_{L^{\frac{2r}{r-2}}}\|\hat{\phi}\|_{L^2} = \|\hat{w}\|_{L^{\frac{2r}{r-2}}}\|\phi\|_{L^2} \quad \text{(by Plancherel).}\end{aligned} \tag{5.10}$$

When either $m = 0$ or $n = 0$, we give the negative derivative $\langle\nabla\rangle^{-\tilde{\beta}}$ to the integral having U and use (5.7) for that term. Then, estimating as above, we can show that

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^d} \mathcal{A}_{m,n}(\phi)(t)U(t,x)dxdt\\ &\leq c^{m+n}\|\hat{w}_1\|^{m+n-1}_{L_x^{\frac{2d}{d-2}}}\|\hat{w}_1\|_{L^\infty}\|\hat{w}_2\|_{L^\infty}\|(1+|\cdot|)^\beta f\|_{L^\infty}\|\phi\|^{m+n}_{L_t^2L_x^2}\|U\|_{L_t^2L_x^2}.\end{aligned} \tag{5.11}$$

Therefore, by duality, we complete the proof of (5.1) for higher order terms.

It remains to consider the case $m = n = 1$. In this case, the inequalities (5.6) and (5.7) does not suffice. Indeed, the first inequality in (5.9) requires $\frac{1}{2d}\cdot(m+n-1)+\frac{d-1}{2d}\cdot 2 \geq 1$, i.e., $m+n \geq 3$. Thus, motivated by Corollary 3.2, in order to upgrade summability, we put negative derivatives on the last term,

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^d} \mathcal{A}_{1,1}(\phi)(t)U(t,x)dxdt\\ &= \mathrm{Tr}\Big(\int_0^\infty\int_0^t\int_0^t e^{-it_1\Delta}W(t_1)e^{it_1\Delta}\gamma_f e^{-it_1'\Delta}W(t_1')e^{it_1'\Delta}e^{-it\Delta}(w_2 * U)(t)e^{it\Delta}dt_1'dt_1dt\Big)\\ &= \mathrm{Tr}\Big(\int_0^\infty\int_0^t\int_0^t \langle\nabla\rangle^{\beta_0}e^{-it_1\Delta}W(t_1)e^{it_1\Delta}\gamma_f e^{-it_1'\Delta}W(t_1')e^{it_1'\Delta}\langle\nabla\rangle^{\beta_0}\\ &\qquad\cdot\langle\nabla\rangle^{-\beta_0}e^{-it\Delta}(w_2 * U)(t)e^{it\Delta}\langle\nabla\rangle^{-\beta_0}dt_1'dt_1dt\Big).\end{aligned} \tag{5.12}$$

We now claim that

$$\begin{aligned}&\mathrm{Tr}\Big(\int_0^\infty\int_0^t\int_0^t e^{-it_1\Delta}V_1(t_1)e^{it_1\Delta}\gamma_f e^{-it_1'\Delta}V_2(t_1')e^{it_1'\Delta}\\ &\qquad\cdot\langle\nabla\rangle^{-\beta_0}e^{-it\Delta}(w_2 * U)(t)e^{it\Delta}\langle\nabla\rangle^{-\beta_0}dt_1'dt_1dt\Big)\\ &\lesssim \|\hat{w}_2\|_{L^{\frac{2d}{d-3}}}\|V_1\|_{L_t^2L_x^2}\|V_2\|_{L_t^2L_x^2}\|U\|_{L_t^2L_x^2}.\end{aligned} \tag{5.13}$$

Indeed, by complex interpolation between (5.6) and (3.10) with $\alpha_1 = \alpha_2 > \frac{d-1}{4}$, we have

$$\Big\|\langle\nabla\rangle^{-\beta_0}\int_0^\infty e^{-it\Delta}V(t)e^{it\Delta}dt\langle\nabla\rangle^{-\beta_0}\Big\|_{\mathfrak{S}^d} \lesssim \|V\|_{L_t^2L_x^{\frac{2d}{3}}}. \tag{5.14}$$

Thus, repeating (5.9) but using (5.14) instead of (5.7),

$$\begin{aligned}
&\mathrm{Tr}\Big(\int_0^\infty\int_0^t\int_0^t e^{-it_1\Delta}V_1(t_1)e^{it_1\Delta}\gamma_f e^{-it_1'\Delta}V_2(t_1')e^{it_1'\Delta}\\
&\qquad\qquad\cdot\langle\nabla\rangle^{-\beta_0}e^{-it\Delta}(w_2*U)(t)e^{it\Delta}\langle\nabla\rangle^{-\beta_0}dt_1'dt_1dt\Big)\\
&\le\Big\|\int_0^\infty e^{-it\Delta}|V_1(t)|e^{it\Delta}dt\langle\nabla\rangle^{-\tilde\beta}\Big\|_{\mathfrak{S}^{\frac{2d}{d-1}}}\|(1-\Delta)^{\tilde\beta}\gamma_f\|_{\mathcal{B}(L^2)}\\
&\quad\cdot\Big\|\langle\nabla\rangle^{-\tilde\beta}\int_0^\infty e^{-it\Delta}|V_2(t)|e^{it\Delta}dt\Big\|_{\mathfrak{S}^{\frac{2d}{d-1}}}\\
&\quad\cdot\Big\|\langle\nabla\rangle^{-\beta_0}\int_0^\infty e^{-it\Delta}|w_2*U(t)|e^{it\Delta}dt\langle\nabla\rangle^{-\beta_0}\Big\|_{\mathfrak{S}^d}\\
&\lesssim\|V_1\|_{L_t^2L_x^2}\|V_2\|_{L_t^2L_x^2}\|(1+|\cdot|)^{\tilde\beta}f\|_{L^\infty}\|w_2*U\|_{L_t^2L_x^{\frac{2d}{3}}}\\
&\lesssim\|\hat w_2\|_{L^{\frac{2d}{d-3}}}\|(1+|\cdot|)^{\tilde\beta}f\|_{L^\infty}\|V_1\|_{L_t^2L_x^2}\|V_2\|_{L_t^2L_x^2}\|U\|_{L_t^2L_x^2},
\end{aligned}\tag{5.15}$$

where in the last step, we used (5.10).

Next, distributing derivatives $1-\Delta=1-\sum_{j=1}^d\partial_{x_j}^2$ and then applying (5.13), we can obtain

$$\begin{aligned}
&\mathrm{Tr}\Big(\int_0^\infty\int_0^t\int_0^t (1-\Delta)e^{-it_1\Delta}V_1(t_1)e^{it_1\Delta}\gamma_f e^{-it_1'\Delta}V_2(t_1')e^{it_1'\Delta}(1-\Delta)\\
&\qquad\qquad\cdot\langle\nabla\rangle^{-\beta_0}e^{-it\Delta}(w_2*U)(t)e^{it\Delta}\langle\nabla\rangle^{-\beta_0}dt_1'dt_1dt\Big)\\
&\lesssim\|(1-\Delta)V_1\|_{L_t^2L_x^2}\|(1-\Delta)V_2\|_{L_t^2L_x^2}\|(1+|\cdot|)^{\tilde\beta+2}f\|_{L^\infty}\|w_2*U\|_{L_t^2L_x^{\frac{2d}{3}}}\\
&\le\|\hat w_2\|_{L^{\frac{2d}{d-3}}}\|(1+|\cdot|)^{\beta}f\|_{L^\infty}\|(1-\Delta)V_1\|_{L_t^2L_x^2}\|(1-\Delta)V_2\|_{L_t^2L_x^2}\|U\|_{L_t^2L_x^2}.
\end{aligned}$$

Hence, interpolating it with (5.13), we get

$$\begin{aligned}
&\mathrm{Tr}\Big(\int_0^\infty\int_0^t\int_0^t \langle\nabla\rangle^{\beta_0}e^{-it_1\Delta}V_1(t_1)e^{it_1\Delta}\gamma_f e^{-it_1'\Delta}V_2(t_1')e^{it_1'\Delta}\langle\nabla\rangle^{\beta_0}\\
&\qquad\qquad\cdot\langle\nabla\rangle^{-\beta_0}e^{-it\Delta}(w_2*U)(t)e^{it\Delta}\langle\nabla\rangle^{-\beta_0}dt_1'dt_1dt\Big)\\
&\lesssim\|V_1\|_{L_t^2H_x^{\beta_0}}\|V_2\|_{L_t^2H_x^{\beta_0}}\|w_2*U\|_{L_t^2L_x^{\frac{2d}{3}}}\\
&\le\|\hat w_2\|_{L^{\frac{2d}{d-3}}}\|(1+|\cdot|)^{\beta}f\|_{L^\infty}\|V_1\|_{L_t^2H_x^{\beta_0}}\|V_2\|_{L_t^2H_x^{\beta_0}}\|U\|_{L_t^2L_x^2}.
\end{aligned}\tag{5.16}$$

Finally, coming back to (5.12), applying this inequality, we prove that

$$\begin{aligned}
&\int_0^\infty\int_{\mathbb{R}^d}\mathcal{A}_{1,1}(\phi)(t)U(t,x)dxdt\\
&\lesssim\|\hat w_2\|_{L^{\frac{2d}{d-3}}}\|(1+|\cdot|)^{\beta}f\|_{L^\infty}\|w_1*\phi\|_{L_t^2H_x^{\beta_0}}\|w_1*\phi\|_{L_t^2H_x^{\beta_0}}\|U\|_{L_t^2L_x^2}\\
&\lesssim\|\langle\cdot\rangle^{\beta_0}\hat w_1\|_{L^\infty}^2\|\hat w_2\|_{L^{\frac{2d}{d-3}}}\|(1+|\cdot|)^{\beta}f\|_{L^\infty}\|\phi\|_{L_t^2L_x^2}^2\|U\|_{L_t^2L_x^2}.
\end{aligned}\tag{5.17}$$

For (5.2), we decompose $\mathcal{W}_{w_1*\phi}^{(n)}(t)-\mathcal{W}_{w_1*\psi}^{(n)}(t)$ into the sum of n integrals,

$$
\begin{aligned}
&(-i)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 e^{-it_n\Delta}\big(w_1*(\phi-\psi)\big)(t_n)e^{it_n\Delta}\\
&\qquad\qquad\qquad \cdot e^{-it_{n-1}\Delta}(w_1*\phi)(t_{n-1})e^{it_{n-1}\Delta}\cdots e^{-it_1\Delta}(w_1*\phi)(t_1)e^{it_1\Delta}\\
&+(-i)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 e^{-it_n\Delta}(w_1*\psi)(t_n)e^{it_n\Delta}\\
&\qquad\qquad\qquad \cdot e^{-it_{n-1}\Delta}\big(w_1*(\phi-\psi)\big)(t_{n-1})e^{it_{n-1}\Delta}\cdots e^{-it_1\Delta}(w_1*\phi)(t_1)e^{it_1\Delta}\\
&+\cdots\\
&+(-i)^n \int_0^t dt_n \int_0^{t_n} dt_{n-1} \cdots \int_0^{t_2} dt_1 e^{-it_n\Delta}(w_1*\psi)(t_n)e^{it_n\Delta}\\
&\qquad\qquad\qquad \cdot e^{-it_{n-1}\Delta}(w_1*\psi)(t_{n-1})e^{it_{n-1}\Delta}\cdots e^{-it_1\Delta}\big(w_1*(\phi-\psi)\big)(t_1)e^{it_1\Delta}.
\end{aligned}
\tag{5.18}
$$

Using this sum, we decompose the difference

$$
\begin{aligned}
&\mathcal{A}_{m,n}(\phi)(t)-\mathcal{A}_{m,n}(\psi)(t)\\
&=w_2*\rho\Big[e^{it\Delta}\big(\mathcal{W}^{(m)}_{w_1*\phi}(t)-\mathcal{W}^{(m)}_{w_1*\psi}(t)\big)\gamma_f\mathcal{W}^{(n)}_{w_1*\phi}(t)^*e^{-it\Delta}\Big]\\
&\quad+w_2*\rho\Big[e^{it\Delta}\mathcal{W}^{(m)}_{w_1*\psi}(t)\gamma_f\big(\mathcal{W}^{(n)}_{w_1*\phi}(t)-\mathcal{W}^{(n)}_{w_1*\psi}(t)\big)e^{-it\Delta}\Big],
\end{aligned}
\tag{5.19}
$$

into $(m+n)$ terms. For each term, we estimate as in the proof of (5.1). Collecting all, we obtain (5.2). □

6. Bounds on $\mathcal{B}(\phi)$

We prove the bounds on the operator

$$\mathcal{B}(\phi)(t)=w_2*\rho\Big[e^{it\Delta}\mathcal{W}_{w_1*\phi}(t)Q_0\mathcal{W}_{w_1*\phi}(t)^*e^{-it\Delta}\Big]$$

introduced in (2.16).

Proposition 6.1 *(Bounds on $\mathcal{B}(\phi)$). Let $d\geq 3$, $\alpha>\frac{d-2}{2}$ and α_0 be given by (3.8) with $\alpha_1=\alpha_2=\alpha$. Suppose that $w=w_1*w_2$, and $|\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1, |\cdot|^{-1/2}\langle\cdot\rangle^{-\alpha_0}\hat{w}_2\in L^\infty$. Then, there exist small $\epsilon_{\mathcal{B}}>0$ and large $C_{\mathcal{B}}, C'_{\mathcal{B}}>0$ such that if $\|\phi\|_{L^2_{t,x}}, \|\psi\|_{L^2_{t,x}}\leq\epsilon_{\mathcal{B}}$, then*

$$
\begin{aligned}
\|\mathcal{B}(\phi)\|_{L^2_{t,x}}&\leq C_{\mathcal{B}}\|Q_0\|_{\mathcal{H}^\alpha},\\
\|\mathcal{B}(\phi)-\mathcal{B}(\psi)\|_{L^2_{t,x}}&\leq C'_{\mathcal{B}}\|Q_0\|_{\mathcal{H}^\alpha}\|\phi-\psi\|_{L^2_{t,x}}.
\end{aligned}
\tag{6.1}
$$

The constants $\epsilon_{\mathcal{B}}$, $C_{\mathcal{B}}$ and $C'_{\mathcal{B}}$ depend only on d, $\||\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1\|_{L^\infty}$ and $\||\cdot|^{-1/2}\langle\cdot\rangle^{-\alpha_0}\hat{w}_2\|_{L^\infty}$.

Proof. For notational convenience, we denote

$$Q_\phi(t):=e^{it\Delta}\mathcal{W}_{w_1*\phi}(t)Q_0\mathcal{W}_{w_1*\phi}(t)^*e^{-it\Delta}=\mathcal{U}_{w_1*\phi}(t)Q_0\mathcal{U}_{w_1*\phi}(t)^*. \tag{6.2}$$

Note that by definition, $\mathcal{B}(\phi)=w_2*\rho_{Q_\phi(t)}$.

Recalling (2.1) and (2.2), we see by differentiating (6.2) in t that Q_ϕ solves the following equation:

$$i\partial_t Q_\phi=[-\Delta+w_1*\phi, Q_\phi]$$

with initial data $Q_\phi(0)=Q_0$, equivalently,

$$Q_\phi(t)=e^{it\Delta}Q_0e^{-it\Delta}-i\int_0^t e^{i(t-s)\Delta}[w_1*\phi, Q_\phi](s)e^{-i(t-s)\Delta}ds. \tag{6.3}$$

Hence, by the Strichartz estimates in Theorem 3.6, we get

$$\begin{aligned}\|Q_\phi\|_{\mathcal{S}^\alpha} &\le c\|Q_0\|_{\mathcal{H}^\alpha} + c\Big\|\langle\nabla_x\rangle^\alpha\langle\nabla_{x'}\rangle^\alpha[w_1*\phi, Q_\phi](t,x,x')\Big\|_{L_t^1L_{x,x'}^2}\\ &\le c\|Q_0\|_{\mathcal{H}^\alpha} + 2c\Big\|\langle\nabla_x\rangle^\alpha\langle\nabla_{x'}\rangle^\alpha\Big((w_1*\phi)(t,x)Q_\phi(t,x,x')\Big)\Big\|_{L_t^1L_{x,x'}^2}\\ &\quad + 2c\Big\|\langle\nabla_x\rangle^\alpha\langle\nabla_{x'}\rangle^\alpha\Big((w_1*\phi)(t,x')Q_\phi(t,x,x')\Big)\Big\|_{L_t^1L_{x,x'}^2},\end{aligned} \tag{6.4}$$

where the time interval $[0,+\infty)$ is omitted in the norms for notational convenience. Moreover, applying the triangle inequality, the Strichartz estimate in Theorem 3.1 with $\alpha_1=\alpha_2=\alpha$ to the density of (6.3), we get

$$\begin{aligned}\||\nabla|^{1/2}\rho_{Q_\phi}\|_{L_t^2H_x^{\alpha_0}} &\le \||\nabla|^{1/2}\rho_{e^{it\Delta}Q_0e^{-it\Delta}}\|_{L_t^2H_x^{\alpha_0}}\\ &\quad+\int_{\mathbb{R}}\||\nabla|^{1/2}\rho_{e^{i(t-s)\Delta}[w_1*\phi,Q_\phi](s)e^{-i(t-s)\Delta}}\|_{L_t^2H_x^{\alpha_0}}ds\\ &\le c\|Q_0\|_{\mathcal{H}^\alpha} + c\int_{\mathbb{R}}\|e^{-is\Delta}[w_1*\phi,Q_\phi](s)e^{is\Delta}\|_{\mathcal{H}_x^\alpha}ds\\ &\le c\|Q_0\|_{\mathcal{H}^\alpha} + 2c\Big\|\langle\nabla_x\rangle^\alpha\langle\nabla_{x'}\rangle^\alpha\Big((w_1*\phi)(t,x)Q_\phi(t,x,x')\Big)\Big\|_{L_t^1L_{x,x'}^2}\\ &\quad + 2c\Big\|\langle\nabla_x\rangle^\alpha\langle\nabla_{x'}\rangle^\alpha\Big((w_1*\phi)(t,x')Q_\phi(t,x,x')\Big)\Big\|_{L_t^1L_{x,x'}^2}.\end{aligned} \tag{6.5}$$

By the fractional Leibniz rule and Sobolev inequalities with the choices of α_0 and α (both are applied only for the x-variable),

$$\begin{aligned}&\big\|\langle\nabla_x\rangle^\alpha\langle\nabla_{x'}\rangle^\alpha\big((w_1*\phi)(t,x)Q_\phi(t,x,x')\big)\big\|_{L_t^1L_{x,x'}^2}\\ &\lesssim \|(w_1*\phi)\|_{L_t^2L_x^d}\|\langle\nabla_x\rangle^\alpha\langle\nabla_{x'}\rangle^\alpha Q_\phi(t,x,x')\|_{L_t^2L_x^{\frac{2d}{d-2}}L_{x'}^2}\\ &\quad + \||\nabla|^\alpha(w_1*\phi)\|_{L_{t,x}^2}\|\langle\nabla_{x'}\rangle^\alpha Q_\phi(t,x,x')\|_{L_t^2L_x^\infty L_{x'}^2}\\ &\lesssim \||\nabla|^{1/2}(w_1*\phi)\|_{L_t^2H_x^{\alpha_0}}\|\langle\nabla_x\rangle^\alpha\langle\nabla_{x'}\rangle^\alpha Q_\phi(t,x,x')\|_{L_t^2L_x^{\frac{2d}{d-2}}L_{x'}^2}\\ &\le \||\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1\|_{L^\infty}\|\phi\|_{L_{t,x}^2}\|Q_\phi\|_{\mathcal{S}^\alpha}.\end{aligned} \tag{6.6}$$

We estimate $(w_1*\phi)(t,x')Q_\phi(t,x,x')$ in a similar way, interchanging x and x'. Thus, we prove that if $\|\phi\|_{L_{t,x}^2}\le\epsilon_{\mathcal{B}}$, then

$$\begin{aligned}\|Q_\phi\|_{\mathcal{S}^\alpha} + \||\nabla|^{1/2}\rho_{Q_\phi}\|_{L_t^2H_x^{\alpha_0}} &\le 2c\|Q_0\|_{\mathcal{H}^\alpha} + 2c'\||\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1\|_{L^\infty}\|\phi\|_{L_{t,x}^2}\|Q_\phi\|_{\mathcal{S}^\alpha}\\ &\le 2c\|Q_0\|_{\mathcal{H}^\alpha} + 2c'\epsilon_{\mathcal{B}}\||\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1\|_{L^\infty}\|Q_\phi\|_{\mathcal{S}^\alpha}.\end{aligned} \tag{6.7}$$

We take $\epsilon_{\mathcal{B}} := \frac{1}{4c'\||\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1\|_{L^\infty}}$. Then, we get

$$\|Q_\phi\|_{\mathcal{S}^\alpha} + \||\nabla|^{1/2}\rho_{Q_\phi}\|_{L_t^2H_x^{\alpha_0}} \le 4c\|Q_0\|_{\mathcal{H}^\alpha}. \tag{6.8}$$

As a result, by (5.10), we conclude that

$$\|\mathcal{B}(\phi)\|_{L_{t,x}^2} = \|w_2*\rho_{Q_\phi}\|_{L_{t,x}^2} \le \||\cdot|^{-1/2}\langle\cdot\rangle^{-\alpha_0}\hat{w}_2\|_{L^\infty}\||\nabla|^{1/2}\rho_{Q_\phi}\|_{L_t^2H_x^{\alpha_0}} \le C_{\mathcal{B}}\|Q_0\|_{\mathcal{H}^\alpha}, \tag{6.9}$$

where $C_{\mathcal{B}} = 4c\||\cdot|^{-1/2}\langle\cdot\rangle^{-\alpha_0}\hat{w}_2\|_{L^\infty}$.

For the difference

$$Q_\phi(t) - Q_\psi(t) = -i\int_0^t e^{i(t-s)\Delta}\big[w_1 * (\phi-\psi), Q_\phi\big](s)e^{-i(t-s)\Delta}ds \\ - i\int_0^t e^{i(t-s)\Delta}\big[w_1*\psi, Q_\phi - Q_\psi\big](s)e^{-i(t-s)\Delta}ds, \tag{6.10}$$

repeating the estimates in the proof of (6.7), we prove that if $\|\phi\|_{L^2_{t,x}}, \|\psi\|_{L^2_{t,x}} \le \epsilon_{\mathcal{B}}$, then

$$\begin{aligned} &\|Q_\phi - Q_\psi\|_{\mathcal{S}^\alpha} + \||\nabla|^{1/2}\rho_{Q_\phi - Q_\psi}\|_{L^2_t H^{\alpha_0}_x} \\ &\le c'\||\nabla|^{1/2}w_1 * (\phi-\psi)\|_{L^2_t H^{\alpha_0}_x}\|Q_\phi\|_{\mathcal{S}^\alpha} + c'\||\nabla|^{1/2}w_1*\psi\|_{L^2_t H^{\alpha_0}_x}\|Q_\phi - Q_\psi\|_{\mathcal{S}^\alpha} \\ &\le c'\||\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1\|_{L^\infty}\|\phi-\psi\|_{L^2_{t,x}}\|Q_\phi\|_{\mathcal{S}^\alpha} \\ &\quad + c'\||\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1\|_{L^\infty}\|\psi\|_{L^2_{t,x}}\|Q_\phi - Q_\psi\|_{\mathcal{S}^\alpha} \\ &\le c'\||\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1\|_{L^\infty}\|\phi-\psi\|_{L^2_{t,x}}\cdot 4c\|Q_0\|_{\mathcal{H}^\alpha} \quad \text{(by (6.8))} \\ &\quad + c'\||\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1\|_{L^\infty}\cdot\epsilon_{\mathcal{B}}\cdot\|Q_\phi - Q_\psi\|_{\mathcal{S}^\alpha}. \end{aligned} \tag{6.11}$$

By the choice of $\epsilon_{\mathcal{B}}$,

$$\||\nabla|^{1/2}\rho_{Q_\phi - Q_\psi}\|_{L^2_t H^{\alpha_0}_x} \le 4cc'\||\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1\|_{L^\infty}\|Q_0\|_{\mathcal{H}^\alpha}\|\phi-\psi\|_{L^2_{t,x}}. \tag{6.12}$$

Thus, by (5.10), we conclude that

$$\begin{aligned} \|\mathcal{B}(\phi) - \mathcal{B}(\psi)\|_{L^2_{t,x}} &= \|w_2 * (\rho_{Q_\phi} - \rho_{Q_\psi})\|_{L^2_{t,x}} \\ &\le \||\cdot|^{-1/2}\langle\cdot\rangle^{-\alpha_0}\hat{w}_2\|_{L^\infty}\||\nabla|^{1/2}\rho_{Q_\phi - Q_\psi}\|_{L^2_t H^{\alpha_0}_x} \\ &\le C'_{\mathcal{B}}\|Q_0\|_{\mathcal{H}^\alpha}\|\phi-\psi\|_{L^2_{t,x}}, \end{aligned} \tag{6.13}$$

where $C'_{\mathcal{B}} = 4cc'\||\cdot|^{1/2}\langle\cdot\rangle^{\alpha_0}\hat{w}_1\|_{L^\infty}\||\cdot|^{-1/2}\langle\cdot\rangle^{-\alpha_0}\hat{w}_2\|_{L^\infty}$. □

7. Proof of the main theorem

First, we prove that

$$\Gamma(\phi) = (1+\mathcal{L})^{-1}\Big\{\sum_{m,n=1}^{\infty}\mathcal{A}_{m,n}(\phi) + \mathcal{B}(\phi)\Big\} \tag{7.1}$$

is contractive in a small ball in $L^2_{t,x}$. Let $\epsilon > 0$ be a sufficiently small number. Suppose that $\|Q_0\|_{\mathcal{H}^\alpha} \le \epsilon$ and

$$\|\phi\|_{L^2_{t,x}}, \|\psi\|_{L^2_{t,x}} \le 2C_{\mathcal{B}}\|1+\mathcal{L}\|^{-1}_{L^2_{t,x}\to L^2_{t,x}}\|Q_0\|_{\mathcal{H}^\alpha} =: R. \tag{7.2}$$

Note that R is also a sufficiently small number, since $\|Q_0\|_{\mathcal{H}^\alpha}$ is assumed to be small. Then, by Proposition 5.1 and 6.1,

$$\|\Gamma(\phi)\|_{L^2_{t,x}} \le \|1+\mathcal{L}\|^{-1}_{L^2_{t,x}\to L^2_{t,x}}\Big\{\sum_{m,n=1}^{\infty}C_{\mathcal{A}}^{m+n+1}\|\langle\cdot\rangle^\alpha f\|_{L^\infty}R^{m+n} + C_{\mathcal{B}}\|Q_0\|_{\mathcal{H}^\alpha}\Big\} \le R, \tag{7.3}$$

where in the second inequality, we used that the sum $\sum_{m,n=1}^\infty C_{\mathcal{A}}^{m+n+1}\|\langle\cdot\rangle^\alpha f\|_{L^\infty}R^{m+n}$ is $O(R^2)$, so it is bounded by $C_{\mathcal{B}}\|Q_0\|_{\mathcal{H}^\alpha} = O(R)$. Similarly, we prove that

$$\begin{aligned}
&\|\Gamma(\phi)-\Gamma(\psi)\|_{L^2_{t,x}}\\
&\le \|1+\mathcal{L}\|^{-1}_{L^2_{t,x}\to L^2_{t,x}}\Big\{\sum_{m,n=1}^{\infty}(m+n)C_{\mathcal{A}}^{m+n+1}\|\langle\cdot\rangle^{\alpha}f\|_{L^\infty}(2R)^{m+n-1}+C'_{\mathcal{B}}\epsilon\Big\}\|\phi-\psi\|_{L^2_{t,x}}\\
&\le \frac{1}{2}\|\phi-\psi\|_{L^2_{t,x}}.
\end{aligned}\tag{7.4}$$

Thus, by the contraction mapping theorem, there exists a unique $\phi\in L^2_{t,x}$ such that $\phi=\Gamma(\phi)$.

Next, we derive the equation (2.6) from $\phi=\Gamma(\phi)$. Precisely, we claim that $Q(t)$, defined by

$$Q(t):=e^{it\Delta}\mathcal{W}_{w_1*\phi}(t)(\gamma_f+Q_0)\mathcal{W}_{w_1*\phi}(t)^*e^{-it\Delta}-\gamma_f,\tag{7.5}$$

is a solution to (2.6). Indeed, it follows from the series expansion for the wave operator (see (2.4)) and its boundedness (see (2.9)) that $Q(t)$ is well-defined in $\mathfrak{S}^{2d}$. Moreover, we have

$$\begin{aligned}
\|w_2*\rho_Q-\phi\|_{L^2_{t,x}}&=\Big\|-\mathcal{L}(\phi)+\sum_{m,n=1}^{\infty}\mathcal{A}_{m,n}(\phi)+\mathcal{B}(\phi)-\phi\Big\|_{L^2_{t,x}}\\
&=\Big\|-\mathcal{L}(\phi)+(1+\mathcal{L})(1+\mathcal{L})^{-1}\Big\{\sum_{m,n=1}^{\infty}\mathcal{A}_{m,n}(\phi)+\mathcal{B}(\phi)\Big\}-\phi\Big\|_{L^2_{t,x}}\\
&=\|-\mathcal{L}(\phi)+(1+\mathcal{L})\Gamma(\phi)-\phi\|_{L^2_{t,x}}\\
&=\|-\mathcal{L}(\phi)+(1+\mathcal{L})\phi-\phi\|_{L^2_{t,x}}=0\quad(\text{by }\Gamma(\phi)=\phi),
\end{aligned}\tag{7.6}$$

where the first identity follows from straightforward calculations using the infinite series expansion of the wave operator and the definitions of $\mathcal{L}$, $\mathcal{A}_{m,n}$ and $\mathcal{B}$. Now, inserting $\phi=w_2*\rho_Q$ into (7.5), we conclude that Q satisfies the equation (2.6),

$$\begin{aligned}
Q(t)&=\mathcal{U}_{w_1*w_2*\rho_Q}(t)(\gamma_f+Q_0)\mathcal{U}_{w_1*w_2*\rho_Q}(t)^*-\gamma_f\\
&=\mathcal{U}_{w*\rho_Q}(t)(\gamma_f+Q_0)\mathcal{U}_{w*\rho_Q}(t)^*-\gamma_f
\end{aligned}\tag{7.7}$$

in $C_t([0,+\infty);\mathfrak{S}^{2d})$.

Finally, by (5.10), we prove the desired global-in-time bound,

$$\begin{aligned}
\|w*\rho_Q\|_{L^2_tL^d_x}&\le\|w_1*w_2*\rho_Q\|_{L^2_tL^d_x}\le\|\hat{w}_1\|_{L^{\frac{2d}{d-2}}}\|w_2*\rho_Q\|_{L^2_{t,x}}\\
&\le\|\hat{w}_1\|_{L^{\frac{2d}{d-2}}}\|\hat{w}_2\|_{L^\infty}\|\phi\|_{L^2_{t,x}}<\infty,
\end{aligned}\tag{7.8}$$

which implies scattering in $\mathfrak{S}^{2d}$ by Lemma 2.1.

Conflict of interest statement

There is no conflict of interest.

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