Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 35 (2018) 357–392

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Doubly nonlocal Cahn–Hilliard equations

Ciprian G. Gal

Florida International University, Department of Mathematics, DM 435B, Miami, FL 33199, USA

Received 23 November 2015; received in revised form 20 December 2016; accepted 9 May 2017
Available online 19 May 2017

Abstract

We consider a doubly nonlocal nonlinear parabolic equation which describes phase-segregation of a two-component material in a bounded domain. This model is a more general version than the recent nonlocal Cahn–Hilliard equation proposed by Giacomin and Lebowitz [26], such that it reduces to the latter under certain conditions. We establish well-posedness results along with regularity and long-time results in the case when the interaction between the two levels of nonlocality is strong-to-weak.

MSC: 35R09; 37L30; 82C24

Keywords: Nonlocal Cahn–Hilliard; phase transition; solutions; doubly nonlocal equation; anomalous transport; fractional Laplace

1. Introduction

Today the Cahn–Hilliard equation is as fundamental to material science as the Navier–Stokes equation is for the theory of fluid dynamics. The Cahn–Hilliard equation was proposed by Cahn and Hilliard [6] in the late 1950's as a model for (isothermal) phase separation phenomena in materials made of two components. Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Scaling all the relevant physical constants so that they are all equal to one, the basic form of this equation is very well-known and reads as follows:

$$\partial_t \varphi + \operatorname{div}(M) = 0, \ \mu = -\Delta\varphi + F'(\varphi), \ \text{in } \Omega \times (0, \infty), \tag{1.1}$$

M is the mass flux evolving according to Fick's law of diffusion, $M = -m(\varphi)\nabla\mu$; m is the density dependent mobility. Here φ is the relative difference of the two phases (or the concentration of one phase), μ is called the chemical potential and is determined as the variational derivative of the free energy

$$E_{loc}(\varphi) = \int_\Omega \left(\frac{1}{2}|\nabla\varphi|^2 + F(\varphi)\right) dx, \tag{1.2}$$

assuming the no-flux boundary conditions

E-mail address: cgal@fiu.edu.

http://dx.doi.org/10.1016/j.anihpc.2017.05.001

$$M \cdot \nu = \nabla\varphi \cdot \nu = 0 \text{ on } \partial\Omega \times (0, \infty)\,; \tag{1.3}$$

ν denotes the outer normal to $\partial\Omega$. We recall that in the framework of phase separation, one has ± 1 corresponding to the pure phases of the material while $\varphi \in (-1, 1)$ corresponds to the transition in the interface between the two material phases. A physically relevant choice for F is a singular logarithmic potential which is often approximated by a regular polynomial potential, typically, the double well-potential $F(r) = (r^2 - 1)^2$. Since the 1950's the Cahn–Hilliard equation (1.1)–(1.3) has also found applications in phenomena arising in various other contexts in biology, ecology and astronomy (see [7,35]). We note that an essential feature of (1.1)–(1.3) is that each material component of the two-phase mixture is actually preserved over time, a property reflected through the fact that

$$\frac{d}{dt}\int_\Omega \varphi(t,x)\,dx = 0 \Rightarrow \int_\Omega \varphi(t,x)\,dx = \int_\Omega \varphi(0,x)\,dx. \tag{1.4}$$

From the view of applications (see [7]), significant examples of a mobility function m are either a positive constant, $m(s) \equiv m_0 > 0$, or the so-called degenerate case, namely, when $m(s) = m_0\left(1 - s^2\right)$, $s \in [-1, 1]$. Concerning (1.1)–(1.3) with constant mobility, a substantial number of references and results can be found in the survey paper of Cherfils, Miranville and Zelik [7]. However, we may quote that well-posedness was first proven in a series of papers that appeared in the 1990's (see [9,14,31]), while the regularity and global longtime behavior were also analyzed recently in [34] and [2], respectively. In the case of degenerate mobility, an existence result for a weak solution has been proved in [13], but other important issues, such as uniqueness and regularity of a weak solution, are still open to this day.

Toward a complete mathematical theory for the nonlocal Cahn–Hilliard equation. Although the derivation of the Cahn–Hilliard equation (1.1)–(1.3) is simple, elegant and appears physically sound, it is still performed from a phenomenological point of view (see [6,12]; and [30], for a derivation based on the second law of thermodynamics). Moreover, it assumes that the interaction between any two particles at the sites $x, y \in \Omega$ of the material to be *only* short-ranged, a property which shows up directly in the energy functional (1.2), through the gradient term. For this reason, (1.1) is called the *local Cahn–Hilliard* equation. However, up until recently there was no microscopic derivation of models of phase-segregation phenomena. In [26], starting from a microscopic model for a lattice gas with long-range Kac (symmetric) potentials of the form $J_{\text{long}}(x - y) = \delta^d J(\delta(x - y))$, for $\delta > 0$ and $J \in C^2\left(\mathbb{R}^d\right)$, Giacomin and Lebowitz have rigorously derived a *nonlocal* version of the Cahn–Hilliard equation (1.1). Note that such a version assumes that the long-range interaction between any two particles at the sites $x, y \in \Omega$ is observed on the spatial scale δ^{-1} for small $\delta > 0$ up to the system size. In this theory the free energy E_{loc} occurs as a first order-approximation of the nonlocal free energy

$$E_{nonloc}(\varphi) = \frac{1}{4}\int_\Omega\int_\Omega J_{\text{long}}(x - y)\,|\varphi(x) - \varphi(y)|^2 dx dy + \int_\Omega F(\varphi)dx. \tag{1.5}$$

On the other hand, the Cahn–Hilliard equation associated with such an internal energy E_{nonloc} takes on the form

$$\partial_t\varphi + \operatorname{div}(M) = 0,\ \mu = a(x)\varphi - J * \varphi + F'(\varphi)\,,\ \text{in } \Omega \times (0, \infty)\,. \tag{1.6}$$

Here, for the sake of mathematical convenience we have fixed the strength of the long-range interaction to a (given) spatial scale $\delta = 1$ (i.e., $J_{\text{long}} \equiv J$), $\mu = \frac{\delta}{\delta\varphi}(E_{nonloc})$ and we have set

$$(J * \varphi)(x) := \int_\Omega J(x - y)\varphi(y)dy, \quad a(x) := \int_\Omega J(x - y)dy. \tag{1.7}$$

It is worth emphasizing that from a mathematical point of view, the derivation contained in [26] yields the same equation (1.6) if instead of $J_{\text{long}}(\cdot)$ in (1.5) one also accounts for quadratic terms involving a short-range interaction kernel $J_{\text{short}}(\cdot)$. The latter, however, is of the form $J_{\text{short}}(x) = 0$ if $|x| \geq r$, for some sufficiently small $r > 0$ independent of δ. Thus, since for now we are only interested in the dynamical properties of the nonlocal equation (1.6) for a fixed $\delta > 0$ without observing its limiting behavior as $\delta \to 0$, we can assume that the interaction kernel $J(\cdot)$ in (1.7) can be written as a sum of $J_{\text{long}}(\cdot)$ and $J_{\text{short}}(\cdot)$ (cf. also [26, Section 3]). As before, if the balance law in (1.6) is accompanied by the usual no-flux boundary condition

$$M \cdot \nu = 0 \text{ on } \partial\Omega \times (0, \infty), \tag{1.8}$$

then conservation of mass (1.4) still holds for the system (1.6)–(1.8). Another important feature of (1.6)–(1.8) is that any sufficiently smooth solution satisfies the following energy identity

$$\frac{d}{dt} E_{nonloc}(\varphi(t)) + \left\| m^{1/2} \nabla \mu(t) \right\|^2_{L^2(\Omega)} = 0, \text{ for all } t \geq 0, \tag{1.9}$$

which is the analogue of the energy identity for the local Cahn–Hilliard equation (1.1), when in place of E_{nonloc} in (1.9) one finds E_{loc} (see (1.2)). In the literature, the well-posedness of (1.6) with a degenerate mobility has been first investigated in [21,27], and then later in [19,16], concerning a coupled system involving (1.6) in fluid dynamics. The long-time behavior of a weak solution has also been analyzed recently in [32,33], and the global asymptotic properties (in terms of finite dimensional global attractors) of this equation has also been established in [24] (see also [19]). On the other hand, in the case of constant mobility, the well posedness of (1.6) has been settled in [3,4, 8,16], and the existence of a finite dimensional attractor for (1.6) as well as the convergence of a weak solution to a single steady state were recently established in [15,24] (see also [17,1] for some related equations and/or working assumptions). Note that in the case of constant mobility ($m_0 = 1$), (1.6) can be written equivalently as a (second-order) quasi-linear equation, namely,

$$\partial_t \varphi + \nabla \cdot (q(x, \varphi) \nabla \varphi + \nabla a \varphi - \nabla J * \varphi) = 0, \ q(x, \varphi) := a(x) + F''(\varphi). \tag{1.10}$$

More precisely, nonlocal effects in (1.10) are modeled by convolution with a sufficiently smooth, fast decaying kernel J (such as, the Newtonian or Bessel potential). Such potentials are of essential interest in phase-segregation phenomena which exhibit competition between nonlocal aggregation and nonlinear (nondegenerate) diffusion $q(x, \cdot) \geq c_0 > 0$. From the analytical viewpoint, the second order of the equation (1.10) plays a major role in the behavior of its solutions when compared to the local one in (1.1); this is mainly due to the fact that the Laplacian term in the definition for μ, in (1.1), induces an additional regularization effect from μ to φ so that weak solutions to (1.1) become even classical for positive times provided that Ω is smooth enough (see, e.g., [7,35]). On the other hand, for sufficiently singular kernels J, there is a lack of regularization in (1.6) due to the form of the equation in (1.10). Indeed, this is the case when J is symmetric and either a Newtonian or Bessel-like potential (see [24]; cf. also [16,22] for coupled systems in fluid dynamics involving (1.10)). No regularization to classical solutions can be expected for (1.10) unless J is smooth and non-singular (see, for instance, [26]).

The doubly nonlocal Cahn–Hilliard equation. In both approaches of [6] and [26], the Cahn–Hilliard equation with constant mobility has been traditionally described as a deterministic process (as given by the first equation of (1.1) in the form $\partial_t \varphi = \Delta\mu$) in which particle transport obeys Fick's law of diffusion, i.e., $M = -\nabla\mu$. However, in many instances when the observed phenomena takes place in a multiscale heterogeneous environment Ω, non-Fickian behavior of the chemical potential may manifest itself through deviations of the mean concentration profile from the shape of a normal (Gaussian) distribution. This kind of behavior is commonly referred in the literature as anomalous transport or diffusion. We refer the reader to [37,38] for some new perspectives on theories of non-Fickian diffusion. In the case of anomalous transport, the mass conservation law must be replaced by a nonlocal formulation employing integral operators rather than differential operators. Nonlocal analysis allows to formulate the equation of motion for mass transport in terms of operators of the form

$$\begin{aligned} \mathcal{L}(\mu) &= 2\text{P.V.} \int_\Omega K(x, y)(\mu(y) - \mu(x))\, dy \\ &= 2 \lim_{\varepsilon \to 0^+} \int_{\Omega \setminus B_\varepsilon(x)} K(x, y)(\mu(y) - \mu(x))\, dy, \end{aligned} \tag{1.11}$$

(provided that the limit exists) by also allowing long-range interactions to occur between any two points $x, y \in \Omega$. For instance, such nonlocal formulation of the classical conservation law $\partial_t \varphi = div(M)$ is appropriate for study of phenomena in heterogeneous environments, and it has also been observed experimentally [37,38]. We note that in this new formulation, the chemical potential is merely a measurable real-valued function which is not assumed to possess any a priori regularity, while the kernel K encodes the physical properties of the environment Ω in a manner in which mass is being transported throughout Ω. The governing system of equations then takes the form

$$\partial_t \varphi = \mathcal{L}(\mu), \ \mu = \int_\Omega J(x-y)(\varphi(x) - \varphi(y))\,dy + F'(\varphi), \tag{1.12}$$

where the chemical potential is defined as before as a variational derivative, $\mu = \frac{\delta}{\delta\varphi}(E_{nonloc})$. For (1.12), the corresponding energy identity reads as

$$\frac{d}{dt} E_{nonloc}(\varphi(t)) + \int_\Omega \int_\Omega K(x,y)(\mu(y,t) - \mu(x,t))^2\,dydx = 0, \tag{1.13}$$

for all $t \geq 0$, provided that K is symmetric and a sufficiently smooth solution for (1.12) exists. Also, the integral in the definition of the chemical potential in (1.12) needs to be understood in a principal value sense if J is non-smooth, say if $J \notin L^1_{loc}(\mathbb{R}^d)$. Interestingly, we can observe at least formally that the second nonlocal interaction term on the left-hand side of (1.13) can be locally approximated by the square of the gradient term (see (1.9)) by taking the approximation $\mu(x) - \mu(y) \approx \nabla\mu(x) \cdot (x-y)$ whenever $|x-y| \ll 1$ provided that K is a radial function, sufficiently supported at zero (i.e., it has a finite second moment) and μ is sufficiently smooth. In this sense, the doubly nonlocal Cahn–Hilliard equation (1.12) is justified and more general than the recent nonlocal system (1.6)–(1.8), although results have not been established yet for (1.12). One reason might be the fact that doubly nonlocal integro-differential nonlinear parabolic equations have not been considered before in bounded domains or otherwise, especially when the interaction between the two levels of nonlocality is a tricky business. To explain further, let us introduce some terminology. Let $B_\sigma = B_\sigma(0)$ be a sufficiently large ball such that it contains the bounded domain Ω. We call a kernel K integrable, if for every $x \in B_\sigma$, the quantity $\int_{B_\sigma} K(x,y)dy$ is finite and the mapping $x \mapsto \int_{B_\sigma} K(x,y)dy$ is locally integrable. We say a kernel is non-integrable if it is not integrable in this sense. We note that the operator in (1.11) is in fact not a principle value integral if the kernel K is integrable in the above sense. Nevertheless, in the case of the doubly nonlinear parabolic equation (1.12) we can distinguish between four different cases according to whether one expects that either kernel K or J is integrable or not. Indeed, for radially symmetric (nonnegative) kernels $K(x,y) = \rho(|x-y|)$ and $J(x) = J(-x)$, we observe the following four interesting cases:

- the *strong-to-weak* interaction case: $\rho \notin L^1_{loc}(\mathbb{R}^d)$ and $J \in L^1_{loc}(\mathbb{R}^d)$;
- the *weak-to-weak* interaction case: both $\rho, J \in L^1_{loc}(\mathbb{R}^d)$;
- The *weak-to-strong* interaction case: $\rho \in L^1_{loc}(\mathbb{R}^d)$ and $J \notin L^1_{loc}(\mathbb{R}^d)$;
- The *strong-to-strong* interaction case: $\rho, J \notin L^1_{loc}(\mathbb{R}^d)$.

In this work we will deal with the *strong-to-weak* interaction case which is also one of the most difficult cases. It is our goal to give a unified analysis of the doubly nonlocal Cahn–Hilliard equation (1.12) for a large class of non-integrable kernels $\rho \notin L^1_{loc}(\mathbb{R}^d)$ and integrable $J \in L^1_{loc}(\mathbb{R}^d)$, to establish sharp results in terms of existence, regularity and stability (with respect to the initial data) of properly-defined solutions. We will also discuss and derive sufficient conditions for problem (1.12) to possess finite dimensional global and exponential attractors, and for solutions to eventually convergence to single steady states as time goes to infinity. We can summarize the main mathematical features of the present work in our case of interest, as follows.

(I) Equation (1.12) does not fall in the category of non-local second order parabolic equations treated in the aforementioned works. This is not surprising due to the full nonlocal nature of the equation. Indeed, in contrast to (1.10), we are dealing with the non-local analogue of a (local) divergence form equation, rather than a uniformly parabolic equation in which the nonlocal terms can be treated as a lower-order perturbation provided that $J \in W^{1,1}_{loc}(\mathbb{R}^d)$. The latter is a minimal assumption in all of the aforementioned contributions. Even the issue of well-posedness for (1.12) (i.e., Theorems 2.7, 2.10) requires devising a new approach based on carefully crafted energy estimates and construction of a globally defined weak solution based on a Galerkin scheme. For the latter, further knowledge of the operator $\mathcal{L}$ according to [23], and its "nonlocal" effect on differences of the form $\mu(x) - \mu(y)$, with μ given by (1.12), are crucial in order to derive even something as elementary as an estimate for φ in $L^\infty_{loc}(\mathbb{R}_+; L^2(\Omega))$. We can achieve it by carefully splitting some of the crucial terms (when $p = 1$), that occur as a result of the double interaction in (1.12), of the form

$$\int_{\Omega}\int_{\Omega} \rho\left(|x-y|\right)\left(\left(J*\varphi\right)(x)-\left(J*\varphi\right)(y)\right) \times \left(|\varphi(x)|^{p-1}\varphi(x)-|\varphi(y)|^{p-1}\varphi(y)\right)dydx, \tag{1.14}$$

into two separate regions of interaction. Namely, the regions are

$$\{(x,y)\in\Omega\times\Omega:\rho\left(|x-y|\right)\leq 1\}$$

and $\{(x,y)\in\Omega\times\Omega:\rho\left(|x-y|\right)\geq 1\}$, respectively, where in the former the source ρ has only a weak effect on differences in (1.14), while in the latter region it has a dominating effect which eventually necessitates that additionally, $\nabla J\in L^1_{loc}\left(\mathbb{R}^d\right)$.

(II) Perhaps, beside the basic question of well-posedness, the main major development in the paper is the key issue of regularity for (1.12). Indeed, it is expected that some kind of "regularity" is actually necessary for the weak solution of (1.12), in order to establish properties in regard to its long-term behavior of its solutions, that are similar to the ones obtained for (1.10) (see [18,24]). For (1.10), under the same assumptions (H3)–(H6) (see below), as in [24], weak solutions are known to regularize to a strong one in $C^{\alpha/2,\alpha}\left([t,\infty)\times\overline{\Omega}\right)\cap L^{\infty}\left([t,\infty);H^1(\Omega)\right)$, for all $t>0$, as a consequence of the divergence structure of the equation. Although we don't know whether weak solutions for (1.12) eventually satisfy $\varphi\in C^{\alpha/2,\alpha}$, a first crucial step in this direction is to determine whether they are at least bounded in $L^{\infty}([t,\infty)\times\Omega)$ (see Lemma 3.1). For (1.10), such a bound is an immediate consequence of the fact (1.10) is a quasi-linear parabolic equation that is strictly non-degenerate (see [3]). Our proof can no longer exploit such basic structure, and uses two main ingredients:

(a) Some crucial functional inequalities, including a version of the Stroock–Varopoulos inequality that was established for the operator $\mathcal{L}$ in [25, Lemma 3.4].

(b) A uniform estimate for the $L^{p+1}(\Omega)$-norm of a weak solution φ of (1.12), by performing a more detailed analysis of the interaction terms (1.14) for *all* $p>1$, followed by a key iteration argument that allows to bootstrap the $L^{\infty}_{loc}\left(\mathbb{R}_+;L^2(\Omega)\right)$-regularity to a uniform $L^{\infty}_{loc}\left(\mathbb{R}_+;L^{p+1}(\Omega)\right)$-regularity estimate for φ. Besides, in our iteration scheme we need to get good control of certain constants in such a way that they do not depend on the *strongly singular* behavior of $\rho\notin L^1_{loc}\left(\mathbb{R}^d\right)$. This final point may be viewed as a generalization of the classical Moser–Alikakos iteration argument, which relies heavily on key Sobolev interpolation inequalities, and which were used extensively in the case of parabolic equations, such as (1.10), cf. [3, Theorem 2.1]. Much in the spirit of [25], rather than considering the non-locality of the equation $\partial_t\varphi=\mathcal{L}(\mu)$, see (1.12), as an additional technical difficulty to the implementation of the scheme, we use it toward our advantage to construct a more elementary proof. We emphasize here that we do not employ interpolation-like inequalities, but only require a basic Poincare–Sobolev inequality, which is usually looked at as the basic tool for variational solution theory associated with the operator $\mathcal{L}$.

(III) Having established that φ is bounded, one then proceeds to derive additional regularity properties for the weak solution. Among many such properties, one such regularity is $\varphi\in L^{\infty}_{loc}(\mathbb{R}_+;V)$ (see Lemma 3.3), for some space V that is densely contained and compactly embedded into $L^2(\Omega)$. As a by-product of this analysis, we obtain the existence of a strong solution with sufficient regularity such that all the equations of (1.12) are satisfied for almost all $(t,x)\in(0,\infty)\times\Omega$. These results also allow us to address the issue of long-term behavior for solutions of (1.12) by performing proofs in the spirit of the approach used in [24], to deal with the lack of (strong) regularization effect in the weak solution (see Section 4, the proofs of Theorems 4.1 and 4.4).

In any event, we find it remarkable, as it can be easily seen by the interested reader in the following sections, that our general assumptions on J and the potential F will be of similar nature as the ones exploited by [24], while including the nontrivial generalization to a large class of nonlocal operators $\mathcal{L}$, including the case of fractional Laplace operators, in addition to that of the classical Laplacian Δ (cf. Section 5).

The article is organized as follows. In Section 2 we define the functional framework needed for our approach and explain their basic properties. We also define what a weak and strong solution of (1.12) shall be and then provide a well-posedness result for weak solutions and some other properties. Section 3 is dedicated to the regularity of weak solutions as well as the existence of strong solutions. Section 4 is devoted to the long-term behavior of weak solutions for (1.12) in terms of finite dimensional attractors. The convergence of each weak solution to a steady state is also discussed in this section. Finally, in Section 5 we provide some examples of nonlocal operators $\mathcal{L}$ and discuss their properties they have on the corresponding solutions for (1.12) (see Sections 5.1, 5.2). Presumably Section 5 is also important for the reader. We conclude the paper with Section 5.3 where we make some final comments and give a long list of interesting open problems.

2. Framework and well-posedness

In this section we derive a (weak) variational formulation of (1.12). We start with the definition of the relevant function framework. If $K : \Omega \times \Omega \to [0, \infty)$ is measurable, the energy functional

$$l(\mu, v) := \int_\Omega \int_\Omega K(x, y)(\mu(x) - \mu(y))(v(x) - v(y))dydx \tag{2.1}$$

associated with $\mathcal{L}$ is symmetric for any (real-valued) $\mu, v \in D(l)$. It is also clearly nonnegative in $L^2(\Omega)$. Here, we have set

$$D(l) = \left\{ \mu \in L^2(\Omega) : \int_\Omega \int_\Omega K(x, y)(\mu(x) - \mu(y))^2 dydx < \infty \right\}.$$

(H1) Assume that l is closed, symmetric and $1 \in \mathrm{Null}(l)$, where

$$\mathrm{Null}(l) = \{u \in V : l(u, v) = 0, \text{ for any } v \in V\}.$$

We have $D(l) =: V$, where

$$\|v\|_V := \sqrt{l(v, v) + \|v\|^2_{L^2(\Omega)}}$$

is a norm and $D(l) = V$ is a Hilbert space.

(H2) Assume the injection $V \subset L^2(\Omega)$ is dense and compact, and that there exists $q > 1$ such that

$$V \subset L^{2q}(\Omega) \text{ is continuous.} \tag{2.2}$$

Then owing to (H1)–(H2) the operator realization $(\mathcal{L}, D(\mathcal{L}))$ of the form l, defined as

$$\begin{cases} D(\mathcal{L}) & := \{\mu \in V, \ \exists\, w \in L^2(\Omega), \ l(\mu, \varphi) = (w, \varphi)_{L^2(\Omega)}, \ \forall \varphi \in V\} \\ \mathcal{L}\mu & = w, \end{cases} \tag{2.3}$$

is self-adjoint, nonnegative on $L^2(\Omega)$ and $(\mathcal{L} + I)^{-1}$ is compact as an operator in $L^2(\Omega)$. In fact, much more can be said about $\mathcal{L}$. Indeed, l is a Dirichlet form in the sense of [23, Definition 2.1] and, therefore, $\mathcal{L}$ satisfies all the conclusions of [23, Theorem 2.9]. This additional information (cf. (2.2)) about $\mathcal{L}$ will become crucial in the proof for the existence of a weak solution (in particular, in connection with the regularity problem for the eigenfunctions of $\mathcal{L}$).

We have the following Poincaré–Young inequality which will become essential in the next section.

Proposition 2.1. *Let (H1)–(H2) hold. Then for every $\varepsilon \in (0, 1)$ there exists $\alpha > 0$ such that*

$$\|v\|^2_{L^2(\Omega)} \le \varepsilon l(v, v) + \varepsilon^{-\alpha} \|v\|^2_{L^1(\Omega)},$$

for every $v \in V$.

Proof. A proof of this statement can be reproduced from [25, Lemma 3.3] with some minor modifications, owing to assumptions (H1)–(H2). □

Remark 2.2. From the physical point of view, the condition $1 \in \mathrm{Null}(l)$ is necessary to ensure that total mass is conserved (see (1.4)) and is a condition about the nature of the space V.

For every $v \in V^*$ (where V^* is the topological dual of V) we denote by $\overline{v}$ the average of v over Ω, i.e., $\overline{v} := |\Omega|^{-1}\langle v, 1\rangle$. As usual $|\Omega|$ stands for the Lebesgue measure of Ω. Then we introduce the spaces

$$V_0 := \{v \in V : \overline{v} = 0\}, \qquad V_0^* := \{v \in V^* : \overline{v} = 0\},$$

and notice that by (H1)–(H2), the operator $\mathcal{L}$ can be extended to a bounded operator from $V \to V^*$ still denoted (for convenience of notation) by $\mathcal{L}$,

$$\langle \mathcal{L}u, v\rangle := l(u, v) \qquad \forall u, v \in V.$$

Since $1 \in$Null(l) we see that $\mathcal{L}$ maps V onto V_0^* and the restriction of $\mathcal{L}$ to V_0 maps V_0 onto V_0^* isomorphically. Let us denote by $\mathcal{L}^{-1} : V_0^* \to V_0$ the inverse map defined by

$$\mathcal{L}\left(\mathcal{L}^{-1}v\right) = v, \quad \forall v \in V_0^* \qquad \text{and} \qquad \mathcal{L}^{-1}(\mathcal{L}u) = u, \quad \forall u \in V_0.$$

By the Lax–Milgram lemma, it is also clear that for every $g \in V_0^*$, $\mathcal{L}^{-1}g$ is the unique solution with zero mean value of the (nonlocal) boundary value problem $\mathcal{L}u = g$. Furthermore, the following relations hold

$$\langle \mathcal{L}u, \mathcal{L}^{-1}v\rangle = \langle v, u\rangle, \qquad \forall u \in V, \quad \forall v \in V_0^*, \tag{2.4}$$

$$\langle u, \mathcal{L}^{-1}v\rangle = \langle v, \mathcal{L}^{-1}u\rangle, \qquad \forall u, v \in V_0^*. \tag{2.5}$$

By definition we also have

$$(\mathcal{L}u, v)_{L^2(\Omega)} = (u, v)_{V_0} = l(u, v), \qquad \forall u \in D(\mathcal{L}), \quad \forall v \in V_0. \tag{2.6}$$

Finally, by assumption (H2) (see also Proposition 2.1), the following Poincare inequality holds: $\|v\|^2_{L^2(\Omega)} \le C_{\mathcal{L}} l(v, v)$, for all $v \in V_0$, for some constant $C_{\mathcal{L}} > 0$. It follows that $\sqrt{l(v, v) + (\overline{v})^2}$ is also an equivalent norm for the Hilbert space V.

Now let us give assumptions on the interaction kernel J and the potential F.

(H3) $J \in W^{1,1}(\mathbb{R}^d)$, $J(x) = J(-x)$, $a(x) = \int_\Omega J(x-y)dy \ge 0$, a.e. $x \in \Omega$.

(H4) $F \in C^2(\mathbb{R})$ and there exists $c_0 > 0$ such that

$$F''(s) + a(x) \ge c_0, \qquad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega.$$

(H5) There exist $c_1 > 0$, $c_2 \ge 0$ and $p \in (1, 2]$ such that

$$|F'(s)|^p \le c_1|F(s)| + c_2, \qquad \forall s \in \mathbb{R}.$$

(H6) There exist $c_F' \ge 0$, $c_F \ge \|a\|_\infty + C_{\mathcal{L}}$, such that

$$F(s) \ge c_F s^2 - c_F', \text{ for any } s \in \mathbb{R}.$$

Concerning the interaction kernel K associated with the form $l(\cdot,\cdot)$ of (2.1), we further assume that

(H7) K is nonnegative and radially symmetric, that is, $K(x, y) = \rho(|x-y|) \ge 0$ for some function $\rho : \mathbb{R}_+ \to \mathbb{R}_+$, with

$$\begin{cases} \int_0^1 \rho(r) r^{d+1} dr = \overline{C}_\rho < \infty, \\ \sup_{r \in [1,\infty)} \rho(r) = C_\rho < \infty, \end{cases} \tag{2.7}$$

for some constants $C_\rho, \overline{C}_\rho \ge 0$.

Remark 2.3. We observe that (H4) implies that the potential F is a quadratic perturbation of a (strictly) convex function $G(s) = F(s) + \frac{\|a\|_\infty}{2}s^2$, such that $G'' \ge c_0$ a.e. in Ω. Assumption (H5) is fulfilled by a potential of arbitrary polynomial growth (of order $p^* = p/(p-1) \in [2, \infty)$). Of course, the assumptions (H4)–(H6) are satisfied by the double-well smooth potentials $F(s) = C_F(s^2-1)^2$, for some $C_F > 0$. The assumptions (H3)–(H6) are quite natural in the analytic theory for the nonlocal Cahn–Hilliard equation (1.6)–(1.8) (cf. [8,24]). For a slightly more general kernel K, refer to Remark 2.9.

Our definition of a weak solution for the full nonlocal Cahn–Hilliard equation (1.12) is as follows.

Definition 2.4. Let $\varphi_0 \in L^2(\Omega)$ with $F(\varphi_0) \in L^1(\Omega)$ and $0 < T < +\infty$ be given. We say φ is a weak solution if

1. The functions φ and μ satisfy

$$\varphi \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;V), \tag{2.8}$$

$$\varphi_t \in L^2(0,T;V^*),\ \ \mu \in L^2(0,T;V), \tag{2.9}$$

$$F(\varphi) \in L^\infty\left(0,T;L^1(\Omega)\right). \tag{2.10}$$

2. Setting $b(x,\varphi) := a(x)\varphi + F'(\varphi)$, then for every $\psi \in V$, a.e. $t \in (0,T)$ we have

$$\langle \varphi_t, \psi \rangle + l(\mu, \psi) = 0, \tag{2.11}$$

$$\mu = b(x,\varphi) - J * \varphi \text{ a.e. in } \Omega. \tag{2.12}$$

3. We have $\varphi(0) = \varphi_0$.

Remark 2.5. Observe that mass is conserved for (1.12). Indeed setting $\psi \equiv 1$ in (2.11) we deduce that $(\varphi(t),1)_{L^2(\Omega)} = (\varphi_0,1)_{L^2(\Omega)}$ for all $t \geq 0$. The initial condition $\varphi(0) = \varphi_0$ also makes sense in a weak sense since by (2.8)–(2.9) we have $\varphi \in C_w([0,T];L^2(\Omega))$, i.e., φ is continuous with respect to the weak topology.

We also define what we mean by a strong solution.

Definition 2.6. Let $\varphi_0 \in L^\infty(\Omega) \cap V$ and $0 < T < +\infty$ be given. We say φ is a strong solution of (1.12) if it is a weak solution in the sense of Definition 2.4, and φ and μ satisfy

$$\varphi \in L^\infty(0,T;L^\infty(\Omega)) \cap L^\infty(0,T;V),$$

$$\varphi_t \in L^2(0,T;L^2(\Omega)),\ \ \mu \in L^2(0,T;D(\mathcal{L})).$$

In particular, for the strong solution we have $\partial_t\varphi = \mathcal{L}(\mu)$, a.e. in $\Omega \times (0,T)$, and $\mu = a(x)\varphi - J * \varphi + F'(\varphi)$ a.e. in $\Omega \times (0,T)$, respectively.

The first result establishes the existence of a weak solution under natural assumptions on J, K and the potential F.

Theorem 2.7. *Let $\varphi_0 \in L^2(\Omega)$ such that $F(\varphi_0) \in L^1(\Omega)$ and suppose that (H1)–(H5), (H7) are satisfied. Then, for every $T > 0$ there exists a weak solution φ to problem (1.12) on $[0,T]$ in the sense of Definition 2.4. Furthermore, the following energy identity holds for any $t \geq 0$,*

$$\mathcal{N}(\varphi(t)) + \int_0^t l(\mu(\tau), \mu(\tau))\, d\tau = \mathcal{N}(\varphi_0) \tag{2.13}$$

and the functions $t \mapsto \|\varphi(t)\|^2_{L^2(\Omega)}$ and $t \mapsto (F(\varphi(t)),1)_{L^2(\Omega)}$ are absolutely continuous on $[0,T]$. Here, we recall that

$$\mathcal{N}(\varphi) = \frac{1}{4} \int_{\Omega\times\Omega} J(x-y)|\varphi(x)-\varphi(y)|^2 dxdy + \int_\Omega F(\varphi)dx.$$

Proof. We give a proof based on a Galerkin approximation scheme. Since the operator $(\mathcal{L}, D(\mathcal{L}))$ is self-adjoint and nonnegative on $L^2(\Omega)$, we can take the family $\{\psi_j\}_{j\geq 1} \subset D(\mathcal{L}) \cap V$ of eigenfunctions for this operator to be a proper Galerkin basis in V. Then, one can define the n-dimensional subspaces V_n spanned by $\{w_1, ..., w_n\}$ and as usual consider the orthogonal projectors P_n on these subspaces in $L^2(\Omega)$. We look for functions of the form

$$\varphi_n(t)=\sum_{k=1}^{n}a_k^{(n)}(t)\psi_k,\quad \mu_n(t)=\sum_{k=1}^{n}b_k^{(n)}(t)\psi_k \tag{2.14}$$

which solve to the following approximate problem:

$$(\partial_t\varphi_n,\psi)_{L^2(\Omega)}+l(\mu_n,\psi)=0, \tag{2.15}$$

$$b(\cdot,\varphi_n):=a(\cdot)\varphi_n+F'(\varphi_n), \tag{2.16}$$

$$\mu_n=P_n(b(x,\varphi_n)-J*\varphi_n) \tag{2.17}$$

$$\varphi_n(0)=\varphi_{0n}, \tag{2.18}$$

for every $\psi\in V_n$, where $\varphi_{0n}=P_n\varphi_0$. The approximate problem (2.15)–(2.18) is equivalent to solving a Cauchy problem for a system of ordinary differential equations in the n-unknowns $a_i^{(n)}$. Since $F'\in C^1(\mathbb{R})$, the Cauchy–Lipschitz theorem ensures that there exists $T_n^*\in(0,+\infty]$ such that this system has a unique maximal solution $\mathbf{a}^{(n)}:=(a_1^{(n)},\cdots,a_n^{(n)})$ on $[0,T_n^*)$ such that $\mathbf{a}^{(n)}\in C^1([0,T_n^*);\mathbb{R}^n)$. The coefficient $\mathbf{b}^{(n)}:=(b_1^{(n)},\cdots,b_n^{(n)})$ associated with the chemical potential μ_n can now be determined through the relation (2.17). Clearly, testing (2.15) with $\psi=1$, we also obtain the conservation of mass: $(\varphi_n(t),1)_{L^2(\Omega)}=(\varphi_{0n},1)_{L^2(\Omega)}$, for all $t\geq 0$. It remains to deduce sufficiently strong a priori uniform estimates to show that $T_n^*=+\infty$ for every $n\geq 1$ and that the sequences of φ_n and μ_n are bounded in suitable functional spaces. By using μ_n as a test function in (2.15), we first deduce

$$(\partial_t\varphi_n,\mu_n)_{L^2(\Omega)}+l(\mu_n,\mu_n)=0,$$

where from (2.1) we recall that

$$l(\mu_n,v)=\int_\Omega\int_\Omega\rho(|x-y|)(\mu_n(x)-\mu_n(y))(v(x)-v(y))\,dxdy.$$

Since

$$\begin{aligned}&(\partial_t\varphi_n,\mu_n)_{L^2(\Omega)}=(\partial_t\varphi_n,a(x)\varphi_n+F'(\varphi_n)-J*\varphi_n)_{L^2(\Omega)}\\&=\frac{d}{dt}\Big(\frac12\|\sqrt{a}\varphi_n\|^2_{L^2(\Omega)}+\int_\Omega F(\varphi_n)dx-\frac12(\varphi_n,J*\varphi_n)_{L^2(\Omega)}\Big)\\&=\frac{d}{dt}\Big(\frac14\int_\Omega\int_\Omega J(x-y)(\varphi_n(x)-\varphi_n(y))^2dxdy+\int_\Omega F(\varphi_n)dx\Big),\end{aligned}$$

owing to (2.16)–(2.17), we deduce

$$\frac{d}{dt}\mathcal{N}(\varphi_n(t))+l(\mu_n(t),\mu_n(t))=0,\ \text{for all } t\geq 0. \tag{2.19}$$

Hence, integrating (2.19) with respect to time on the interval $(0,t)$ for $t\in(0,T_n^*)$, we immediately get the equality:

$$\mathcal{N}(\varphi_n(t))+\int_0^t l(\mu_n(\tau),\mu_n(\tau))\,d\tau=\mathcal{N}(\varphi_{0n}). \tag{2.20}$$

It follows that $\mathcal{N}(\varphi_n)\in L^\infty(0,T)$, as well as $l(\mu_n,\mu_n)\in L^1(0,T)$ uniformly with respect to $n\geq 1$; in particular it follows that

$$F(\varphi_n)\in L^\infty\left(0,T;L^1(\Omega)\right) \tag{2.21}$$

uniformly in n (in this case the corresponding bounds depend only on $\varphi_0\in L^2(\Omega)$, $F(\varphi_0)\in L^1(\Omega)$, and on $J\in L^1(\mathbb{R}^d)$).

Our next goal is to derive uniform bounds for φ_n in the space

$$L^\infty\left(0,T;L^2(\Omega)\right)\cap L^2(0,T;V).$$

To this end, we test equation (2.15) with $\psi = \varphi_n$ to find that

$$\frac{1}{2}\frac{d}{dt}\|\varphi_n\|^2_{L^2(\Omega)} + l(\mu_n, \varphi_n) = 0. \tag{2.22}$$

Let us now set $\alpha_n(\cdot, \varphi_n) := a(\cdot)\varphi_n + F'(\varphi_n) - J * \varphi_n$ such that $\mu_n = P_n\alpha_n$. In order to get rid of the projector P_n in the quadratic form $l(P_n\alpha_n, \varphi_n)$ of (2.22) we essentially use the fact that by [23, Theorem 2.9, (3)], the family of eigenfunctions $\{\psi_j\}_{j\geq 1} \subset L^\infty(\Omega)$; this yields, owing to (2.14), that also $\varphi_n(t) \in L^\infty(\Omega)$ and, in particular, $F^{(j)}(\varphi_n(t)) \in L^\infty(\Omega)$, for any $j \in \{0, 1, 2\}$. This implies, by virtue of the mean-value theorem for F' and the estimates (2.26)–(2.30) below, that $\alpha_n \in L^\infty(\Omega) \cap V$. Note that this is the only place where assumption (2.2) is used in this proof (according to [23, Theorem 2.9]). Therefore, on account of the fact that $\mathcal{L}$ is self-adjoint according to the representation (2.3), (2.6), we have

$$\begin{aligned} l(\mu_n, \varphi_n) = (\mu_n, \mathcal{L}(\varphi_n))_{L^2(\Omega)} &= \left(\alpha_n, \sum_{k=1}^{n} a_k^{(n)}(t)\lambda_k\psi_k\right)_{L^2(\Omega)} \\ &= (\alpha_n, \mathcal{L}(\varphi_n))_{L^2(\Omega)} = l(\alpha_n, \varphi_n), \end{aligned}$$

where $\{\lambda_k\}_{k\in\mathbb{N}} \subset [0, \infty)$ is such that $\mathcal{L}(\psi_k) = \lambda_k\psi_k$. On the other hand, we have

$$\begin{aligned} l(\alpha_n, \varphi_n) &= \int_\Omega\int_\Omega \rho(|x-y|)(\alpha_n(x) - \alpha_n(y))(\varphi_n(x) - \varphi_n(y))\,dydx \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where

$$\begin{cases} I_1 := \int_\Omega\int_\Omega \rho(|x-y|)(a(x) + q_F(\varphi_n))(\varphi_n(x) - \varphi_n(y))^2\,dydx, \\ I_2 := \int_\Omega\int_\Omega \rho(|x-y|)(a(x) - a(y))\varphi_n(y)(\varphi_n(x) - \varphi_n(y))\,dydx, \\ I_3 := \int_\Omega\int_\Omega \rho(|x-y|)((J*\varphi_n)(x) - (J*\varphi_n)(y))(\varphi_n(x) - \varphi_n(y))\,dydx \end{cases}$$

and we have set

$$q_F(\varphi_n) := \frac{F'(\varphi_n(x)) - F'(\varphi_n(y))}{\varphi_n(x) - \varphi_n(y)}. \tag{2.23}$$

Owing to the mean value theorem for $F \in C^2(\mathbb{R})$, we observe that assumption (H4) then implies $I_1 \geq c_0 l(\varphi_n, \varphi_n)$, and thus (2.22) yields the inequality

$$\frac{1}{2}\frac{d}{dt}\|\varphi_n\|^2_{L^2(\Omega)} + c_0 l(\varphi_n, \varphi_n) \leq |I_2 + I_3|; \tag{2.24}$$

so it remains to estimate the integrals on the right-hand side. We show the estimate for I_3 in detail and sketch it for I_2 since it follows in the same fashion. First, Cauchy–Schwarz and Young inequalities give

$$\begin{aligned} I_3 \leq{}& \frac{c_0}{4}\int_\Omega\int_\Omega \rho(|x-y|)(\varphi_n(x) - \varphi_n(y))^2\,dydx \\ &+ \frac{1}{c_0}\int_\Omega\int_\Omega \rho(|x-y|)((J*\varphi_n)(x) - (J*\varphi_n)(y))^2\,dydx \\ ={}& \frac{c_0}{4} l(\varphi_n, \varphi_n) + \frac{1}{c_0}\int_\Omega\int_\Omega \rho(|x-y|)((J*\varphi_n)(x) - (J*\varphi_n)(y))^2\,dydx. \end{aligned} \tag{2.25}$$

Secondly, we split the last integral into two parts:

$$\int_\Omega\int_\Omega \rho(|x-y|)((J*\varphi_n)(x) - (J*\varphi_n)(y))^2\,dydx = A + B, \tag{2.26}$$

where

$$\begin{cases} A := \int_\Omega \int_{\Omega:|x-y|\geq 1} \rho(|x-y|)\left((J*\varphi_n)(x)-(J*\varphi_n)(y)\right)^2 dydx, \\ B := \int_\Omega \int_{\Omega:|x-y|<1} \rho(|x-y|)\left((J*\varphi_n)(x)-(J*\varphi_n)(y)\right)^2 dydx. \end{cases}$$

Then, it follows

$$\begin{aligned} |A| &\leq 2\int_\Omega \int_{\Omega:|x-y|\geq 1} \rho(|x-y|)\left(|(J*\varphi_n)(x)|^2+|(J*\varphi_n)(y)|^2\right) dydx \\ &\leq 2\|\rho\|_{L^\infty[1,\infty)}\left(2|\Omega|\,\|J\|_{L^1}^2\,\|\varphi_n\|_{L^2}^2\right) \\ &= 4C_\rho\,|\Omega|\,\|J\|_{L^1}^2\,\|\varphi_n\|_{L^2}^2, \end{aligned} \tag{2.27}$$

owing to the fact that $J \in L^1(\mathbb{R}^d)$ and exploiting the Young convolution theorem. Next, let us consider the trivial extension $\widetilde{\varphi}$ of φ from Ω into $\mathbb{R}^d$ (that is, $\widetilde{\varphi}_{|\Omega} = \varphi$ and $\widetilde{\varphi}_{|\mathbb{R}^d\setminus\Omega} = 0$) such that

$$\begin{aligned} |B| &\leq \int_\Omega \int_{B_1} \rho(|z|)\left((J*\varphi_n)(x)-(J*\varphi_n)(z+x)\right)^2 dzdx \\ &\leq \int_\Omega \int_{B_1} \left(\frac{|(J*\varphi_n)(x)-(J*\varphi_n)(z+x)|}{|z|}\right)^2 |z|^2\rho(|z|)\,dzdx \\ &\leq \int_\Omega \int_{B_1} \left(\int_0^1 |\nabla J*\varphi_n(x+tz)|\,dt\right)^2 |z|^2\rho(|z|)\,dzdx \\ &\leq \int_{\mathbb{R}^d} \int_{B_1} \int_0^1 |\nabla J*\widetilde{\varphi}_n(x+tz)|^2\,|z|^2\rho(|z|)\,dtdzdx \\ &\leq \|\nabla J\|_{L^1(\mathbb{R}^d)}^2 \int_{B_1} \int_0^1 |z|^2\rho(|z|)\,\|\widetilde{\varphi}_n\|_{L^2(\mathbb{R}^d)}^2\,dtdz \\ &\leq |S_{d-1}|\,\|J\|_{W^{1,1}(\mathbb{R}^d)}^2 \left(\int_0^1 r^{d+1}\rho(r)\,dr\right)\|\varphi_n\|_{L^2(\Omega)}^2 \\ &= \overline{C}_\rho\,|S_{d-1}|\,\|J\|_{W^{1,1}(\mathbb{R}^d)}^2\,\|\varphi_n\|_{L^2(\Omega)}^2, \end{aligned} \tag{2.28}$$

where in the final steps we have exploited the Young inequality for convolutions in $\mathbb{R}^d$ (here, $|S_{d-1}|$ denotes the surface measure of the unit ball). As far as the integral I_2 is concerned we observe that due assumption (H3), there holds $a = J*1 \in W^{1,\infty}(\mathbb{R}^d)$ and therefore, splitting I_2 in a similar fashion as in (2.25)–(2.26), we find

$$I_2 \leq \frac{c_0}{4} l(\varphi_n,\varphi_n) + \frac{1}{c_0}\int_\Omega \int_\Omega \rho(|x-y|)\,(a(x)-a(y))^2\,(\varphi_n(y))^2\,dydx, \tag{2.29}$$

and

$$\int_\Omega \int_{\Omega:|x-y|\geq 1} \rho(|x-y|)\,(a(x)-a(y))^2\,(\varphi_n(y))^2\,dydx \leq C\,\|\varphi_n\|_{L^2}^2,$$

for some constant $C > 0$ which depends only on $C_\rho \in (0,\infty)$, $|\Omega|$ and $\|J\|_{L^1}$. Moreover, arguing exactly as in the estimate (2.28) it holds

$$\int_{\Omega}\int_{\Omega:|x-y|<1}\rho\left(|x-y|\right)\left(a\left(x\right)-a\left(y\right)\right)^{2}\left(\varphi_{n}\left(y\right)\right)^{2}dydx \tag{2.30}$$
$$\leq\|\nabla a\|_{L^{\infty}\left(\mathbb{R}^{d}\right)}\overline{C}_{\rho}\left\|\varphi_{n}\right\|_{L^{2}(\Omega)}^{2}.$$

Combining together all estimates from (2.25)–(2.30) into (2.24), we obtain

$$\frac{d}{dt}\left\|\varphi_{n}\right\|_{L^{2}(\Omega)}^{2}+c_{0}l\left(\varphi_{n},\varphi_{n}\right)\leq C\left\|\varphi_{n}\right\|_{L^{2}(\Omega)}^{2}, \tag{2.31}$$

for some constant $C>0$ which is independent of n,φ_n and time, but depends only on the structural parameters of the problem. Hence integrating (2.31) in time over the interval $(0,t)$ for any $t\in(0,T)$, allows one to deduce that

$$\|\varphi_{n}\|_{L^{\infty}(0,T;L^{2}(\Omega))}\leq C_{T},\ \ \|l\left(\varphi_{n},\varphi_{n}\right)\|_{L^{1}(0,T)}\leq C_{T} \tag{2.32}$$

and, moreover,

$$\|\varphi_{n}\|_{L^{2}(0,T;V)}^{2}\leq C_{T}, \tag{2.33}$$

owing to assumption (H2) and using again (2.32). The next step is to deduce an uniform estimate for the sequence of μ_n in $L^2(0,T;V)$. To this aim we first observe that (H5) implies that $|F'(s)|\leq c|F(s)|+c$ for every $s\in\mathbb{R}$ and therefore we have

$$\left|\overline{\mu}_{n}\right|=\left|(\mu_{n},1)_{L^{2}(\Omega)}\right|=|(a\varphi_{n}+F'(\varphi_{n})-J*\varphi_{n},1)_{L^{2}(\Omega)}|$$
$$\leq\int_{\Omega}|F'(\varphi_{n})|dx+C_{T}$$
$$\leq C_{T}\left(\int_{\Omega}|F(\varphi_{n})|dx+1\right)$$
$$\leq C_{T}$$

by virtue of the uniform bounds in (2.21) and (2.32). Hence, by means of the Poincaré inequality and assumption (H2), we get

$$\|\mu_{n}\|_{L^{2}(0,T;V)}\leq C_{T}. \tag{2.34}$$

This final estimate implies from (2.15) that

$$\|\partial_{t}\varphi_{n}\|_{L^{2}(0,T;V^{*})}\leq C_{T}. \tag{2.35}$$

The estimates (2.35) and (2.32)–(2.33) and the compact embedding

$$L^{2}(0,T;V)\cap H^{1}(0,T;V')\subset L^{2}(0,T;L^{2}\left(\Omega\right))$$

allows to find a (limit) function φ with the following properties

$$\varphi\in L^{\infty}(0,T;H)\cap L^{2}(0,T;V)\cap H^{1}\left(0,T;V^{*}\right). \tag{2.36}$$

In particular, without relabeling subsequences, we deduce

$$\varphi_{n}\rightharpoonup\varphi\quad\text{weakly}^{*}\text{ in }L^{\infty}(0,T;L^{2}\left(\Omega\right)), \tag{2.37}$$
$$\varphi_{n}\rightharpoonup\varphi\quad\text{weakly in }L^{2}(0,T;V), \tag{2.38}$$
$$\varphi_{n}\rightarrow\varphi\quad\text{strongly in }L^{2}(0,T;L^{2}\left(\Omega\right)), \tag{2.39}$$
$$\mu_{n}\rightharpoonup\mu\quad\text{weakly in }L^{2}(0,T;V), \tag{2.40}$$

and

$$\partial_{t}\varphi_{n}\rightharpoonup\partial_{t}\varphi\quad\text{weakly in }L^{2}(0,T;V^{*}). \tag{2.41}$$

We can now pass to the limit in (2.15)–(2.18) in a standard fashion in order to prove that the limit function φ yields a weak solution to problem (1.12) in the sense of Definition 2.4. The argument to recover (2.11) is immediate owing to (2.40) and (2.41). The pointwise convergence (2.39) also implies that we have $b(\cdot,\varphi_n) \to a(x)\varphi + F'(\varphi)$ almost everywhere in $\Omega \times (0,T)$ and therefore from (2.40) we also have $b(\cdot,\varphi) = a\varphi + F'(\varphi)$ a.e. in $\Omega \times (0,T)$. Finally, this also allows us to further identify the weak limit μ in (2.40) through the relation (2.17), as $\mu = b(x,\varphi) - J*\varphi$, owing once more to the strong convergence (2.39). Hence we also recover (2.12).

Finally, to show that the energy equality (2.13) holds for the weak solution φ (as it does for a Galerkin truncation φ_n, see (2.20)), it turns out that checking this fact reduces to making sense of the duality product $\langle \varphi_t, \mu \rangle$ in (2.11), with a chemical potential $\mu = a(x)\varphi + F'(\varphi) - J*\varphi$, as defined by (2.12). Recalling that $\varphi_t \in L^2(0,T;V^*)$ and $\varphi \in L^2(0,T;V) \cap L^\infty\left(0,T;L^2(\Omega)\right)$, we are then led to the duality $\langle \varphi_t, F'(\varphi)\rangle$ which can be rewritten for a.e. $t \in (0,T)$,

$$\langle \varphi_t, F'(\varphi)\rangle = \frac{d}{dt}\int_\Omega F(\varphi)$$

owing once again to the fact that F is a quadratic perturbation of strictly convex potential G (see the proof of [8, Corollary 2] for further details). Henceforth, on account of this identity, from (2.11) we also obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{a}\varphi\|^2_{L^2(\Omega)} - (\varphi, J*\varphi)_{L^2(\Omega)} + 2\int_\Omega F(\varphi)dx\right) + l(\mu,\mu) = 0. \tag{2.42}$$

Integrating this relation over $(0,t)$ gives the desired energy identity (2.13). The proof is now complete □

Remark 2.8. We observe that the estimates (2.25)–(2.30) performed in the proof of Theorem 2.7 immediately give the following regularity: $J*\varphi \in L^2(0,T;V)$ and

$$b(x,\varphi) = a(x)\varphi + F'(\varphi) \in L^2(0,T;V).$$

The energy equality (2.13) (cf. also (2.42)) implies in a standard way the continuity of energy functional $\mathcal{N}(\varphi(\cdot))$ on $[0,\infty)$, and in particular, $\varphi \in C([0,T];L^2(\Omega))$ owing to inequality (2.31) which also holds for the solution φ on $[0,T]$.

Remark 2.9. The reader can easily check that condition (H7) can be slightly weakened as follows: assume K is nonnegative and symmetric, and $K(x,y) \le c_K\rho(|x-y|)$, for all $x \neq y \in \Omega$, for some $c_K > 0$, with ρ satisfying (2.7).

The uniqueness of the weak solution follows from the following.

Theorem 2.10. *Under the assumptions of Theorem 2.7, the weak solution φ corresponding to the initial datum φ_0 is unique. More precisely, let φ_i $(i=1,2)$ be two weak solutions corresponding to two initial data $\varphi_{0i} \in L^2(\Omega)$ such that $F(\varphi_{0i}) \in L^1(\Omega)$. Then the following estimate holds:*

$$\begin{aligned}&\|\varphi_2(t)-\varphi_1(t)\|^2_{V^*} + c_0\int_0^t \|\varphi_2(\tau)-\varphi_1(\tau)\|^2_{L^2(\Omega)}d\tau \\ &\le C\|\varphi_2(0)-\varphi_1(0)\|^2_{V^*}e^{Ct} + \left|\overline{\varphi_2(0)} - \overline{\varphi_1(0)}\right| Q_T\left(\mathcal{N}(\varphi_{01}),\mathcal{N}(\varphi_{02})\right)e^{Ct},\end{aligned} \tag{2.43}$$

for all $t \in [0,T]$, where the function $Q_T : \mathbb{R}^2 \to \mathbb{R}_+$ and constant $C > 0$ depend on F, J, K and Ω.

Proof. As usual we set $\varphi := \varphi_2 - \varphi_1$. Then, the difference φ satisfies the system:

$$\langle \varphi_t, \psi\rangle + l(\mu_d,\psi) = 0, \tag{2.44}$$

$$\mu_d = a(x)\varphi + F'(\varphi_1) - F'(\varphi_2) - J*\varphi \text{ a.e. in } \Omega, \tag{2.45}$$

a.e. in $(0,T)$, for any $\psi \in V$. Let us now test (2.44) by $\psi = \mathcal{L}^{-1}(\varphi - \overline{\varphi}) \in V_0$ (notice that we also have $\overline{\varphi} = \overline{\varphi}_{01} - \overline{\varphi}_{02}$ by the conservation of masses: $\overline{\varphi}_i = \overline{\varphi}_{0i}$). We get from (2.4)–(2.5),

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\|^2_{L^2(\Omega)} + (a\varphi + F'(\varphi_1) - F'(\varphi_2), \varphi)_{L^2(\Omega)}$$
$$= (J * \varphi, \varphi)_{L^2(\Omega)} + |\Omega|\,(\overline{\varphi})\,(\overline{\mu_d}).$$

Exploiting the assumption (H4), we further find

$$\frac{1}{2}\frac{d}{dt}\|\mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\|^2_{L^2(\Omega)} + c_0\|\varphi\|^2_{L^2(\Omega)} \le |(J * \varphi, \varphi)_{L^2(\Omega)}| + |\Omega|\overline{\varphi}\overline{\mu}_d. \tag{2.46}$$

The first term on the right-hand side of (2.46) can be easily controlled as follows:

$$\begin{aligned}
&\left|(J * \varphi, \varphi - \overline{\varphi})_{L^2(\Omega)}\right| + \left|(J * \varphi, \overline{\varphi})_{L^2(\Omega)}\right| \\
&= \left|\left(\mathcal{L}^{1/2}(J * \varphi - \overline{J * \varphi}), \mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\right)_{L^2(\Omega)}\right| + |(J * \varphi, \overline{\varphi})_{L^2(\Omega)}| \\
&\le \frac{c_0}{4}\|\varphi\|^2_{L^2(\Omega)} + C\|\mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\|^2_{L^2(\Omega)} + \frac{c_0}{4}\|\varphi\|^2_{L^2(\Omega)} + C\overline{\varphi}^2,
\end{aligned} \tag{2.47}$$

where we have used the fact that $\|\mathcal{L}^{1/2}v\|^2_{L^2(\Omega)} = l(v,v) = \|v\|^2_{V_0}$, for all $v \in D(\mathcal{L})$ (see (2.6)) (hence, $\|\mathcal{L}^{1/2}v\|^2_{L^2(\Omega)} = \|v\|^2_{V_0}$ also holds by density for all $v \in V_0$, due to (H2)). In particular, the V_0-norm bound for $J * \varphi - \overline{J * \varphi}$ is performed exactly as in the estimates (2.25)–(2.28). Indeed, we have

$$\begin{aligned}
&\left|\left(\mathcal{L}^{1/2}(J * \varphi - \overline{J * \varphi}), \mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\right)_{L^2(\Omega)}\right| \\
&\le \left\|J * \varphi - \overline{J * \varphi}\right\|_{V_0} \left\|\mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\right\|_{L^2(\Omega)} \\
&\le C\,\|\varphi\|_{L^2(\Omega)} \left\|\mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\right\|_{L^2(\Omega)} \\
&\le \frac{c_0}{4}\|\varphi\|^2_{L^2(\Omega)} + C\left\|\mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\right\|^2_{L^2(\Omega)},
\end{aligned}$$

for some constant $C > 0$ that depends only on $C_\rho, \overline{C}_\rho, \|J\|_{W^{1,1}(\mathbb{R}^d)}$ and Ω. Next, from (2.45) we have

$$\begin{aligned}
|\Omega||\overline{\mu}_d| &\le \int_\Omega \left(|F'(\varphi_2)| + |F'(\varphi_1)|\right)dx \le C\int_\Omega (|F(\varphi_2)| + |F(\varphi_1)|)\,dx + C \\
&\le Q_T\,(\mathcal{N}(\varphi_{01}), \mathcal{N}(\varphi_{02}))\,,
\end{aligned} \tag{2.48}$$

for all $t \in [0,T]$, where we have used (H5) (which implies that $|F'(s)| \le cF(s) + c$, for all $s \in \mathbb{R}$) and (2.8)–(2.10). Therefore, (2.46)–(2.47) can be rewritten as

$$\begin{aligned}
&\frac{d}{dt}\|\mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\|^2_{L^2(\Omega)} + c_0\|\varphi\|^2_{L^2(\Omega)} \\
&\le C\left\|\mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\right\|^2_{L^2(\Omega)} + |\overline{\varphi}|\,Q_T\,(\mathcal{N}(\varphi_{01}), \mathcal{N}(\varphi_{02})) + C\overline{\varphi}^2.
\end{aligned}$$

Integrating the foregoing inequality over $(0,t)$ for $t \in [0,T]$, and applying the Gronwall lemma we deduce

$$\begin{aligned}
&\|\mathcal{L}^{-1/2}(\varphi(t) - \overline{\varphi}(t))\|^2_{L^2(\Omega)} + c_0\int_0^t \|\varphi(\tau)\|^2_{L^2(\Omega)}d\tau \\
&\le C\|\mathcal{L}^{-1/2}(\varphi(0) - \overline{\varphi}(0))\|^2_{L^2(\Omega)}e^{Ct} + \left|\overline{\varphi}_{01} - \overline{\varphi}_{02}\right| Q_T\,(\mathcal{N}(\varphi_{01}), \mathcal{N}(\varphi_{02}))\,e^{Ct},
\end{aligned} \tag{2.49}$$

since $\overline{\varphi} = \overline{\varphi}_{01} - \overline{\varphi}_{02}$. Finally recalling that $\|\mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\|^2_{L^2(\Omega)} \approx \|\varphi - \overline{\varphi}\|^2_{V_0^*}$ and $\|\mathcal{L}^{-1/2}(\varphi - \overline{\varphi})\|^2_{L^2(\Omega)} + (\overline{\varphi})^2 \approx \|\varphi\|^2_{V^*}$, in the sense of equivalent norms, we deduce (2.43) from (2.49) owing to the fact that $\partial_t\overline{\varphi} = 0$ on $[0,T]$. Finally, we observe that if φ_i $(i = 1,2)$ are two weak solutions corresponding to the same initial datum $\varphi_{01} \equiv \varphi_{02}$, then we have $\overline{\varphi} = \overline{\varphi}_{01} - \overline{\varphi}_{02} = 0$ and so the continuous dependence estimate (2.49) also yields $\varphi(t) \equiv 0$ on $[0, T]$. Hence the uniqueness of the weak solution follows as well. □

We finish this section with a dissipative estimate enjoyed by the (unique) weak solution of Theorem 2.7.

Theorem 2.11. *Let the assumptions of Theorem 2.7 be satisfied and further assume (H6). Then the following dissipative estimate holds:*

$$\mathcal{N}(\varphi(t)) \leq \mathcal{N}(\varphi(0))\, e^{-L_0 t} + L_1(m), \qquad \forall t \geq 0, \tag{2.50}$$

where $m > 0$ is such that $\left|\overline{\varphi}_0\right| \leq m$ and L_0, L_1 are two positive constants which are independent of the initial data and time.

Proof. To show (2.50), let us test $\mu = a\varphi - J * \varphi + F'(\varphi)$ by φ in $L^2(\Omega)$. We obtain

$$(\mu, \varphi)_{L^2(\Omega)} = \frac{1}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(x)-\varphi(y))^2 dydx + (F'(\varphi), \varphi)_{L^2(\Omega)}. \tag{2.51}$$

By the convexity of $G(s) = F(s) + \frac{\|a\|_\infty}{2}s^2$ (and therefore, $G''(s) \geq c_0$ a.e. in Ω, owing to (H4); see also [15, Proof of Corollary 2]), we have

$$F'(s)s \geq F(s) - \frac{\|a\|_\infty}{2}s^2 - F(0), \text{ for any } s \in \mathbb{R}.$$

Therefore, from (2.51) we get

$$(\mu, \varphi)_{L^2(\Omega)} \geq \frac{1}{2}\int_\Omega\int_\Omega J(x-y)(\varphi(x)-\varphi(y))^2 dydx + \int_\Omega F(\varphi(t))dx$$
$$- \frac{\|a\|_\infty}{2}\|\varphi\|^2_{L^2(\Omega)} - F(0)|\Omega|. \tag{2.52}$$

On the other hand, we can exploit the Poincare inequality $\|\mu - \overline{\mu}\|^2_{L^2(\Omega)} \leq C_{\mathcal{L}} l(\mu, \mu)$, $C_{\mathcal{L}} > 0$, and the conservation of mass $\overline{\varphi} = \overline{\varphi}_0$, to observe that

$$(\mu, \varphi)_{L^2(\Omega)} = (\mu - \overline{\mu}, \varphi)_{L^2(\Omega)} \leq C_{\mathcal{L}}^{1/2}\sqrt{l(\mu, \mu)}\|\varphi\|_{L^2(\Omega)},$$

assuming for simplicity (for now) that $\overline{\varphi}_0(=\overline{\varphi}) = 0$. Thus, by virtue of assumption (H6) we rewrite (2.52) and estimate in a simple fashion, in order to get

$$C_{\mathcal{L}}^{1/2}\sqrt{l(\mu, \mu)}\|\varphi\|_{L^2(\Omega)} \geq \frac{1}{2}\mathcal{N}(\varphi(t)) + \frac{c_F}{2}\|\varphi\|^2_{L^2(\Omega)}$$
$$- \left(\frac{c_F'}{2}|\Omega| + \frac{\|a\|_\infty}{2}\|\varphi\|^2_{L^2(\Omega)} + F(0)|\Omega|\right).$$

Application of Young's inequality yields

$$\frac{1}{2}\mathcal{N}(\varphi(t)) + \frac{c_F}{2}\|\varphi\|^2_{L^2(\Omega)} - \left(\frac{c_F'}{2}|\Omega| + \frac{\|a\|_\infty}{2}\|\varphi\|^2_{L^2(\Omega)} + F(0)|\Omega|\right)$$
$$\leq \frac{1}{2}l(\mu, \mu) + \frac{C_{\mathcal{L}}}{2}\|\varphi\|^2_{L^2(\Omega)}.$$

We thus easily deduce $\frac{1}{2}\mathcal{N}(\varphi(t)) \leq \frac{1}{2}l(\mu, \mu) + C$, provided that c_F is sufficiently large, i.e., $c_F \geq \|a\|_\infty + C_{\mathcal{L}}$, for some constant $C > 0$ which depends only on c_F', $F(0)$ and $|\Omega|$. It follows by virtue of the foregoing inequality and the energy identity for φ that we have

$$\frac{d}{dt}\mathcal{N}(\varphi(t)) + \mathcal{N}(\varphi(t)) \leq 2C, \tag{2.53}$$

for all $t \geq 0$. By means of the Gronwall inequality we obtain

$$\mathcal{N}(\varphi(t)) \leq \mathcal{N}(\varphi(0))\, e^{-t} + L_1, \text{ with } L_1 = 2C. \tag{2.54}$$

If $\overline{\varphi}_0 \neq 0$ is such that $\overline{\varphi} = \overline{\varphi}_0 \in [-m, m]$ for some $m > 0$, observe that if φ is a weak solution with initial datum φ_0 for the problem with potential F, then $\widetilde{\varphi} = \varphi - \overline{\varphi}_0$ is a weak solution with initial datum $\widetilde{\varphi}(0) = \varphi_0 - \overline{\varphi}_0$ for the same problem with potential $\widetilde{F}(s) := F(s + \overline{\varphi}_0) - F(\overline{\varphi}_0)$. Since now $\overline{\widetilde{\varphi}} = 0$, we can employ the dissipative estimate (2.54) for the solution $\widetilde{\varphi}$ and easily arrive at the final inequality (2.50) owing to the fact that $|\overline{\varphi}_0| \leq m$. The proof is complete. □

Remark 2.12. Under the assumptions of Theorem 2.11, $\varphi \in L^\infty\left(0, T; L^2(\Omega)\right)$ is also uniform in the final time $T > 0$, improving the estimate (2.32), owing to (2.50) and assumption (H6). Therefore the function Q_T from the statement of Theorem 2.10 (see (2.43)) is also independent of the final time $T > 0$.

Remark 2.13. The conclusion of Theorem 2.11 also holds if we replace (H6) by the *stronger* condition:

$$F''(s) + a(x) \geq c_0 |s|^{2q} - \overline{c}_0, \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega, \tag{2.55}$$

for some $q > 0$, $c_0 > 0$, $\overline{c}_0 \geq 0$. In particular, owing to (2.54), it yields that $\varphi \in L^\infty\left(\mathbb{R}_+; L^{2+2q}(\Omega)\right)$. We recall that (2.55) is another natural condition which occurs in the analytic theory for the nonlocal Cahn–Hilliard equation (1.6)–(1.8) (see [3,15,24]).

3. Regularity of weak solution

Let us define

$$\mathcal{Y}_m := \left\{\psi \in L^2(\Omega) : F(\psi) \in L^1(\Omega),\ |\overline{\varphi}| \leq m\right\},$$

for some (given) $m > 0$ and endow $\mathcal{Y}_m$ with the following metric

$$d(\psi_1, \psi_2) := \|\psi_1 - \psi_2\|_{L^2(\Omega)} + \left|\int_\Omega F(\psi_1)\, dx - \int_\Omega F(\psi_2)\, dx\right|^{1/2}, \tag{3.1}$$

for any $\psi_1, \psi_2 \in \mathcal{Y}_m$. Thanks to the statements of Theorem 2.7 and Theorem 2.10, we can then associate with problem (1.12) the (closed) solution semiflow

$$S(t) : \mathcal{Y}_m \to \mathcal{Y}_m, \quad \varphi_0 \mapsto S(t)\varphi_0 = \varphi(t), \tag{3.2}$$

where $\varphi(t)$ is the unique weak solution of (1.12). Furthermore, under the assumptions of Theorem 2.11, it follows in a straight-forward fashion that S is also dissipative in $\mathcal{Y}_m$, in the following sense:

$$d^2(\varphi(t), 0) \leq C\mathcal{N}(\varphi(0))\, e^{-L_0 t} + L_2(m),$$

for some $C, L_2 > 0$, both independent of φ, t.

Our main goal of this section is to investigate whether the weak solution is eventually bounded in a (stronger) Banach space. Next, we prove a fundamental regularity result for the (unique) weak solution $\varphi(t) = S(t)\varphi_0$.

Lemma 3.1. *Let $\varphi_0 \in \mathcal{Y}_m$ and assume that (H1)–(H6), (H7) are satisfied. The following statements are satisfied:*

1. *For every $\tau > 0$, there exists a constant $C_{m,\tau} > 0$ such that*

$$\sup_{t \geq 2\tau} \|\varphi(t)\|_{L^\infty(\Omega)} \leq C_{m,\tau}. \tag{3.3}$$

2. *There exists $R_0 > 0$ (independent of time, τ and initial data) such that $S(t)$ possesses an absorbing ball $\mathcal{B}_{L^\infty(\Omega)}(R_0)$, bounded in $L^\infty(\Omega)$.*

Before we give a complete proof of this crucial lemma, we state a simple inequality.

Proposition 3.2. *Set* $z \vee t := \max\{z, t\}$ *and let* $p \geq 2$. *We have*

$$\left(|z|^{p-1} z - |t|^{p-1} t\right)^2 \leq D_p \left(|z| \vee |t|\right)^{p-1} \left(|z|^{\frac{p-1}{2}} z - |t|^{\frac{p-1}{2}} t\right)^2,$$

for all $z, t \in \mathbb{R}$. *Here,* $D_p = (2p/(p+1))^2$.

Proof. Since the inequality is clearly satisfied for $z = t$, it suffices to prove it for $z > t$; the remaining case $z < t$ is treated in the same way. Indeed, we have

$$\begin{aligned} p^{-2}\left(|z|^{p-1} z - |t|^{p-1} t\right)^2 &= \left(\int_t^z |\tau|^{p-1} d\tau\right)^2 = \left(\int_t^z |\tau|^{\frac{p-1}{2}} |\tau|^{\frac{p-1}{2}} d\tau\right)^2 \\ &\leq \left(|z|^{\frac{p-1}{2}} \int_t^z |\tau|^{\frac{p-1}{2}} d\tau\right)^2 \\ &= \frac{4}{(p+1)^2} |z|^{p-1} \left(|z|^{\frac{p-1}{2}} z - |t|^{\frac{p-1}{2}} t\right)^2. \end{aligned}$$

The desired inequality is thus proved. □

Proof of Lemma 3.1. The scheme we employ is to choose a sequence of datum $\varphi_{0\epsilon} \in L^\infty(\Omega) \cap V$ such that $\varphi_{0\epsilon} \to \varphi_0$ in $L^2(\Omega)$, and such that $F(\varphi_{0\epsilon}) \to F(\varphi_0)$ in $L^1(\Omega)$ as $\epsilon \to 0$. We can also take advantage of the approximate scheme employed in the previous section. As we shall see below, the assumed regularity of $\varphi_{0\epsilon}$ will allow us to derive an a priori estimate such that $\varphi_\epsilon \in L^\infty(0, T; L^\infty(\Omega))$ for any $T > 0$ and $\epsilon > 0$. Then in a second step we give additional arguments leading to the desired (uniform with respect to $\epsilon > 0$ and $\varphi_{0\epsilon} \in L^\infty(\Omega) \cap V$) bound in (3.3). For practical purposes, in this proof $C > 0$ denotes a positive constant that is independent of t, ϵ, φ, p and initial data, but which only depends on the other structural parameters. Such a constant may vary even from line to line. Further dependencies of this constant on other parameters will be pointed out as needed. We shall also omit the subscript $\epsilon > 0$ from the solution for the sake of simplicity of notation.

For $p > 1$, we test equation (2.11) with $\psi = |\varphi|^{p-1}\varphi$ and integrate over Ω, then use (2.12) in order to obtain

$$\begin{aligned} &\frac{1}{p+1}\frac{d}{dt}\int_\Omega |\varphi|^{p+1} dx \\ &= -\int_\Omega\int_\Omega \rho(|x-y|)(\mu(x) - \mu(y))\left(|\varphi(x)|^{p-1}\varphi(x) - |\varphi(y)|^{p-1}\varphi(y)\right) dydx \\ &=: -(J_1 + J_2 + J_3), \end{aligned} \tag{3.4}$$

where $q_F(\varphi)$ is given by (2.23) and we have set

$$\begin{aligned} J_1 := &\int_\Omega\int_\Omega \rho(|x-y|)(a(x) + q_F(\varphi))(\varphi(x) - \varphi(y)) \\ &\times \left(|\varphi(x)|^{p-1}\varphi(x) - |\varphi(y)|^{p-1}\varphi(y)\right) dydx, \end{aligned}$$

$$\begin{aligned} J_2 := &\int_\Omega\int_\Omega \rho(|x-y|)(a(x) - a(y))\varphi(y) \\ &\times \left(|\varphi(x)|^{p-1}\varphi(x) - |\varphi(y)|^{p-1}\varphi(y)\right) dydx \end{aligned}$$

and

$$J_3 := \int_\Omega \int_\Omega \rho(|x-y|)\left((J*\varphi)(x) - (J*\varphi)(y)\right) \times \left(|\varphi(x)|^{p-1}\varphi(x) - |\varphi(y)|^{p-1}\varphi(y)\right) dydx.$$

We also recall that $a(x) + q_F(\varphi) \geq c_0$ for all $\varphi \in \mathbb{R}$, a.e. in Ω, on account of assumption (H4) and the application of the mean value theorem for $F \in C^2$. Then, owing to the essential [25, Lemma 3.4], we find

$$\begin{aligned} J_1 &\geq c_0 \int_\Omega \int_\Omega \rho(|x-y|)(\varphi(x)-\varphi(y))\left(|\varphi(x)|^{p-1}\varphi(x) - |\varphi(y)|^{p-1}\varphi(y)\right) dydx \\ &\geq \frac{4c_0 p}{(p+1)^2} \int_\Omega \int_\Omega \rho(|x-y|)\left(|\varphi(x)|^{\frac{p-1}{2}}\varphi(x) - |\varphi(y)|^{\frac{p-1}{2}}\varphi(y)\right)^2 dydx \\ &= \frac{4c_0 p}{(p+1)^2} l\left(|\varphi|^{\frac{p-1}{2}}\varphi, |\varphi|^{\frac{p-1}{2}}\varphi\right), \end{aligned} \tag{3.5}$$

where in the last step we have used the definition for the form $l(\cdot,\cdot)$. Next, it remains to estimate J_2 and J_3, respectively. For the latter, we split it over two integrals that account for total contribution over $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$, where we set

$$\begin{cases} \Omega_0 := \{y \in \Omega : \varphi(y) = \varphi(x)\}, \\ \Omega_1 := \{y \in \Omega : \varphi(y) > \varphi(x)\}, \\ \Omega_2 := \{y \in \Omega : \varphi(y) < \varphi(x)\}. \end{cases}$$

We have

$$\begin{aligned} J_3 &= \sum_{j=1}^{2} \int_\Omega \int_{\Omega_j} \rho(|x-y|)\left((J*\varphi)(x) - (J*\varphi)(y)\right) \\ &\quad \times \left(|\varphi(x)|^{p-1}\varphi(x) - |\varphi(y)|^{p-1}\varphi(y)\right) dydx \\ &= \sum_{j=1}^{2} \int_\Omega \int_{\Omega_j} \rho(|x-y|)\left((J*\varphi)(x) - (J*\varphi)(y)\right) \\ &\quad \times \left(|\varphi(x)|^{\frac{p-1}{2}}\varphi(x) - |\varphi(y)|^{\frac{p-1}{2}}\varphi(y)\right) \\ &\quad \times \left(\frac{|\varphi(x)|^{p-1}\varphi(x) - |\varphi(y)|^{p-1}\varphi(y)}{|\varphi(x)|^{\frac{p-1}{2}}\varphi(x) - |\varphi(y)|^{\frac{p-1}{2}}\varphi(y)}\right) dydx \\ &\leq \frac{c_0 p}{(p+1)^2} \int_\Omega \int_\Omega \rho(|x-y|)\left(|\varphi(x)|^{\frac{p-1}{2}}\varphi(x) - |\varphi(y)|^{\frac{p-1}{2}}\varphi(y)\right)^2 dydx \\ &\quad + \frac{(p+1)^2 D_p}{4c_0 p} \sum_{j=1}^{2} \int_\Omega \int_{\Omega_j} \rho(|x-y|)\left((J*\varphi)(x) - (J*\varphi)(y)\right)^2 \\ &\quad \times \left(|\varphi(x)| \vee |\varphi(y)|\right)^{p-1} dydx \\ &\leq \frac{c_0 p}{(p+1)^2} \int_\Omega \int_{\Omega_j} \rho(|x-y|)\left(|\varphi(x)|^{\frac{p-1}{2}}\varphi(x) - |\varphi(y)|^{\frac{p-1}{2}}\varphi(y)\right)^2 dydx \\ &\quad + \frac{p}{c_0}(J_{31} + J_{32}), \end{aligned} \tag{3.6}$$

owing to Cauchy–Schwarz and Young inequalities, as well as the application of Proposition 3.2 and the fact that $\Omega \supset \Omega_i$ $(i = 1, 2)$. Above, we have set

$$J_{31} := \int_{\Omega}\int_{\Omega_1} \rho(|x-y|)\left((J*\varphi)(x)-(J*\varphi)(y)\right)^2 |\varphi(y)|^{p-1}\,dydx,$$

$$J_{32} := \int_{\Omega}\int_{\Omega_2} \rho(|x-y|)\left((J*\varphi)(x)-(J*\varphi)(y)\right)^2 |\varphi(x)|^{p-1}\,dydx.$$

A similar argument yields

$$\begin{aligned} J_2 &\le \frac{c_0 p}{(p+1)^2}\int_{\Omega}\int_{\Omega} \rho(|x-y|)\left(|\varphi(x)|^{\frac{p-1}{2}}\varphi(x)-|\varphi(y)|^{\frac{p-1}{2}}\varphi(y)\right)^2 dydx \\ &+ \frac{(p+1)^2 D_p}{4c_0 p}\sum_{j=1}^{2}\int_{\Omega}\int_{\Omega_j} \rho(|x-y|)(a(x)-a(y))^2(\varphi(y))^2 \\ &\times (|\varphi(x)|\vee|\varphi(y)|)^{p-1}\,dydx \\ &\le \frac{c_0 p}{(p+1)^2}\int_{\Omega}\int_{\Omega} \rho(|x-y|)\left(|\varphi(x)|^{\frac{p-1}{2}}\varphi(x)-|\varphi(y)|^{\frac{p-1}{2}}\varphi(y)\right)^2 dydx \\ &+ \frac{p}{c_0}(J_{21}+J_{22}), \end{aligned} \tag{3.7}$$

where

$$J_{21} := \int_{\Omega}\int_{\Omega_1} \rho(|x-y|)(a(x)-a(y))^2(\varphi(y))^2|\varphi(x)|^{p-1}\,dydx,$$

$$J_{22} := \int_{\Omega}\int_{\Omega_2} \rho(|x-y|)(a(x)-a(y))^2|\varphi(y)|^{p+1}\,dydx.$$

First, we have

$$J_{31} = \int_{\Omega}\int_{\Omega_1:|x-y|<1} (\cdot)\,dydx + \int_{\Omega}\int_{\Omega_1:|x-y|\ge 1} (\cdot)\,dydx$$

where we estimate each part as follows:

$$\begin{aligned} &\left|\int_{\Omega}\int_{\Omega_1:|x-y|\ge 1} \rho(|x-y|)\left((J*\varphi)(x)-(J*\varphi)(y)\right)^2|\varphi(y)|^{p-1}\,dydx\right| \\ &\le C_\rho\int_{\Omega}\int_{\Omega:|x-y|\ge 1} \left(|(J*\varphi)(x)|+((J*\varphi)(y))\right)^2|\varphi(y)|^{p-1}\,dydx \\ &\le 2C_\rho\int_{\Omega}|(J*\varphi)(x)|^2\,dx\int_{\Omega}|\varphi(y)|^{p-1}\,dy + 2C_\rho|\Omega|\int_{\Omega:|x-y|\ge 1}|(J*\varphi)(y)|^2|\varphi(y)|^{p-1}\,dy \\ &\le 2C_\rho\left(\|J\|_{L^1}^2\|\varphi\|_{L^1(\Omega)}^2\|\varphi\|_{L^{p-1}(\Omega)}^{p-1} + |\Omega|\,\|J*\varphi\|_{L^{p+1}}^2\|\varphi\|_{L^{p+1}(\Omega)}^{p-1}\right) \\ &\le 2C_\rho\left(\|J\|_{L^1}^2\|\varphi\|_{L^{p+1}(\Omega)}^{p+1} + |\Omega|\,\|J\|_{L^1}^2\|\varphi\|_{L^{p+1}}^2\|\varphi\|_{L^{p+1}(\Omega)}^{p-1}\right) \\ &= 2C_\rho(1+|\Omega|)\|J\|_{L^1}^2\|\varphi\|_{L^{p+1}(\Omega)}^{p+1}, \end{aligned} \tag{3.8}$$

on account of the second condition of (2.7), which states that

$$C_\rho = \sup_{r\in[1,\infty)} \rho(r) < \infty.$$

If we denote by $\widetilde{\varphi}$ the trivial extension of φ from Ω to all of $\mathbb{R}^d$, we also have

$$\left| \int_{\Omega} \int_{\Omega_1 : |x-y|<1} \rho(|x-y|) ((J*\varphi)(x) - (J*\varphi)(y))^2 |\varphi(y)|^{p-1} dy dx \right| \tag{3.9}$$
$$\leq \int_{\Omega} \int_{B_1} |z|^2 \rho(|z|) \frac{|(J*\varphi)(x) - (J*\varphi)(z+x)|^2}{|z|^2} |\varphi(z+x)|^{p-1} dz dx$$
$$\leq \int_{\Omega} \int_{B_1} \int_0^1 |\nabla J * \varphi(x+tz)|^2 |z|^2 \rho(|z|) dt \, |\varphi(z+x)|^{p-1} dz dx$$
$$\leq \int_{\mathbb{R}^d} \int_{B_1} \int_0^1 |\nabla J * \widetilde{\varphi}(x+tz)|^2 |z|^2 \rho(|z|) dt \, |\widetilde{\varphi}(z+x)|^{p-1} dz dx$$
$$\leq \int_{B_1} \int_0^1 \|\nabla J * \widetilde{\varphi}\|^2_{L^{p+1}(\mathbb{R}^d)} \|\widetilde{\varphi}\|^{p-1}_{L^{p+1}(\mathbb{R}^d)} |z|^2 \rho(|z|) dt dz$$
$$\leq \|\nabla J\|^2_{L^1(\mathbb{R}^d)} \|\widetilde{\varphi}\|^{p+1}_{L^{p+1}(\mathbb{R}^d)} \left(\int_{B_1} |z|^2 \rho(|z|) dz \right)$$
$$\leq |S_{d-1}| \, \|\nabla J\|^2_{L^1(\mathbb{R}^d)} \|\varphi\|^{p+1}_{L^{p+1}(\Omega)} \left(\int_0^1 \rho(r) r^{d+1} dr \right)$$
$$= \overline{C}_\rho |S_{d-1}| \, \|\nabla J\|^2_{L^1(\mathbb{R}^d)} \|\varphi\|^{p+1}_{L^{p+1}(\Omega)},$$

owing to the Young convolution theorem and Holder's inequality in $\mathbb{R}^d$. Concerning J_{32}, we have

$$\left| \int_{\Omega} \int_{\Omega_2 : |x-y|\geq 1} \rho(|x-y|) ((J*\varphi)(x) - (J*\varphi)(y))^2 |\varphi(x)|^{p-1} dy dx \right| \tag{3.10}$$
$$\leq 2C_\rho \int_{\Omega} |\varphi(x)|^{p-1} |(J*\varphi)(x)|^2 dx \int_{\Omega} dy$$
$$+ 2C_\rho \int_{\Omega} |(J*\varphi)(y)|^2 dy \int_{\Omega} |\varphi(x)|^{p-1} dx$$
$$\leq 4C_\rho (1+|\Omega|) \|J\|^2_{L^1} \|\varphi\|^{p+1}_{L^{p+1}(\Omega)},$$

exactly as before (see (3.8)). The argument leading to the following bound follows verbatim from that of (3.9):

$$\left| \int_{\Omega} \int_{\Omega_2 : |x-y|<1} \rho(|x-y|) ((J*\varphi)(x) - (J*\varphi)(y))^2 |\varphi(x)|^{p-1} dy dx \right| \tag{3.11}$$
$$\leq 2\overline{C}_\rho |S_{d-1}| \, \|\nabla J\|^2_{L^1(\mathbb{R}^d)} \|\varphi\|^{p+1}_{L^{p+1}(\Omega)}.$$

Summarizing in (3.8)–(3.11), one can thus find a constant $C > 0$, depending only on J, ρ, Ω and on the dimension d, such that

$$|J_{31} + J_{32}| \leq C \|\varphi\|^{p+1}_{L^{p+1}(\Omega)} \tag{3.12}$$

As for J_2, we estimate J_{22} over the sets $\Omega\times(\Omega_2\cap\{|x-y|\geq 1\})$ and $\Omega\times(\Omega_2\cap\{|x-y|<1\})$, respectively. First,

$$\begin{aligned}&\left|\int_{\Omega}\int_{\Omega_2:|x-y|\geq 1}\rho(|x-y|)(a(x)-a(y))^2|\varphi(y)|^{p+1}dydx\right| \\ &\leq 2C_\rho\left(\int_\Omega |a(x)|^2dx\int_\Omega|\varphi(y)|^{p+1}dy+\int_\Omega dx\int_\Omega |a(y)|^2|\varphi(y)|^{p+1}dy\right)\\ &\leq 4C_\rho\|a\|^2_{L^\infty(\Omega)}|\Omega|\,\|\varphi\|^{p+1}_{L^{p+1}(\Omega)},\end{aligned}\tag{3.13}$$

due to the fact that $a=J*1\in L^\infty(\Omega)$ by virtue of $J\in L^1(\mathbb{R}^d)$. On the other hand, as in the proof of (3.9) we have

$$\begin{aligned}&\left|\int_{\Omega}\int_{\Omega_2:|x-y|<1}\rho(|x-y|)(a(x)-a(y))|\varphi(y)|^{p+1}dydx\right| \\ &\leq \int_{\mathbb{R}^d}\int_{B_1}\int_0^1\rho(|z|)|z|^2|\nabla a(x+tz)|^2|\widetilde{\varphi}(z+x)|^{p+1}dtdzdx\\ &\leq \overline{C}_\rho|S_{d-1}|\,\|\nabla a\|^2_{L^\infty(\mathbb{R}^d)}\|\varphi\|^{p+1}_{L^{p+1}(\Omega)},\end{aligned}\tag{3.14}$$

since $\nabla a=\nabla J*1\in L^\infty(\mathbb{R}^d)$, owing to $J\in W^{1,1}(\mathbb{R}^d)$. The final term J_{21} can be bounded in the following fashion:

$$\begin{aligned}&\left|\int_{\Omega}\int_{\Omega_1:|x-y|\geq 1}\rho(|x-y|)(a(x)-a(y))^2(\varphi(y))^2|\varphi(x)|^{p-1}dydx\right| \\ &\leq 2C_\rho\int_\Omega |a(x)|^2|\varphi(x)|^{p-1}dx\int_\Omega|\varphi(y)|^2dy\\ &+2C_\rho\int_\Omega|\varphi(x)|^{p-1}dx\int_\Omega|a(y)|^2|\varphi(y)|^2dy\\ &\leq 4C_\rho\|a\|^2_{L^\infty(\Omega)}\|\varphi\|^{p+1}_{L^{p+1}(\Omega)}\end{aligned}\tag{3.15}$$

and

$$\begin{aligned}&\left|\int_{\Omega}\int_{\Omega_1:|x-y|<1}\rho(|x-y|)(a(x)-a(y))^2(\varphi(y))^2|\varphi(x)|^{p-1}dydx\right| \\ &\leq \int_{\mathbb{R}^d}\int_{B_1}\int_0^1\rho(|z|)|z|^2|\nabla a(x+tz)|^2|\widetilde{\varphi}(z+x)|^2|\widetilde{\varphi}(x)|^{p-1}dtdzdx\\ &\leq \|\nabla a\|^2_{L^\infty(\mathbb{R}^d)}\|\widetilde{\varphi}\|^2_{L^{p+1}(\mathbb{R}^d)}\|\widetilde{\varphi}\|^{p-1}_{L^{p+1}(\mathbb{R}^d)}\left(\int_{B_1}\rho(|z|)|z|^2dz\right)\\ &\leq \overline{C}_\rho|S_{d-1}|\,\|\nabla a\|^2_{L^\infty(\mathbb{R}^d)}\|\varphi\|^{p+1}_{L^{p+1}(\Omega)},\end{aligned}\tag{3.16}$$

owing to the Holder's inequality in $\mathbb{R}^d$. Combining the estimates from (3.13)–(3.16), we have found a constant $C>0$, depending only on a, ρ, Ω and on the dimension d, such that

$$|J_{21}+J_{22}| \leq C\,\|\varphi\|_{L^{p+1}(\Omega)}^{p+1}. \tag{3.17}$$

Hence, putting together (3.12), (3.17) with (3.5)–(3.7) into (3.4), we deduce

$$\frac{d}{dt}\|\varphi\|_{L^{p+1}(\Omega)}^{p+1}+\frac{2c_0p}{p+1}l\left(|\varphi|^{\frac{p-1}{2}}\varphi,|\varphi|^{\frac{p-1}{2}}\varphi\right)\leq C\,(p+1)^2\,\|\varphi\|_{L^{p+1}(\Omega)}^{p+1}, \tag{3.18}$$

for all $t\geq 0$. We now note that (3.18) is key in proving the desired estimate (3.3). To this end, setting $p\,(=p_k)=2^k-1$, $k\geq 0$, and

$$x_k\,(t):=\int_\Omega |\varphi\,(t)|^{2^k}\,dx,\quad k\geq 0,$$

and having established (3.18), we obtain

$$\frac{d}{dt}x_k\,(t)+\frac{2c_0p_k}{p_k+1}l\left(|\varphi\,(t)|^{p_{k-1}}\varphi\,(t)\,,|\varphi\,(t)|^{p_{k-1}}\varphi\,(t)\right)\leq C2^{2k}x_k\,(t)\,,\,\forall t\geq 0. \tag{3.19}$$

As usual, our goal is to derive a recursive inequality for x_k using (3.19). In order to do so, for $q>1$ fixed such that $V\subset L^{2q}\,(\Omega)$ continuously, we define

$$\overline{p}_k:=\frac{p_k-p_{k-1}}{q\,(1+p_k)-(1+p_{k-1})}=\frac{1}{2q-1}<1,\ \overline{q}_k:=1-\overline{p}_k=2\frac{q-1}{2q-1}.$$

We aim to estimate the x_k-term on the right-hand side of (3.19) in terms of x_{k-1}. Next, the Hölder inequality, the Sobolev inequality $V\subset L^{2q}\,(\Omega)$ together with the Poincare inequality (see Proposition 2.1) yield

$$\begin{aligned} x_k&=\int_\Omega|\varphi|^{1+p_k}\,dx\leq\left(\int_\Omega|\varphi|^{(1+p_k)q}\,dx\right)^{\overline{p}_k}\left(\int_\Omega|\varphi|^{1+p_{k-1}}\,dx\right)^{\overline{q}_k} \\ &\leq C\left[l(|\varphi|^{\frac{p_k-1}{2}}\varphi,|\varphi|^{\frac{p_k-1}{2}}\varphi)+\left(\int_\Omega|\varphi|^{1+p_{k-1}}\,dx\right)^2\right]^{\overline{s}_k} \\ &\quad\times\left(\int_\Omega|\varphi|^{1+p_{k-1}}\,dx\right)^{\overline{q}_k}, \end{aligned} \tag{3.20}$$

with $\overline{s}_k=\overline{p}_kq\equiv q/\,(2q-1)\in(0,1)$ (also note that $(1+p_k)\,/2=1+p_{k-1}$). Here, note that we have also used the fact that

$$l(|\varphi|^{1+p_{k-1}},|\varphi|^{1+p_{k-1}})\leq l(|\varphi|^{\frac{p_k-1}{2}}\varphi,|\varphi|^{\frac{p_k-1}{2}}\varphi),\ \text{for}\ p_k=2^k-1,\ k\geq 2,$$

which holds as a consequence of the basic inequality

$$\left(z^{\frac{p_k+1}{2}}-t^{\frac{p_k+1}{2}}\right)^2\leq\left(|z|^{\frac{p_k-1}{2}}z-|t|^{\frac{p_k-1}{2}}t\right)^2,\ \text{for all}\ z,t\in\mathbb{R}.$$

Applying Young's inequality on the right-hand side of (3.20), we get for every $\eta>0$,

$$\begin{aligned}(p_k+1)^2\int_\Omega|\varphi|^{1+p_k}\,dx\leq&\,\eta l(|\varphi|^{\frac{p_k-1}{2}}\varphi,|\varphi|^{\frac{p_k-1}{2}}\varphi)+\eta\left(\int_\Omega|\varphi|^{1+p_{k-1}}\,dx\right)^2\\ &+Q_\eta\,(p_k+1)\left(\int_\Omega|\varphi|^{1+p_{k-1}}\,dx\right)^2,\end{aligned} \tag{3.21}$$

for some function $Q_\eta(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of k, owing to the fact that $z_k := \overline{q}_k/(1-\overline{s}_k) \equiv 2$ (indeed, $Q_\eta(y) = C_\eta y^{4/\overline{q}_k} = C_\eta y^{2(2q-1)/(q-1)}$, for some constant $C_\eta > 0$). Therefore, inserting (3.21) into the inequality (3.19), choosing a sufficiently small $0 < \eta \leq \eta_0 := \frac{1}{2}$ independent of k, we obtain for $t \geq 0$,

$$\frac{d}{dt}x_k(t) + \frac{\eta_0}{2} l\left(|\varphi(t)|^{\frac{p_k-1}{2}}\varphi(t), |\varphi(t)|^{\frac{p_k-1}{2}}\varphi(t)\right) \leq Q_\eta\left(2^k\right)(x_{k-1}(t))^2, \tag{3.22}$$

for some positive function $Q_\eta > 0$ independent of k, depending only on J, a, ρ and Ω. Next, we can apply Proposition 2.1 to infer that

$$\begin{aligned}\eta_0 l\left(|\varphi|^{\frac{p_k-1}{2}}\varphi, |\varphi|^{\frac{p_k-1}{2}}\varphi\right) &\geq \int_\Omega |\varphi|^{p_k+1}dx - \eta_0^{-\alpha}\left(\int_\Omega |\varphi|^{1+p_{k-1}}dx\right)^2 \\ &= x_k - \eta_0^{-\alpha}(x_{k-1})^2,\end{aligned} \tag{3.23}$$

for some $\alpha > 0$ independent of φ, k. We can now combine (3.23) with (3.22) to deduce

$$\frac{d}{dt}x_k(t) + \frac{1}{2}x_k(t) \leq Q_\eta\left(2^k\right)(x_{k-1})^2, \ \forall t \geq 0. \tag{3.24}$$

Integrating (3.24) over $(0,t)$, we infer from Gronwall–Bernoulli's inequality [5, Lemma 1.2.4] that there exists yet another constant $C > 0$, independent of k, such that

$$x_k \leq \max\left\{\int_\Omega |\varphi_{0\epsilon}|^{2^k}dx, C\left(2^k\right)^{\frac{2(2q-1)}{q-1}}\sup_{t\geq 0}(x_{k-1})^2\right\}, \text{ for all } k \geq 2. \tag{3.25}$$

On the other hand, let us observe that there exists a positive constant $C_{\infty,\epsilon} = C_\infty(\|\varphi_{0\epsilon}\|_{L^\infty(\Omega)}) \geq 1$, independent of k, such that $\|\varphi_{0\epsilon}\|_{L^{2^k}(\Omega)} \leq C_{\infty,\epsilon}$. Taking the 2^k-th root on both sides of (3.25), and defining $X_k := \sup_{t\geq 0}\|\varphi(t)\|_{L^{2^k}(\Omega)}$, we easily arrive at

$$X_k \leq \max\left\{C_{\infty,\epsilon}, \left(C\left(2^k\right)^{\frac{2(2q-1)}{q-1}}\right)^{\frac{1}{2^k}}X_{k-1}\right\}, \text{ for all } k \geq 2. \tag{3.26}$$

We can now iterate in (3.26) (see, e.g., [5, Lemma 9.3.1]), to finally obtain

$$\begin{aligned}\sup_{t\geq 0}\|\varphi_\epsilon(t)\|_{L^\infty(\Omega)} &\leq \lim_{k\to+\infty} X_k \\ &\leq C\max\left(C_{\infty,\epsilon}, \sup_{t\geq 0}\|\varphi_\epsilon(t)\|_{L^2(\Omega)}\right),\end{aligned} \tag{3.27}$$

for some $C > 0$ independent of $\epsilon > 0$. Thus, since $\varphi_\epsilon \in L^\infty\left(0,T; L^2(\Omega)\right)$ (uniformly with respect to $\epsilon > 0$ and time), owing to estimate (2.50) it follows from (3.27) that $\varphi_\epsilon \in L^\infty(0,T; L^\infty(\Omega))$ uniformly with respect to time.

In the final step, we recall how the inequality (3.24) yields (3.3) uniformly with respect to the initial datum $\varphi_{0\epsilon}$. Indeed, using (3.24) and (3.22) one can exploit the scheme of [24, Lemma 2.10] (cf. also [25, Theorem 3.9]) to derive

$$x_k(t) \leq C_\xi\left(2^k\right)^\sigma\left(\sup_{s\geq t-\xi/2^k} x_{k-1}(s)\right)^2, \quad \forall k \geq 1, \tag{3.28}$$

where t, ξ are two positive constants such that $t - \xi/2^k > 0$, and C_ξ, σ are positive constants independent of k; the constant C_ξ is bounded if ξ is bounded away from zero. For instance, one may choose any numbers $\tau' > \tau > 0$ such that $\xi = (\tau' - \tau)$, $t_0 = \tau'$ and $t_k = t_{k-1} - \xi/2^k$, $k \geq 1$. In particular, iterating in (3.28) with respect to $k \geq 1$ yields

$$\sup_{t\geq t_0=\tau'}\|\varphi(t)\|_{L^\infty(\Omega)} \leq \lim_{k\to+\infty}\sup_{t\geq t_0}(x_k(t))^{1/2^k} \leq C_\xi\left(C_{L^2}\right), \tag{3.29}$$

for some positive constant C_ξ independent of t, k, φ, ε and initial data, with

$$C_{L^2} := \sup_{t\geq \frac{3\tau}{2}} \|\varphi(t)\|^2_{L^2(\Omega)} \leq C_m \left(1+\mathcal{N}(\varphi_0)\right). \tag{3.30}$$

The desired bound (3.3) follows by setting $\tau' = 2\tau$ above, and hence the first claim of the lemma follows. Finally, the statement of Theorem 2.11 yields the existence of a bounded absorbing ball in $L^2(\Omega)$, owing to $\mathcal{N}(\varphi) \geq c_F \|\varphi\|^2_{L^2(\Omega)} - c'_F$. More precisely, for any bounded set $\mathcal{B} \subset \mathcal{Y}_m$, there exists a time $t_* = t_*(\mathcal{B}) > 0$ such that $S(t)\mathcal{B} \subset L^2(\Omega)$, for all $t \geq t_*$. Next, we can choose $\tau' = \tau + 2\xi$ with $\tau = t_*$ and $\xi = 1$, so that both C_{L^2} and C_ξ are bounded uniformly with respect to the initial data $\varphi_0 \in \mathcal{Y}_m$ as $t \geq t_*$. Hence, the smoothing property (3.29) immediately entails the second assertion of lemma. □

The previous lemma allows us to show a further improved regularity result for the weak solution of Theorem 2.7.

Lemma 3.3. *Let the assumptions of Lemma 3.1 be satisfied. Then, for every $\tau > 0$, there exists a constant $C_{m,\tau} > 0$ such that*

$$\sup_{t\geq 3\tau} \left[\|\varphi(t)\|_V + \|\partial_t\varphi\|_{L^2([t,t+1]\times\Omega)} + \|\mu(t)\|_V\right] \leq C_{m,\tau}. \tag{3.31}$$

Moreover, for any bounded set $\mathcal{B} \subset \mathcal{Y}_m$, there exists a time $t_\# = t_\#(\mathcal{B}) > 0$ such that $S(t)\mathcal{B} \subset V$, for all $t \geq t_\#$.

Proof. First, from (2.31) we recall the energy inequality:

$$\frac{d}{dt}\|\varphi\|^2_{L^2(\Omega)} + c_0 l(\varphi(t), \varphi(t)) \leq C\|\varphi(t)\|^2_{L^2(\Omega)},$$

where $C > 0$ depends only on J, ρ, c_0 and Ω. Using the first conclusion of Lemma 3.1, the application of the uniform Gronwall lemma then gives

$$\sup_{t\geq 2\tau} \int_t^{t+1} \|\varphi(\tau)\|^2_V \, d\tau \leq C_{m,\tau}, \tag{3.32}$$

due to assumption (H1). Recalling also that $b(x,\varphi) = a(x)\varphi + F'(\varphi)$, from the above estimates (3.32) and (3.3), we easily arrive at the uniform (in the initial data) estimate

$$\sup_{t\geq 2\tau} \int_t^{t+1} \|b(x,\varphi(\tau))\|^2_V \, d\tau \leq C_{m,\tau} \tag{3.33}$$

since $a \in L^\infty$ and $F^{(i)} \in L^\infty([\tau,\infty); L^\infty(\Omega))$, for any $i = 1, 2$. Next, we exploit the weak formulation (2.11)–(2.12) by setting $\psi = \partial_t b(x,\varphi)$ (recall that such a test function is always allowed within a Galerkin approximation scheme). We deduce

$$\begin{aligned} &\frac{d}{dt}E(t) + \int_\Omega \left(a(x) + F''(\varphi)\right)|\partial_t\varphi|^2 dx \\ &\leq -\int_\Omega\int_\Omega \rho(|x-y|)\left((J*\partial_t\varphi)(x) - (J*\partial_t\varphi)(y)\right)\left(b(x,\varphi(x)) - b(y,\varphi(y))\right), \end{aligned} \tag{3.34}$$

where we have set

$$\begin{aligned} E(t) := &\frac{1}{2}\int_\Omega\int_\Omega \rho(|x-y|)\left(b(x,\varphi(x)) - b(y,\varphi(y))\right)^2 dydx \\ &- \int_\Omega\int_\Omega \rho(|x-y|)\left((J*\varphi)(x) - (J*\varphi)(y)\right)\left(b(x,\varphi(x)) - b(y,\varphi(y))\right). \end{aligned}$$

We observe that, owing to (3.3), and Cauchy–Schwarz and Young inequalities we have

$$\begin{aligned}\frac{1}{4}l(b,b)+C_{m,\tau} &\geq E(t) \\ &\geq \frac{1}{4}\int_\Omega\int_\Omega \rho(|x-y|)(b(x,\varphi(x))-b(y,\varphi(y)))^2dydx - C_{m,\tau} \\ &= \frac{1}{4}l(b,b)-C_{m,\tau},\end{aligned} \tag{3.35}$$

by arguing as in the proof of Theorem 2.7 (see, in particular, (2.26)–(2.28)). Moreover, for every $\varepsilon>0$ it follows once again as in (2.26)–(2.28) that

$$\begin{aligned}&\left|\int_\Omega\int_\Omega \rho(|x-y|)((J*\partial_t\varphi)(x)-(J*\partial_t\varphi)(y))(b(x,\varphi(x))-b(y,\varphi(y)))\right| \\ &\leq \varepsilon\int_\Omega\int_\Omega \rho(|x-y|)((J*\partial_t\varphi)(x)-(J*\partial_t\varphi)(y))^2dydx \\ &+C_\varepsilon\int_\Omega\int_\Omega \rho(|x-y|)(b(x,\varphi(x))-b(y,\varphi(y)))^2dydx \\ &\leq \varepsilon C_*\|\partial_t\varphi\|^2_{L^2(\Omega)}+C_\varepsilon l(b,b),\end{aligned} \tag{3.36}$$

for some constant $C_*>0$, which depends only on Ω, C_ρ, $\overline{C}_\rho$ and $\|J\|_{W^{1,1}(\mathbb{R}^d)}$. Exploiting also assumption (H4), from (3.34) it follows

$$\begin{aligned}\frac{d}{dt}E(t)+c_0\int_\Omega|\partial_t\varphi|^2dx &\leq \varepsilon C_*\|\partial_t\varphi\|^2_{L^2(\Omega)}+C_\varepsilon l(b,b) \\ &\leq \frac{c_0}{2}\|\partial_t\varphi\|^2_{L^2(\Omega)}+C(c_0,C_*)l(b,b),\end{aligned}$$

if we choose $\varepsilon=c_0/2C_*>0$. The foregoing inequality then yields

$$\frac{d}{dt}E(t)+\frac{c_0}{2}\|\partial_t\varphi(t)\|^2_{L^2}\leq E(t)+C_{m,\tau}, \tag{3.37}$$

for some $C_{m,\tau}>0$ independent of time and the initial data. We can now apply the uniform Gronwall lemma to infer that

$$\sup_{t\geq 3\tau}E(t)\leq C_{m,\tau}, \tag{3.38}$$

which yields the first part of the claim in (3.31), owing to (3.35) and the fact that

$$\begin{aligned}(b(x)-b(y))^2 &\geq \frac{1}{2}(a(x)+q_F(\varphi))^2(\varphi(x)-\varphi(y))^2 \\ &-(a(x)-a(y))^2(\varphi(y))^2 \\ &\geq \frac{c_0}{2}(\varphi(x)-\varphi(y))^2-(a(x)-a(y))^2(\varphi(y))^2;\end{aligned} \tag{3.39}$$

(indeed, use assumption (H4) and the basic inequality $(A+B)^2\geq\frac{1}{2}A^2-B^2$). Finally, in order to obtain the second part of (3.31), one integrates (3.37) over the interval $(t,t+1)$, and exploits once again the inequality (3.38). The final claim then also follows on account of the second statement of Lemma 3.1. The reader can also immediately check that the chemical potential has in fact more regularity; in particular, $\mu\in L^2([t,t+1];D(\mathcal{L}))$ for all $t\geq 3\tau>0$, owing to the second of (3.31) and the first of (1.12). The proof is now finished. □

We conclude this section with the following result on the existence of strong solutions. Indeed, this statement is a consequence of the proofs of Lemmas 3.1 and 3.3, respectively.

Theorem 3.4. *Let the assumptions of Lemma 3.1 be satisfied and assume $\varphi_0\in L^\infty(\Omega)\cap V$. Then problem (1.12) has a unique strong solution in the sense of Definition 2.6.*

4. Long-term behavior

In this section, we show how to modify the whole scheme of [24, Section 2.2], applied to the nonlocal Cahn–Hilliard equation (1.6)–(1.8), to prove the following.

Theorem 4.1. *Let the assumptions of Lemma 3.1 be satisfied. For every fixed $m \geq 0$, there exists an exponential attractor $\mathcal{E}_m = \mathcal{E}(m)$ bounded in $V \cap L^\infty(\Omega)$ for the dynamical system $(\mathcal{Y}_m, S(t))$ which satisfies the following properties:*

(i) Semi-invariance: $S(t)\mathcal{E} \subseteq \mathcal{E}$, for every $t \geq 0$.

(ii) Exponential attraction:

$$dist_{L^\infty(\Omega)}(S(t)B, \mathcal{E}) \leq C_m e^{-\kappa t}, \quad \forall t \geq 0,$$

for any bounded $B \subset \mathcal{Y}_m$, for some positive constants C_m and κ.

(iii) Finite dimensionality:

$$\dim_F\left(\mathcal{E}, L^\infty(\Omega)\right) \leq C_m < \infty.$$

Consequently, with the assumptions of Theorem 4.1 it holds the following

Corollary 4.2. *The dynamical system $(\mathcal{Y}_m, S(t))$ possesses the connected global attractor $\mathcal{G}$, bounded in $V \cap L^\infty(\Omega)$ and of finite fractal dimension:*

$$\dim_F\left(\mathcal{G}, L^\infty(\Omega)\right) < \infty.$$

Proof of Theorem 4.1. We only briefly explain the (actual) differences with respect to the proof of [24, Theorem 2.8]. The main observation here is that while the proof of [24, Theorem 2.8] uses the $C^{\alpha/2,\alpha}\left([t, t+1] \times \overline{\Omega}\right)$-regularity ($t > 0$) of any weak solution of (1.10) in a crucial way, we can still adapt that proof to our situation when much less regularity is actually available for (1.12), according to the statements of Lemma 3.2 and Lemma 3.3. Their statements imply in particular, that the ball $B_0 := B_{L^\infty(\Omega)\cap V}(R_0)$ will be absorbing for $S(t)$, provided that $R_0 > 0$ is sufficiently large. The main ingredients in this proof are as follows:

1. We define the set $B_1 = \left[\cup_{t\geq 0} S(t) B_0\right]_{V_m'}$, where $[\cdot]_{V_m^*}$ denotes closure in the space

   $$V_m^* := \left\{\psi \in V^* : \frac{1}{|\Omega|}\langle \psi, 1\rangle \in [-m, m]\right\},$$

   and then set $\mathbb{B} = S(1) B_1$. Thus, $\mathbb{B}$ is a semi-invariant and closed (for the metric $d_{V_m^*}(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\|_{V^*} + |\langle \varphi_1 - \varphi_2\rangle|^{1/2}$) subset of the phase space $\mathcal{Y}_m$. On the other hand, a comparison argument in (2.11) together with the uniform bounds established in these lemmas, gives

   $$\sup_{t\geq 0}\left(\|\varphi(t)\|_{L^\infty\cap V} + \|\mu(t)\|_V + \|\partial_t \varphi(t)\|_{V^*}\right) \leq C_m, \tag{4.1}$$

   for every trajectory φ originating from $\varphi_0 = \varphi(0) \in \mathbb{B}$, for some positive constant C_m which is independent of the choice of $\varphi_0 \in \mathbb{B}$.
2. In light of (4.1) and the application of the Gronwall inequality (on (2.46)), for $\varphi_1(0), \varphi_2(0) \in \mathbb{B}$ we have the estimate

   $$\begin{aligned} &\|\varphi_1(t) - \varphi_2(t)\|_{V^*}^2 + C\left|\overline{\varphi}_1 - \overline{\varphi}_2\right| \\ &\leq e^{-\kappa t}\left(\|\varphi_1(0) - \varphi_2(0)\|_{V^*}^2 + C\left|\overline{\varphi}_1 - \overline{\varphi}_2\right|\right) \\ &+ C_m \int_0^t \left(\|\varphi_1(s) - \varphi_2(s)\|_{V^*}^2 + \left|\overline{\varphi}_1 - \overline{\varphi}_2\right|\right) ds, \end{aligned} \tag{4.2}$$

for all $t \geq 0$. Furthermore, in view of (2.44)–(2.45) (for any test function $\psi \in D(\mathcal{L}) \subset V$ and $\varphi = \varphi_1 - \varphi_2$), the smoothing property for the difference of two solutions also holds, owing once again to (4.1),

$$\|\partial_t \varphi_1 - \partial_t \varphi_2\|^2_{L^2([0,t];D(\mathcal{L})^*)} + c_0 \int_0^t \|\varphi_1(s) - \varphi_2(s)\|^2_{L^2(\Omega)} ds \tag{4.3}$$
$$\leq C_m e^{\kappa t} \|\varphi_1(0) - \varphi_2(0)\|^2_{V^*} + C e^{\kappa t} \left|\overline{\varphi}_1 - \overline{\varphi}_2\right|,$$

for all $t \geq 0$. Here, C_m, C and $\kappa > 0$ depend also on c_0, Ω, ρ and J.

3. It can be easily verified that the map $(t, \varphi_0) \mapsto S(t)\varphi_0$ is also uniformly Hölder continuous on $[0, T] \times \mathbb{B}$, where $\mathbb{B}$ is endowed with the metric topology of V_m^* (as well as, the strong $L^2(\Omega)$-topology).

Finally, one can define the map $\mathbb{S} = S(T) : \mathbb{B} \to \mathbb{B}$ and $\mathcal{H} = V_m^*$, for a fixed $T > 0$ such that $e^{-\kappa T} < \frac{1}{2}$, where $\kappa > 0$ is the same as in (4.2). Next, one introduces the functional spaces

$$\mathcal{V}_1 := L^2\left([0, T]; L^2(\Omega)\right) \cap H^1\left([0, T]; D(\mathcal{L})^*\right), \quad \mathcal{V} := L^2\left([0, T]; V_m^*\right), \tag{4.4}$$

and the operator $\mathbb{T} : \mathbb{B} \to \mathcal{V}_1$, by $\mathbb{T}\varphi_0 := \varphi \in \mathcal{V}_1$, where φ solves our fully nonlocal problem (1.12) with $\varphi(0) = \varphi_0 \in \mathbb{B}$. Here we observe that $\mathcal{V}_1$ is compactly embedded into $\mathcal{V}$, on account of (H2). With this choice of spaces and operators, it follows that all the hypotheses of the abstract result of [24, Proposition 2.18] can be verified, and, subsequently, we can infer the existence of (discrete) exponential attractor associated with the semigroup $\mathbb{S}(n) = S(nT)$. Arguing in a similar fashion as in the last part of the proof of [24, Theorem 2.8], we may conclude the proof of Theorem 4.1. □

Our final goal in this section is to show that once each (unique) weak solution φ enters a small neighborhood of a nonzero stationary state φ_*, then it must remain there for all time $t \geq t_*$, t_* large enough and, consequently, φ fully converges to φ_*. We first check that every weak solution given by Theorem 2.11 has a non-empty (weak) ω-limit set $\omega(\varphi)$, where $\omega(\varphi)$ is defined by

$$\omega(\varphi) = \left\{\varphi_* : \exists t_n \to \infty \text{ such that } \varphi(t_n) \to \varphi_* \text{ weakly in } L^2(\Omega)\right\}.$$

Lemma 4.3. *Let φ be the (unique) weak solution obtained from Theorem 2.11. Then, any divergent sequence $\{t_n\} \subset [0, \infty)$ admits a subsequence, denoted by $\{t_{n_k}\}$, such that*

$$\lim_{t_{n_k} \to \infty} \varphi\left(t_{n_k}\right) = \varphi_* \text{ weakly in } L^2(\Omega), \text{ strongly in } V^*, \tag{4.5}$$

for some $\varphi_* \in \mathcal{Y}_m$ which is a solution of

$$\begin{cases} a(x)\varphi_* - J * \varphi_* + F'(\varphi_*) = \mu_*, & \text{a.e. in } \Omega, \\ \mu_* = \text{constant}, \quad \overline{\varphi}_* = \overline{\varphi}_0. \end{cases} \tag{4.6}$$

Proof. A proof for (4.5), based on the energy identity (2.13), together with the assumptions (H1)–(H7), and the regularity properties stated in Lemmas 3.1, 3.3, can be adapted from [3,24] with some minor modifications, and it may be left to the reader. □

We have the following convergence result which is the main result of this section.

Theorem 4.4. *Let the assumptions of Lemma 3.1 hold and, in addition, assume that F is real analytic. Then, for any (given) $\varphi_0 \in \mathcal{Y}_m$ the corresponding weak solution φ satisfies*

$$\lim_{t \to \infty} \|\varphi(t) - \varphi_*\|_{L^\infty(\Omega)} = 0, \tag{4.7}$$

where φ_* is (some) solution to (4.6). Namely, $\omega(\varphi) = \{\varphi_*\}$.

Proof. The main ingredients in the proof are the following:

1. Observe that all stationary solutions $\varphi_* \in \omega(\varphi)$ are continuous in $\overline{\Omega}$ and bounded in V (and, therefore, $\omega(\varphi)$ is also strong in the $L^2(\Omega)$-topology; cf. also Lemma 3.3), owing to the fact that $\varphi \mapsto J * \varphi : L^\infty(\Omega) \to C\left(\overline{\Omega}\right)$ is compact (due to $J \in W^{1,1}\left(\mathbb{R}^d\right)$).
2. The energy identity (2.13) implies that

$$\frac{d}{dt}\mathcal{N}(\varphi(t)) = -l(\mu(t), \mu(t)), \text{ for } t \geq 0, \tag{4.8}$$

and that $\mathcal{N}(\varphi(t))$ is non-increasing in $t \in [0, \infty)$ and bounded below, $\mathcal{N}(\varphi(t)) \geq -C_F$, owing to (H6). Integrating (4.8) over (t, ∞) yields

$$\int_t^\infty l(\mu(s), \mu(s))\, ds = \mathcal{N}(\varphi(t)) - \mathcal{N}_\infty = \mathcal{N}(\varphi(t)) - \mathcal{N}(\varphi_*). \tag{4.9}$$

3. One can apply the Lojasiewicz–Simon theorem of [24, Lemma 2.20] to infer the existence of some constants $\theta \in (0, \frac{1}{2})$, $C, C_{\mathcal{L}} > 0$, $\varepsilon > 0$, such that

$$|\mathcal{N}(\varphi(t)) - \mathcal{N}(\varphi_*)|^{1-\theta} \leq C \|\mu(t) - \overline{\mu}(t)\|_{L^2(\Omega)} \leq C_{\mathcal{L}}\sqrt{l(\mu(t), \mu(t))} \tag{4.10}$$

provided that $\|\varphi - \varphi_*\|_{L^2(\Omega)} \leq \varepsilon$. The last inequality in (4.10) is a Poincare inequality.

Combining (4.10) with (4.9) yields

$$\left(\int_t^\infty l(\mu(s), \mu(s))\, ds\right)^{2(1-\theta)} \leq Cl(\mu(t), \mu(t)), \tag{4.11}$$

for all $t > 0$, for as long as $\|\varphi - \varphi_*\|_{L^2(\Omega)} \leq \varepsilon$ holds. Let us now set $M = \cup \mathcal{I}$, where $\mathcal{I}$ is an open interval on which $\|\varphi - \varphi_*\|_{L^2(\Omega)} \leq \varepsilon$ holds. This set is nonempty since $\varphi_* \in \omega(\varphi)$. One can then use (4.11), the fact that $Z(t) := \sqrt{l(\mu(t), \mu(t))} \in L^2(0, \infty)$ (cf. (4.8)), and exploit [20, Lemma 7.1] with $\alpha = 2(1-\theta)$ to deduce that $Z(\cdot) \in L^1(M)$ and

$$\int_M Z(s)\, ds = \int_M \sqrt{l(\mu(s), \mu(s))} ds \leq C(\varphi_*) < \infty. \tag{4.12}$$

Consequently, using (4.12) and (2.11), one also obtain $\partial_t \varphi \in L^1(M; V^*)$. This fact together with a simple contradiction argument (see, for instance, [24, Theorem 2.21]) allows to conclude that $\varphi(t) \to \varphi_*$ in the strong topology of V^*, as well as in that of $L^2(\Omega)$, owing to compactness (see (H2)). The claim follows by appealing once again to the L^2-$(L^\infty \cap V)$ smoothing property of φ. □

5. Applications and concluding remarks

Although the nonlocal operator $\mathcal{L}$ from (1.12) has an abstract nature due to assumptions (H1)–(H2), as well as (H7), we shall now show that these assumptions are indeed verified for a large class of interesting non-local operators. Some of them will be discussed below.

5.1. The regional fractional Laplacian

We start with the case of the so-called *anomalous mass diffusion* in Ω, i.e., we let $\mathcal{L}$ be the operator associated with the *regional fractional Laplacian* A^s_Ω, $s \in (0, 1)$, with (homogeneous) fractional-type Neumann type boundary conditions (see [24,28,39] for further details). To this end, set

$$\mathbb{L}(\Omega) := \{\mu : \Omega \to \mathbb{R} \text{ measurable}, \int_\Omega \frac{|\mu(x)|}{(1+|x|)^{d+2s}} dx < \infty\}$$

and define for $\mu \in \mathbb{L}(\Omega)$, $x \in \Omega$, the operator

$$A_{\Omega}^{s}\mu(x) = C_{d,s}\text{P.V.}\int_{\Omega} \frac{\mu(x)-\mu(y)}{|x-y|^{d+2s}}dy$$
$$= \lim_{\varepsilon\downarrow 0} C_{d,s} \int_{\{y\in\Omega, |y-x|>\varepsilon\}} \frac{\mu(x)-\mu(y)}{|x-y|^{d+2s}}dy, \quad x \in \Omega,$$

provided that the limit exists. Note that $\mathbb{L}(\Omega)$ is nonempty since it contains at least $L^{\infty}(\Omega)$ and $C_{d,s}$ is a normalizing constant (see [25]). Letting

$$K(x,y) = \rho(|x-y|) := \frac{C_{d,s}}{|x-y|^{d+2s}}, \; s \in (0,1)$$

and, for $\mu, v \in V = W^{s,2}(\Omega)$, the bilinear form

$$l(\mu,v) := \frac{C_{d,s}}{2}\int_{\Omega}\int_{\Omega} \frac{(\mu(x)-\mu(y))(v(x)-v(y))}{|x-y|^{N+2s}}dydx$$

it turns out that l is symmetric, nonnegative and closed with domain $D(l) = V$ (see [24,28,29,39]). It is also well-known that $V = W^{s,2}(\Omega)$ is compactly and densely contained in $L^2(\Omega)$, for any $s \in (0,1)$ provided that Ω is a bounded domain with Lipschitz boundary $\partial\Omega$. Next, $V = W^{s,2}(\Omega)$ is continuously embedded into $L^{2_*}(\Omega)$, for $2_* := \frac{2d}{d-2s}$ if $d > 2s$ and into $L^r(\Omega)$, for any $r \in (2,\infty)$ if $d = 2s$, while $V \subset C^{0,\alpha}(\overline{\Omega})$ with $\alpha := s - \frac{d}{2}$ provided that $d < 2s$. Hence, the embedding $V \subset L^{2q_d}(\Omega)$ is continuous provided that $q_d = d/(d-2s)$ if $d = 3$, $q_d = r/2$, for any (arbitrary) $r \in (2,\infty)$ if $d = 1, 2$. Hence, the assumption (2.2) is also satisfied if we set $q = q_d$. It remains to check assumption (H7) for K as defined above. Indeed, it is easy to see that, for $\rho(r) = C_{d,s}/r^{d+2s}$, $r > 0$, we easily obtain in (2.7) that $C_\rho = C_{d,s} < \infty$ and $\overline{C}_\rho = C_{d,s}/(2-2s) < \infty$, provided that $s \in (0,1)$. Hence, both assumptions (H1)–(H2) are satisfied. Then, $\mathcal{L} = \mathcal{L}_s$ is the closed linear self-adjoint operator associated with the form $l(\cdot,\cdot)$ in the sense of (2.3). In particular, one may call $\mathcal{L}_s$ the realization of the regional fractional Laplace operator A_{Ω}^{s} on $L^2(\Omega)$ with (homogeneous) fractional Neumann type boundary conditions. The reason for this is due to the fact that a further (explicit) characterization of $\mathcal{L}_s$ may be given for bounded domains of class $C^{1,1}$ (cf. [24,28,39]). To define the "fractional" normal derivative $\mathcal{N}^{\alpha}$, consider a function $u \in C^1(\Omega)$, $z \in \partial\Omega$ and $0 \le \alpha < 2$ and let

$$\mathcal{N}^{\alpha}u(z) = -\lim_{t\downarrow 0} \frac{du(z+\vec{n}(z)t)}{dt}t^{\alpha}, \tag{5.1}$$

whenever the limit exists (cf. [28, Definition 2.1]). We emphasize that if $\alpha = 0$, then $\mathcal{N}^{0}u(z) = -\nabla u \cdot \vec{n}(z) = \frac{\partial u(z)}{\partial \nu}$ for every $u \in C^1(\overline{\Omega})$ and $z \in \partial\Omega$; here $n(z)$ denotes the inner normal vector to $\partial\Omega$ at the point $z \in \partial\Omega$ and $\nu(z) = -n(z)$ is the outer normal vector to $\partial\Omega$ at the point z. Clearly, if $0 < \alpha < 2$, then $\mathcal{N}^{\alpha}u(z) = 0$ for every $u \in C^1(\overline{\Omega})$ and $z \in \partial\Omega$. Let us now denote by $\Omega_\delta = \{x \in \Omega : \; 0 < \rho(x) < \delta\}$, where $\rho(x) = \text{dist}(x, \partial\Omega)$ for some $\delta > 0$. Next, let $\beta > 0$ and recall that there exist a real number $\delta > 0$ (depending only on Ω) and a function $h_\beta \in C^2(\Omega)$ (depending on Ω and β) such that

$$h_\beta(x) = \begin{cases} \rho(x)^{\beta-1}, & \forall\, x \in \Omega_\delta, \; \text{when} \; \beta \in (0,1)\cup(1,\infty); \\ \ln(\rho(x)), & \forall\, x \in \Omega_\delta, \; \text{when} \; \beta = 1, \end{cases} \tag{5.2}$$

see e.g., [28, p. 294]. For $\beta > 0$, define the space

$$C_\beta^2(\overline{\Omega}) = \{u : \; u(x) = f(x)h_\beta(x) + g(x), \; \forall x \in \Omega, \; \text{ for some } \; f, g \in C^2(\overline{\Omega})\}.$$

When $\beta > 1$, we always assume that $u \in C_\beta^2(\overline{\Omega})$ is defined on $\overline{\Omega}$ by continuous extension. For $\frac{1}{2} < s \le 1$ and $u \in C_{2s}^2(\overline{\Omega})$, then one may call the function $\mathcal{N}^{2-2s}u$ *the fractional normal derivative* of the function u in direction of the outer normal vector (cf. also [39]). An explicit characterization of $\mathcal{L} = \mathcal{L}_s$, given by (2.3), when Ω is a bounded open set of class $C^{1,1}$ was shown in [28,29,39]. Indeed, for $\mu \in C_{2s}^2(\overline{\Omega}) \cap D(\mathcal{L})$ one has $\mathcal{L}\mu = A_{\Omega}^{s}\mu$ and $\mathcal{N}^{2-2s}\mu = 0$ on $\partial\Omega$, for as long as $1/2 < s < 1$. In the case $s \in (0, 1/2]$, the operator $\mathcal{L} = \mathcal{L}_s$ turns out to be the self-adjoint realization of the regional fractional Laplacian A_{Ω}^{s} with (homogeneous) Dirichlet boundary conditions on $\partial\Omega$ (see, again [24,39]). Henceforth, in order to ensure that there is conservation of mass (see Remark 2.2), the only interesting case for this operator is when $s \in (1/2, 1)$ although the results below also hold for the case $s \in (0, 1/2]$, albeit without any conservation of mass.

Thus, on account of Theorems 2.7, 2.10, we can verify the following statements when $\mathcal{L} = \mathcal{L}_s$ is the self-adjoint realization of the regional fractional Laplacian with (homogeneous) fractional Neumann type boundary conditions.

Corollary 5.1. *Let $\varphi_0 \in L^2(\Omega)$ such that $F(\varphi_0) \in L^1(\Omega)$ and suppose that (H3)–(H5) are satisfied by the potential F and interaction kernel J. Then, for every $T > 0$ there exists a unique weak solution φ to problem (1.12) on $[0, T]$ in the sense of Definition 2.4, and φ also satisfies the energy identity (2.13).*

Corollary 5.2. *If in addition to the assumptions of Corollary 5.1, the potential F also obeys (H6) then every week solution φ satisfies the dissipative estimate (2.50).*

Proof. It follows from Theorem 2.11. □

Finally, the Theorems 4.1 and 4.4 are also verified in the following case.

Corollary 5.3. *Assume that (H3)–(H6) are satisfied by the potential F and kernel J. Let $\mathcal{L} = \mathcal{L}_s$, $s \in (0, 1)$. Then, the conclusion of Theorem 4.1 is satisfied for this operator, and if further F is real analytic then the conclusion of Theorem 4.4 is satisfied as well.*

5.2. Other nonlocal operators

We now consider an example of nonlocal form $l(\cdot, \cdot)$ which is also motivated by a series of bond-based peridynamic models in continuum mechanics (see [10,11] and the references therein). To this end, let

$$l(\mu, v) := \int_\Omega \int_\Omega K(x, y)(\mu(x) - \mu(y))(v(x) - v(y))dydx, \tag{5.3}$$

for all $\mu, v \in V := H^s(\Omega)$, $s \in (0, 1)$, where the kernel $K(x, y) = K(y, x)$ (for all $x, y \in \Omega$), also satisfies

$$c_{K,\Omega} \le K(x, y)|x - y|^{d+2s} \le C_{K,\Omega}, \text{ for all } x, y \in \Omega,\ x \neq y, \tag{5.4}$$

Then it follows from [23, Proposition 2.11] that the form l, given by (5.3), is closed and symmetric in $L^2(\Omega)$. Henceforth, the assumptions (H1)–(H2) of Section 2 are satisfied for the form in (5.3). We also note that assumption (H7) is satisfied as a consequence of (5.4). Let now $\mathcal{L} = \mathcal{L}(K)$ be the operator realization of (5.3) in the sense of (2.3). Then $(\mathcal{L}, D(\mathcal{L}))$ is self-adjoint, nonnegative on $L^2(\Omega)$ and $(\mathcal{L} + I)^{-1}$ is compact as an operator in $L^2(\Omega)$. Unfortunately, there aren't any explicit representations of $(\mathcal{L}, D(\mathcal{L}))$ in the literature, as in the previous subsection, due to the general assumption (5.4) on K. Even so, in view of Remark 2.9 assumption (H7) holds, and so all the statements of the previous subsection (namely, Corollaries 5.1 and 5.3) are satisfied for the operator $\mathcal{L} = \mathcal{L}(K)$. We point out that other classes of (unbounded) kernels ρ (see [10]) can be considered by our analysis, namely, when ρ is given by

$$\rho(|\xi|) = \begin{cases} \rho_0(|\xi|), & \text{for } |\xi| \le 1, \\ 0, & \text{for } |\xi| > 1, \end{cases}, \quad \rho(|\xi|) = -\frac{\ln(|\xi|)}{|\xi|^{d+2s}}, \tag{5.5}$$

for $s \in [0, 1)$ with $d \ge 2$ (which implies in particular that $\rho(|\xi|) \notin L^1(\mathbb{R}^d)$, but $|\xi|^2 \rho(|\xi|) \in L^1(\mathbb{R}^d)$). Sufficient conditions were given in [10,11] such that the corresponding form $l(\cdot, \cdot)$ associated with this kind of kernel, is a Dirichlet form in the sense of [23, Definition 2.1], and that $D(l) := V$ is dense and compact in $L^2(\Omega)$. Although, we do not further pursue these issues in the generality provided here, our next goal is to derive sufficient conditions on ρ such that V *embeds continuously into* $L^{2q}(\Omega)$, for some $q > 1$. Indeed, according to the analysis performed in Sections 2–4, such an embedding seems to be necessary if some kind of "regularity" is to be expected for the weak solution of (1.12) with an unbounded (in $\mathbb{R}$) potential F. In particular, we recall that Lemma 3.1 was the crucial tool in the derivation of further properties (such as, the existence of finite dimensional attractors and convergence to steady states) for our doubly nonlocal Cahn–Hilliard equation (1.12) (see Section 4).

Our final goal is then to derive sufficient conditions on ρ such that (2.2) holds as well. Although this question is of independent interest and is worth studying separately, we derive a partial result that gives sufficient conditions for the case of strictly positive kernels $\rho(r)$ having (possibly) different growths as $r \to 0^+$ and as $r \to \infty$, respectively. To this end, we extend the general procedure developed in [36, Chapter 6] for the fractional Sobolev space $W^{s,p}(\Omega)$, $s \in (0,1)$, $p > 1$ to derive such sufficient conditions on ρ.

Theorem 5.4. *Let Ω be a bounded domain with Lipschitz continuous boundary. Assume the following hypotheses:*

1. *ρ is non-increasing in $|\xi|$ and supp $(\rho) = (0,\infty)$, $|\xi|^2 \rho(|\xi|) \in L^1_{loc}(\mathbb{R}^d)$ and*
$$A(\alpha) := \int_\alpha^\infty \rho(r) r^{d-1} dr < \infty, \text{ for } \alpha > 0.$$
2. *There exists a real number $q > 1$ and a constant $C = C(d,q) > 0$, independent of α, such that $A\left(\alpha^{1/d}\right)\alpha^{\frac{q-1}{q}} \geq C$, for all $\alpha > 0$.*

Then $V \subset L^{2q}(\Omega)$ boundedly.

Remark 5.5. If $\rho(r) = C_{d,s} r^{-d-2s}$ for $s \in (0,1)$, then both hypotheses of Theorem 5.4 are satisfied for ρ with $q = d/(d-2s)$ for $d > 2s$. In this case $V = W^{s,2}(\Omega)$ if $s \in (1/2, 1)$, and $V = W_0^{s,2}(\Omega)$ if $s \in (0, 1/2]$, and for both we recover the standard embedding result $V \subset L^{2d/(d-2s)}(\Omega)$.

The second hypothesis of Theorem 5.4 seems reasonable although it is not clear whether it is optimal (see Section 5.3). Indeed, besides the obvious choice $\rho(r) = r^{-d-2s}$ it also allows for kernels with different growths as $r \to 0^+$ and $r \to \infty$, respectively. As an example, we can take

$$\rho(r) = \begin{cases} \frac{1-\ln(r)}{r^{d+2p}}, & \text{for } 0 < r \leq 1, \\ \frac{1}{r^{d+2s}}, & \text{for } r > 1 \end{cases} \tag{5.6}$$

for $s, p \in (0,1)$ with $p \geq s$. Observe that $|\xi|^2 \rho(|\xi|) \in L^1_{loc}(\mathbb{R}^d)$, ρ is nonincreasing on $(0,\infty)$ and

$$A(\alpha) = \alpha^{-2p}\left(\frac{1}{2p} + \frac{1}{(2p)^2}\right) + \frac{1}{2s} - \left(\frac{1}{2p} + \frac{1}{(2p)^2}\right) - \frac{1}{2p}\ln(\alpha)\alpha^{-2p},$$

if $\alpha \in (0,1)$, and

$$A(\alpha) = \frac{1}{2s}\alpha^{-2s}, \text{ if } \alpha \geq 1.$$

Thus setting $q = d/(d-2s)$ if $d > 2s$, it can be easily checked that

$$A\left(\alpha^{1/d}\right)\alpha^{\frac{q-1}{q}} \geq (2s)^{-1},$$

for any $\alpha > 0$, whenever $p \geq s$. We leave the details to check (H1)–(H2) for a kernel like (5.6) to the curious mind of the reader.

Before we conclude this subsection with a proof of Theorem 5.4 we need a series of preliminary estimates; the first one is an extended version of [36, Lemma 6.1].

Lemma 5.6. *Fix $x \in \mathbb{R}^d$. Assume $\rho(r)$ is non-increasing in $r > 0$ and $A(\alpha) := \int_\alpha^\infty \rho(r) r^{d-1} dr < \infty$, for $\alpha > 0$. Let $E \subset \mathbb{R}^d$ be a measurable set such that $|E| < \infty$. Then,*

$$\int_{\complement E} \rho(|x-y|)\, dy \geq A((|E|/\omega_d)^{1/d}),$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

Proof. As in the proof of [36, Lemma 6.1], one sets $\alpha := (|E|/\omega_d)^{\frac{1}{d}}$ and then observes that $|(\mathcal{C}E) \cap B_\alpha(x)| = |E \cap \mathcal{C}B_\alpha(x)|$. Owing to the fact that ρ is non-increasing on $(0, \infty)$, it follows that

$$\begin{aligned}\int_{\mathcal{C}E} \rho(|x-y|)\,dy &= \int_{(\mathcal{C}E)\cap B_\alpha(x)} \rho(|x-y|)\,dy + \int_{(\mathcal{C}E)\cap \mathcal{C}B_\alpha(x)} \rho(|x-y|)\,dy \\ &\geq |(\mathcal{C}E)\cap B_\alpha(x)|\rho(\alpha) + \int_{(\mathcal{C}E)\cap \mathcal{C}B_\alpha(x)} \rho(|x-y|)\,dy \\ &= |E\cap \mathcal{C}B_\alpha(x)|\rho(\alpha) + \int_{(\mathcal{C}E)\cap \mathcal{C}B_\alpha(x)} \rho(|x-y|)\,dy \\ &\geq \int_{E\cap \mathcal{C}B_\alpha(x)} \rho(|x-y|)\,dy + \int_{(\mathcal{C}E)\cap \mathcal{C}B_\alpha(x)} \rho(|x-y|)\,dy \\ &= \int_{\mathcal{C}B_\alpha(x)} \rho(|x-y|)\,dy. \end{aligned} \tag{5.7}$$

Finally, using polar coordinates centered at $x \in \mathbb{R}^d$, we observe that the final term on the right-hand side of (5.7) can be computed in terms of $A(\alpha) < \infty$, which gives the desired inequality. □

The following lemma is also a straight-forward generalization of [36, Lemma 6.2].

Lemma 5.7. *Fix $T, q > 1$ and let $N \in \mathbb{Z}$ and a_k be a bounded, nonnegative, decreasing sequence, with $a_k = 0$ for any $k \geq N$. Then*

$$\sum_{k\in\mathbb{Z}} a_k^{1/q} T^k \leq C \sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} a_{k+1} a_k^{-\frac{q-1}{q}} T^k,$$

for a suitable constant $C = C(q, T) > 0$ independent of N.

Proof. Indeed this follows essentially from the proof of [36, Lemma 6.2] exploiting the Holder inequality with exponents $(q, q/(q-1))$ for $q > 1$. □

We are now ready to give the

Proof of Theorem 5.4. We divide the proof into several key steps.

Step 1 *(Extension property).* For any function $f \in V$ there exists $\overline{f} \in L^2(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(|x-y|)(\overline{f}(x) - \overline{f}(y))^2 dy dx < \infty \tag{5.8}$$

such that $\overline{f}_{|\Omega} = f$ and

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \rho(|x-y|)(\overline{f}(x) - \overline{f}(y))^2 dy dx + \int_{\mathbb{R}^d} (\overline{f}(x))^2 dx \leq C \|f\|_V^2,$$

for some $C = C(d, \Omega) > 0$. Indeed, any function $f \in V$ may be extended to a function in $\overline{f}$ in all of $\mathbb{R}^d$, owing to the following facts: ρ is non-increasing in $|\xi|$, $|\xi|^2 \rho(|\xi|)$ is summable for $|\xi| \leq 1$, and $\rho(|\xi|)$ is summable for $|\xi| \geq 1$. We recall that the latter are already implied consequences of the first hypothesis of the theorem, allowing one to extend [36, Lemmas 5.1–5.3] to our situation in a straight-forward fashion. Thus the extension property is an immediate consequence of the proof of [36, Theorem 5.4] which relies mainly on these essential lemmas.

Step 2 *(The inequality in* $\mathbb{R}^d$*).* Due to the extension property is suffices to show the inequality for $\overline{f} \in L^{2q}\left(\mathbb{R}^d\right) \cap L^{\infty}\left(\mathbb{R}^d\right)$. Namely, we wish to show the following inequality:

$$\|\overline{f}\|^2_{L^{2q}(\mathbb{R}^d)} \leq C \int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} \rho\left(|x-y|\right)\left(\overline{f}(x)-\overline{f}(y)\right)^2 dy dx, \tag{5.9}$$

for some $C = C(d,q) > 0$. We shall comment in the final Step 3 how to extend it for measurable functions that belong only to $L^{2q}\left(\mathbb{R}^d\right)$, for some $q > 1$. In what follows, we drop the bar from $\overline{f}$ for the sake of simplicity, and assume without loss of generality that $0 \leq f \in L^{\infty}\left(\mathbb{R}^d\right)$ is compactly supported. To this end, for any $k \in \mathbb{Z}$ let

$$A_k := \{x \in \mathbb{R}^d : |f(x)| > 2^k\} \text{ and } a_k := |A_k|. \tag{5.10}$$

We also define

$$D_k := A_k \setminus A_{k+1} = \{x \in \mathbb{R}^d : 2^k < f(x) \leq 2^{k+1}\} \qquad \text{and} \qquad d_k := |D_k|.$$

We remark as in the proof of [36, Lemma 6.3] that $A_{k+1} \subseteq A_k$, $a_{k+1} \leq a_k$ for all $k \in \mathbb{Z}$, and $\{d_k\}$ and $\{a_k\}$ are bounded and they become zero when k is large enough. Also, one may easily observe that the D_k's are disjoint, and

$$\bigcup_{\substack{l\in\mathbb{Z}\\ l\leq k}} D_l = \complement A_{k+1}, \bigcup_{\substack{l\in\mathbb{Z}\\ l\geq k}} D_l = A_k \tag{5.11}$$

as well as

$$a_k = \sum_{\substack{l\in\mathbb{Z}\\ l\geq k}} d_l,\ d_k = a_k - \sum_{\substack{l\in\mathbb{Z}\\ l\geq k+1}} d_l. \tag{5.12}$$

We stress that each series in (5.12) is convergent as well as the following series

$$S := \sum_{\substack{l\in\mathbb{Z}\\ a_{l-1}\neq 0}} 2^{2l} a_{l-1}^{-\frac{q-1}{q}} d_l \tag{5.13}$$

is convergent. Another key property is that $D_k \subseteq A_k \subseteq A_{k-1}$, and therefore $a_{i-1}^{-\frac{q-1}{q}} d_l \leq a_{i-1}^{-\frac{q-1}{q}} a_{l-1}$. Henceforth, as in [36, Lemma 6.3],

$$\begin{aligned}\Big\{(i,l) \in \mathbb{Z}^2 : a_{i-1} \neq 0 \text{ and } a_{i-1}^{-\frac{q-1}{q}} d_l \neq 0\Big\}& \\ \subseteq \Big\{(i,\ell) \in \mathbb{Z}^2 : a_{l-1} \neq 0\Big\}&.\end{aligned} \tag{5.14}$$

We claim that

$$\int\limits_{\mathbb{R}^d}\int\limits_{\mathbb{R}^d} |f(x)-f(y)|^2 \rho\left(|x-y|\right) dx\, dy \geq C \sum_{\substack{k\in\mathbb{Z}\\ a_k\neq 0}} a_{k+1} a_k^{-\frac{q-1}{q}} 2^{2k}, \tag{5.15}$$

for some $C = C(d,q) > 0$, and

$$\|f\|^2_{L^{2q}(\mathbb{R}^d)} \leq 2^2 \sum_{k\in\mathbb{Z}} a_k^{1/q} 2^{2k}. \tag{5.16}$$

Indeed, to check (5.16) one notices that

$$\begin{aligned}\|f\|^{2q}_{L^{2q}(\mathbb{R}^d)} &= \sum_{k\in\mathbb{Z}} \int\limits_{D_k} |f(x)|^{2q} dx \leq \sum_{k\in\mathbb{Z}} \int\limits_{D_k} \left(2^{k+1}\right)^{2q} dx \\ &\leq \sum_{k\in\mathbb{Z}} \left(2^{k+1}\right)^{2q} a_k,\end{aligned}$$

and in particular, (5.16) holds owing to the fact that $q > 1$ and by taking the $(\cdot)^{1/q}$ on both sides of the foregoing inequality. We now verify (5.15). First, one can use (5.14) and the fact that $a_{k+1} \leq a_k$, to show as in [36, (6.14)] the following elementary computation:

$$\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \sum_{\substack{l\in\mathbb{Z}\\ l\geq i+1}} 2^{2i} a_{i-1}^{-\frac{q-1}{q}} d_l \leq S = \sum_{\substack{l\in\mathbb{Z}\\ a_{l-1}\neq 0}} 2^{2l} a_{l-1}^{-\frac{q-1}{q}} d_l .$$

Secondly, fixing $i \in \mathbb{Z}$ and $x \in D_i$, then, for any $j \in \mathbb{Z}$ with $j \leq i-2$ and any $y \in D_j$ we have that $|f(x) - f(y)| \geq 2^{i-1}$, and therefore, on account of (5.11),

$$\begin{aligned} \sum_{\substack{j\in\mathbb{Z}\\ j\leq i-2}} \int_{D_j} |f(x)-f(y)|^2 \rho(|x-y|)\, dy &\geq 2^{2(i-1)} \sum_{\substack{j\in\mathbb{Z}\\ j\leq i-2}} \int_{D_j} \rho(|x-y|)\, dy \\ &= 2^{2(i-1)} \int_{\mathcal{C}A_{i-1}} \rho(|x-y|)\, dy \\ &\geq C 2^{2i} A((a_{i-1}/\omega_d)^{1/d}), \end{aligned} \tag{5.17}$$

owing to Lemma 5.6, for some constant $C > 0$. Inequality (5.17) together with the second hypothesis of the theorem (i.e., $A\left(\alpha^{1/d}\right)\alpha^{\frac{q-1}{q}} \geq C$, for all $\alpha > 0$) implies that, for any $i \in \mathbb{Z}$ and any $x \in D_i$,

$$\sum_{\substack{j\in\mathbb{Z}\\ j\leq i-2}} \int_{D_j} |f(x)-f(y)|^2 \rho(|x-y|)\, dy \geq C 2^{2i} a_{i-1}^{-\frac{q-1}{q}},$$

for a suitable constant $C > 0$. Consequently, for any $i \in \mathbb{Z}$

$$\sum_{\substack{j\in\mathbb{Z}\\ j\leq i-2}} \int_{D_i}\int_{D_j} |f(x)-f(y)|^2 \rho(|x-y|)\, dy dx \geq C 2^{2i} a_{i-1}^{-\frac{q-1}{q}} d_i . \tag{5.18}$$

We can now exploit (5.18) together with (5.11)–(5.13) in a similar fashion as in the proof of [36, (6.15)–(6.18)], to deduce

$$\sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \sum_{\substack{j\in\mathbb{Z}\\ j\leq i-2}} \int_{D_i}\int_{D_j} |f(x)-f(y)|^2 \rho(|x-y|)\, dy\, dx \geq C \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} 2^{2i} a_{i-1}^{-\frac{q-1}{q}} a_i, \tag{5.19}$$

up to relabeling the constant $C > 0$.

On the other hand, by symmetry one has

$$\begin{aligned} &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x)-f(y)|^2 \rho(|x-y|)\, dx\, dy \\ &\geq 2 \sum_{\substack{i\in\mathbb{Z}\\ a_{i-1}\neq 0}} \sum_{\substack{j\in\mathbb{Z}\\ j\leq i-2}} \int_{D_i}\int_{D_j} |f(x)-f(y)|^2 \rho(|x-y|)\, dy\, dx. \end{aligned} \tag{5.20}$$

The desired inequality (5.15) then follows immediately from (5.19) and (5.20). We can now conclude that (5.9) is actually verified for a compactly supported function $f \in L^\infty\left(\mathbb{R}^d\right)$, by combining estimates (5.15), (5.16), after we have applied Lemma 5.7 with the value $T = 2^2$.

Step 3 *(Eliminating the condition $f \in L^\infty\left(\mathbb{R}^d\right)$).* Since (5.8) holds for bounded functions, it also holds for the level functions $f_N = \max\{\min\{f, N\}, -N\}$, for any $N \in \mathbb{N}$. Indeed, by the dominated convergence theorem and (5.8), one has

$$\lim_{N\to\infty}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(|x-y|)(f_N(x)-f_N(y))^2dydx$$
$$=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho(|x-y|)(f(x)-f(y))^2dydx,$$

while the application of [36, Lemma 6.4] yields

$$\lim_{N\to\infty}\|f_N\|^2_{L^{2q}(\mathbb{R}^d)}=\|f\|^2_{L^{2q}(\mathbb{R}^d)}.$$

Therefore, it follows that (5.9) also holds for measurable functions $f\in L^{2q}\left(\mathbb{R}^d\right)$, $q>1$. The proof of the theorem is now complete. □

5.3. Concluding remarks and open questions

Well-posedness of a weak solution was proven for the doubly nonlocal Cahn–Hilliard equation in any space dimension for bounded domains with Lipschitz continuous boundary. These results hold in the case when F is a regular unbounded potential (in $\mathbb{R}$) and the nonlocal interaction for the parabolic problem is a *strong-to-weak* one in the sense that $\rho\notin L^1_{loc}\left(\mathbb{R}^d\right)$ and $J\in W^{1,1}_{loc}\left(\mathbb{R}^d\right)$ are both symmetric. We have also chosen to give a unified approach by stating sufficiently general conditions on (ρ,J) such that a suitable dynamical system generated by all weak solutions of the doubly nonlocal Cahn–Hilliard equation possesses a finite dimensional global attractor. Finally, the global asymptotic stability of any weak solution to a steady state was also proved under the same hypotheses.

We list a series of questions that we hope will be of some interest to the reader. Besides the strong-to-weak interaction case for (ρ,J), the following cases remain open for further study:

- The *weak-to-weak* interaction case when both $\rho,J\in L^1_{loc}\left(\mathbb{R}^d\right)$.
- The *weak-to-strong* interaction case when $\rho\in L^1_{loc}\left(\mathbb{R}^d\right)$ and $J\notin L^1_{loc}\left(\mathbb{R}^d\right)$.
- The *strong-to-strong* interaction case when $\rho\notin L^1_{loc}\left(\mathbb{R}^d\right)$ and $J\notin L^1_{loc}\left(\mathbb{R}^d\right)$. This case was also investigated in [23], but some questions remain open in the case when $F:[-1,1]\to\mathbb{R}$ is a singular potential.
- Other open questions involves the treatment of all these cases when F is a singular potential, and/or the nonlocal operators associated with the form l are also φ-dependent and degenerate at the pure phases $\varphi(\pm1)=0$.
- It would be interesting to know whether the continuous embedding $V\subset L^{2q}(\Omega)$, $q>1$, holds under other (more general) conditions on ρ, and whether it can be adapted for compactly supported kernels. We conjecture that the conclusion of Theorem 5.4 still holds if in the second hypothesis, we ask instead that $A\left(\alpha^{1/d}\right)\alpha^{\frac{q-1}{q}}\geq C$, for all $\alpha\in\{r>0:\rho(r)>0\}$.
- The further regularity of weak solutions in all these cases is an ongoing research question. See [24] for the standard nonlocal Cahn–Hilliard equation, where in fact the $C^{\beta,\beta/2}\left(\overline{\Omega}\times(\tau,T)\right)$-regularity of the weak solution is well-known for all $T\geq\tau>0$ and for some $\beta\in(0,1)$.
- It would be also interesting to find out what happens if either one of the nonnegative kernels K, J is *not* symmetric.
- Equation (1.12) is also interesting to investigate in the case when $\Omega=\mathbb{R}^d$.
- As far as we know the doubly nonlocal Cahn–Hilliard equation has not yet been investigated from the numerical point of view in any of the proposed cases.

Conflict of interest statement

There is none.

Acknowledgements

The author is indebted to the anonymous referees for their careful reading of the manuscript and many of the improvements they have suggested.

References

[1] H. Abels, S. Bosia, M. Grasselli, Cahn–Hilliard equation with nonlocal singular free energies, Ann. Mat. Pura Appl. 194 (2015) 1071–1106.
[2] H. Abels, M. Wilke, Convergence to equilibrium for the Cahn–Hilliard equation with a logarithmic free energy, Nonlinear Anal. 67 (2007) 3176–3193.
[3] P.W. Bates, J. Han, The Neumann boundary problem for a nonlocal Cahn–Hilliard equation, J. Differ. Equ. 212 (2005) 235–277.
[4] P.W. Bates, J. Han, The Dirichlet boundary problem for a nonlocal Cahn–Hilliard equation, J. Math. Anal. Appl. 311 (2005) 289–312.
[5] J.W. Cholewa, T. Dlotko, Global Attractors in Abstract Parabolic Problems, Cambridge University Press, 2000.
[6] J.W. Cahn, J.E. Hilliard, Free energy of a nonuniform system. I. Interfacial free energy, J. Chem. Phys. 28 (1958) 258–267.
[7] L. Cherfils, A. Miranville, S. Zelik, The Cahn–Hilliard equation with logarithmic potentials, Milan J. Math. 79 (2011) 561–596.
[8] P. Colli, S. Frigeri, M. Grasselli, Global existence of weak solutions to a nonlocal Cahn–Hilliard–Navier–Stokes system, J. Math. Anal. Appl. 386 (2012) 428–444.
[9] A. Debussche, L. Dettori, On the Cahn–Hilliard equation with a logarithmic free energy, Nonlinear Anal. 24 (1995) 1491–1514.
[10] Q. Du, T. Mengesha, Analysis of a scalar nonlocal peridynamic model with a sign changing kernel, Discrete Contin. Dyn. Syst., Ser. B 5 (2013) 1415–1437.
[11] Q. Du, T. Mengesha, The bond-based peridynamic system with Dirichlet-type volume constraint, Proc. R. Soc. Edinb. A 144 (2014) 161–186.
[12] C.M. Elliott, The Cahn–Hilliard model for the kinetics of phase separation, in: Mathematical Models for Phase Change Problems, Obidos, 1988, in: Int. Ser. Numer. Math., vol. 88, Birkhäuser, Basel, 1989, pp. 35–73.
[13] C.M. Elliott, H. Garcke, On the Cahn–Hilliard equation with degenerate mobility, SIAM J. Math. Anal. 27 (1996) 404–423.
[14] C.M. Elliott, S. Luckhaus, A Generalized Diffusion Equation for Phase Separation of a Multi-Component Mixture with Interfacial Energy, SFB 256 Preprint No. 195, University of Bonn, 1991.
[15] S. Frigeri, M. Grasselli, Global and trajectory attractors for a nonlocal Cahn–Hilliard system, J. Dyn. Differ. Equ. 24 (2012) 827–856.
[16] S. Frigeri, C.G. Gal, M. Grasselli, On nonlocal Cahn–Hilliard–Navier–Stokes systems in two dimensions, J. Nonlinear Sci. 26 (2016) 847–893.
[17] S. Frigeri, M. Grasselli, Nonlocal Cahn–Hilliard–Navier–Stokes systems with singular potentials, Dyn. Partial Differ. Equ. 9 (2012) 273–304.
[18] S. Frigeri, M. Grasselli, P. Krejčí, Strong solutions for two-dimensional nonlocal Cahn–Hilliard–Navier–Stokes systems, J. Differ. Equ. 255 (2013) 2587–2614.
[19] S. Frigeri, M. Grasselli, E. Rocca, A diffuse interface model for two-phase incompressible flows with nonlocal interactions and nonconstant mobility, Nonlinearity 28 (2015) 1257–1293.
[20] E. Feireisl, F. Simondon, Convergence for semilinear degenerate parabolic equations in several space dimensions, J. Dyn. Differ. Equ. 12 (2000) 647–673.
[21] H. Gajewski, K. Zacharias, On a nonlocal phase separation model, J. Math. Anal. Appl. 286 (2003) 11–31.
[22] C.G. Gal, On an inviscid model for incompressible two-phase flows with nonlocal interaction, J. Math. Fluid Mech. 18 (2016) 659–677.
[23] C.G. Gal, On the strong-to-strong interaction case for doubly nonlocal Cahn–Hilliard equations, Discrete Contin. Dyn. Syst. 37 (2017) 131–167.
[24] C.G. Gal, M. Grasselli, Longtime behavior of nonlocal Cahn–Hilliard equations, Discrete Contin. Dyn. Syst. 34 (1) (2014) 145–179.
[25] C.G. Gal, M. Warma, Reaction–diffusion equations with fractional diffusion on non-smooth domains with various boundary conditions, Discrete Contin. Dyn. Syst. 36 (2016) 1279–1319.
[26] G. Giacomin, J.L. Lebowitz, Phase segregation dynamics in particle systems with long range interactions, I. Macroscopic limits, J. Stat. Phys. 87 (1997) 37–61.
[27] G. Giacomin, J.L. Lebowitz, Phase segregation dynamics in particle systems with long range interactions, II. Phase motion, SIAM J. Appl. Math. 58 (1998) 1707–1729.
[28] Q.Y. Guan, Integration by parts formula for regional fractional Laplacian, Commun. Math. Phys. 266 (2006) 289–329.
[29] Q.Y. Guan, Z.M. Ma, Reflected symmetric α-stable processes and regional fractional Laplacian, Probab. Theory Relat. Fields 134 (2006) 649–694.
[30] M.E. Gurtin, Generalized Ginzburg–Landau and Cahn–Hilliard equations based on a microforce balance, Phys. D 92 (1996) 178–192.
[31] N. Kenmochi, M. Niezgodka, I. Pawlow, Subdifferential operator approach to the Cahn–Hilliard equation with constraint, J. Differ. Equ. 117 (1995) 320–356.
[32] S.-O. Londen, H. Petzeltová, Convergence of solutions of a non-local phase-field system, Discrete Contin. Dyn. Syst., Ser. S 4 (2011) 653–670.
[33] S.-O. Londen, H. Petzeltová, Regularity and separation from potential barriers for a non-local phase-field system, J. Math. Anal. Appl. 379 (2011) 724–735.
[34] A. Miranville, S. Zelik, Robust exponential attractors for Cahn–Hilliard type equations with singular potentials, Math. Methods Appl. Sci. 27 (2004) 545–582.
[35] A. Novick-Cohen, The Cahn–Hilliard equation, in: Evolutionary Equations, vol. IV, in: Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, pp. 201–228.
[36] E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, Bull. Sci. Math. 136 (2012) 521–573.
[37] S.P. Neumana, D.M. Tartakovsky, Perspective on theories of non-Fickian transport in heterogeneous media, Adv. Water Resour. 32 (2009) 670–680.
[38] L. Vlahos, H. Isliker, Y. Kominis, K. Hizonidis, Normal and anomalous diffusion: a tutorial, in: T. Bountis (Ed.), Order and Chaos, vol. 10, Patras University Press, 2008.
[39] M. Warma, The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets, Potential Anal. 42 (2015) 499–547.