A family of degenerate elliptic operators: Maximum principle and its consequences ☆

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Abstract

In this paper we investigate the validity and the consequences of the maximum principle for degenerate elliptic operators whose higher order term is the sum of k eigenvalues of the Hessian. In particular we shed some light on some very unusual phenomena due to the degeneracy of the operator. We prove moreover Lipschitz regularity results and boundary estimates under convexity assumptions on the domain. As a consequence we obtain the existence of solutions of the Dirichlet problem and of principal eigenfunctions.
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1. Introduction

In this paper we shall study solutions of Dirichlet problem for degenerate elliptic operators whose higher order term is given by some sort of "truncated Laplacian", i.e.

$$\mathcal{P}_k^-(D^2 u) = \sum_{i=1}^{k} \lambda_i(D^2 u) \quad \text{and} \quad \mathcal{P}_k^+(D^2 u) = \sum_{i=N-k+1}^{N} \lambda_i(D^2 u),$$

where $\lambda_1(D^2 u) \leq \lambda_2(D^2 u) \leq \cdots \leq \lambda_N(D^2 u)$ are the ordered eigenvalues of the Hessian of u. These operators have lately been investigated in various contexts e.g. [1,12–14,20,21,31,32]. We are interested in the case $N \geq 2$ and $k < N$

since $\mathcal{P}_N^-(D^2u) = \mathcal{P}_N^+(D^2u) = \Delta u$. In the whole paper solutions are meant in the viscosity sense, see e.g. [16] and Definition 2.1.

Clearly, for any symmetric matrix X, $\mathcal{P}_k^+(X) = -\mathcal{P}_k^-(-X)$ hence we will mainly state the results for $\mathcal{P}_k^-$ with obvious equivalents when the operator $\mathcal{P}_k^+$ is considered. Such operators are positively homogeneous of degree one and degenerate elliptic.

In the following we propose to consider the Dirichlet problem

$$\begin{cases} \mathcal{P}_k^\pm(D^2u) + H(x, \nabla u) + \mu u = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ and the Hamiltonian $H \in C(\Omega \times \mathbb{R}^N; \mathbb{R})$ is assumed to satisfy the structure condition:

$$\exists b \in \mathbb{R}_+ \text{ s.t. } |H(x, \xi)| \le b|\xi| \quad \forall(x, \xi) \in \Omega \times \mathbb{R}^N. \tag{SC 1}$$

The prototype we have in mind is $H(x, \nabla u) = b(x)|\nabla u|$ or $H(x, \nabla u) = b(x) \cdot \nabla u$ with $b(x)$ a bounded continuous function in Ω.

In particular, in bounded domains Ω, we want to raise and partially answer the following questions, which are very intertwined:

(1) Under which conditions do the operators $\mathcal{P}_k^\pm(D^2u) + H(x, \nabla u) + \mu u$ satisfy the maximum principle, be it weak or strong?
(2) What are the regularity of the solutions of the Dirichlet problem?
(3) Do the principal eigenvalues and corresponding eigenfunctions exist?

In order to be more specific, let us describe what we call maximum or minimum principle in the sense of the *sign propagation property*.

Definition 1.1. F satisfies the maximum or weak maximum principle in Ω if

$$F[u] \ge 0 \text{ in } \Omega, \quad \limsup_{x \to \partial\Omega} u \le 0 \implies u \le 0 \text{ in } \Omega.$$

It satisfies the strong maximum principle if

$$F[u] \ge 0 \text{ in } \Omega, \quad u \le 0 \text{ in } \Omega \implies \text{either } u < 0 \text{ or } u \equiv 0.$$

Respectively, F satisfies the minimum or weak minimum principle in Ω if

$$F[u] \le 0 \text{ in } \Omega, \quad \liminf_{x \to \partial\Omega} u \ge 0 \implies u \ge 0 \text{ in } \Omega.$$

It satisfies the strong minimum principle if

$$F[u] \le 0 \text{ in } \Omega, \quad u \ge 0 \text{ in } \Omega \implies \text{either } u > 0 \text{ or } u \equiv 0.$$

Of course when F is odd then the notions of maximum and minimum principle are equivalent, but here we shall see that they differ quite a lot.

Just to give a flavor of the kind of results that we shall obtain, let us begin by saying that for any $k < N$, the Hopf Lemma, the Harnack inequality and the strong minimum principle do not hold in general for solutions of (1.1). On the other hand, if $bR \le k$, the weak minimum principle holds in any domain $\Omega \subset B_R$. For subsolutions, instead, the strong maximum principle will be a consequence of the Hopf Lemma. The condition $bR \le k$ has been shown to be optimal in a previous work of the second named author with Vitolo [18]. Other phenomena which are unusual with respect to the uniformly elliptic case will be described in subsection 4.2.

Historically, the maximum (or minimum) principle for degenerate elliptic operators has been mostly studied when the degeneracy depends on the points where the operator acts, e.g.

$$Lu = \text{tr}(A(x)D^2u) \text{ with } A \ge 0$$

or

$$Lu = \sum_{i=1}^{k} X_i^2 u,$$

where the X_i are vector fields that may fail to generate the whole space, see e.g. the fundamental works of Bony [10] or Kohn and Nirenberg [25]. We shall not even try to enumerate the results in these sub-elliptic contexts.

Other class of degenerate operators are the quasilinear operators such as the p-Laplacian or the ∞-Laplacian, whose degeneracy depends on the solution itself, but more precisely on the gradient of the solution. Here also, for the truncated Laplacian, the "direction" of the degeneracy depends on the solution but through the eigenvectors of the Hessian of the solution. Let us furthermore remark that these operators are neither linear nor variational.

The operators $\mathcal{P}_k^{\pm}$ have initially been introduced in connection with Riemannian manifolds. In particular when the manifolds are k convex this was studied by Sha in [31], the case of partially positive curvature was seen by Wu in [32]. Later they can be found in [16, Example 1.8], as examples of fully nonlinear degenerate elliptic operators, and [1], where Ambrosio and Soner have investigated the mean curvature flow with arbitrary codimension through a level set approach. More recently, in a PDE context, we wish to recall the works of Harvey and Lawson [20,21] that have given a new geometric interpretation of solutions, while Caffarelli, Li and Nirenberg in [12,13] in their study of degenerate elliptic equations, give some results concerning removable singularities along smooth manifolds for Dirichlet problems associated to $\mathcal{P}_k^{-}$. See also [2] for the extended version of the maximum principle and [14] in the case of entire solutions.

In order to describe the results contained in this work let us introduce the generalized principal eigenvalues à la Berestycki, Nirenberg, Varadhan [5]. For the following equation

$$\mathcal{P}_k^{-}(D^2 u) + H(x, \nabla u) + \mu u = 0 \text{ in } \Omega, \tag{1.2}$$

we define the following "generalized principal eigenvalues":

$$\overline{\mu}_k^{+} = \sup\{\mu \in \mathbb{R} : \exists w > 0 \text{ in } \overline{\Omega} \text{ a supersolution of (1.2)}\},$$
$$\mu_k^{+} = \sup\{\mu \in \mathbb{R} : \exists w > 0 \text{ in } \Omega \text{ a supersolution of (1.2)}\}$$

and

$$\overline{\mu}_k^{-} = \sup\{\mu \in \mathbb{R} : \exists w < 0 \text{ in } \overline{\Omega} \text{ a subsolution of (1.2)}\},$$
$$\mu_k^{-} = \sup\{\mu \in \mathbb{R} : \exists w < 0 \text{ in } \Omega \text{ a subsolution of (1.2)}\}.$$

When we say that w is a supersolution of (1.2) and $w > 0$ in $\overline{\Omega}$ as in the definition of $\overline{\mu}_k^{+}$ above, it is implicit that the function w is defined, as a real-valued function, and lower semicontinuous in $\overline{\Omega}$. Similar assumptions are made in the definition of $\overline{\mu}_k^{-}$ above.

It is immediate that $\overline{\mu}_k^{\pm} \leq \mu_k^{\pm}$ and also, using the sub-additivity of $\mathcal{P}_k^{-}$ (see (2.2)), that $\mu_k^{-} \leq \mu_k^{+}$ and $\overline{\mu}_k^{-} \leq \overline{\mu}_k^{+}$ if H is odd in the gradient. What we prove in section 4 is that these values are thresholds for the validity of the weak maximum or the weak minimum principle, precisely below $\overline{\mu}_k^{-}$ and below $\overline{\mu}_k^{+}$ the minimum principle and respectively the maximum principle holds.

In order to be able to reach the values μ_k^{+} and μ_k^{-}, which are the standard upper bounds in the uniformly elliptic case, we shall need some further conditions. Precisely, if $\Omega \subset B_R$ with $bR < k$ the maximum principle holds for any μ since, we prove in Proposition 4.3 that $\overline{\mu}_k^{+} = \mu_k^{+} = +\infty$. For the minimum principle the situation is more delicate. The weak minimum principle holds up to μ_k^{-} if, beside the above condition on R, we shall require that Ω satisfies a convexity type assumption, precisely that it is the intersection of a family of balls of same radius; in that case we say that Ω is a "hula hoop" domain. In particular a C^2 strictly convex domain is a hula hoop domain, see Proposition 2.7. A similar notion of strict convexity has been previously considered in e.g. [28].

Under these hypotheses, in Proposition 4.5, we prove that for $\mu = \overline{\mu}_k^{-}$ the minimum principle does not hold. This implies also that $\overline{\mu}_k^{-} = \mu_k^{-}$, see Theorem 4.4; let us emphasize that the hula hoop condition does not imply the regularity of the domain e.g. the intersection of two balls of same radius. In general the question of whether μ_k^{-} and $\overline{\mu}_k^{-}$ coincide is an open problem.

In the recent paper [4] that had a great influence on this research, Berestycki, Capuzzo Dolcetta, Porretta and Rossi have studied the validity of the maximum principle for degenerate elliptic operators. For that aim they introduce another value

$$\mu^* := \sup\{\mu \in \mathbb{R} : \exists \Omega' \supset \overline{\Omega}, \, w > 0 \text{ in } \Omega', \, F[u] + \mu u \le 0 \text{ in } \Omega'\}.$$

Observe that for $F[u] := \mathcal{P}_k^-(D^2 u) + H(x, \nabla u)$, the value $\mu^* \le \overline{\mu}_k^+$. In [4] they prove that $F[\cdot] + \mu \cdot$ satisfies the maximum principle in Ω in the viscosity sense if and only if $\mu < \mu^*$. In section 4 of that paper, they also study the equality between the different definitions of generalized principal eigenvalues, but the sufficient conditions require that the domain be regular.

The existence of solutions for Dirichlet problems are proved in Section 5 when Ω is a hula hoop domain. When the operators concern $\mathcal{P}_k^-$ for general k, the existence and uniqueness is given provided that the Hamiltonian is Lipschitz in the gradient variable and $\mu < \mu_{k,b}^- \le \mu_k^-$, where $\mu_{k,b}^-$ refers to the generalized principal eigenvalue of $\mathcal{P}_k^-(D^2 \cdot) - b|\nabla \cdot|$ with b the Lipschitz constant of H. Instead, thanks to the Lipschitz estimates, for $k = 1$ the existence is given without the extra condition on the Hamiltonian and for any $\mu < \mu_1^-$. In the particular case $f \le 0$ the existence holds for any μ. Some questions concerning existence remain open, e.g. does the existence of solutions holds when $\mu > \mu_1^-$ for a more general class of forcing terms f? Is the hula hoop condition optimal?

Of course a natural question is whether these generalized principal eigenvalues correspond to an eigenfunction. In the case of uniformly elliptic fully nonlinear operators, this has been proved to be the case in different context (see [3,6,11,22,26,30]). We are able to give a positive answer to this question when $k = 1$ and Ω is a hula hoop domain. This will be somehow an application of the global Lipschitz results that are proved in section 3. The proof of the Lipschitz regularity is extremely sleek.

It is quite clear that there are a number of open problems. Maybe the most important one is whether the global Lipschitz or Hölder regularity of the solutions holds also for $k \ge 2$. This would in particular lead to the existence of the principal eigenfunction in that case as well. On one hand it is not surprising that the case of $\mathcal{P}_1^-$ is simpler since, when the lower order term is zero, solutions of $\mathcal{P}_1^-(D^2 u) = f(x)$ are semiconvex. On the other hand, it is also the most degenerate of these operators, so it would be very surprising that the case $k = 1$ and the case $k = N$ give rise to smooth solutions and that it is not the case for the values of k in between.

Still concerning the regularity, let us recall that in the context of convex analysis, Oberman and Silvestre in [27] prove the $C^{1,\alpha}$ regularity of solutions of

$$\mathcal{P}_1^-(D^2 u) = 0 \text{ in } \Omega, \, u(x) = g(x) \text{ on } \partial\Omega,$$

under some regularity condition on g. The solution of this problem is the convex envelope, of given boundary data g. They proved that the solutions of the Dirichlet problem with $C^{1,\gamma}$ boundary data, are $C^{1,\gamma}$ in the interior. When f is not zero and there is a first order term the question of the Hölder regularity of the gradient is to our knowledge completely open.

Of course there are other open questions concerning these generalized principal eigenvalues. One concerns the simplicity of the eigenvalue. Observe that this is not so obvious, as can be seen in [8], for other degenerate fully nonlinear operators. Another question is whether the symmetry of the domain decreases the eigenvalues. Since the eigenvalues are not characterized by a sort of "Rayleigh quotient" standard techniques are not available. We refer to [9] for analogous questions concerning the Pucci's operators. These problems will be the object of subsequent papers.

In the next section, beside recalling a few standard facts, we give estimates near the boundary that will be crucial along the paper. In section 3, using those bounds, we prove global Lipschitz regularity of solutions when $k = 1$. Section 4 is divided into two subsections, in the first one we prove that the generalized principal eigenvalues bound the validity of the maximum and minimum principle; in the second subsection we describe some unusual phenomena. Section 5 is dedicated to the existence of solutions for the Dirichlet problem and existence of the principal eigenfunction. In the last section we prove that C^2 strictly convex domains are "hula hoop domains".

2. Barrier functions, bounds, Hopf lemma

For convenience of the reader, we begin this section by recalling the definition of viscosity solution and some facts concerning the operators $\mathcal{P}_k^-$ and $\mathcal{P}_k^+$.

Let us denote by $\mathbb{S}^N$ the set of $N \times N$ real symmetric matrices, endowed with the standard partial order: $X \le Y$ in $\mathbb{S}^N$ if $\langle X\xi, \xi \rangle \le \langle Y\xi, \xi \rangle \, \forall \xi \in \mathbb{R}^N$. The identity matrix will be denoted by I and the trace of $X \in \mathbb{S}^N$ by $\text{tr}(X)$. A continuous mapping $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{S}^N \mapsto \mathbb{R}$ is *degenerate elliptic* if it is nondecreasing in the matrix argument: for any $(x, r, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$

$$F(x, r, \xi, X) \leq F(x, r, \xi, Y) \quad \text{whenever } X \leq Y. \tag{2.1}$$

Definition 2.1. u is a viscosity supersolution of

$$F(x, u, \nabla u, D^2 u) = 0 \text{ in } \Omega$$

if it is lower semicontinuous in Ω and for any x in Ω, for any C^2 function φ touching u from below at x then

$$F(x, u, \nabla\varphi(x), D^2\varphi(x)) \leq 0.$$

Analogously, u is a viscosity subsolution if it is upper semicontinuous in Ω and for any x in Ω, for any C^2 function φ touching u from above at x then

$$F(x, u, \nabla\varphi(x), D^2\varphi(x)) \geq 0.$$

A continuous function u is a viscosity solution if it is both a subsolution and a supersolution.

If $X \leq Y$ in $\mathbb{S}^N$, the Courant's min-max representation formula for eigenvalues implies that $\lambda_i(X) \leq \lambda_i(Y)$, for $i = 1, \ldots, N$. In particular the operators $\mathcal{P}_k^-$ and $\mathcal{P}_k^+$ satisfy (2.1). Moreover the representation formula

$$\mathcal{P}_k^-(X) = \min \left\{ \sum_{i=1}^k \langle X\xi_i, \xi_i \rangle \mid \xi_i \in \mathbb{R}^N \text{ and } \langle \xi_i, \xi_j \rangle = \delta_{ij}, \text{ for } i, j = 1, \ldots, k \right\},$$

see [12, Lemma 8.1], allows us to obtain easily the inequalities

$$\mathcal{P}_k^-(Y) \leq \mathcal{P}_k^\pm(X + Y) - \mathcal{P}_k^\pm(X) \leq \mathcal{P}_k^+(Y) \tag{2.2}$$

and deduce the superadditivity (subadditivity) property of $\mathcal{P}_k^-$ ($\mathcal{P}_k^+$).

We will consider a couple of radial barrier functions in the paper and hence we recall the following elementary Lemma that can be found e.g. in [17].

Lemma 2.2. *Let $\eta \in C^2([0, b])$, with $0 < b$ such that $\eta'(0) = 0$. Set $v(x) = \eta(|x|)$ in $\overline{B}_b$. Then, v is $C^2(\overline{B}_b)$ and, for $x \neq 0$, the eigenvalues of $D^2 v(x)$ are $\eta''(|x|)$ and $\eta'(|x|)/|x|$, and the (algebraic) multiplicity of $\eta'(|x|)/|x|$ is equal to $N - 1$, if $\eta''(|x|) \neq \eta'(|x|)/|x|$, and N otherwise. For $x = 0$, they are all equal to $\eta''(|x|)$.*

We start with a computation that leads to a remark on the Hopf lemma for the operator $\mathcal{P}_k^-(D^2 \cdot) + H(x, \nabla \cdot)$. In $B_R = B_R(0)$, the ball of radius R and center the origin, let

$$w(x) = (R^2 - |x|^2)^\gamma \quad \text{with } \gamma > 1. \tag{2.3}$$

By Lemma 2.2 or a straightforward computation, the eigenvalues of the Hessian of w are

$$\lambda_i(D^2 w) = -2\gamma(R^2 - |x|^2)^{\gamma-1} < 0 \text{ for } i = 1, \ldots, N-1$$

while

$$\begin{aligned}
\lambda_N(D^2 w) &= -2\gamma(R^2 - |x|^2)^{\gamma-1} + 4|x|^2\gamma(\gamma - 1)(R^2 - |x|^2)^{\gamma-2} \\
&= 2\gamma(R^2 - |x|^2)^{\gamma-2}((2(\gamma - 1) + 1)|x|^2 - R^2).
\end{aligned}$$

In this way, from (SC 1)

$$\begin{aligned}
\mathcal{P}_k^-(D^2 w) + H(x, \nabla w) &\leq \mathcal{P}_k^-(D^2 w) + b|\nabla w| \\
&= 2\gamma(R^2 - |x|^2)^{\gamma-1}(b|x| - k) \leq 0 \qquad \text{if } bR \leq k,
\end{aligned}$$

so that w is a positive supersolution, for $k < N$, of $\mathcal{P}_k^-(D^2 w) + H(x, \nabla w) = 0$ in B_R, which is zero on the boundary and such that the outer normal derivative $\partial_\nu w(x) = 0$ for x on ∂B_R. This proves the following remark.

Remark 2.3. For any $k < N$, the Hopf lemma does not hold in general for supersolutions of $\mathcal{P}_k^-(D^2\cdot) + H(x, \nabla\cdot)$, i.e. there exists a positive supersolution in B_R which is zero together with its gradient at the boundary.

Moreover the extension

$$\bar{w}(x) = \begin{cases} w(x) & \text{if } |x| < R \\ 0 & \text{otherwise} \end{cases}$$

yields, for $\gamma > 2$, a counterexample of C^2-function invalidating the strong minimum principle.

In [18] the authors dealt with the removable singularities issue for second order elliptic operators whose principal part is a weighted version of $\mathcal{P}_k^\pm$. By means of an explicit counterexample they deduced the sharpness of the condition $bR \leq k < N$ for the validity of the weak maximum/minimum principle in the cases $H(x, \nabla u) = \pm b|\nabla u|$. For the reader's convenience we report the proof in the case of the minimum principle. Assume $bR \leq k$ and by contradiction let v be a lower semicontinuous function such that

$$\begin{cases} \mathcal{P}_k^-(D^2 v) + H(x, \nabla v) \leq 0 & \text{in } \Omega \subset B_R \\ \liminf_{x \to \partial\Omega} v(x) \geq 0 \end{cases}$$

and $v(x_0) < 0$ for some $x_0 \in \Omega$.

Set $\varphi(x) = \varepsilon|x|^2$ and $0 < \varepsilon < -\frac{v(x_0)}{R^2}$. Since

$$\liminf_{x \to \partial\Omega}(v - \varphi)(x) \geq -\varepsilon R^2 > v(x_0) \geq (v - \varphi)(x_0)$$

then

$$\inf_{x \in \Omega}(v - \varphi)(x) = (v - \varphi)(x_\varepsilon), \quad x_\varepsilon \in \Omega.$$

Using φ as test function we get

$$\begin{aligned} 0 &\geq \mathcal{P}_k^-(D^2\varphi(x_\varepsilon)) + H(x_\varepsilon, \nabla\varphi(x_\varepsilon)) \\ &\geq 2\varepsilon k - 2\varepsilon b|x_\varepsilon| \\ &> 2\varepsilon(k - bR) \end{aligned}$$

a contradiction. For the sharpness of the condition see Example 4.9.

Summarizing we can assert that for H fulfilling (SC 1).

Proposition 2.4. $\mathcal{P}_k^-(D^2\cdot) + H(x, \nabla\cdot)$ *does not satisfy the strong minimum principle in any bounded domain* Ω.

On the other hand, the weak minimum principle holds true in $\Omega \subset B_R$ *if* $bR \leq k$ *and in the case* $H(x, \nabla u) = -b|\nabla u|$, *the condition* $bR \leq k$ *is sharp.*

For later purposes we need to compare the distance function to the boundary of Ω i.e. $d(x) = \inf_{y \in \partial\Omega}|y - x|$ with subsolutions of (1.1). This is the content of the next propositions.

Proposition 2.5 (Hopf for subsolutions). *Let* Ω *be a bounded* C^2-*domain and let* u *satisfy*

$$\begin{cases} \mathcal{P}_k^-(D^2 u) + H(x, \nabla u) \geq 0 & \text{in } \Omega \\ u < 0 & \text{in } \Omega. \end{cases}$$

Then there exists a positive constant $C = C(\Omega, u, k, b)$ *such that*

$$u(x) \leq -Cd(x).$$

Proof. The proof is quite standard. We report it for the sake of completeness. The conditions on Ω imply the existence of a positive constant δ, depending on Ω, such that for any $x \in \Omega_\delta = \{x \in \Omega \mid d(x) < \delta\}$ there are a unique $y \in \partial\Omega$ for which $d(x) = |y - x|$ and a ball $B_{2\delta}(\bar{y}) \subset \Omega$ such that $\overline{B_{2\delta}(\bar{y})} \cap (\mathbb{R}^N \setminus \Omega) = \{y\}$ (see [19, Lemma 14.16] for details).

Let us fix an arbitrary $x_0 \in \Omega_\delta$ and consider the smooth negative radial function

$$v(x) = \beta \left(e^{-2\alpha\delta} - e^{-\alpha|x-\overline{y_0}|} \right)$$

in the annular region $A = B_{2\delta}(\overline{y_0}) \backslash B_\delta(\overline{y_0})$. For $\alpha > \left(\frac{k-1}{\delta} + b \right)$ and $\beta = \dfrac{\sup_{\Omega \backslash \Omega_\delta} u}{\left(e^{-2\alpha\delta} - e^{-\alpha\delta} \right)}$, a direct calculation (or [Lemma 2.2](#)) yields

$$\mathcal{P}_k^-(D^2 v(x)) + H(x, \nabla v(x)) \leq \mathcal{P}_k^-(D^2 v(x)) + b\,|\nabla v(x)|$$
$$= \alpha\beta e^{-\alpha|x-\overline{y_0}|} \left(\frac{k-1}{|x - \overline{y_0}|} + b - \alpha \right) < 0 \quad \text{in } A$$

and

$$\limsup_{x \to \partial A}(u - v)(x) \leq 0.$$

Using the comparison principle between a classical strict supersolution and a viscosity subsolution, we get

$$u(x_0) \leq v(x_0) = \beta \left(e^{-\alpha|y_0 - \overline{y_0}|} - e^{-\alpha|x_0 - \overline{y_0}|} \right) \leq -\alpha\beta e^{-2\alpha\delta} d(x_0).$$

Moreover since $\max_{\Omega \backslash \Omega_\delta} \dfrac{u(x)}{d(x)} < 0$ we conclude by taking C small enough. □

Remark 2.6. Standard procedures allow us to deduce from the above computation that the strong maximum principle holds for $\mathcal{P}_k^-(D^2 \cdot) + H(x, \nabla \cdot) + \mu \cdot$ for any $\mu \in \mathbb{R}$.

In [Proposition 2.8](#), we shall prove that for any $\gamma \in (0, 1)$ and any subsolution u of $\mathcal{P}_1^-(D^2 u) + H(x, \nabla u) = f(x)$ in Ω, the ratio $\frac{u(x)}{d(x)^\gamma}$ is bounded from above by a constant C, without requiring further assumptions on Ω. The constant C depends in particular on γ and blows up for $\gamma \to 1$. In order to obtain a similar bound with $\gamma = 1$ and in the general case of subsolutions of the equation [(1.1)](#), we restrict to convex domains Ω satisfying the following assumption: there exist $R > 0$ and $Y \subset \mathbb{R}^N$, depending on Ω, such that

$$\Omega = \bigcap_{y \in Y} B_R(y). \tag{2.4}$$

For any $R > 0$ we define the class $\mathcal{C}_R$ of such domains, i.e.

$$\mathcal{C}_R := \left\{ \Omega \subset \mathbb{R}^N : \text{representation formula (2.4) holds} \right\}, \quad \text{and set } \mathcal{C} = \bigcup_{R>0} \mathcal{C}_R.$$

The class $\mathcal{C}$ includes the set of bounded domains with C^2-boundary which are strictly convex in the sense that all the principal curvatures of the surface $\partial\Omega$ are positive everywhere. Indeed, we shall give, in section [6](#), the proof of the following

Proposition 2.7. *Let Ω be a bounded domain with C^2-boundary. Let $\kappa_i(x)$ denote the principal curvatures of $\partial\Omega$ at x for $i = 1, \ldots, N - 1$, set*

$$\underline{\kappa} = \min\{\kappa_i(x) : i = 1, \ldots, N - 1, \ x \in \partial\Omega\},$$

and assume that $\underline{\kappa} > 0$. If $R \geq 1/\underline{\kappa}$, then $\Omega \in \mathcal{C}_R$.

By means of [(2.4)](#) we show that the distance function $d(x)$ is an upper barrier for any subsolution of [(1.1)](#).

Proposition 2.8. *Let m be a positive constant and let u satisfy*

$$\begin{cases} \mathcal{P}_1^-(D^2 u) + H(x, \nabla u) \geq -m & \text{in } \Omega \\ u \leq 0 & \text{on } \partial\Omega. \end{cases}$$

Then for any $\gamma \in (0, 1)$ there exists $C = C(\gamma, b, m, \|u^+\|_\infty)$ such that

$$u(x) \leq Cd(x)^\gamma.$$

Let $R > 0$, $\Omega \in \mathcal{C}_R$ and u be a solution of

$$\begin{cases} \mathcal{P}_k^+(D^2u) + H(x, \nabla u) \geq -m & \text{in } \Omega \\ u \leq 0 & \text{on } \partial\Omega. \end{cases}$$

If H satisfies (SC 1) *and $bR < k$ then there exists $C = C(\Omega, b, k, m)$ such that*

$$u(x) \leq Cd(x). \tag{2.5}$$

Proof. Let $\Omega_\delta = \{x \in \Omega \,|\, d(x) < \delta\}$ with

$$\delta = \min\left(\frac{1-\gamma}{2b}, \left(\frac{\gamma(1-\gamma)}{4m} \|u^+\|_\infty \right)^{\frac{1}{2}} \right) \tag{2.6}$$

and without loss of generality we may assume $u^+ \not\equiv 0$. For $x_0 \in \Omega_\delta$, take $y_0 \in \partial\Omega$ such that $d(x_0) = |x_0 - y_0|$ and consider the function $v(x) = C|x - y_0|^\gamma$, where $C = \frac{\|u^+\|_\infty}{\delta^\gamma}$. Then $v(x)$ satisfies in $B_\delta(y_0) \cap \Omega$

$$\mathcal{P}_1^-(D^2v(x)) + H(x, \nabla v(x)) \leq C\gamma|x - y_0|^{\gamma-2}(\gamma - 1 + b|x - y_0|)$$

$$\leq -C\gamma\frac{1-\gamma}{2}\delta^{\gamma-2} < -m.$$

Moreover

$$u(x) \leq v(x) \quad \text{for any } x \in \partial(B_\delta(y_0) \cap \Omega)$$

and by comparison $u(x_0) \leq v(x_0) = Cd(x_0)^\gamma$. Since x_0 is arbitrary we obtain the desired inequality $u(x) \leq Cd(x)^\gamma$ in Ω_δ and the same conclusion is still true in $\Omega \setminus \Omega_\delta$ by the choice of the constant C.

For the second inequality, fix any $y \in Y$ and consider the function $v_y(x) = M(R^2 - |x - y|^2)$, where $M = \frac{m}{(k-bR)}$. Note that $v_y(x) \geq 0$ for all $x \in \overline{B}_R(y)$ and hence $v_y(x) \geq 0$ in $\overline{\Omega}$. Then

$$\mathcal{P}_k^+(D^2v_y(x)) + H(x, \nabla v_y(x)) \leq 2M(-k + b|x - y|)$$

$$\leq -2M(k - bR) < -m \quad \text{in } B_R(y)$$

and by comparison

$$u(x) \leq v_y(x) \quad \text{in } \overline{\Omega}. \tag{2.7}$$

We will show that this implies that

$$u(x) \leq Cd(x) \quad \text{for all } x \in \overline{\Omega}$$

with $C = 2MR$. Indeed, let $x \in \Omega$ and select $z \in \partial\Omega$ so that $d(x) = |x - z|$. Then select $y \in Y$ so that $z \notin B_R(y)$. Since $x \in B_R(y)$, we have

$$R^2 - |x - y|^2 = (R - |x - y|)(R + |x - y|) \leq 2R(R - |x - y|)$$

$$= 2Rd(x, \partial B_R(y)) = 2R|x - z| = 2Rd(x)$$

and we conclude by (2.7). $\quad\square$

We end this section by observing that the upper bound (2.5) fails to be true if the boundary $\partial\Omega$ is flat, at least if Ω is unbounded. Indeed in the case of the half space

$$\Omega = \left\{ x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_1 > 0 \right\},$$

the function $u(x) = x_1^\gamma$ is a solution in Ω of $\mathcal{P}_k^+(D^2u) = 0$ for any $\gamma \in (0, 1)$ and $k < N$, but on the other hand the ratio $\frac{u(x)}{d(x)} = \frac{1}{x_1^{1-\gamma}}$ is unbounded near $x_1 = 0$.

3. Lipschitz regularity, compactness

In this section we will study the Lipschitz regularity of viscosity solutions of

$$\begin{cases} \mathcal{P}_1^-(D^2u) + H(x, \nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.1}$$

and, in a dual fashion, of

$$\begin{cases} \mathcal{P}_1^+(D^2u) + H(x, \nabla u) = f(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

where f is continuous and bounded in Ω.

Proposition 3.1. *Let $\Omega \in \mathcal{C}_R$. If H satisfies* (SC 1) *and $bR < 1$, then the solutions u of* (3.1) *and* (3.2) *are Lipschitz continuous in $\overline{\Omega}$. The Lipschitz norm of u can be bounded by a constant depending only on Ω, b and the L^∞ norms of u and f.*

Proof. We shall write the proof in the case $\mathcal{P}_1^-$, since if v is a solutions of (3.2), then $u = -v$ is a solution of $\mathcal{P}_1^-(D^2u) + \tilde{H}(x, \nabla u) = -f(x)$ in Ω, where $\tilde{H}(x, \xi) = -H(x, -\xi)$ satisfies in turn (SC 1).

Let u be a solution of (3.1). It is sufficient to show that for any $x, y \in \overline{\Omega}$ such that $|x - y| < \delta$, where δ is a positive constant to be determined, then

$$u(x) - u(y) \leq L|x - y|$$

with $L = L(\Omega, b, \|u\|_\infty, \|f\|_\infty)$.

Fix $\theta \in (1, 2)$ and consider

$$v(x) = |x| - |x|^\theta, \quad x \in B_1.$$

The function v is strictly positive for $x \neq 0$ and satisfies the inequality

$$\mathcal{P}_1^-(D^2v(x)) + H(x, \nabla v(x)) \leq -\theta(\theta - 1)|x|^{\theta-2} + b(1 + \theta|x|^{\theta-1}), \quad x \in B_1 \setminus \{0\}. \tag{3.3}$$

Since the right hand side in (3.3) tends to $-\infty$ as $|x| \to 0$, we can then pick a $\delta = \delta(b, \theta, \|f\|_\infty) \in (0, 1)$ such that

$$\mathcal{P}_1^-(D^2v(x)) + H(x, \nabla v(x)) < -\|f\|_\infty \quad \text{in} \in B_\delta \setminus \{0\}.$$

Moreover, in view of Proposition 2.8, there exists a positive constant $C = C(\Omega, b, \|f\|_\infty)$ such that

$$-u(x) \leq Cd(x) \quad \forall x \in \overline{\Omega}. \tag{3.4}$$

For $x_0, y_0 \in \overline{\Omega}$, with $|x_0 - y_0| < \delta$ and $L = \max\left(\frac{2\|u\|_\infty}{\delta - \delta^\theta}, \frac{C}{1 - \delta^{\theta-1}}\right)$, let

$$v_{y_0}(x) := u(y_0) + Lv(x - y_0), \quad x \in B_\delta(y_0). \tag{3.5}$$

By construction

$$\mathcal{P}_1^-(D^2v_{y_0}(x)) + H(x, \nabla v_{y_0}(x)) < -\|f\|_\infty \quad \text{in } B_\delta(y_0) \setminus \{y_0\}$$

and

$$v_{y_0}(y_0) = u(y_0).$$

We claim that

$$u(x) \leq v_{y_0}(x) \quad \text{on } \partial(B_\delta(y_0) \cap \Omega), \tag{3.6}$$

so that the comparison principle yields the conclusion

$$u(x_0) \leq v_{y_0}(x_0) \leq u(y_0) + L|x_0 - y_0|.$$

To prove the inequality (3.6) we note that for any $x \in \partial B_\delta(y_0) \cap \Omega$

$$v_{y_0}(x) = u(y_0) + L(\delta - \delta^\theta) \geq u(y_0) + 2\,\|u\|_\infty \geq u(x),$$

while if $x \in \overline{B}_\delta(y_0) \cap \partial\Omega$, we obtain in view of (3.4), together with the choice of L,

$$u(x) = 0 \leq u(y_0) + Cd(y_0) \leq u(y_0) + C|x - y_0|$$
$$\leq u(y_0) + L(|x - y_0| - |x - y_0|^\theta) = v_{y_0}(x)$$

as we wanted to show. $\square$

The conditions concerning the geometry of Ω and the smallness of the Hamiltonian in the Proposition 3.1, i.e.

$$\Omega \in \mathcal{C}_R \quad \text{and} \quad bR < 1, \tag{3.7}$$

are only used to get the inequality (3.4), in order to apply comparison principle up to the boundary. For this reason and following the arguments of the previous proof, it is easy to obtain interior Lipschitz regularity for any bounded domain Ω and any H satisfying (SC 1), assuming u to be merely a subsolution of (3.1).

Moreover the assumptions (3.7) can be dropped if we require that the subsolution u satisfies (3.4). These observations are summarized as follows.

Proposition 3.2. *Suppose that Ω is a bounded domain and H satisfies condition (SC 1). The following holds:*

  i) *any subsolution u of (3.1) is a locally Lipschitz continuous function in Ω;*
 ii) *any subsolution u of (3.1) that satisfies (3.4) for some constant C is Lipschitz continuous in $\overline{\Omega}$.*
    The Lipschitz norm of u can be estimated by a constant which depends on b and the L^∞ norms of u and f.

Finally the same conclusion holds for supersolutions u of (3.2), with (3.4) replaced by the inequality $u \leq Cd$ in $\overline{\Omega}$.

This globally Lipschitz regularity result for nonnegative subsolutions of (3.1), a consequence of Proposition 3.2 ii), is quite surprising, considering that the global $C^{0,\gamma}$-regularity may fail for any $\gamma \in (0, 1]$ in the class of nonpositive subsolutions of (3.1). Here below an example: the nonpositive radial function

$$u(x) = \begin{cases} \frac{1}{\log(1-\delta)} & \text{if } |x| \leq \delta \\ \frac{1}{\log(1-|x|)} & \text{if } \delta < |x| < 1 \\ 0 & \text{if } |x| = 1, \end{cases}$$

is convex for $\delta \in (0, 1)$ close to 1 and

$$\mathcal{P}_1^-(D^2 u(x)) \geq 0 \quad \text{in } B_1.$$

On the other hand, for any $\gamma \in (0, 1]$,

$$\sup_{\substack{x,y \in \overline{B}_1 \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\gamma} = +\infty.$$

4. Demi-eigenvalues

4.1. Maximum and minimum principle

We now investigate the relationship between the generalized principal eigenvalues $\overline{\mu}_k^\pm$ and $\mu_k^\pm$ given in the introduction and the validity of the maximum and minimum principle.

In the following we shall sometimes need to reinforce the assumptions on the Hamiltonian H. In particular:

$$H(x, t\xi) = tH(x, \xi) \quad \forall (x, t, \xi) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}^N, \tag{SC 2}$$

$$\exists\, \omega \text{ modulus of continuity s.t. } |H(x, \xi) - H(y, \xi)| \leq \omega\,(|x - y|\,(1 + |\xi|))\,. \tag{SC 3}$$

Observe that (SC 2) implies (SC 1) with $b = \sup_{(x,\xi) \in \Omega \times B_1} |H(x,\xi)|$ hence this will be the meaning of b under condition (SC 2). Furthermore (SC 2) and (SC 3) imply that H is Lipschitz continuous in the following sense:

$$|H(x,\xi) - H(y,\xi)| \leq C|x - y||\xi|$$

for some constant $C > 0$. Indeed, for $\eta = \frac{\xi}{|\xi||x-y|}$,

$$|H(x,\xi) - H(y,\xi)| = |H(x,\eta) - H(y,\eta)||\xi||x - y|$$

and

$$|H(x,\xi) - H(y,\xi)| \leq \omega(|x - y|(1 + |\eta|))|\xi||x - y| \leq \omega(1 + \text{diam}(\Omega))|x - y||\xi|.$$

Theorem 4.1. *Let Ω be a bounded domain. Under the assumption* (SC 2)–(SC 3)*, the operator*

$$\mathcal{P}_k^-(D^2 \cdot) + H(x, \nabla \cdot) + \mu \cdot$$

satisfies

i) *the minimum principle in Ω for $\mu < \overline{\mu}_k^-$,*
ii) *the maximum principle in Ω for $\mu < \overline{\mu}_k^+$.*

Proof. The proof follows the argument of [6].

Without loss of generality we can suppose that $\mu \geq 0$, because otherwise the results are well known. We shall detail the case i) of the minimum principle, since with minor changes the arguments prove ii) as well. We argue by contradiction by assuming that v is a solution of

$$\begin{cases} \mathcal{P}_k^-(D^2 v) + H(x, \nabla v) + \mu v \leq 0 & \text{in } \Omega \\ \liminf_{x \to \partial\Omega} v(x) \geq 0 \end{cases} \tag{4.1}$$

and $v(x_0) < 0$ for some $x_0 \in \Omega$.

By the definition of $\overline{\mu}_k^-$ there exists $\rho \in (\mu, \overline{\mu}_k^-)$ and $u < 0$ in $\overline{\Omega}$, a solution of

$$\mathcal{P}_k^-(D^2 u) + H(x, \nabla u) + \rho u \geq 0 \quad \text{in } \Omega. \tag{4.2}$$

The function $\frac{v}{u}$ is upper semicontinuous in the compact set

$$K = \left\{ x \in \Omega : \frac{v(x)}{u(x)} \geq \frac{v(x_0)}{u(x_0)} \right\}$$

and if $\gamma := \sup_{x \in \Omega} \frac{v(x)}{u(x)}$, then

$$\gamma = \sup_{x \in K} \frac{v(x)}{u(x)} < +\infty \quad \text{and} \quad 0 < \frac{v(x_0)}{u(x_0)} \leq \gamma. \tag{4.3}$$

For $0 < \varepsilon < \gamma$ the lower semicontinuous function $v - (\gamma - \varepsilon)u$ reaches its negative minimum in Ω, say

$$\min_{x \in \Omega} (v(x) - (\gamma - \varepsilon)u(x)) = v(x_\varepsilon) - (\gamma - \varepsilon)u(x_\varepsilon), \quad x_\varepsilon \in \Omega,$$

since

$$\liminf_{x \to \partial\Omega} (v(x) - (\gamma - \varepsilon)u(x)) \geq \liminf_{x \to \partial\Omega} v(x) \geq 0$$

and by definition of the supremum there exists $y_\varepsilon \in \Omega$ such that

$$v(y_\varepsilon) - (\gamma - \varepsilon)u(y_\varepsilon) < 0.$$

Moreover, by lower semicontinuity, we can find a subdomain $\Omega' \subset\subset \Omega$, depending on ε and containing x_ε, for which

$$\min_{\partial\Omega'} (v(x) - (\gamma - \varepsilon)u(x)) > v(x_\varepsilon) - (\gamma - \varepsilon)u(x_\varepsilon) \tag{4.4}$$

and a sequence $(x_k, y_k) \in \overline{\Omega'} \times \overline{\Omega'}$ such that

$$v(x_k) - (\gamma - \varepsilon)u(y_k) + \frac{k}{2}|x_k - y_k|^2 = \min_{(x,y)\in\overline{\Omega'}\times\overline{\Omega'}} \left(v(x) - (\gamma - \varepsilon)u(y) + \frac{k}{2}|x - y|^2 \right).$$

Using [16, Lemma 3.1], up to subsequences, we have

$$\frac{k}{2}|x_k - y_k|^2 \to 0, \quad (x_k, y_k) \to (\hat{x}_\varepsilon, \hat{x}_\varepsilon) \text{ for some } \hat{x}_\varepsilon \in \Omega'$$

and

$$(v(x_k), (\gamma - \varepsilon)u(y_k)) \to (v(\hat{x}_\varepsilon), (\gamma - \varepsilon)u(\hat{x}_\varepsilon)) \quad \text{for } k \to +\infty.$$

Hence $(x_k, y_k) \in \Omega' \times \Omega'$ for large k and in view of [16, Theorem 3.2] there exist X_k and Y_k, $N \times N$ symmetric matrices, such that

$$X_k \geq Y_k, \ (k(y_k - x_k), X_k) \in \overline{J}^{2,-} v(x_k), \ (k(y_k - x_k), Y_k) \in \overline{J}^{2,+}(\gamma - \varepsilon)u(y_k).$$

Since the function $(\gamma - \varepsilon)u(x)$ is still a solution of (4.2) by the homogeneity assumption (SC 2), we have from (4.1)–(4.2)–(SC 3), that

$$\begin{aligned}
\mu v(x_k) &\leq -\mathcal{P}_k^-(X_k) - H(x_k, k(y_k - x_k)) \\
&\leq -\mathcal{P}_k^-(Y_k) - H(y_k, k(y_k - x_k)) + \omega(|x_k - y_k|(1 + k|x_k - y_k|)) \\
&\leq \rho(\gamma - \varepsilon)u(y_k) + \omega(|x_k - y_k|(1 + k|x_k - y_k|)).
\end{aligned}$$

Sending $k \to +\infty$

$$\mu v(\hat{x}_\varepsilon) \leq \rho(\gamma - \varepsilon)u(\hat{x}_\varepsilon). \tag{4.5}$$

If $\mu = 0$ this is a contradiction. Otherwise, for $\mu > 0$, since $\gamma u(\hat{x}_\varepsilon) \leq v(\hat{x}_\varepsilon)$ we deduce from (4.5) that

$$1 < \frac{\rho}{\mu} \leq \frac{\gamma}{\gamma - \varepsilon};$$

which is a contradiction for small ε. $\quad \square$

The same proof as above works for general, positively homogeneous of degree one, degenerate elliptic operators $F(x, \nabla\cdot, D^2\cdot)$, to which the proof of comparison principle applies (see [16, Theorem 3.3]).

Theorem 4.1 implies the following

Corollary 4.2. *Under the assumption* (SC 2)–(SC 3)*, if* $B_{R_1} \subset \Omega$*, then*

$$\overline{\mu}_k^- \leq \frac{2(k + bR_1)(2 + k + bR_1)}{R_1^2}. \tag{4.6}$$

Moreover if $\Omega \subset B_{R_2}$ *and* $bR_2 \leq k$*, then*

$$\overline{\mu}_k^- \geq \frac{2(k - bR_2)}{R_2^2}. \tag{4.7}$$

Proof. For $B_{R_1} \subset \Omega$ consider the function

$$w(x) = -(R_1^2 - |x|^2)^2$$

extended to zero outside of B_{R_1}, as in [5]. Then

$$\begin{aligned}
\sup_{|x|<R_1} \frac{\mathcal{P}_k^-(D^2 w) + H(x, \nabla w)}{-w} &\leq \sup_{|x|<R_1} \frac{\mathcal{P}_k^-(D^2 w) + b|\nabla w|}{-w} \\
&\leq 4 \sup_{|x|<R_1} \left(\frac{k + bR_1}{(R_1^2 - |x|^2)} - \frac{2|x|^2}{(R_1^2 - |x|^2)^2} \right).
\end{aligned}$$

In the set $\Omega_1 = \left\{ x \in B_{R_1} : |x|^2 \geq \frac{R_1^2(k+bR_1)}{2+k+bR_1} \right\}$ we have

$$\frac{k+bR_1}{(R_1^2 - |x|^2)} - \frac{2|x|^2}{(R_1^2 - |x|^2)^2} \leq 0,$$

while in $\Omega_2 = B_{R_1} \backslash \Omega_1$

$$\frac{k+bR_1}{(R_1^2 - |x|^2)} - \frac{2|x|^2}{(R_1^2 - |x|^2)^2} \leq \frac{k+bR_1}{(R_1^2 - |x|^2)} \leq \frac{(k+bR_1)(2+k+bR_1)}{2R_1^2}.$$

Hence v is a negative solution in Ω of

$$\mathcal{P}_k^-(D^2 w) + H(x, \nabla w) + \frac{2(k+bR_1)(2+k+bR_1)}{R_1^2} w \leq 0,$$

which is zero on the boundary $\partial\Omega$. This contradicts the minimum principle and, by Theorem 4.1,

$$\overline{\mu_k^-} \leq \frac{2(k+bR_1)(2+k+bR_1)}{R_1^2},$$

leading to (4.6).

Let $\Omega \subset B_{R_2}$ and $w(x) = -(R_2^2 - |x|^2)$. For $bR_2 < k$ (the case $bR_2 = k$ is trivial) we may assume as in the proof of Proposition 4.3 that $\overline{\Omega} \subset B_{R_2}$, so $w < 0$ in $\overline{\Omega}$ and

$$\begin{aligned}
\mathcal{P}_k^-(D^2 w) + H(x, \nabla w) + \mu w &\geq \mathcal{P}_k^-(D^2 w) - b|\nabla w| + \mu w \\
&= 2(k - b|x|) + \mu\left(|x|^2 - R_2^2\right) \\
&\geq 2(k - bR_2) - \mu R_2^2 = 0
\end{aligned}$$

if $\mu = \frac{2(k-bR_2)}{R_2^2}$ and therefore

$$\overline{\mu_k^-} \geq \frac{2(k-bR_2)}{R_2^2}. \quad \square$$

We now impose some conditions on the domain Ω. For the maximum principle we get

Proposition 4.3. *Under the assumption* (SC 1)*, if $\Omega \subset B_R$ then, for any $k < N$,*

$$bR < k \implies \mu_k^+ = \overline{\mu_k^+} = +\infty. \tag{4.8}$$

In particular, in the case $H \equiv 0$, for any bounded domain Ω, $\mu_k^+ = \overline{\mu_k^+} = +\infty$ and the operator $\mathcal{P}_k^-(D^2 \cdot) + \mu \cdot$ satisfies the maximum principle for any μ.

Proof. Choose any $\mu > 0$ and assume without loss of generality that $\overline{\Omega} \subset B_R$ and $\gamma := \frac{\mu R^2}{2(k-bR)} > 1$, replacing if necessary R with $R' > R$ in order that $k - bR'$ is positive and sufficiently close to 0. Let w be the function introduced in section 2, then $w(x) > 0$ in $\overline{\Omega}$ and

$$\begin{aligned}
&\mathcal{P}_k^-(D^2 w(x)) + H(x, \nabla w(x)) + \mu w(x) \\
&\leq \mathcal{P}_k^-(D^2 w(x)) + b|\nabla w(x)| + \mu w(x) \\
&= -2\gamma k(R^2 - |x|^2)^{\gamma-1} + 2\gamma b|x|(R^2 - |x|^2)^{\gamma-1} + \mu(R^2 - |x|^2)^\gamma \\
&\leq (R^2 - |x|^2)^{\gamma-1}(-2\gamma(k - bR) + \mu R^2) \\
&= 0.
\end{aligned}$$

By definition, we have obtained that $\mu_k^+ \geq \overline{\mu_k^+} = +\infty$. $\quad \square$

For the minimum principle, the assumptions are slightly stronger.

Theorem 4.4. *Let $\Omega \in \mathcal{C}_R$, and assume (SC 2)–(SC 3) and that $bR < k$. Then,*

$$\mu_k^- = \overline{\mu}_k^-,$$

and the minimum principle holds true if and only if $\mu < \mu_k^-$.

In order to prove Theorem 4.4 we shall need the following proposition which proves that if Ω is a hula hoop domain, the bound $\overline{\mu}_k^-$ of Theorem 4.1 is sharp. We indeed exhibit a supersolution v at level $\overline{\mu}_k^-$ which will invalidate the minimum principle. The result has been inspired by [4, Proposition 3.2].

Proposition 4.5. *Assume (SC 2)–(SC 3). Then $\overline{\mu}_k^-$ is finite and, if $\Omega \in \mathcal{C}_R$ and $bR < k$, there exists a nonpositive supersolution $v \not\equiv 0$ of*

$$\begin{cases} \mathcal{P}_k^-(D^2 v) + H(x, \nabla v) + \overline{\mu}_k^- v = 0 & in \ \Omega \\ v = 0 & on \ \partial\Omega. \end{cases}$$

For the proof of the proposition above, we need the following existence result that will be used also in the next section.

Proposition 4.6. *Assume (SC 2)–(SC 3). Let $\Omega \in \mathcal{C}_R$ and $\mu < \overline{\mu}_k^-$, and assume that $bR < k$. Then, for f bounded, there exist a subsolution v and a supersolution w of*

$$\mathcal{P}_k^-(D^2 u) + H(x, \nabla u) + \mu u = f(x) \quad in \ \Omega \tag{4.9}$$

that satisfy $w \leq v$ in $\overline{\Omega}$ and $w = v = 0$ on $\partial\Omega$.

Proof of Proposition 4.6. Fix $\rho \in (\mu, \overline{\mu}_k^-)$, and, in view of the definition of $\overline{\mu}_k^-$, we may select a real valued subsolution ψ of

$$\mathcal{P}_k^-(D^2 \psi) + H(x, \nabla \psi) + \rho \psi = 0 \quad in \ \Omega$$

such that $\psi < 0$ in $\overline{\Omega}$. We may assume by multiplying ψ by a positive constant if necessary that $(\rho - \mu)\psi \leq -\|f\|_\infty$ in $\overline{\Omega}$. It is now clear that ψ is a subsolution of (4.9) or more precisely

$$\mathcal{P}_k^-(D^2 \psi) + H(x, \nabla \psi) + \mu \psi = \|f\|_\infty \quad in \ \Omega.$$

By translation, we may assume that $0 \in \Omega$. Since Ω is a bounded, open, convex set, for any $\varepsilon > 0$, there is $\delta > 0$ such that

$$(1 + \varepsilon)\Omega \supset \Omega^\delta := \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < \delta\}.$$

We select such a $\delta = \delta(\varepsilon)$ so that $0 < \delta < \varepsilon$.

Define $\psi_\varepsilon(x) = \psi((1 + \varepsilon)^{-1} x)$ for $x \in (1 + \varepsilon)\Omega$ and note that ψ_ε is a subsolution of

$$(1 + \varepsilon)^2 \mathcal{P}_k^-(D^2 \psi_\varepsilon(x)) + (1 + \varepsilon)H((1 + \varepsilon)^{-1} x, \nabla \psi_\varepsilon(x)) + \mu \psi_\varepsilon(x) = \|f\|_\infty \quad in \ (1 + \varepsilon)\Omega.$$

Thus, setting $H_\varepsilon(x, \xi) = (1 + \varepsilon)^{-1} H((1 + \varepsilon)^{-1} x, \xi)$ and $\mu_\varepsilon = (1 + \varepsilon)^{-2} \mu$, we see that ψ_ε is a subsolution of

$$\mathcal{P}_k^-(D^2 \psi_\varepsilon) + H_\varepsilon(x, \nabla \psi_\varepsilon) + \mu_\varepsilon \psi_\varepsilon = (1 + \varepsilon)^{-2} \|f\|_\infty \quad in \ \Omega^\delta.$$

For each $z \in B_\delta$, we define functions ψ_ε^z in $\overline{\Omega}$ and $\widetilde{H}_\varepsilon$ in $\overline{\Omega} \times \mathbb{R}^N$, respectively, by

$$\psi_\varepsilon^z(x) = \psi_\varepsilon(x + z), \quad \text{and} \quad \widetilde{H}_\varepsilon(x, \xi) = \sup_{z \in B_\delta} H_\varepsilon(x + z, \xi),$$

and note that ψ_ε^z is a subsolution of

$$\mathcal{P}_k^-(D^2 \psi_\varepsilon^z) + \widetilde{H}_\varepsilon(x, \nabla \psi_\varepsilon^z) + \mu_\varepsilon \psi_\varepsilon^z = (1 + \varepsilon)^{-2} \|f\|_\infty \quad in \ \Omega. \tag{4.10}$$

Set

$$W_\varepsilon(x) := \max_{z \in \overline{B}_{\delta/2}} \psi_\varepsilon^z(x) = \max_{y \in \overline{B}_{\delta/2}(x)} \psi_\varepsilon(y) = \max_{y \in \overline{B}_{\delta/2}(x)} \psi((1+\varepsilon)^{-1}y) \quad \text{for } x \in \overline{\Omega},$$

and observe that W_ε is upper semicontinuous in $\overline{\Omega}$ and it is a subsolution of (4.10), that $W_\varepsilon \le \max_{\overline{\Omega}} \psi < 0$ in $\overline{\Omega}$, and that the function

$$\widetilde{H}_\varepsilon(x, \xi) = \sup_{z \in B_\delta} (1+\varepsilon)^{-1} H((1+\varepsilon)^{-1}(x+z), \xi)$$

satisfies (SC 2) and (SC 3), with constant $(1+\varepsilon)^{-1}b$ in place of b.

Fix any $\varepsilon > 0$. We show that W_ε is bounded from below in $\overline{\Omega}$. For this, we argue by contradiction and thus suppose that there is a sequence $(x_n)_{n \in \mathbb{N}} \subset \overline{\Omega}$ such that $W_\varepsilon(x_n) < -n$ for all $n \in \mathbb{N}$. We may assume up to extracting a subsequence that (x_n) converges to some $x_0 \in \overline{\Omega}$. Moreover, we may assume that $x_n \in \overline{B}_{\delta/2}(x_0)$ for all n, which implies that, for any $n \in \mathbb{N}$, $x_0 \in \overline{B}_{\delta/2}(x_n)$ and

$$\psi_\varepsilon(x_0) \le W_\varepsilon(x_n),$$

which gives a lower bound of the sequence $(W_\varepsilon(x_n))$, a contradiction.

Next, we choose a sequence $(\varepsilon_n)_{n \in \mathbb{N}}$ of positive numbers converging to zero, and, for $n \in \mathbb{N}$, set $V_n = W_{\varepsilon_n}$, $H_n = \widetilde{H}_{\varepsilon_n}$, $\mu_n = \mu_{\varepsilon_n}$, and observe that, as $n \to +\infty$, $H_n \to H$ in $C(\overline{\Omega} \times \mathbb{R}^N)$, $\mu_n \to \mu$.

Fix any $n \in \mathbb{N}$, and let $f_n(x) = (1+\varepsilon_n)^{-2} f(x)$. The above computations show that V_n is a subsolution of

$$\mathcal{P}_k^-(D^2u) + H_n(x, \nabla u) + \mu_n u = f_n(x) \quad \text{in } \Omega, \tag{4.11}$$

while the standard construction of barrier functions for elliptic PDE yields a supersolution $W \in C(\overline{\Omega})$ of (4.11) that satisfies $W = 0$ on $\partial\Omega$ and $W \ge 0$ in $\overline{\Omega}$ (see e.g. [15]). If $f \ge 0$ then just take $W \equiv 0$. We define the function z_n in $\overline{\Omega}$ by

$$z_n(x) = \inf\{u(x) : u \text{ supersolution of (4.11)}, \ V_n \le u \le W \text{ in } \overline{\Omega}, \ u = 0 \text{ on } \partial\Omega\}.$$

By Perron procedure, the function z_n is a "viscosity solution" of (4.11) in the sense that the upper semicontinuous envelope $(z_n)^*$ of z_n, given by

$$(z_n)^*(x) = \inf_{r > 0} \sup\{z_n(y) : y \in \overline{\Omega}, \ |y - x| < r\},$$

is a subsolution of (4.11) and the lower semicontinuous envelope $(z_n)_*$ of z_n, given by

$$(z_n)_*(x) = \sup_{r > 0} \inf\{z_n(y) : y \in \overline{\Omega}, \ |y - x| < r\},$$

is a supersolution of (4.11). It is clear that $\inf_{\overline{\Omega}} V_n \le (z_n)_* \le (z_n)^* \le W$ in $\overline{\Omega}$. If u is a supersolution of (4.11) and if $V_n \le u \le W$ in $\overline{\Omega}$ and $u = 0$ on $\partial\Omega$, then u is supersolution of

$$\mathcal{P}_k^-(D^2u) + H_n(x, \nabla u) = f_n(x) - |\mu_n| \inf_{\overline{\Omega}} V_n \quad \text{in } \Omega.$$

Proposition 2.8, applied to $-u$, yields an inequality $u(x) \ge -C_n d(x)$ for all $x \in \overline{\Omega}$ and some $C_n > 0$, where C_n is independent of the choice of u. This implies that $-C_n d \le (z_n)_* \le (z_n)^* \le W$ in $\overline{\Omega}$, which, in particular, ensures that $(z_n)_* = (z_n)^* = 0$ on $\partial\Omega$.

Now, we intend to send $n \to +\infty$. We claim that $\sup \|(z_n)_*\|_\infty < +\infty$. To check this, we argue by contradiction and suppose that $\sup \|(z_n)_*\|_\infty = +\infty$. We may assume up to a subsequence that $\lim_{n \to +\infty} \|(z_n)_*\|_\infty = +\infty$. Since the sequence $(z_n)_*$ is uniformly bounded from above by W in $\overline{\Omega}$, this in particular implies that $\lim_{n \to +\infty} \inf_{x \in \overline{\Omega}} (z_n)_*(x) = -\infty$. Set

$$Z_n(x) = \frac{(z_n)_*(x)}{\|(z_n)_*\|_\infty} \quad \text{for } x \in \overline{\Omega}, \ n \in \mathbb{N},$$

and note that if we set

$$M_0 = \sup_{n \in \mathbb{N}} \frac{\|f\|_\infty}{\|(z_n)_*\|_\infty} + |\mu|,$$

then Z_n is a supersolution of

$$\mathcal{P}_k^-(D^2 Z_n) + H_n(x, \nabla Z_n) = M_0 \quad \text{in } \Omega.$$

Since $bk < R$, by applying Proposition 2.8 to $-Z_n$, we get, for some constant $M_1 > 0$,

$$Z_n(x) \geq -M_1 d(x) \quad \text{for all } x \in \overline{\Omega}, \ n \in \mathbb{N}. \tag{4.12}$$

We take the lower relaxed limit of $(Z_n)_{n \in \mathbb{N}}$, that is, we set

$$Z^-(x) = \liminf_{n \to +\infty}{}_* Z_n(x) = \sup_{r > 0} \inf\{Z_n(y) : y \in \overline{\Omega}, \ |y - x| < r, \ n > r^{-1}\}.$$

It is a standard observation (see, e.g., [16, Chapter 6]) that Z^- is lower semicontinuous in $\overline{\Omega}$ and a supersolution of (1.2). It is clear that $\min_{\overline{\Omega}} Z^- = -1$. Moreover, it follows from (4.12) that $Z^- = 0$ on $\partial\Omega$. According to Theorem 4.1, the minimum principle holds for (1.2), but this contradicts that $\min_\Omega Z^- = -1$. Thus, we have $\sup_{n \in \mathbb{N}} \|(z_n)_*\|_\infty < +\infty$.

For the sequence (z_n), which is uniformly bounded in $\overline{\Omega}$, we consider the upper and lower relaxed limits z^+ and z^- defined, respectively, by

$$z^+(x) = \limsup_{n \to \infty}{}^* z_n(x) = \inf_{r > 0} \sup\{z_n(y) : |y - x| < r, \ n > r^{-1}\},$$

and

$$z^-(x) = \liminf_{n \to \infty}{}_* z_n(x) = \sup_{r > 0} \inf\{z_n(y) : |y - x| < r, \ n > r^{-1}\},$$

and observe that $-\sup_{n \in \mathbb{N}} \|(z_n)_*\|_\infty \leq z^- \leq z^+ \leq W$ in $\overline{\Omega}$ and that z^+ and z^- are a subsolution and a supersolution of (4.9), respectively.

Similarly to (4.12) for Z_n, since (z_n) is uniformly bounded in $\overline{\Omega}$, we deduce that there is a constant $M_2 > 0$ such that $(z_n)_*(x) \geq -M_2 d(x)$ for all $x \in \overline{\Omega}$ and $n \in \mathbb{N}$, which implies that $z^- = z^+ = 0$ on $\partial\Omega$. The proof is now complete. $\quad\square$

We remark that defining W^ε from ψ^ε in the proof above is a sort of supconvolution (see [24] for the use of this supconvolution in a different situation).

Proof of Proposition 4.5. The finiteness of $\overline{\mu}_k^-$ is a consequence of Corollary 4.2 which gives a precise estimate.

For $n \in \mathbb{N}$ let us consider the equation

$$\mathcal{P}_k^-(D^2 w) + H(x, \nabla w) + \left(\overline{\mu}_k^- - \frac{1}{n}\right) w = 1 \quad \text{in } \Omega. \tag{4.13}$$

For each $n \in \mathbb{N}$, by Proposition 4.6, there are a subsolution v_n and a supersolution w_n of

$$\mathcal{P}_k^-(D^2 u) + H(x, \nabla u) + \left(\overline{\mu}_k^- - \frac{1}{n}\right) u = 1 \quad \text{in } \Omega, \tag{4.14}$$

satisfying $w_n \leq v_n \leq 0$ in $\overline{\Omega}$ and $w_n = v_n = 0$ on $\partial\Omega$.

We claim that $\sup_{n \in \mathbb{N}} \|w_n\|_\infty = +\infty$. Suppose by contradiction that $\sup_{n \in \mathbb{N}} \|w_n\|_\infty < +\infty$. We choose $j \in \mathbb{N}$ large enough so that

$$\frac{1}{j} \left(2 \sup_{n \in \mathbb{N}} \|w_n\|_\infty + \overline{\mu}_k^- + \frac{1}{j}\right) \leq 1,$$

which implies that, since $w_j \leq v_j \leq 0$,

$$\frac{2}{j} v_j - \frac{1}{j}\left(\overline{\mu}_k^- + \frac{1}{j}\right) \geq -1 \quad \text{in } \overline{\Omega},$$

and, hence, $v_j - 1/j$ is a subsolution of

$$\mathcal{P}_k^-(D^2 u) + H(x, \nabla u) + (\overline{\mu}_k^- + \frac{1}{j})u = 0 \quad \text{in } \Omega.$$

Since $v_j - 1/j < 0$ in $\overline{\Omega}$, this contradicts the definition of $\overline{\mu}_k^-$ and proves that $\sup_{n \in \mathbb{N}} \|w_n\|_\infty = +\infty$.

Up to extracting a subsequence, we may assume that

$$\lim_{n \to +\infty} \|w_n\|_\infty = +\infty.$$

We introduce bounded functions $z_n = \frac{w_n}{\|w_n\|_\infty}$, solutions of

$$\mathcal{P}_k^-(D^2 z_n) + H(x, \nabla z_n) + \left(\overline{\mu}_k^- - \frac{1}{n}\right) z_n \le \frac{1}{\|w_n\|_\infty} \quad \text{in } \Omega.$$

We set

$$v(x) := \liminf_{n \to +\infty}{}_* z_n(x) \quad \text{for } x \in \overline{\Omega}.$$

This is the lower half relaxed limit of (z_n) and is a supersolution of $\mathcal{P}_k^-(D^2 v) + H(x, \nabla v) + \mu_k^- v \le 0$ in Ω. Moreover, it is clear that $v \le 0$ in $\overline{\Omega}$ and $\min_{\overline{\Omega}} v = -1$. Using again the bound (2.5), we deduce that $v = 0$ on $\partial\Omega$, and the proof is complete. $\square$

Proof of Theorem 4.4. We begin by proving the following

Claim. *For $\mu < \mu_k^-$ the operator $\mathcal{P}_k^-(D^2 \cdot) + H(x, \nabla \cdot) + \mu \cdot$ satisfies the minimum principle.*

The proof proceeds like the proof of Theorem 4.1, the only difference is that for $\rho \in (\mu, \mu_k^-)$, the $\limsup_{x \to z} u(x)$ could be zero for some $z \in \partial\Omega$. But using (SC 2) and the negativity of $u(x)$ we get

$$\mathcal{P}_k^-(D^2 u) + H(x, \nabla u) \ge 0 \quad \text{in } \Omega$$

while

$$\mathcal{P}_k^+(D^2(-v)) - H(x, -\nabla(-v)) \ge -\mu \|v^-\|_\infty \quad \text{in } \Omega.$$

In view of Propositions 2.5–2.8, with $m = \mu \|v^-\|_\infty$, there exist two positive constants C_1 and C_2 such that

$$u(x) \le -C_1 d(x) \quad \text{and} \quad -v(x) \le C_2 d(x) \quad \text{for any } x \in \Omega.$$

Hence

$$0 < \frac{v(x_0)}{u(x_0)} \le \gamma := \sup_{x \in \Omega} \frac{v(x)}{u(x)} \le \frac{C_2}{C_1} < +\infty.$$

Now we can proceed exactly as in the proof of Theorem 4.1 in order to complete the proof of the claim.

To finish the proof of Theorem 4.4 we observe that Proposition 4.5 and the claim imply that $\overline{\mu}_k^- \ge \mu_k^-$, but the reverse inequality is true by definition. $\square$

Remark 4.7. The bound (4.6) clearly holds for μ_k^- under the assumptions of Theorem 4.4.

Since $\mu_k^- \ge \overline{\mu}_k^-$, by definition, the inequality (4.7) is a fortiori true for μ_k^-. Moreover (4.7) is trivial for $bR_2 \ge k$. We show in the Example 4.9 that $\overline{\mu}_k^-$ can be zero.

Remark 4.8. The equality $\mu_k^- = \overline{\mu}_k^-$ holds true also in some non-convex case, for instance if Ω is a star-shaped domain, i.e.

$$\overline{\Omega - \{x_0\}} \subseteq (1 + \varepsilon)(\Omega - \{x_0\}) \tag{4.15}$$

for some $x_0 \in \Omega$ and all $\varepsilon > 0$. That was noticed e.g. in [29] in the case of the Pucci's extremal uniformly elliptic operators. Supposing $x_0 = 0$, for any $\varepsilon > 0$ there exists, by definition, $w_\varepsilon < 0$ in Ω satisfying

$$\mathcal{P}_k^-(D^2 w_\varepsilon) + H(\nabla w_\varepsilon) + (\mu_k^- - \varepsilon)w_\varepsilon \geq 0.$$

Hence $v_\varepsilon(x) = w_\varepsilon\left(\frac{x}{1+\varepsilon}\right)$ is negative in $\overline{\Omega}$ and if

$$H = H(\xi) = H^+(\xi), \tag{4.16}$$

then

$$\mathcal{P}_k^-(D^2 v_\varepsilon) + H(\nabla v_\varepsilon) + \frac{\mu_k^- - \varepsilon}{(1+\varepsilon)^2} v_\varepsilon \geq 0 \quad \text{in } \Omega.$$

In this way

$$\frac{\mu_k^- - \varepsilon}{(1+\varepsilon)^2} \leq \overline{\mu}_k^-$$

and $\mu_k^- = \overline{\mu}_k^-$ in the limit $\varepsilon \to 0$. The same holds true for μ_k^+ and $\overline{\mu}_k^+$ when $H = H(\xi) = -H^-(\xi)$.

Note that on one hand the class of the bounded domains satisfying (4.15) strictly includes $\mathcal{C}$, but on the other hand the equality $\mu_k^- = \overline{\mu}_k^-$ is here realized under the restriction (4.16), while in Theorem 4.4 the Hamiltonian H is allowed to be negative and dependent on the x-variable.

4.2. Some unusual phenomena

It is well known (see e.g. [5]) that in the uniformly elliptic case the principal eigenvalues tend to infinity when the measure of the domain tends to zero; the next example shows that this is not necessarily the case for $\mathcal{P}_k^-$.

Example 4.9. We show that in an annulus $\overline{\mu}_k^- = 0$, even if the measure of the annulus tends to zero, as long as the diameter is sufficiently large. For $k < N$, the radial function

$$v(x) = \sin|x| + \cos\varepsilon$$

is a supersolution of the problem

$$\begin{cases} \mathcal{P}_k^-(D^2 v) - b|\nabla v| = 0 & \text{in } A_\varepsilon = B_{\frac{3}{2}\pi+\varepsilon} \setminus \overline{B}_{\frac{3}{2}\pi-\varepsilon} \\ v = 0 & \text{on } \partial A_\varepsilon \end{cases}$$

where $b = \frac{k}{\frac{3}{2}\pi}$ and ε is small enough (see [18]). Since v violates the minimum principle, being negative in the annulus A_ε, we deduce from i) of Theorem 4.1 that

$$\overline{\mu}_k^- = 0.$$

In the next example we show how the definition of $\overline{\mu}_k^+$ is strongly unstable with respect to perturbations both of the operator and the domain.

Example 4.10. Let $\Omega = B_R$. For $k < N$ and $n \in \mathbb{N}$, the values $\overline{\mu}_k^+$ associated to the operators

$$\mathcal{P}_k^-(\cdot) + \frac{k}{R + \frac{1}{n}}|\cdot|$$

blows-up to $+\infty$ in view of Proposition 4.3, since in this case $\frac{k}{R+\frac{1}{n}}R < k$. Moreover

$$\mathcal{P}_k^-(\cdot) + \frac{k}{R + \frac{1}{n}}|\cdot| \longrightarrow \mathcal{P}_k^-(\cdot) + \frac{k}{R}|\cdot|$$

as $n \to +\infty$ locally uniformly in $\mathbb{S}^N \times \mathbb{R}^N$. On the other hand, taking the function $w(x) = \left(R^2 - |x|^2\right)^\gamma$ with $\gamma > 1$, it turns out that

$$\mathcal{P}_k^-(D^2 w) + \frac{k}{R}|\nabla w| + \frac{2\gamma k}{R^2}w = \left(R^2 - |x|^2\right)^{\gamma-1}\left(-2\gamma k + 2\frac{k}{R}\gamma|x| + \frac{2\gamma k}{R^2}(R^2 - |x|^2)\right) \geq 0;$$

moreover $w = 0$ on $\partial\Omega$, $w > 0$ in Ω and so $\overline{\mu}_k^+ \leq \frac{2\gamma k}{R^2}$ by ii) of Theorem 4.1.

Concerning the instability with respect to small perturbations of Ω, we consider the sequence of expanding subdomains $\Omega_n = B_{R-\frac{1}{n}}$ and the operator $\mathcal{P}_k^-(\cdot) + \frac{k}{R}|\cdot|$. As before, for any Ω_n one has $\overline{\mu}_k^+ = +\infty$, while in $\overline{\mu}_k^+ \leq 1$ in the limit case $\Omega = \cup_{n \in \mathbb{N}} \Omega_n$.

Notice that in [5] the stability of the principal eigenvalue with respect to interior perturbations of the domain is proved by means of the Krylov–Safonov Harnack inequality. It is not surprising therefore to expect the failure of the Harnack inequality in our degenerate setting, which is indeed the case as can be seen in the following very simple example. The nonnegative function $u(x_1, \dots, x_N) = x_N^2$ is clearly a solution of $\mathcal{P}_k^-(D^2 u) = 0$ in B_1 for $k < N$, but $\sup_{B_1} u = 1$ and $\inf_{B_1} u = 0$.

Other examples of instability are provided in [4] for first order operators.

5. Existence

In this section we shall prove existence results for Dirichlet problems

$$
\begin{cases}
\mathcal{P}_k^-(D^2 u) + H(x, \nabla u) + \mu u = f(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{5.1}
$$

with Ω in the class $\mathcal{C}_R$. We start with the case where k is any number between 1 and N.

Proposition 5.1. *Assume* (SC 2)–(SC 3). *Assume that* $\Omega \in \mathcal{C}_R$, $bR < k$ *and* $\mu < \mu_k^-$. *If* f *is bounded and* H *satisfies, for all* $x \in \Omega$ *and for all* ξ, η *in* $\mathbb{R}^N$,

$$
|H(x, \xi) - H(x, \eta)| \leq b|\xi - \eta|,
\tag{5.2}
$$

then for all

$$
\mu < \mu_{k,b}^- := \sup\{\mu \in \mathbb{R} : \exists w < 0 \text{ in } \Omega, \ \mathcal{P}_k^-(D^2 w) - b|\nabla w| + \mu w \geq 0 \text{ in } \Omega\},
$$

there exists a unique solution of (5.1).

Proof. Let v and w be as in Proposition 4.6. By (5.2), the nonpositive function $u = w - v$ is a supersolution of $\mathcal{P}_k^-(D^2 u) - b|\nabla u| + \mu u = 0$ (see [18]). Using Theorem 4.4 we have that $\mu_k^- = \overline{\mu}_k^-$ then by Theorem 4.1 i), we get that $u \geq 0$. Hence $v = w$ is the required solution. $\quad\square$

In the rest of the section we shall only consider the case $k = 1$, in that case beside the existence below the generalized eigenvalue we can also prove existence of the eigenfunction. The proofs somehow follow the schemes of [6,7].

Theorem 5.2. *Let* $\Omega \in \mathcal{C}_R$, *let* H *satisfying* (SC 2)–(SC 3) *and let* f *be a bounded continuous function in* Ω. *Assume* $bR < 1$. *Then there exists a solution* $u \in \mathrm{Lip}(\overline{\Omega})$ *of*

$$
\begin{cases}
\mathcal{P}_1^-(D^2 u) + H(x, \nabla u) + \mu u = f(x) & \text{in } \Omega \\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{5.3}
$$

in the following two cases:

i) *for* $\mu < \mu_1^-$;
ii) *for any* μ *if* $f \leq 0$.

The proof uses the construction in Proposition 4.6 and the global Lipschitz regularity obtained in Proposition 3.2 for subsolutions.

Proof of Theorem 5.2. We first consider the case where $\mu < \mu_1^-$. By Proposition 4.6, we see that there are a subsolution v and a supersolution w of (5.3) such that $w \leq v$ in $\overline{\Omega}$. By estimate (2.5), there is a constant $C > 0$ such that $-Cd \leq w \leq v$ in $\overline{\Omega}$.

As in Proposition 4.6, the standard construction of barrier functions for elliptic PDE yields a supersolution $W \in C(\overline{\Omega})$ of (5.3) that satisfies $W = 0$ on $\partial\Omega$ and $W \geq 0$ in $\overline{\Omega}$. If $f \geq 0$ then just take $W \equiv 0$.

We define u in $\overline{\Omega}$ through the Perron procedure, that is,

$$u(x) = \sup\{z(x) : z \text{ subsolution of } (5.3), \ v \leq u \leq W \text{ in } \overline{\Omega}\}.$$

The upper semicontinuous envelope u^* is a subsolution of (5.3) and satisfies $v \leq u^* \leq W$ in $\overline{\Omega}$, which implies that $u = u^*$ in $\overline{\Omega}$ and, hence, u is upper semicontinuous in $\overline{\Omega}$. Since $u \geq -Cd$ and $u = 0$ on $\partial\Omega$, by Proposition 3.2, we see that u is Lipschitz continuous in $\overline{\Omega}$. Hence $u = u_*$ and it is a supersolution of (5.3), this ends the proof of i).

For the proof of ii), we can treat the case where $f \leq 0$ in $\overline{\Omega}$ in the same way. The only difference is that, when $f \leq 0$, the constant function 0 is a subsolution of (5.3) and replaces v in the argument above. Thus, the bound on μ is not needed and the resulting solution u is nonnegative. $\quad\square$

Theorem 5.3. *Let Ω, H and b as in the Theorem 5.2. Then there exists a negative function $\psi_1 \in \mathrm{Lip}(\overline{\Omega})$ such that*

$$\begin{cases} \mathcal{P}_1^-(D^2\psi_1) + H(x, \nabla\psi_1) + \mu_1^-\psi_1 = 0 & \text{in } \Omega \\ \psi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.4}$$

Proof. Let $\mu_n \nearrow \mu_1^-$ and use Theorem 5.2 to build $u_n \in \mathrm{Lip}(\overline{\Omega})$ a solution of

$$\begin{cases} \mathcal{P}_1^-(D^2 u_n) + H(x, \nabla u_n) + \mu_n u_n = 1 & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.5}$$

Observe that u_n are nonnegative because the forcing term being positive in Perron's construction we can use "zero" as the supersolution that bounds the u_n from above.

We claim that $\lim_{n\to\infty} \|u_n\|_\infty = +\infty$. Assume by contradiction that $\sup_{n\in\mathbb{N}} \|u_n\|_\infty < +\infty$. By Proposition 3.1 the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded in $\mathrm{Lip}(\overline{\Omega})$ and converges, up to some subsequence, to a nonpositive solution u of

$$\begin{cases} \mathcal{P}_1^-(D^2 u) + H(x, \nabla u) + \mu_1^- u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The function u is negative in Ω, otherwise if $\max_{x\in\overline{\Omega}} u = u(x_0) = 0$ and $x_0 \in \Omega$, then $\varphi(x) = 0$ should be a test function touching u from above in x_0 and therefore $0 \geq 1$.

Hence, for small positive ε, we have

$$\mathcal{P}_1^-(D^2 u) + H(x, \nabla u) + (\mu_1^- + \varepsilon)u \geq 0 \quad \text{in } \Omega$$

contradicting the maximality of μ_1^-.

For $n \in \mathbb{N}$ the functions $v_n = \frac{u_n}{\|u_n\|_\infty}$ satisfy

$$\begin{cases} \mathcal{P}_1^-(D^2 v_n) + H(x, \nabla v_n) + \mu_n v_n = \frac{1}{\|u_n\|_\infty} & \text{in } \Omega \\ v_n = 0 & \text{on } \partial\Omega \end{cases} \tag{5.6}$$

and are bounded in $\mathrm{Lip}(\overline{\Omega})$, again by means of Proposition 3.1. Extracting a subsequence if necessary, $(v_n)_{n\in\mathbb{N}}$ converges uniformly to a nonpositive function ψ_1 such that $\|\psi_1\|_\infty = 1$. Taking the limit as $n \to +\infty$ in (5.6) we have

$$\begin{cases} \mathcal{P}_1^-(D^2\psi_1) + H(x, \nabla\psi_1) + \mu_1^-\psi_1 = 0 & \text{in } \Omega \\ \psi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

By the strong maximum principle (see Remark 2.6), we conclude $\psi_1 < 0$ in Ω as we wanted to show. $\quad\square$

We end this section by computing explicitly the principal eigenvalue and eigenfunction for $\mathcal{P}_1^-$, with $H = 0$, in the ball B_R. We first note that $\overline{\mu}_1^- = \mu_1^-$, as a consequence of Theorem 4.4 or, equivalently, of Remark 4.8.

The function

$$\psi_1(x) = -\cos\left(\frac{\pi}{2R}|x|\right)$$

is twice differentiable everywhere, negative in B_R and zero on ∂B_R. The ordered eigenvalues of the Hessian matrix are

$$\lambda_1\left(D^2\psi_1(x)\right) = \left(\frac{\pi}{2R}\right)^2\cos\left(\frac{\pi}{2R}|x|\right)$$

$$\lambda_2\left(D^2\psi_1(x)\right) = \ldots = \lambda_N\left(D^2\psi_1(x)\right) = \left(\frac{\pi}{2R}\right)\frac{\sin\left(\frac{\pi}{2R}|x|\right)}{|x|},$$

if $x \neq 0$ and

$$\lambda_1\left(D^2\psi_1(0)\right) = \ldots = \lambda_N\left(D^2\psi_1(0)\right) = \left(\frac{\pi}{2R}\right)^2,$$

so that

$$\mathcal{P}_1^-\left(D^2\psi_1(x)\right) + \left(\frac{\pi}{2R}\right)^2\psi_1(x) = 0 \quad \text{in } \Omega.$$

In particular ψ_1 is a negative subsolution of $\mathcal{P}_1^-(D^2\cdot) + \left(\frac{\pi}{2R}\right)^2\cdot = 0$, hence by definition of μ_1^- we have $\mu_1^- \geq \left(\frac{\pi}{2R}\right)^2$. On the other hand the function ψ_1 invalidates the minimum principle and we get also the reversed inequality $\mu_1^- \leq \left(\frac{\pi}{2R}\right)^2$ by means of Theorem 4.4. In this way

$$\mu_1^- = \left(\frac{\pi}{2R}\right)^2$$

and ψ_1 is a negative radial eigenfunction.

It is worth to point out that for the 1-homogeneous infinity Laplacian $\Delta_\infty u = \left\langle D^2 u \frac{\nabla u}{|\nabla u|}, \frac{\nabla u}{|\nabla u|}\right\rangle$, one has

$$\mu_1^- = \mu_1^+ = \left(\frac{\pi}{2R}\right)^2$$

with $\varphi_1(x) = \cos\left(\frac{\pi}{2R}|x|\right)$ positive eigenfunction (see [23, Section 4]). In our framework we have on the contrary $\overline{\mu}_1^+ = +\infty$ in view of Proposition 4.3.

6. Strictly convex domains, a characterization

In this section we give the proof of Proposition 2.7 which we like to refer to as Proposition hula hoop. We begin with a technical lemma.

Lemma 6.1. *Let Ω be a non-empty bounded and open subset of $\mathbb{R}^N$, with C^2-boundary, and $p \in \partial\Omega$. Let $\nu(x)$ denote the outward normal unit vector of Ω at $x \in \partial\Omega$. Assume that $N > 2$, and let $H \subset \mathbb{R}^N$ be a 2-dimensional plane passing through p which is not perpendicular to $\nu(p)$. Set $\Delta = \Omega \cap H$. Let H have the Euclidean structure induced by $\mathbb{R}^N$.*

  i) *Then, Δ is a non-empty bounded and open subset, with C^2-boundary, of the plane H.*
  ii) *Assume in addition that the principal curvatures, $\kappa_1, \ldots, \kappa_{N-1}$, of $\partial\Omega$ at p are positive.*

Then, the curvature of the planar curve $\partial_H\Delta$ at p is bounded from below by $\min_{1 \leq i < N} \kappa_i$, where $\partial_H A$ denotes the boundary of $A \subset H$, relative to H.

In the above, the perpendicularity of H and $\nu(p)$ may be expressed as the condition that $\nu(p) \cdot (q - p) = 0$ for all $q \in H$.

Proof. We first prove i). We choose two orthonormal vectors e_1, $e_2 \in \mathbb{R}^N$ so that $H = \{p + x_1 e_1 + x_2 e_2 : x_1, x_2 \in \mathbb{R}\}$. By the non-perpendicularity of H and $\nu(p)$, we may assume that $\nu(p) \cdot e_1 < 0$.

Since Ω has C^2-boundary, if $\delta > 0$ is small enough, then $p + \delta e_1 \in \Omega$ and $p + \delta e_1$ is an interior point of Δ, relative to H. Since Ω is open, Δ is open relative to H. Hence, Δ is a non-empty open subset of H. It is clear that Δ is convex since it is an intersection of two convex sets and also that Δ is bounded.

Now, we show that Δ is a domain, with C^2-boundary, in H. It is obvious that $\partial_H \Delta \subset H \cap \partial\Omega$. Fix any $q \in H \cap \partial\Omega$. We consider the function $\rho \in C(\mathbb{R}^N)$ given by

$$\rho(x) = \begin{cases} \operatorname{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\operatorname{dist}(x, \partial\Omega) & \text{if } x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

This function ρ is C^2 near the boundary $\partial\Omega$ and $\nabla\rho(x) = -\nu(x)$ for all $x \in \partial\Omega$. Set $p_\delta = p + \delta e_1 \in \Delta$, note that $\rho(p_\delta) > 0$, and choose $(a, b) \in \mathbb{R}^2$ so that $q = p_\delta + a e_1 + b e_2$. By the concavity of ρ, we find that for any $t \in [0, 1]$,

$$\rho(p_\delta + t(a e_1 + b e_2)) = \rho((1 - t)p_\delta + tq) \geq (1 - t)\rho(p_\delta) + t\rho(q) = (1 - t)\rho(p_\delta),$$

and, hence,

$$\frac{d}{dt}\rho(p_\delta + t(a e_1 + b e_2))\Big|_{t=1} \leq -\rho(p_\delta) < 0,$$

which shows that

$$0 > \nabla\rho(q) \cdot (a e_1 + b e_2) = -\nu(q) \cdot (a e_1 + b e_2).$$

Noting that

$$H \cap \partial\Omega = \{p_\delta + x_1 e_1 + x_2 e_2 : (x_1, x_2) \in \mathbb{R}^2, \ \rho(p_\delta + x_1 e_1 + x_2 e_2) = 0\}$$

and applying the implicit function theorem to the function: $\mathbb{R}^2 \ni (x_1, x_2) \mapsto \rho(p_\delta + x_1 e_1 + x_2 e_2)$, we see that, in a neighborhood of q, $H \cap \partial\Omega$ is a C^2-curve in H and that $q \in \partial_H \Delta$. Because of the arbitrariness of $q \in H \cap \partial\Omega$, we find that $H \cap \partial\Omega$ is a C^2-curve in H and also that $H \cap \partial\Omega \subset \partial_H \Delta$. Thus, we conclude that $\partial_H \Delta = H \cap \partial\Omega$ and that Δ has C^2-boundary in H.

Next, we prove (ii). We may assume by translation and orthogonal transformation that $p = 0$ and $\nu(p) = (0, \ldots, 0, -1)$. We can choose a neighborhood $V \subset \mathbb{R}^N$ of $p = 0$, a neighborhood $U \subset \mathbb{R}^{N-1}$ of $0 \in \mathbb{R}^{N-1}$ and a function $g \in C^2(U)$ such that for any $x = (x_1, \ldots, x_N) \in V$,

$$x \in \Omega \quad \text{if and only if} \quad (x_1, \ldots, x_{N-1}) \in U \text{ and } x_N > g(x_1, \ldots, x_{N-1}).$$

We have $g(0) = 0$, $\nabla g(0) = 0$ and we may assume further that $D^2 g(0) = \operatorname{diag}(\kappa_1, \ldots, \kappa_{N-1})$. We choose $R > 0$ so that $1/R < \min_{1 \leq i < N} \kappa_i$, and consider the open ball B with center at $-R\nu(p) = (0, \ldots, 0, R)$ and radius R. We may assume by replacing U and V by smaller ones (in the sense of inclusion), if necessary, that for any $x \in V$,

$$x \in B \quad \text{if and only if} \quad (x_1, \ldots, x_{N-1}) \in U \text{ and } x_N > f(x_1, \ldots, x_{N-1}),$$

where $f(x_1, \ldots, x_{N-1}) = R - \sqrt{R^2 - (x_1^2 + \cdots + x_{N-1}^2)}$. Note that $\nabla f(0) = 0$ and $D^2 f(0) = (1/R)I$, where I denotes the identity matrix of order $n - 1$. By Taylor's theorem, we may assume again by replacing U and V by smaller ones, if necessary, that $f(y) < g(y)$ for all $y \in U \setminus \{0\}$. This yields

$$V \cap \Omega \subset V \cap B,$$

which shows that

$$V \cap \Delta \subset V \cap B \cap H.$$

Thus, observing that $\partial B \cap H = \partial_H (B \cap H)$, which is a special case of the identity, $\partial\Omega \cap H = \partial_H \Delta$, with B in place of Ω, that $B \cap H$ is a non-empty, planar, open disk with radius smaller than or equal to R and that $p = 0 \in \partial_H \Delta \cap \partial_H (B \cap H)$, we conclude that the curvature of the planar curve $\partial_H \Delta$ at p is larger than or equal to $1/R$. This completes the proof. $\quad\square$

Lemma 6.2. *Let Ω be a non-empty bounded and open subset, with C^2-boundary, of $\mathbb{R}^N$. Let $\kappa > 0$ be a lower bound of the principal curvatures of $\partial\Omega$ at every point $x \in \partial\Omega$. Set $R = 1/\kappa$. Then, for any $z \in \partial\Omega$, we have*

$$\Omega \subset B_R(z - R\nu(z)). \tag{6.1}$$

Clearly, (6.1) shows that $\Omega \in \mathcal{C}_R$. Indeed we have proved that

$$\Omega \subset \bigcap_{z\in\partial\Omega} B_R(z - R\nu(z)).$$

On the other hand, by the convexity of Ω, we have

$$\Omega = \bigcap_{z\in\partial\Omega} \{x \in \mathbb{R}^N : (x - z) \cdot \nu(z) < 0\}.$$

Observe that for any $z \in \partial\Omega$,

$$B_R(z - R\nu(z)) \subset \{x \in \mathbb{R}^N : (x - z) \cdot \nu(z) < 0\}.$$

Indeed, if $x \in B_R(z - R\nu(z))$, then

$$R^2 > |x - z + R\nu(z)|^2 = |x - z|^2 + 2R(x - z) \cdot \nu(z) + R^2 > 2R(x - z) \cdot \nu(z) + R^2,$$

and

$$(x - z) \cdot \nu(z) < 0.$$

Thus,

$$\Omega \supset \bigcap_{z\in\partial\Omega} B_R(z - R\nu(z)).$$

In conclusion the Lemma 6.2 above proves Proposition 2.7.

Proof. It is enough to show that for any $M > R$ and $z \in \partial\Omega$,

$$\Omega \subset B_M(z - M\nu(z)). \tag{6.2}$$

We fix any $M > R$ and $p \in \partial\Omega$. To show (6.2), we suppose to the contrary that (6.2) does not hold, and will get a contradiction.

We can thus choose a point $q \in \Omega \setminus B_M(p - M\nu(p))$.

Select $m > 0$ so small that $r := p - m\nu(p) \in \Omega \cap B_M(p - M\nu(p))$. Note that the line segment $[r, q] := \{(1 - t)r + tq : 0 \le t \le 1\}$ is contained in the set Ω and that $r \in B_M(p - M\nu(p))$ and $q \notin B_M(p - M\nu(p))$. These imply that, for some $\tau \in (0, 1]$,

$$(1 - \tau)r + \tau q \in \Omega \cap \partial B_M(p - M\nu(p)).$$

Replacing q by $(1 - \tau)r + \tau q$ if $\tau < 1$, we may assume that $q \in \partial B_M(p - M\nu(p))$.

Since Ω is open, we may assume by replacing q by a nearby point, if needed, that two vectors $\nu(p)$ and $q - p$ are linearly independent. In particular, we have $q \ne p$ and $q \ne p - 2M\nu(p)$. Let H be the plane passing through three points p, q, $p - M\nu(p)$. We set $\Delta = \Omega \cap H$ and $B_H = B_M(p - M\nu(p)) \cap H$. Since $p - M\nu(p) \in H$, it is clear that B_H is the planar open disk with center $p - M\nu(p)$ and radius M.

Fix $Q \in (R, M)$, so that $\kappa > 1/Q$. As in the proof of Lemma 6.1 (ii), we can choose a neighborhood V of p so that

$$\Omega \cap V \subset B_Q(p - Q\nu(p)) \cap V,$$

from which we find that

$$\Delta \cap V \subset H \cap B_Q(p - Q\nu(p)) \cap V. \tag{6.3}$$

We put $e_1 = -\nu(p)$ and select a unit vector $e_2 \in \mathbb{R}^N$, orthogonal to e_1, so that two vectors e_1, e_2 parallel to the plane H, that is, $H = \{p + x_1 e_1 + x_2 e_2 : x_1, x_2 \in \mathbb{R}\}$.

We select $(a, b) \in \mathbb{R}^2$ so that $q = p + ae_1 + be_2$. Since $q \in \partial B_M(p - Mv(p)) \setminus \{p - 2Mv(p), \ p\}$, it follows that $0 < a < 2M$ and $b \neq 0$. We may assume by replacing e_2 by $-e_2$, if needed, that $b < 0$.

We set

$$\Delta_2 = \{(x_1, x_2) \in \mathbb{R}^2 : p + x_1 e_1 + x_2 e_2 \in \Delta\},$$

$$g(x_1) = \inf\{x_2 \in \mathbb{R} : (x_1, x_2) \in \Delta_2\} \quad \text{for } x_1 \in (0, a].$$

It is easily seen that Δ_2 is a strictly convex, bounded and open set, with C^2-boundary, in $\mathbb{R}^2$, that the line segment $\{t(a, b) : (0, 1]\}$, connecting the origin and the point (a, b), lies in the set Δ_2, that g is locally Lipschitz continuous, convex function on $(0, a]$, and that the graph $\{(x_1, g(x_1)) : x_1 \in (0, a]\}$ is a subset of $\partial \Delta_2$. The last two remarks together with the smoothness of Δ_H implies that $g \in C^2((0, a])$.

We consider the function $f_M \in C([0, a])$ defined by

$$f_M(x_1) = -\sqrt{M^2 - (x_1 - M)^2}.$$

Obviously we have, for $(x_1, x_2) \in (0, a] \times \mathbb{R}$,

$$x_2 > f_M(x_1) \quad \text{if } p + x_1 e_1 + x_2 e_2 \in B_H.$$

Similarly, we define $f_Q \in C([0, 2Q])$ by

$$f_Q(x_1) = -\sqrt{Q^2 - (x_1 - Q)^2}.$$

By (6.3), if we define the function h on $[0, a]$ by

$$h(x) = \begin{cases} 0 & \text{if } x = 0, \\ g(x) & \text{if } x \in (0, a], \end{cases}$$

then $f_M(0) = f_Q(0) = h(0) = 0$ and $f_M(x_1) < f_Q(x_1) \leq h(x_1)$ for all $x_1 \in (0, \delta]$ and some small $\delta > 0$. On the other hand, since $\{x_1(a, b) : x_1 \in (0, 1]\} \subset \Delta_2$, we have $h(x_1) = g(x_1) \leq (b/a)x_1$ for all $x_1 \in (0, a]$. It is now clear that $h \in C([0, 1])$.

Since $q = p + ae_1 + be_2 \in \Delta \cap \partial B_M(p - Mv(p))$, we have $h(a) = g(a) < b = f_M(a)$. Consider the function $\phi \in C([0, a])$ given by

$$\phi(x) = h(x) - f_M(x).$$

It follows that $\phi(0) = 0$, $\phi(a) < 0$ and $\phi(\delta) > 0$. Accordingly, ϕ has a positive maximum at a point $d \in (0, a)$. Hence, $\phi'(d) = 0$ and $\phi''(d) \leq 0$. That is, we have $f_M'(d) = g'(d)$ and $f_M''(d) \geq g''(d)$, which shows that the curvature of the graph g at $(d, g(d))$ is smaller than or equal to that of f_M, which is $1/M$. This shows that the planar curve $\partial_H \Delta$ has curvature smaller than $1/R$ at $p + de_1 + g(d)e_2$. Since the planar curve $\partial_H \Delta$ has curvature larger than or equal to $\kappa = 1/R$ by Lemma 6.1, this is a contradiction. □

Conflict of interest statement

There is no conflict of interest.

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