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Regularity of the Eikonal equation with two vanishing entropies

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Abstract

Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain. The Eikonal equation $|\nabla u| = 1$ for a function $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ has very little regularity, examples with singularities of the gradient existing on a set of positive H^1 measure are trivial to construct. With the mild additional condition of two vanishing entropies we show ∇u is locally Lipschitz outside a locally finite set. Our condition is motivated by a well known problem in Calculus of Variations known as the Aviles–Giga problem. The two entropies we consider were introduced by Jin, Kohn [26], Ambrosio, DeLellis, Mantegazza [2] to study the Γ-limit of the Aviles–Giga functional. Formally if u satisfies the Eikonal equation and if

$$\nabla \cdot \left(\widetilde{\Sigma}_{e_1 e_2}(\nabla u^{\perp})\right) = 0 \text{ and } \nabla \cdot \left(\widetilde{\Sigma}_{\epsilon_1 \epsilon_2}(\nabla u^{\perp})\right) = 0 \text{ distributionally in } \Omega, \tag{1}$$

where $\widetilde{\Sigma}_{e_1 e_2}$ and $\widetilde{\Sigma}_{\epsilon_1 \epsilon_2}$ are the entropies introduced by Jin, Kohn [26], and Ambrosio, DeLellis, Mantegazza [2], then ∇u is locally Lipschitz continuous outside a locally finite set.

Condition (1) is motivated by the zero energy states of the Aviles–Giga functional. The zero energy states of the Aviles–Giga functional have been characterized by Jabin, Otto, Perthame [25]. Among other results they showed that if $\lim_{n\to\infty} I_{\epsilon_n}(u_n) = 0$ for some sequence $u_n \in W_0^{2,2}(\Omega)$ and $u = \lim_{n\to\infty} u_n$ then ∇u is Lipschitz continuous outside a finite set. This is essentially a corollary to their theorem that if u is a solution to the Eikonal equation $|\nabla u| = 1$ a.e. and if for every "entropy" Φ (in the sense of [18], Definition 1) function u satisfies $\nabla \cdot \left[\Phi(\nabla u^{\perp})\right] = 0$ distributionally in Ω then ∇u is locally Lipschitz continuous outside a locally finite set. In this paper we generalize this result in that we require only two entropies to vanish.

The method of proof is to transform any solution of the Eikonal equation satisfying (1) into a differential inclusion $DF \in K$ where $K \subset M^{2\times 2}$ is a connected compact set of matrices without Rank-1 connections. Equivalently this differential inclusion can be written as a constrained non-linear Beltrami equation. The set K is also non-elliptic in the sense of Sverak [32]. By use of this transformation and by utilizing ideas from the work on regularity of solutions of the Eikonal equation in fractional Sobolev space by Ignat [23], DeLellis, Ignat [15] as well as methods of Sverak [32], regularity is established.

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1. Introduction

The Eikonal equation is a much studied equation whose more general form $|\nabla u| = f$ occurs in numerous areas of physics (geometric optics, wave propagation) and applied mathematics. Historically there has been great interest in first uniqueness and then subsequently regularity of the Eikonal equation. Uniqueness was largely resolved by the development of the theory of viscosity solutions [14], and subsequent regularity results have been established by a number of authors, [10,11]. Indeed, regularity and uniqueness of viscosity solutions of the Eikonal equation was one of the early triumphs that followed the development of the theory of viscosity solutions. Without additional hypotheses solutions of the Eikonal equation need have little regularity, it is easy to construct examples whose gradient is singular on a set of positive H^1 measure. One of the simplest Eikonal equations is

$$|\nabla u(x)| = 1 \text{ for } a.e.\ x \in \Omega, \tag{2}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded simply-connected domain. Our main theorem is a strong regularity result for solutions of equation (2) with an additional condition that is best described as having two vanishing entropies. The two entropies we consider were introduced into the study of the Aviles–Giga functional by Jin, Kohn [26], Ambrosio, DeLellis, Mantegazza [2], later works by DeSimone, Kohn, Müller, Otto [19], Jabine, Otto, Perthame [25] and Otto, DeLellis [16] characterized a wide class of entropies and used this characterization in a fundamental way to prove the strongest results known for the functional. In truth our main motivation also came from the Aviles–Giga functional and for this reason we will introduce it in some detail:

The Aviles–Giga functional is the second order functional

$$I_\epsilon(u) = \int_\Omega \frac{\left|1 - |\nabla u|^2\right|^2}{\epsilon} + \epsilon \left|\nabla^2 u\right|^2 dx$$

minimized over the space of functions $W_0^{2,2}(\Omega;\mathbb{R})$ or $W_0^{2,2}(\Omega;\mathbb{R}) \cap \{u : \nabla u(x) = \eta_x \text{ on } \partial\Omega\}$ where η_x is the inward pointing unit normal to $\partial\Omega$, where $\Omega \subset \mathbb{R}^2$ is a simply-connected Lipschitz domain. The Aviles–Giga functional I_ϵ forms a model for blistering and (in certain regimes) a model for liquid crystals [6,26,21]. In addition there is a closely related functional modeling thin magnetic films known as the *micromagnetics functional* [18,19,13,30,31,1,3]. For function $u \in W_0^{2,2}(\Omega)$ we refer to $I_\epsilon(u)$ as the *Aviles–Giga energy* of u. The Aviles–Giga functional is the most natural higher order generalization of the Modica–Mortola functional [28].

The biggest open problem in the study of the Aviles–Giga functional is the characterization of its Γ-limit, [6,7,26, 2]. Given the structure of I_ϵ it is not a surprise that the conjectured limiting function class is a subspace of functions that satisfy the Eikonal equation (2). By analogy to the Modica–Mortola functional, it might be expected that the limiting function space is also a subspace of $\{v : \nabla v \in BV\}$ and the limiting energy is related to $\|D[\nabla u]\|$. However this is completely *false*; see the example following Theorem 3.9 of [2]. It is necessary to build a function class that is in a sense analogous to the function class $\{v : \nabla v \in BV\}$ that is tailored to the functional I_ϵ. This is done by introducing certain *entropies* on the space of solutions of the Eikonal equation. The divergence of these entropies will (by virtue of the structure of I_ϵ) form measures that in regular examples pick up the jump in the gradient ∇u. Specifically it can be shown [2,18] that if $u_n \in W_0^{2,2}(\Omega)$ with the property that $\limsup_{n\to\infty} I_{\epsilon_n}(u_n) < \infty$ then for some subsequence $\{n_k\}$ we have $u_{n_k} \overset{W^{1,3}(\Omega)}{\rightarrow} u$. This allows us to show that if the vector field $\Sigma_{\xi\eta} u$ is defined by

$$\Sigma_{\xi\eta} u := u_\xi \left(1 - u_\eta^2 - \frac{1}{3}u_\xi^2\right)\xi - u_\eta \left(1 - u_\xi^2 - \frac{1}{3}u_\eta^2\right)\eta, \tag{3}$$

(where u_ξ and u_η are the partial derivatives along ξ and η respectively) then $\nabla \cdot \left(\Sigma_{\xi\eta} u\right)$ is a measure. So instead of having that the gradient of the gradient is a measure (as would be the case if $u \in \{v : \nabla v \in BV\}$) we have that the divergence of a vector field made up of first order partial gradients is a measure, which "morally" is not that far away.

Following [2], we denote by (e_1, e_2) the canonical basis of $\mathbb{R}^2$, and by

$$\epsilon_1 := \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \epsilon_2 = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \tag{4}$$

the basis obtained from (e_1, e_2) under an anticlockwise rotation of $\frac{\pi}{4}$. It is straightforward to check that

$$\Sigma_{e_1e_2}u = \left(u_{,1}\left(1-u_{,2}^2-\frac{u_{,1}^2}{3}\right), -u_{,2}\left(1-u_{,1}^2-\frac{u_{,2}^2}{3}\right)\right) \tag{5}$$

and

$$\Sigma_{\epsilon_1\epsilon_2}u = \left(u_{,2}\left(1-\frac{2u_{,2}^2}{3}\right), u_{,1}\left(1-\frac{2u_{,1}^2}{3}\right)\right). \tag{6}$$

It has been shown in [2] that the measure

$$S \to \left\|\begin{pmatrix}\nabla\cdot\left(\Sigma_{e_1e_2}u\right)\\ \nabla\cdot\left(\Sigma_{\epsilon_1\epsilon_2}u\right)\end{pmatrix}\right\|(S) \text{ for any } S\subset\mathbb{R}^2$$

forms a lower bound on the energy $I_{\epsilon_n}(u_n)$ of any sequence $\{u_n\}$ such that $\lim_{n\to\infty}u_n = u$. As such the functional

$$u \to \left\|\begin{pmatrix}\nabla\cdot\left(\Sigma_{e_1e_2}u\right)\\ \nabla\cdot\left(\Sigma_{\epsilon_1\epsilon_2}u\right)\end{pmatrix}\right\|(\Omega) \tag{7}$$

was conjectured in [2] to be the Γ-limiting energy of the Aviles–Giga functional.

Following [18,16] we say $\Phi\in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ is an entropy if

$$z\cdot D\Phi(z)z^\perp = 0 \text{ for all } z\in\mathbb{R}^2,\ \Phi(0)=0,\ D\Phi(0)=0, \tag{8}$$

where $z^\perp = (-z_2, z_1)$ is the anticlockwise rotation of z by $\frac{\pi}{2}$. Note that in [18], entropies are applied to divergence free vector fields $m:\Omega\to S^1$, in our paper they will be applied to $m=\nabla u^\perp$ where u satisfies (2). Vector fields

$$\tilde{\Sigma}_{e_1e_2}(x,y) := \left(y\left(1-x^2-\frac{y^2}{3}\right), x\left(1-y^2-\frac{x^2}{3}\right)\right) \text{ and } \tilde{\Sigma}_{\epsilon_1\epsilon_2}(x,y) := \left(-x\left(1-\frac{2x^2}{3}\right), y\left(1-\frac{2y^2}{3}\right)\right) \tag{9}$$

satisfy

$$z\cdot D\tilde{\Sigma}_{e_1e_2}(z)z^\perp = 0 \text{ for all } z\in\mathbb{S}^1,\ \ \tilde{\Sigma}_{e_1e_2}(0)=0 \tag{10}$$

and

$$z\cdot D\tilde{\Sigma}_{\epsilon_1\epsilon_2}(z)z^\perp = 0 \text{ for all } z\in\mathbb{S}^1,\ \ \tilde{\Sigma}_{\epsilon_1\epsilon_2}(0)=0. \tag{11}$$

Note that $\Sigma_{e_1e_2}u \overset{(5),(9)}{=} \tilde{\Sigma}_{e_1e_2}(\nabla u^\perp)$ and $\Sigma_{\epsilon_1\epsilon_2}u \overset{(6),(9)}{=} \tilde{\Sigma}_{\epsilon_1\epsilon_2}(\nabla u^\perp)$, where $\nabla u^\perp = (-u_{,2}, u_{,1})$. Since we are applying $\tilde{\Sigma}_{e_1e_2}$, $\tilde{\Sigma}_{\epsilon_1\epsilon_2}$ to gradient vector fields ∇u that satisfy $|\nabla u|=1$ a.e., for simplicity, and following the convention of [16], we will call them entropies even though they only satisfy (10), (11). However this is just a naming convenience and is not important to the mathematics that follows. Whenever we use any results about entropies from [18] we will mean vector fields $\Phi\in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ that satisfy (8). The main point about entropies is that given a sequence $\{u_n\}$ that satisfies $\limsup_{n\to\infty} I_{\epsilon_n}(u_n)<\infty$ and $u=\lim_{n\to\infty}u_n$, if Φ is an entropy then $\nabla\cdot\left[\Phi\left(\nabla u^\perp\right)\right]$ is a measure.

The characterization of this class of entropies is one of the main achievements of [18] and it leads to many further developments. It was the main tool used in [18] to prove pre-compactness in $W^{1,3}(\Omega)$ of a sequence of functions $\{u_n\}$ of bounded Aviles–Giga energy (an alternative proof just using *two* entropies $\tilde{\Sigma}_{e_1e_2}$, $\tilde{\Sigma}_{\epsilon_1\epsilon_2}$ is provided in [2]). More importantly it allows for the classification achieved by Jabin, Otto, Perthame in [25] of all functions u and all domains Ω for which there exists a sequence $\{u_n\}\subset W_0^{2,2}(\Omega)$ such that $u=\lim_{n\to\infty}u_n$ and $\lim_{n\to\infty}I_{\epsilon_n}(u_n)=0$. Functions u with this property are called zero energy states. It was shown in [25] that if $\Omega\neq\mathbb{R}^2$ then Ω is a ball and (after possibly change of sign) u is just the distance function away from the boundary of the ball. The characterization of entropies also permitted the deep work on the structure of solutions of the Eikonal equation u that arise as limits of sequences of *finite* Aviles–Giga energy [17].

While the works [17,25] are impressive achievements and indeed represent the state of the art with respect to the structure of solutions of the Eikonal equation that arise as limits of sequences of finite (or converging to zero) Aviles–Giga energy, when these results are formulated simply in terms of the Eikonal equation, the statements can appear a bit technical.

Theorem 1 *([25]). Let Ω be any open set in $\mathbb{R}^2$. Let $m\colon \Omega \to \mathbb{R}^2$ be a measurable function that satisfies $|m(x)| = 1$ for a.e. $x \in \Omega$ and*

$$\xi \cdot \nabla \chi(\cdot, \xi) = 0 \text{ distributionally in } \Omega \text{ for all } \xi \in \mathbb{S}^1, \tag{12}$$

where

$$\chi(x,\xi) := \begin{cases} 1 & \text{for } m(x)\cdot\xi > 0, \\ 0 & \text{for } m(x)\cdot\xi \leq 0. \end{cases} \tag{13}$$

Then m is locally Lipschitz outside a locally finite set of points.

It turns out that $\xi\chi(\cdot,\xi)$ is the pointwise limit of a sequence of entropies $\{\Phi_n\}$ (see the proof of Lemma 4, [18]), so if vector field m is such that

$$\nabla \cdot [\Phi(m)] = 0 \text{ distributionally in } \Omega \text{ for all entropies } \Phi, \tag{14}$$

then m satisfies (12). Hence by Theorem 1 any vector field m satisfying (14) is locally Lipschitz outside a locally finite set of points. This is the main result needed by Jabin, Otto, Perthame [25] to characterize all zero energy states of the Aviles–Giga energy.

Corollary 2 *([25]). Let u be a limit of a sequence $\{u_n\} \subset W_0^{2,2}(\Omega)$ with $\lim_{n\to\infty} I_{\epsilon_n}(u_n) = 0$ then ∇u is Lipschitz outside a finite set of points.*

Actually in [25] a more general result is proved that includes zero energy states of the micromagnetic functional, but since our interest is focused on the Aviles–Giga functional we do not state their result in full generality.

What is achieved in this paper is a proof of the regularity result under the much weaker condition that only the divergence of two entropies $\widetilde{\Sigma}_{e_1e_2}$ and $\widetilde{\Sigma}_{\epsilon_1\epsilon_2}$ applied to $\nabla u^\perp$ vanishes.

Theorem 3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain and u be a solution to the Eikonal equation (2). Suppose*

$$\nabla \cdot \left(\Sigma_{e_1e_2} u\right) = 0 \text{ and } \nabla \cdot \left(\Sigma_{\epsilon_1\epsilon_2} u\right) = 0 \text{ distributionally in } \Omega. \tag{15}$$

Then ∇u is locally Lipschitz outside a locally finite set of points S. Moreover, in any convex neighborhood $\mathcal{O} \subset\subset \Omega$ of a point $\zeta \in S$ there exists $\alpha \in \{-1, 1\}$ such that

$$u(x) = \alpha\,|x - \zeta| \text{ for any } x \in \mathcal{O}. \tag{16}$$

This result also includes Corollary 2 as a consequence in the case that Ω satisfies the assumptions of Theorem 3. The value of this result is twofold. The Eikonal equation with the additional assumption of two vanishing entropies seems to us a fairly natural condition and as such the statement of Theorem 3 is of interest purely from the perspective of the Eikonal equation alone. Essentially our theorem says that the dimension of the set of singularities of solutions of the Eikonal equation is reduced by one under the additional hypothesis of having two vanishing entropies. On this topic we mention the recent powerful results of Ignat [23] and DeLellis, Ignat [15] on regularity of solutions of the Eikonal equation in fractional Sobolev spaces. We learned a great deal and took numerous ideas from these works.

Remark 1. The choice of $\Sigma_{e_1e_2}$, $\Sigma_{\epsilon_1\epsilon_2}$ is arbitrary among the class defined by (3). If (ξ_1,η_1), (ξ_2,η_2) are any two different orthonormal bases for $\mathbb{R}^2$ then we could instead have the hypothesis

$$\nabla \cdot \left(\Sigma_{\xi_1\eta_1} u\right) = 0 \text{ and } \nabla \cdot \left(\Sigma_{\xi_2\eta_2} u\right) = 0 \text{ distributionally in } \Omega. \tag{17}$$

This follows from Theorem 3.2 [2] where it is shown that $\Sigma_{\xi_i\eta_i} u = \cos(2\theta_i)\Sigma_{e_1e_2}u + \sin(2\theta_i)\Sigma_{\epsilon_1\epsilon_2}u$ where $\xi_i = (\cos(\theta_i), \sin(\theta_i))$, $\eta_i = (-\sin(\theta_i), \cos(\theta_i))$. Thus

$$\sin(2\theta_1 - 2\theta_2)\,\Sigma_{\epsilon_1\epsilon_2}u = \cos(2\theta_2)\Sigma_{\xi_1\eta_1}u - \cos(2\theta_1)\Sigma_{\xi_2\eta_2}u \tag{18}$$

and hence (17), (18) implies $\nabla \cdot \left(\Sigma_{\epsilon_1\epsilon_2} u\right) = 0$ distributionally in Ω. In the same way

$$\sin(2\theta_1 - 2\theta_2)\,\Sigma_{e_1e_2}u = \sin(2\theta_1)\Sigma_{\xi_2\eta_2}u - \sin(2\theta_2)\Sigma_{\xi_1\eta_1}u \tag{19}$$

and so (17), (19) implies $\nabla\cdot\left(\Sigma_{e_1e_2}u\right)=0$ distributionally in Ω. Thus (17) implies (15).

Remark 2. Under the hypothesis that $\nabla\cdot\left(\Sigma_{\xi\eta}u\right)=0$ for a single entropy then Theorem 3 is false. Since we have just one entropy we can assume without loss of generality that $(\eta,\xi)=(e_1,e_2)$. And finding a scalar function u that satisfies

$$|\nabla u| = 1 \text{ a.e. and } \nabla\cdot\left(\Sigma_{e_1e_2}u\right)=0 \tag{20}$$

is equivalent to finding a function $w:\Omega\to\mathbb{R}^2$ that satisfies the differential inclusion

$$Dw \in \left\{\begin{pmatrix} \cos(\psi) & \sin(\psi) \\ \frac{2}{3}\sin^3(\psi) & \frac{2}{3}\cos^3(\psi)\end{pmatrix} : \psi\in(-\pi,\pi]\right\} =: \Pi \text{ a.e. in } \Omega. \tag{21}$$

Now Π has non-trivial rank-1 connections because

$$\begin{aligned}
&\det\begin{pmatrix} \cos(\psi_2)-\cos(\psi_1) & \sin(\psi_2)-\sin(\psi_1) \\ \frac{2}{3}\left(\sin^3(\psi_2)-\sin^3(\psi_1)\right) & \frac{2}{3}\left(\cos^3(\psi_2)-\cos^3(\psi_1)\right)\end{pmatrix}\\
&=\frac{2}{3}\left(\left(\cos^3(\psi_2)-\cos^3(\psi_1)\right)(\cos(\psi_2)-\cos(\psi_1)) - \left(\sin^3(\psi_2)-\sin^3(\psi_1)\right)(\sin(\psi_2)-\sin(\psi_1))\right)\\
&=0
\end{aligned}$$

has non-trivial solutions, for example given by $\psi_2=\pi n-\psi_1-\frac{\pi}{2}$ for $n\in\mathbb{Z}$. Thus we can construct non trivial functions w satisfying (21) for which Dw is singular along lines running through Ω, such functions are called *laminates* (see [29], Section 2.1). This gives us solutions to (20) whose gradient is singular along lines in Ω.

As stated previously our principal interest is in the Aviles–Giga functional. As described the original conjectured Γ-limiting energy from [2] is given by (7). As the study of the Aviles–Giga functional evolved it was increasingly understood that to make progress the conjectured Γ-limiting energy had to be an energy that incorporated all the entropies, not simply $\widetilde{\Sigma}_{e_1e_1}$ and $\widetilde{\Sigma}_{\epsilon_1\epsilon_2}$. As mentioned the proof of Corollary 2 requires the use of a sequence of entropies $\{\Phi_n\}$ that approximates $\xi\chi(\cdot,\xi)$. In [16] DeLellis, Otto proved many strong structural results on a class of solutions of the Eikonial equation denoted by $A(\Omega)$ that includes all $W^{1,3}(\Omega)$ limits of sequences $\{u_n\}\subset W_0^{2,2}(\Omega)$ that have equibounded Aviles–Giga energy. Among the results they proved was that for any $u\in A(\Omega)$ there exists a set of σ-finite H^1 measure J on which ∇u has jumps and has traces in exactly the way it would have if $\nabla u\in BV$. What would be most natural is if J was the singular set of vector valued measure that is the Γ-limiting energy of I_ϵ. However this is not exactly the case and J has to be defined as the singular set of measure into the dual space of all entropies (see [16], proof of Proposition 1). It is in some sense a singular set of an infinite set of entropies simultaneously.

While utilizing the information available from all entropies is in our opinion the best way to progress with the study of the Aviles–Giga functional, it does have the disadvantage that the statements of the theorems proved are less transparent. It is for example not clear what the conjecture for the Γ-limiting energy of the Aviles–Giga energy is. What Theorem 3 does is to raise the possibility of reformulating the structure results of [16,25] in terms of the two entropies $\widetilde{\Sigma}_{e_1e_2}$, $\widetilde{\Sigma}_{\epsilon_1\epsilon_2}$. Were this to be accomplished it would return the measure $S\to\left\|\begin{pmatrix}\nabla\cdot\left(\Sigma_{e_1e_2}u\right)\\ \nabla\cdot\left(\Sigma_{\epsilon_1\epsilon_2}u\right)\end{pmatrix}\right\|(S)$ as the natural conjecture for Γ-limiting energy for the Aviles–Giga functional.

1.1. Reduction to differential inclusions

We denote

$$E(\Omega) := \{u\in W^{1,\infty}(\Omega) : |\nabla u| = 1 \text{ a.e. and (15) is satisfied}\}. \tag{22}$$

The starting point for our work is the transformation of functions $u\in E(\Omega)$ into functions $F_u:\Omega\to\mathbb{R}^2$ that satisfy the differential inclusions $DF_u\in K$, where $K\subset M^{2\times 2}$ is a compact connected set defined by (42). This can be done because (15) can be rewritten as

$$\operatorname{curl}\left((\Sigma_{e_1e_2}u)^{\perp}\right)=0 \text{ and } \operatorname{curl}\left((\Sigma_{\epsilon_1\epsilon_2}u)^{\perp}\right)=0 \text{ distributionally in } \Omega.$$

Hence we can find some potential F_u^1 such that $\nabla F_u^1=(\Sigma_{e_1e_2}u)^{\perp}$ and F_u^2 such that $\nabla F_u^2=(\Sigma_{\epsilon_1\epsilon_2}u)^{\perp}$.[1] The structure of $\Sigma_{e_1e_2}$, $\Sigma_{\epsilon_1\epsilon_2}$ implies that $DF_u\in K$ a.e. in Ω. It is a calculation to see that K does not have rank-1 connections, i.e., there do not exist $A, B\in K$, $A\neq B$ with $\mathrm{Rank}(A-B)=1$. Regularity of differential inclusions into sets without rank-1 connections has been studied by Sverak in his seminar paper [32]. He showed that if function v satisfies $Dv\in S\subset M^{2\times 2}$ where S has no rank-1 connections and is elliptic in the sense that if $A, B\in S$, then $\det(A-B)\geq c\,|A-B|^2$, then v is smooth. The set K defined by (42) is *not elliptic* in the sense of Sverak, but it turns out that for some constant $c_0>0$, $\det(A-B)\geq c_0\,|A-B|^4$ for any $A, B\in K$. This is not enough to establish smoothness of F_u (indeed since $\nabla u^{\perp}$ could be a vortex, smoothness of F_u could not be true) by using the methods of [32], but is enough to establish fractional Sobolev regularity. The fact that an inequality of this form is enough to establish fractional Sobolev regularity was previously noted by Faraco and Kristensen [20], Proposition 1. The differential inclusion $DF_u\in K$ can be reformulated as a constrained non-linear Beltrami equation and our proof of fractional Sobolev regularity can hence be formulated as the following theorem.

Theorem 4. *Given a bounded simply-connected domain $\widetilde{\Omega}\subset\mathbb{C}$, and $f\in W^{1,\infty}(\widetilde{\Omega};\mathbb{C})$ that satisfies $f(0)=0$ (assuming $0\in\widetilde{\Omega}$) and the non-linear Beltrami system*

$$\frac{\partial f}{\partial\bar z}(z)=\frac{4}{3}\left(\frac{\partial f}{\partial z}(z)\right)^3,\ \left|\frac{\partial f}{\partial z}(z)\right|=\frac{1}{2}\quad \text{for a.e. } z\in\widetilde{\Omega}, \tag{23}$$

we have that

$$Df\in W^{\sigma,4}_{loc}(\widetilde{\Omega})\quad \text{for all } \sigma\in\left(0,\frac{1}{3}\right). \tag{24}$$

In addition, given $\widetilde{\Omega}'\subset\subset\widetilde{\Omega}$, for all $\epsilon\in\left(0,\frac{1}{2}\mathrm{dist}(\widetilde{\Omega}',\partial\widetilde{\Omega})\right)$, we have that

$$\int_{\widetilde{\Omega}'}\int_{B_\epsilon(0)}\frac{|Df(z+y)-Df(z)|^4}{\epsilon^{2+\frac{4}{3}}}dydz<C \tag{25}$$

for some constant C independent of ϵ.

Now if we define $\mathcal{H}_0(\xi):=\frac{4}{3}\xi^3$ then (23) can be written as $\frac{\partial f}{\partial\bar z}(z)=\mathcal{H}_0\left(\frac{\partial f}{\partial z}(z)\right)$, $\left|\frac{\partial f}{\partial z}\right|=\frac{1}{2}$. We will call this a constrained non-linear Beltrami equation. The study of equations of the form $\frac{\partial f}{\partial\bar z}=\mathcal{H}\left(z,\frac{\partial f}{\partial z}\right)$ has flourished in the last few years. Under the assumptions that

(I) $z\to\mathcal{H}(z,w)$ is measurable
(II) And for $w_1, w_2\in\mathbb{C}$, $|\mathcal{H}(z,w_1)-\mathcal{H}(z,w_2)|\leq k\,|w_1-w_2|$ for some $k<1$

the existence and regularity theory of non-linear Beltrami equations resembles that of the linear theory; see [8,24,9,5,4]. But note when restricted to the circle $\partial B_{\frac{1}{2}}(0)$ the Lipschitz constant of $\mathcal{H}_0$ is exactly 1, so in some sense $\mathcal{H}_0$ is a critical case.[2] We are not aware of any other regularity results for non-linear Beltrami equations without the assumptions (I), (II). While Theorem 4 is essentially a regularity result for differential inclusions, we formulate it in the language of non-linear Beltrami equations because these are much better known and more studied objects. We also find the connection to this area is interesting and potentially worth further investigation.

The connection between Theorem 3 and Theorem 4 is made by the following result.

[1] The idea to study F_u comes from [2], see the proof of Proposition 4.6.

[2] If instead we had $\mathcal{H}_0(\xi)=\frac{4}{3}\xi^3$ and $\left|\frac{\partial f}{\partial z}\right|=\alpha$ for some $\alpha\in(0,\frac{1}{2})$ we believe the standard methods of [4] would give regularity.

Theorem 5. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain. Define $\widetilde{\Omega} := \{x_1 + ix_2 \in \mathbb{C} : (x_1, x_2) \in \Omega\}$ and define $B(\widetilde{\Omega})$ as the set of functions $f \in W^{1,\infty}(\widetilde{\Omega}; \mathbb{C})$ that satisfy $f(0) = 0$ (assuming $0 \in \Omega$) and the constrained non-linear Beltrami equation*

$$\frac{\partial f}{\partial \bar{z}}(z) = \frac{4}{3}\left(\frac{\partial f}{\partial z}(z)\right)^3, \left|\frac{\partial f}{\partial z}(z)\right| = \frac{1}{2} \quad \text{for a.e. } z \in \widetilde{\Omega}. \tag{26}$$

Then there exists an injective transformation

$$\Gamma : \left[E(\Omega)/\mathbb{R}\right] \to B(\widetilde{\Omega}),$$

where $E(\Omega)$ is defined in (22) and two functions $u_1, u_2 \in E(\Omega)$ satisfy $u_1 = u_2$ in $\left[E(\Omega)/\mathbb{R}\right]$ if and only if $u_1 = u_2 + C$ for some constant C. Further Γ restricted to $\left[E(\Omega)/\mathbb{R}\right] \cap W^{2,1}(\Omega)$ forms a bijective transformation onto $B(\widetilde{\Omega}) \cap W^{2,1}(\widetilde{\Omega})$.

However Theorem 4 and Theorem 5 will only give us fractional Sobolev regularity. Ignat [23] studied regularity of solutions of the Eikonal equation in fractional Sobolev space, and showed that if u is a solution of the Eikonal equation and $\nabla u \in W^{\frac{1}{p},p}_{loc}(\Omega)$ for some $p \in [1, 2]$ then ∇u is locally Lipschitz outside a locally finite set of points. Note that if ∇u is smooth and Φ is an entropy it follows from properties of entropies from [18] (see Lemma 10 of this paper) that $\nabla \cdot \left[\Phi\left(\nabla u^{\perp}\right)\right] = 0$. The proof of [23] carefully exploits the structure of entropies to weaken the hypothesis on ∇u to that of fractional Sobolev space. Following this work DeLellis and Ignat [15] substantially weakened the hypothesis to $\nabla u \in W^{\frac{1}{p},p}_{loc}(\Omega)$ for some $p \in [1, 3]$. It again was achieved by very careful work using the structure of entropies and by use of an estimate of Constantin, E and Titi [12]. However close though it is, this result is not quite what we need because it requires a full $1/3$ of a derivative and with the methods of [32], a $1/3$ of a derivative is not available – Theorem 4 just stops short of what is required.

An interesting question that we were not able to answer is whether or not the transformation Γ from Theorem 5 is actually a bijection. If this were so then Theorem 3 would also yield local Lipschitz regularity of the gradient DF outside a locally finite set of points in Ω for the differential inclusion $DF \in K$. This would be a very attractive result and would hint at the possibility of a regularity theory for differential inclusions into sets S that do not have rank-1 connections but are not elliptic.

Acknowledgments. The first author would like to thank Camillo DeLellis for pointing out that the Hilbert Schmidt norm of the matrix $M_h(x)$ (of the proof of Proposition 4.6. [2]) tends towards $\frac{10}{36}$ as $h \to 0$. Roughly speaking $M_h(x)$ is (in the limit) analogous to $DF_u(x)$ from this paper and hence from this calculation it is clear that F_u is a quasiregular mapping. Our desire to further investigate this observation was the starting point of this paper. We would also like to thank the referee for many clarifying comments that have greatly improved the paper. The first author would also like to acknowledge the support of a Simons Foundation collaborative grant, award number 426900.

2. Sketch of the proof

As explained in the introduction, via the reduction to differential inclusions we get fractional Sobolev regularity $\nabla u \in W^{\sigma,4}(\Omega')$ for all $\sigma \in (0, \frac{1}{3})$ and all $\Omega' \subset\subset \Omega$. In particular we have estimate (25). The main thing we gain from this is the following estimate (see Lemma 14, (103)) which is one of our key technical tools

$$\int_{\Omega'} \left|1 - |\nabla u_\epsilon|^2\right| \left|u_{\epsilon,mn}\right| |g|\, dx \leq C\|g\|_{L^r(\Omega')} \text{ for any } g \in L^r\left(\Omega'\right),\ m, n \in \{1, 2\}, r \geq 4. \tag{27}$$

We will use (27) repeatedly.

Our strategy will be to show that for

$$\Phi^{\xi}(z) := \begin{cases} |z|^2\xi & \text{for } z \cdot \xi > 0, \\ 0 & \text{for } z \cdot \xi \leq 0, \end{cases} \tag{28}$$

we have

$$\nabla\cdot\left[\Phi^{\xi}\left(\nabla u^{\perp}\right)\right]=0 \text{ distributionally in } \Omega,\ \text{ for any } \xi\in\mathbb{S}^1\backslash\{e_1,-e_1,e_2,-e_2\}. \tag{29}$$

Regularity then follows by Theorem 1 because any $\Phi^{\xi}\left(\nabla u(x)^{\perp}\right)=\xi\chi(x,\xi)$ for $|\nabla u(x)|=1$, hence $\xi\cdot\nabla\chi(x,\xi)=0$ distributionally in Ω. This is a somewhat similar strategy to that of Ignat [23] and DeLellis, Ignat [15] except that in [23,15] it was shown that $\nabla\cdot\left[\Phi\left(\nabla u^{\perp}\right)\right]=0$ distributionally in Ω for all entropies Φ, they then conclude (12) using (as explained in the introduction) the fact that $\xi\chi(\cdot,\xi)$ is the limit of a sequence of entropies. We will build toward establishing (29) in a couple of steps.

Step 1. Harmonic entropies vanish: In this step we identify a class of entropies whose divergence vanishes when applied to $\nabla u^{\perp}$ as consequences of (15) holding. From Lemma 3 [18] (see Lemma 11 in this paper) we know there is a one to one correspondence between entropies Φ and functions $\varphi\in C_c^{\infty}(\mathbb{R}^2)$ via the formula

$$\Phi(z)=\varphi(z)z+\left(\nabla\varphi(z)\cdot z^{\perp}\right)z^{\perp}. \tag{30}$$

As we will sketch, it will turn out that under the assumption of (15), if φ is harmonic then $\nabla\cdot\left[\Phi(\nabla u^{\perp})\right]=0$. We will call entropies Φ that come from (30) via a harmonic φ, *harmonic entropies*.

To see this we argue as follows. One of the key lemmas on entropies is Lemma 2 [18] (see Lemma 10 in this paper), says that we can write

$$\nabla\cdot[\Phi(m)]=\Psi(m)\cdot\nabla(1-|m|^2) \text{ for some } \Psi\in C_c^{\infty}(\mathbb{R}^2;\mathbb{R}^2). \tag{31}$$

Now let $g_\epsilon:=g*\rho_\epsilon$ where $\rho_\epsilon(z)=\rho\left(\frac{z}{\epsilon}\right)\epsilon^{-2}$ and ρ is the standard convolution kernel. Let $w=(w^1,w^2)=\nabla u^{\perp}$. For $\Omega'\subset\subset\Omega$, let $\zeta\in C_c^{\infty}(\Omega')$ be a test function, so integrating by parts we have

$$\begin{aligned}\int_{\Omega'}\nabla\cdot[\Phi(w_\epsilon)]\,\zeta\,dx&\approx-\int_{\Omega'}(1-|w_\epsilon|^2)\nabla\cdot[\Psi(w_\epsilon)]\,\zeta\,dx\\&=-\int_{\Omega'}(1-|w_\epsilon|^2)\left(\Psi_{1,1}(w_\epsilon)w^1_{\epsilon,1}+\Psi_{1,2}(w_\epsilon)w^2_{\epsilon,1}+\Psi_{2,1}(w_\epsilon)w^1_{\epsilon,2}+\Psi_{2,2}(w_\epsilon)w^2_{\epsilon,2}\right)\zeta\,dx.\end{aligned}$$

The key point is that if Φ is a harmonic entropy then it is a calculation to see that $\Psi_{1,2}=\Psi_{2,1}$. Now we have

$$\nabla\cdot\left[\widetilde{\Sigma}_{e_1e_2}(\nabla u_\epsilon^{\perp})\right]\overset{(111)}{=}(u_{\epsilon,11}-u_{\epsilon,22})(1-|\nabla u_\epsilon|^2)=(w^2_{\epsilon,1}+w^1_{\epsilon,2})(1-|w_\epsilon|^2) \tag{32}$$

and

$$\nabla\cdot\left[\widetilde{\Sigma}_{\epsilon_1\epsilon_2}(\nabla u_\epsilon^{\perp})\right]\overset{(112)}{=}2u_{\epsilon,12}(1-|\nabla u_\epsilon|^2)=-2w^1_{\epsilon,1}(1-|w_\epsilon|^2)=2w^2_{\epsilon,2}(1-|w_\epsilon|^2). \tag{33}$$

Proceeding formally and absorbing $\Psi_{1,1}(w_\epsilon)$ into the test function ζ (strictly speaking we can not do this because $\Psi_{1,1}(w_\epsilon)$ depends on ϵ, however this can be overcome with estimate (27)) we have that since $\nabla\cdot\left[\widetilde{\Sigma}_{\epsilon_1\epsilon_2}(\nabla u^{\perp})\right]$ vanishes so

$$\int_{\Omega'}(1-|w_\epsilon|^2)\Psi_{1,1}(w_\epsilon)w^1_{\epsilon,1}\zeta\,dx\approx 0.$$

In the same way $\int_{\Omega'}(1-|w_\epsilon|^2)\Psi_{2,2}(w_\epsilon)w^2_{\epsilon,2}\zeta\,dx\approx 0$ and, since $\nabla\cdot\left[\widetilde{\Sigma}_{e_1e_2}(\nabla u^{\perp})\right]=0$ and $\Psi_{1,2}=\Psi_{2,1}$,

$$\int_{\Omega'}(1-|w_\epsilon|^2)\left(\Psi_{1,2}(w_\epsilon)w^2_{\epsilon,1}+\Psi_{2,1}(w_\epsilon)w^1_{\epsilon,2}\right)\zeta\,dx=\int_{\Omega'}(1-|w_\epsilon|^2)\Psi_{1,2}(w_\epsilon)\left(w^2_{\epsilon,1}+w^1_{\epsilon,2}\right)\zeta\,dx\approx 0.$$

Thus $\nabla\cdot\left[\Phi(\nabla u^{\perp})\right]=0$ for all harmonic entropies.

Step 2. Estimate (29) holds: As we can see the real issue of getting the divergences of entropies to vanish from hypothesis (15) is the term $(1-|w_\epsilon|^2)\left(\Psi_{1,2}(w_\epsilon)w^2_{\epsilon,1}+\Psi_{2,1}(w_\epsilon)w^1_{\epsilon,2}\right)$. Given that we started with just two entropies $\widetilde{\Sigma}_{e_1e_2}$ and $\widetilde{\Sigma}_{\epsilon_1\epsilon_2}$ whose divergence vanishes (when applied to $\nabla u^{\perp}$) and end up with an entire class of entropies (what we call *harmonic entropies*) whose divergence vanishes, the natural way to proceed is to attempt to use our class of harmonic entropies to further expand into a larger class of vanishing entropies. So what we need is a harmonic entropy to deal with terms of the form $\Psi_{1,2}(w_\epsilon)w^2_{\epsilon,1} + \Psi_{2,1}(w_\epsilon)w^1_{\epsilon,2}$. It turns out there is a harmonic entropy that serves this purpose.

Now notice that

$$\begin{aligned}
&\int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \left[\Psi_{1,2}(w_\epsilon)w^2_{\epsilon,1} + \Psi_{2,1}(w_\epsilon)w^1_{\epsilon,2}\right] \zeta \, dx \\
&\quad = \int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \frac{\left(\Psi_{1,2}(w_\epsilon) + \Psi_{2,1}(w_\epsilon)\right)}{2} \left(w^2_{\epsilon,1} + w^1_{\epsilon,2}\right) \zeta \, dx \\
&\qquad + \int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \frac{\left(\Psi_{1,2}(w_\epsilon) - \Psi_{2,1}(w_\epsilon)\right)}{2} \left(w^2_{\epsilon,1} - w^1_{\epsilon,2}\right) \zeta \, dx.
\end{aligned}$$

The first term can be dealt with by absorbing $\frac{(\Psi_{1,2}(w_\epsilon)+\Psi_{2,1}(w_\epsilon))}{2}$ into ζ as before then applying (32). So the term we have to deal with is the latter term. Now if φ is related to Φ by (30) it is a calculation to see that $\Psi_{1,2}(z) - \Psi_{2,1}(z) \overset{(97)}{=} \frac{1}{2}\nabla(\Delta\varphi(z)) \cdot z^\perp =: \psi(z)$. So

$$\begin{aligned}
&\int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \frac{\left(\Psi_{1,2}(w_\epsilon) - \Psi_{2,1}(w_\epsilon)\right)}{2} \left(w^2_{\epsilon,1} - w^1_{\epsilon,2}\right) \zeta \, dx \\
&\qquad = \frac{1}{2} \int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \psi(w_\epsilon) \left(w^2_{\epsilon,1} - w^1_{\epsilon,2}\right) \zeta \, dx \\
&\qquad = \frac{1}{2} \int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \psi(w_\epsilon) \, \Delta u_\epsilon \zeta \, dx.
\end{aligned} \tag{34}$$

Thus what we need is a harmonic entropy that includes the term Δu_ϵ. Now taking $\varphi(z) = z_1^2 - z_2^2$, via formula (30) we obtain entropy $\Phi_0(z) = (z_1^3 + 3z_1z_2^2, -3z_1^2z_2 - z_2^3)$ and a short calculation gives

$$\nabla \cdot [\Phi_0(w_\epsilon)] = -6\left(u_{\epsilon,1}u_{\epsilon,2}\Delta u_\epsilon + |\nabla u_\epsilon|^2 u_{\epsilon,12}\right).$$

Now it is a calculation (see (152)) using (31) to write

$$\begin{aligned}
&\int_{\Omega'} \left(1 - |\nabla u_\epsilon|^2\right) \left(u_{\epsilon,1}u_{\epsilon,2}\Delta u_\epsilon + |\nabla u_\epsilon|^2 u_{\epsilon,12}\right) \zeta \, dx \\
&= -\frac{1}{12} \int_{\Omega'} \nabla \cdot \left[\Psi_0(w_\epsilon)\left(1 - |w_\epsilon|^2\right)^2\right] \zeta \, dx + \frac{1}{12} \int_{\Omega'} \left(1 - |w_\epsilon|^2\right)^2 \nabla \cdot [\Psi_0(w_\epsilon)] \zeta \, dx.
\end{aligned}$$

The first term can be dealt with by integration by parts, and the second can be controlled via estimate (27). It follows that

$$\int_{\Omega'} \left(1 - |\nabla u_\epsilon|^2\right) \left(u_{\epsilon,1}u_{\epsilon,2}\Delta u_\epsilon + |\nabla u_\epsilon|^2 u_{\epsilon,12}\right) \zeta \, dx \to 0 \text{ as } \epsilon \to 0.$$

Now as $|\nabla u| = 1$ a.e. we have

$$\begin{aligned}
\int_{\Omega'} \left(1 - |\nabla u_\epsilon|^2\right) |\nabla u_\epsilon|^2 u_{\epsilon,12} \zeta \, dx \;&\approx\; \int_{\Omega'} \left(1 - |\nabla u_\epsilon|^2\right) u_{\epsilon,12} \zeta \, dx \\
&\overset{(33)}{=} \frac{1}{2} \int_{\Omega'} \nabla \cdot \left[\Sigma_{\epsilon_1\epsilon_2} u_\epsilon\right] \zeta \, dx \to 0 \quad \text{as } \epsilon \to 0.
\end{aligned}$$

So

$$\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,1}u_{\epsilon,2}\Delta u_\epsilon \zeta\, dx \to 0 \quad \text{ as } \epsilon \to 0. \tag{35}$$

Now from (34), using $w_\epsilon = \nabla u_\epsilon^\perp$, we can write

$$\int_{\Omega'} \left(1-|w_\epsilon|^2\right) \frac{\left(\Psi_{1,2}(w_\epsilon)-\Psi_{2,1}(w_\epsilon)\right)}{2} \left(w^2_{\epsilon,1}-w^1_{\epsilon,2}\right)\zeta\, dx = \frac{1}{2}\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,1}u_{\epsilon,2}\Delta u_\epsilon \frac{\psi(w_\epsilon)\,\zeta}{u_{\epsilon,1}u_{\epsilon,2}} dx.$$

It turns out that if

$$\frac{\psi(w_\epsilon)}{u_{\epsilon,1}u_{\epsilon,2}} \text{ remains uniformly bounded for small } \epsilon > 0, \tag{36}$$

then it can be absorbed (via estimate (27)) into ζ, and as a result of (35) we have

$$\int_{\Omega'} \left(1-|w_\epsilon|^2\right) \frac{\left(\Psi_{1,2}(w_\epsilon)-\Psi_{2,1}(w_\epsilon)\right)}{2} \left(w^2_{\epsilon,1}-w^1_{\epsilon,2}\right)\zeta\, dx \to 0 \quad \text{ as } \epsilon \to 0. \tag{37}$$

Hence the estimate (37) holds as long as (36) holds true. So we need to restrict ourselves to a class of entropies for which (36) is true. The key point is that for the sequence of entropies $\{\Phi^k\}$ that approximates Φ^ξ (for $\xi \in \mathbb{S}^1 \setminus \{e_1, -e_1, e_2, -e_2\}$) we can guarantee that (36) holds true. Thus we can establish (29).

Sketch of proof completed. The choice of coordinate system axis $\{e_1, e_2\}$ in (29) is completely arbitrary. We could have carried out the proof with the coordinate system axis $\{\epsilon_1, \epsilon_2\}$ and could then conclude (29) for any $\xi \in \mathbb{S}^1 \setminus \{\epsilon_1, -\epsilon_1, \epsilon_2, -\epsilon_2\}$. Thus (29) holds from any $\xi \in \mathbb{S}^1$ and therefore (12) holds true and regularity follows by Theorem 1.

3. Background

In this section we provide some background. Any two by two matrix can be uniquely decomposed into conformal and anticonformal parts as follows

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a_{11}+a_{22} & -(a_{21}-a_{12}) \\ a_{21}-a_{12} & a_{11}+a_{22} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} a_{11}-a_{22} & a_{21}+a_{12} \\ a_{21}+a_{12} & -(a_{11}-a_{22}) \end{pmatrix}.$$

So for a matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$, define

$$[A]_c := \frac{1}{2}\begin{pmatrix} a_{11}+a_{22} & -(a_{21}-a_{12}) \\ a_{21}-a_{12} & a_{11}+a_{22} \end{pmatrix} \text{ and } [A]_a := \frac{1}{2}\begin{pmatrix} a_{11}-a_{22} & a_{21}+a_{12} \\ a_{21}+a_{12} & -(a_{11}-a_{22}) \end{pmatrix}. \tag{38}$$

It is easy to see that

$$\det(A) = \det([A]_c) + \det([A]_a). \tag{39}$$

Given $w : \Omega \to \mathbb{R}^2$ such that $w(x_1, x_2) = (u(x_1, x_2), v(x_1, x_2))$, for $z = x_1 + i x_2$, let $\varpi(z) = u(x_1, x_2) + i v(x_1, x_2)$. Note that $\frac{\partial \varpi}{\partial \bar{z}}(z) = \frac{1}{2}(\frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2})\varpi = \frac{1}{2}(u_{,1} - v_{,2}) + \frac{i}{2}(v_{,1} + u_{,2})$ and $\frac{\partial \varpi}{\partial z}(z) = \frac{1}{2}(\frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2})\varpi = \frac{1}{2}(u_{,1} + v_{,2}) + \frac{i}{2}(v_{,1} - u_{,2})$. Now identifying complex numbers with conformal matrices in the standard way

$$[x_1 + i x_2]_M = \begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix}, \tag{40}$$

we have that

$$[Dw(x)]_a = \left[\frac{\partial \varpi}{\partial \bar{z}}(z)\right]_M \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } [Dw(x)]_c = \left[\frac{\partial \varpi}{\partial z}(z)\right]_M. \tag{41}$$

4. Proof of Theorem 5

Lemma 6. *Let Ω and $\widetilde{\Omega}$ be as in Theorem 5. Define*

$$K := \left\{ \begin{pmatrix} \frac{2}{3}\sin^3(\theta) & \frac{2}{3}\cos^3(\theta) \\ -\cos(\theta)\left(1-\frac{2}{3}\cos^2(\theta)\right) & \sin(\theta)\left(1-\frac{2}{3}\sin^2(\theta)\right) \end{pmatrix} : \theta \in [0, 2\pi) \right\}. \tag{42}$$

Let a map $F = (F_1, F_2) \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ and a function $f \in W^{1,\infty}(\widetilde{\Omega}; \mathbb{C})$ be related by $f(x_1 + ix_2) = F_1(x_1, x_2) + iF_2(x_1, x_2)$. Then $DF \in K$ at $x \in \Omega$ if and only if f satisfies the following non-linear Beltrami equation and constraint at $z = x_1 + ix_2 \in \widetilde{\Omega}$:

$$\frac{\partial f}{\partial \bar{z}}(z) = \frac{4}{3}\left(\frac{\partial f}{\partial z}(z)\right)^3, \quad \left|\frac{\partial f}{\partial z}(z)\right| = \frac{1}{2}. \tag{43}$$

Proof of Lemma 6. First assume $x \in \Omega$ is such that $DF(x) \in K$. We show that f satisfies (43) at $z = x_1 + ix_2$. Note that since $DF(x) \in K$, there exists $\theta \in [0, 2\pi)$ such that

$$DF(x) = \begin{pmatrix} \frac{2}{3}\sin^3(\theta) & \frac{2}{3}\cos^3(\theta) \\ -\cos(\theta)\left(1-\frac{2}{3}\cos^2(\theta)\right) & \sin(\theta)\left(1-\frac{2}{3}\sin^2(\theta)\right) \end{pmatrix}. \tag{44}$$

As described in Section 3, for any matrix A, we decompose $A = [A]_c + [A]_a$, where $[A]_c$ and $[A]_a$ are the conformal and anticonformal parts of A, respectively. Using (38) and (44) we have

$$[DF(x)]_c = \frac{1}{2}\begin{pmatrix} \sin(\theta) & \cos(\theta) \\ -\cos(\theta) & \sin(\theta) \end{pmatrix}. \tag{45}$$

Now recalling the trigonometry

$$\sin(3\theta) = -4\sin^3(\theta) + 3\sin(\theta), \ \cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta). \tag{46}$$

Note that

$$\begin{aligned} [DF(x)]_a &= \frac{1}{2}\begin{pmatrix} \frac{4}{3}\sin^3(\theta) - \sin(\theta) & \frac{4}{3}\cos^3(\theta) - \cos(\theta) \\ \frac{4}{3}\cos^3(\theta) - \cos(\theta) & -\left(\frac{4}{3}\sin^3(\theta) - \sin(\theta)\right) \end{pmatrix} \\ &\overset{(46)}{=} \frac{1}{2}\begin{pmatrix} -\frac{1}{3}\sin(3\theta) & \frac{1}{3}\cos(3\theta) \\ \frac{1}{3}\cos(3\theta) & \frac{1}{3}\sin(3\theta) \end{pmatrix} \\ &= \frac{1}{6}\begin{pmatrix} -\sin(3\theta) & \cos(3\theta) \\ \cos(3\theta) & \sin(3\theta) \end{pmatrix}. \end{aligned} \tag{47}$$

Recall that $f(x_1 + ix_2) = F_1(x_1, x_2) + iF_2(x_1, x_2)$. It follows from (41) that

$$\left[\frac{\partial f}{\partial z}(z)\right]_M \overset{(41)}{=} [DF(x)]_c \quad \text{and} \quad \left[\frac{\partial f}{\partial \bar{z}}(z)\right]_M \overset{(41)}{=} [DF(x)]_a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{48}$$

Thus

$$\frac{\partial f}{\partial z}(z) \overset{(40),(45),(48)}{=} \frac{1}{2}\left(\sin(\theta) - i\cos(\theta)\right), \tag{49}$$

$$[DF(x)]_a \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overset{(47)}{=} \frac{1}{6}\begin{pmatrix} -\sin(3\theta) & -\cos(3\theta) \\ \cos(3\theta) & -\sin(3\theta) \end{pmatrix}. \tag{50}$$

Therefore it follows that

$$\begin{aligned}\frac{\partial f}{\partial \bar z}(z) &\overset{(48),(50),(40)}{=} -\frac{1}{6}\left(\sin(3\theta)-i\cos(3\theta)\right)\\ &= \frac{1}{6}\left(\sin(\theta)-i\cos(\theta)\right)^3\\ &\overset{(49)}{=} \frac{1}{6}\left(2\frac{\partial f}{\partial z}(z)\right)^3\\ &= \frac{4}{3}\left(\frac{\partial f}{\partial z}(z)\right)^3. \end{aligned}\tag{51}$$

We obtain from (51) and (49) that f satisfies the constrained non-linear Beltrami equation (43) at $z \in \widetilde{\Omega}$.

Conversely, suppose the function $f \in W^{1,\infty}(\widetilde{\Omega};\mathbb{C})$ satisfies (43) at $z = x_1 + ix_2$. Recall that $F(x_1,x_2) = (\mathrm{Re}(f(x_1+ix_2)), \mathrm{Im}(f(x_1+ix_2)))$. We will show that $DF(x) \in K$. Indeed, we have

$$\frac{\partial f}{\partial z} = \frac{1}{2}\Big[\left(F_{1,1}+F_{2,2}\right) + i\left(F_{2,1}-F_{1,2}\right)\Big]\tag{52}$$

and

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\Big[\left(F_{1,1}-F_{2,2}\right) + i\left(F_{2,1}+F_{1,2}\right)\Big].\tag{53}$$

Since $\left|\frac{\partial f}{\partial z}(z)\right| = \frac{1}{2}$, there exists $\theta \in [0, 2\pi)$ such that

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\cos(\theta)+i\sin(\theta)\right).\tag{54}$$

Now since f satisfies (43) at z, we have

$$\begin{aligned}\frac{\partial f}{\partial \bar z}(z) &= \frac{4}{3}\left(\frac{1}{2}\left(\cos(\theta)+i\sin(\theta)\right)\right)^3\\ &= \frac{1}{6}\Big(\cos(3\theta)+i\sin(3\theta)\Big).\end{aligned}\tag{55}$$

Now we obtain from (52)–(55) that

$$\begin{aligned} F_{1,1}+F_{2,2} &= \cos(\theta),\\ F_{2,1}-F_{1,2} &= \sin(\theta),\\ F_{1,1}-F_{2,2} &= \frac{1}{3}\cos(3\theta),\\ F_{2,1}+F_{1,2} &= \frac{1}{3}\sin(3\theta).\end{aligned}\tag{56}$$

So solving (56) for $F_{1,1}, F_{1,2}, F_{2,1}, F_{2,2}$, we obtain

$$\begin{aligned} F_{1,1} &= \frac{1}{2}\cos(\theta)+\frac{1}{6}\cos(3\theta) \overset{(46)}{=} \frac{2}{3}\cos^3(\theta),\\ F_{1,2} &= \frac{1}{6}\sin(3\theta)-\frac{1}{2}\sin(\theta) \overset{(46)}{=} -\frac{2}{3}\sin^3(\theta),\\ F_{2,1} &= \frac{1}{2}\sin(\theta)+\frac{1}{6}\sin(3\theta) \overset{(46)}{=} \sin(\theta)-\frac{2}{3}\sin^3(\theta),\\ F_{2,2} &= \frac{1}{2}\cos(\theta)-\frac{1}{6}\cos(3\theta) \overset{(46)}{=} \cos(\theta)-\frac{2}{3}\cos^3(\theta).\end{aligned}$$

Now letting $\tilde\theta = \frac{\pi}{2}+\theta$, we have $\cos(\tilde\theta) = -\sin(\theta)$ and $\sin(\tilde\theta) = \cos(\theta)$. One can check immediately that $DF \in K$ at $x = (x_1, x_2)$ with the phase function $\widetilde{\theta}$. $\square$

4.1. Proof of Theorem 5 completed

Firstly given $u \in E(\Omega)$ we can define $F_u : \Omega \to \mathbb{R}^2$ by $F_u(0,0) = (0,0)$ and

$$DF_u = \begin{pmatrix} u_{,2}\left(1 - u_{,1}^2 - \frac{u_{,2}^2}{3}\right) & u_{,1}\left(1 - u_{,2}^2 - \frac{u_{,1}^2}{3}\right) \\ -u_{,1}\left(1 - \frac{2u_{,1}^2}{3}\right) & u_{,2}\left(1 - \frac{2u_{,2}^2}{3}\right) \end{pmatrix}. \tag{57}$$

The existence of F_u over bounded simply-connected Lipschitz domains in the classical L^2 framework can be found in [22]. We provide a proof of the existence of F_u over bounded simply-connected domains in Lemma 23 in the Appendix. Such results might be well-known to experts, but we were not able to find a reference. Therefore we include a proof for the convenience of the readers. Since $|\nabla u| = 1$ a.e. in Ω, it is clear that $F_u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ is a mapping that satisfies

$$DF_u \in \left\{ \begin{pmatrix} \sin(\theta)(1 - \cos^2(\theta) - \frac{\sin^2(\theta)}{3}) & \cos(\theta)(1 - \sin^2(\theta) - \frac{\cos^2(\theta)}{3}) \\ -\cos(\theta)\left(1 - \frac{2}{3}\cos^2(\theta)\right) & \sin(\theta)\left(1 - \frac{2}{3}\sin^2(\theta)\right) \end{pmatrix} : \theta \in [0, 2\pi) \right\} \overset{(42)}{=} K \text{ a.e. in } \Omega.$$

Thus applying Lemma 6 we have that $f_u(x_1 + ix_2) := F_u^1(x_1, x_2) + iF_u^2(x_1, x_2)$ satisfies the non-linear Beltrami system (26). So defining

$$\Gamma(u) := f_u, \tag{58}$$

we have that Γ forms a transformation of $\left[E(\Omega)/\mathbb{R}\right]$ into $B(\widetilde{\Omega})$. Now we show that Γ is injective. Given $u, w \in \left[E(\Omega)/\mathbb{R}\right]$ such that $\Gamma(u) = \Gamma(w)$, we have $DF_u = DF_w$. Note that for all $x \in \Omega$ such that $|\nabla u(x)| = 1$, we deduce from (57) that

$$u_{,1} = F_{u,2}^1 - F_{u,1}^2 \quad \text{ and } \quad u_{,2} = F_{u,1}^1 + F_{u,2}^2. \tag{59}$$

The same relations hold for ∇w. This implies $\nabla u = \nabla w$ a.e. in Ω and hence $u = w$ in $\left[E(\Omega)/\mathbb{R}\right]$. Thus we have shown that Γ is injective.

Now for the second part of the theorem, given a function $f \in B(\widetilde{\Omega}) \cap W^{2,1}(\widetilde{\Omega})$ we need to show that there exists some $u \in \left[E(\Omega)/\mathbb{R}\right] \cap W^{2,1}(\widetilde{\Omega})$ such that $\Gamma(u) = f$. Let us define

$$F(x_1, x_2) = (\mathrm{Re}(f(x_1 + ix_2)), \mathrm{Im}(f(x_1 + ix_2))).$$

By Lemma 6 we have

$$DF \in K \ a.e. \text{ in } \Omega.$$

We have that $DF \in W^{1,1}(\Omega)$ and there exists $\theta(x) : \Omega \to [0, 2\pi)$ such that

$$DF(x) = \begin{pmatrix} \sin(\theta(x))\left(1 - \cos^2(\theta(x)) - \frac{\sin^2(\theta(x))}{3}\right) & \cos(\theta(x))\left(1 - \sin^2(\theta(x)) - \frac{\cos^2(\theta(x))}{3}\right) \\ -\cos(\theta(x))\left(1 - \frac{2}{3}\cos^2(\theta(x))\right) & \sin(\theta(x))\left(1 - \frac{2}{3}\sin^2(\theta(x))\right) \end{pmatrix} \tag{60}$$

for a.e. $x \in \Omega$. Similar to (59), we deduce from (60) that

$$\cos(\theta(x)) = F_{1,2}(x) - F_{2,1}(x) \quad \text{ and } \quad \sin(\theta(x)) = F_{1,1}(x) + F_{2,2}(x) \quad \text{a.e. in } \Omega.$$

Hence $\alpha(x) := \cos(\theta(x))$ and $\beta(x) := \sin(\theta(x))$ are such that $\alpha, \beta \in W^{1,1}(\Omega)$. Now we have, for a.e. $x \in \Omega$,

$$\begin{aligned} 0 = \mathrm{curl}\,(\nabla F_1) &= \mathrm{curl}\left(\beta(x)\left(1 - \alpha(x)^2 - \frac{\beta(x)^2}{3}\right), \alpha(x)\left(1 - \beta(x)^2 - \frac{\alpha(x)^2}{3}\right)\right) \\ &= \left(1 - \alpha(x)^2 - \beta(x)^2\right)\left(\alpha_{,1}(x) - \beta_{,2}(x)\right) + 2\alpha(x)\beta(x)\left(\alpha_{,2}(x) - \beta_{,1}(x)\right) \\ &= 2\alpha(x)\beta(x)\left(\alpha_{,2}(x) - \beta_{,1}(x)\right), \end{aligned} \tag{61}$$

and

$$
\begin{aligned}
0 = \operatorname{curl}(\nabla F_2) &= \operatorname{curl}\left(-\alpha(x)\left(1-\frac{2}{3}\alpha(x)^2\right), \beta(x)\left(1-\frac{2}{3}\beta(x)^2\right)\right) \\
&= \beta_{,1}(x)\left(1-2\beta(x)^2\right) + \alpha_{,2}(x)\left(1-2\alpha(x)^2\right) \\
&= \beta_{,1}(x)\left(1-2\beta(x)^2\right) + \beta_{,1}(x)\left(1-2\alpha(x)^2\right) + \left(\alpha_{,2}(x)-\beta_{,1}(x)\right)\left(1-2\alpha(x)^2\right) \\
&= 2\beta_{,1}(x)\left(1-\beta(x)^2-\alpha(x)^2\right) + \left(\alpha_{,2}(x)-\beta_{,1}(x)\right)\left(\beta(x)^2-\alpha(x)^2\right) \\
&= \left(\alpha_{,2}(x)-\beta_{,1}(x)\right)\left(\beta(x)^2-\alpha(x)^2\right).
\end{aligned} \tag{62}
$$

Taking the squares of (61) and (62) and adding, and using the fact that $\alpha(x)^2+\beta(x)^2=1$, we have

$$
\begin{aligned}
0 &= \left[\left(\alpha(x)^2-\beta(x)^2\right)^2 + 4\alpha(x)^2\beta(x)^2\right]|\operatorname{curl}(\alpha(x),\beta(x))|^2 \\
&= \left(\alpha(x)^2+\beta(x)^2\right)^2 |\operatorname{curl}(\alpha(x),\beta(x))|^2 = |\operatorname{curl}(\alpha(x),\beta(x))|^2 \text{ for a.e. } x\in\Omega.
\end{aligned}
$$

Therefore, we have

$$
\operatorname{curl}(\alpha(x),\beta(x)) = 0 \quad \text{for a.e. } x\in\Omega.
$$

Since $(\alpha(x),\beta(x))\in L^\infty(\Omega;\mathbb{R}^2)$, by Lemma 23 in the Appendix, there exists $u\in H^1(\Omega)$ such that $\nabla u=(\alpha,\beta)=(\cos(\theta),\sin(\theta))$. Since $\alpha(x)^2+\beta(x)^2=1$, it is clear that u also belongs to $W^{1,\infty}(\Omega)$. This along with (60) and the fact that $\alpha,\beta\in W^{1,1}(\Omega)$ implies that $u\in\left[E(\Omega)/\mathbb{R}\right]\cap W^{2,1}(\Omega)$. Now looking at (60) and the definition of Γ in (58), it is clear that $\Gamma(u)=f$. Hence, this completes the proof of the bijective part of the theorem.

5. Proof of Theorem 4

Define $F(x_1,x_2)=(\operatorname{Re}(f(x_1+ix_2)),\operatorname{Im}(f(x_1+ix_2)))$. By Lemma 6 the function F satisfies the differential inclusion $DF\in K$ a.e. in Ω, where K is the subset of all two by two matrices defined by (42). Let $\Omega=\left\{(x_1,x_2): x_1+ix_2\in\widetilde{\Omega}\right\}$. Define

$$
M(\theta) := \begin{pmatrix} \frac{2}{3}\sin^3(\theta) & \frac{2}{3}\cos^3(\theta) \\ -\cos(\theta)\left(1-\frac{2}{3}\cos^2(\theta)\right) & \sin(\theta)\left(1-\frac{2}{3}\sin^2(\theta)\right) \end{pmatrix}.
$$

By Lemma 6, there exists $\psi:\Omega\to[0,2\pi)$ such that

$$
DF(x) = M(\psi(x)) \text{ for a.e. } x\in\Omega.
$$

Given $\Omega'\subset\subset\Omega$, denote $\gamma:=\operatorname{dist}(\Omega',\partial\Omega)>0$. Let $h\in B_\gamma(0)$ and define

$$
\alpha_h(x) = \psi(x+h)-\psi(x) \text{ for } x\in\Omega'. \tag{63}
$$

First, we prove the following lemma.

Lemma 7. *For all $x\in\Omega'$ and $h\in B_\gamma(0)$ such that $DF(x), DF(x+h)\in K$, we have that*

$$
\det(DF(x+h)-DF(x)) > c_0\,|DF(x+h)-DF(x)|^4, \tag{64}
$$

where the constant c_0 is independent of x and h.

Proof. Given $x\in\Omega'$ and $h\in B_\gamma(0)$ such that $DF(x), DF(x+h)\in K$, we will show the estimate (64) in several steps.

Step 1. We have

$$
\det(DF(x+h)-DF(x)) = \frac{4}{9}-\frac{2}{3}\cos(\alpha_h(x))+\frac{2}{9}\cos^3(\alpha_h(x)) = \frac{\alpha_h^4}{6}+o(\alpha_h^4). \tag{65}
$$

Proof of Step 1. We know

$$\begin{aligned} DF(x) &= [DF(x)]_c + [DF(x)]_a \\ &\overset{(45),(47)}{=} \frac{1}{2}\begin{pmatrix} \sin(\psi(x)) & \cos(\psi(x)) \\ -\cos(\psi(x)) & \sin(\psi(x)) \end{pmatrix} + \frac{1}{6}\begin{pmatrix} -\sin(3\psi(x)) & \cos(3\psi(x)) \\ \cos(3\psi(x)) & \sin(3\psi(x)) \end{pmatrix}. \end{aligned} \tag{66}$$

It follows that

$$\begin{aligned} & DF(x+h) - DF(x) \\ &\overset{(66)}{=} \frac{1}{2}\begin{pmatrix} \sin(\psi(x+h)) - \sin(\psi(x)) & \cos(\psi(x+h)) - \cos(\psi(x)) \\ -\cos(\psi(x+h)) + \cos(\psi(x)) & \sin(\psi(x+h)) - \sin(\psi(x)) \end{pmatrix} \\ &\quad + \frac{1}{6}\begin{pmatrix} -\sin(3\psi(x+h)) + \sin(3\psi(x)) & \cos(3\psi(x+h)) - \cos(3\psi(x)) \\ \cos(3\psi(x+h)) - \cos(3\psi(x)) & \sin(3\psi(x+h)) - \sin(3\psi(x)) \end{pmatrix}. \end{aligned} \tag{67}$$

So using (39) we have

$$\begin{aligned} & \det(DF(x+h) - DF(x)) \\ &= \frac{1}{4}\left((\sin(\psi(x+h)) - \sin(\psi(x)))^2 + (\cos(\psi(x+h)) - \cos(\psi(x)))^2\right) \\ &\quad - \frac{1}{36}\left((\sin(3\psi(x+h)) - \sin(3\psi(x)))^2 + (\cos(3\psi(x+h)) - \cos(3\psi(x)))^2\right) \\ &= \frac{1}{4}\left(2 - 2\sin(\psi(x+h))\sin(\psi(x)) - 2\cos(\psi(x+h))\cos(\psi(x))\right) \\ &\quad - \frac{1}{36}\left(2 - 2\sin(3\psi(x+h))\sin(3\psi(x)) - 2\cos(3\psi(x+h))\cos(3\psi(x))\right). \end{aligned} \tag{68}$$

Recall that $\alpha_h(x)$ is defined by (63), so $\psi(x+h) = \psi(x) + \alpha_h(x)$. Now

$$\sin(\psi(x+h)) = \sin(\psi(x))\cos(\alpha_h(x)) + \cos(\psi(x))\sin(\alpha_h(x)) \tag{69}$$

and

$$\cos(\psi(x+h)) = \cos(\psi(x))\cos(\alpha_h(x)) - \sin(\psi(x))\sin(\alpha_h(x)). \tag{70}$$

Thus

$$\begin{aligned} & 2 - 2\sin(\psi(x+h))\sin(\psi(x)) - 2\cos(\psi(x+h))\cos(\psi(x)) \\ &\overset{(69),(70)}{=} 2 - 2\left(\sin(\psi(x))\cos(\alpha_h(x)) + \cos(\psi(x))\sin(\alpha_h(x))\right)\sin(\psi(x)) \\ &\quad -2\left(\cos(\psi(x))\cos(\alpha_h(x)) - \sin(\psi(x))\sin(\alpha_h(x))\right)\cos(\psi(x)) \\ &= 2\left(1 - \cos(\alpha_h(x))\right). \end{aligned} \tag{71}$$

Note that $3\psi(x+h) - 3\psi(x) = 3\alpha_h(x)$, so $3\psi(x+h) = 3\psi(x) + 3\alpha_h(x)$. Thus

$$\begin{aligned} & 2 - 2\sin(3\psi(x+h))\sin(3\psi(x)) - 2\cos(3\psi(x+h))\cos(3\psi(x)) \\ &= 2 - 2\left(\sin(3\psi(x))\cos(3\alpha_h(x)) + \cos(3\psi(x))\sin(3\alpha_h(x))\right)\sin(3\psi(x)) \\ &\quad -2\left(\cos(3\psi(x))\cos(3\alpha_h(x)) - \sin(3\psi(x))\sin(3\alpha_h(x))\right)\cos(3\psi(x)) \\ &= 2\left(1 - \cos(3\alpha_h(x))\right). \end{aligned} \tag{72}$$

Thus putting (71) and (72) together with (68) we have that

$$\det(DF(x+h) - DF(x)) = \frac{1}{2}\left(1 - \cos(\alpha_h(x))\right) - \frac{1}{18}\left(1 - \cos(3\alpha_h(x))\right).$$

Now $\cos(3\alpha_h(x)) \overset{(46)}{=} 4\cos^3(\alpha_h(x)) - 3\cos(\alpha_h(x))$. So

$$\begin{aligned}\det\left(DF(x+h)-DF(x)\right) &= \frac{1}{2}\left(1-\cos(\alpha_h(x))\right)-\frac{1}{18}\left(1-4\cos^3(\alpha_h(x))+3\cos(\alpha_h(x))\right)\\ &= \frac{4}{9}-\frac{2}{3}\cos(\alpha_h(x))+\frac{2}{9}\cos^3(\alpha_h(x)).\end{aligned}$$

Now since $\cos(\alpha_h(x)) = 1-\frac{\alpha_h^2}{2}+\frac{\alpha_h^4}{24}+o(\alpha_h^4)$, we have

$$\begin{aligned}\frac{4}{9}-\frac{2}{3}\cos(\alpha_h(x))+\frac{2}{9}\cos^3(\alpha_h(x)) &= \frac{4}{9}-\frac{2}{3}\left(1-\frac{\alpha_h^2}{2}+\frac{\alpha_h^4}{24}\right)+\frac{2}{9}\left(1-\frac{\alpha_h^2}{2}+\frac{\alpha_h^4}{24}\right)^3+o(\alpha_h^4)\\ &= -\frac{2}{9}+\frac{\alpha_h^2}{3}-\frac{\alpha_h^4}{36}+\frac{2}{9}\left(1-\frac{3}{2}\alpha_h^2+\frac{7}{8}\alpha_h^4\right)+o(\alpha_h^4)\\ &= \frac{\alpha_h^4}{6}+o(\alpha_h^4)\end{aligned}$$

for $\alpha_h > 0$ sufficiently small.

Step 2. We have

$$|DF(x+h)-DF(x)|^2 = \frac{10}{9}-\frac{2}{3}\cos(\alpha_h(x))-\frac{4}{9}\cos^3(\alpha_h(x)) = \alpha_h^2+o(\alpha_h^2). \tag{73}$$

Proof of Step 2. Now looking at (67), it is clear that the two matrices in the decomposition are orthogonal when they are identified as vectors in $\mathbb{R}^4$. Therefore, using similar calculations as in Step 1, we have

$$\begin{aligned}&|DF(x+h)-DF(x)|^2 \overset{(67)}{=}\\ &\quad \frac{1}{2}\left((\sin(\psi(x+h))-\sin(\psi(x)))^2+(\cos(\psi(x+h))-\cos(\psi(x)))^2\right)\\ &\quad +\frac{1}{18}\left((\sin(3\psi(x+h))-\sin(3\psi(x)))^2+(\cos(3\psi(x+h))-\cos(3\psi(x)))^2\right)\\ &\quad = \frac{1}{2}\left(2-2\sin(\psi(x+h))\sin(\psi(x))-2\cos(\psi(x+h))\cos(\psi(x))\right)\\ &\qquad +\frac{1}{18}\left(2-2\sin(3\psi(x+h))\sin(3\psi(x))-2\cos(3\psi(x+h))\cos(3\psi(x))\right)\\ &\quad \overset{(71),(72)}{=} \left(1-\cos(\alpha_h(x))\right)+\frac{1}{9}\left(1-\cos(3\alpha_h(x))\right)\\ &\quad \overset{(46)}{=} \frac{10}{9}-\frac{2}{3}\cos(\alpha_h(x))-\frac{4}{9}\cos^3(\alpha_h(x)).\end{aligned}$$

When α_h is sufficiently small, we have

$$\frac{10}{9}-\frac{2}{3}\cos(\alpha_h(x))-\frac{4}{9}\cos^3(\alpha_h(x)) = \frac{10}{9}-\frac{2}{3}\left(1-\frac{\alpha_h^2}{2}\right)-\frac{4}{9}\left(1-\frac{\alpha_h^2}{2}\right)^3+o(\alpha_h^2) = \alpha_h^2+o(\alpha_h^2).$$

Step 3. We have

$$\det\left(DF(x+h)-DF(x)\right) > c_0\,|DF(x+h)-DF(x)|^4$$

for some constant c_0 independent of x and h.

Proof of Step 3. It follows from (65) and (73) that there exist $\delta > 0$ and $c_1 > 0$, such that, for all $0 \leq \alpha_h < \delta$, we have

$$\det\left(DF(x+h)-DF(x)\right) > c_1\,|DF(x+h)-DF(x)|^4. \tag{74}$$

Let $\varrho(t) = \frac{4}{9}-\frac{2}{3}\cos(t)+\frac{2}{9}\cos^3(t)$. Then $\varrho'(t) = \frac{2}{3}\sin^3(t)$. Note that $\varrho(0) = 0$ and $\int_0^t \frac{2}{3}\sin^3(s)ds > 0$ for all $t \in (0, 2\pi)$, since $\sin^3(t) > 0$ and $\sin^3(t) = -\sin^3(t+\pi)$ for $t \in (0,\pi)$. Therefore, for all $t \in (0, 2\pi)$, we have $\varrho(t) = \varrho(0)+\int_0^t \frac{2}{3}\sin^3(s)ds > 0$. By periodicity of the function ϱ, it is clear that $\varrho(t) \geq 0$ for all $t \in \mathbb{R}$, and $\varrho(t) = 0$ if and only if $t = 2k\pi$, $k \in \mathbb{Z}$. Similarly, given $0 < \delta < \pi$, by the odd symmetry of $\varrho'(t)$ with respect to $t = \pi$, we have $\int_\delta^t \varrho'(s)ds \geq 0$ for all $\delta \leq t \leq 2\pi - \delta$. As a consequence, for all $\delta \leq t \leq 2\pi - \delta$, we have $\varrho(t) \geq \varrho(\delta)$. Thus for all $\delta \leq \alpha_h \leq 2\pi - \delta$ we have

$$\frac{4}{9} - \frac{2}{3}\cos(\alpha_h(x)) + \frac{2}{9}\cos^3(\alpha_h(x)) \geq \frac{4}{9} - \frac{2}{3}\cos(\delta) + \frac{2}{9}\cos^3(\delta) > 0. \tag{75}$$

Note that $|DF(x+h) - DF(x)|^4$ is uniformly bounded for all x and h such that $DF(x), DF(x+h) \in K$. Therefore, it follows from (65) and (75) that there exists some $c_2 > 0$ such that for all $\delta \leq \alpha_h \leq 2\pi - \delta$

$$\det(DF(x+h) - DF(x)) > c_2 |DF(x+h) - DF(x)|^4. \tag{76}$$

Since $\varrho(t)$ is even with respect to $t = 0$ and periodic with period 2π, and so is the function $\tau(t) = \frac{10}{9} - \frac{2}{3}\cos(t) - \frac{4}{9}\cos^3(t)$, it is clear that the estimate (74) also holds for $2\pi - \delta < \alpha_h < 2\pi$. Combining (74) with (76), and using the periodicity of the functions ϱ and τ, we conclude that

$$\det(DF(x+h) - DF(x)) > c_0 |DF(x+h) - DF(x)|^4,$$

where $c_0 = \min\{c_1, c_2\} > 0$ is independent of x and h. $\square$

Proof of Theorem 4 completed. Here we follow the idea in the proof of Theorem 3 in [32] to show the regularity of Df. Let $\Omega' \subset\subset \Omega$ and $\gamma := \mathrm{dist}(\Omega', \partial\Omega) > 0$. Let $\eta \in C_c^\infty(\Omega)$ be such that $\eta \equiv 1$ on Ω' and $\mathrm{dist}(\mathrm{Spt}(\eta), \partial\Omega) \geq \frac{\gamma}{2}$. Given $e \in \mathbb{S}^1$ and $h \in \mathbb{R}$ satisfying $0 < h < \frac{\gamma}{2}$, we have $DF(x), DF(x+he) \in K$ for a.e. $x \in \mathrm{Spt}(\eta)$. It follows from Lemma 7 that

$$\begin{aligned}\det\left(\eta(x)\frac{DF(x+he) - DF(x)}{h}\right) &= \frac{\eta(x)^2}{h^2}\det(DF(x+he) - DF(x)) \\ &\overset{(64)}{\geq} c_0\eta(x)^2\frac{|DF(x+he) - DF(x)|^4}{h^2} \quad \text{for a.e. } x \in \mathrm{Spt}(\eta).\end{aligned} \tag{77}$$

Using the identity $\det(A+B) = \det(A) + \det(B) + A : \mathrm{Cof}(B)$, where $\mathrm{Cof}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = \begin{pmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11}\end{pmatrix}$, we have

$$\begin{aligned}0 &= \int_\Omega \det\left(D\left(\eta(x)\left(\frac{F(x+he) - F(x)}{h}\right)\right)\right)dx \\ &= \int_\Omega \det\left(D\eta(x) \otimes \left(\frac{F(x+he) - F(x)}{h}\right) + \eta(x)\left(\frac{DF(x+he) - DF(x)}{h}\right)\right)dx \\ &= \int_\Omega \det\left(D\eta(x) \otimes \left(\frac{F(x+he) - F(x)}{h}\right)\right) \\ &\qquad + D\eta(x) \otimes \left(\frac{F(x+he) - F(x)}{h}\right) : \mathrm{Cof}\left(\eta(x)\left(\frac{DF(x+he) - DF(x)}{h}\right)\right) \\ &\qquad + \det\left(\eta(x)\left(\frac{DF(x+he) - DF(x)}{h}\right)\right)dx.\end{aligned}$$

Since $\det(a \otimes b) = 0$ for any $a, b \in \mathbb{R}^2$, the above simplifies to

$$\begin{aligned}0 = \int_\Omega & D\eta(x) \otimes \left(\frac{F(x+he) - F(x)}{h}\right) : \mathrm{Cof}\left(\eta(x)\left(\frac{DF(x+he) - DF(x)}{h}\right)\right) \\ & + \det\left(\eta(x)\left(\frac{DF(x+he) - DF(x)}{h}\right)\right)dx.\end{aligned} \tag{78}$$

Using (77), (78) and Hölder's inequality, we have

$$
\begin{aligned}
&\int_\Omega \eta(x)^2 \frac{|DF(x+he)-DF(x)|^4}{h^2}dx \\
&\overset{(77)}{\leq} \frac{1}{c_0}\int_\Omega \det\left(\eta(x)\left(\frac{DF(x+he)-DF(x)}{h}\right)\right)dx \\
&\overset{(78)}{\leq} \frac{1}{c_0}\int_\Omega \frac{|D\eta(x)|}{\sqrt{h}}\left|\frac{F(x+he)-F(x)}{h}\right||\eta(x)|\left|\frac{DF(x+he)-DF(x)}{\sqrt{h}}\right|dx \\
&\leq \frac{C(\Omega)}{c_0}\|D\eta\|_{L^\infty(\Omega)}\|\sqrt{\eta}\|_{L^\infty(\Omega)}\mathrm{Lip}(F)\frac{1}{\sqrt{h}}\left(\int_\Omega \eta(x)^2\frac{|DF(x+he)-DF(x)|^4}{h^2}dx\right)^{\frac14} \\
&\leq \frac{C(\Omega,\gamma)}{\sqrt{h}}\left(\int_\Omega \eta(x)^2\frac{|DF(x+he)-DF(x)|^4}{h^2}dx\right)^{\frac14},
\end{aligned}
\tag{79}
$$

where the constant $C(\Omega,\gamma)$ depends only on Ω and γ.

Given $\beta\in(0,\frac43)$, it follows from (79) that

$$
\begin{aligned}
&\frac{1}{h^\beta}\int_\Omega \eta(x)^2\frac{|DF(x+he)-DF(x)|^4}{h^2}dx \\
&\overset{(79)}{\leq} \frac{1}{h^\beta}\frac{C(\Omega,\gamma)}{\sqrt{h}}\left(\int_\Omega \eta(x)^2\frac{|DF(x+he)-DF(x)|^4}{h^2}dx\right)^{\frac14} \\
&= \frac{C(\Omega,\gamma)}{h^{\frac12}}\frac{1}{h^{\frac{3\beta}{4}}}\left(\int_\Omega \eta(x)^2\frac{|DF(x+he)-DF(x)|^4}{h^{2+\beta}}dx\right)^{\frac14}.
\end{aligned}
\tag{80}
$$

Note that the above estimate (80) holds for all $e\in\mathbb{S}^1$ and for all $0<h<\frac{\gamma}{2}$. So for $0<R<\frac{\gamma}{2}$ we have

$$
\begin{aligned}
&\int_{B_R}\int_\Omega \eta(x)^2\frac{|DF(x+y)-DF(x)|^4}{|y|^{2+\beta}}dxdy \\
&\overset{(80)}{\leq} C(\Omega,\gamma)\int_{B_R}\frac{1}{|y|^{\frac{3\beta+2}{4}}}\left(\int_\Omega \eta(x)^2\frac{|DF(x+y)-DF(x)|^4}{|y|^{2+\beta}}dx\right)^{\frac14}dy.
\end{aligned}
\tag{81}
$$

Now by Holder's inequality

$$
\begin{aligned}
&\int_{B_R}\frac{1}{|y|^{\frac{3\beta+2}{4}}}\left(\int_\Omega \eta(x)^2\frac{|DF(x+y)-DF(x)|^4}{|y|^{2+\beta}}dx\right)^{\frac14}dy \\
&\leq \left(\int_{B_R}\frac{1}{|y|^{\beta+\frac23}}dy\right)^{\frac34}\left(\int_{B_R}\int_\Omega \eta(x)^2\frac{|DF(x+y)-DF(x)|^4}{|y|^{2+\beta}}dxdy\right)^{\frac14}.
\end{aligned}
\tag{82}
$$

As $\beta\in(0,\frac43)$, let $\delta:=2-(\beta+\frac23)>0$, then $\beta+\frac23=2-\delta$. We have

$$
\int_{B_R}\frac{1}{|y|^{2-\delta}}dy=2\pi\int_0^R\frac{1}{r^{2-\delta}}rdr=2\pi\int_0^R r^{-1+\delta}dr=\frac{2\pi}{\delta}R^\delta.
\tag{83}
$$

Putting this together with (81)–(82) we have

$$
\begin{aligned}
&\int_{B_R}\int_{\Omega} \eta(x)^2 \frac{|DF(x+y)-DF(x)|^4}{|y|^{2+\beta}} dx dy \\
&\overset{(81),(82),(83)}{\leq} C(\Omega,\gamma)\left(\frac{2\pi}{\delta}R^{\delta}\right)^{\frac{3}{4}} \left(\int_{B_R}\int_{\Omega} \eta(x)^2 \frac{|DF(x+y)-DF(x)|^4}{|y|^{2+\beta}} dx dy\right)^{\frac{1}{4}}.
\end{aligned} \tag{84}
$$

Thus, noting $\delta = 2-(\beta+\frac{2}{3})>0$, we deduce from (84) that

$$
\int_{B_R}\int_{\Omega} \eta(x)^2 \frac{|DF(x+y)-DF(x)|^4}{|y|^{2+\beta}} dx dy < C(\Omega,\gamma,\beta) \tag{85}
$$

for some constant $C(\Omega,\gamma,\beta)$ depending only on Ω, γ and β. Note that $\eta(x)\equiv 1$ for $x\in\Omega'$. Therefore, we deduce from (85) that

$$
\int_{B_R}\int_{\Omega'} \frac{|DF(x+y)-DF(x)|^4}{|y|^{2+\beta}} dx dy < C(\Omega,\gamma,\beta).
$$

It follows that

$$
\begin{aligned}
&\int_{\Omega'}\int_{\Omega'} \frac{|DF(x)-DF(w)|^4}{|x-w|^{2+\beta}} dw dx \\
&\leq \int_{\Omega'}\int_{B_R(x)} \frac{|DF(x)-DF(w)|^4}{|x-w|^{2+\beta}} dw dx + \int_{\Omega'}\int_{\Omega'\backslash B_R(x)} \frac{|DF(x)-DF(w)|^4}{|x-w|^{2+\beta}} dw dx \\
&< C(\Omega,\gamma,\beta) + \frac{1}{R^{2+\beta}} \int_{\Omega'}\int_{\Omega'\backslash B_R(x)} |DF(x)-DF(w)|^4 dw dx < C.
\end{aligned}
$$

So this implies $DF\in W^{\frac{\beta}{4},4}(\Omega')$. Recall that $F(x_1,x_2)=(\mathrm{Re}\,(f(x_1+ix_2)),\mathrm{Im}\,(f(x_1+ix_2)))$. Therefore we have established (24).

Now from (79) we have that for any $y\in B_{\frac{\gamma}{2}}(0)$

$$
\int_{\Omega'} \frac{|DF(x+y)-DF(x)|^4}{|y|^2} dx \leq \int_{\Omega} \eta(x)^2 \frac{|DF(x+y)-DF(x)|^4}{|y|^2} dx \overset{(79)}{\leq} \left(\frac{C(\Omega,\gamma)}{\sqrt{|y|}}\right)^{\frac{4}{3}}.
$$

It follows that

$$
\int_{\Omega'} \frac{|DF(x+y)-DF(x)|^4}{|y|^{\frac{4}{3}}} dx \leq \widetilde{C}(\Omega,\gamma). \tag{86}
$$

Given $0<\epsilon<\frac{\gamma}{2}$, integrating the above with respect to y over $B_\epsilon(0)$ and using the fact that $|y|\leq\epsilon$ for all $y\in B_\epsilon(0)$, we obtain

$$\int_{\Omega'}\int_{B_\epsilon(0)} \frac{|DF(x+y)-DF(x)|^4}{\epsilon^{2+\frac{4}{3}}}dydx$$
$$\leq \frac{1}{\epsilon^2}\int_{\Omega'}\int_{B_\epsilon(0)} \frac{|DF(x+y)-DF(x)|^4}{|y|^{\frac{4}{3}}}dydx$$
$$\overset{(86)}{\leq} \frac{\widetilde{C}(\Omega,\gamma)}{\epsilon^2}\int_{B_\epsilon(0)} 1dy = \pi\widetilde{C}(\Omega,\gamma).$$

This establishes (25). □

As a corollary of Theorem 4, we have

Corollary 8. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain and $u \in E(\Omega)$, where $E(\Omega)$ is defined in (22). Then $\nabla u \in W^{\sigma,4}_{loc}(\Omega)$ for all $0<\sigma<\frac{1}{3}$. Further, for any $\Omega' \subset\subset \Omega$, there exists a constant C such that*

$$\int_{\Omega'}\int_{B_\epsilon(0)} \frac{|\nabla u(x+y)-\nabla u(x)|^4}{\epsilon^{2+\frac{4}{3}}}dydx < C \tag{87}$$

for all ϵ sufficiently small, where the above constant C is independent of ϵ.

Proof of Corollary 8. Since $u \in E(\Omega)$, it follows from Theorem 5 that there exists F_u such that

$$DF_u = \begin{pmatrix} u_{,2}\left(1-u_{,1}^2-\frac{u_{,2}^2}{3}\right) & u_{,1}\left(1-u_{,2}^2-\frac{u_{,1}^2}{3}\right) \\ -u_{,1}\left(1-\frac{2u_{,1}^2}{3}\right) & u_{,2}\left(1-\frac{2u_{,2}^2}{3}\right) \end{pmatrix}$$

and therefore $DF_u \in K$ a.e. in Ω, where the space K is defined in (42). Using (59) we have that

$$u_{,1} = F^1_{u,2} - F^2_{u,1} \quad \text{and} \quad u_{,2} = F^1_{u,1} + F^2_{u,2} \quad \text{a.e. in } \Omega. \tag{88}$$

From Theorem 4, we have $DF_u \in W^{\sigma,4}_{loc}(\Omega)$ for all $\sigma < \frac{1}{3}$, and for any $\Omega' \subset\subset \Omega$,

$$\int_{\Omega'}\int_{B_\epsilon(0)} \frac{|DF_u(x+y)-DF_u(x)|^4}{\epsilon^{2+\frac{4}{3}}}dydx < C \tag{89}$$

for some constant C independent of ϵ. It follows from (88) that $\nabla u \in W^{\sigma,4}_{loc}(\Omega)$ for all $\sigma < \frac{1}{3}$.

By the inequality $|A+B|^4 \leq 8(|A|^4+|B|^4)$ we have that

$$\begin{aligned} \left|u_{,1}(x+y)-u_{,1}(x)\right|^4 &\overset{(88)}{=} \left|\left(F^1_{u,2}(x+y)-F^2_{u,1}(x+y)\right)-\left(F^1_{u,2}(x)-F^2_{u,1}(x)\right)\right|^4 \\ &\leq 8\left|F^1_{u,2}(x+y)-F^1_{u,2}(x)\right|^4 + 8\left|F^2_{u,1}(x+y)-F^2_{u,1}(x)\right|^4 \\ &\leq C\,|DF_u(x+y)-DF_u(x)|^4. \end{aligned} \tag{90}$$

In the same way we can show that

$$\left|u_{,2}(x+y)-u_{,2}(x)\right| \leq C\,|DF_u(x+y)-DF_u(x)|^4. \tag{91}$$

Thus

$$|\nabla u(x+y)-\nabla u(x)|^4 \overset{(90),(91)}{\leq} C\,|DF_u(x+y)-DF_u(x)|^4 \tag{92}$$

for some pure constant C. Finally, putting (89) and (92) together, we immediately obtain (87). □

6. Vanishing of harmonic entropies

Recall the definition of entropies in (8). We first recall a few lemmas from [18].

Lemma 9 *([18], Lemma 1). Let $\Phi \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ be an entropy. Then there exists a $\Psi \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ such that*

$$D\Phi(z) + 2\Psi(z) \otimes z \quad \text{is isotropic for all } z. \tag{93}$$

Consequently, we have

$$\Psi_1(z) = -\frac{1}{2z_2}\Phi_{1,2}(z) \quad \text{and} \quad \Psi_2(z) = -\frac{1}{2z_1}\Phi_{2,1}(z). \tag{94}$$

Lemma 10 *([18], Lemma 2). Let $\Phi \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and $\Psi \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ satisfy (93). Let $m \in H^1(\Omega;\mathbb{R}^2)$ satisfy*

$$\nabla \cdot m = 0 \quad \text{a.e. in } \Omega.$$

Then

$$\nabla \cdot [\Phi(m)] = \Psi(m) \cdot \nabla \left(1 - |m|^2\right) \quad \text{a.e. in } \Omega. \tag{95}$$

Lemma 11 *([18], Lemma 3). There is a one-to-one correspondence between entropies $\Phi \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ and functions $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\varphi(0) = 0$ via*

$$\Phi(z) = \varphi(z)z + \left(\nabla\varphi(z) \cdot z^\perp\right) z^\perp, \tag{96}$$

where $z^\perp = (-z_2, z_1)$ is the anticlockwise rotation of z by $\frac{\pi}{2}$.

Using the above lemmas, we have the following relationship between Ψ and φ.

Lemma 12. *Let $\Phi \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ be an entropy, and Ψ and φ be the functions related to Φ through Lemmas 9 and 11, respectively. Then we have*

$$\Psi_{1,2}(z) - \Psi_{2,1}(z) = \frac{1}{2}\nabla(\Delta\varphi) \cdot z^\perp. \tag{97}$$

Proof. Using formula (96), we have

$$\Phi_{1,2}(z) = 2z_2\varphi_{,1}(z) + z_2^2\varphi_{,12}(z) - z_1z_2\varphi_{,22}(z), \quad \Phi_{2,1}(z) = 2z_1\varphi_{,2}(z) + z_1^2\varphi_{,12}(z) - z_1z_2\varphi_{,11}(z).$$

Putting the above into (94), we obtain

$$\Psi_1(z) = -\varphi_{,1}(z) - \frac{z_2}{2}\varphi_{,12}(z) + \frac{z_1}{2}\varphi_{,22}(z), \quad \Psi_2(z) = -\varphi_{,2}(z) - \frac{z_1}{2}\varphi_{,12}(z) + \frac{z_2}{2}\varphi_{,11}(z).$$

By direct calculations, we have

$$\Psi_{1,2}(z) = -\frac{3}{2}\varphi_{,12}(z) - \frac{z_2}{2}\varphi_{,122}(z) + \frac{z_1}{2}\varphi_{,222}(z), \quad \Psi_{2,1}(z) = -\frac{3}{2}\varphi_{,12}(z) - \frac{z_1}{2}\varphi_{,112}(z) + \frac{z_2}{2}\varphi_{,111}(z).$$

Hence, we have

$$\Psi_{1,2}(z) - \Psi_{2,1}(z) = \frac{1}{2}\left(\varphi_{,122} + \varphi_{,111}, \varphi_{,112} + \varphi_{,222}\right) \cdot (-z_2, z_1) = \frac{1}{2}\nabla(\Delta\varphi) \cdot z^\perp. \qquad \square$$

Given a function $u \in W^{1,\infty}(\Omega)$, for all $\epsilon > 0$, we denote $u_\epsilon = u * \rho_\epsilon$, where ρ_ϵ is the standard convolution kernel supported in $B_\epsilon(0) \subset \mathbb{R}^2$. Very often in this paper, we use the following notation

$$w := (\nabla u)^\perp = \left(-u_{,2}, u_{,1}\right). \tag{98}$$

Then we have

$$w_\epsilon = \left(w_\epsilon^1, w_\epsilon^2\right) = \left(-u_{\epsilon,2}, u_{\epsilon,1}\right). \tag{99}$$

The main result of this section is the following theorem.

Theorem 13. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain. Let $\Phi \in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ be an entropy, and $\varphi \in C_c^\infty(\mathbb{R}^2)$ with $\varphi(0)=0$ be the smooth function related to Φ through (96). In addition, we assume that*

$$\nabla(\Delta\varphi)\cdot z^\perp = 0 \quad \textit{for all } z \in \mathbb{R}^2. \tag{100}$$

Then, for all $u \in E(\Omega)$, where $E(\Omega)$ is defined in (22), we have

$$\nabla \cdot \left[\Phi(\nabla u^\perp)\right] = 0$$

in the sense of distributions.

Proof of Theorem 13. Given $\Omega' \subset\subset \Omega$, let $\zeta \in C_c^\infty(\Omega')$ be a test function. Recall $w \overset{(98)}{=} (\nabla u)^\perp = (-u_{,2}, u_{,1})$. So as in Step 6 of the proof of Proposition 3 [15] we have

$$\begin{aligned}
&\int_{\Omega'} \zeta(x)\nabla\cdot[\Phi(w_\epsilon)]\,dx \\
&\quad\overset{(95)}{=} \int_{\Omega'} \zeta(x)\Psi(w_\epsilon)\cdot\nabla\left(1-|w_\epsilon|^2\right)dx \\
&\quad= \overbrace{\int_{\Omega'} \zeta(x)\nabla\cdot\left[\Psi(w_\epsilon)\left(1-|w_\epsilon|^2\right)\right]dx}^{I_\epsilon} - \overbrace{\int_{\Omega'} \zeta(x)\left(1-|w_\epsilon|^2\right)\nabla\cdot[\Psi(w_\epsilon)]\,dx}^{\Pi_\epsilon}.
\end{aligned} \tag{101}$$

Since $\Psi(w_\epsilon)\left(1-|w_\epsilon|^2\right) \xrightarrow{L^1} 0$, integrating by parts we see that

$$I_\epsilon \to 0. \tag{102}$$

In the following, we show that, under the additional assumption (100), we have

$$\Pi_\epsilon \to 0.$$

Thus, we have

$$\int_{\Omega'} \Phi(w)\cdot\nabla\zeta\,dx = \lim_{\epsilon\to 0}\int_{\Omega'} \Phi(w_\epsilon)\cdot\nabla\zeta\,dx = -\lim_{\epsilon\to 0}\int_{\Omega'} \nabla\cdot[\Phi(w_\epsilon)]\,\zeta\,dx = 0,$$

from which Theorem 13 will follow.

We will need several lemmas. First, we provide the following lemma, which will be used repeatedly.

Lemma 14. *Let Ω be as in Theorem 13 and $u \in E(\Omega)$. Given $\Omega' \subset\subset \Omega$, there exists a constant $\epsilon_0 = \epsilon_0(\Omega')$ such that, for all $\epsilon < \epsilon_0$, and for all $r \geq 4$ and $g \in L^r(\Omega')$, we have*

$$\int_{\Omega'} \left|1-|\nabla u_\epsilon|^2\right|\left|u_{\epsilon,mn}\right||g|\,dx \leq C\|g\|_{L^r(\Omega')} \tag{103}$$

for all $m=1,2$ and $n=1,2$, and for some constant C independent of ϵ.

Consequently, if $g_j \to g$ in $L^r(\Omega)$, then for any sequence $\{\epsilon_j\}$ such that $0<\epsilon_j<\epsilon_0$ for all j, we have

$$\int_{\Omega'} \left|1-|\nabla u_{\epsilon_j}|^2\right|\left|u_{\epsilon_j,mn}\right|\left|g-g_j\right|dx \to 0 \quad \textit{as } j\to\infty \tag{104}$$

for all $m=1,2$ and $n=1,2$.

Proof. Given $r \geq 4$ and $g \in L^r(\Omega')$, by Hölder's inequality, we have

$$\int_{\Omega'} \left|1 - |\nabla u_\epsilon|^2\right| \left|u_{\epsilon,mn}\right| |g|\, dx \leq \left(\int_{\Omega'} \left|1 - |\nabla u_\epsilon|^2\right|^{r'} \left|u_{\epsilon,mn}\right|^{r'} dx \right)^{\frac{1}{r'}} \|g\|_{L^r(\Omega')}, \tag{105}$$

where $\frac{1}{r} + \frac{1}{r'} = 1$. Now as in (i) and (ii) of Step 6 of the proof of Proposition 3 [15], we have that

$$1 - |w_\epsilon(x)|^2 \leq \frac{2\|\rho\|_{L^\infty(\mathbb{R}^2)}}{\epsilon^2} \int_{B_\epsilon} |w(x-z) - w(x)|^2\, dz, \tag{106}$$

and

$$\left|w_{\epsilon,j}(x)\right| \leq \frac{\|\nabla \rho\|_{L^\infty(\mathbb{R}^2)}}{\epsilon^3} \int_{B_\epsilon} |w(x-z) - w(x)|\, dz, \tag{107}$$

where recall that we defined the vector fields w and w_ϵ in (98) and (99), respectively. For the convenience of the reader, we take the proofs of (106)–(107) in [15] and put them into Lemma 22 in the Appendix.

Note that since $r \geq 4$, we have $r' \leq \frac{4}{3}$ and, therefore, $\frac{3r'}{4} \leq 1$. Now arguing very similarly to (iii) of Step 6 of Proposition 3 [15], we have

$$\begin{aligned}
&\int_{\Omega'} \left|1 - |\nabla u_\epsilon|^2\right|^{r'} \left|u_{\epsilon,mn}\right|^{r'} dx \\
&\overset{(106),(107)}{\leq} \frac{C}{\epsilon^{r'}} \int_{\Omega'} \left(\fint_{B_\epsilon} |w(x-z) - w(x)|^2\, dz \right)^{r'} \left(\fint_{B_\epsilon} |w(x-z) - w(x)|\, dz \right)^{r'} dx \\
&\leq \frac{C}{\epsilon^{r'}} \int_{\Omega'} \left(\fint_{B_\epsilon} |w(x-z) - w(x)|^4\, dz \right)^{\frac{r'}{2}} \left(\fint_{B_\epsilon} |w(x-z) - w(x)|^4\, dz \right)^{\frac{r'}{4}} dx \\
&= \frac{C}{\epsilon^{r'}} \int_{\Omega'} \left(\fint_{B_\epsilon} |w(x-z) - w(x)|^4\, dz \right)^{\frac{3r'}{4}} dx \\
&\leq \frac{C}{\epsilon^{r'}} \left(\int_{\Omega'} \fint_{B_\epsilon} |w(x-z) - w(x)|^4\, dz dx \right)^{\frac{3r'}{4}} \\
&= C \left(\int_{\Omega'} \int_{B_\epsilon} \frac{|w(x-z) - w(x)|^4}{\epsilon^{2+\frac{4}{3}}} dz dx \right)^{\frac{3r'}{4}} \overset{(87)}{\leq} C.
\end{aligned} \tag{108}$$

Putting (108) into (105), we immediately obtain (103). The estimate (104) is a direct consequence of (103). □

Lemma 15. *Let Ω be as in Theorem 13 and $u \in E(\Omega)$. Denote $w := (\nabla u)^\perp$. Given $\Omega' \subset\subset \Omega$, for all $\zeta \in C_c^\infty(\Omega')$, we have*

$$\int_{\Omega'} (1 - |w_\epsilon(x)|^2) w^1_{\epsilon,1}(x) \zeta(x) dx \to 0, \quad \int_{\Omega'} (1 - |w_\epsilon(x)|^2) w^2_{\epsilon,2}(x) \zeta(x) dx \to 0, \tag{109}$$

and

$$\int_{\Omega'} (1 - |w_\epsilon(x)|^2) \left(w^2_{\epsilon,1}(x) + w^1_{\epsilon,2}(x) \right) \zeta(x) dx \to 0 \tag{110}$$

as $\epsilon \to 0$.

Proof. Given a smooth function v, by direct calculations, we have

$$\nabla \cdot \left[\Sigma_{e_1 e_2} v\right] \overset{(5)}{=} \left(v_{,11} - v_{,22}\right)\left(1 - |\nabla v|^2\right) \tag{111}$$

and

$$\nabla \cdot \left[\Sigma_{\epsilon_1 \epsilon_2} v\right] \overset{(6)}{=} 2v_{,12}\left(1 - |\nabla v|^2\right). \tag{112}$$

Recall the definition of w_ϵ in (99). In particular, we have

$$w^1_{\epsilon,1} = -u_{\epsilon,12}, \quad w^2_{\epsilon,2} = u_{\epsilon,12}, \quad w^2_{\epsilon,1} + w^1_{\epsilon,2} = u_{\epsilon,11} - u_{\epsilon,22}. \tag{113}$$

Thus, using (99), (112) and (113), we have

$$\begin{aligned}
\int_{\Omega'} \left(1 - |w_\epsilon(x)|^2\right) w^1_{\epsilon,1}(x)\zeta(x)dx &\overset{(99),(113)}{=} -\int_{\Omega'} \left(1 - |\nabla u_\epsilon(x)|^2\right) u_{\epsilon,12}(x)\zeta\, dx \\
&\overset{(6),(112)}{=} -\frac{1}{2}\int_{\Omega'} \nabla \cdot \left(u_{\epsilon,2}\left(1 - \frac{2u^2_{\epsilon,2}}{3}\right), u_{\epsilon,1}\left(1 - \frac{2u^2_{\epsilon,1}}{3}\right)\right)\zeta\, dx \\
&= \frac{1}{2}\int_{\Omega'} \left(u_{\epsilon,2}\left(1 - \frac{2u^2_{\epsilon,2}}{3}\right), u_{\epsilon,1}\left(1 - \frac{2u^2_{\epsilon,1}}{3}\right)\right) \cdot \nabla\zeta\, dx.
\end{aligned} \tag{114}$$

Since $u \in W^{1,\infty}(\Omega)$, it follows from (114) and (15) that

$$\begin{aligned}
&\int_{\Omega'} \left(1 - |w_\epsilon(x)|^2\right) w^1_{\epsilon,1}(x)\zeta(x)dx \\
&\overset{(114)}{\to} \frac{1}{2}\int_{\Omega'} \left(u_{,2}\left(1 - \frac{2u^2_{,2}}{3}\right), u_{,1}\left(1 - \frac{2u^2_{,1}}{3}\right)\right) \cdot \nabla\zeta\, dx = \frac{1}{2}\int_{\Omega'} \Sigma_{\epsilon_1\epsilon_2} u(x) \cdot \nabla\zeta(x)dx \overset{(15)}{=} 0.
\end{aligned} \tag{115}$$

Similarly, as $w^2_{\epsilon,2} = u_{\epsilon,12}$, we have

$$\int_{\Omega'} \left(1 - |w_\epsilon(x)|^2\right) w^2_{\epsilon,2}(x)\zeta(x)dx \to 0.$$

Next, using (99), (111) and (113), we have

$$\begin{aligned}
&\int_{\Omega'} \left(1 - |w_\epsilon(x)|^2\right)\left(w^2_{\epsilon,1}(x) + w^1_{\epsilon,2}(x)\right)\zeta(x)dx \\
&\overset{(99),(113)}{=} \int_{\Omega'} \left(1 - |\nabla u_\epsilon(x)|^2\right)\left(u_{\epsilon,11}(x) - u_{\epsilon,22}(x)\right)\zeta(x)dx \\
&\overset{(5),(111)}{=} \frac{1}{2}\int_{\Omega'} \nabla \cdot \left(u_{\epsilon,1}\left(1 - u^2_{\epsilon,2} - \frac{u^2_{\epsilon,1}}{3}\right), -u_{\epsilon,2}\left(1 - u^2_{\epsilon,1} - \frac{u^2_{\epsilon,2}}{3}\right)\right)\zeta\, dx.
\end{aligned}$$

By (15) and the same arguments as in (115), we conclude (110). □

Lemma 16. *Let Ω be as in Theorem 13 and $u \in E(\Omega)$. Denote $w := (\nabla u)^\perp$. Given $\Omega' \subset\subset \Omega$ and any $F \in C^\infty_c(\mathbb{R}^2)$, we have*

$$\int_{\Omega'} \left(1 - |w_\epsilon|^2\right) w^n_{\epsilon,m}\left(F(w) - F(w_\delta)\right) dx \to 0 \quad \text{as } \epsilon, \delta \to 0 \tag{116}$$

for all $m = 1, 2$ and $n = 1, 2$. As a consequence, we have

$$\int_{\Omega'} \left(1 - |w_\epsilon|^2\right) w^n_{\epsilon,m} \left(F(w_\epsilon) - F(w_\delta)\right) dx \to 0 \quad \text{as } \epsilon, \delta \to 0. \tag{117}$$

Proof. First, it is clear that by applying Lemma 14, we have

$$\begin{aligned} &\left| \int_{\Omega'} (1 - |w_\epsilon(x)|^2) \left(F(w(x)) - F(w_\delta(x))\right) w^n_{\epsilon,m}(x) dx \right| \\ &\leq \sup_{\mathbb{R}^2} |DF| \int_{\Omega'} (1 - |w_\epsilon(x)|^2) \left|w^n_{\epsilon,m}(x)\right| |w(x) - w_\delta(x)| \, dx \\ &\overset{(103)}{\leq} C \sup_{\mathbb{R}^2} |DF| \, \|w - w_\delta\|_{L^r(\Omega')} \end{aligned} \tag{118}$$

for all $r \geq 4$. Now as $w \in L^\infty(\Omega) \subset L^r(\Omega)$ so $w_\delta \overset{L^r(\Omega)}{\to} w$. Applying this to (118), equation (116) follows.

To show (117), we write

$$\begin{aligned} &\int_{\Omega'} \left(1 - |w_\epsilon|^2\right) w^n_{\epsilon,m} \left(F(w_\epsilon) - F(w_\delta)\right) dx \\ &\quad = \int_{\Omega'} \left(1 - |w_\epsilon|^2\right) w^n_{\epsilon,m} \left(F(w_\epsilon) - F(w)\right) dx + \int_{\Omega'} \left(1 - |w_\epsilon|^2\right) w^n_{\epsilon,m} \left(F(w) - F(w_\delta)\right) dx. \end{aligned}$$

By applying (116) to the above two terms on the right side, we obtain (117). □

Lemma 17. *Let Ω be as in Theorem 13 and $u \in E(\Omega)$. Denote $w := (\nabla u)^\perp$. Given $\Omega' \subset\subset \Omega$, for any $F \in C^\infty_c(\mathbb{R}^2)$ and any $\zeta \in C^\infty_c(\Omega')$, We have*

$$\int_{\Omega'} \left(1 - |w_\epsilon(x)|^2\right) w^1_{\epsilon,1}(x) F(w_\epsilon(x)) \zeta(x) dx \to 0, \tag{119}$$

$$\int_{\Omega'} \left(1 - |w_\epsilon(x)|^2\right) w^2_{\epsilon,2}(x) F(w_\epsilon(x)) \zeta(x) dx \to 0, \tag{120}$$

and

$$\int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \left(w^2_{\epsilon,1} + w^1_{\epsilon,2}\right) F(w_\epsilon(x)) \zeta(x) dx \to 0 \tag{121}$$

as $\epsilon \to 0$.

Proof. We write

$$\begin{aligned} &\int_{\Omega'} \left(1 - |w_\epsilon(x)|^2\right) F(w_\epsilon(x)) w^1_{\epsilon,1}(x) \zeta(x) dx \\ &= \int_{\Omega'} \left(1 - |w_\epsilon(x)|^2\right) F(w_\delta(x)) w^1_{\epsilon,1}(x) \zeta(x) dx \\ &\quad + \int_{\Omega'} \left(1 - |w_\epsilon(x)|^2\right) \left(F(w_\epsilon(x)) - F(w_\delta(x))\right) w^1_{\epsilon,1}(x) \zeta(x) dx, \end{aligned}$$

where $w_\delta = \rho_\delta * w$. It follows from Lemma 15 (109) that, for any fixed $\delta > 0$,

$$\int_{\Omega'} (1-|w_\epsilon(x)|^2) F(w_\delta(x)) w^1_{\epsilon,1}(x)\zeta(x)dx \to 0 \quad \text{as } \epsilon \to 0. \tag{122}$$

On the other hand, we obtain from Lemma 16 that

$$\int_{\Omega'} (1-|w_\epsilon(x)|^2)\left(F(w_\epsilon(x)) - F(w_\delta(x))\right) w^1_{\epsilon,1}(x)\zeta(x)dx \to 0 \quad \text{as } \epsilon, \delta \to 0. \tag{123}$$

Given $\alpha > 0$, it follows from (123) that there exist $\delta_0 = \delta_0(\alpha)$, $\epsilon_0 = \epsilon_0(\alpha) > 0$ sufficiently small such that, for all $\epsilon < \epsilon_0$, we have

$$\left| \int_{\Omega'} (1-|w_\epsilon(x)|^2)\left(F(w_\epsilon(x)) - F(w_{\delta_0}(x))\right) w^1_{\epsilon,1}(x)\zeta(x)dx \right| < \frac{\alpha}{2}. \tag{124}$$

By (122), there exists $\epsilon_1 (= \epsilon_1(\alpha))$ such that, for all $\epsilon < \epsilon_1$, we have

$$\left| \int_{\Omega'} (1-|w_\epsilon(x)|^2) F(w_{\delta_0}(x)) w^1_{\epsilon,1}(x)\zeta(x)dx \right| < \frac{\alpha}{2}. \tag{125}$$

Define $\epsilon_\alpha := \min\{\epsilon_0, \epsilon_1\}$. Combining (124), (125), we have that, for all $\epsilon < \epsilon_\alpha$

$$\left| \int_{\Omega'} \left(1-|w_\epsilon(x)|^2\right) F(w_\epsilon(x)) w^1_{\epsilon,1}(x)\zeta(x)dx \right| < \alpha. \tag{126}$$

This implies (119). The estimates (120) and (121) follow exactly the same lines. □

Proof of Theorem 13 completed. Now we return to (101). We have

$$\Pi_\epsilon = \int_{\Omega'} \left(1-|w_\epsilon|^2\right)\left[\Psi_{1,1}(w_\epsilon)w^1_{\epsilon,1} + \Psi_{1,2}(w_\epsilon)w^2_{\epsilon,1} + \Psi_{2,1}(w_\epsilon)w^1_{\epsilon,2} + \Psi_{2,2}(w_\epsilon)w^2_{\epsilon,2}\right]\zeta(x)dx. \tag{127}$$

By Lemma 17 (119)–(120), we have

$$\int_{\Omega'} \left(1-|w_\epsilon|^2\right)\left[\Psi_{1,1}(w_\epsilon)w^1_{\epsilon,1} + \Psi_{2,2}(w_\epsilon)w^2_{\epsilon,2}\right]\zeta(x)dx \to 0 \text{ as } \epsilon \to 0. \tag{128}$$

By Lemma 12, the assumption (100) implies $\Psi_{1,2}(w_\epsilon) = \Psi_{2,1}(w_\epsilon)$. Therefore, we have

$$\int_{\Omega'} \left(1-|w_\epsilon|^2\right)\left[\Psi_{1,2}(w_\epsilon)w^2_{\epsilon,1} + \Psi_{2,1}(w_\epsilon)w^1_{\epsilon,2}\right]\zeta(x)dx = \int_{\Omega'} \left(1-|w_\epsilon|^2\right)\Psi_{1,2}(w_\epsilon)\left(w^2_{\epsilon,1} + w^1_{\epsilon,2}\right)\zeta(x)dx. \tag{129}$$

Now applying Lemma 17 (121) to (129) implies that

$$\int_{\Omega'} \left(1-|w_\epsilon|^2\right)\left[\Psi_{1,2}(w_\epsilon)w^2_{\epsilon,1} + \Psi_{2,1}(w_\epsilon)w^1_{\epsilon,2}\right]\zeta(x)dx \to 0 \text{ as } \epsilon \to 0. \tag{130}$$

Finally, putting (128) and (130) into (127), we obtain $\Pi_\epsilon \to 0$ as $\epsilon \to 0$. This together with (101) and (102) completes the proof of Theorem 13. □

7. Vanishing of the special entropies

Given $\xi \in \mathbb{S}^1$, recall the definition of the function Φ^ξ in (28). The main result of this section is the following theorem.

Theorem 18. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain and $u \in E(\Omega)$, where $E(\Omega)$ is defined in (22). Then for every $\xi \in \mathbb{S}^1 \setminus \{e_1, -e_1, e_2, -e_2\}$, we have that*

$$\nabla \cdot [\Phi^\xi(w)] = 0 \quad \text{in the sense of distributions,} \tag{131}$$

where $w(x) = (-u_{,2}(x), u_{,1}(x))$.

We first recall the following lemma from [18].

Lemma 19 *([18], Lemma 4). For a fixed $\xi \in \mathbb{S}^1$, the map Φ^ξ defined in (28) is a generalized entropy in the sense that there exists a sequence $\{\Phi_\nu\}_{\nu\to\infty}$ of entropies in $C_c^\infty(\mathbb{R}^2; \mathbb{R}^2)$ such that*

$$\{\Phi_\nu(z)\}_{\nu\to\infty} \text{ is bounded uniformly for bounded } z,$$

$$\Phi_\nu(z) \to \Phi^\xi(z) \text{ for all } z. \tag{132}$$

For the convenience of the reader, we include the proof of Lemma 19 in the Appendix. Now we provide the proof of Theorem 18.

Proof of Theorem 18. Given $\xi \in \mathbb{S}^1 \setminus \{e_1, -e_1, e_2, -e_2\}$, we may approximate Φ^ξ by smooth entropies Φ^k as in Lemma 19. We prove that

$$\nabla \cdot [\Phi^k(w)] = 0 \text{ in the sense of distributions}$$

for all k sufficiently large. As a result, we have (131).

As can be understood from the proof of Theorem 13, by virtue of Lemma 17 the only thing we need to show is

$$\int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \left[\Psi^k_{1,2}(w_\epsilon) w^2_{\epsilon,1} + \Psi^k_{2,1}(w_\epsilon) w^1_{\epsilon,2}\right] \zeta\, dx \to 0, \tag{133}$$

where the function Ψ^k is related to Φ^k through Lemma 9 and $\zeta \in C_c^\infty(\Omega')$ is any test function. Let us write

$$\begin{aligned}
&\int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \left[\Psi^k_{1,2}(w_\epsilon) w^2_{\epsilon,1} + \Psi^k_{2,1}(w_\epsilon) w^1_{\epsilon,2}\right] \zeta\, dx \\
&= \int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \frac{\Psi^k_{1,2}(w_\epsilon) + \Psi^k_{2,1}(w_\epsilon)}{2} \left(w^2_{\epsilon,1} + w^1_{\epsilon,2}\right) \zeta\, dx \\
&\qquad + \int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \frac{\Psi^k_{1,2}(w_\epsilon) - \Psi^k_{2,1}(w_\epsilon)}{2} \left(w^2_{\epsilon,1} - w^1_{\epsilon,2}\right) \zeta\, dx.
\end{aligned}$$

We deduce from Lemma 17 (121) that

$$\int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \left(w^2_{\epsilon,1} + w^1_{\epsilon,2}\right) \frac{\Psi^k_{1,2}(w_\epsilon) + \Psi^k_{2,1}(w_\epsilon)}{2} \zeta\, dx \to 0 \quad \text{as } \epsilon \to 0. \tag{134}$$

In the following, we show

$$\int_{\Omega'} \left(1 - |w_\epsilon|^2\right) \frac{\Psi^k_{1,2}(w_\epsilon) - \Psi^k_{2,1}(w_\epsilon)}{2} \left(w^2_{\epsilon,1} - w^1_{\epsilon,2}\right) \zeta\, dx \to 0 \quad \text{as } \epsilon \to 0. \tag{135}$$

Let us denote $\psi^k(z) = \frac{\nabla(\Delta\varphi^k)\cdot z^\perp}{4}$, where the function φ^k is related to Φ^k through Lemma 11. Using this new function ψ^k and the calculation (97), we write

$$\begin{aligned}
&\int_{\Omega'} \left(1-|w_\epsilon|^2\right) \frac{\Psi^k_{1,2}(w_\epsilon)-\Psi^k_{2,1}(w_\epsilon)}{2} \left(w^2_{\epsilon,1}-w^1_{\epsilon,2}\right)\zeta\, dx \\
&= \int_{\Omega'} \left(1-|w_\epsilon|^2\right) \psi^k(w) \left(w^2_{\epsilon,1}-w^1_{\epsilon,2}\right)\zeta\, dx + \int_{\Omega'} \left(1-|w_\epsilon|^2\right)\left(\psi^k(w_\epsilon)-\psi^k(w)\right)\left(w^2_{\epsilon,1}-w^1_{\epsilon,2}\right)\zeta\, dx.
\end{aligned} \tag{136}$$

Recall the definition of w_ϵ in (99). For the above first term, we further write

$$\begin{aligned}
&\int_{\Omega'} \left(1-|w_\epsilon|^2\right) \psi^k(w) \left(w^2_{\epsilon,1}-w^1_{\epsilon,2}\right)\zeta\, dx \\
&= \int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) \psi^k(w) \left(u_{\epsilon,11}+u_{\epsilon,22}\right)\zeta\, dx \\
&= \int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) \psi^k(w) \left(u_{\epsilon,11}+u_{\epsilon,22}+\frac{u_{\epsilon,12}}{u_{,1}u_{,2}}\right)\zeta\, dx - \int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,12}\frac{\psi^k(w)}{u_{,1}u_{,2}}\zeta\, dx.
\end{aligned} \tag{137}$$

In the following, we will establish

$$\int_{\Omega'} \left(1-|w_\epsilon|^2\right)\left(\psi^k(w_\epsilon)-\psi^k(w)\right)\left(w^2_{\epsilon,1}-w^1_{\epsilon,2}\right)\zeta\, dx \to 0, \tag{138}$$

$$\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) \psi^k(w) \left(u_{\epsilon,11}+u_{\epsilon,22}+\frac{u_{\epsilon,12}}{u_{,1}u_{,2}}\right)\zeta\, dx \to 0, \tag{139}$$

and

$$\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,12}\frac{\psi^k(w)}{u_{,1}u_{,2}}\zeta\, dx \to 0 \tag{140}$$

as $\epsilon \to 0$, respectively. Putting (136)–(140) together, we obtain (135), which together with (134) gives us (133). This will conclude the proof of the theorem.

First note (138) follows as a direct consequence of Lemma 16. Equations (139), (140) will be established in the following two lemmas. □

Lemma 20. *We have for sufficiently large k*

$$\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,12}\frac{\psi^k(w)}{u_{,1}u_{,2}}\zeta\, dx \to 0 \text{ as } \epsilon \to 0. \tag{141}$$

Proof. A key observation in the proof is that, for k sufficiently large, $\chi^k := \frac{\psi^k(w)}{u_{,1}u_{,2}}$ is an L^∞ function. Indeed, we use smooth entropies Φ^k to approximate the entropy Φ^ξ in the way that is given in the proof of Lemma 19 in the Appendix. In particular, for k sufficiently large, the function φ^k satisfies $D^2\varphi^k = 0$ outside a sufficiently small neighborhood of the line $z\cdot\xi = 0$ inside the ball $B_k(0)$. Consequently, *on* $\mathbb{S}^1$, $\psi^k(z) = \frac{\nabla(\Delta\varphi^k)\cdot z^\perp}{4}$ is supported in a sufficiently small neighborhood of the points $z\cdot\xi = 0$ with $|z| = 1$. Since we have chosen $\xi \in \mathbb{S}^1$ to be such that ξ is not parallel to the axes, for k sufficiently large, the support of $\psi^k(z)$ *on* $\mathbb{S}^1$ is bounded away from the axes. Indeed, let $\alpha > 0$ denote the distance between the support of ψ^k on $\mathbb{S}^1$ and the axes. Then, either $|u_{,i}| < \frac{\alpha}{2}$ for some $i = 1, 2$, so $\psi^k(w) = 0$, or $|u_{,1}| \geq \frac{\alpha}{2}$ and $|u_{,2}| \geq \frac{\alpha}{2}$, so $|\chi^k| \leq \frac{4\|\psi^k\|_\infty}{\alpha^2}$. Therefore, for all $x \in \Omega$ such that $|\nabla u(x)| = 1$, we have $\chi^k(x) \leq C_k$ for some constant C_k depending only on Φ^k. Since $|\nabla u| = 1$ a.e. in Ω, we have $\chi^k \in L^\infty(\Omega)$.

In particular, we have $\chi^k \in L^4(\Omega)$. Let $\{\chi_j\}$ be a sequence of smooth functions such that

$$\chi_j \to \chi^k \quad \text{in } L^4(\Omega).$$

Then we have

$$\begin{aligned}&\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,12} \frac{\psi^k(w)}{u_{,1}u_{,2}} \zeta\, dx \\ &= \int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,12} \chi_j \zeta\, dx + \int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,12} \left(\chi^k - \chi_j\right) \zeta\, dx.\end{aligned} \tag{142}$$

It follows from Lemma 14 that

$$\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,12} \left(\chi^k - \chi_j\right) \zeta\, dx \to 0 \quad \text{as } \epsilon \to 0,\ j \to \infty. \tag{143}$$

On the other hand, we have $\chi_j \zeta \in C_c^\infty(\Omega')$. It follows from Lemma 15 (noting the relationship between w_ϵ and u_ϵ as in (99)) that

$$\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,12} \chi_j \zeta\, dx \to 0 \quad \text{as } \epsilon \to 0. \tag{144}$$

Putting (142)–(144) together and using arguments similar to those in (124)–(126), we obtain (141). □

Lemma 21. *We have*

$$\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) \psi^k(w) \left(u_{\epsilon,11} + u_{\epsilon,22} + \frac{u_{\epsilon,12}}{u_{,1}u_{,2}}\right) \zeta\, dx \to 0 \textit{ as } \epsilon \to 0.$$

Proof. Recall that we defined $\chi^k := \frac{\psi^k(w)}{u_{,1}u_{,2}} \in L^\infty(\Omega)$. We write

$$\begin{aligned}&\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) \psi^k(w) \left(u_{\epsilon,11} + u_{\epsilon,22} + \frac{u_{\epsilon,12}}{u_{,1}u_{,2}}\right) \zeta\, dx \\ &= \int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) \left(u_{,1}u_{,2}\left(u_{\epsilon,11}+u_{\epsilon,22}\right) + u_{\epsilon,12}\right) \overbrace{\frac{\psi^k(w)}{u_{,1}u_{,2}}}^{\chi^k} \zeta\, dx \\ &= \overbrace{\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) \left(u_{\epsilon,1}u_{\epsilon,2}\left(u_{\epsilon,11}+u_{\epsilon,22}\right) + u_{\epsilon,12}\right) \chi^k \zeta\, dx}^{I} \\ &\quad + \overbrace{\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) \left(u_{\epsilon,11}+u_{\epsilon,22}\right) \left(u_{,1}u_{,2} - u_{\epsilon,1}u_{\epsilon,2}\right) \chi^k \zeta\, dx}^{II} \\ &= \overbrace{\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) \left(u_{\epsilon,1}u_{\epsilon,2}\left(u_{\epsilon,11}+u_{\epsilon,22}\right) + |\nabla u_\epsilon|^2 u_{\epsilon,12}\right) \chi^k \zeta\, dx + \int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right)^2 u_{\epsilon,12} \chi^k \zeta\, dx}^{I} \\ &\quad + \overbrace{\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right) \left(u_{\epsilon,11}+u_{\epsilon,22}\right) \left(u_{,1}u_{,2} - u_{\epsilon,1}u_{\epsilon,2}\right) \chi^k \zeta\, dx}^{II}.\end{aligned} \tag{145}$$

First, we have (noting $|\nabla u| = 1$ a.e.)

$$\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right)^2 u_{\epsilon,12}\chi^k \zeta\, dx = \int_{\Omega'} \left(|\nabla u|^2 - |\nabla u_\epsilon|^2\right)\left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,12}\chi^k\zeta\, dx. \tag{146}$$

Since $|\nabla u_\epsilon| \le 1$ and $|\nabla u| = 1$, and $\chi^k\zeta \in L^\infty(\Omega)$, we have

$$\left|\int_{\Omega'} \left(|\nabla u|^2 - |\nabla u_\epsilon|^2\right)\left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,12}\chi^k\zeta\, dx\right| \le C\int_{\Omega'} |\nabla u - \nabla u_\epsilon|\left(1-|\nabla u_\epsilon|^2\right)\left|u_{\epsilon,12}\right| dx. \tag{147}$$

Since $\|\nabla u - \nabla u_\epsilon\|_{L^4(\Omega')} = \|w - w_\epsilon\|_{L^4(\Omega')} \to 0$, we deduce from (147) and Lemma 14 that

$$\int_{\Omega'} \left(|\nabla u|^2 - |\nabla u_\epsilon|^2\right)\left(1-|\nabla u_\epsilon|^2\right) u_{\epsilon,12}\chi^k\zeta\, dx \to 0 \quad \text{as } \epsilon \to 0. \tag{148}$$

Combining (146) with (148), we obtain

$$\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right)^2 u_{\epsilon,12}\chi^k\zeta\, dx \to 0 \quad \text{as } \epsilon \to 0. \tag{149}$$

For the last term in (145), since $\|w - w_\epsilon\|_{L^p(\Omega')} \to 0$ for all $p \ge 1$, it is clear that $\|u_{,1}u_{,2} - u_{\epsilon,1}u_{\epsilon,2}\|_{L^4(\Omega')} \to 0$. It follows from the fact that $\chi^k\zeta \in L^\infty(\Omega)$ and Lemma 14 again that

$$\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right)\left(u_{\epsilon,11}+u_{\epsilon,22}\right)\left(u_{,1}u_{,2} - u_{\epsilon,1}u_{\epsilon,2}\right)\chi^k\zeta\, dx \to 0 \quad \text{as } \epsilon \to 0. \tag{150}$$

Finally, we look at the first term in (145). Following the arguments in Lemma 20, we choose a sequence of smooth functions $\{\chi_j\}$ such that

$$\chi_j \to \chi^k \quad \text{in } L^4(\Omega).$$

Note that we have $|u_{\epsilon,1}u_{\epsilon,2}| \le 1$ and $|\nabla u_\epsilon| \le 1$. Therefore, we have

$$\begin{aligned}&\left|\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right)\left(u_{\epsilon,1}u_{\epsilon,2}\left(u_{\epsilon,11}+u_{\epsilon,22}\right) + |\nabla u_\epsilon|^2 u_{\epsilon,12}\right)\left(\chi^k - \chi_j\right)\zeta\, dx\right| \\ &\qquad \le \int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right)\left(\left|u_{\epsilon,11}\right| + \left|u_{\epsilon,22}\right| + \left|u_{\epsilon,12}\right|\right)\left|\chi^k - \chi_j\right| |\zeta|\, dx.\end{aligned}$$

Let $\alpha > 0$. By Lemma 14 there exists some $j_0 \in \mathbb{N}$ such that

$$\left|\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right)\left(u_{\epsilon,1}u_{\epsilon,2}\left(u_{\epsilon,11}+u_{\epsilon,22}\right) + |\nabla u_\epsilon|^2 u_{\epsilon,12}\right)\left(\chi^k - \chi_j\right)\zeta\, dx\right| \le \frac{\alpha}{2} \text{ for all } \epsilon \in (0,\epsilon_0),\ j \ge j_0 \tag{151}$$

where ϵ_0 is the small constant as in Lemma 14.

Using the harmonic polynomial $\widetilde{\varphi}(z) = z_1^2 - z_2^2$ and the formula (96), we obtain $\widetilde{\Phi}(z) = (z_1^3 + 3z_1z_2^2, -3z_1^2z_2 - z_2^3)$. Let $\eta \in C_c^\infty(\mathbb{R}^2)$ be a cut-off function such that $\eta \equiv 1$ on $B_2(0)$ and define $\varphi := \widetilde{\varphi}\eta \in C_c^\infty(\mathbb{R}^2)$. Let Φ be the entropy obtained from the function φ through formula (96). As noted in Lemma 11, Φ is an entropy in the sense of (8) and hence we can apply Lemma 9. Since $\varphi = \widetilde{\varphi}$ on $B_2(0)$, we have $\Phi = \widetilde{\Phi}$ on $B_2(0)$. Since $|w| = 1$ a.e. and $|w_\epsilon| \le 1$, we see that $\Phi(w) = \widetilde{\Phi}(w)$ for a.e. $x \in \Omega$ and $\Phi(w_\epsilon) = \widetilde{\Phi}(w_\epsilon)$ for all $x \in \Omega'$. By direct calculations, we have

$$\begin{aligned}\nabla \cdot [\Phi(w_\epsilon)] &= \nabla \cdot \left(-u_{\epsilon,2}^3 - 3u_{\epsilon,2}u_{\epsilon,1}^2, -3u_{\epsilon,2}^2u_{\epsilon,1} - u_{\epsilon,1}^3\right) \\ &= -6\left(u_{\epsilon,1}u_{\epsilon,2}\left(u_{\epsilon,11}+u_{\epsilon,22}\right) + |\nabla u_\epsilon|^2 u_{\epsilon,12}\right).\end{aligned}$$

Let us apply (101) to our particular entropy Φ:

$$\begin{aligned}
&\int_{\Omega'} \left(1-|\nabla u_\epsilon|^2\right)\left(u_{\epsilon,1}u_{\epsilon,2}\left(u_{\epsilon,11}+u_{\epsilon,22}\right)+|\nabla u_\epsilon|^2 u_{\epsilon,12}\right)\chi_{j_0}\zeta\,dx \\
&= -\frac{1}{6}\int_{\Omega'}\left(1-|w_\epsilon|^2\right)\chi_{j_0}\zeta\nabla\cdot[\Phi(w_\epsilon)]\,dx \\
&\overset{(95)}{=} -\frac{1}{6}\int_{\Omega'}\left(1-|w_\epsilon|^2\right)\chi_{j_0}\zeta\Psi(w_\epsilon)\cdot\nabla\left(1-|w_\epsilon|^2\right)dx \\
&= -\frac{1}{12}\int_{\Omega'}\chi_{j_0}\zeta\Psi(w_\epsilon)\cdot\nabla\left(1-|w_\epsilon|^2\right)^2 dx \\
&= -\frac{1}{12}\int_{\Omega'}\chi_{j_0}\zeta\nabla\cdot\left[\Psi(w_\epsilon)\left(1-|w_\epsilon|^2\right)^2\right]dx+\frac{1}{12}\int_{\Omega'}\chi_{j_0}\zeta\left(1-|w_\epsilon|^2\right)^2\nabla\cdot[\Psi(w_\epsilon)]\,dx,
\end{aligned} \tag{152}$$

where $\Psi\in C_c^\infty(\mathbb{R}^2;\mathbb{R}^2)$ is related to the particular entropy Φ via Lemma 9. It is clear that $\{\sup|\Psi(w_\epsilon)|\}$ is uniformly bounded. It follows from integration by parts that

$$\int_{\Omega'}\chi_{j_0}\zeta\nabla\cdot\left[\Psi(w_\epsilon)\left(1-|w_\epsilon|^2\right)^2\right]dx\to 0 \quad \text{as } \epsilon\to 0. \tag{153}$$

Now we write out the other term in (152)

$$\begin{aligned}
&\int_{\Omega'}\chi_{j_0}\zeta\left(1-|w_\epsilon|^2\right)^2\nabla\cdot[\Psi(w_\epsilon)]\,dx \\
&=\int_{\Omega'}\chi_{j_0}\zeta\left(1-|w_\epsilon|^2\right)^2\left[\Psi_{1,1}(w_\epsilon)w^1_{\epsilon,1}+\Psi_{1,2}(w_\epsilon)w^2_{\epsilon,1}+\Psi_{2,1}(w_\epsilon)w^1_{\epsilon,2}+\Psi_{2,2}(w_\epsilon)w^2_{\epsilon,2}\right]dx.
\end{aligned} \tag{154}$$

For all $m,n\in\{1,2\}$, using $|w|=1$ a.e., we have

$$\int_{\Omega'}\chi_{j_0}\zeta\left(1-|w_\epsilon|^2\right)^2\Psi_{m,n}(w_\epsilon)w^n_{\epsilon,m}dx=\int_{\Omega'}\chi_{j_0}\zeta\left(|w|^2-|w_\epsilon|^2\right)\left(1-|w_\epsilon|^2\right)\Psi_{m,n}(w_\epsilon)w^n_{\epsilon,m}dx. \tag{155}$$

Since $\|w-w_\epsilon\|_{L^4(\Omega')}\to 0$ and $\{\sup|\Psi_{m,n}(w_\epsilon)|\}$ is uniformly bounded, an application of Lemma 14 yields

$$\int_{\Omega'}\chi_{j_0}\zeta\left(|w|^2-|w_\epsilon|^2\right)\left(1-|w_\epsilon|^2\right)\Psi_{m,n}(w_\epsilon)w^n_{\epsilon,m}dx\le C\int_{\Omega'}|w-w_\epsilon|\left(1-|w_\epsilon|^2\right)w^n_{\epsilon,m}dx\to 0. \tag{156}$$

Putting (155)–(156) together, we obtain

$$\int_{\Omega'}\chi_{j_0}\zeta\left(1-|w_\epsilon|^2\right)^2\Psi_{m,n}(w_\epsilon)w^n_{\epsilon,m}dx\to 0.$$

Taking the sum over all m,n, we deduce from (154) that

$$\int_{\Omega'}\chi_{j_0}\zeta\left(1-|w_\epsilon|^2\right)^2\nabla\cdot[\Psi(w_\epsilon)]\,dx\to 0. \tag{157}$$

Combining (157) with (153) and (152), we have

$$\int_{\Omega'}\left(1-|\nabla u_\epsilon|^2\right)\left(u_{\epsilon,1}u_{\epsilon,2}\left(u_{\epsilon,11}+u_{\epsilon,22}\right)+|\nabla u_\epsilon|^2u_{\epsilon,12}\right)\chi_{j_0}\zeta\,dx\to 0 \text{ as } \epsilon\to 0.$$

So there exists some $\epsilon_1 \in (0, \epsilon_0)$ such that

$$\int_{\Omega'} \left(1 - |\nabla u_\epsilon|^2\right) \left(u_{\epsilon,1} u_{\epsilon,2} \left(u_{\epsilon,11} + u_{\epsilon,22}\right) + |\nabla u_\epsilon|^2 u_{\epsilon,12}\right) \chi_{j_0} \zeta \, dx < \frac{\alpha}{2} \text{ for any } \epsilon \in (0, \epsilon_1) . \tag{158}$$

Inequality (158) together with (151) yield

$$\left| \int_{\Omega'} \left(1 - |\nabla u_\epsilon|^2\right) \left(u_{\epsilon,1} u_{\epsilon,2} \left(u_{\epsilon,11} + u_{\epsilon,22}\right) + |\nabla u_\epsilon|^2 u_{\epsilon,12}\right) \chi^k \zeta \, dx \right| < \alpha \text{ for any } \epsilon \in (0, \epsilon_1).$$

As this is true for any $\alpha > 0$ we have shown

$$\int_{\Omega'} \left(1 - |\nabla u_\epsilon|^2\right) \left(u_{\epsilon,1} u_{\epsilon,2} \left(u_{\epsilon,11} + u_{\epsilon,22}\right) + |\nabla u_\epsilon|^2 u_{\epsilon,12}\right) \chi^k \zeta \, dx \to 0 \text{ as } \epsilon \to 0. \tag{159}$$

Finally, putting (149), (150) and (159) into (145) concludes the proof of Lemma 21. □

8. Proof of Theorem 3

By Theorem 18, (131) we have that

$$\nabla \cdot \left[\Phi^\xi \left(\nabla u^\perp\right)\right] = 0 \text{ distributionally in } \Omega, \text{ for any } \xi \in \mathbb{S}^1 \setminus \{e_1, -e_1, e_2, -e_2\}. \tag{160}$$

As explained in the sketch of the proof, we could carry out the argument that establishes (160) for a coordinate axis $\{\epsilon_1, \epsilon_2\}$ (see (4)) and this gives (160) for all $\xi \in \mathbb{S}^1 \setminus \{\epsilon_1, -\epsilon_1, \epsilon_2, -\epsilon_2\}$ and hence (160) holds for any $\xi \in \mathbb{S}^1$.

Now defining $w(x) = \nabla u(x)^\perp$ we have that $\Phi^\xi \left(\nabla u(x)^\perp\right) \overset{(13)}{=} \xi \chi(x, \xi)$ for a.e. $x \in \Omega$ and so

$$0 = \nabla \cdot \left[\Phi^\xi \left(\nabla u^\perp\right)\right] = \nabla \cdot \left[\xi \chi(\cdot, \xi)\right] = \xi \cdot \nabla \chi(\cdot, \xi) \text{ in } \mathcal{D}'(\Omega)$$

and thus applying Theorem 1 we have that ∇u is locally Lipschitz outside a locally finite set of points.

It has been observed in [23] that the results of [25] imply that under the hypothesis of Theorem 1, if $\mathcal{O} \subset\subset \Omega$ is a convex neighborhood of a point $\zeta \in S$ (where $w = \nabla u^\perp$ is locally Lipschitz outside of S) then there exists $\alpha \in \{1, -1\}$ such that

$$w(z) = \alpha \frac{(z - \zeta)^\perp}{|z - \zeta|} \text{ for any } z \in \mathcal{O}. \tag{161}$$

Since we have shown that w satisfies (12), this implies (16). For the convenience of the reader, we note that (161) follows from the results of [25] in the following way. Firstly by Lemma 5.1 [25] for any $x_0, y_0 \in \mathcal{O}$ that are Lebesgue points of w we have

$$|w(x_0) - \alpha w(y_0)| \le \frac{|x_0 - y_0|}{d} \text{ for some } \alpha \in \{1, -1\}, \tag{162}$$

where $d = \text{dist}(\mathcal{O}, \partial\Omega) > 0$. In the proof of Theorem 1.3 (that follows the proof of Lemma 5.1) the estimate (162) is strengthened in that it is shown that $\alpha = 1$. Thus w is $\frac{1}{d}$-Lipschitz in $\mathcal{O}$. This contradicts the fact that $\zeta \in \mathcal{O}$ and hence (161) follows. □

Conflict of interest statement

There is no conflict of interest.

Appendix A. Some auxiliary results

We have used in a fundamental way a couple of estimates from [15], these in turns were inspired by a commutator estimate of Constantin, E, Titi [12]. For convenience of the reader we repeat the proof from [15].

Lemma 22 *([15]). Let $\Omega \subset \mathbb{R}^2$ be a bounded domain and $w \in L^3(\Omega; \mathbb{R}^2)$ satisfy $|w| = 1$ a.e. in Ω. Given $\Omega' \subset\subset \Omega$, let $\gamma := \operatorname{dist}(\Omega', \partial\Omega) > 0$. Then, for all $x \in \Omega'$ and $0 < \epsilon < \gamma$, denoting $w_\epsilon = w * \rho_\epsilon$, we have*

$$1 - |w_\epsilon(x)|^2 \leq \frac{2\|\rho\|_{L^\infty}}{\epsilon^2} \int_{B_\epsilon} |w(x-z) - w(x)|^2 \, dz, \tag{163}$$

and

$$|\partial_j w_\epsilon(x)| \leq \frac{\|\nabla\rho\|_{L^\infty}}{\epsilon^3} \int_{B_\epsilon} |w(x-z) - w(x)| \, dz. \tag{164}$$

Proof. First, for $x \in \Omega'$ and for $0 < \epsilon < \gamma$, using $|w| = 1$ a.e., we have

$$\begin{aligned}
1 - |w_\epsilon(x)|^2 &= |w|^2 * \rho_\epsilon(x) - |w * \rho_\epsilon|^2 \\
&= \int_{\mathbb{R}^2} |w(x-z)|^2 \rho_\epsilon(z) dz \\
&\qquad - \left(\int_{\mathbb{R}^2} w(x-z)\rho_\epsilon(z) dz \right) \cdot \left(\int_{\mathbb{R}^2} w(x-y)\rho_\epsilon(y) dy \right) \\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} w(x-z)\left(w(x-z) - w(x-y)\right)\rho_\epsilon(z)\rho_\epsilon(y) dz dy \\
&\overset{z:=y,\, y:=z}{=} \frac{1}{2}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |w(x-z) - w(x-y)|^2 \rho_\epsilon(z)\rho_\epsilon(y) dz dy \\
&\leq 2\int_{\mathbb{R}^2} |w(x-z) - w(x)|^2 \rho_\epsilon(z) dz \\
&\leq \frac{2\|\rho\|_{L^\infty}}{\epsilon^2} \int_{B_\epsilon} |w(x-z) - w(x)|^2 \, dz.
\end{aligned}$$

This establishes (163).

To show (164), note that $\int_{B_\epsilon} \partial_j \rho(\frac{z}{\epsilon}) dz = 0$ for $j = 1, 2$. Therefore, we have

$$\begin{aligned}
\left|\partial_j w_\epsilon(x)\right| = \left|w * \partial_j \rho_\epsilon(x)\right| &= \left| \frac{1}{\epsilon^3} \int_{B_\epsilon} w(x-z)\partial_j \rho(\frac{z}{\epsilon}) dz \right| \\
&= \left| \frac{1}{\epsilon^3} \int_{B_\epsilon} \left(w(x-z) - w(x)\right) \partial_j \rho(\frac{z}{\epsilon}) dz \right| \\
&\leq \frac{\|\nabla\rho\|_{L^\infty}}{\epsilon^3} \int_{B_\epsilon} |w(x-z) - w(x)| \, dz. \qquad \square
\end{aligned}$$

Lemma 23. *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected domain and $v \in L^\infty(\Omega;\mathbb{R}^2)$ be such that* $\mathrm{curl} v = 0$ *weakly. Then there exists some potential $f \in W^{1,\infty}(\Omega)$ such that $\nabla f = v$ a.e. on Ω.*

Proof of Lemma 23. We follow some of the ideas in the proof of Theorem 2.9 in [22]. The proof goes in two steps.

Step 1. We can find a sequence $\{\Omega_k\}_k$ of open simply-connected sets with the following properties:

(1) $\Omega_k \subset\subset \Omega$;
(2) $\Omega_k \subset \Omega_{k+1}$;
(3) $\bigcup_k \Omega_k = \Omega$.

Proof of Step 1. Define $O_k := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > 2^{-k}\}$. We start with some k_0 sufficiently large such that O_{k_0} is nonempty. Define Ω_{k_0} to be any connected component of O_{k_0}. For all $k > k_0$, define Ω_k to be the connected component of O_k that contains Ω_{k_0}. It is clear that the sequence $\{\Omega_k\}_{k\geq k_0}$ satisfies (1) and (2). To see (3), we claim that for any $k_1 \geq k_0$, $O_{k_1} \subset \Omega_k$ for all k sufficiently large. Indeed, let $\{O^j_{k_1}\}_{j=1}^m$ be the connected components of O_{k_1}. Without loss of generality, assume $O^1_{k_1} = \Omega_{k_1}$. For each $j = 1, 2, ..., m$, we fix a point $a_j \in O^j_{k_1}$. Since Ω is connected, we can find continuous paths $\gamma^j_1 \subset \Omega$ connecting a_1 and a_j for $j = 2, 3, ..., m$. Denote $\delta_j = \mathrm{dist}(\gamma^j_1, \partial\Omega) > 0$, and let T_j be a tubular neighborhood of γ^j_1 of size $\frac{\delta_j}{2}$ for $j = 2, 3, ..., m$. Now denote $\delta_{k_1} := \min\{\delta_j\} > 0$. It is clear that $O_{k_1} \bigcup \left(\bigcup_{j=2}^m T_j\right) \subset \Omega$ is connected, and for any $x \in O_{k_1} \bigcup \left(\bigcup_{j=2}^m T_j\right)$, $\mathrm{dist}(x, \partial\Omega) > \min\{2^{-k_1}, \frac{\delta_{k_1}}{2}\}$. Therefore $O_{k_1} \bigcup \left(\bigcup_{j=2}^m T_j\right) \subset O_k$ for k sufficiently large. Since $O_{k_1} \bigcup \left(\bigcup_{j=2}^m T_j\right)$ is connected and $\Omega_{k_0} \subset O_{k_1}$, by definition of Ω_k, we have $O_{k_1} \bigcup \left(\bigcup_{j=2}^m T_j\right) \subset \Omega_k$. Since $\Omega = \bigcup O_k$, it follows that (3) is satisfied.

Now we claim that each Ω_k is simply-connected. We argue by contradiction. Suppose Ω_k is not simply-connected for some k. Then we can find some closed curve $\Gamma \subset \Omega_k$ such that there exists $x \in \mathrm{Int}(\Gamma) \cap (O_k)^c$. By the definition of O_k, we have $\mathrm{dist}(x, \partial\Omega) \leq 2^{-k}$. Let $y \in \partial\Omega$ be such that $|x - y| = \mathrm{dist}(x, \partial\Omega)$, and let z be the intersection of Γ with the line segment joining x and y. Then clearly we have $|z - y| \leq |x - y| \leq 2^{-k}$. On the other hand, since $z \in \Gamma \subset \Omega_k$, we have $|z - y| \geq \mathrm{dist}(z, \partial\Omega) > 2^{-k}$. This is a contradiction. It follows that Ω_k is simply-connected.

Step 2: proof of Lemma completed. Without loss of generality we can assume $0 \in \Omega_k$ for all k. For any $\epsilon \in (0, 2^{-k})$, $v_\epsilon = v * \rho_\epsilon$ is such that $\mathrm{curl} v_\epsilon = 0$ on Ω_k. Since Ω_k is simply-connected, there exists f_ϵ such that $\nabla f_\epsilon = v_\epsilon$ on Ω_k and $f_\epsilon(0) = 0$. Now take some sequence $\epsilon_n \to 0$. By basic properties of convolutions, we know $\nabla f_{\epsilon_n} \overset{L^p(\Omega_k)}{\rightarrow} v$ as $n \to \infty$, for all $1 \leq p < \infty$.

Since $v \in L^\infty(\Omega;\mathbb{R}^2)$, we have $\|v_\epsilon\|_\infty \leq \|v\|_\infty$, and hence $\{f_{\epsilon_n}\}$ is a sequence of equicontinuous functions on $\overline{\Omega}_k$ with $f_{\epsilon_n}(0) = 0$. It follows from the Arzelà–Ascoli Theorem that for some subsequence (not relabeled) $f_{\epsilon_n} \overset{L^\infty(\Omega_k)}{\rightarrow} f_k$ for some Lipschitz function f_k with $f_k(0) = 0$. Therefore $\nabla f_k = v$ a.e. on Ω_k.

We claim

$$f_l = f_k \text{ on } \Omega_k \text{ for all } l > k. \tag{165}$$

Indeed, the equation (165) follows from the facts that $f_l - f_k$ is Lipschitz and $f_l(0) = f_k(0)$ and $\nabla(f_l - f_k) = 0$ a.e. on Ω_k. Thus by (165) we can define

$$f(x) = \left\{ \begin{array}{ll} f_1(x) & \text{on } \Omega_1 \\ f_2(x) & \text{on } \Omega_2 \\ & \dots \\ f_k(x) & \text{on } \Omega_k \\ & \dots \end{array} \right\}.$$

And finally $\nabla f = v$ a.e. on Ω. $\square$

Finally, we provide the proof of Lemma 19.

Proof of Lemma 19. We mostly follow the proof of Lemma 4 in [18]. Let us consider the function φ defined by

$$\varphi(z) = \begin{cases} z \cdot \xi & \text{for } z \cdot \xi > 0, \\ 0 & \text{for } z \cdot \xi \le 0, \end{cases}$$

and the map F given by

$$F(z) = \begin{cases} \xi & \text{for } z \cdot \xi > 0, \\ 0 & \text{for } z \cdot \xi \le 0. \end{cases}$$

Note that F is the gradient of φ whenever φ is differentiable.

Now we construct a sequence $\{\varphi_k\}_k$ in $C_c^\infty(\mathbb{R}^2)$ such that

$$\{(\varphi_k(z), \nabla\varphi_k(z))\}_k \quad \text{is bounded uniformly for bounded } z, \tag{166}$$

$$(\varphi_k(z), \nabla\varphi_k(z)) \overset{k\to\infty}{\to} (\varphi(z), F(z)) \quad \text{for all } z. \tag{167}$$

Here we use an approximation that was used by the first author in [27] to make the proof more transparent than that in [18]. Clearly there exists a monotone smooth function $s_0 : \mathbb{R} \to \mathbb{R}$ such that $s_0(x) \equiv 0$ for $x \le 0$ and $s_0(x) = x$ for $x \ge 1$. Given $k \in \mathbb{N}^+$, define $s_k(x) := \frac{1}{k} s_0(kx)$. It is easy to check that s_k is a smooth function satisfying

$$\{(s_k(x), s_k'(x))\} \quad \text{is bounded uniformly for bounded } x, \tag{168}$$

$$(s_k(x), s_k'(x)) \overset{k\to\infty}{\to} (s(x), f(x)) \quad \text{for all } x, \tag{169}$$

where

$$s(x) = \begin{cases} x & \text{for } x > 0, \\ 0 & \text{for } x \le 0, \end{cases}$$

and

$$f(x) = \begin{cases} 1 & \text{for } x > 0, \\ 0 & \text{for } x \le 0. \end{cases}$$

Now we define $\varphi_k(z) = s_k(z \cdot \xi)\chi_k$, where $\chi_k \in C_c^\infty(\mathbb{R}^2)$ satisfies $\mathrm{Spt}(\chi_k) \subset\subset B_{k+1}(0)$ and $\chi_k \equiv 1$ on $B_k(0)$. It is clear that $\varphi_k \in C_c^\infty(\mathbb{R}^2)$ and $\nabla\varphi_k(z) = s_k'(z \cdot \xi)\xi$ for $z \in B_k(0)$. One can check directly that the properties (168)–(169) for s_k translate to (166)–(167).

According to Lemma 11,

$$\Phi_k(z) := \varphi_k(z)z + \left(\nabla\varphi_k(z) \cdot z^\perp\right) z^\perp$$

is an entropy. It is clear that (166) implies that $\{\Phi_k(z)\}$ is bounded uniformly for bounded z. According to (167),

$$\Phi_k(z) \to \varphi(z)z + \left(F(z) \cdot z^\perp\right) z^\perp = \begin{cases} |z|^2\xi & \text{for } z \cdot \xi > 0, \\ 0 & \text{for } z \cdot \xi \le 0, \end{cases}$$

which is (132). □

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