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Self-similar solutions with fat tails for Smoluchowski's coagulation equation with singular kernels

Solutions auto-similaire avec queues lourdes pour l'équation de coagulation de Smoluchowski avec noyaux singuliers

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Abstract

We show the existence of self-similar solutions with fat tails for Smoluchowski's coagulation equation for homogeneous kernels satisfying $C_1\left(x^{-a}y^b+x^by^{-a}\right)\le K(x,y)\le C_2\left(x^{-a}y^b+x^by^{-a}\right)$ with $a>0$ and $b<1$. This covers especially the case of Smoluchowski's classical kernel $K(x,y)=(x^{1/3}+y^{1/3})(x^{-1/3}+y^{-1/3})$.

For the proof of existence we take a self-similar solution h_ε for a regularized kernel K_ε and pass to the limit $\varepsilon\to 0$ to obtain a solution for the original kernel K. The main difficulty is to establish a uniform lower bound on h_ε. The basic idea for this is to consider the time-dependent problem and to choose a special test function that solves the dual problem.

Résumé

Nous démontrons l'existence des solutions auto-similaires avec queues lourdes pour l'équation de coagulation de Smoluchowski avec un noyau K satisfaisant $C_1\left(x^{-a}y^b+x^by^{-a}\right)\le K(x,y)\le C_2\left(x^{-a}y^b+x^by^{-a}\right)$ avec $a>0$ et $b<1$. Cela contient en particulier le noyau classique de Smoluchowski $K(x,y)=(x^{1/3}+y^{1/3})(x^{-1/3}+y^{-1/3})$.

Pour la démonstration de l'existence nous prenons une solution auto-similaire h_ε pour un noyau régularisé K_ε et nous obtenons une solution pour le noyau original K en passant à la limite $\varepsilon\to 0$. La difficulté principale consiste à établir une borne inférieure

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pour h_ε. La clé ici est de considérer le problème dépendant du temps et choisir une solution du problème dual comme fonction test dans la formulation faible de l'équation auto-similaire.

Keywords: Smoluchowski's coagulation equation; Self-similar solution; Singular kernel; Fat tail; Dual problem

1. Introduction

1.1. Smoluchowski's equation and self-similarity

Smoluchowski's coagulation equation [13] describes irreversible aggregation of clusters through binary collisions by a mean-field model for the density $f(\xi,t)$ of clusters of mass ξ. It is assumed that the rate of coagulation of clusters of size ξ and η is given by a rate kernel $K = K(\xi,\eta)$, such that the evolution of f is determined by

$$\partial_t f(\xi,t) = \frac{1}{2}\int_0^\xi K(\xi-\eta,\eta)f(\xi-\eta,t)f(\eta,t)\mathrm{d}\eta - f(\xi,t)\int_0^\infty K(\xi,\eta)f(\eta,t)\mathrm{d}\eta. \tag{1}$$

Applications in which this model has been used are numerous and include, for example, aerosol physics, polymerization, astrophysics and mathematical biology (see e.g. [1,3]).

A topic of particular interest in the theory of coagulation is the scaling hypothesis on the long-time behaviour of solutions to (1). Indeed, for homogeneous kernels one expects that solutions converge to a uniquely determined self-similar profile. This issue is however only well-understood for the solvable kernels $K(x,y)=2$, $K(x,y)=x+y$ and $K(x,y)=xy$. In these cases it is known [9] that (1) has one fast-decaying self-similar solution with finite mass and a family of so-called fat-tail or heavy-tailed self-similar solutions with power-law decay. Such solutions with certain infinite moments have been studied extensively in probability theory and are of considerable interest since they predict a high probability of the occurrence of extreme events. Furthermore, in [9] the domains of attraction of all these self-similar solutions under the evolution (1) have been completely characterized. For non-solvable kernels much less is known and it is exclusively for the case of kernels with homogeneity $\gamma<1$. In [5,6] existence of self-similar solutions with finite mass has been established for a large range of kernels and some properties of those solutions have been investigated in [2,4,7]. More recently, the first existence results of self-similar solutions with fat tails have been proved, first for the diagonal kernel [11], then for kernels that are bounded by $C(x^\gamma+y^\gamma)$ for $\gamma\in[0,1)$ [12]. It is the goal of this paper to extend the results in [12] to singular kernels, such as Smoluchowski's classical kernel $K(x,y)=(x^{1/3}+y^{1/3})(x^{-1/3}+y^{-1/3})$.

In order to describe our results in more detail, we first derive the equation for self-similar solutions. Such solutions to (1) for kernels of homogeneity $\gamma<1$ are of the form

$$f(\xi,t)=\frac{\beta}{t^\alpha}g(x)\,,\qquad \alpha=1+(1+\gamma)\beta,\qquad x=\frac{\xi}{t^\beta}, \tag{2}$$

where the self-similar profile g solves

$$-\frac{\alpha}{\beta}g-xg'(x)=\frac{1}{2}\int_0^x K(x-y,y)g(x-y)g(y)\mathrm{d}y - g(x)\int_0^\infty K(x,y)g(y)\mathrm{d}y. \tag{3}$$

It is known that for some kernels the self-similar profiles are singular at the origin, so that the integrals on the right-hand side are not finite and it is necessary to rewrite the equation in a weaker form. Multiplying the equation by x and rearranging we obtain that a weak self-similar solution g solves

$$\partial_x(x^2g(x))=\partial_x\left[\int_0^x\int_{x-y}^\infty yK(y,z)g(z)g(y)\,\mathrm{d}z\,\mathrm{d}y\right]+\left((1-\gamma)-\frac{1}{\beta}\right)xg(x) \tag{4}$$

in a distributional sense. If one in addition requires that the solution has finite first moment, then this also fixes $\beta = 1/(1-\gamma)$ and in this case the second term on the right hand side of (4) vanishes.

For the following it is convenient to go over to the monomer density function $h(x,t) = xg(x,t)$ and to introduce the parameter $\rho = \gamma + \frac{1}{\beta}$. Then Eq. (4) becomes

$$\partial_x \left[\int_0^x \int_{x-y}^\infty \frac{K(y,z)}{z} h(z) h(y)\, dz\, dy \right] - [\partial_x (xh) + (\rho - 1) h](x) = 0. \tag{5}$$

Our approach to find a solution to (5) requires to work with the corresponding evolution equation. Using as new time variable $\log(t)$ which will be denoted as t from now on, the time dependent version of Eq. (5) becomes

$$\partial_t h(x,t) + \partial_x \left[\int_0^x \int_{x-y}^\infty \frac{K(y,z)}{z} h(z,t) h(y,t)\, dz\, dy \right] - [\partial_x (xh) + (\rho - 1) h](x,t) = 0, \tag{6}$$

with initial data

$$h(x,0) = h_0(x). \tag{7}$$

1.2. Assumptions on the kernel and main result

We now formulate our assumptions on the kernel K. We assume that

$$K \in C^1((0,\infty) \times (0,\infty)), \qquad K(x,y) = K(y,x) \geq 0 \qquad \text{for all } x, y \in (0,\infty), \tag{8}$$

K is homogeneous of degree $\gamma \in (-\infty, 1)$, that is

$$K(\lambda x, \lambda y) = \lambda^\gamma K(x,y) \qquad \text{for all } x, y \in (0,\infty), \tag{9}$$

and satisfies the growth condition

$$C_1 \left(x^{-a} y^b + x^b y^{-a} \right) \leq K(x,y) \leq C_2 \left(x^{-a} y^b + x^b y^{-a} \right) \qquad \text{for all } x, y \in (0,\infty), \tag{10}$$

where $a > 0$, $b < 1$, $\gamma = b - a$, and C_1, C_2 are positive constants. We need to assume here that $b < 1$ since for $b > 1$ we could have gelation and $b = 1$ is a borderline case that can also not be treated with our methods. The same assumption has also been made in related work, where, for example in [2], regularity of self-similar solutions with finite mass has been investigated.

Furthermore we assume the following locally uniform bound on the partial derivative: for each interval $[d, D] \subset (0,\infty)$ there exists a constant $C_3 = C_3(d,D) > 0$ such that

$$|\partial_x K(x,y)| \leq C_3 \left(y^{-a} + y^b \right) \quad \text{for all } x \in [d,D] \text{ and } y \in (0,\infty). \tag{11}$$

Our goal in this paper is to prove the existence of self-similar solutions with fat tails. Let us first discuss what we can expect on the possible decay behaviours of self-similar solutions. If $h(x) \sim Cx^{-\rho}$ as $x \to \infty$, then in order for $\int_1^\infty \frac{K(x,y)}{y} h(y)\, dy < \infty$ we need

$$\rho > b = \gamma + a \qquad \text{and} \qquad \rho + a > 0. \tag{12}$$

Note that since γ can be negative, $-a$ can be larger than b. While these conditions are natural it turns out that in addition we have to assume $\rho > 0$ (cf. Lemma 2.10). It is not clear to us whether this is just a technical restriction or whether there is really an obstacle to the existence of such solutions for the kernels considered in this paper. In fact, it is known that for the diagonal kernel with homogeneity $\gamma \in (-\infty, 1)$ fat tail solutions exist for $\rho \in (\gamma, 1)$ [11], but the diagonal kernel certainly represents a very special case. Finally, recall that the decay behaviour of h implies that the corresponding number density $g = h/x$ has infinite first moment for all ρ.

Our main result can now be formulated as follows

Theorem 1.1. *Let K be a kernel that satisfies assumptions (8)–(11) for some $b \in (-\infty, 1)$ and $a > 0$. Then for any $\rho \in (\max(-a, b, 0), 1) = (\max(b, 0), 1)$ there exists a non-negative measure $h \in \mathcal{M}([0, \infty))$ that solves (5) in the sense of distributions. Furthermore this measure h has a continuous density and satisfies $h(x) \sim (1-\rho)x^{-\rho}$ as $x \to \infty$.*

1.3. Strategy of the proof

The idea of the proof is to first consider the regularized kernel

$$K_\varepsilon(y, z) := K(y + \varepsilon, z + \varepsilon).$$

Proposition 1.2. *For any $\rho \in (\max(b, 0), 1)$ there exists a continuous function $h_\varepsilon : (0, \infty) \to [0, \infty)$ that is a weak solution to (5) with K replaced by K_ε. This solution satisfies*

$$\int_0^r h_\varepsilon(x)\,dx \le r^{1-\rho} \qquad \textit{and} \qquad \lim_{r\to\infty} \frac{\int_0^r h_\varepsilon(x)\,dx}{r^{1-\rho}} = 1.$$

The proof of Proposition 1.2 follows closely the proof of Theorem 1.1 in [12] and is based on the idea to construct a stationary solution to (6) by using the following variant of Tikhonov's fixed point theorem.

Theorem 1.3. *(See Theorem 1.2 in [5,8].) Let X be a Banach space and $(S_t)_{t\ge 0}$ be a continuous semi-group on X. Assume that S_t is weakly sequentially continuous for any $t > 0$ and that there exists a subset $\mathcal{Y}$ of X that is nonempty, convex, weakly sequentially compact and invariant under the action of S_t. Then there exists $z_0 \in \mathcal{Y}$ which is stationary under the action of S_t.*

In order to apply Theorem 1.3 to find a self-similar solution for the coagulation equation with kernel K_ε we denote for $\rho \in (b, 1)$ as $\mathcal{X}_\rho$ the set of measures $h\mathrm{d}x \in \mathcal{M}_+([0, \infty))$ such that

$$\|h\| := \sup_{R\ge 0} \frac{\int_{[0,R]} h\mathrm{d}x}{R^{1-\rho}} < \infty. \tag{13}$$

The set $\mathcal{Y}$ is defined by the family of $h \in \mathcal{X}_\rho$ that satisfy

$$\int_{[0,r]} h\mathrm{d}x \le r^{1-\rho}, \quad \text{for all } r \ge 0 \tag{14}$$

$$\int_{[0,r]} h\mathrm{d}x \ge r^{1-\rho}\left(1 - \frac{R_0^\delta}{r^\delta}\right)_+ \quad \text{for all } r > 0. \tag{15}$$

The key part of the analysis is then to show that $\mathcal{Y}$ is left invariant under the evolution of (6). Many of the estimates are very similar to the ones in [12] apart from some parts in the proof of the lower bound (15). To keep this paper at a reasonable length we do not give the details here but the interested reader may find them in the Preprint version [10].

In order to obtain a solution to our original problem we want to pass to the limit $\varepsilon \to 0$. The problem is that the lower bound (15) that we obtain for h_ε is not uniform in ε and thus we cannot exclude a priori that we obtain the trivial solution in the limit. Accordingly, our first goal is to obtain a uniform lower bound on h_ε. In order to do so, we again investigate the corresponding dual problem and look for a suitable subsolution, that is for a function ψ that satisfies

$$\partial_t \psi \le (\rho - 1)\psi - x\partial_x \psi + \int_0^\infty \frac{K_\varepsilon(x, y)}{y} h_\varepsilon(y) \left[\psi(x + y) - \psi(x)\right] dy \tag{16}$$

(cf. (22) below) with suitably described data $\psi(0,\cdot)$. However, to obtain uniform bounds for ψ we see from (16) that we need uniform bounds on $\frac{K_\varepsilon(x,y)}{y}h_\varepsilon(y)$. In particular, we need to have uniform bounds on

$$\mu_\varepsilon = \int_0^1 h_\varepsilon(x)\,(x+\varepsilon)^{-a}\,dx, \qquad \lambda_\varepsilon = \int_0^1 h_\varepsilon(x)\,(x+\varepsilon)^{b}\,dx, \qquad L_\varepsilon := \max\left(\lambda_\varepsilon^{\frac{1}{1+a}}, \mu_\varepsilon^{\frac{1}{1-b}}\right) \tag{17}$$

that are a priori not available. To circumvent this difficulty we rescale h_ε suitably with L_ε and aim for a lower bound on the rescaled solution H_ε. The proof of the corresponding Proposition 2.1 is the contents of Section 2.1. In a next step we need to exclude that $L_\varepsilon \to \infty$ as $\varepsilon \to 0$. In order to do so we derive, assuming that $L_\varepsilon \to \infty$ a limit equation for $H := \lim_{\varepsilon\to 0} H_\varepsilon$ and show that this equation does not have a solution with the properties satisfied by H. It is in this step that we need the additional assumption $\rho > 0$ (cf. Lemma 2.10). The bound on L_ε implies that the lower bound for H_ε transfers to the corresponding one for h_ε. In order to pass to the limit in the equation for h_ε we in addition need to get a better control at the origin (see Lemma 2.12) which allows us to show that a weak limit h of h_ε (in the sense of measures) solves the coagulation equation. This is the contents of Section 2.4. Note that here and in the following we denote with some abuse of notation a measure $h\mathrm{d}x$ by h, even though we a priori may not know that $h\mathrm{d}x$ has a density.

In Section 3 we finally show that the limit measure h has a continuous density that decays pointwise in the expected manner. The proof is similar to the corresponding one in [12], but somewhat more technical.

2. Existence of a self-similar solution

As described above, our starting point is that we have a continuous positive function h_ε that is a weak solution to

$$\partial_x I_\varepsilon[h_\varepsilon] = \partial_x(xh_\varepsilon) + (\rho-1)h_\varepsilon, \qquad \text{with} \qquad I_\varepsilon[h_\varepsilon] = \int_0^x \int_{x-y}^\infty \frac{K_\varepsilon(y,z)}{z} h_\varepsilon(y)h_\varepsilon(z)\,dz\,dy. \tag{18}$$

Furthermore we have the estimates

$$\int_0^r h_\varepsilon(x)\,dx \le r^{1-\rho} \qquad \text{and} \qquad \lim_{r\to\infty} \int_0^r h_\varepsilon(x)\,dx / r^{1-\rho} = 1. \tag{19}$$

Up to passing to a subsequence we can in the following assume that either L_ε, as defined in (17) above, converges or $L_\varepsilon \to \infty$ for $\varepsilon \to 0$. Furthermore as the case $L_\varepsilon \to 0$ behaves slightly differently, we use from now on the following notation: we define $L := L_\varepsilon$ if $L_\varepsilon \not\to 0$ and $L := 1$ if $L_\varepsilon \to 0$ and thus (up to passing maybe to another subsequence) we may assume $L > 0$. For the following let

$$X = \frac{x}{L}, \qquad h_\varepsilon(x) = H_\varepsilon(X)L^{-\rho}.$$

We will first derive a uniform lower integral bound for H_ε.

2.1. *Uniform lower bound for H_ε*

The main result of this subsection is the following lower bound on H_ε.

Proposition 2.1. *For any $\delta > 0$ there exists $R_\delta > 0$ such that*

$$\int_0^R H_\varepsilon(X)\mathrm{d}X \ge (1-\delta)R^{1-\rho} \qquad \textit{for all } R \ge R_\delta. \tag{20}$$

Our goal is to prove such a lower bound by a suitable subsolution of the corresponding dual problem. After some changes of variables this reduces to finding a subsolution to a function W satisfying (23) below. The idea is now to replace the terms $\frac{K_\varepsilon(x,y)}{y}h_\varepsilon(y)$ by corresponding power laws. However, in the limit $\varepsilon \to 0$ these functions become too singular at the origin, such that we have to make a more complicated ansatz, as described in Lemma 2.2. After some surgery (cutting W at the origin) we then obtain a suitable subsolution. For this function we obtain a first lower bound in Lemma 2.6 that we can iterate (cf. Lemma 2.7) to finally obtain Proposition 2.1.

In the following we always denote for $\kappa \in (0,1)$ by φ_κ a non-negative, symmetric standard mollifier with $\operatorname{supp}\varphi_\kappa \subset [-\kappa,\kappa]$.

2.1.1. Construction of a suitable test function

We start by constructing a special test function and therefore notice that for $\psi = \psi(x,t)$ with $\psi \in C^1$ and compact support in $[0,T]\times[0,\infty)$ we obtain from the equation on h_ε that

$$0 = \int_0^T\int_0^\infty \partial_x\psi I_\varepsilon[h_\varepsilon]\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_0^\infty x\partial_x\psi h_\varepsilon \mathrm{d}x\mathrm{d}t + (\rho-1)\int_0^T\int_0^\infty \psi h_\varepsilon \mathrm{d}x\mathrm{d}t$$
$$= \int_0^T\int_0^\infty \partial_t\psi h_\varepsilon \mathrm{d}x\mathrm{d}t + \int_0^\infty \psi(\cdot,0)h_\varepsilon \mathrm{d}x - \int_0^\infty \psi(\cdot,T)h_\varepsilon \mathrm{d}x.$$

Choosing ψ such that

$$\int_0^T\int_0^\infty \partial_x\psi I_\varepsilon[h_\varepsilon]\,\mathrm{d}x\mathrm{d}t - \int_0^T\int_0^\infty x\partial_x\psi h_\varepsilon \mathrm{d}x\mathrm{d}t + (\rho-1)\int_0^T\int_0^\infty \psi h_\varepsilon \mathrm{d}x\mathrm{d}t - \int_0^T\int_0^\infty \partial_t\psi h_\varepsilon \mathrm{d}x\mathrm{d}t \geq 0 \tag{21}$$

we obtain

$$\int_0^\infty \psi(\cdot,0)h_\varepsilon \mathrm{d}x \geq \int_0^\infty \psi(\cdot,T)h_\varepsilon \mathrm{d}x.$$

Rewriting (21) we obtain

$$\int_0^T\int_0^\infty h_\varepsilon(x)\left\{\int_0^\infty \frac{K_\varepsilon(x,y)}{y}h_\varepsilon(y)\left[\psi(x+y)-\psi(x)\right]\mathrm{d}y - x\partial_x\psi(x) + (\rho-1)\psi(x) - \partial_t\psi(x)\right\}\mathrm{d}x\mathrm{d}t \geq 0. \tag{22}$$

Defining W by $\psi(x,t) = \mathrm{e}^{-(1-\rho)t}W(\xi,t)$ with $\xi = \frac{x}{L\mathrm{e}^t}$ we can rewrite the term in brackets and obtain that it suffices to construct W such that

$$\partial_t W\left(\frac{x}{L\mathrm{e}^t},t\right) \leq \int_0^\infty \frac{K_\varepsilon(x,y)}{y}h_\varepsilon(y)\left[W\left(\frac{x+y}{L\mathrm{e}^t},t\right) - W\left(\frac{x}{L\mathrm{e}^t},t\right)\right]\mathrm{d}y. \tag{23}$$

For further use we also note that we only need this in weak form, i.e. we need that

$$\int_0^T\int_0^\infty \mathrm{e}^{-(1-\rho)t}h_\varepsilon(x)\left\{\partial_t W\left(\frac{x}{L\mathrm{e}^t},t\right) - \int_0^\infty \frac{K_\varepsilon(x,y)}{y}h_\varepsilon(y)\left[W\left(\frac{x+y}{L\mathrm{e}^t},t\right) - W\left(\frac{x}{L\mathrm{e}^t},t\right)\right]\mathrm{d}y\right\}\mathrm{d}x\mathrm{d}t \leq 0, \tag{24}$$

provided that we can justify the change from ψ to W.

We furthermore list here some parameters that are frequently used in the following. For given $\nu \in (0,1)$ that will be fixed later we define

$$\beta := \begin{cases} b & b \geq 0 \\ \nu b & b < 0 \end{cases}, \qquad \omega_1 := \min\{\rho - b, \rho\}, \qquad \omega_2 := \rho, \qquad \tilde{b} := \max\{0, b\}.$$

As described above, we want to replace the integral kernel K_ε and h_ε by corresponding power laws, but need to consider the region near the origin separately.

Lemma 2.2. *For any constant $\tilde{C} > 0$ there exists a function $\tilde{W} \in C^1([0,T], C^\infty(\mathbb{R}))$ solving*

$$\partial_t \tilde{W}(\xi,t) - C\int_0^1 \frac{h_\varepsilon(z)}{z}\left(L^b A^\beta (z+\varepsilon)^{-a} + L^{-a}A^{-\nu a}(z+\varepsilon)^b\right)\left[\tilde{W}\left(\xi+\frac{z}{L}\right) - \tilde{W}(\xi)\right]\mathrm{d}z$$
$$- \frac{C}{L^{\rho+a-\max\{0,b\}}}\int_0^\infty \frac{A^{-\nu a}}{\eta^{1+\omega_1}}\left[\tilde{W}(\xi+\eta) - \tilde{W}(\xi)\right]\mathrm{d}\eta - \frac{C}{L^{\rho-b}}\int_0^\infty \frac{A^\beta}{\eta^{1+\omega_2}}\left[\tilde{W}(\xi+\eta) - \tilde{W}(\xi)\right]\mathrm{d}\eta = 0$$

with $\tilde{W}(\cdot,0) = \chi_{(-\infty,A-\kappa]} *^3 \varphi_{\kappa/3}(\cdot)$.

Proof. This is shown in Proposition B.8. □

Remark 2.3. As shown in the appendix $\tilde{W}$ is non-increasing, has support in $(-\infty, A]$, is non-negative and bounded by 1.

As K_ε might get quite singular at the origin for $\varepsilon \to 0$, we define now W as the function $W(\xi,t) := \tilde{W}(\xi,t)\,\chi_{[A^\nu,\infty)}(\xi)$, i.e. we cut $\tilde{W}$ at $\xi = A^\nu$ in order to avoid integrating near the origin. Obviously W is not in C^1 and thus the corresponding ψ is also not differentiable. But as already mentioned it is enough to show that (24) holds, provided we can justify the change from ψ to W (and reverse). The latter is not difficult and we omit the details here (see Lemma 3.5 in [10]). Thus, it suffices to show that (23) holds for all $\xi \neq A^\nu$ and this is done in the following Lemma.

Lemma 2.4. *For sufficiently large $\tilde{C}$, inequality (23) holds pointwise for all $\xi \neq A^\nu$.*

Proof. From the non-negativity of W the claim follows immediately for $\xi < A^\nu$ (where W is identically zero). Thus it suffices to consider $\xi > A^\nu$. Using furthermore that $\operatorname{supp} W \subset (-\infty, A]$ it suffices to consider $\xi \in (A^\nu, A]$. As W is non-increasing on $(A^\nu, A]$ we can estimate $-\left[W\left(\xi + \frac{y}{Le^t}\right) - W(\xi)\right] \le -\left[W\left(\xi+\frac{y}{L}\right) - W(\xi)\right]$. On the other hand using the estimates on the kernel K we obtain

$$\begin{aligned}
&-\int_0^\infty \frac{K_\varepsilon\left(Le^t\xi, y\right)}{y} h_\varepsilon(y)\left[W\left(\xi+\frac{y}{Le^t}\right) - W(\xi)\right]\mathrm{d}y\\
&\le -C_2\int_0^\infty \frac{\left(Le^t\xi+\varepsilon\right)^{-a}(y+\varepsilon)^b + \left(Le^t\xi+\varepsilon\right)^b (y+\varepsilon)^{-a}}{y} h_\varepsilon(y)\left[W\left(\xi+\frac{y}{L}\right) - W(\xi)\right]\mathrm{d}y\\
&\le -C\int_0^\infty \frac{L^{-a}A^{-\nu a}(y+\varepsilon)^b + L^b A^\beta (y+\varepsilon)^{-a}}{y} h_\varepsilon(y)\left[W\left(\xi+\frac{y}{L}\right) - W(\xi)\right]\mathrm{d}y\\
&\le -C\int_0^1 \frac{h_\varepsilon(y)}{y}\left[L^b A^\beta (y+\varepsilon)^{-a} + L^{-a}A^{-\nu a}(y+\varepsilon)^b\right]\left[W\left(\xi+\frac{y}{L}\right) - W(\xi)\right]\mathrm{d}y\\
&\quad - \frac{C}{L^\rho}\int_{1/L}^\infty \frac{H_\varepsilon(\eta)}{\eta}\left[L^b A^\beta + L^{-a}A^{-\nu a}(L\eta+\varepsilon)^b\right]\left[W(\xi+\eta) - W(\xi)\right]\mathrm{d}\eta\\
&\le -C\int_0^1 \frac{h_\varepsilon(y)}{y}\left[L^b A^\beta (y+\varepsilon)^{-a} + L^{-a}A^{-\nu a}(y+\varepsilon)^b\right]\left[W\left(\xi+\frac{y}{L}\right) - W(\xi)\right]\mathrm{d}y\\
&\quad - \frac{CA^\beta}{L^{\rho-b}}\int_0^\infty \frac{H_\varepsilon(\eta)}{\eta}\left[W(\xi+\eta) - W(\xi)\right]\mathrm{d}\eta - \frac{CA^{-\nu a}}{L^{\rho+a-\max\{0,b\}}}\int_0^\infty \frac{H_\varepsilon(\eta)}{\eta^{1-\max\{0,b\}}}\left[W(\xi+\eta) - W(\xi)\right]\mathrm{d}\eta.
\end{aligned} \tag{25}$$

As $\xi > A^{\nu}$ we have by construction

$$\partial_t W(\xi,t) = \partial_t \tilde{W}(\xi,t) = \tilde{C}\int_0^1 \frac{h_\varepsilon(z)}{z}\left[L^b A^\beta (z+\varepsilon)^{-a} + L^{-a}A^{-\nu a}(z+\varepsilon)^b\right]\left[\tilde{W}\left(\xi+\frac{z}{L}\right) - \tilde{W}(\xi)\right]\mathrm{d}z$$
$$+\tilde{C}\frac{A^\beta}{L^{\rho-b}}\int_0^\infty \frac{1}{\eta^{1+\omega_2}}\left[\tilde{W}(\xi+\eta) - \tilde{W}(\xi)\right]\mathrm{d}\eta$$
$$+\tilde{C}\frac{A^{-\nu a}}{L^{\rho+a-\max\{0,b\}}}\int_0^\infty \frac{1}{\eta^{1+\omega_1}}\left[\tilde{W}(\xi+\eta) - \tilde{W}(\xi)\right]\mathrm{d}\eta. \tag{26}$$

Thus, using that W and $\tilde{W}$ coincide on (A^ν,∞), in order to show (23), i.e.

$$\partial_t W(\xi,t) - \int_0^\infty \frac{K_\varepsilon\left(L\mathrm{e}^t\xi, y\right)}{y} h_\varepsilon(y)\left[W\left(\xi+\frac{y}{L\mathrm{e}^t},t\right) - W(\xi,t)\right]\mathrm{d}y \le 0$$

it is sufficient to show

$$\left(\tilde{C}-C\right)\int_0^1 \frac{h_\varepsilon(z)}{z}\left[L^b A^\beta (z+\varepsilon)^{-a} + L^{-a}A^{-\nu a}(z+\varepsilon)^b\right]\left[\tilde{W}\left(\xi+\frac{z}{L}\right) - \tilde{W}(\xi)\right]\mathrm{d}z$$
$$+\frac{A^\beta}{L^{\rho-b}}\int_0^\infty \left(\frac{\tilde{C}}{\eta^{1+\omega_2}} - C\frac{H_\varepsilon(\eta)}{\eta}\right)\left[\tilde{W}(\xi+\eta) - \tilde{W}(\xi)\right]\mathrm{d}\eta$$
$$+\frac{A^{-\nu a}}{L^{\rho+a-\max\{0,b\}}}\int_0^\infty \left(\frac{\tilde{C}}{\eta^{1+\omega_1}} - C\frac{H_\varepsilon(\eta)}{\eta^{1-\max\{0,b\}}}\right)\left[\tilde{W}(\xi+\eta) - \tilde{W}(\xi)\right]\mathrm{d}\eta \le 0. \tag{27}$$

We prove that the three terms on the left-hand side are non-positive individually for sufficiently large $\tilde{C}$. This is obvious for the first term as $\tilde{W}$ is non-increasing. It remains to consider the other two terms. Defining $V(\eta) := C\int_\eta^\infty \frac{H_\varepsilon(r)}{r}\mathrm{d}r$, rewriting the second term on the left-hand side and integrating by parts, we obtain

$$\frac{A^\beta}{L^{\rho-b}}\int_0^\infty \left(\frac{\tilde{C}}{\eta^{1+\omega_2}} - C\frac{H_\varepsilon(\eta)}{\eta}\right)\left[\tilde{W}(\xi+\eta) - \tilde{W}(\xi)\right]\mathrm{d}\eta$$
$$= \frac{A^\beta}{L^{\rho-b}}\int_0^\infty -\partial_\eta\left(\frac{\tilde{C}}{\omega_2}\eta^{-\omega_2} - V(\eta)\right)\left[\tilde{W}(\xi+\eta) - \tilde{W}(\xi)\right]\mathrm{d}\eta$$
$$= \frac{A^\beta}{L^{\rho-b}}\int_0^\infty \left(\frac{\tilde{C}}{\omega_2}\eta^{-\omega_2} - V(\eta)\right)\partial_\eta\tilde{W}(\xi+\eta)\,\mathrm{d}\eta.$$

As $\tilde{W}$ is non-increasing we have $\partial_\eta\tilde{W}(\xi+\eta) \le 0$. Furthermore $\frac{A^\beta}{L^{\rho-b}} \ge 0$ and thus it suffices to show $\frac{\tilde{C}}{\omega_2}\eta^{-\omega_2} - V(\eta) \ge 0$ for all $\eta > 0$. Due to the rescaling the estimates from Lemma A.1 also hold for H_ε and we obtain $V(\eta) \le C\eta^{-\rho}$. Using furthermore that $\omega_2 = \rho$ it remains to show that $\left(\frac{\tilde{C}}{\rho} - C\right)\eta^{-\rho} \ge 0$ which certainly holds for $\tilde{C}$ sufficiently large. The third term on the left-hand side of (27) can be estimated in the same way. This concludes the proof. □

Remark 2.5. The change of variables from ψ to W (and reverse) can be justified rigorously (see Lemma 3.5 in [10]) and inequality (24) holds.

In the following we will derive a lower estimate for W.

2.1.2. Lower bound on W

Lemma 2.6. *There exists* $\sigma \in (\max\{b,\nu\},1)$ *and* $\theta > 0$ *such that*

$$1 - W(A - A^{\sigma}) \le C t A^{-\theta}$$

for sufficiently large A.

Proof. From the construction in Section B.1 we know that $\tilde{W}$ can be written as $\tilde{W}(\xi,t) = \int_{\xi}^{\infty} G(\eta,t)\,\mathrm{d}\eta$ with $G = -\partial_{\xi}\tilde{W} = G_{1,1} * G_{1,2} * G_2$, where $G_{1,1}$, $G_{1,2}$ and G_2 solve

$$\begin{aligned}
\partial_t G_{1,1} &= \frac{CA^{-\nu a}}{L^{\rho+a-\max\{0,b\}}} \int_0^{\infty} \frac{1}{\eta^{1+\omega_1}} \left[G_{1,1}(\xi+\eta) - G_{1,1}(\xi)\right] \mathrm{d}\eta \\
G_{1,1}(\cdot,0) &= \delta(\cdot - A + \kappa) * \varphi_{\kappa/3} \\
\partial_t G_{1,2} &= \frac{CA^{\beta}}{L^{\rho-b}} \int_0^{\infty} \frac{1}{\eta^{1+\omega_2}} \left[G_{1,2}(\xi+\eta) - G_{1,2}(\xi)\right] \mathrm{d}\eta \\
G_{1,2}(\cdot,0) &= \delta(\cdot) * \varphi_{\kappa/3} \\
\partial_t G_2 &= C \int_0^1 \frac{h_{\varepsilon}(z)}{z} \left[\frac{A^{-\nu a}}{L^a}(z+\varepsilon)^b + A^{\beta} L^{\beta}(z+\varepsilon)^{-a}\right] \cdot \left[G_2\left(\xi + \frac{z}{L}\right) - G_2(\xi)\right] \mathrm{d}z \\
G_2(\cdot,0) &= \delta(\cdot) * \varphi_{\kappa/3}.
\end{aligned} \tag{28}$$

Then one has from Lemma B.10 for any $\mu \in (0,1)$:

$$\int_{-\infty}^{-D} G_{1,2}(\xi,t)\,\mathrm{d}\xi \le C\left(\frac{\kappa}{D}\right)^{\mu} + \frac{CA^{\beta}t}{L^{\rho-b}D^{\omega_2}}$$

and

$$\begin{aligned}
\int_{-\infty}^{-D+A} G_{1,1}(\xi,t)\,\mathrm{d}\xi &= \int_{-\infty}^{(A-\kappa)-(D-\kappa)} G_{1,1}(\xi,t)\,\mathrm{d}\xi \le C\left(\frac{\kappa}{D-\kappa}\right)^{\mu} + \frac{CA^{-\nu a}t}{L^{\rho+a-\max\{0,b\}}(D-\kappa)^{\omega_1}} \\
&\le C\left(\frac{\kappa}{D}\right)^{\mu} + \frac{CA^{-\nu a}t}{L^{\rho+a-\max\{0,b\}}D^{\omega_1}}.
\end{aligned}$$

In the last step we used that for any $\delta \in (0,1)$ and $D \ge 1$, $\kappa \le 1/2$ it holds $(D-\kappa)^{-\delta} \le 2^{\delta} D^{-\delta}$. One thus needs an estimate for G_2. This will be quite similar to the proof of Lemma B.10 but due to the different behaviour for $L_{\varepsilon} \to 0$ and $L_{\varepsilon} \not\to 0$ we sketch this here again. Defining $\tilde{G}_2(p,t) := \int_{\mathbb{R}} G_2(\xi,t)\,\mathrm{e}^{p(\xi-\kappa/3)}\mathrm{d}\xi$ and multiplying the equation for G_2 in (28) by $\mathrm{e}^{p(\xi-\kappa/3)}$ and integrating one obtains

$$\begin{aligned}
\partial_t \tilde{G}_2(p,t) &= C \int_0^1 \frac{h_{\varepsilon}(z)}{z} \left[(z+\varepsilon)^{-a} L^b A^{\beta} + (z+\varepsilon)^b L^{-a} A^{-\nu a}\right] \cdot \left[\mathrm{e}^{-\frac{pz}{L}} - 1\right] \mathrm{d}z\, \tilde{G}_2(p,t) \\
&=: M(p,L)\,\tilde{G}_2(p,t).
\end{aligned}$$

Thus $\tilde{G}_2(p,t) = \int_{\mathbb{R}} \varphi_{\kappa/3}(\xi)\,\mathrm{e}^{p(\xi-\kappa/3)}\mathrm{d}\xi \exp(-t\,|M(p,L)|)$ and one can estimate:

$$|M(p,L)| \le C\int_0^1 \frac{h_\varepsilon(z)}{z}\left[(z+\varepsilon)^{-a}L^bA^\beta + (z+\varepsilon)^{-a}L^{-a}A^{-\nu a}\right]\cdot\frac{pz}{L}\mathrm{d}z$$
$$= Cp\left(L^{b-1}A^\beta\mu_\varepsilon + L^{-a-1}A^{-\nu a}\lambda_\varepsilon\right) \le Cp\left(A^\beta + A^{-\nu a}\right).$$

For the last step note that due to our notation either $L = L_\varepsilon$ (in the case $L_\varepsilon \not\to 0$) and then the estimate is due to the definition of L_ε. If $L = 1$ (in the case $L_\varepsilon \to 0$) one can assume without loss of generality that ε is so small that $L_\varepsilon \le 1$ (and thus by definition also $\lambda_\varepsilon, \mu_\varepsilon \le 1$). Using this and inserting $p := \frac{1}{D}$ we obtain in the same way as in the proof of Lemma B.10:

$$\int_{-\infty}^{-D} G_2(\xi,t)\,\mathrm{d}\xi \le C\left(\left(\frac{\kappa}{D}\right)^\mu + \frac{t}{D}\left(A^\beta + A^{-\nu a}\right)\right).$$

Using these estimates on $G_{1,1}$, $G_{1,2}$ and G_2 one obtains from Lemma B.11 (note also Remark B.12):

$$1-\tilde{W}(A-D) = \int_{-\infty}^{A-D}\left(G_{1,1}*G_{1,2}*G_2\right)\mathrm{d}\xi \le \int_{-\infty}^{A-\frac{D}{4}} G_{1,1}\mathrm{d}\xi + \int_{-\infty}^{-\frac{D}{4}} G_{1,2}\mathrm{d}\xi + \int_{-\infty}^{-\frac{D}{4}} G_2\mathrm{d}\xi$$
$$\le C\frac{\kappa^\mu}{D^\mu} + \frac{CA^{-\nu a}t}{L^{\rho+a-\max\{0,b\}}D^{\omega_1}} + \frac{CA^\beta t}{L^{\rho-b}D^{\omega_2}} + \frac{Ct\left(A^\beta + A^{-\nu a}\right)}{D}.$$

Choosing $D = A^\sigma$ (with $A \ge 1$) one has

$$1-\tilde{W}\left(A-A^\sigma\right) \le C\kappa^\mu A^{-\mu\sigma} + \frac{Ct}{L^{a+\omega_1}}A^{-\nu a-\sigma\omega_1} + \frac{Ct}{L^{\rho-b}}A^{\beta-\sigma\omega_2} + Ct\left(A^{\beta-\sigma} + A^{-\nu a-\sigma}\right).$$

In the case $L = 1$ (i.e. $L_\varepsilon \to 0$) it suffices to consider the exponents of A:

- $-\mu\sigma < 0$, as $\mu, \sigma > 0$,
- $-\nu a - \sigma\omega_1 < 0$, as $\omega_1, a > 0$,
- $\beta - \sigma\omega_2 = \beta - \sigma\rho = \begin{cases} b-\sigma\rho & b \ge 0 \\ \nu b - \sigma\rho & b < 0 \end{cases} < 0$, independently of the sign of b if we choose σ sufficiently close to 1 and $\sigma > \nu$ (as $b < \rho$).
- $\beta - \sigma = \begin{cases} b-\sigma & b \ge 0 \\ \nu b - \sigma & b < 0 \end{cases} < 0$, independently of the sign of b if we choose $\sigma > b$ as $\nu < 1$ and $b < 1$ (note that this choice of σ does not collide with the choice made before)
- $-\nu a - \sigma < 0$, as $a, \sigma > 0$.

Thus, taking $-\theta$ to be the maximum of the (negative) exponents proves the claim in this case.

If $L = L_\varepsilon$ (i.e. $L_\varepsilon \not\to 0$) one has to consider also the exponents of L:

- $a + \omega_1 > 0$, as $a, \omega_1 > 0$,
- $\rho - b > 0$, as by assumption $b < \rho$.

Thus either the two terms containing $L = L_\varepsilon$ are bounded (if L_ε is bounded) or converge to zero (if $L_\varepsilon \to \infty$) and so in both cases with the same $\theta > 0$ as above the claim follows. □

2.1.3. The iteration argument

In this section we will prove Proposition 2.1. We therefore define

$$F_\varepsilon(X) := \int_0^X H_\varepsilon(Y)\,\mathrm{d}Y \quad \text{while for } L_\varepsilon \to 0 \text{ this reduces to} \quad F_\varepsilon(x) = \int_0^x h_\varepsilon(y)\,\mathrm{d}y.$$

We first show the following Lemma, that will be the key in the proof of Proposition 2.1.

Lemma 2.7. *There exists $\theta > 0$ such that*

$$F_\varepsilon(A) \geq -CA^{\nu(1-\rho)} + F_\varepsilon\left(\left(A - A^\sigma\right) e^T\right) e^{-(1-\rho)T}\left(1 - \frac{C}{A^\theta}\right)$$

for A sufficiently large.

Proof. From the choice of ψ and W respectively (using also the non-negativity and monotonicity properties of W)

$$\begin{aligned}
F_\varepsilon(A) &= \int_0^A H_\varepsilon(X)\,dX \geq \int_0^\infty W(X,0)\,H_\varepsilon(X)\,dX \geq e^{-(1-\rho)T}\int_0^\infty H_\varepsilon(X)\,W\left(\frac{X}{e^T}, T\right) dX \\
&\geq e^{-(1-\rho)T}\int_{A^\nu}^\infty \partial_X F_\varepsilon(X)\,W\left(\frac{X}{e^T}, T\right) dX \\
&= -e^{-(1-\rho)T} F_\varepsilon\left(A^\nu\right) W\left(\frac{A^\nu}{e^T}, T\right) - \int_{A^\nu}^\infty e^{-(1-\rho)T} e^{-T} F_\varepsilon(X)\,\partial_\xi W\left(\frac{X}{e^T}, T\right) dX \\
&\geq -CA^{\nu(1-\rho)} + e^{-(1-\rho)T}\int_{A^\nu e^{-T}}^\infty F_\varepsilon\left(Xe^T\right)\left(G_{1,1} * G_{1,2} * G_2\right)(X,T)\,dX
\end{aligned}$$

where we changed variables in the last step and used that W is bounded, $\partial_\xi W = -G_{1,1} * G_{1,2} * G_2$ on (A^ν, ∞) as well as $F_\varepsilon(A^\nu) \leq A^{\nu(1-\rho)}$. Noting that for $\sigma > \nu$ we have $A^\nu e^{-T} \leq A - A^\sigma$ for sufficiently large A and using also the monotonicity of F_ε we can further estimate

$$\begin{aligned}
F_\varepsilon(A) &\geq -CA^{\nu(1-\rho)} + e^{-(1-\rho)T}\int_{A-A^\sigma}^\infty F_\varepsilon\left(Xe^T\right)\left(G_{1,1} * G_{1,2} * G_2\right)(X,T)\,dX \\
&\geq -CA^{\nu(1-\rho)} + e^{-(1-\rho)T} F_\varepsilon\left(\left(A - A^\sigma\right) e^T\right)\int_{A-A^\sigma}^\infty \left(G_{1,1} * G_{1,2} * G_2\right)(X,T)\,dX \\
&= -CA^{\nu(1-\rho)} + F_\varepsilon\left(\left(A - A^\sigma\right) e^T\right) e^{-(1-\rho)T} W\left(A - A^\sigma\right) \\
&\geq -CA^{\nu(1-\rho)} + F_\varepsilon\left(\left(A - A^\sigma\right) e^T\right) e^{-(1-\rho)T}\left(1 - \frac{C}{A^\theta}\right),
\end{aligned}$$

while in the last step Lemma 2.6 was applied. □

We are now prepared to prove Proposition 2.1. This will be done by some iteration argument using recursively Lemma 2.7.

Proof of Proposition 2.1. Let $\alpha := e^T > 1$. For any $\delta > 0$ there exists $R_{\epsilon,\delta} > 0$ such that $F_\varepsilon(R) \geq R^{1-\rho}(1-\delta)$ for all $R \geq R_{\varepsilon,\delta}$. For $A_0 > \left(\frac{\alpha}{\alpha-1}\right)^{\frac{1}{1-\sigma}}$ we define a sequence $\{A_k\}_{k\in\mathbb{N}_0}$ by $A_{k+1} := \alpha\left(A_k - A_k^\sigma\right)$. From the choice of A_0 one obtains that A_k is strictly increasing and one has $A_k \to \infty$ as $k \to \infty$. Furthermore $\alpha A_k = A_{k+1} + \alpha A_k^\sigma$ and thus

$$A_k = \frac{A_{k+1}}{\alpha}\left(1 + \alpha\frac{A_k^\sigma}{A_{k+1}}\right).$$

By iteration one obtains for any $N \in \mathbb{N}$:

$$A_0 = \frac{A_N}{\alpha^N}\prod_{k=0}^{N-1}\left(1 + \alpha\frac{A_k^\sigma}{A_{k+1}}\right). \tag{29}$$

For any $N \in \mathbb{N}$ and $0 \le k < N$ applying Lemma 2.7 one gets by induction:

$$F_\varepsilon(A_k) \ge F_\varepsilon(A_N)\,\alpha^{-(N-k)(1-\rho)} \prod_{n=k}^{N-1}\left(1-\frac{C}{A_n^\theta}\right) - C\sum_{m=k}^{N-1}\alpha^{-(m-k)(1-\rho)}\left(\prod_{n=k}^{m-1}\left(1-\frac{C}{A_n^\theta}\right)\right)A_m^{\nu(1-\rho)}, \tag{30}$$

where we use the convention $\sum_{k=l}^{u} a_k = 0$ and $\prod_{k=l}^{u} a_k = 1$ if $u < l$. Thus for $k = 0$ one particularly obtains

$$\begin{aligned} F_\varepsilon(A_0) \ \ge\ & F_\varepsilon(A_N)\,\alpha^{-N(1-\rho)} \prod_{n=0}^{N-1}\left(1-\frac{C}{A_n^\theta}\right) - C\sum_{m=0}^{N-1}\alpha^{-m(1-\rho)}\left(\prod_{n=0}^{m-1}\left(1-\frac{C}{A_n^\theta}\right)\right)A_m^{\nu(1-\rho)} \\ =\ & F_\varepsilon(A_N)\,\alpha^{-N(1-\rho)} \prod_{n=0}^{N-1}\left(1-\frac{C}{A_n^\theta}\right) - C\sum_{m=0}^{N-1}\alpha^{(\nu-1)(1-\rho)m}\left(\prod_{n=0}^{m-1}\left(1-\frac{C}{A_n^\theta}\right)\right)\left(\alpha^{-m}A_m\right)^{\nu(1-\rho)} \\ =:\ & (I)-(II)\,. \end{aligned} \tag{31}$$

We now estimate the two terms separately.

Let $\delta_* := \delta/2$. Choosing N sufficiently large such that $A_N \ge R_{\epsilon,\delta_*}$ one has, using also (29)

$$\begin{aligned} (I) &\ge (1-\delta_*)\,A_N^{1-\rho}\alpha^{-N(1-\rho)}\prod_{n=0}^{N-1}\left(1-\frac{C}{A_n^\theta}\right) = (1-\delta_*)\left(\frac{A_N}{\alpha^N}\right)^{1-\rho}\prod_{n=0}^{N-1}\left(1-\frac{C}{A_n^\theta}\right) \\ &= (1-\delta_*)\,A_0^{1-\rho}\frac{\prod_{n=0}^{N-1}\left(1-\frac{C}{A_n^\theta}\right)}{\left(\prod_{k=0}^{N-1}\left(1+\alpha\frac{A_k^\sigma}{A_{k+1}}\right)\right)^{1-\rho}}. \end{aligned} \tag{32}$$

Let $0 < D_0 < D$ be parameters to be fixed later and assume $A_0 > D$. One has $A_{k+1} = \alpha\left(A_k - A_k^\sigma\right)$. Thus using the monotonicity of A_k

$$\frac{A_{k+1}}{A_k} = \alpha\left(1 - A_k^{\sigma-1}\right) > \alpha\left(1 - D_0^{\sigma-1}\right) =: \beta_0 > 1$$

if we fix D_0 sufficiently large as $\alpha > 1$. Using this, one has $A_{k+1} > \beta_0 A_k$ and thus by iteration $A_{k+1} > \beta_0^{k+1}A_0$.

We continue to estimate (*I*) and thus consider first $\prod_{n=0}^{N-1}\left(1-\frac{C}{A_n^\theta}\right)$ while we assume that D_0 is sufficiently large such that $\frac{C}{D^\theta} < 1$ and thus also $\frac{C}{A_n^\theta} < 1$ by the monotonicity of A_n. Taking the logarithm of the product one has using the estimate $\log(1-x) \ge -\frac{x}{1-x}$:

$$\begin{aligned} \sum_{n=0}^{N-1}\log\left(1-\frac{C}{A_n^\theta}\right) &\ge \sum_{n=0}^{N-1} -\frac{C}{A_n^\theta}\cdot\frac{1}{1-\frac{C}{A_n^\theta}} = -C\sum_{n=0}^{N-1}\frac{1}{A_n^\theta - C} \ge -C\sum_{n=0}^{N-1}\frac{1}{\beta_0^{n\theta}A_0^\theta - C} \\ &\ge -C\sum_{n=0}^{N-1}\frac{1}{\beta_0^{\theta n}}\frac{1}{D^\theta - C} \ge -C\frac{\beta_0^\theta}{\beta_0^\theta - 1}\frac{1}{D^\theta - C} =: -\frac{C_\beta}{D^\theta - C}. \end{aligned}$$

Thus one obtains using $\exp(-x) \ge 1 - x$:

$$\prod_{n=0}^{N-1}\left(1-\frac{C}{A_n^\theta}\right) \ge \exp\left(-\frac{C_\beta}{D^\theta - C}\right) \ge 1 - \frac{C_\beta}{D^\theta - C}. \tag{33}$$

Considering $\prod_{k=0}^{N-1}\left(1+\alpha\frac{A_k^\sigma}{A_{k+1}}\right)$ and applying again first the logarithm on the product and then using $\log(1+x) \le x$ one obtains

$$\begin{aligned} \sum_{k=0}^{N-1}\log\left(1+\alpha\frac{A_k^\sigma}{A_{k+1}}\right) &\le \sum_{k=0}^{N-1}\alpha\frac{A_k^\sigma}{A_{k+1}} \le \alpha\sum_{k=0}^{N-1}A_k^{\sigma-1} \le \alpha A_0^{\sigma-1}\sum_{k=0}^{N-1}\left(\beta_0^{\sigma-1}\right)^k \le \alpha D^{\sigma-1}\sum_{k=0}^{\infty}\left(\beta_0^{\sigma-1}\right)^k \\ &=: C_\gamma D^{\sigma-1}. \end{aligned}$$

Thus one can estimate

$$\left(\prod_{k=0}^{N-1}\left(1+\alpha\frac{A_k^\sigma}{A_{k+1}}\right)\right)^{1-\rho} \le \exp\left[\left(C_\gamma D^{\sigma-1}\right)(1-\rho)\right] = 1+\sum_{n=1}^{\infty}\frac{(1-\rho)^n C_\gamma^n D^{n(\sigma-1)}}{n!}$$
$$\le 1 + D^{\sigma-1}\sum_{n=1}^{\infty}\frac{C_\gamma^n (1-\rho)^n}{n!} \le 1+\frac{\exp\left(C_\gamma(1-\rho)\right)}{D^{1-\sigma}}. \tag{34}$$

Combining the estimates of (33) and (34) one obtains

$$\frac{\prod_{n=0}^{N-1}\left(1-\frac{C}{A_n^\theta}\right)}{\left(\prod_{k=0}^{N-1}\left(1+\alpha\frac{A_k^\sigma}{A_{k+1}}\right)\right)^{1-\rho}} \ge \frac{1-\frac{C_\beta}{D^\theta - C}}{1+\frac{\exp(C_\gamma(1-\rho))}{D^{1-\sigma}}} = 1-\frac{\frac{C_\beta}{D^\theta-C}+\frac{\exp(C_\gamma(1-\rho))}{D^{1-\sigma}}}{1+\frac{\exp(C_\gamma(1-\rho))}{D^{1-\sigma}}}$$
$$\ge 1-\frac{C_\beta}{D^\theta - C}-\frac{\exp\left(C_\gamma(1-\rho)\right)}{D^{1-\sigma}}. \tag{35}$$

Together with (32) this shows

$$(I) \ge (1-\delta_*)\,A_0^{1-\rho}\left(1-\frac{C_\beta}{D^\theta-C}-\frac{\exp\left(C_\gamma(1-\rho)\right)}{D^{1-\sigma}}\right). \tag{36}$$

To estimate (II) we first note that one has $\prod_{n=0}^{m-1}\left(1-\frac{C}{A_n^\theta}\right)\le 1$ as well as $\prod_{k=0}^{m-1}\left(1+\alpha\frac{A_k^\sigma}{A_{k+1}}\right)\ge 1$ for any $m\in\mathbb{N}_0$. Using this as well as (29) for $N=m$ and $m=0,\dots,N-1$ one obtains

$$(II) = C\sum_{m=0}^{N-1}\alpha^{(\nu-1)(1-\rho)m}\left(\prod_{n=0}^{m-1}\left(1-\frac{C}{A_n^\theta}\right)\right)\left(\alpha^{-m}A_m\right)^{\nu(1-\rho)} \le C\sum_{m=0}^{N-1}\alpha^{(\nu-1)(1-\rho)m}\left(\alpha^{-m}A_m\right)^{\nu(1-\rho)}$$
$$= C\sum_{m=0}^{N-1}\alpha^{(\nu-1)(1-\rho)m}\frac{A_0^{\nu(1-\rho)}}{\left(\prod_{k=0}^{m-1}\left(1+\alpha\frac{A_k^\sigma}{A_{k+1}}\right)\right)^{\nu(1-\rho)}} \le C\sum_{m=0}^{N-1}\alpha^{(\nu-1)(1-\rho)m}A_0^{\nu(1-\rho)}$$
$$\le A_0^{\nu(1-\rho)}C\sum_{m=0}^{\infty}\alpha^{(\nu-1)(1-\rho)m} =: C_\nu A_0^{\nu(1-\rho)}. \tag{37}$$

Combining (36), (37) and (31) yields

$$F_\varepsilon(A_0) \ge (1-\delta_*)\left(1-\frac{C_\beta}{D^\theta-C}-\frac{\exp\left(C_\gamma(1-\rho)\right)}{D^{1-\sigma}}\right)A_0^{1-\rho} - C_\nu A_0^{\nu(1-\rho)}$$
$$\ge A_0^{1-\rho}\left((1-\delta_*)\left(1-\frac{C_\beta}{D^\theta-C}-\frac{\exp\left(C_\gamma(1-\rho)\right)}{D^{1-\sigma}}\right)-\frac{C_\nu}{D^{(1-\nu)(1-\rho)}}\right). \tag{38}$$

We choose now D sufficiently large such that one has

- $D\ge\left(3C_\beta\frac{2-\delta}{\delta}+C\right)^{1/\theta}$ which is equivalent to $\frac{C_\beta}{D^\theta-C}\le\frac{1}{3}\frac{\delta}{2-\delta}$
- $D\ge\left(3\exp\left(C_\gamma(1-\rho)\right)\frac{2-\delta}{\delta}\right)^{\frac{1}{1-\sigma}}$ which is equivalent to $\frac{\exp(C_\gamma(1-\rho))}{D^{1-\sigma}}\le\frac{1}{3}\frac{\delta}{2-\delta}$
- $D\ge\left(\frac{3C_\nu}{1-\delta/2}\frac{2-\delta}{\delta}\right)^{\frac{1}{(1-\nu)(1-\rho)}}$ which is equivalent to $C_\nu\frac{1}{D^{(1-\nu)(1-\rho)}}\le(1-\delta/2)\frac{1}{3}\frac{\delta}{2-\delta}$.

Inserting these estimates into (38) together with $\delta_*=\delta/2$ one gets

$$F_\varepsilon(A_0) \geq A_0^{1-\rho}\left(\left(1-\frac{\delta}{2}\right)\left(1-\frac{2}{3}\frac{\delta}{2-\delta}\right)-\left(1-\frac{\delta}{2}\right)\frac{1}{3}\frac{\delta}{2-\delta}\right)$$
$$= A_0^{1-\rho}\left(1-\frac{\delta}{2}\right)\left(1-\frac{\delta}{2-\delta}\right) = (1-\delta)\, A_0^{1-\rho}.$$

This proves the claim with $R_\delta = D$. □

2.2. *Excluding $L_\varepsilon \to \infty$*

Before we can pass to the limit $\varepsilon \to 0$ we have to exclude the case $L_\varepsilon \to \infty$ as $\varepsilon \to 0$ in order to obtain Proposition 2.1 also for h_ε (instead of H_ε). This will be done by some contradiction argument. We furthermore remark here that throughout this section we frequently use that we can bound

$$\int_0^1 (x+\varepsilon)^{-a} h_\varepsilon(x)\, dx \leq L_\varepsilon^{1-b} \quad \text{and} \quad \int_0^1 (x+\varepsilon)^{b} h_\varepsilon(x)\, dx \leq L_\varepsilon^{1+a} \tag{39}$$

due to the definition of L_ε in (17).

2.2.1. *Deriving a limit equation for H_ε*

We first show the following lemma, stating the convergence of a certain integral occurring later.

Lemma 2.8. *Assume $L_\varepsilon \to \infty$ as $\varepsilon \to 0$. Let Q_ε be given by*

$$Q_\varepsilon(X) := \int_0^1 \frac{h_\varepsilon(y)}{L_\varepsilon} K_\varepsilon(y, L_\varepsilon X)\, dy.$$

Then there exists a (continuous) function Q such that $Q_\varepsilon \to Q$ locally uniformly up to a subsequence.

Proof. It suffices to show that both Q_ε as well as Q_ε' are uniformly bounded on each fixed interval $[d, D] \subset \mathbb{R}_+$. One has using (39)

$$\begin{aligned}
|Q_\varepsilon(X)| &\leq C_2 \int_0^1 \frac{h_\varepsilon(z)}{L_\varepsilon}\left((L_\varepsilon X+\varepsilon)^{-a}(z+\varepsilon)^b + (L_\varepsilon X+\varepsilon)^b (z+\varepsilon)^{-a}\right) dz \\
&\leq \frac{C_2}{L_\varepsilon}(L_\varepsilon X+\varepsilon)^{-a}\int_0^1 (z+\varepsilon)^b h_\varepsilon(z)\, dz + \frac{C_2}{L_\varepsilon}(L_\varepsilon X+\varepsilon)^{b}\int_0^1 (z+\varepsilon)^{-a} h_\varepsilon(z)\, dz \\
&\leq C_2 L_\varepsilon^a (L_\varepsilon X+\varepsilon)^{-a} + C_2 L_\varepsilon^{-b}(L_\varepsilon X+\varepsilon)^b \\
&= C_2\left(X+\frac{\varepsilon}{L_\varepsilon}\right)^{-a} + C_2\left(X+\frac{\varepsilon}{L_\varepsilon}\right)^{b}
\end{aligned} \tag{40}$$

while the right hand side is clearly locally uniformly bounded under the given assumptions. Rewriting Q_ε one obtains

$$Q_\varepsilon(X) = \int_0^1 \frac{h_\varepsilon(z)}{L_\varepsilon} K_\varepsilon(L_\varepsilon X, z)\, dz = L_\varepsilon^\gamma \int_0^1 \frac{h_\varepsilon(z)}{L_\varepsilon} K_{\frac{\varepsilon}{L_\varepsilon}}\left(X, \frac{z}{L_\varepsilon}\right) dz.$$

Furthermore from (11) one has

$$\left|\partial_y K_\varepsilon(y,z)\right| \leq C\left((z+\varepsilon)^{-a} + (z+\varepsilon)^b\right) \quad \text{for all } y \in [a, A]$$

and hence, similarly as before

$$\left|Q'_\varepsilon(X)\right| \le C L_\varepsilon^\gamma \int_0^1 \frac{h_\varepsilon(z)}{L_\varepsilon}\left[\left(\frac{z+\varepsilon}{L_\varepsilon}\right)^{-a} + \left(\frac{z+\varepsilon}{L_\varepsilon}\right)^{b}\right] \mathrm{d}z$$

$$= C L_\varepsilon^{\gamma-1+a} \int_0^1 (z+\varepsilon)^{-a} h_\varepsilon(z)\,\mathrm{d}z + C L_\varepsilon^{\gamma-1-b} \int_0^1 (z+\varepsilon)^{b} h_\varepsilon(z)\,\mathrm{d}z \le C$$

where we used also $\gamma = b - a$. □

Lemma 2.9. *Let $\rho \in (\max\{0, b\}, 1)$ and assume $L_\varepsilon \to \infty$ as $\varepsilon \to 0$. Then there exists a measure H such that (up to a subsequence) $H_\varepsilon \overset{*}{\rightharpoonup} H$ and H satisfies*

$$\partial_X(XH) + (\rho - 1)H - \partial_X(Q(X)H(X)) + \frac{H(X)Q(X)}{X} = 0 \tag{41}$$

in the sense of distributions with $Q(X) = \lim_{\varepsilon\to 0}\int_0^1 \frac{h_\varepsilon(y)}{L_\varepsilon} K_\varepsilon(y, L_\varepsilon X)\,\mathrm{d}y$.

Proof. Transforming the equation

$$\partial_x\left(\int_0^x \int_{x-y}^\infty \frac{K_\varepsilon(y,z)}{z} h_\varepsilon(y) h_\varepsilon(z)\,\mathrm{d}z\mathrm{d}y\right) = \partial_x(x h_\varepsilon(x)) + (\rho-1) h_\varepsilon(x)$$

to the rescaled variables $X = \frac{x}{L_\varepsilon}$ one obtains

$$\frac{1}{L_\varepsilon}\partial_X\left(\int_0^{L_\varepsilon X} \int_{L_\varepsilon X - y}^\infty \frac{K_\varepsilon(y,z)}{z} h_\varepsilon(y) h_\varepsilon(z)\,\mathrm{d}z\mathrm{d}y\right) = \frac{1}{L_\varepsilon}\partial_X\left(L_\varepsilon X \frac{H(X)}{L_\varepsilon^\rho}\right) + (\rho-1)\frac{H_\varepsilon(X)}{L_\varepsilon^\rho}.$$

Testing with $\psi \in C_c^\infty(\mathbb{R}_+)$ (in the rescaled X-variable), splitting the integral and interchanging the order of integration we can rewrite this as

$$\int_0^\infty \left(X\partial_X\psi(X) - (\rho-1)\psi(X)\right) H_\varepsilon(X)\,\mathrm{d}X$$

$$= \frac{1}{L_\varepsilon^{\rho-\gamma}} \int_{\frac{1}{L_\varepsilon}}^\infty \int_{\frac{1}{L_\varepsilon}}^\infty \frac{K_{\frac{\varepsilon}{L_\varepsilon}}(Y,Z)}{Z} H_\varepsilon(Y) H_\varepsilon(Z)\left[\psi(Y+Z) - \psi(Y)\right]\mathrm{d}Z\mathrm{d}Y$$

$$+ \int_{\frac{1}{L_\varepsilon}}^\infty \int_0^1 \frac{K_\varepsilon(L_\varepsilon Y, z)}{z} h_\varepsilon(z) H_\varepsilon(Y)\left[\psi\left(Y + \frac{z}{L_\varepsilon}\right) - \psi(Y)\right]\mathrm{d}z\mathrm{d}Y$$

$$+ \int_0^1 \int_0^\infty \left(\int_{\frac{y}{L_\varepsilon}}^{\frac{y}{L_\varepsilon}+Z} \partial_X\psi(X)\,\mathrm{d}X\right) \frac{K_\varepsilon(y, L_\varepsilon Z)}{L_\varepsilon Z} h_\varepsilon(y) H_\varepsilon(Z)\,\mathrm{d}Z\mathrm{d}y$$

$$= (I) + (II) + (III).$$

In the following we assume that $\operatorname{supp}\psi \subset [d, D]$ with $d > 0$ and $D > 1$. Furthermore we can assume that $L_\varepsilon > 1$ is sufficiently large and that $\varepsilon < 1$ is sufficiently small (as we are assuming $L_\varepsilon \to \infty$ for $\varepsilon \to 0$). We first show that $(I) \to 0$ as $\varepsilon \to 0$:

$$|(I)| \leq \frac{C_2}{L_\varepsilon^{\rho-\gamma}} \int_{\frac{1}{L_\varepsilon}}^{D} \int_{\frac{1}{L_\varepsilon}}^{\infty} \frac{\left(Y+\frac{\varepsilon}{L_\varepsilon}\right)^{-a}\left(Z+\frac{\varepsilon}{L_\varepsilon}\right)^{b}}{Z} H_\varepsilon(Y) H_\varepsilon(Z) |\psi(Y+Z)-\psi(Y)| \,\mathrm{d}Z\mathrm{d}Y$$

$$+ \frac{C_2}{L_\varepsilon^{\rho-\gamma}} \int_{\frac{1}{L_\varepsilon}}^{D} \int_{\frac{1}{L_\varepsilon}}^{\infty} \frac{\left(Y+\frac{\varepsilon}{L_\varepsilon}\right)^{b}\left(Z+\frac{\varepsilon}{L_\varepsilon}\right)^{-a}}{Z} H_\varepsilon(Y) H_\varepsilon(Z) |\psi(Y+Z)-\psi(Y)| \,\mathrm{d}Z\mathrm{d}Y$$

$$\leq \frac{2^{\tilde{b}} C}{L_\varepsilon^{\rho-\gamma}} \int_{\frac{1}{L_\varepsilon}}^{D} \int_{\frac{1}{L_\varepsilon}}^{\infty} \frac{Y^{-a}Z^{b}+Y^{b}Z^{-a}}{Z} H_\varepsilon(Y) H_\varepsilon(Z) |\psi(Y+Z)-\psi(Y)| \,\mathrm{d}Z\mathrm{d}Y$$

$$\leq \frac{C\left\|\psi'\right\|_{L^\infty}}{L_\varepsilon^{\rho-\gamma-a}} \int_{\frac{1}{L_\varepsilon}}^{D} \int_{\frac{1}{L_\varepsilon}}^{1} \left(Z^{b}+Y^{b}\right) H_\varepsilon(Y) H_\varepsilon(Z) \,\mathrm{d}Z\mathrm{d}Y + \frac{C\left\|\psi\right\|_{L^\infty}}{L_\varepsilon^{\rho-\gamma}}$$

$$\cdot \left[L_\varepsilon^{a} \int_{\frac{1}{L_\varepsilon}}^{D} H_\varepsilon(Y)\,\mathrm{d}Y \int_{1}^{\infty} Z^{b-1} H_\varepsilon(Z)\,\mathrm{d}Z + \max\left\{L_\varepsilon^{-b}, D^{b}\right\} \int_{\frac{1}{L_\varepsilon}}^{D} H_\varepsilon(Y)\,\mathrm{d}Y \int_{1}^{\infty} Z^{-a-1} H_\varepsilon(Z)\,\mathrm{d}Z \right]$$

$$\leq C L_\varepsilon^{\gamma+a-\rho} \max\left\{1, L_\varepsilon^{-b}, D^{b}\right\} \int_{\frac{1}{L_\varepsilon}}^{D} H_\varepsilon(Y)\,\mathrm{d}Y \int_{\frac{1}{L_\varepsilon}}^{1} H_\varepsilon(Z)\,\mathrm{d}Z + C L_\varepsilon^{\gamma-\rho}\left[L_\varepsilon^{a} D^{1-\rho} + \max\left\{L_\varepsilon^{-b}, D^{b}\right\} D^{1-\rho}\right]$$

$$= C D^{1-\rho} L_\varepsilon^{b-\rho} \max\left\{D^{b}, L_\varepsilon^{-b}\right\} + C D^{1-\rho} L_\varepsilon^{b-\rho} + C D^{1-\rho} L_\varepsilon^{b-a-\rho} \max\left\{L_\varepsilon^{-b}, D^{b}\right\} \to 0$$

as $L_\varepsilon \to \infty$ (i.e. for $\varepsilon \to 0$ by assumption).

Next we show $(II) \to \int_0^\infty \partial_X \psi(X) H(X) Q(X)\,\mathrm{d}X$. As H_ε is a sequence of locally uniformly bounded (non-negative Radon) measures there exists a (non-negative Radon) measure H such that $H_\varepsilon \overset{*}{\rightharpoonup} H$ in the sense of measures. Using now Taylor's formula for ψ one obtains

$$\psi\left(Y+\frac{z}{L_\varepsilon}\right) - \psi(Y) = \psi'(Y)\cdot\frac{z}{L_\varepsilon} + \frac{z^2}{L_\varepsilon^2}\int_0^z (z-t)\,\psi''\left(Y+\frac{t}{L_\varepsilon}\right)\mathrm{d}t.$$

Using this in (II) one gets

$$(II) = \int_{\frac{1}{L_\varepsilon}}^{\infty} \int_0^1 \frac{K_\varepsilon(L_\varepsilon Y, z)}{z} h_\varepsilon(z) H_\varepsilon(Y) \cdot \frac{z}{L_\varepsilon}\psi'(Y)\,\mathrm{d}z\mathrm{d}Y$$

$$+ \int_{\frac{1}{L_\varepsilon}}^{\infty} \int_0^1 \frac{K_\varepsilon(L_\varepsilon Y, z)}{z} h_\varepsilon(z) H_\varepsilon(Y) \cdot \frac{z^2}{L_\varepsilon^2} \int_0^z (z-t)\,\psi''\left(Y+\frac{t}{L_\varepsilon}\right)\mathrm{d}t\mathrm{d}z\mathrm{d}Y$$

$$= (II)_a + (II)_b\,.$$

We consider terms separately beginning with $(II)_b$ (and assuming L_ε to be sufficiently large, i.e. $\frac{1}{L_\varepsilon} < d$):

$$\left|(II)_b\right| \leq \frac{1}{L_\varepsilon^2} \int_{\frac{1}{L_\varepsilon}}^{\infty} \int_0^1 z^2 h_\varepsilon(z) K_\varepsilon(L_\varepsilon Y, z) H_\varepsilon(Y) \int_0^z \left|\psi''\left(Y+\frac{t}{L_\varepsilon}\right)\right| \mathrm{d}t\mathrm{d}z\mathrm{d}Y$$

$$
\begin{aligned}
&\leq \frac{C\left\|\psi''\right\|_\infty}{L_\varepsilon^2} \int_{d-\frac{1}{L_\varepsilon}}^{D} \int_0^1 h_\varepsilon(z) H_\varepsilon(Y) \left[(L_\varepsilon Y+\varepsilon)^{-a}(z+\varepsilon)^b + (L_\varepsilon Y+\varepsilon)^b (z+\varepsilon)^{-a}\right] \mathrm{d}z\mathrm{d}Y \\
&\leq \frac{CD^{1-\rho}}{L_\varepsilon^2}\left[(L_\varepsilon d-1+\varepsilon)^{-a} L_\varepsilon^{1+a} + \max\left\{(L_\varepsilon D+\varepsilon)^b, (L_\varepsilon d-1+\varepsilon)^b\right\} L_\varepsilon^{1-b}\right] \\
&= CD^{1-\rho}\left[\frac{\left(d-\frac{1}{L_\varepsilon}+\frac{\varepsilon}{L_\varepsilon}\right)^{-a}}{L_\varepsilon} + \frac{\max\left\{\left(D+\frac{\varepsilon}{L_\varepsilon}\right)^b, \left(d-\frac{1}{L_\varepsilon}+\frac{\varepsilon}{L_\varepsilon}\right)^b\right\}}{L_\varepsilon}\right] \\
&\to 0
\end{aligned}
$$

as $\varepsilon \to 0$ (and $L_\varepsilon \to \infty$). On the other hand (using the symmetry of K_ε)

$$
(II)_a = \int_{\frac{1}{L_\varepsilon}}^{\infty} H_\varepsilon(Y)\psi'(Y) \int_0^1 \frac{h_\varepsilon(z)}{L_\varepsilon} K_\varepsilon(L_\varepsilon Y, z)\,\mathrm{d}z\mathrm{d}Y = \int_{\frac{1}{L_\varepsilon}}^{\infty} H_\varepsilon(Y)\psi'(Y) Q_\varepsilon(Y)\,\mathrm{d}Y.
$$

Thus one obtains $(II)_a \to \int_0^\infty H(Y)Q(Y)\psi'(Y)\,\mathrm{d}Y$ directly from Lemma 2.8. It remains to show that $(III) \to \int_0^\infty H(Y)\frac{Q(Y)}{Y}\psi(Y)\,\mathrm{d}Y$. We first rewrite (III) as

$$
\begin{aligned}
(III) &= \int_0^\infty \int_0^1 \int_{\frac{y}{L_\varepsilon}}^{\frac{y}{L_\varepsilon}+Z} \frac{K_\varepsilon(y, L_\varepsilon Z)}{L_\varepsilon Z} h_\varepsilon(y) H_\varepsilon(Z)\,\partial_X \psi(X)\,\mathrm{d}X\mathrm{d}y\mathrm{d}Z \\
&= \int_0^\infty \int_0^1 \frac{K_\varepsilon(y, L_\varepsilon Z)}{L_\varepsilon Z} h_\varepsilon(y) H_\varepsilon(Z)\left[\psi(Z) + \psi\left(Z+\frac{y}{L_\varepsilon}\right) - \psi(Z) - \psi\left(\frac{y}{L_\varepsilon}\right)\right]\mathrm{d}y\mathrm{d}Z.
\end{aligned}
$$

Due to $\operatorname{supp}\psi \subset [d, D]$ we obtain for L_ε sufficiently large (i.e. $L_\varepsilon \geq \frac{2}{d}$) that $\psi\left(\frac{y}{L_\varepsilon}\right) = 0$ for all $y \in [0,1]$. Thus using also the definition of Q_ε we can rewrite (III) as

$$
\begin{aligned}
(III) &= \int_0^\infty \psi(Z) H_\varepsilon(Z) \frac{Q_\varepsilon(Z)}{Z}\mathrm{d}Z + \int_0^\infty \int_0^1 \frac{K_\varepsilon(y, L_\varepsilon Z)}{L_\varepsilon Z} h_\varepsilon(y) H_\varepsilon(Z)\left[\psi\left(Z+\frac{y}{L_\varepsilon}\right) - \psi(Z)\right]\mathrm{d}y\mathrm{d}Z \\
&=: (III)_a + (III)_b.
\end{aligned}
$$

The integral $(III)_a$ converges (up to a subsequence) to $\int_0^\infty \psi(Z)H(Z)\frac{Q(Z)}{Z}\mathrm{d}Z$ according to Lemma 2.8. It thus remains to show that $(III)_b$ converges to zero. To see this note that as ψ is smooth and compactly supported we have for $y \in [0,1]$:

$$
\left|\psi\left(Z+\frac{y}{L_\varepsilon}\right) - \psi(Z)\right| \leq C(\psi)\frac{y}{L_\varepsilon}\chi_{\left[d-\frac{1}{L_\varepsilon},\infty\right)}(Z) \leq \frac{C(\psi)}{L_\varepsilon}\chi_{\left[d-\frac{1}{L_\varepsilon},\infty\right)}(Z).
$$

Using this we can estimate

$$
(III)_b \leq \frac{C(\psi)}{L_\varepsilon} \int_{d-1/L_\varepsilon}^{\infty} H_\varepsilon(Z)\frac{Q_\varepsilon(Z)}{Z}\mathrm{d}Z.
$$

From the estimates on Q_ε in (40) we obtain that the integral on the right hand side is bounded uniformly in ε and thus for $L_\varepsilon \to \infty$ the right hand side converges to zero, concluding the proof. □

2.2.2. Non-solvability of the limit equation

Lemma 2.10. *For $\rho \in (0,1)$ there exists no solution H to (41) satisfying the lower bound (20) and $\int_0^R H(X)\,\mathrm{d}X \le R^{1-\rho}$ for each $R \ge 0$.*

Proof. Assuming such a solution exists and rewriting (41) one has

$$\partial_X \left((X - Q(X))\, H(X) \right) = \left((1-\rho) - \frac{Q(X)}{X} \right) H(X).$$

Defining $F(X) := (X - Q(X))\, H(X)$ this is equivalent to

$$\partial_X F(X) = \frac{(1-\rho) - \frac{Q(X)}{X}}{X - Q(X)} F(X) \quad \text{and thus} \quad F(X) = C \cdot \exp\left(\int_A^X \frac{(1-\rho) - \frac{Q(Y)}{Y}}{Y - Q(Y)} \mathrm{d}Y \right)$$

for some constant C. Considering Q one can assume (up to passing to a subsequence of ε again denoted by ε) that either $\lambda_\varepsilon \ge \mu_\varepsilon$ or $\mu_\varepsilon \ge \lambda_\varepsilon$ for all ε, while both cases will be denoted as $\lambda \ge \mu$ or $\mu \ge \lambda$ either. One can then estimate (using the definition of λ_ε and μ_ε):

$$\begin{aligned} 0 \le Q(X) &\le \lim_{\varepsilon \to 0} C_2 \int_0^1 \frac{h_\varepsilon(y)}{L_\varepsilon} \left((y+\varepsilon)^{-a} (L_\varepsilon X + \varepsilon)^b + (y+\varepsilon)^b (L_\varepsilon X + \varepsilon)^{-a} \right) \mathrm{d}y \\ &\le \lim_{\varepsilon \to 0} \frac{C_2}{L_\varepsilon} \left(L_\varepsilon^{1-b} (L_\varepsilon X + \varepsilon)^b + L_\varepsilon^{a+1} (L_\varepsilon X + \varepsilon)^{-a} \right) = \lim_{\varepsilon \to 0} C_2 \left(\left(X + \frac{\varepsilon}{L_\varepsilon} \right)^b + \left(X + \frac{\varepsilon}{L_\varepsilon} \right)^{-a} \right) \\ &= C_2 \left(X^b + X^{-a} \right). \end{aligned}$$

On the other hand

$$\begin{aligned} Q(X) &\ge \lim_{\varepsilon \to 0} C_1 \int_0^1 \frac{h_\varepsilon(y)}{L_\varepsilon} \left((y+\varepsilon)^{-a} (L_\varepsilon X + \varepsilon)^b + (y+\varepsilon)^b (L_\varepsilon X + \varepsilon)^{-a} \right) \mathrm{d}y \\ &\ge \lim_{\varepsilon \to 0} \frac{C_1}{L_\varepsilon} \begin{cases} L_\varepsilon^{1-b} (L_\varepsilon X + \varepsilon)^b & \mu \ge \lambda \\ L_\varepsilon^{1+a} (L_\varepsilon X + \varepsilon)^{-a} & \lambda \ge \mu \end{cases} \\ &= C_1 \begin{cases} X^b & \mu \ge \lambda \\ X^{-a} & \lambda \ge \mu. \end{cases} \end{aligned}$$

This shows in particular that for sufficiently large A one has $Q(X) < X$ for all $X \ge A$ and F is well defined for $X \ge A$. We claim now $C > 0$. To see this assume $C \le 0$. Then $F \le 0$ and as just shown $X - Q(X) \ge 0$ for $X \ge A$. As $H(X) = \frac{F(X)}{X - Q(X)}$ one has (using $\int_0^R H\mathrm{d}X \ge \frac{R^{1-\rho}}{2}$ for sufficiently large R due to Proposition 2.1 and $\int_0^A H\mathrm{d}X \le A^{1-\rho}$):

$$0 \ge \int_A^R H(X)\,\mathrm{d}X = \int_0^R H(X)\,\mathrm{d}X - \int_0^A H(X)\,\mathrm{d}X \ge \frac{1}{2} R^{1-\rho} - A^{1-\rho} = R^{1-\rho} \left(\frac{1}{2} - \left(\frac{A}{R} \right)^{1-\rho} \right) > 0$$

for sufficiently large R and thus a contradiction. Therefore we have $C > 0$.

We choose now $X_0 < A$ such that $Q(X_0) = X_0$ and $Q(X) < X$ for all $X > X_0$ which is possible due to the lower and upper bound for Q. We get that $\frac{X - Q(X)}{X - X_0}$ is bounded on $[X_0, \infty)$ and thus we have $X - Q(X) \le K(X - X_0)$ for some $K > 0$. Furthermore as $Q(Y) \sim Y$ for $Y \to X_0$ we obtain

$$-(1-\rho) + \frac{Q(Y)}{Y} = \rho - 1 + \frac{Q(Y)}{Y} \ge \rho - \delta > 0$$

on $\left[X_0, \overline{X} \right]$ for some $\overline{X} > X_0$ close to X_0 and $\delta > 0$. We then get

$$\int_{\overline{X}}^{A} \frac{(1-\rho) - \frac{Q(Y)}{Y}}{Y - Q(Y)} \mathrm{d}Y := C\left(\overline{X}, A\right) < \infty.$$

Using the definition of F we can then write H as

$$\begin{aligned} H &= \frac{C}{X - Q(X)} \exp\left(-\int_{X}^{A} \frac{(1-\rho) - \frac{Q(Y)}{Y}}{Y - Q(Y)} \mathrm{d}Y\right) \\ &\geq \frac{C}{K(X - X_0)} \exp\left(-\int_{X}^{\overline{X}} \frac{(1-\rho) - \frac{Q(Y)}{Y}}{Y - Q(Y)} \mathrm{d}Y - \int_{\overline{X}}^{A} \frac{(1-\rho) - \frac{Q(Y)}{Y}}{Y - Q(Y)} \mathrm{d}Y\right) \\ &\geq \frac{C}{K(X - X_0)} \exp\left(-C\left(\overline{X}, A\right)\right) \exp\left((\rho - \delta) \int_{X}^{\overline{X}} \frac{1}{K(Y - X_0)} \mathrm{d}Y\right) \\ &= \frac{C}{K} \exp\left(-C\left(\overline{X}, A\right)\right) \left(\overline{X} - X_0\right)^{\frac{\rho-\delta}{K}} \frac{1}{(X - X_0)^{1+\frac{\rho-\delta}{K}}} = C\left(A, \overline{X}, X_0, K\right) \frac{1}{(X - X_0)^{1+\alpha}} \end{aligned}$$

with $\alpha = \frac{\rho-\delta}{K} > 0$, contradicting the local integrability of H. □

This shows that L_ε has to be bounded and thus by scale invariance we obtain from Proposition 2.1 also the lower bound for h_ε, i.e. we have

Proposition 2.11. *For any $\delta > 0$ there exists $R_\delta > 0$ such that*

$$\int_0^R h_\varepsilon(x)\, dx \geq (1-\delta) R^{1-\rho} \qquad \textit{for all } R \geq R_\delta. \tag{42}$$

2.3. Exponential decay at the origin

We will show in this section that h_ε decays exponentially near zero in an averaged sense, a property that will be crucial when passing to the limit $\varepsilon \to 0$.

Lemma 2.12. *There exist constants C and c independent of ε such that*

$$\int_0^D h_\varepsilon(y)\, \mathrm{d}y \leq C D^{1-\rho} \exp\left(-c(D+\varepsilon)^{-a}\right)$$

for any $D \in (0, 1]$ and all $\varepsilon > 0$.

Proof. Let $\delta = \frac{1}{2}$, then due to Proposition 2.11 there exists $R_* > 0$ such that $\int_0^{B_2 R_*} h_\varepsilon(z)\, \mathrm{d}z \geq \frac{(B_2 R_*)^{1-\rho}}{2}$ for any $B_2 \geq 1$. On the other hand one has $\int_0^{B_1 R_*} h_\varepsilon(z)\, \mathrm{d}z \leq (B_1 R_*)^{1-\rho}$ for any $B_1 \geq 0$. Thus one has

$$\int_{B_1 R_*}^{B_2 R_*} h_\varepsilon(z)\, \mathrm{d}z = \int_0^{B_2 R_*} h_\varepsilon(z)\, \mathrm{d}z - \int_0^{B_1 R_*} h_\varepsilon(z)\, \mathrm{d}z \geq \frac{(B_2 R_*)^{1-\rho}}{2} - (B_1 R_*)^{1-\rho} \geq 1$$

for sufficiently large B_2 (depending on B_1). Thus one can estimate

$$\begin{aligned}
&\int_0^R \int_{R-y}^\infty \frac{K_\varepsilon(y,z)}{z} h_\varepsilon(y) h_\varepsilon(z)\,\mathrm{d}z\mathrm{d}y \\
&\geq C_1 \int_0^R \int_{B_1R_*}^{B_2R_*} \frac{(y+\varepsilon)^{-a}(z+\varepsilon)^b + (y+\varepsilon)^b (z+\varepsilon)^{-a}}{z} h_\varepsilon(y) h_\varepsilon(z)\,\mathrm{d}z\mathrm{d}y \\
&\geq \frac{C_1}{(R+\varepsilon)^a} \int_0^R h_\varepsilon(y)\,\mathrm{d}y \int_{B_1R_*}^{B_2R_*} \frac{(z+\varepsilon)^b}{z} h_\varepsilon(z)\,\mathrm{d}z \\
&\geq \frac{C}{(R+\varepsilon)^a} \int_0^R h_\varepsilon(y)\,\mathrm{d}y\,(B_2R_*)^{b-1} \int_{B_1R_*}^{B_2R_*} h_\varepsilon(z)\,\mathrm{d}z \geq \frac{C}{(R+\varepsilon)^a}(B_2R_*)^{b-1} \int_0^R h_\varepsilon(y)\,\mathrm{d}y.
\end{aligned}$$

Using this and taking $\chi_{(-\infty,R]}$ (restricted to $[0,\infty)$) by some approximation argument as test function in the equation $(1-\rho)h_\varepsilon(x) - \partial_x(xh_\varepsilon(x)) + \partial_x I_\varepsilon[h_\varepsilon](x) = 0$ we obtain

$$\begin{aligned}
0 &= (1-\rho)\int_0^R h_\varepsilon(x)\,\mathrm{d}x - Rh_\varepsilon(R) + \int_0^R \int_{R-y}^\infty \frac{K_\varepsilon(y,z)}{z} h_\varepsilon(y) h_\varepsilon(z)\,\mathrm{d}z\mathrm{d}y \\
&\geq (1-\rho)\int_0^R h_\varepsilon(x)\,\mathrm{d}x - Rh_\varepsilon(R) + \frac{C}{(R+\varepsilon)^a}(B_2R_*)^{b-1}\int_0^R h_\varepsilon(x)\,\mathrm{d}x.
\end{aligned}$$

Thus one has

$$(1-\rho)\int_0^R h_\varepsilon(x)\,\mathrm{d}x + \frac{C(B_2R_*)^{b-1}}{(R+\varepsilon)^a}\int_0^R h_\varepsilon(x)\,\mathrm{d}x \leq Rh_\varepsilon(R) = R\partial_R \int_0^R h_\varepsilon(x)\,\mathrm{d}x$$

or equivalently

$$\partial_R\left(\int_0^R h_\varepsilon(x)\,\mathrm{d}x\right) \geq \left(\frac{1-\rho}{R} + \frac{C(B_2R_*)^{b-1}}{R(R+\varepsilon)^a}\right)\int_0^R h_\varepsilon(x)\,\mathrm{d}x \geq \left(\frac{1-\rho}{R} + \frac{C(B_2R_*)^{b-1}}{(R+\varepsilon)^a}\right)\int_0^R h_\varepsilon(x)\,\mathrm{d}x,$$

where we used $\frac{1}{R(R+\varepsilon)^a} \geq \frac{1}{(R+\varepsilon)^a}$ for $R \in [D,1]$. Integrating this inequality over $[D,1]$ and using $\int_0^1 h_\varepsilon \mathrm{d}x \leq 1$ as well as $(1+\varepsilon)^{-a} \leq 1$ gives

$$\int_0^D h_\varepsilon(x)\,\mathrm{d}x \leq \exp\left(\frac{C(B_2R_*)^{b-1}}{a}\right) D^{1-\rho} \exp\left(-\frac{C(B_2R_*)^{b-1}}{a}(D+\varepsilon)^{-a}\right). \qquad \square$$

Lemma 2.13. *For $D \leq 1$ and any $\alpha \in \mathbb{R}$ one has the following estimate*

$$\int_0^D (x+\varepsilon)^\alpha h_\varepsilon(x)\,\mathrm{d}x \leq CD^{1-\rho}(D+\varepsilon)^\alpha \exp\left(-c(D+\varepsilon)^{-a}\right) \quad \textit{if } \alpha \geq 0$$

$$\int_0^D (x+\varepsilon)^\alpha h_\varepsilon(x)\,\mathrm{d}x \leq \tilde{C}D^{1-\rho} \exp\left(-\frac{c}{2}(D+\varepsilon)^{-a}\right) \quad \textit{if } \alpha < 0.$$

Proof. The case $\alpha \geq 0$ follows immediately from Lemma 2.12. The case $\alpha < 0$ follows from Lemma 2.12 using a dyadic decomposition as in Lemma A.1. □

As $\{h_\varepsilon\}_{\varepsilon>0}$ is a locally bounded sequence of non-negative Radon measures one can extract a subsequence (again denoted by ε) such that $h_\varepsilon \overset{*}{\rightharpoonup} h$ in the sense of measures and h is non-trivial due to Proposition 2.11. As a direct consequence of Lemma 2.13 one obtains:

Lemma 2.14. *For $D \leq 1$ and $\alpha \in \mathbb{R}$ one has*

$$\int_0^D x^\alpha h(x)\,\mathrm{d}x \leq \tilde{C} D^{1-\rho} \exp\left(-\frac{c}{2} D^{-a}\right) \quad \text{if } \alpha < 0$$

$$\int_0^D x^\alpha h(x)\,\mathrm{d}x \leq C D^{1+\alpha-\rho} \exp\left(-c D^{-a}\right) \quad \text{if } \alpha \geq 0.$$

Proof. This follows from Lemma 2.13. □

As a consequence of Lemma 2.14 together with Lemma A.1 we obtain

Corollary 2.15. *For any $\alpha \in \mathbb{R}$ and $D > 0$ each limit h satisfies*

1. $\int_0^\infty x^\alpha h(x)\,\mathrm{d}x < \infty$ *if* $\alpha < \rho - 1$,
2. $\int_0^D x^\alpha h(x)\,\mathrm{d}x < C(D)$.

Remark 2.16. We obtain corresponding results for h_ε and h with x^α replaced by $(x+\varepsilon)^\alpha$.

2.4. Passing to the limit $\varepsilon \to 0$

In this section we will finally conclude the proof of existence of a solution to (5) as stated in Theorem 1.1 by passing to the limit $\varepsilon \to 0$ in (18). Before doing this we first show that $I[h]$ is locally integrable:

Lemma 2.17. *For h as given above one has $I[h] \in L^1_{\mathrm{loc}}([0,\infty))$.*

Proof. Let $D > 0$. Then one has

$$\begin{aligned}\int_0^D I[h](x)\,\mathrm{d}x &= \int_0^D \int_0^x \int_{x-y}^\infty \frac{K(y,z)}{z} h(y) h(z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &\leq C \int_0^D \int_0^D \int_0^\infty \left(y^{-a} z^{b-1} + y^b z^{-a-1}\right) h(y) h(z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &\leq C \int_0^D \int_0^D \left(y^{-a} + y^b\right) h(y)\,\mathrm{d}y\mathrm{d}x \leq C(D)\end{aligned}$$

where Corollary 2.15 was used. One similarly gets $\int_N I[h]\,\mathrm{d}x = 0$ for bounded null sets $N \subset [0,\infty)$. □

To show that h is a (weak) self-similar solution it only remains to pass to the limit in the weak form of the equation

$$\partial_x I_\varepsilon[h_\varepsilon] = \partial_x (x h_\varepsilon) + (\rho - 1) h_\varepsilon.$$

Thus let $\varphi \in C_c^\infty([0,\infty))$. Then the weak form reads as

$$\int_0^\infty \partial_x \varphi(x) \int_0^x \int_{x-y}^\infty \frac{K_\varepsilon(y,z)}{z} h_\varepsilon(y) h_\varepsilon(z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x = \int_0^\infty \partial_x \varphi(x)\, x h_\varepsilon(x)\,\mathrm{d}x + (1-\rho)\int_0^\infty \varphi(x) h_\varepsilon(x)\,\mathrm{d}x.$$

One can easily pass to the limit in the right hand side. To prove Theorem 1.1 it thus remains to show that one can also take the limit in the left hand side of this equation. This will be done in the following Proposition.

Proposition 2.18. *For any $\varphi \in C_c^\infty([0,\infty))$ one has*

$$\int_0^\infty \partial_x \varphi(x) \int_0^x \int_{x-y}^\infty \frac{K_\varepsilon(y,z)}{z} h_\varepsilon(z) h_\varepsilon(y)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x \longrightarrow \int_0^\infty \partial_x \varphi(x) \int_0^x \int_{x-y}^\infty \frac{K(y,z)}{z} h(z) h(y)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x$$

as $\varepsilon \to 0$.

Proof. Taking the difference of the two integrals and rewriting one obtains

$$\begin{aligned}
&\left|\int_0^\infty \partial_x\varphi(x)\left(\int_0^x\int_{x-y}^\infty \frac{K(y,z)}{z}h(y)h(z) - \frac{K_\varepsilon(y,z)}{z}h_\varepsilon(y)h_\varepsilon(z)\,\mathrm{d}y\mathrm{d}z\right)\mathrm{d}x\right| \\
&\quad\le \left|\int_0^\infty \partial_x\varphi(x)\left(\int_0^x\int_{x-y}^\infty \frac{K(y,z)-K_\varepsilon(y,z)}{z}h(y)h(z)\,\mathrm{d}z\mathrm{d}y\right)\mathrm{d}x\right| \\
&\qquad+ \left|\int_0^\infty \partial_x\varphi(x)\left(\int_0^x\int_{x-y}^\infty \frac{K_\varepsilon(y,z)}{z}h(y)\left(h(z)-h_\varepsilon(z)\right)\mathrm{d}z\mathrm{d}y\right)\mathrm{d}x\right| \\
&\qquad+ \left|\int_0^\infty \partial_x\varphi(x)\left(\int_0^x\int_{x-y}^\infty \frac{K_\varepsilon(y,z)}{z}h_\varepsilon(z)\left(h(y)-h_\varepsilon(y)\right)\mathrm{d}z\mathrm{d}y\right)\mathrm{d}x\right| =: (I)+(II)+(III).
\end{aligned}$$

We estimate the three terms separately and take $D>0$ such that $\operatorname{supp}\varphi \subset [0,D]$. Then due to Lebesgue's Theorem (using also Corollary 2.15 and Lemma A.1) we obtain

$$(I) \le \int_0^\infty |\partial_x\varphi(x)|\left(\int_0^x\int_{x-y}^\infty \frac{|K(y,z)-K_\varepsilon(y,z)|}{z}h(y)h(z)\,\mathrm{d}z\mathrm{d}y\right)\mathrm{d}x \to 0 \quad \text{as } \varepsilon\to 0.$$

To estimate the other two terms we will need some cutoff functions. Let $M,N\in\mathbb{N}$ and $\zeta_1^N,\zeta_2^N,\xi_1^M,\xi_2^M \in C^\infty([0,\infty))$ such that

$$\zeta_1^N = 0 \text{ on } \left[0,\frac{1}{N}\right]\cup[N+1,\infty), \quad \zeta_1^N = 1 \text{ on } \left[\frac{2}{N},N\right], \quad 0\le\zeta_1^N\le 1, \quad \zeta_2^N := 1-\zeta_1^N,$$

$$\xi_1^M = 0 \text{ on } \left[0,\frac{1}{M}\right], \quad \xi_1^M = 1 \text{ on } \left[\frac{2}{M},\infty\right), \quad 0\le\xi_1^M\le 1, \quad \xi_2^M := 1-\xi_1^M.$$

Defining $K_\varepsilon^{i,N}(y,z) := K_\varepsilon(y,z)\cdot\zeta_i^N(z)$ for $i=1,2$ one obtains using also Fubini's Theorem:

$$\begin{aligned}
(II) &\le \left|\int_0^\infty\int_0^\infty \frac{K_\varepsilon^{1,N}(y,z)}{z}h(y)\left(h(z)-h_\varepsilon(z)\right)\int_y^{y+z}\partial_x\varphi(x)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\right| \\
&\quad+ \left|\int_0^\infty\int_0^\infty \frac{K_\varepsilon^{2,N}(y,z)}{z}h(y)\left(h(z)-h_\varepsilon(z)\right)\int_y^{y+z}\partial_x\varphi(x)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\right| =: (II)_a+(II)_b.
\end{aligned}$$

We consider again terms separately and without loss of generality we assume $\varepsilon < 1$. Then using Corollary 2.15 and Lemma A.1 we obtain

$$\begin{aligned}(II)_b &\leq C \|\partial_x \varphi\|_\infty \int_0^{\frac{2}{N}} \int_0^D \left[(y+\varepsilon)^{-a}(z+\varepsilon)^b + (y+\varepsilon)^b (z+\varepsilon)^{-a}\right] h(y)(h(z)+h_\varepsilon(z))\,\mathrm{d}y\mathrm{d}z \\ &\quad + C\|\varphi\|_\infty \int_N^\infty \int_0^D \frac{(y+\varepsilon)^{-a}(z+\varepsilon)^b + (y+\varepsilon)^b(z+\varepsilon)^{-a}}{z} h(y)(h(z)+h_\varepsilon(z))\,\mathrm{d}y\mathrm{d}z \\ &\leq \|\partial_x\varphi\|_\infty C(D) \int_0^{\frac{2}{N}} \left((z+\varepsilon)^b + (z+\varepsilon)^{-a}\right)(h(z)+h_\varepsilon(z))\,\mathrm{d}z \\ &\quad + C(D)\|\varphi\|_\infty \int_N^\infty \left(2^{\tilde{b}} z^{b-1} + z^{-a-1}\right)(h(z)+h_\varepsilon(z))\,\mathrm{d}z \\ &\leq \|\partial_x\varphi\|_\infty C(D)\left[\frac{1}{N^{1-\rho}}\left(\frac{2}{N}+\varepsilon\right)^{\tilde{b}} + \frac{1}{N^{1-\rho}}\right] + C(D)\|\varphi\|_\infty \left[N^{b-\rho} + N^{-a-\rho}\right] \longrightarrow 0, \end{aligned} \tag{43}$$

for $N \to \infty$. Furthermore one has

$$(II)_a = \left| \int_0^\infty (h(z) - h_\varepsilon(z))\, \psi_\varepsilon^N(z)\,\mathrm{d}z \right| \tag{44}$$

with $\psi_\varepsilon^N(z) := \int_0^D \frac{K_\varepsilon^{1,N}(y,z)}{z} h(y)\left[\varphi(y+z) - \varphi(y)\right]\mathrm{d}y$. We claim that $\psi_\varepsilon^N \to \psi^N$ strongly in $C([0,\infty))$ with $\psi^N(z) := \int_0^D \frac{K(y,z)}{z} h(y)\,\zeta_1^N(z)\left[\varphi(y+z)-\varphi(y)\right]\mathrm{d}y$. Note that by construction we have $\operatorname{supp}\psi_\varepsilon^N \subset \left[\frac{1}{N}, N+1\right]$ for all $\varepsilon > 0$. To show (uniform) convergence we have to use a cutoff also in y, i.e. one can estimate

$$\begin{aligned}\left|\psi^N(z) - \psi_\varepsilon^N(z)\right| &\leq \left| \int_0^D \frac{K(y,z)-K_\varepsilon(y,z)}{z} \zeta_1^N(z)\,\xi_1^M(y)\,h(y)\left[\varphi(y+z)-\varphi(y)\right]\mathrm{d}y \right| \\ &\quad + \left| \int_0^D \frac{K(y,z)-K_\varepsilon(y,z)}{z} \zeta_1^N(z)\,\xi_2^M(y)\,h(y)\left[\varphi(y+z)-\varphi(y)\right]\mathrm{d}y \right| \\ &=: (II)_{a,1} + (II)_{a,2}\,. \end{aligned}$$

Using similar arguments as in (43) we get

$$(II)_{a,2} \leq C(N,\varphi)\left[\frac{1}{M^{1+\tilde{b}-\rho}} + \frac{1}{M^{1-\rho}}\right] \longrightarrow 0 \tag{45}$$

for $M \to \infty$ and N fixed. As K is continuous on $\left[\frac{1}{M}, D\right] \times \left[\frac{1}{N}, N+1\right]$ for $M, N \in \mathbb{N}$ fixed, one has $K_\varepsilon \to K$ uniformly on $\left[\frac{1}{M}, D\right] \times \left[\frac{1}{N}, N+1\right]$ for $\varepsilon \to 0$. Thus we get $(II)_{a,1} \to 0$ for $\varepsilon \to 0$ (with M, N fixed). Together with (45) this shows that $\psi_\varepsilon^N \to \psi^N$ strongly. Thus one can pass to the limit in (44) to obtain together with (43): $(II) \to 0$ as $\varepsilon \to 0$.

In a similar way we can show that $(III) \to 0$ for $\varepsilon \to 0$. □

3. Regularity and large-mass behaviour

In this section we will finish the proof of Theorem 1.1 by showing that the measure h solving (5) obtained in Section 2.4 has a continuous density. Furthermore we will also show that this density satisfies $h(x) \sim (1-\rho)x^{-\rho}$ for $x \to \infty$, i.e. h has the expected pointwise decay behaviour at infinity.

3.1. Continuity of the self-similar solution

Lemma 3.1. *The solution h of (5) obtained in Section 2.4 is locally bounded, i.e. $h \in L^\infty_{\mathrm{loc}}((0,\infty))$.*

Proof. We first show that $I[h] \in L^\infty_{\mathrm{loc}}([0,\infty))$. To see this let $D > 0$ and $x \in [0, D]$. Using (10) and Corollary 2.15 we can estimate

$$|I[h](x)| \le C \int_0^x \int_{x-y}^\infty \left(y^{-a} z^{b-1} + y^b z^{-a-1}\right) h(z) h(y)\,\mathrm{d}z\mathrm{d}y \le C \int_0^D \left(y^{-a} + y^b\right) h(y)\,\mathrm{d}y \le C(D).$$

Next we show $xh(x) \in L^\infty_{\mathrm{loc}}([0,\infty))$ from which the claim directly follows. As h solves (5) in the sense of distributions it holds

$$\int_0^\infty xh(x)\,\partial_x\varphi(x)\,\mathrm{d}x = \int_0^\infty I[h](x)\,\partial_x\varphi(x)\,\mathrm{d}x - (1-\rho)\int_0^\infty \varphi(x) h(x)\,\mathrm{d}x \tag{46}$$

for all $\varphi \in C_c^\infty([0,\infty))$. Let now $R > 0$ and $\varphi \in C_c^\infty([0,\infty))$ being such that $\operatorname{supp}\varphi \subset [0,R]$. Defining $\Phi(x) := -\int_x^\infty \varphi(y)\,\mathrm{d}y$ it holds $\operatorname{supp}\Phi \subset [0,R]$ and $\partial_x\Phi = \varphi$. Substituting φ by Φ in (46) we can estimate

$$\begin{aligned}\left|\int_0^\infty xh(x)\varphi(x)\,\mathrm{d}x\right| &= \left|\int_0^\infty h(x)\,\partial_x\Phi(x)\,\mathrm{d}x\right| \le \left|\int_0^\infty I[h](x)\,\partial_x\Phi(x)\,\mathrm{d}x\right| + (1-\rho)\left|\int_0^\infty \Phi(x) h(x)\,\mathrm{d}x\right| \\ &\le \left|\int_0^\infty I[h](x)\varphi(x)\,\mathrm{d}x\right| + (1-\rho)\int_0^R\int_x^R |\varphi(y)|\,\mathrm{d}y h(x)\,\mathrm{d}x \\ &\le \|I[h]\|_{L^\infty([0,R])}\|\varphi\|_{L^1([0,R])} + (1-\rho)R^{1-\rho}\|\varphi\|_{L^1([0,R])} \le C(R)\|\varphi\|_{L^1([0,R])}.\end{aligned}$$

Thus we have $xh(x) \in \left(L^1([0,R])\right)' \cong L^\infty([0,R])$. As $R > 0$ was arbitrary this shows $xh(x) \in L^\infty_{\mathrm{loc}}([0,\infty))$. □

Lemma 3.2. *For h being the solution obtained in Section 2.4 the expression $I[h]$ is continuous in $[0,\infty)$.*

Proof. We first show continuity at $x = 0$, i.e. $I[h](x) \to 0$ as $x \to 0$. Using (10) and Corollary 2.15 we directly obtain

$$I[h](x) = \int_0^x \int_{x-y}^\infty \frac{K(y,z)}{z} h(y) h(z)\,\mathrm{d}z\mathrm{d}y \le C\int_0^x \left(y^{-a} + y^b\right) h(y)\,\mathrm{d}y \longrightarrow 0 \quad \text{as } x \longrightarrow 0.$$

Next we show continuity on $(0,\infty)$. Let $0 < x_1 < x_2 < \infty$ and fix $r, R > 0$ such that $0 < r < x_1 < x_2 < R$ and let $0 < \delta < r/2$. Rewriting the expression $I[h](x_2)$ we obtain

$$\begin{aligned}|I[h](x_2) - I[h](x_1)| &= \left|\int_0^{x_2}\int_{x_2-y}^\infty \frac{K(y,z)}{z} h(y) h(z)\,\mathrm{d}z\mathrm{d}y - \int_0^{x_1}\int_{x_1-y}^\infty \frac{K(y,z)}{z} h(y) h(z)\,\mathrm{d}z\mathrm{d}y\right| \\ &= \left|\int_0^{x_1}\int_{x_2-y}^\infty (\cdots)\,\mathrm{d}z\mathrm{d}y + \int_{x_1}^{x_2}\int_{x_2-y}^\infty (\cdots)\,\mathrm{d}z\mathrm{d}y - \int_0^{x_1}\int_{x_1-y}^{x_2-y} (\cdots)\,\mathrm{d}z\mathrm{d}y - \int_0^{x_1}\int_{x_2-y}^\infty (\cdots)\,\mathrm{d}z\mathrm{d}y\right|\end{aligned}$$

$$\leq \left|\int_{x_1}^{x_2}\int_0^{\infty}(\cdots)\,\mathrm{d}z\mathrm{d}y\right| + C\left|\int_0^{x_1}\int_{x_1-y}^{x_2-y}(\cdots)\,\mathrm{d}z\mathrm{d}y\right| =: (I) + (II)\,.$$

Estimating the two terms separately we first obtain together with (10)

$$(I) \leq C\int_{x_1}^{x_2}\left(y^{-a}+y^{b}\right)h(y)\,\mathrm{d}y \leq C\,\|h\|_{L^\infty([r,R])}\left(r^{-a}+\max\left\{r^b, R^b\right\}\right)|x_2-x_1| \longrightarrow 0 \quad \text{as } |x_2-x_1|\to 0.$$

To estimate (II) we split the first integral in parts that are close and far to 0 and x_1 respectively, to obtain

$$(II) \leq C\int_0^{x_1}\int_{x_1-y}^{x_2-y}\left(y^{-a}z^{b-1}+y^{b}z^{-a-1}\right)h(z)\,h(y)\,\mathrm{d}z\mathrm{d}y$$
$$= C\left(\int_0^{\delta}\int_0^{\infty}(\cdots)\,\mathrm{d}z\mathrm{d}y + \int_{\delta}^{x_1-\delta}\int_{x_1-y}^{x_2-y}(\cdots)\,\mathrm{d}z\mathrm{d}y + \int_{x_1-\delta}^{x_1}\int_0^{\infty}(\cdots)\,\mathrm{d}z\mathrm{d}y\right).$$

Using now Lemma 2.14 as well as Corollary 2.15 we obtain for sufficiently small $\delta < 1$ that

$$(II) \leq C\left(\delta^{1-\rho} + \|h\|^2_{L^\infty([\delta,R])}\left(\delta^{b-a-1}+\max\left\{R^b,\delta^b\right\}\delta^{-a-1}\right)R\,|x_2-x_1|\right.$$
$$\left. + \left(\left(\frac{r}{2}\right)^{-a}+\max\left\{x_1^b,\left(\frac{r}{2}\right)^b\right\}\right)\|h\|_{L^\infty([\frac{r}{2},x_1])}\,\delta\right)$$
$$\leq C\left(\delta^{1-\rho}+\left(r^{-a}+\max\left\{R^b,\left(\frac{r}{2}\right)^b\right\}\right)\|h\|_{L^\infty([\frac{r}{2},R])}\,\delta\right)+C(r,R,\delta)\,|x_2-x_1|$$
$$\leq C(r,R)\left(\delta^{1-\rho}+\delta\right)+C(r,R,\delta)\,|x_2-x_1|\,.$$

Thus choosing first δ small and then $|x_2-x_1|$ small the right hand side can be made arbitrarily small, showing the continuity of $I[h]$. □

As a consequence we can now show the continuity of h on $[0,\infty)$ as well as pointwise exponential decay near $x=0$.

Lemma 3.3. *The solution h is continuous on $[0,\infty)$ and there exists $\alpha_0 \in \mathbb{R}$ and $\beta > 0$ such that $|h(x)| \leq Cx^{\alpha_0}\exp\left(-\beta x^{-a}\right)$ for all $x \in [0,1]$.*

Proof. We first show continuity on $(0,\infty)$. As h solves (5) we have $\partial_x\left(I[h](x) - xh(x)\right) = -(1-\rho)\,h(x)$ in the sense of distributions. Furthermore $h \in L^\infty_{\mathrm{loc}}((0,\infty))$ according to Lemma 3.1. Thus $I[h](x) - xh(x)$ is of bounded variation and for some $x_0 > 0$ and a.e. $x \in (0,\infty)$ it holds

$$xh(x) = I[h](x) + (1-\rho)\int_{x_0}^{x} h(y)\,\mathrm{d}y - I[h](x_0) + x_0 h(x_0)\,. \tag{47}$$

As the right hand side is continuous we obtain that $xh(x)$ and thus also h is continuous on $(0,\infty)$.

We now show that h is bounded on $[0,1]$ which will allow us to take $x_0 = 0$ in (47). Rearranging (5) we obtain that h satisfies $\partial_x h(x) = \frac{1}{x}\partial_x I[h](x) - \rho\frac{h(x)}{x}$ in the sense of distributions. Taking as test function (after approximation) the characteristic function of $[z,1]$ and using the continuity of h and $I[h]$ we obtain for $z \in (0,1)$:

$$h(z) = \frac{1}{z}I[h](z) + \int_z^1 \frac{I[h](x)}{x^2}\,\mathrm{d}x + \rho\int_z^1 \frac{h(x)}{x}\,\mathrm{d}x - h(1) - I[h](1)\,. \tag{48}$$

From Corollary 2.15 we obtain directly $\int_z^1 \frac{h(x)}{x}\mathrm{d}x \leq \int_0^1 \frac{h(x)}{x}\mathrm{d}x \leq C$. Furthermore considering $I[h]$ we obtain due to Lemma 2.14 and Corollary 2.15 for $x \in (0,1)$

$$I[h](x) \leq C \int_0^x \left(y^{-a} + y^b\right) h(y)\,\mathrm{d}y \leq C x^{1-\rho} \exp\left(-\tilde{c} x^{-a}\right). \tag{49}$$

This shows that both $\frac{1}{z} I[h](z)$ and $\int_z^1 \frac{I[h](x)}{x^2}\mathrm{d}x$ are bounded for all $z \in (0,1)$. Thus we obtain from (48) that also h is bounded in $(0,1)$ and we can take $x_0 = 0$ in (47) to get $xh(x) = I[h](x) + (1-\rho)\int_0^x h(y)\,\mathrm{d}y$ for all $x \in [0,1]$. Finally combining this with Lemma 2.14 and (49) we also obtain $|h(x)| \leq C x^{\alpha_0} \exp\left(-\beta x^{-a}\right)$ for suitable constants $\alpha_0 \in \mathbb{R}$ and $\beta > 0$. $\square$

3.2. Large-mass behaviour

We first prove some moment estimate that will be used several times in the following and which is a consequence of Corollary 2.15 and Lemma A.1.

Lemma 3.4. *Let $r_0 > 0$ be a constant and $\alpha < \rho - 1$. There exists a constant $C > 0$ such that the solution h of (5) obtained in Section 2.4 satisfies*

$$\int_r^\infty x^\alpha h(x)\,\mathrm{d}x \leq C(r_0 + r)^{1+\alpha-\rho} \quad \textit{for all } r > 0.$$

Proof. For $r \leq 1$ we use that due to Corollary 2.15 we have $\int_r^\infty x^\alpha h(x)\,\mathrm{d}x \leq \int_0^\infty x^\alpha h(x)\,\mathrm{d}x \leq \hat{C}$. Thus it suffices to take $C \geq \hat{C}(r_0+1)^{\rho-1-\alpha}$. On the other hand for $r \geq 1$ we use that due to Lemma A.1 we have $\int_r^\infty x^\alpha h(x)\,\mathrm{d}x \leq \tilde{C} r^{1+\alpha-\rho}$. Taking $C \geq \tilde{C}(r_0+1)^{\rho-1-\alpha}$ we then obtain the claim using $C(r_0+r)^{1+\alpha-\rho} \geq C(r_0+1)^{1+\alpha-\rho} r^{1+\alpha-\rho} \geq \tilde{C} r^{1+\alpha-\rho}$. $\square$

Before proving that h has the expected pointwise decay behaviour for large cluster sizes, let us recall that h also satisfies the lower bound obtained in Proposition 2.11, as this property is preserved under the limit procedure in Section 2.4, i.e.

Remark 3.5. For each $\delta > 0$ the solution h satisfies $\int_0^R h(x)\,\mathrm{d}x \geq (1-\delta) R^{1-\rho}$ for all $R \geq R_\delta$.

Proposition 3.6. *The solution h of (5) obtained in Section 2.4 satisfies $h(x) \sim (1-\rho) x^{-\rho}$ as $x \to \infty$.*

Proof. We have to show that $\frac{h(R)}{(1-\rho)R^{-\rho}} - 1 \to 0$ as $R \to \infty$. We can estimate this expression in the following way:

$$\begin{aligned}
\left|\frac{h(R)}{(1-\rho)R^{-\rho}} - 1\right| &\leq \left|\frac{h(R)}{(1-\rho)R^{-\rho}} - \frac{Rh(R)}{(1-\rho)\int_0^R h(x)\,\mathrm{d}x}\right| + \left|\frac{Rh(R)}{(1-\rho)\int_0^R h(x)\,\mathrm{d}x} - 1\right| \\
&\leq \frac{Rh(R)}{(1-\rho)\int_0^R h(x)\,\mathrm{d}x}\left|1 - R^{\rho-1}\int_0^R h(x)\,\mathrm{d}x\right| + \omega(R) \\
&\leq \delta(\omega(R)+1) + |\omega(R)|
\end{aligned}$$

for all $R \geq R_\delta$, where we used the lower bound on h from Remark 3.5 and we denoted $\omega(R) := \frac{Rh(R)}{(1-\rho)\int_0^R h(x)\mathrm{d}x} - 1$. Thus it suffices to show that $\omega(R) \to 0$ as $R \to \infty$. Furthermore we assume $\delta < 1/2$ and $R \geq 2$ in the following. Using (47) with $x_0 = 0$, i.e. $Rh(R) = (1-\rho)\int_0^R h(x)\,\mathrm{d}x + I[h](R)$ we can rewrite $\omega(R)$ to obtain

$$|\omega(R)| = \left|\frac{I[h](R)}{(1-\rho)\int_0^R h(x)\,\mathrm{d}x}\right| \leq \frac{I[h](R)}{(1-\rho)(1-\delta)R^{1-\rho}} \leq \frac{2}{1-\rho} R^{\rho-1} I[h](R).$$

Thus it suffices to show $I[h](R) \leq CR^{\nu}$ with $\nu < 1-\rho$ to finish the proof. This will be done in the following and it turns out that we have to consider the two cases $b<0$ and $b \geq 0$ separately, where the latter will be more complicated and require more work. We have using Lemma 3.4 that

$$\begin{aligned} I[h](R) &= \int_0^R \int_{R-y}^{\infty} \frac{K(y,z)}{z} h(y)h(z)\,\mathrm{d}z\mathrm{d}y = \int_0^{R/2}\int_{R-y}^{\infty} (\cdots)\,\mathrm{d}z\mathrm{d}y + \int_{R/2}^{R}\int_{R-y}^{\infty}(\cdots)\,\mathrm{d}z\mathrm{d}y \\ &\leq C\int_0^{R/2}\int_{R/2}^{\infty}\left(y^{-a}z^{b-1}+y^b z^{-a-1}\right)h(y)h(z)\,\mathrm{d}z\mathrm{d}y + C\int_{R/2}^{R}\int_0^{\infty} y^{-a}z^{b-1}h(y)h(z)\,\mathrm{d}z\mathrm{d}y \\ &\quad + C\int_{R/2}^{R}\int_{R-y}^{\infty} y^b z^{-a-1}h(y)h(z)\,\mathrm{d}z\mathrm{d}y \\ &\leq C\int_0^{R/2}\left(y^{-a}R^{b-\rho}y^bR^{-a-\rho}\right)h(y)\,\mathrm{d}y + C\int_{R/2}^{R} y^{-a}h(y)\,\mathrm{d}y + C\int_{R/2}^{R} y^b\frac{h(y)}{(2+(R-y))^{a+\rho}}\,\mathrm{d}y \\ &\leq CR^{1-\rho}\left[R^{b-\rho}+R^{-a-\rho+\max\{0,b\}}+R^{-a}\right] + C\int_{R/2}^{R} y^b\frac{h(y)}{(2+(R-y))^{a+\rho}}\,\mathrm{d}y. \end{aligned}$$

Note that due to our assumptions we have $b-\rho$, $-a-\rho+\max\{0,b\}$, $-a<0$ it suffices to consider only the integral expression on the right-hand side. Here we have to consider the two cases $b<0$ and $b\geq 0$ and we start with $b<0$ which is the easier one. It holds

$$C\int_{R/2}^{R} y^b\frac{h(y)}{(2+(R-y))^{a+\rho}}\,\mathrm{d}y \leq CR^{1-\rho}R^b$$

and as we assume $b<0$ the claim then follows as before.

Consider now the case $b\geq 0$: By splitting the integral in equidistant pieces we can estimate

$$\begin{aligned} C\int_{R/2}^{R} y^b\frac{h(y)}{(2+(R-y))^{a+\rho}}\,\mathrm{d}y &\leq CR^b\int_{R/2}^{R}\frac{h(y)}{(2+(R-y))^{a+\rho}}\,\mathrm{d}y \\ &\leq CR^b\sum_{k=\left\lfloor\frac{R}{2}\right\rfloor+1}^{\lceil R\rceil}\int_{k-1}^{k}\frac{h(y)}{(2+(R-y))^{a+\rho}}\,\mathrm{d}y \leq CR^b\sum_{k=\left\lfloor\frac{R}{2}\right\rfloor+1}^{\lceil R\rceil}\frac{a_k}{(2+(R-k))^{a+\rho}}. \end{aligned}$$

Here we use $\lfloor x\rfloor := \max\{(-\infty,x]\cap\mathbb{Z}\}$ and $\lceil x\rceil := \min\{[x,\infty)\cap\mathbb{Z}\}$. Furthermore for $k\in\mathbb{N}$ we denote by $a_k := \int_{k-1}^{k} h(x)\,\mathrm{d}x$. As shown in Lemma 3.7 it holds $a_k \leq C_0 k^{-\rho}$ for $b\geq 0$ and thus using this and estimating the sum by an integral we obtain

$$\begin{aligned} C\int_{R/2}^{R} y^b\frac{h(y)}{(2+(R-y))^{a+\rho}}\,\mathrm{d}y &\leq CR^b\sum_{k=\left\lfloor\frac{R}{2}\right\rfloor+1}^{\lceil R\rceil}\frac{k^{-\rho}}{(2+(R-k))^{a+\rho}} \leq CR^{b-\rho}\int_{\left\lfloor\frac{R}{2}\right\rfloor+1}^{\lceil R\rceil+1}(2+(R-x))^{-\rho}\,\mathrm{d}x \\ &= \frac{C}{1-\rho}R^{b-\rho}\left[\left(1+R-\left\lfloor\frac{R}{2}\right\rfloor\right)^{1-\rho}-(1+R-\lceil R\rceil)^{1-\rho}\right] \end{aligned}$$

$$\leq CR^{b-\rho}\left(1+R-\left\lfloor\frac{R}{2}\right\rfloor\right)^{1-\rho}$$
$$\leq CR^{1-\rho}R^{b-\rho}.$$

As $b-\rho<0$ we can argue as before and this finishes the proof. □

It thus remains only to prove the following Lemma.

Lemma 3.7. *Assume $b\geq 0$. For $R\geq 1$ we define $a_R := \int_{R-1}^{R} h(x)\,\mathrm{d}x$. There exists a constant $C_0>0$ such that $a_R\leq C_0 R^{-\rho}$ holds for all $R\geq 1$.*

Remark 3.8. This result also holds for $b<0$ but as we can estimate directly in this case as seen above we do not need this here.

Proof of Lemma 3.7. Note first that it suffices to prove the claim only for $R\in\mathbb{N}$. To see this assume that $a_n\leq \tilde{C}_0 n^{-\rho}$ for all $n\in\mathbb{N}$ and let $R\in[1,\infty)\setminus\mathbb{Z}$. We can then estimate

$$a_R=\int_{R-1}^{R} h(y)\,\mathrm{d}y\leq\int_{\lfloor R\rfloor-1}^{\lceil R\rceil} h(y)\,\mathrm{d}y=\int_{\lfloor R\rfloor-1}^{\lfloor R\rfloor} h(y)\,\mathrm{d}y+\int_{\lfloor R\rfloor}^{\lceil R\rceil} h(y)\,\mathrm{d}y\leq \tilde{C}_0\left(\lfloor R\rfloor^{-\rho}+\lceil R\rceil^{-\rho}\right)$$
$$\leq \tilde{C}_0R^{-\rho}\left(\left(\frac{\lfloor R\rfloor}{R}\right)^{-\rho}+\left(\frac{\lceil R\rceil}{R}\right)^{-\rho}\right)\leq \tilde{C}_0\left(2^{\rho}+1\right)R^{-\rho}.$$

Here we used that $\frac{\lfloor R\rfloor}{R}\geq\frac{1}{2}$ for $R\geq 1$. Thus choosing $C_0=\tilde{C}_0\left(2^{\rho}+1\right)$ the claim holds for general $R\geq 1$.

We now prove the claim for a_n with $n\in\mathbb{N}$ by induction. By choosing $\tilde{C}_0$ sufficiently large, i.e. $\int_0^1 h(x)\,\mathrm{d}x\leq\tilde{C}_0$, the claim already holds for $n=1$ and we can assume $n>1$ in the following. Taking Eq. (47) with $x_0=0$, dividing by x and integrating from $n-1$ to n we obtain

$$\begin{aligned}
a_n&=\int_{n-1}^{n} h(x)\,\mathrm{d}x=(1-\rho)\int_{n-1}^{n}\frac{1}{x}\int_0^x h(y)\,\mathrm{d}y\mathrm{d}x+\int_{n-1}^{n}\frac{1}{x}\int_0^x\int_{x-y}^{\infty}\frac{K(y,z)}{z}h(y)h(z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\\
&\leq(1-\rho)\int_{n-1}^{n}x^{-\rho}\mathrm{d}x+C\int_{n-1}^{n}\frac{1}{x}\int_0^x\int_{x-y}^{\infty}\left(y^{-a}z^{b-1}+y^bz^{-a-1}\right)h(y)h(z)\,\mathrm{d}z\mathrm{d}y\mathrm{d}x\\
&\leq(1-\rho)(n-1)^{-\rho}+\frac{C}{n-1}\int_{n-1}^{n}\int_0^x\frac{y^{-a}h(y)}{(2+(x-y))^{\rho-b}}\mathrm{d}y\mathrm{d}x+\frac{C}{n-1}\int_{n-1}^{n}\int_0^x\frac{y^{b}h(y)}{(2+(x-y))^{a+\rho}}\mathrm{d}y\mathrm{d}x\\
&=:(I)+(II)+(III),
\end{aligned}\tag{50}$$

where we used Lemma 3.4 in the last step. We estimate the three terms separately:

$$(I)\leq(1-\rho)n^{-\rho}\left(1-\frac{1}{n}\right)^{-\rho}\leq 2^{\rho}(1-\rho)n^{-\rho},\tag{51}$$

where we used $n\geq 2$. Furthermore using Corollary 2.15 we have

$$\begin{aligned}
(II)&\leq\frac{C}{n-1}\int_{n-1}^{n}\left(\int_0^1\frac{y^{-a}h(y)}{(2+(x-y))^{\rho-b}}\mathrm{d}y+\int_1^x\frac{y^{-a}h(y)}{(2+(x-y))^{\rho-b}}\mathrm{d}y\right)\leq\frac{C}{n-1}\int_{n-1}^{n}1+x^{1-\rho}\mathrm{d}x\\
&\leq\frac{C}{n-1}\left(1+n^{1-\rho}\right)\leq Cn^{-\rho},
\end{aligned}\tag{52}$$

using $n \geq 2$ and $\rho < 1$ in the last step. The third term cannot be estimated directly as $b \geq 0$ and will require some iteration argument. Choosing $0 < \delta < a$ such that $\delta + \rho < 1$ we first of all get

$$(III) \leq \frac{C}{n-1} \int_{n-1}^{n} \int_{0}^{x} \frac{y^b h(y)}{(1+n-y)^{a+\rho}} \mathrm{d}y\mathrm{d}x \leq \frac{C}{n-1} \sum_{k=1}^{n} \int_{k-1}^{k} \frac{y^b h(y)}{(1+(n-y))^{\delta+\rho}} \mathrm{d}y$$
$$\leq \frac{C}{n-1} \sum_{k=1}^{n} \frac{k^b}{(1+(n-k))^{\delta+\rho}} a_k. \tag{53}$$

Summarising (50)–(53) we have shown so far that there exists a constant $C_1 > 0$ that does not depend on n such that

$$a_n \leq C_1 n^{-\rho} + \frac{C_1}{n-1} \sum_{k=1}^{n} \frac{k^b}{(1+(n-k))^{\delta+\rho}} a_k = C_1 n^{-\rho} + \frac{C_1}{n-1} \sum_{k=1}^{n-1} \frac{k^b}{(1+(n-k))^{\delta+\rho}} a_k + C_1 \frac{n^b}{n-1} a_n.$$

As $b < 1$ there exists $N_0 \in \mathbb{N}$ such that $C_1 \frac{n^b}{n-1} < \frac{1}{2}$ for all $n \geq N_0$ (i.e. $N_0 > (2C_1)^{\frac{1}{1-b}} + 1$ suffices). Thus for $n \geq N_0$ we get

$$a_n \leq 2C_1 n^{-\rho} + \frac{2C_1}{n-1} \sum_{k=1}^{n-1} \frac{k^b}{(1+(n-k))^{\delta+\rho}} a_k. \tag{54}$$

Choosing now $\tilde{C}_0$ sufficiently large such that $\int_0^{N_0} h(y)\,\mathrm{d}y \leq \tilde{C}_0 N_0^{-\rho}$ we immediately have $a_n \leq \tilde{C}_0 n^{-\rho}$ for all $n \leq N_0$. We now finish the proof by induction, i.e. we show $a_n \leq \tilde{C}_0 n^{-\rho}$ for all $n \in \mathbb{N}$. The constant $\tilde{C}_0$ depends on N_0 that will be both fixed below.

For $n \leq N_0$ the claim holds by the choice of $\tilde{C}_0$. Assuming now that $a_k \leq \tilde{C}_0 k^{-\rho}$ holds for all $k \leq n$ we have due to (54)

$$a_{n+1} \leq 2C_1 (n+1)^{-\rho} + \frac{2C_1}{n} \sum_{k=1}^{n} \frac{k^b}{(2+n-k)^{\delta+\rho}} a_k \leq 2C_1 (n+1)^{-\rho} + \frac{2C_1\tilde{C}_0}{n} \sum_{k=1}^{n} \frac{k^{b-\rho}}{(2+n-k)^{\delta+\rho}}$$
$$\leq 2C_1 (n+1)^{-\rho} + \frac{2C_1\tilde{C}_0}{n} \sum_{k=0}^{n-1} (2+k)^{-\delta-\rho} \leq 2C_1 (n+1)^{-\rho} + \frac{2C_1\tilde{C}_0}{n} \int_{-1}^{n-1} (2+x)^{-\delta-\rho}$$
$$\leq 2C_1 (n+1)^{-\rho} + \frac{2C_1\tilde{C}_0}{1-\delta-\rho} \left(\frac{(n+1)^{1-\delta-\rho}}{n} - 1 \right) \leq \left(2C_1 + \tilde{C}_0 \frac{4C_1}{1-\delta-\rho} (1+n)^{-\delta} \right) (n+1)^{-\rho}. \tag{55}$$

We fix now first $N_0 \in \mathbb{N}$ such that $N_0 > \max\left\{ (2C_1)^{\frac{1}{1-b}} + 1, \left(\frac{8C_1}{1-\delta-\rho}\right)^{1/\delta} - 1 \right\}$, which in particular gives $\frac{4C_1}{1-\delta-\rho}(1+n)^{-\delta} < \frac{1}{2}$ for all $n \geq N_0$. Then we fix $\tilde{C}_0 > \max\left\{ 4C_1, N_0^{\rho} \int_0^{N_0} h(y)\,\mathrm{d}y \right\}$, which in particular ensures that $2C_1 < \frac{\tilde{C}_0}{2}$. Using this in (55) we finally obtain

$$a_{n+1} \leq \left(\frac{\tilde{C}_0}{2} + \frac{\tilde{C}_0}{2} \right) (n+1)^{-\rho} = \tilde{C}_0 (n+1)^{-\rho},$$

finishing the proof. □

Conflict of interest statement

The authors declare that there is no known conflict of interest.

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Appendix A. Moment estimates

Lemma A.1. *Let $h \in \mathcal{X}_\rho$ and $\alpha \in \mathbb{R}$. Then one has the following estimates*

1. $\int_0^D x^\alpha h(x)\,\mathrm{d}x \le C\|h\| D^{1-\rho+\alpha}$ *for all $D>0$ if $\rho - 1 < \alpha$,*
2. $\int_D^\infty x^\alpha h(x)\,\mathrm{d}x \le C\|h\| D^{1-\rho+\alpha}$ *for all $D>0$ if $\alpha < \rho - 1$,*

where $\|h\|$ is defined in (13).

Proof.

1. The case $\alpha \ge 0$ is clear by definition of $\mathcal{X}_\rho$. For $\alpha \in (\rho-1, 0)$ one has, using a dyadic decomposition, that

$$\begin{aligned}\int_0^D x^\alpha h(x)\,\mathrm{d}x &= \sum_{n=0}^\infty \int_{2^{-(n+1)}D}^{2^{-n}D} x^\alpha h(x)\,\mathrm{d}x \le \sum_{n=0}^\infty 2^{-\alpha(n+1)} D^\alpha \int_{2^{-(n+1)}D}^{2^{-n}D} h(x)\,\mathrm{d}x \\ &\le \|h\| \sum_{n=0}^\infty 2^{-\alpha(n+1)} D^\alpha \left(2^{-n}D\right)^{1-\rho} = 2^{-\alpha}\|h\| D^{1+\alpha-\rho} \sum_{n=0}^\infty \left(2^{1+\alpha-\rho}\right)^{-n} \\ &= C(\alpha,\rho)\|h\| D^{1+\alpha-\rho}.\end{aligned}$$

2. This follows similarly using again a dyadic decomposition. □

Appendix B. Dual problems

B.1. Existence results

In this section we show the existence of solutions to some dual problems arising in the proof of the lower bounds. Throughout this section we will use the following notation: $\mathcal{M}^{\text{fin}}$ will denote the space of finite measures, $\mathcal{M}_+^{\text{fin}}$ is the space on non-negative finite measures. Furthermore C_b^n denotes the space of bounded n-times differentiable functions with bounded derivatives. Let $\omega \in (0,1)$, $A \in \mathbb{R}$ and consider the equation

$$\partial_t f(x,t) - P\int_0^\infty \frac{1}{y^{1+\omega}}\left[f(x+y) - f(x)\right]\mathrm{d}y = 0 \tag{B.1}$$

together with initial value $f(x,0) = \delta(\cdot - A)$.

Proposition B.1. *There exists a (weak) solution $f \in C\left([0,T], \mathcal{M}_+^{\text{fin}}\right)$ of (B.1) with initial value $f_0 = \delta(\cdot - A)$. Furthermore this f satisfies* $\operatorname{supp} f(\cdot,t) \subset (-\infty, A]$ *and $\int_{\mathbb{R}} f(\cdot,t)\,\mathrm{d}x = 1$ for all $t \in [0,T]$.*

Proof (Sketch). First we consider the regularized equation

$$\begin{aligned}\partial_t f(x,t) &= P\int_0^\infty \frac{1}{y^{1+\omega}+\nu}\left[f(x+y,t) - f(x,t)\right]\mathrm{d}y \\ f(\cdot,0) &= \delta(\cdot - A)\end{aligned} \tag{B.2}$$

with $\nu > 0$. In the second step we will pass to the limit $\nu \to 0$. We can reformulate (B.2) as the following fixed-point problem:

$$f^{\nu}(x,t) = \delta(x-A)\,\mathrm{e}^{-P\int_0^\infty \frac{1}{y^{1+\omega}+\nu}\mathrm{d}y} + \int_0^t \mathrm{e}^{-(t-s)\int_0^\infty \frac{1}{y^{1+\omega}+\nu}\mathrm{d}y} \int_0^\infty \frac{1}{y^{1+\omega}+\nu} f(x+y)\,\mathrm{d}y\mathrm{d}s. \tag{B.3}$$

It is straightforward, applying the contraction mapping theorem, to obtain a solution $f \in C\left([0,T],\mathcal{M}_+^{\mathrm{fin}}\right)$ for any $T > 0$. Furthermore, one obtains $\int_{\mathbb{R}} f^{\nu}(x,t)\,\mathrm{d}x = 1$ for all $t > 0$ and $\nu > 0$ (by integrating the equation, see below). In addition f^{ν} satisfies Eq. (B.2) in weak form, i.e.

$$\int_{\mathbb{R}} f^{\nu}(x,t)\,\psi(x)\,\mathrm{d}x = \psi(A) + \int_0^t \int_{\mathbb{R}} \int_0^\infty \frac{1}{y^{1+\omega}+\nu} f^{\nu}(x,s)\left[\psi(x-y)-\psi(x)\right]\mathrm{d}y\mathrm{d}x\mathrm{d}s \tag{B.4}$$

for all $\psi \in C_b(\mathbb{R})$ and for $0 < \tilde{\omega} < \omega$ taking $\psi(x) = \frac{|x|}{1+|x|^{1-\tilde{\omega}}}$ and using $|\psi(x-y)-\psi(x)| \le C\min\left\{|y|^{\tilde{\omega}},|y|\right\}$ we obtain (by approximation)

$$\begin{aligned}\int_{\mathbb{R}} f^{\nu}(x,t)\,\psi(x)\,\mathrm{d}x &\le |A|^{\tilde{\omega}} + \int_0^t \int_{\mathbb{R}} \int_0^\infty \frac{|\psi(x-y)-\psi(x)|}{y^{1+\omega}+\nu} f^{\nu}(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s \\ &\le |A|^{\tilde{\omega}} + C\int_0^t \int_{\mathbb{R}} \int_0^\infty \frac{\min\left\{|y|^{\tilde{\omega}},|y|\right\}}{y^{1+\omega}+\nu} f^{\nu}(x,s)\,\mathrm{d}y\mathrm{d}x\mathrm{d}s \le C(T,\omega,\tilde{\omega},A).\end{aligned}$$

Thus $\int_{\mathbb{R}} \frac{|x|}{1+|x|^{1-\tilde{\omega}}} f^{\nu}(x,t)\,\mathrm{d}x$ is uniformly bounded (i.e. independent of ν and t).

Using this and that $\{f^{\nu}\}_{\nu>0}$ is uniformly bounded by 1, we can extract a subsequence $\{f^{\nu_n}\}_{n\in\mathbb{N}}$ (denoted in the following as $\{f^n\}_{n\in\mathbb{N}}$) such that $f^n(\cdot,t_k)$ converges in the sense of measures to some $f(\cdot,t_k)$ for all $k \in \mathbb{N}$, where $\{t_k\}_{k\in\mathbb{N}} = [0,T]\cap\mathbb{Q}$.

We next show that f^n is equicontinuous in t as a distribution, i.e. from (B.4) we obtain for any $\psi \in C_c^1(\mathbb{R})$:

$$\begin{aligned}\left|\int_{\mathbb{R}} \left(f^n(x,t)-f^n(x,s)\right)\psi(x)\,\mathrm{d}x\right| &= \left|\int_s^t \int_{\mathbb{R}} f^n(x,r)\int_0^\infty \frac{1}{y^{1+\omega}+\nu}\left[\psi(x-y)-\psi(x)\right]\mathrm{d}y\mathrm{d}x\mathrm{d}r\right| \\ &\le \int_s^t \int_{\mathbb{R}} f^n(x,r)\left[\int_0^1 \frac{\|\psi'\|_{L^\infty}\,y}{y^{1+\omega}+\nu}\mathrm{d}y + \int_1^\infty \frac{2\|\psi\|_{L^\infty}}{y^{1+\omega}+\nu}\mathrm{d}y\right]\mathrm{d}x\mathrm{d}r \\ &\le C(\psi)\,|t-s|,\end{aligned}$$

where $C(\psi)$ is a constant independent of ν but depending on ψ and ψ'. Using the equicontinuity of f^n (as a distribution) one can show that f^n converges to some limit f (in the sense of distributions) for all $t \in [0,T]$.

Using furthermore the uniform boundedness of $\int_{\mathbb{R}} |x|^{\tilde{\omega}} f^n(x,t)\,\mathrm{d}x$ one can show that f^n converges already in the sense of measures by approximating and cutting the test function for large values of $|x|$.

Using similar arguments we can also show that for the limit $f^n \rightharpoonup f$ we have $f \in C\left([0,T],\mathcal{M}_+^{\mathrm{fin}}\right)$ and taking the limit $n \to \infty$ in (B.4), f satisfies

$$\int_{\mathbb{R}} f(x,t)\,\psi(x)\,\mathrm{d}x = \psi(A) + \int_0^t \int_{\mathbb{R}} \int_0^\infty \frac{1}{y^{1+\omega}} f(x,s)\left[\psi(x-y)-\psi(x)\right]\mathrm{d}y\mathrm{d}x\mathrm{d}s \tag{B.5}$$

for each $\psi \in C_b^1(\mathbb{R})$ and all $t \in [0,T]$.

From the construction of f using the contraction mapping principle we immediately get $\operatorname{supp} f(\cdot,t) \subset (-\infty,A]$ for all $t \in [0,T]$. To see $\int_{\mathbb{R}} f(\cdot,t)\,\mathrm{d}x = 1$ for all $t \in [0,T]$ we integrate Eq. (B.1) over $\mathbb{R}$ and use Fubini's theorem to obtain $\partial_t \int_{\mathbb{R}} G(\cdot,t)\,\mathrm{d}x = 0$. Thus together with the initial condition the claim follows. □

Remark B.2. The analogous result holds true if $f_0 = -\delta(\cdot - A)$.

As a direct consequence of Proposition B.1 we also obtain smooth solutions for smoothed initial data. Therefore for $\kappa > 0$ we denote in the following by φ_κ a non-negative, symmetric standard mollifier with $\operatorname{supp}\varphi_\kappa \subset [-\kappa, \kappa]$.

Proposition B.3. *Let $f_0 := \delta(\cdot - A)$. Then there exists a solution $f \in C^1([0,T], C^\infty(\mathbb{R}))$ to (B.1) with initial datum $f_0 * \varphi_\kappa = \varphi_\kappa(\cdot - A)$.*

Proof. This follows directly by convolution in x from Proposition B.1. □

Proposition B.4. *There exists a strong solution $f \in C^1([0,T], C^\infty(\mathbb{R}))$ to (B.1) with initial datum $f_0 := \chi_{(-\infty,A]} * \varphi_\kappa$.*

Proof. Let G be the solution given by Proposition B.3 for $G_0 := \delta(\cdot - A) * \varphi_\kappa$. Then $f(x,t) := \int_x^\infty G(y,t)\,\mathrm{d}y$ solves (B.1) with the desired initial condition. □

In the same way as in the proofs of Proposition B.4 and Proposition B.4 we obtain the following existence result:

Proposition B.5. *Let $\varepsilon > 0$, $L > 0$ and $\lambda_1, \lambda_2 > 0$ be two constants (depending on some parameters). Then there exists a weak solution $G \in C\left([0,T], \mathcal{M}_+^{\text{fin}}\right)$ and a strong solution $W \in C([0,T], C^\infty)$ of the equation*

$$\partial_t W(\xi,t) - \int_0^1 \frac{h_\varepsilon(z)}{z}\left[\lambda_1 (z+\varepsilon)^{-a} + \lambda_2 (z+\varepsilon)^b\right]\left[W\left(\xi + \frac{z}{L}, t\right) - W(\xi,t)\right]\mathrm{d}z = 0 \tag{B.6}$$

together with initial condition $G(\cdot,0) = \delta(\cdot - A)$ and $W(\cdot,0) = \chi_{(-\infty,A]} * \varphi_\kappa$.

Remark B.6. The measure G has the same properties as the measure f in Proposition B.1.

Remark B.7. By convolution we also obtain a strong solution $G \in C([0,T], C^\infty)$ of (B.6) with initial condition $G(\cdot,0) = \delta(\cdot - A) * \varphi_\kappa$.

For further use we denote the integral kernels occurring in Proposition B.1 and Proposition B.5 by

$$N_\omega(z) := z^{-1-\omega} \quad \text{and} \quad N_\varepsilon(z) := \frac{h_\varepsilon(z)}{z}\left[\lambda_1 (z+\varepsilon)^{-a} + \lambda_2 (z+\varepsilon)^b\right]. \tag{B.7}$$

Proposition B.8. *Let $n \in \mathbb{N}$, $R \in \mathbb{R}$ and $N_i\colon (0,\infty) \to \mathbb{R}_{\geq 0}$ either of the form N_{ω_i} for some $\omega_i \in (0,1)$ or N_ε given by (B.7) (and then continued by 0 to $(0,\infty)$) for $i = 1, \dots n$. Let $N := \sum_{i=1}^n N_i$. Then there exists a solution $f \in C^1([0,T], C^\infty(\mathbb{R}))$ to the equation*

$$\partial_t f(x,t) = \int_0^\infty N(z)\left[f(x+z) - f(x)\right]\mathrm{d}z \tag{B.8}$$

either with initial datum $f_0 = \chi_{(-\infty,R]} *^n \varphi_\kappa$ or $f_0 = \delta(\cdot - R) *^n \varphi_\kappa$, where $*^n$ denotes the n-fold convolution with φ_κ.

Proof. It suffices to consider the case $n = 2$ (otherwise argue by induction). Then by Proposition B.3 and Proposition B.4 there exist solutions f^i to Eq. (B.8) with N replaced by N_i and initial datum $f_0^1 = \delta(\cdot) * \varphi_\kappa$ and $f_0^2 = \chi_{(-\infty,R]} * \varphi_\kappa$ (or $f_0^2 = \delta(\cdot - R) * \varphi_\kappa$). A straightforward computation shows that the convolution $f := f^1 * f^2$ satisfies (B.8) together with the correct initial condition. □

Remark B.9. Let G_κ and f_κ be the solutions given by Proposition B.8 with initial condition $G_\kappa(\cdot,0) = \delta(\cdot - A) *^n \varphi_\kappa$ and $f(\cdot,0) = \chi_{(-\infty,A]} *^n \varphi_\kappa$. Then from the construction in the proof of Proposition B.8 and Proposition B.1 we obtain:

1. $G_\kappa \geq 0$ on $\mathbb{R}$ (in the sense of measures) and $0 \leq f_\kappa \leq 1$ for all $t \in [0, T]$,
2. $\operatorname{supp} G_\kappa(\cdot, t)$, $\operatorname{supp} f_\kappa(\cdot, t) \subset (-\infty, A + n\kappa]$ for all $t \in [0, T]$,
3. $\int_{\mathbb{R}} G(\cdot, t)\,\mathrm{d}x = 1$ for all $t \in [0, T]$,
4. f_κ is non-increasing.

B.2. Integral estimates for subsolutions

In this section we will always assume that the integral kernel N is given as the sum of kernels of the form N_{ω_i} or N_ε and we will prove some integral estimates that are frequently used.

Lemma B.10. *Let $\omega \in (0, 1)$ and G the solution of*

$$\partial_t G(x,t) = P \int_0^\infty N_\omega(z)\,[G(x+z) - G(x)]\,\mathrm{d}z$$
$$G(\cdot, 0) = \delta(\cdot - A) * \varphi_\kappa = \varphi_\kappa(x - A) \tag{B.9}$$

given by Proposition B.3, where P is a constant. Then for any $\mu \in (0, 1)$ one has

$$\int_{-\infty}^{A-D} G(x,t)\,\mathrm{d}x \leq C\left(\frac{\kappa}{D}\right)^\mu + C\frac{Pt}{D^\omega} \quad \textit{for all } D > 0.$$

Proof. By shifting with A we can assume $A = 0$. Let $Z > 0$. Then testing Eq. (B.9) with $\mathrm{e}^{Z(x-\kappa)}$ (note that this is possible as $\operatorname{supp} G \subset (-\infty, \kappa]$) one obtains

$$\begin{aligned}\partial_t \int_{\mathbb{R}} G(x,t)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}x &= P\int_{\mathbb{R}}\int_0^\infty N_\omega(y)\left[G(x+y) - G(x)\right]\mathrm{e}^{Z(x-\kappa)}\mathrm{d}y\mathrm{d}x \\ &= P\int_0^\infty N_\omega(y)\left(\mathrm{e}^{-Zy} - 1\right)\mathrm{d}y \int_{\mathbb{R}} G(x,t)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}y =: M_\omega(Z)\int_{\mathbb{R}} G(x,t)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}x.\end{aligned}$$

Furthermore

$$\int_{\mathbb{R}} G_\kappa(x,0)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}x = \int_{\mathbb{R}} \varphi_\kappa(x)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}x.$$

Thus we obtain $\int_{\mathbb{R}} G(x,t)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}x = \int_{\mathbb{R}} \varphi_\kappa(x)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}x \exp(-t\,|M_\omega(Z)|)$. Estimating $M_\omega(Z)$ we obtain

$$\begin{aligned}|M_\omega(Z)| &\leq P\int_0^\infty \frac{1 - \mathrm{e}^{-Zy}}{y^{1+\omega}}\mathrm{d}y = -\frac{P}{\omega}\int_0^\infty \left(1 - \mathrm{e}^{-Zy}\right)\frac{\partial}{\partial y}\left(y^{-\omega}\right)\mathrm{d}y \\ &= \frac{PZ}{\omega}\int_0^\infty \frac{\mathrm{e}^{-Zy}}{y^\omega}\mathrm{d}y = \frac{PZ^\omega}{\omega}\int_0^\infty y^{-\omega}\mathrm{e}^{-y}\mathrm{d}y = P\frac{\Gamma(1-\omega)}{\omega}Z^\omega \\ &= CPZ^\omega.\end{aligned}$$

Using that $G = 0$ on (κ, ∞) we get

$$\int_{-\infty}^{\kappa} G(x,t)\left(1 - \mathrm{e}^{Z(x-\kappa)}\right)\mathrm{d}x$$

$$= \int_{\mathbb{R}} G(x,t)\,\mathrm{d}x - \int_{\mathbb{R}} G(x,t)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}x = 1 - \int_{\mathbb{R}} \varphi_\kappa(x)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}x \exp(-t\,|M_\omega(Z)|)$$
$$\leq \left[\left(1 - \int_{\mathbb{R}} \varphi_\kappa(x)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}x \right) + \int_{\mathbb{R}} \varphi_\kappa(x)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}x\,|M_\omega(Z)|\,t \right].$$

As $\operatorname{supp}\varphi \subset [-\kappa,\kappa]$ we can estimate $\mathrm{e}^{-2Z\kappa} \leq \int_{\mathbb{R}} \varphi_\kappa(x)\,\mathrm{e}^{Z(x-\kappa)}\mathrm{d}x \leq 1$. Then choosing $Z = \frac{1}{D}$ and using also the estimate for M_ω we obtain

$$\int_{-\infty}^{-D} G(x,t)\,\mathrm{d}x \leq \int_{-\infty}^{-D} G(x,t)\,\frac{1-\mathrm{e}^{\frac{x-\kappa}{D}}}{1-\mathrm{e}^{-1-\frac{\kappa}{D}}}\mathrm{d}x \leq \int_{-\infty}^{\kappa} (\cdots)\,\mathrm{d}x$$
$$\leq \frac{1}{1-\mathrm{e}^{-1-\frac{\kappa}{D}}}\left[\left(1-\mathrm{e}^{-\frac{2\kappa}{D}}\right) + CPt\frac{1}{D^\omega} \right] \leq C\left(\frac{\kappa}{D}\right)^\mu + C\frac{Pt}{D^\omega}. \qquad \square$$

We now consider the situation of Proposition B.8 where the integral kernel is given as the sum of different kernels

Lemma B.11. *In the situation of Proposition B.8 with $n = 2$ one has*

$$\int_{-\infty}^{A-D} G(x,t)\,\mathrm{d}x \leq \int_{-\infty}^{A-D/2} G_1(x,t)\,\mathrm{d}x + \int_{-\infty}^{-D/2} G_2(x,t)\,\mathrm{d}x$$

for all $D > 0$.

Proof. We consider again only the case $A = 0$, while the general result follows by shifting. One has

$$\int_{-\infty}^{-D} G(x,t)\,\mathrm{d}x = \int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{(-\infty,-D]}(x+y)\,G_1(x,t)\,G_2(y,t)\,\mathrm{d}x\mathrm{d}y$$
$$= \int_{\mathbb{R}} \int_{-\infty}^{-D-y} G_1(x,t)\,G_2(y,t)\,\mathrm{d}x\mathrm{d}y$$
$$= \int_{-\frac{D}{2}}^{\infty} \int_{-\infty}^{-D-y} G_1(x,t)\,G_2(y,t)\,\mathrm{d}x\mathrm{d}y + \int_{-\infty}^{-\frac{D}{2}} \int_{-\infty}^{-D-y} G_1(x,t)\,G_2(y,t)\,\mathrm{d}x\mathrm{d}y$$
$$\leq \int_{-\infty}^{-\frac{D}{2}} G_1(x,t)\,\mathrm{d}x \int_{\mathbb{R}} G_2(y,t)\,\mathrm{d}y + \int_{-\infty}^{-\frac{D}{2}} G_2(y,t)\,\mathrm{d}y \int_{\mathbb{R}} G_1(x,t)\,\mathrm{d}x$$
$$\leq \int_{-\infty}^{-D/2} G_1(x,t)\,\mathrm{d}x + \int_{-\infty}^{-D/2} G_2(x,t)\,\mathrm{d}x$$

where in the last step we used that G_i is normalized for $i = 1, 2$. $\square$

Remark B.12. By induction we can prove the corresponding estimate for $n > 2$ with $D/2$ replaced by $D/2^{n-1}$ and κ replaced by $n\kappa$ (and of course summing over all G_i, $i = 1, \ldots, n$ on the right hand side).

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