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Regularity of solutions to fully nonlinear elliptic and parabolic free boundary problems ☆

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Abstract

We consider fully nonlinear obstacle-type problems of the form

$$\begin{cases} F(D^2u,x)=f(x) & \text{a.e. in } B_1\cap\Omega, \\ |D^2u|\le K & \text{a.e. in } B_1\backslash\Omega, \end{cases}$$

where Ω is an open set and $K>0$. In particular, structural conditions on F are presented which ensure that $W^{2,n}(B_1)$ solutions achieve the optimal $C^{1,1}(B_{1/2})$ regularity when f is Hölder continuous. Moreover, if f is positive on $\overline{B}_1$, Lipschitz continuous, and $\{u\neq 0\}\subset\Omega$, we obtain interior C^1 regularity of the free boundary under a uniform thickness assumption on $\{u=0\}$. Lastly, we extend these results to the parabolic setting.

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1. Introduction

Obstacle-type problems appear in several mathematical disciplines such as minimal surface theory, potential theory, mean field theory of superconducting vortices, optimal control, fluid filtration in porous media, elasto-plasticity, and financial mathematics [1–5]. The classical obstacle problem involves minimizing the Dirichlet energy on a given domain in the space of square integrable functions with square integrable gradient constrained to remain above a fixed

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obstacle function and with prescribed boundary data. Due to the structure of the Dirichlet integral, this minimization process leads to the free boundary problem

$$\Delta u = f\chi_{\{u>0\}} \quad \text{in } B_1,$$

where $B_1 \subset \mathbb{R}^n$ is the unit ball centered at the origin. A simple one-dimensional example shows that even if $f \in C^\infty$, u is not more regular than $C^{1,1}$, and Lipschitz continuity of f yields this optimal regularity via the Harnack inequality.

An obstacle-type problem is a free boundary problem of the form

$$\Delta u = f\chi_\Omega \quad \text{in } B_1, \tag{1}$$

where Ω is an (unknown) open set. If $\Omega = \{u \neq 0\}$ and f is Lipschitz continuous, monotonicity formulas may be used to prove $C^{1,1}$ regularity of u. Nevertheless, this method strongly depends on the Lipschitz continuity of f. Recently, a harmonic analysis technique was developed in [6] to prove optimal regularity under the weakest possible assumption: if the Newtonian potential of f is $C^{1,1}$, then u is uniformly $C^{1,1}$ in $B_{1/2}$, where the bound on the Hessian depends on $\|u\|_{L^\infty(B_1)}$.

Fully nonlinear analogs of (1) have been considered by several researchers. The case

$$F(D^2u) = f\chi_\Omega \quad \text{in } B_1$$

has been studied in [7] for $\Omega = \{u > 0\}$ and in [8] when $\Omega = \{u \neq 0\}$. Moreover, a fully nonlinear version of the method in [6] was developed in [9] and applied to

$$\begin{cases} F(D^2u) = 1 & \text{a.e. in } B_1 \cap \Omega, \\ |D^2u| \le K & \text{a.e. in } B_1 \backslash \Omega, \end{cases}$$

where Ω is an open set, $K > 0$, and $u \in W^{2,n}(B_1)$. The idea is to replace the projection on second-order harmonic polynomials carried out in [6] with a projection involving the BMO estimates in [10]. For convex operators, this tool is employed to prove that u is $C^{1,1}$ in $B_{1/2}$ and, under a standard thickness assumption, that the free boundary is locally C^1.

Our main result is Theorem 1 and establishes optimal regularity for the more general free boundary problem

$$\begin{cases} F(D^2u, x) = f(x) & \text{a.e. in } B_1 \cap \Omega, \\ |D^2u| \le K & \text{a.e. in } B_1 \backslash \Omega, \end{cases} \tag{2}$$

where Ω is an open set, $K > 0$, f is Hölder continuous, and under certain structural conditions on F (see §1.1). As a direct consequence, we obtain optimal regularity for general operators $F(D^2u, Du, u, x)$ and thereby address [9, Remark 1.1], see Corollary 2. Free boundary problems of this type appear in the mean field theory of superconducting vortices [3, Introduction] and optimal switching problems [11].

The underlying principle in the proof is to locally apply Caffarelli's elliptic regularity theory [12] to rescaled variants of (2) in order to obtain a bound on D^2u. The main difficulty lies in verifying an average L^n decay of the right-hand side in question. However, one may exploit that $u \in C^{1,\alpha}(B_1)$, D^2u is bounded in $B_1 \backslash \Omega$, and the BMO estimates in [10] to prove that locally around a free boundary point, the coincidence set $B_1 \backslash \Omega$ decays fast enough to ensure the L^n decay. Our assumptions on F involve conditions which enable us to utilize standard tools such as the maximum principle and Evans–Krylov theorem.

Moreover, once we establish that $u \in C^{1,1}$ in $B_{1/2}$, the corresponding regularity theory for the free boundary follows in a standard way through the classification of blow-up solutions and is carried out in §3. Indeed, non-degeneracy holds if f is positive on $\overline{B}_1$ and $\{|\nabla u| \neq 0\} \subset \Omega$. Furthermore, blow-up solutions around thick free boundary points are half-space solutions, and this fact combines with a directional monotonicity result to yield C^1 regularity of the free boundary, see Theorem 15 for a precise statement.

Finally, we generalize the above-mentioned results to the parabolic setting (see also [13]) in §4 by considering the free boundary problem

$$\begin{cases} \mathcal{H}(u(X), X) = f(X) & \text{a.e. in } Q_1 \cap \Omega, \\ |D^2u| \le K & \text{a.e. in } Q_1 \backslash \Omega, \end{cases}$$

where $X = (x,t) \in \mathbb{R}^n \times \mathbb{R}$, $\mathcal{H}(u(X), X) := F(D^2u(X), X) - \partial_t u(X)$, Q_1 is the parabolic cylinder $B_1(0) \times (-1, 0)$, $\Omega \subset Q_1$ is some open set, and $K > 0$.

1.1. Setup

In what follows, we record the structural conditions on the operator F that will be employed throughout this paper. The first three conditions are well known in the study of free boundary problems and provide tools such as the maximum principle and Evans–Krylov theorem. The last condition, which we denote by (H4), controls the oscillation of the operator in the x-variable and enables the application of Caffarelli's regularity theory in our general framework. Moreover, we note that throughout the paper the constants of proportionality in our estimates may change from line to line while still being denoted by the same symbol C.

(H1) $F(0, x) = 0$ for all $x \in B_1$.

(H2) The operator F is uniformly elliptic with ellipticity constants $\lambda_0, \lambda_1 > 0$ such that

$$\mathcal{P}^-(M - N) \le F(M, x) - F(N, x) \le \mathcal{P}^+(M - N) \quad \forall x \in B_1,$$

where M and N are symmetric matrices and $\mathcal{P}^\pm$ are the Pucci operators

$$\mathcal{P}^-(M) := \inf_{\lambda_0 \,\mathrm{Id} \le N \le \lambda_1 \,\mathrm{Id}} \mathrm{Tr}\, NM, \qquad \mathcal{P}^+(M) := \sup_{\lambda_0 \,\mathrm{Id} \le N \le \lambda_1 \,\mathrm{Id}} \mathrm{Tr}\, NM.$$

(H3) $F(M, x)$ will be assumed to be concave or convex in M for all x in B_1.

(H4)

$$|F(M, x) - F(M, y)| \le C(|M| + 1)|x - y|^\alpha,$$

for some $\alpha \in (0, 1]$.

Remark 1. Note that (H1) is not restrictive since we can consider $G(M, x) := F(M, x) - F(0, x)$ which fulfills (H2) with the same ellipticity constants as well as (H3) and (H4). The uniform ellipticity also implies Lipschitz regularity,

$$|F(M, x) - F(N, x)| \le \max\{|\mathcal{P}^-(M - N)|, |\mathcal{P}^+(M - N)|\} \le n\lambda_1 |M - N|. \tag{3}$$

In particular,

$$|F(M, x) - F(M, y)| \le |F(M, x) - F(0, x)| + |F(M, y) - F(0, y)| \le 2n\lambda_1 |M|. \tag{4}$$

Remark 2. Let

$$\tilde{\beta}(x) = \sup_{M \in \mathcal{S}} \frac{|F(M, x) - F(M, 0)|}{|M| + 1},$$

where $\mathcal{S}$ is the space of symmetric matrices. Note that (H4) implies the Hölder continuity of $\tilde{\beta}$ at the origin.

2. Interior $C^{1,1}$ regularity

In this section, we prove optimal regularity in the interior for $W^{2,n}(B_1)$ solutions of the free boundary problem (2):

Theorem 1. *Let $f \in C^\alpha(B_1)$ be a given function and Ω a domain such that $u : B_1 \to \mathbb{R}$ is a $W^{2,n}(B_1)$ solution of*

$$\begin{cases} F(D^2u, x) = f(x) & \text{a.e. in } B_1 \cap \Omega, \\ |D^2u| \le K & \text{a.e. in } B_1 \setminus \Omega. \end{cases}$$

Assume F satisfies (H1)–(H4). Then there exists a constant $\overline{C} > 0$, depending on $\|u\|_{W^{2,n}(B_1)}$, $\|f\|_{C^\alpha(B_1)}$, the dimension, and the ellipticity constants such that

$$|D^2u| \le \overline{C}, \quad \text{a.e. in } B_{1/2}.$$

Since $W^{2,n}(B_1)$ solutions of (2) are $C^{1,\alpha}(B_1)$, one may utilize the above theorem to deduce an optimal regularity result for more general operators and thereby address [9, Remark 1.1]:

Corollary 2. *Let $f \in C^\alpha(B_1)$ be a given function and Ω a domain such that $u : B_1 \to \mathbb{R}$ is a $W^{2,n}(B_1)$ solution of*

$$\begin{cases} F(D^2u, Du, u, x) = f(x) & \text{a.e. in } B_1 \cap \Omega, \\ |D^2u(x)| \le K & \text{a.e. in } B_1 \backslash \Omega, \end{cases}$$

and assume that: $F(0, v, t, x) = 0$ for all $v \in \mathbb{R}^n$, $t \in \mathbb{R}$, and $x \in B_1$; F satisfies (H1)–(H3) in the matrix variable (keeping all other variables fixed); and

$$|F(M, w_1, s_1, x_1) - F(M, w_2, s_2, x_2)| \le C(|M| + 1)(|w_1 - w_2|^{\alpha_1} + |s_1 - s_2|^{\alpha_2} + |x_1 - x_2|^{\alpha_3}),$$

for some $\alpha_i \in (0, 1]$. Then there exists a constant $\overline{C} > 0$, depending on $\|u\|_{W^{2,n}(B_1)}$, $\|f\|_{C^\alpha(B_1)}$, the dimension, and the ellipticity constants such that

$$|D^2u| \le \overline{C}, \quad \text{a.e. in } B_{1/2}.$$

Proof. Define

$$\tilde{F}(M, x) := F(M, Du(x), u(x), x),$$

and simply note that the assumptions on F together with the fact that $u \in C^{1,\alpha}(B_1)$ imply that $\tilde{F}$ satisfies the assumptions of Theorem 1. □

Standing assumptions. Unless otherwise stated, we let $x_0 \in B_{1/2} \cap \overline{\Omega}$ and assume without loss of generality that $u(x_0) = |\nabla u(x_0)| = 0$ (otherwise we can replace $u(x)$ with $\tilde{u}(x) := u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)$).

Moreover, set

$$A_r(x_0) := \frac{(B_r(x_0)\backslash\Omega) - x_0}{r} = B_1 \backslash ((\Omega - x_0)/r).$$

Whenever we refer to a solution u of (2), it is implicit that $u \in W^{2,n}(B_1)$ and F satisfies (H1)–(H4).

The theorem will be established through several key lemmas. The first step consists of finding an approximation for the Hessian of u at x_0 through the following projection lemma.

Lemma 3. *Let $f \in L^\infty(B_1)$ and u be a solution to (2). Then there exists a constant $C = C(\|u\|_{W^{2,n}(B_1)}, \|f\|_{L^\infty(B_1)}, n, \lambda_0) > 0$ such that*

$$\min_{F(P,x_0)=f(x_0)} \fint_{B_r(x_0)} |D^2u(y) - P|^2\, dy \le C, \quad \forall r \in (0, 1/4).$$

Proof. Let $Q_r(x_0) := (D^2u)_{r,x_0} = \fint_{B_r(x_0)} D^2u(y)\, dy$ and note that for $t \in \mathbb{R}$, the ellipticity and boundedness of F implies

$$\begin{aligned} &\mathcal{P}^-(t\,\mathrm{Id}) \le F(Q_r(x_0) + t\,\mathrm{Id}, x_0) - F(Q_r(x_0), x_0) \le \mathcal{P}^+(t\,\mathrm{Id}) \\ &\quad \Rightarrow \lambda_0 tn - C \le F(Q_r(x_0) + t\,\mathrm{Id}, x_0) \le \lambda_1 tn + C. \end{aligned}$$

Thus, there exists $\xi_r(x_0) \in \mathbb{R}$ such that $F(Q_r(x_0) + \xi_r(x_0)\,\mathrm{Id}, x_0) = f(x_0)$ (by continuity). With this in mind,

$$\begin{aligned} \min_{F(P,x_0)=f(x_0)} \fint_{B_r(x_0)} |D^2u(y) - P|^2\, dy &\le \fint_{B_r(x_0)} |D^2u(y) - Q_r(x_0) - \xi_r(x_0)\,\mathrm{Id}\,|^2\, dy \\ &\le 2 \fint_{B_r(x_0)} |D^2u(y) - Q_r(x_0)|^2\, dy + 2\xi_r(x_0)^2 \\ &\le 2C_{BMO} + 2\xi_r(x_0)^2, \end{aligned}$$

where we have used the BMO estimate in [9]. It remains to find a uniform bound on $\xi_r(x_0)$: applying (3), (4), Hölder's inequality, and the BMO estimate again, we obtain

$$\begin{aligned}
|F(Q_r(x_0), x_0)| &= \left| \fint_{B_r(x_0)} F(Q_r(x_0) - D^2u(y) + D^2u(y), x_0)\, dy \right| \\
&\le \fint_{B_r(x_0)} |F(D^2u(y), x_0)| + n\lambda_1 |D^2u(y) - Q_r(x_0)|\, dy \\
&\le \fint_{B_r(x_0)} \big(|F(D^2u(y), x_0) - F(D^2u(y), y)| + |F(D^2u(y), y)| \\
&\quad + n\lambda_1 |D^2u(y), x_0) - Q_r(x_0)| \big) dy \\
&\le \fint_{B_r(x_0)} |F(D^2u(y), x_0) - F(D^2u(y), y)|\, dy \\
&\quad + \max\{\|f\|_\infty, n\lambda_1 K\} \\
&\quad + n\lambda_1 \sqrt{\fint_{B_r(x_0)} |D^2u(y) - Q_r(x_0)|^2\, dy} \\
&\le 2n\lambda_1 \|D^2u\|_{W^{2,n}(B_1)} + \max\{\|f\|_\infty, n\lambda_1 K\} + C_{BMO} =: C.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathcal{P}^-(\xi_r(x_0)\,\mathrm{Id}) \le F(Q_r(x_0) + \xi_r(x_0)\,\mathrm{Id}, x_0) - F(Q_r(x_0), x_0) \le \mathcal{P}^+(\xi_r(x_0)\,\mathrm{Id}) \\
&\quad \Rightarrow \lambda_0 \xi_r(x_0) n - C \le F(Q_r(x_0) + \xi_r(x_0)\,\mathrm{Id}, x_0) \le \lambda_1 \xi_r(x_0) n + C \\
&\quad \Rightarrow \lambda_0 \xi_r(x_0) n - C \le f(x_0) \le \lambda_1 \xi_r(x_0) n + C.
\end{aligned}$$

In particular, $|\xi_r(x_0)| \le \frac{\|f\|_\infty + C}{\lambda_0 n}$ and this concludes the proof. □

In what follows, let $P_r(x_0)$ denote any minimizer of

$$\min_{F(P, x_0) = f(x_0)} \fint_{B_r(x_0)} |D^2u(y) - P|^2\, dy,$$

for $r \in (0, 1/4)$. Lemma 3 and the triangle inequality readily imply that the growth of $P_r(x_0)$ is controlled in r:

Corollary 4. *Let $f \in L^\infty(B_1)$ and u be a solution to (2). Then there exists a constant $C_0 = C_0(\|D^2u\|_{W^{2,n}(B_1)}, \|f\|_{L^\infty(B_1)}, n, \lambda_0)$ such that*

$$|P_{2r}(x_0) - P_r(x_0)| \le C_0 \quad \forall r \in (0, 1/8).$$

Remark 3. (H4) is not needed in the proofs of Lemma 3 and Corollary 4.

Next we verify that $P_r(x_0)$ is an approximation to $D^2u(x_0)$ in the following sense.

Lemma 5. *Let $f \in L^\infty(B_1)$ and u be a solution to (2). Then there exists a constant $C_1 = C_1(K, \|f\|_{L^\infty(B_1)}, n, \lambda_0, \lambda_1, \|u\|_{W^{2,n}(B_1)})$ such that*

$$\sup_{x \in B_r(x_0)} \left| u(x) - \frac{1}{2} \langle P_r(x_0)(x - x_0), (x - x_0) \rangle \right| \le C_1 r^2 \quad \forall r \in (0, 1/8).$$

Proof. Assume without loss of generality that F is concave (otherwise, consider $\tilde{F}(M, x) := -F(-M, x)$ and $v = -u$) and define

$$u_{r,x_0}(y) := \frac{u(ry + x_0)}{r^2} - \frac{1}{2}\langle P_r(x_0)y, y\rangle,$$
$$G(Q) := G(Q, x_0) := F(P_r(x_0) + Q, x_0) - f(x_0).$$

Then, $G(0) = 0$ and

$$\begin{aligned} G(D^2u_{r,x_0}(y)) &= F(D^2u(ry + x_0), x_0) - f(x_0) \\ &= F(D^2u(ry + x_0), ry + x_0) - f(x_0) + h(y) \end{aligned}$$

where $h(y) := F(D^2u(ry + x_0), x_0) - F(D^2u(ry + x_0), ry + x_0)$. Thus, u_{r,x_0} solves

$$\begin{cases} G(D^2u_{r,x_0}(y)) = f(ry + x_0) - f(x_0) + h(y), & \text{in } B_1 \setminus A_r(x_0), \\ G(D^2u_{r,x_0}(y)) = F(D^2u(ry + x_0), ry + x_0) - f(x_0) + h(y), & \text{in } A_r(x_0). \end{cases}$$

Next note that if $ry + x_0 \notin \Omega$, then $F(D^2u(ry + x_0), ry + x_0)$ is essentially bounded, so by letting

$$\begin{cases} \phi(y) := f(ry + x_0) - f(x_0) & \text{in } B_1 \setminus A_r(x_0), \\ \phi(y) := F(D^2u(ry + x_0), ry + x_0) - f(x_0) & \text{in } A_r(x_0), \end{cases}$$

it follows that ϕ has an L^∞ bound depending only on the given data and

$$G(D^2u_{r,x_0}(y)) = \phi(y) + h(y) \quad \text{a.e. in } B_1. \tag{5}$$

Moreover, $\bar{u}_{r,x_0}(y) := u_{r,x_0}(y) - (u_{r,x_0})_{1,0} - y \cdot (\nabla u_{r,x_0})_{1,0}$ solves the same equation (recall $(g)_{r,x_0} = \fint_{B_r(x_0)} g(y)\,dy$). Since

$$u_{r,x_0}(0) = |\nabla u_{r,x_0}(0)| = 0$$

($u(x_0) = |\nabla u(x_0)| = 0$ by assumption) it follows that

$$(u_{r,x_0})_{1,0} = -\bar{u}_{r,x_0}(0),$$
$$(\nabla u_{r,x_0})_{1,0} = -\nabla \bar{u}_{r,x_0}(0),$$

and we may write $u_{r,x_0}(y) = \bar{u}_{r,x_0}(y) - \bar{u}_{r,x_0}(0) - y \cdot \nabla \bar{u}_{r,x_0}(0)$. Next we wish to apply Theorem 2 in [12]. First note that our assumptions on F imply the required interior a priori estimates for G; moreover, G has no spatial dependence so it remains to verify the L^n condition of $\phi + h$. Since ϕ has an L^∞ bound depending only on the given data, we need to verify it solely for h. Indeed, let $s \leq 1$ and note that thanks to (H4),

$$\begin{aligned} \int_{B_s} |h(y)|^n dy &\leq (rs)^{\alpha n} \int_{B_s} (|D^2u(ry + x_0)| + 1)^n dy \\ &\leq C(\|u\|_{W^{2,n}(B_1)} + 1)s^{\alpha n}. \end{aligned} \tag{6}$$

Therefore, applying the theorem yields

$$\begin{aligned} \|u_{r,x_0}\|_{L^\infty(B_{1/2})} &= \|\bar{u}_{r,x_0} - \bar{u}_{r,x_0}(0) - y \cdot \nabla \bar{u}_{r,x_0}(0)\|_{L^\infty(B_{1/2})} \\ &\leq C(\|\bar{u}_{r,x_0}\|_{L^\infty(B_{1/2})} + 1), \end{aligned} \tag{7}$$

where C does not depend on r. Moreover, due to the concavity of G (which is inherited from F), there is a linear functional L so that $L(Q) \geq G(Q, x_0)$ and $L(0) = 0$ (this linear functional depends on x_0). In particular,

$$L(D^2\bar{u}_{r,x_0}(y)) \geq G(D^2\bar{u}_{r,x_0}(y), x_0) = \phi(y) + h(y),$$

a.e. in B_1 (recall (5)); this fact together with (6), Theorem 9.20 in [14] applied to the subsolution $\bar{u}^+_{r,x_0}$, and Theorem 4 in [12] applied to $v(y) = \sup \bar{u}_{r,x_0} - \bar{u}_{r,x_0}(y)$ implies

$$\|\bar{u}_{r,x_0}\|_{L^\infty(B_{1/2})} \leq C\|\bar{u}_{r,x_0}\|_{L^2(B_1)} + \|\phi + h\|_{L^n(B_1)}$$
$$\leq C\|\bar{u}_{r,x_0}\|_{L^2(B_1)} + C(K, \|f\|_{L^\infty(B_1)}, n, \lambda_0, \lambda_1, \|u\|_{W^{2,n}(B_1)}),$$

and applying the Poincaré inequality twice yields

$$\|\bar{u}_{r,x_0}\|_{L^2(B_1)} \leq C\|D^2 u_{r,x_0}\|_{L^2(B_1)} = C \fint_{B_r(x_0)} |D^2 u(y) - P_r(x_0)|^2\, dy \leq C,$$

where Lemma 3 is used in the last inequality. This combined with (7) implies

$$\|u_{r,x_0}\|_{L^\infty(B_{1/2})} \leq C;$$

thus,

$$\sup_{B_{r/2}(x_0)} \left| \frac{u(x) - \frac{1}{2}\langle P_r(x_0)(x-x_0), (x-x_0)\rangle}{r^2} \right| \leq C.$$

The result now follows by replacing $r/2$ with r and utilizing Corollary 4. □

Lemma 6. *Let $f \in C^0(B_1)$ and u be a solution to (2). Then there exists a constant $M = M(K, \|f\|_{L^\infty(B_1)}, n, \lambda_0)$ such that, for any $r \in (0, 1/8)$,*

$$|A_{r/2}(x_0)| \leq \frac{|A_r(x_0)|}{2^n}$$

if $|P_r(x_0)| > M$.

Proof. Let $u_{r,x_0}(y) := \frac{u(ry+x_0)}{r^2} - \frac{1}{2}\langle P_r(x_0)y, y\rangle$ and

$$\tilde{G}(Q, y) := F(P_r(x_0) + Q, ry + x_0) - f(x_0).$$

Remark 1 below Theorem 8.1 in [15] implies the existence of a solution v_{r,x_0} to the equation

$$\begin{cases} \tilde{G}(D^2 v_{r,x_0}(y), y) = f(ry + x_0) - f(x_0) & \text{in } B_1, \\ v_{r,x_0} = u_{r,x_0} & \text{on } \partial B_1; \end{cases} \tag{8}$$

set

$$w_{r,x_0} := u_{r,x_0} - v_{r,x_0},$$

and note that by definition

$$\tilde{G}(D^2 u_{r,x_0}(y), y) = F(D^2 u(ry + x_0), ry + x_0) - f(x_0).$$

Therefore,

$$\begin{aligned} \tilde{G}(D^2 u_{r,x_0}(y), y) - \tilde{G}(D^2 v_{r,x_0}(y), y) &= \big(F(D^2 u(ry + x_0), ry + x_0) - f(x_0)\big) - (f(ry + x_0) - f(x_0)) \\ &= \big(F(D^2 u(ry + x_0), ry + x_0) - f(ry + x_0)\big)\chi_{A_r(x_0)} \\ &=: \tilde{\phi}(y)\chi_{A_r(x_0)}, \end{aligned}$$

where $\tilde{\phi} \in L^\infty(B_1)$. Combining this information with (H2) and the definition of $\tilde{G}$ yields

$$\mathcal{P}^-(D^2 w_{r,x_0}(y)) \leq \tilde{G}(D^2 u_{r,x_0}(y), y) - \tilde{G}(D^2 v_{r,x_0}(y), y)$$
$$= \tilde{\phi}(y)\chi_{A_r(x_0)} \leq \mathcal{P}^+(D^2 w_{r,x_0}).$$

Since $\tilde{\phi} \in L^\infty(B_1)$ with bounds depending only on the given data and $A_r(x_0)$ is relatively closed in B_1 (recall that Ω is open), we may apply the ABP estimate to obtain

$$\|w_{r,x_0}\|_{L^\infty(B_1)} \leq C(K, f, n, \lambda_0, \lambda_1)|A_r(x_0)|^{1/n}. \tag{9}$$

Since (H4) holds, we may combine Remark 3 following Theorem 8.1 in [15] with a standard covering argument to deduce

$$\|D^2 v_{r,x_0}\|_{C^{0,\alpha}(\overline{B}_{4/5})} \le C(\|v_{r,x_0}\|_{L^\infty(B_{4/5})} + C);$$

now by applying Lemma 5 and the maximum principle for (8) we obtain

$$\begin{aligned}\|v_{r,x_0}\|_{L^\infty(B_{4/5})} &\le \|v_{r,x_0}\|_{L^\infty(\partial B_1)} + 2C_0\|f\|_{L^\infty(B_1)} \\ &= \|u_{r,x_0}\|_{L^\infty(\partial B_1)} + 2C_0\|f\|_{L^\infty(B_1)} \le C. \end{aligned} \tag{10}$$

In particular, since $f \in C^0(B_1)$, $H(M, y) := \tilde{G}(D^2 v_{r,x_0}(y) + M, y) + f(x_0) - f(ry + x_0)$ is continuous in y on $\overline{B}_{4/5}$ and has the same ellipticity constants as F (note also that $H(0, y) = 0$ in $B_{4/5}$). Moreover, w_{r,x_0} solves the equation

$$H(D^2 w_{r,x_0}(y), y) = \phi(y)\chi_{A_r(x_0)} \quad y \in B_{4/5},$$

where ϕ has uniform bounds. The operator H also has interior $C^{1,1}$ estimates since it is concave. Thus, by applying Theorem 1 in [12] (cf. Theorem 7.1 in [15]) and a standard covering argument (again utilizing (H4)), we obtain $w_{r,x_0} \in W^{2,p}(B_{1/2})$ for any $p > n$; selecting $p = 2n$, it follows that

$$\begin{aligned}\int_{B_{1/2}} |D^2 w_{r,x_0}(y)|^{2n}\, dy &\le C(\|w_{r,x_0}\|_{L^\infty(B_{3/4})} + \|\phi\chi_{A_r(x_0)}\|_{L^{2n}(B_{3/4})})^{2n} \\ &\le C|A_r(x_0)| \end{aligned} \tag{11}$$

(note that the last inequality follows from (9) and the fact that $|A_r(x_0)| \le |B_1|$). Since $|D^2 u| \le K$ a.e. in $A_r(x_0)$ and

$$P_r(x_0) = D^2 u(ry + x_0) - D^2 v_{r,x_0}(y) - D^2 w_{r,x_0}(y),$$

by utilizing (10) and (11) we obtain

$$\begin{aligned}|A_r(x_0) \cap B_{1/2}||P_r(x_0)|^{2n} &= \int_{A_r(x_0)\cap B_{1/2}} |P_r(x_0)|^{2n} dy \\ &= \int_{A_r(x_0)\cap B_{1/2}} |D^2 u(ry + x_0) - D^2 v_{r,x_0}(y) - D^2 w_{r,x_0}(y)|^{2n} dy \\ &\le C \int_{A_r(x_0)\cap B_{1/2}} |D^2 v_{r,x_0}|^{2n} + |D^2 w_{r,x_0}|^{2n} + |D^2 u(ry + x_0)|^{2n} dy \\ &\le C(|A_r(x_0) \cap B_{1/2}|\|D^2 v_{r,x_0}\|^{2n}_{L^\infty(B_{1/2})} + C|A_r(x_0)| + K^{2n}|A_r(x_0) \cap B_{1/2}|) \\ &\le C(|A_r(x_0) \cap B_{1/2}| + |A_r(x_0)|) \le C|A_r(x_0)|. \end{aligned}$$

Next note that

$$\begin{aligned}A_{r/2}(x_0) &= B_1 \backslash ((\Omega - x_0)/(r/2)) = 2(B_{1/2} \backslash ((\Omega - x_0)/r)) \\ &= 2(B_{1/2} \cap (B_1 \backslash ((\Omega - x_0)/r))) = 2(B_{1/2} \cap A_r(x_0)); \end{aligned}$$

thus, if $|P_r(x_0)| \ge (4^n C)^{\frac{1}{2n}}$,

$$\begin{aligned}|A_{r/2}(x_0)||P_r(x_0)|^{2n} &= 2^n|A_r(x_0) \cap B_{1/2}||P_r(x_0)|^{2n} \le 2^n C|A_r(x_0)| \\ &\le \frac{|P_r(x_0)|^{2n}}{2^n}|A_r(x_0)|, \end{aligned}$$

which immediately gives the conclusion of the lemma. □

In other words, Lemma 6 says that the free boundary has a cusp-like behavior at x_0 if $|P_r(x_0)|$ is large, see Fig. 1. We now have all the ingredients to prove interior $C^{1,1}$ regularity of the solution u.

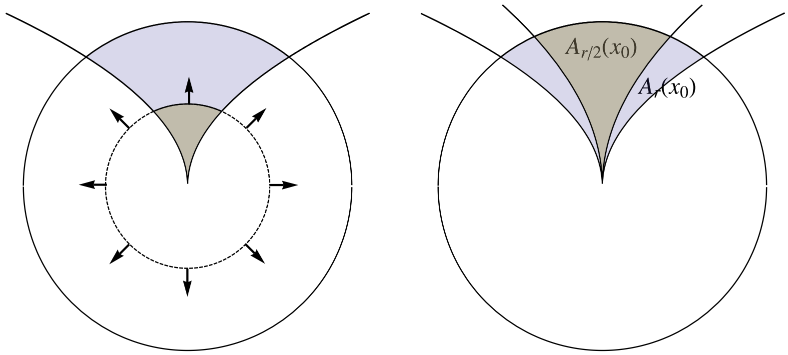

Fig. 1. $B_r(x_0)\backslash\Omega$ and $B_{r/2}(x_0)\backslash\Omega$ are translated to the origin and placed on the same scale and generate $A_r(x_0)$ and $A_{r/2}(x_0)$, respectively. Here, x_0 is the tip of a cusp.

Proof of Theorem 1. By assumption, $|D^2u|$ is bounded a.e. in $B_1\backslash\Omega$. Therefore, consider a point $x_0 \in \overline{\Omega} \cap B_{1/2}$ which is a Lebesgue point for D^2u and where u is twice differentiable (such points differ from Ω by a set of measure zero). Take $M > 0$ as in Lemma 6. If $\liminf_{k\to\infty} |P_{2^{-k}}(x_0)| \le 3M$, then Lemma 3 implies

$$\begin{aligned}|D^2u(x_0)| &= \lim_{k\to\infty} \fint_{B_{2^{-k}}(x_0)} |D^2u(y)|\,dy \\ &\le \liminf_{k\to\infty} \fint_{B_{2^{-k}}(x_0)} |D^2u(y) - P_{2^{-k}}(x_0)|\,dy + \fint_{B_{2^{-k}}(x_0)} |P_{2^{-k}}(x_0)|\,dy \\ &\le C_1 + 3M.\end{aligned}$$

In the case $\liminf_{k\to\infty} P_{2^{-k}}(x_0) > 3M$, let $k_0 \ge 3$ be such that $|P_{2^{-k_0-1}}| \le 2M$ and $|P_{2^{-k}}| \ge 2M$ for all $k \ge k_0$ (k_0 can be assumed to exist by taking M bigger if necessary). Then Corollary 4 implies $|P_{2^{-k_0}}(x_0)| \le 2M + C_0$. Now let

$$\bar{u}_0(y) := 4^{k_0}u(2^{-k_0}y + x_0) - \frac{1}{2}\langle P_{2^{-k_0}}(x_0)y, y\rangle$$

and

$$\tilde{F}(Q, y) := F(P_{2^{-k_0}}(x_0) + Q, 2^{-k_0}y + x_0) - f(2^{-k_0}y + x_0);$$

note that $\tilde{F}(0,0) = 0$ by the definition of $P_{2^{-k_0}}(x_0)$ and $\bar{u}_0(y)$ solves the equation

$$\tilde{F}(D^2u(y), y) = \tilde{f}(y) \quad y \in B_1, \tag{12}$$

where

$$\tilde{f}(y) := g(y)\chi_{A_{2^{-k_0}}(x_0)},$$

and

$$g(y) := F(D^2u(2^{-k_0}y + x_0), 2^{-k_0}y + x_0) - f(2^{-k_0}y + x_0) \in L^\infty(B_1),$$

with uniform bounds. Our goal is to apply Theorem 3 in [12] (cf. Theorem 8.1 in [15]) to (12); thus, we verify the required conditions: Lemma 6 implies

$$|A_{2^{-k_0-j}}(x_0)| \le 2^{-jn}|A_{2^{-k_0}}(x_0)|, \quad \forall j \ge 0,$$

from which it follows that

$$\fint_{B_r} |g\chi_{A_{2^{-k_0}}(x_0)}|^n \le C \fint_{B_r} |\chi_{A_{2^{-k_0}}(x_0)}|^n \le Cr^n, \quad \forall r \in (0, 1/8);$$

indeed, take j so that $2^{-j-1} < r \le 2^{-j}$ and let $A_r(x_0)$ be denoted by A_r so that

$$\begin{aligned}
\fint_{B_r} |\chi_{A_{2^{-k_0}}(x_0)}|^n &\le \frac{|A_{2^{-k_0}} \cap B_r|}{2^{n(-j-1)}} = 2^n 2^{jn} \cdot 2^{-n} |2(A_{2^{-k_0}} \cap B_r)| \\
&= 2^n 2^{jn} \cdot 2^{-n} |2(A_{2^{-k_0}} \cap B_{1/2} \cap B_r)| \\
&= 2^n 2^{jn} \cdot 2^{-n} |2(A_{2^{-k_0}} \cap B_{1/2}) \cap B_{2r}| \\
&= 2^n 2^{jn} \cdot 2^{-n} |A_{2^{-k_0-1}} \cap B_{2r}| \\
&\le \cdots \\
&\le 2^n 2^{jn} \cdot 2^{-jn} |A_{2^{-k_0-j}} \cap B_{2^j r}| \\
&\le 2^n |A_{2^{-k_0-j}}| \le 2^n \cdot 2^{-jn} |A_{2^{-k_0}}| \le C 2^{2n} (2^{-j-1})^n \\
&\le C 2^{2n} r^n.
\end{aligned}$$

We are left with verifying the condition on the oscillation of $\tilde{F}$. To this aim, note that one may work with $\tilde{\beta}_{\tilde{F}}(y)$ (see e.g. (8.3) of Theorem 8.1 in [15]). With this in mind, and for $P = P_{2^{-k_0}}(x_0)$,

$$\begin{aligned}
\tilde{\beta}_{\tilde{F}}(y) &= \sup_{Q \in \mathcal{S}} \frac{\left|\tilde{F}(Q, y) - \tilde{F}(Q, 0)\right|}{|Q| + 1} \\
&= \sup_{Q \in \mathcal{S}} \frac{\left|F(P + Q, \frac{y}{2^{k_0}} + x_0) - f(\frac{y}{2^{k_0}} + x_0) - (F(P + Q, x_0) - f(x_0))\right|}{|Q| + 1} \\
&= \sup_{Q \in \mathcal{S}} \frac{\left|F(P + Q, \frac{y}{2^{k_0}} + x_0) - F(P + Q, x_0) + (f(x_0) - f(\frac{y}{2^{k_0}} + x_0))\right|}{|Q| + 1} \\
&\le C|y|^\alpha
\end{aligned}$$

(the last inequality follows from (H4), the Hölder continuity of f, and the boundedness of P). Thus, the condition on the oscillation of $\tilde{F}$ is verified. Therefore $\bar{u}_0$ is $C^{2,\alpha}$ at the origin with the bound

$$|D^2 \bar{u}_0(0)| \le C$$

for some constant $C > 0$. This in turn implies

$$|D^2 u(x_0)| \le |D^2 \bar{u}_0(0)| + |P_{2^{-k_0}}(x_0)| \le C,$$

and we conclude. □

3. Free boundary regularity

The aim in this section is to prove free boundary regularity for (2). In general singularities may develop, see e.g. [16]. Nevertheless, the free boundary is locally C^1 under a uniform thickness assumption and if $f \ge c > 0$.

3.1. Non-degeneracy and classification of blow-ups

The first step in the free boundary analysis is non-degeneracy (i.e. at least quadratic growth) of the solution near a free boundary point. In general this fails even in the one-dimensional problem $u'' = \chi_{\{u'' \neq 0\}}$ (see e.g. [9, §3.1]). However, for $\{|\nabla u| \neq 0\} \subset \Omega$, non-degeneracy follows from a uniform positivity assumption on the right hand side: if $0 < c \le \inf_{x \in B_1} f(x)$, then by letting $v(x) := u(x) - \frac{c|x - x_0|^2}{2n\lambda_1}$, one may check that v is a subsolution for F in $\Omega \cap B_1$ and apply the argument in [9, Lemma 3.1].

Lemma 7 *(Non-degeneracy). Suppose $0 < c \leq \inf_{x\in B_1} f(x)$ and let u be a $W^{2,n}(B_1)$ solution to (2). If $\{|\nabla u| \neq 0\} \subset \Omega$ and $x_0 \in \overline{\Omega} \cap B_{1/2}$, then for any $r > 0$ such that $B_r(x_0) \Subset B_1$,*

$$\sup_{\partial B_r(x_0)} u \geq u(x_0) + \frac{c}{2n\lambda_1} r^2.$$

The previous result immediately implies a linear growth estimate on the gradient (this is usually referred to as non-degeneracy of the gradient).

Corollary 8. *Suppose $0 < c \leq \inf_{x\in B_1} f(x)$ and let u be a $W^{2,n}(B_1)$ solution to (2). If $\{|\nabla u| \neq 0\} \subset \Omega$ and $x_0 \in \overline{\Omega} \cap B_{1/2}$, then for any $r > 0$ such that $B_r(x_0) \Subset B_1$,*

$$\sup_{B_r(x_0)} |\nabla u| \geq \frac{c}{4n\lambda_1} r.$$

Proof. From the non-degeneracy,

$$\sup_{\partial B_r(x_0)} u \geq u(x_0) + \frac{c}{2n\lambda_1} r^2.$$

Therefore there is a point $x \in \partial B_r(x_0)$ such that $u(x) - u(x_0) \geq \frac{c}{4n\lambda_1} r^2$. Also,

$$u(x) - u(x_0) \leq \sup_{B_r(x_0)} |\nabla u||x - x_0| = \sup_{B_r(x_0)} |\nabla u| r,$$

i.e., $\sup_{B_r(x_0)} |\nabla u| \geq \frac{c}{4n\lambda_1} r$. □

Non-degeneracy of the gradient and the optimal regularity result of Theorem 1 imply the porosity of the free boundary inside $B_{1/4}$, i.e. there is a $0 < \delta < 1$ such that every ball $B_r(x)$ inside $B_{1/2}$ contains a smaller ball $B_{\delta r}(y)$ for which $B_{\delta r}(y) \subset B_r(x) \backslash (\partial\Omega \cap B_{1/4})$.

Lemma 9 *(Porosity of the free boundary). Suppose $0 < c \leq \inf_{x\in B_1} f(x)$ and let u be a $W^{2,n}(B_1)$ solution to (2). If $\{|\nabla u| \neq 0\} \subset \Omega$, then $\partial\Omega \cap B_{1/4}$ is porous.*

Proof. Let $x_0 \in \partial\Omega \cap B_{1/4}$ and $B_r(x_0) \Subset B_{1/2}$. From the non-degeneracy of the gradient, there is a point $x \in \overline{B}_{r/2}(x_0)$ so that

$$|\nabla u(x)| \geq Cr.$$

Let $\bar{C}$ be the constant from Theorem 1 and choose $0 < \delta \leq \min\{\frac{C}{2\bar{C}}, 1/2\}$. If $y \in B_{\delta r}(x)$, then

$$\begin{aligned}|\nabla u(y)| &\geq |\nabla u(x)| - |\nabla u(y) - \nabla u(x)| \geq Cr - \|D^2 u\|_{L^\infty(B_{1/2})}|x - y| \\ &\geq Cr - \|D^2 u\|_{L^\infty(B_{1/2})}\delta r \geq \frac{C}{2} r.\end{aligned}$$

In particular, $y \in \Omega$ and so $B_{\delta r}(x) \subset B_r(x_0) \cap \Omega \subset B_r(x_0) \backslash (\partial\Omega \cap B_{1/4})$. □

A well known consequence of the porosity is the Lebesgue negligibility of the free boundary, see e.g. [17].

Corollary 10. *Suppose $0 < c \leq \inf_{x\in B_1} f(x)$ and let u be a $W^{2,n}(B_1)$ solution to (2). If $\{|\nabla u| \neq 0\} \subset \Omega$, then $\partial\Omega$ has Lebesgue measure zero in $B_{1/4}$.*

Next, we turn our attention to blow-ups of solutions. By $\limsup E_j$ we mean the set of all limit points of sequences $\{x^{j_k}\}$, $x^{j_k} \in E_{j_k}$.

Lemma 11 *(Blow-up). Suppose* $0 < c \le \inf_{x \in B_1} f(x)$ *and let* u *be a* $W^{2,n}(B_1)$ *solution to* (2)*, and assume* f *to be Hölder continuous. If* $\{|\nabla u| \neq 0\} \subset \Omega$*, then for any* $x_0 \in \partial\Omega(u) \cap B_{1/4}$ *there is a sequence* $\{r_j\}$ *such that*

$$u_{r_j}(y) := \frac{u(x_0 + r_j y) - u(x_0)}{r_j^2} \to u_0(y)$$

as $r_j \to 0$ *in* $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$*, and* $u_0 \in C^{1,1}(\mathbb{R}^n)$ *solves*

$$\begin{cases} F(D^2u(y), x_0) = f(x_0) & \text{a.e. in } \Omega(u_0), \\ |D^2u| \le K & \text{a.e. in } \mathbb{R}^n \backslash \Omega(u_0), \end{cases}$$

where $\Omega(u_0) := \mathbb{R}^n \backslash \limsup(B_{1/r_j}((-x_0)/r_j) \backslash \Omega(u_{r_j}))$*,* $\Omega(u_{r_j}) := (\Omega - x_0)/r_j$*. Moreover,* $\{|\nabla u_0| \neq 0\} \subset \Omega(u_0)$*.*

Proof. Theorem 1 implies $u \in C^{1,1}(B_{1/2})$; since $x_0 \in B_{1/4}$, if $r > 0$ it follows that $u_r \in C^{1,1}(B_{1/4r})$. Let $E \Subset \mathbb{R}^n$ and note that since $C^{1,1}(E) \hookrightarrow C^{1,\alpha}_{\text{loc}}(E)$ compactly for any $\alpha \in [0, 1)$, there is a subsequence $\{u_{r_j}\}$ converging in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^n)$ to a function $u_0 \in C^{1,1}(\mathbb{R}^n)$ which is not identically zero by Lemma 7. Note that $|D^2u_0|$ is bounded a.e. in $\mathbb{R}^n \backslash \Omega(u_0)$ (in fact, $|D^2u_0| = 0$ a.e. there since $|D^2u_{r_j}(y)| = 0$ a.e. on $\{|\nabla u_{r_j}| = 0\}$). Next, let $y_0 \in \{|\nabla u_0| \neq 0\}$ and select $\delta > 0$ such that $B_\delta(y_0) \subset \Omega(u_{r_j})$ for j large enough (for j large $|\nabla u_{r_j}| \neq 0$ in a neighborhood of a point x where $|\nabla u_0(x)| \neq 0$, so $\{|\nabla u_0| \neq 0\} \subseteq \Omega(u_0)$ in particular); note that u_{r_j} is $C^{2,\alpha}(B_\delta(y_0))$ (by [15, Theorem 8.1]). We can therefore, without loss of generality, assume strong convergence of u_{r_j} to u_0 in $C^2(B_\delta(y_0))$. Therefore

$$\begin{aligned} F(D^2u_0(y), x_0) &= \lim_{j\to\infty} F(D^2u_{r_j}(y), x_0 + r_j y) \\ &= \lim_{j\to\infty} f(x_0 + r_j y) = f(x_0), \qquad y \in B_\delta(y_0). \end{aligned}$$

We end the proof by showing that $\{|\nabla u_0| \neq 0\} = \Omega(u_0)$ up to a set of measure zero. If there exists a point x in $\Omega(u_0)$ some distance d away from $\{|\nabla u_0| \neq 0\}$, i.e. $\text{dist}(x, \{|\nabla u_0| \neq 0\}) \ge d > 0$, then there are points $B_{d/4}(x) \ni x^j \to x$ such that $|\nabla u_{r_j}(x^j)| > 0$. From nondegeneracy of the gradient,

$$\max_{\overline{B}_{3d/4}(x)} |\nabla u_{r_j}| \ge \max_{\overline{B}_{d/2}(x^j)} |\nabla u_{r_j}| \ge cd,$$

for some uniform constant c. On the other hand we know that, for j large,

$$|\nabla u_0 - \nabla u_{r_j}| \le cd/2$$

due to $C^{1,\alpha}$ convergence. Therefore

$$\begin{aligned} \max_{\overline{B}_{3d/4}(x)} |\nabla u_0| &\ge \max_{\overline{B}_{3d/4}(x)} \left(|\nabla u_{r_j}| - |\nabla u_0 - \nabla u_{r_j}|\right) \\ &\ge \max_{\overline{B}_{3d/4}(x)} |\nabla u_{r_j}| - cd/2 \ge cd/2, \end{aligned}$$

a contradiction to the assumption that x is a distance d away from $\{|\nabla u_0| \neq 0\}$. Therefore $\text{dist}(\Omega(u_0), \{|\nabla u_0| \neq 0\}) = 0$ so $\Omega(u_0) \subseteq \overline{\{|\nabla u_0| \neq 0\}}$. However,

$$\begin{aligned} |\{|\nabla u_0| \neq 0\}| &\le |\Omega(u_0)| \le |\overline{\{|\nabla u_0| \neq 0\}}| \\ &= |\{|\nabla u_0| \neq 0\}| + |\partial\{|\nabla u_0| \neq 0\}| = |\{|\nabla u_0| \neq 0\}| \end{aligned}$$

by Corollary 10, from which we infer that $\Omega(u_0)$ is $\{|\nabla u_0| \neq 0\}$ up to a measure zero set. □

Since blow-up solutions are solutions to a free boundary problem on $\mathbb{R}^n$, one may consider the classification of these global solutions. To this aim, one introduces

$$\delta_r(u, x) := \frac{\text{MD}(\lambda \cap B_r(x))}{r},$$

where $\lambda := B_1 \setminus \Omega$ (recall that MD(E) is the smallest possible distance between two hyperplanes containing E). Note that δ is well-behaved under scaling and thus with respect to the blow-up procedure: $\delta_1(u_r, 0) = \delta_r(u, x)$, where $u_r(y) = (u(x + ry) - u(x))/r^2$. Now after blow-up, even for general operators, the operator will solely be a function of the matrix variable and if f is a positive function bounded away from zero, by letting $G(M) := F(M, x_0)/f(x_0)$, the problem of classifying global solutions reduces to the content of [9, Proposition 3.2].

Proposition 12. *Suppose $0 < c \leq \inf_{x\in B_1} f(x)$, fix $x_0 \in B_1$, and let u_0 be a $W^{2,n}(\mathbb{R}^n)$ solution to*

$$\begin{cases} F(D^2u(y), x_0) = f(x_0) & \text{a.e. in } \Omega(u_0), \\ |D^2u| \leq K & \text{a.e. in } \mathbb{R}^n \setminus \Omega(u_0), \end{cases}$$

with $\{|\nabla u_0| \neq 0\} \subset \Omega(u_0)$. If F is convex and there exists $\epsilon_0 > 0$ such that

$$\delta_r(u, x) \geq \epsilon_0, \quad \forall r > 0,\ \forall x \in \partial\Omega(u_0),$$

then u_0 is a half-space solution, $u_0(x) = \gamma_{x_0}[((x - v_{x_0}) \cdot e_{x_0})^+]^2/2 + c$, where $e_{x_0} \in \mathbb{S}^{n-1}$ and $\gamma_{x_0} \in [1/\lambda_1, 1/\lambda_0]$ are such that $F(\gamma_{x_0} e_{x_0} \otimes e_{x_0}, x_0) = f(x_0)$.

3.2. Directional monotonicity and C^1 regularity of the free boundary

In what follows, two technical monotonicity lemmas will be established and utilized in proving that the free boundary is C^1.

Lemma 13. *Let u be a $W^{2,n}(B_1)$ solution of*

$$\begin{cases} F(D^2u(x), rx) = f(rx) & \text{a.e. in } B_1 \cap \Omega, \\ |D^2u| \leq K & \text{a.e. in } B_1 \setminus \Omega, \end{cases} \tag{13}$$

and assume f is $C^{0,1}$, $\inf_{B_1} f > 0$, and F is convex in the matrix variable and satisfies (H4) with $\alpha = 1$. If $\{u \neq 0\} \subset \Omega$ and $C_0\partial_e u - u \geq -\epsilon_0$ in B_1, then

$$C_0\partial_e u - u \geq 0$$

in $B_{1/2}$ provided

$$\epsilon_0 \leq \inf_{B_1} f/(64n\lambda_1),$$

and

$$0 < r \leq \min\{\|f\|_{L^\infty(B_1)}/(2C_0\|\nabla f\|_{L^\infty(B_1)} + 2C_0\overline{C}), 1\}.$$

Proof. Let $x \in \Omega$ and $\partial F(M, x)$ denote the subdifferential of F at the point (M, x) and note that convexity implies $\partial F(M, x) \neq \emptyset$. Consider a measurable function P^M mapping (M, x) to $P^M(x) \in \partial F(M, x)$. Since $u \in C^{2,\alpha}_{\text{loc}}(\Omega)$ (see e.g. [15, Theorem 8.1]), we can define the measurable coefficients $a_{ij}(x) := (P^{D^2u(x)}(rx))_{ij} \in \partial F(D^2u(x), rx)$. By convexity of $F(\cdot, x)$ and the fact that $F(0, x) \equiv 0$, we have

$$a_{ij}(x)\partial_{ij}u(x) = F(0, rx) + a_{ij}(x)\partial_{ij}u \geq F(D^2u(x), rx).$$

Hence,

$$a_{ij}(x)\frac{\partial_{ij}u(x + he) - \partial_{ij}u(x)}{h} \leq \frac{F(D^2u(x + he), rx) - F(D^2u(x), rx)}{h}, \tag{14}$$

provided $x + he \in \Omega$. Note that by (14) and [18, Theorem 3.8] (uniform limits of viscosity solutions are viscosity solutions), we have

$$
\begin{aligned}
a_{ij}(x)\partial_{ij}\partial_e u(x) &\le \limsup_{h\to 0} \frac{F(D^2u(x+he),rx)-F(D^2u(x),rx)}{h}\\
&= \limsup_{h\to 0} \frac{F(D^2u(x+he),rx)-f(rx)}{h}\\
&= \limsup_{h\to 0} \frac{F(D^2u(x+he),rx)-F(D^2u(x+he),rx+rhe)}{h}\\
&\quad + \frac{f(rx+rhe)-f(rx)}{h}\\
&= r\limsup_{h\to 0} \frac{F(D^2u(x+he),rx)-F(D^2u(x+he),rx+rhe)}{h}\\
&\quad + \frac{f(rx+rhe)-f(rx)}{rh}\\
&= r(\partial_e f)(rx) - r(\partial_{x,e}F)(D^2u(x),rx),
\end{aligned}
$$

where $\partial_{x,e}$ denotes the spatial directional derivative in the direction e. If there is $y_0 \in B_{1/2}\cap\Omega$ such that $C_0\partial_e u(y_0) - u(y_0) < 0$, then consider the auxiliary function

$$
w(x) = C_0\partial_e u(x) - u(x) + c\frac{|x-y_0|^2}{2n\lambda_1},
$$

where $c = \inf_{B_1} f/2 > 0$. Note that for $r \le \min\{c/(C_0\|\nabla f\|_{L^\infty(B_1)} + C_0\overline{C}), 1\}$,

$$
\begin{aligned}
a_{ij}(x)\partial_{ij}w(x) &\le rC_0(\partial_e f)(rx) - rC_0(\partial_{x,e}F)(D^2u(x),rx) - f(rx) + c\\
&\le rC_0\|\nabla f\|_{L^\infty(B_1)} + rC_0\overline{C} - f(rx) + c \le 2c - f(rx) \le 0.
\end{aligned}
$$

Hence w is a supersolution in $B_{1/4}(y_0)\cap\Omega$ and therefore attains its minimum on the boundary of $B_{1/4}(y_0)\cap\Omega$. However on $\partial\Omega$, w is positive (since both u and $\partial_e u$ are zero); thus, the minimum is attained on $\partial B_{1/4}(y_0)$, and this implies

$$
0 > \min_{\overline{B}_{1/4}(y_0)\cap\overline{\Omega}} w \ge -\epsilon_0 + \frac{c}{32n\lambda_1},
$$

a contradiction if $\epsilon_0 \le c/(32n\lambda_1)$. □

Lemma 14. *Let u be a $W^{2,n}(B_1)$ solution of (13) where f is $C^{0,1}$, F is C^1, convex and satisfies (H1), (H2) and (H4) with $\alpha = 1$. Assume further that $\{\nabla u \ne 0\} \subset \Omega$ and $\inf_{B_1} f > 0$. If $C_0\partial_e u - |\nabla u|^2 \ge -\epsilon_0$ in B_1 for some $C_0, \epsilon_0 > 0$, then*

$$
C_0\partial_e u - |\nabla u|^2 \ge 0
$$

in $B_{1/2}$ provided that $\epsilon_0 \le \mu_1$ and $0 < r \le \mu_2$, where $\mu_1 > 0$ and $\mu_2 > 0$ are constants depending on given bounds.

Proof. By differentiating (13), it follows that

$$
F_{ij}(D^2u(y),ry)\partial_{ij}\nabla u = r\nabla f(ry) - r\nabla_x F(D^2u(y),ry) \quad \text{weakly in } \Omega.
$$

Since F is C^1, $u \in C^{2,\alpha}_{\text{loc}}(\Omega)$ (by [15, Theorem 8.1]), and the right hand side of the equation above is in $L^\infty(\Omega)$ (hence, $L^p(\Omega)$ for any $p > 0$), it follows by elliptic regularity theory that $\nabla u \in W^{2,p}_{\text{loc}}(\Omega)$ for any $p < \infty$ (see e.g. [14, Corollary 9.18]). By applying the operator $F_{ij}(D^2u(y),ry)\partial_{ij}$ to $|\nabla u|^2$, we obtain

$$
\begin{aligned}
&F_{ij}(D^2u(y),ry)\partial_{ij}|\nabla u(y)|^2\\
&\quad = 2F_{ij}(D^2u(y),ry)\partial_{ijk}u(y)\partial_k u(y) + 2F_{ij}(D^2u(y),ry)\partial_{ik}u(y)\partial_{jk}u(y)\\
&\quad = 2r(\nabla f(ry) - \nabla_x F(D^2u(y),ry))\cdot\nabla u(y) + 2F_{ij}(D^2u(y),ry)\partial_{ik}u(y)\partial_{jk}u(y). \qquad (15)
\end{aligned}
$$

For differentiable operators, the ellipticity condition can be written as

$$F_{ij}(D^2u(y), ry)\xi_i\xi_j \geq \lambda_0|\xi|^2;$$

thus, (15) yields

$$F_{ij}(D^2u(y), ry)\partial_{ij}|\nabla u(y)|^2 \geq 2r(\nabla f(ry) - \nabla_x F(D^2u(y), ry)) \cdot \nabla u(y) + 2\lambda_0|D^2u(y)|^2. \tag{16}$$

Now (H1)–(H2) and the positivity of f imply

$$0 < c \leq f(ry) = |F(D^2u(y), ry) - F(0, ry)| \leq 2n\lambda_1|D^2u|, \tag{17}$$

where $c := \inf_{B_1} f$. By combining (16) and (17), it follows that

$$F_{ij}(D^2u(y), ry)\partial_{ij}|\nabla u(y)|^2 \geq 2r(\nabla f(ry) - \nabla_x F(D^2u(y), ry)) \cdot \nabla u(y) + \frac{c^2\lambda_0}{2n^2\lambda_1^2}.$$

The proof now follows as in Lemma 13: assume by contradiction that there is a point $y_0 \in B_{1/2} \cap \Omega$ such that $C_0\partial_e u(y_0) - |\nabla u(y_0)|^2 < 0$ (outside Ω we have $|\nabla u| = 0$). Let $d = \frac{c^2\lambda_0}{4n^2\lambda_1^2}$ and

$$w(y) = C_0\partial_e u(y) - |\nabla u(y)|^2 + d\frac{|y - y_0|^2}{2n\lambda_1}.$$

Next note that for r small enough, w is a supersolution of $F_{ij}(D^2u(y), ry)\partial_{ij}$ in $B_{1/4}(y_0) \cap \Omega$. Indeed,

$$\begin{aligned} F_{ij}(D^2u(y), ry)\partial_{ij}w &\leq rC_0\|\nabla f\|_{L^\infty(B_1)} + rC_0\overline{C} \\ &\qquad - 2r(\nabla f(ry) - \nabla_x F(D^2u(y), ry)) \cdot \nabla u(y) - \frac{c^2\lambda_0}{2n^2\lambda_1^2} + d \\ &\leq r\overline{C}_1 - \frac{c^2\lambda_0}{2n^2\lambda_1^2} + d \leq 0, \end{aligned}$$

where for the last inequality we require $r \leq \frac{c^2\lambda_0}{4n^2\lambda_1^2\overline{C}_1}$. Therefore w attains a minimum on the boundary of $B_{1/4}(y_0) \cap \Omega$. However, on $\partial\Omega$, w is non-negative since both u and $\partial_e u$ are zero, so the minimum has to be attained on $\partial B_{1/4}(y_0)$, and this implies

$$0 > \min_{\overline{B}_{1/4}(y_0)\cap\overline{\Omega}} w \geq -\epsilon_0 + \frac{d}{32n\lambda_1}$$

which is a contradiction if $\epsilon_0 \leq d/(32n\lambda_1)$. □

We are now in a position to prove that under a suitable thickness assumption, the free boundary is locally C^1.

Theorem 15. *Let $u : B_1 \to \mathbb{R}$ be a $W^{2,n}(B_1)$ solution of (2). Let F be a convex operator satisfying (H1), (H2), and (H4) with $\alpha = 1$, and assume further that f is $C^{0,1}$. If $\{u \neq 0\} \subset \Omega$ and there exists $\epsilon > 0$ such that*

$$\delta_r(u, x) > \epsilon, \quad \forall r < 1/4,\ x \in \partial\Omega \cap B_r,$$

then there exists $r_0 > 0$ depending only on ϵ and given bounds such that $\partial\Omega \cap B_{r_0}(x)$ is a C^1-graph.

Proof. Let $x \in \partial\Omega \cap B_{1/8}$ and $u_r(y) := \frac{u(ry+x)}{r^2}$, and note that it suffices to consider the case when $|\nabla u(x)| = 0$. By Theorem 1 we have a uniform $C^{1,1}$-estimate with respect to r and can therefore find a subsequence $\{u_{r_j}\}$ converging in $C^1_{\text{loc}}(\mathbb{R}^n)$ to a global solution u_0, where $u_0(0) = 0$. The thickness assumption implies $\delta_r(u, x) > \epsilon$ for all $r > 0$, hence $u_0(y) = \gamma\frac{((y\cdot e_x)_+)^2}{2}$ according to Proposition 12 (since $\{|\nabla u| \neq 0\} \subseteq \{u \neq 0\}$ up to a set of measure zero), where $\gamma \in [1/\lambda_1, 1/\lambda_0]$ and $e_x \in \partial B_1$. Now let $0 < s \leq 1$. Then

$$\frac{\partial_e u_0}{s} - u_0 \geq 0$$

in B_1 for any direction $e \in \partial B_1$ such that $e \cdot e_x \geq s$. From the C^1-convergence of $\{u_{r_j}\}$ we have

$$\frac{\partial_e u_{r_j}}{s} - u_{r_j} \geq -\epsilon_0$$

in B_1 for $j \geq k(s,x)$ and ϵ_0 as in Lemma 13. Therefore u_{r_j} fulfills the assumptions of this lemma and the above inequality can be improved to

$$\frac{\partial_e u_{r_j}(y)}{s} - u_{r_j}(y) \geq 0, \quad y \in B_{1/2}. \tag{18}$$

For $s = 1$, i.e. $e = e_x$, multiplying (18) by $\exp(e \cdot y)$ implies

$$\partial_e[\exp(-e \cdot y) u_{r_j}(y)] = \exp(-e \cdot y)(\partial_e u_{r_j}(y) - u_{r_j}(y)) \geq 0.$$

Integrating this expression yields

$$\exp(-e \cdot y) u_{r_j}(y) - \underbrace{u_{r_j}(0)}_{=0} = \int_0^{e \cdot y} \partial_e[\exp(-e \cdot z) u_{r_j}(z)] d(e \cdot z) \geq 0,$$

so $u_{r_j}(y) \geq 0$ in $B_{1/2}$ and $\partial_e u_{r_j}(y) \geq 0$ follows from (18). In particular, we have shown that if $x \in \partial\Omega \cap B_{1/8}$ and $e \cdot e_x \geq s$, then $\partial_e u(z) \geq 0$ for all $z \in B_{r_j/2}(x)$, where $r_j = r_j(s,x)$. Now

$$\partial\Omega \cap \overline{B}_{1/16} \subset \bigcup_{x \in \partial\Omega \cap \overline{B}_{1/16}} B_{r_j/2}(x),$$

so by extracting a finite subcover and relabeling the radii, it follows that

$$\partial\Omega \cap \overline{B}_{1/16} \subset \bigcup_{k=1}^{N} B_{\eta_k}(x_k),$$

where $\eta_k = \eta_k(x_k, s)$; set $\eta = \eta(s) := \min_k \eta_k$. Thus, for all $x \in \partial\Omega \cap \overline{B}_{1/16}$, we have $\partial_e u(z) \geq 0$ for all $z \in B_\eta(x)$, where η only depends on s and the given data (via the C^1 convergence of u_{r_j}). Therefore, if $s_0 \in (0,1)$, by letting $r_0 := \eta(s_0)$, it follows that the free boundary $\partial\Omega \cap B_{r_0}(x)$ is s_0-Lipschitz. Moreover, note that in a small neighborhood of the origin, by picking s sufficiently small, the Lipschitz constant of the free boundary can be made arbitrarily small (the neighborhood only depends on $\eta(s)$). This shows that the free boundary is locally C^1 at x, and the same reasoning applies to any other point in $\partial\Omega \cap \overline{B}_{r_0}(x)$. □

Remark 4. In view of Lemma 14, we can replace the condition $\{u \neq 0\} \subset \Omega$ by $\{\nabla u \neq 0\} \subset \Omega$ in Theorem 15 whenever F is C^1.

4. Parabolic case

In this section we generalize the former results regarding optimal regularity of the solution as well as C^1 regularity of the free boundary to the non-stationary setting. Since the parabolic case is very similar to the elliptic one, we mostly outline the proofs. The setup of the problem is as follows.

- Let $Q_r(X) := B_r(x) \times (t - r^2, t)$, where $X = (x,t)$. For convenience, $Q_r := Q_r(0)$.
- Instead of (2) we consider the following problem,

$$\begin{cases} \mathcal{H}(u(X), X) = f(X) & \text{a.e. in } Q_1 \cap \Omega, \\ |D^2 u| \leq K & \text{a.e. in } Q_1 \backslash \Omega, \end{cases} \tag{19}$$

where $\mathcal{H}(u(X), X) := F(D^2u(X), X) - \partial_t u(X)$, $\Omega \subset Q_1$ is some unknown set, and K is a positive constant as before. We still assume F to satisfy (H1)–(H3) for all $X \in Q_1$ and

$$F(M, x, t) - F(M, y, s) \leq C(|M| + 1)(|x - y|^{\alpha_1} + |t - s|^{\alpha_2}), \tag{20}$$

with $\alpha_1, \alpha_2 \in (0, 1]$.

- We assume f to be at least Hölder continuous in both the spatial and time coordinates.
- Let $A_r(X^0) := \{(x, t) \in Q_1 : (rx, r^2t) \in Q_r \backslash \Omega - x_0\}$.
- $\tilde{D}^2u := (D^2u, \partial_t u)$ denotes the parabolic Hessian.
- Let

$$\delta_r(u, X^0) := \inf_{t\in[t_0-r^2,t_0+r^2]} \frac{\mathrm{MD}(\mathrm{Proj}_x(A \cap (B_r(x^0) \times \{t\})))}{r},$$

where $\mathrm{MD}(E)$ stands for the minimal diameter, i.e., the smallest distance between two parallel hyperplanes that trap the set E, and Proj_x is the projection on the spatial coordinates.

The main theorems corresponding to Theorems 1 and 15 are now stated for the parabolic case; the first giving the optimal regularity of solutions.

Theorem 16 *(Interior $C_x^{1,1} \cap C_t^{0,1}$ regularity). Let $u : Q_1 \to \mathbb{R}$ be a $W_x^{2,n} \cap W_t^{1,n}$ solution of (19). Then there is $C = C(n, \lambda_0, \lambda_1, \|u\|_\infty, \|f\|_\alpha) > 0$ such that*

$$|\tilde{D}^2u| \leq C, \quad \text{in } Q_{1/2}.$$

The second theorem gives C^1 regularity of the free boundary if we add some additional assumptions on δ_r, f and F, as in the elliptic setting.

Theorem 17 *(C^1 regularity of the free boundary). Let $u : Q_1 \to \mathbb{R}$ be a $W_x^{2,n} \cap W_t^{1,n}$ solution of (19), and assume $\{u \neq 0\} \subset \Omega$. Suppose that f is Lipschitz in (x, t) and $f \geq c > 0$. Let F be convex in the matrix variable and suppose F satisfies (H1), (H2), and (20) with $\alpha_1 = \alpha_2 = 1$. Then there exists an $\epsilon > 0$ such that if*

$$\delta_r(u, X^0) > \epsilon$$

uniformly in r and $X^0 \in \partial\Omega \cap Q_r$, then $\partial\Omega \cap Q_{r_0}$ is a C^1-graph in space–time, where r_0 depends only on ϵ and the data.

Theorem 16 follows from results corresponding to [13, Lemma 2.1 and Proposition 2.2] which readily generalize to the parabolic setting thanks to our results in the elliptic case and [13, Remark 6.3]. Indeed, we can show the inequality

$$\sup_{Q_r(0)} |u - P_r| \leq Cr^2, \quad r \in (0, 1)$$

for some parabolic polynomials P_r that solve the homogeneous equation

$$\mathcal{H}(P_r, 0) = 0,$$

a result that is in the same vein as Lemma 5. Moreover, the above inequality together with an argument similar to the proof of Lemma 6 imply the geometric decay of the coincidence sets,

$$|A_{r/2}| \leq \frac{|A_r|}{2^{n+1}}.$$

Theorem 16 is then proven in the same way as in the elliptic case.

Regarding the regularity of the free boundary, Lemma 7 is easily generalized since the maximum principle holds in our case as well (see [19, Corollary 3.20]), and the rest of the results are extended with the following parabolic blow-up lemma.

Lemma 18. *Let u be a $W_x^{2,n} \cap W_t^{1,n}$ solution to (19). If $\{u \neq 0\} \subset \Omega$, then for any $(x_0, t_0) \in \partial\Omega(u) \cap Q_{1/4}$ there is a sequence $\{r_j\}$ such that*

$$u_{r_j}(y,t) := \frac{u(x_0 + r_j y, t_0 + r_j^2 t)}{r_j^2} \to u_0(y,t)$$

locally uniformly as $r_j \to 0$, and u_0 solves

$$\begin{cases} F(D^2 u(y,t), x_0, t_0) - \partial_t u(y,t) = f(x_0,t_0) & \text{a.e. in } \Omega(u_0), \\ |\tilde{D}^2 u| \le K & \text{a.e. in } \mathbb{R}^{n+1} \setminus \Omega(u_0), \end{cases} \tag{21}$$

where $\{u_0 \neq 0\} \subset \Omega(u_0)$.

Proof. By $C_x^{1,1} \cap C_t^{0,1}$ regularity of u and the fact that $u = 0$ on $\partial\Omega$, it follows that, up to a subsequence, $u_{r_j} \to u_0$ locally uniformly. Define $\Omega(u_0)$ to be the limit involving the open sets $\Omega_j := \{(x,t) : (x_0 + r_j x, t_0 + r_j^2 t) \in \Omega\}$ (as in the elliptic case), and note that $\tilde{D}^2 u$ is bounded on the complement of $\Omega(u_0)$ (since $\{u \neq 0\} \subset \Omega$). Moreover, u_0 is not identically zero by non-degeneracy. Next, let $(y,t) \in \{u \neq 0\}$ and select $\delta > 0$ such that $Q_\delta(y,t) \subset \Omega_j$ for j large enough; note that u_{r_j} is $C_x^{2,\alpha} \cap C_t^{1,\alpha}$ in this set (by the parabolic Evans–Krylov theorem [20]). We can therefore, without loss of generality, assume $C_x^2 \cap C_t^1$ convergence of u_{r_j} to u_0 in $Q_\delta(y,t)$. In particular,

$$\begin{aligned} F(D^2 u_0(y,t), x_0, t_0) &= \lim_{j\to\infty} \Big(F(D^2 u_{r_j}(y,t), x_0 + r_j y, t_0 + r_j^2 t) - \partial_t u_{r_j}(y,t) \Big) \\ &= \lim_{j\to\infty} f(x_0 + r_j y, t_0 + r_j^2 t) = f(x_0,t_0), \quad y \in Q_\delta(y,t). \end{aligned}$$

Finally, non-degeneracy implies that the equation in (21) is satisfied a.e. in $\Omega(u_0)$. □

Since blow-up solutions are solutions to a free boundary problem on $\mathbb{R}^{n+1}$, one may consider the classification of these global solutions just like in the elliptic case. By letting $\mathcal{G}(M) := \mathcal{H}(M, x_0, t_0)/f(x_0,t_0)$, the problem reduces to the content of [13, Proposition 3.2].

Proposition 19. *Fix $X_0 := (x_0, t_0)$. If u_0 is a solution to*

$$\begin{cases} \mathcal{H}(D^2 u(y), X_0) = f(X_0) & \text{a.e. in } \Omega(u_0), \\ |D^2 u| \le K & \text{a.e. in } \mathbb{R}^{n+1} \setminus \Omega(u_0), \end{cases}$$

with $\{u_0 \neq 0\} \subset \Omega(u_0)$, and there exists $\epsilon_0 > 0$ such that

$$\delta_r(u,x) \ge \epsilon_0, \quad \forall r > 0, \ \forall x \in \partial\Omega(u_0),$$

then u_0 is time-independent and of the form $u_0(x) = \gamma_{X_0}[((x - v_{X_0}) \cdot e_{X_0})^+]^2/2$, where $e_{X_0} \in \mathbb{S}^n$ and $\gamma_{X_0} \in [1/\lambda_1, 1/\lambda_0]$ are such that $F(\gamma_{X_0} e_{X_0} \otimes e_{X_0}, X_0) = f(X_0)$.

This proposition can, in turn, be used to prove that the time derivative $\partial_t u$ vanishes on the free boundary. The proof follows the same line of reasoning as in [13] except that Proposition 3.2 is replaced in their proof with Proposition 19. The result is stated in the following lemma.

Lemma 20. *Let u, f, F and δ_r be as in Theorem 17 and $\{u \neq 0\} \subset \Omega$. Then*

$$\lim_{\Omega \ni X \to \partial\Omega} \partial_t u(X) = 0.$$

The parabolic counterpart of Lemma 13 follows by replacing w given in that proof with

$$C\partial_e u(X) - u(X) + \tilde{c}\frac{|x - x_0|^2 - (t - t_0)}{2n\lambda_1 + 1},$$

where $\tilde{c} := \inf_{Q_1} f/2$; this is where the additional assumptions on F and f come into play. With this in mind, the proof of Theorem 17 follows as in the elliptic case.

Conflict of interest statement

There is no conflict of interest.

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