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Gelfand type quasilinear elliptic problems with quadratic gradient terms

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Abstract

In this paper, for $0 < m_1 \leqslant m(x) \leqslant m_2$ and positive parameters λ and p, we study the existence of positive solution for the quasilinear model problem

$$\begin{cases} -\Delta u + m(x)\dfrac{|\nabla u|^2}{1+u} = \lambda(1+u)^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

We prove that the maximal set of λ for which the problem has at least one positive solution is an interval $(0, \lambda^*]$, with $\lambda^* > 0$, and there exists a minimal regular positive solution for every $\lambda \in (0, \lambda^*)$. We also prove, under suitable conditions depending on the dimension N and the parameters p, m_1, m_2, that for $\lambda = \lambda^*$ there exists a minimal regular positive solution. Moreover we characterize minimal solutions as those solutions satisfying a stability condition in the case $m_1 = m_2$.

Keywords: Gelfand problem; Quasilinear elliptic equations; Quadratic gradient; Stability condition; Extremal solutions

1. Introduction

Let Ω be an open and bounded set in $\mathbb{R}^N$ ($N \geqslant 3$) and $\lambda > 0$. We study the existence of positive solution for the following problem

$$\begin{cases} -\Delta u + H(x, u, \nabla u) = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{P_λ}$$

where f is a continuous nonnegative function in $[0, +\infty)$ with $f(0) > 0$ and H is a Carathédory function defined on $\Omega \times [0, +\infty) \times \mathbb{R}^N$, i.e. $H(\cdot, s, \xi)$ is a measurable function for every $(s, \xi) \in [0, +\infty) \times \mathbb{R}^N$ and $H(x, \cdot, \cdot)$ is a continuous function for a.e. $x \in \Omega$.

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We point out that (P_λ) provides a general framework including, as a particular case, semilinear problems which have been studied in the literature. Namely, motivated by different applications (thermal self-ignition in combustion theory [18], temperature distribution in an object heated by a uniform electric current [19,20], etc.) there is a vast amount of works, among others [4,13,15,21], concerned with the problem

$$\begin{cases} -\Delta w = \lambda f(w) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is a smooth function satisfying $f(0) > 0$, $f'(0) > 0$ and $f''(s) > 0$ for every $s > 0$. Specifically, for a general linear second order differential operator and a nonlinearity f depending also on $x \in \Omega$, it is proved in [15] that there exists a parameter $\lambda^* > 0$ such that problem (1.1) has a minimal classical solution $w_\lambda \in C^2(\overline{\Omega})$ provided that $0 \leqslant \lambda < \lambda^*$, and no solution if $\lambda > \lambda^*$. The set $\{w_\lambda\colon\ 0 \leqslant \lambda < \lambda^*\}$ is a branch and w_λ is increasing in λ. These results have been extended in [21] to the case in which f is only assumed to be strictly convex. In addition, it is proved that the pointwise limit $w^*(x) := \lim_{\lambda\to\lambda^*} w_\lambda(x)$ is also a weak solution (usually called *extremal solution*) of (1.1) with $\lambda = \lambda^*$. To prove this, it is essential the fact that the minimal solutions w_λ are stable, i.e. they satisfy

$$\int_\Omega |\nabla\phi|^2 \geqslant \lambda \int_\Omega f'(w_\lambda)\phi^2 \quad \text{for every } \phi \in H_0^1(\Omega).$$

Observe that the above condition is nothing but the nonnegative definiteness of the second variation in $w = w_\lambda$ of the associated energy functional

$$E_\Omega(w) = \int_\Omega \left(\frac{1}{2}|\nabla w|^2 - \lambda F(w)\right), \qquad F' = f.$$

In addition, it implies that the least eigenvalue of $-\Delta - \lambda f'(w_\lambda)$ is nonnegative (see Proposition 1.2.1 in [16]). Even more, for $0 \leqslant \lambda < \lambda^*$ it is always positive and, as it was point out in [15, Proposition 2.15], this fact is also important to prove that an equilibrium solution for the corresponding parabolic problem is stable. Moreover in [21, Théorème 1 and Proposition 1] it is proved that this characterizes the minimal regular solutions.

This stability condition plays also a crucial role in order to study when the extremal solution w^* is regular. Indeed, it can be used to show that, for a few values of N, w_λ is bounded in $C(\overline{\Omega})$ uniformly in $\lambda \in (0, \lambda^*)$ and then $w^* \in L^\infty(\Omega)$. This has been proved in the case $f(s) = e^s$ (Gelfand problem) for $3 \leqslant N < 10$ (see [15, Example 1.12]) and $f(s) = (1+s)^p$ with $p > 1$ and $3 \leqslant N < 4 + 2(1 - 1/p) + 4\sqrt{1 - 1/p}$ (see [15, p. 213]). The cases of nonlinearities f having an asymptote like $f(s) = 1/(1-s)^k$, with $k > 0$, are also covered in [21]. A characterization of singular H^1 extremal solutions appears in [13] in terms of the stability condition, pointing out that the stability condition is a version of the classical Hardy inequality. In fact, an improved inequality with best constant, generalizing the classical Hardy and Poincaré inequalities is used in that paper to determine, in some particular cases, the dimensions for which any H^1 extremal solution is singular. Regularity of extremal solutions of semilinear elliptic problems up to dimension 4 has been proved in [14].

For clarity, we present in this introduction the results only in the particular case

$$\begin{cases} -\Delta u + m(x)\dfrac{|\nabla u|^2}{1+u} = \lambda(1+u)^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with $0 \leqslant m_1 \leqslant m(x) \leqslant m_2 < p$ and $1 < p$. The reader is referred to Theorems 3.1, 4.4, 4.6 and 4.7 below for the corresponding results for the general problem (P_λ). The study of general quasilinear problems with quadratic growth in the gradient ∇u like (1.2) provides a suitable unified framework for all the previously cited results. In fact, it handles as a particular case, taking $m_2 = 0$, the problem (1.1) for $f(s) = (1+s)^p$ with $p > 1$. Moreover it also handles, at least formally, the cases $f(s) = e^s$ and $f(s) = 1/(1-s)^k$, with $k > 0$. Indeed, if w is a positive solution of

$$\begin{cases} -\Delta w = \lambda e^w & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

then the function u given by the change $w = \ln(1+u)$ solves (1.2) in the particular case $m(x) = 1$ and $p = 2$. Similarly, via the change $w = 1 - 1/(u+1)^\alpha$, solutions of the equation $-\Delta w = \lambda/(1-w)^k$ ($k > 0$) are related to solutions of (1.2) with $m(x) = \alpha + 1$ and $p = \alpha k + \alpha + 1$.

Notice that the quasilinear differential operator in (1.2) falls into the framework of the pioneering works by Boccardo, Murat and Puel [9,10]. They have extensively studied this operator confronted with a right-hand side not depending on u. The case in which the right-hand side is nonlinear has been less studied. A power-like right-hand side has been studied by Orsina and Puel in [22] and recently in [2,11]. In [5] and [6] are proved some comparison principles for general differential operators including the left-hand side of the equation in (P_λ) if $H(x,s,\xi)$ is nondecreasing in s. However, in order to deal with the model problem (1.2), where $H(x,s,\xi)$ is decreasing on s, we prove a comparison principle for positive solutions including this case (see Section 2).

We show that the solution set of problem (1.2) behaves as in the semilinear case described above. That is, we give sufficient conditions on m_1, m_2 and p in order to show the existence of $\lambda^* > 0$ such that (1.2) admits a minimal solution u_λ if $0 < \lambda < \lambda^*$ and no solution if $\lambda > \lambda^*$. In order to prove the existence of the minimal solutions we use the quasilinear comparison stated in Section 2.

As regard to the extremal solution, in order to prove that the minimal solutions are bounded in $H_0^1(\Omega)$ uniformly in $\lambda \in [0,\lambda^*)$, we need to generalize the stability condition. Specifically, if $p > m_2 \geqslant m_1 > 1$, we prove that

$$\int_\Omega |\nabla\phi|^2 \geqslant \lambda \frac{(p-m_2)(m_1-1)}{m_2-1} \int_\Omega (1+u_\lambda)^{p-1}\phi^2,$$

for every $\phi \in H_0^1(\Omega)$.

As in the semilinear case, this extension of the stability condition is the keystone to prove, for $0 < \lambda < \lambda^*$ and $(p-m_2)(m_1+1) > \frac{m_2-1}{m_1-1}$, a uniform estimate in the Sobolev space of the minimal solutions and as a corollary the existence of *extremal* solution u^* of (1.2) with $\lambda = \lambda^*$. In the particular case $m_1 = m_2$ we show that the above inequality characterizes minimal solutions (see Theorem 4.6). We point out that, up to the authors' knowledge, stability condition is unknown in the literature of quasilinear elliptic equations with quadratic growth in the gradient. We remark that problem (1.2) does not have variational characterization.

Moreover, we establish whether u^* is regular (i.e. $u^* \in L^\infty(\Omega)$) in terms of N, p, m_1 and m_2. We gather the results for (1.2) in the following theorem.

Theorem 1.1. *Assume that* $1 < m_1 \leqslant m(x) \leqslant m_2 < p$. *Then there exists* $\lambda^* > 0$ *such that* (1.2) *has a minimal regular solution* u_λ *for every* $\lambda < \lambda^*$ *and no solution for every* $\lambda > \lambda^*$. *Moreover, if* $(p-m_2)(m_1+1) > \frac{m_2-1}{m_1-1}$ *then* $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ *is an extremal solution. Even more,* u^* *is regular if*

$$3 \leqslant N < 4\frac{(p-m_2)(m_1-1)}{(p-1)(m_2-1)} + 2 + \frac{4(m_1-1)}{m_2-1}\sqrt{\frac{p-m_2}{p-1}}. \tag{1.3}$$

We point out that in the case $m_1 = m_2$ the condition $m_1 > 1$ can be overcome (see Remark 4.9 below).

Notice that, the particular case $p = 2$ and $m(x) = 1$, which, as it has been observed, handles Example 1.12 of [15] (Gelfand problem), the above condition (1.3) it is nothing but $3 \leqslant N < 10$. If $m_2 = 0$, the dimension condition is $3 \leqslant N < 4 + 2(1-1/p) + 4\sqrt{1-1/p}$ (see [15, p. 213]). Other particular case (related to $-\Delta w = \lambda/(1-w)^k$ $(k > 0)$) is $p = \alpha k + \alpha + 1$ and $m(x) = \alpha + 1$ where condition (1.3) reduces to $3 \leqslant N < 4\frac{k}{k+1} + 2 + 4\sqrt{\frac{k}{k+1}}$. Therefore, Theorem 1.1 is an extension of the cited semilinear results of [15].

The plan of the paper is the following: in Section 2 we deduce a comparison principle for the general problem (P_λ). In Section 3 we prove the existence of a minimal solution of (P_λ) for λ in a bounded interval and the extension of the stability condition for such minimal solutions. Section 4 is devoted to study the properties of the minimal solutions and the existence as well as the regularity of the extremal solutions.

Notation. As usual for every $s \in \mathbb{R}$ we consider the positive and negative parts given by $s^+ = \max\{s,0\}$ and $s^- = \min\{s,0\}$. We denote by T_k the usual truncature function given by $T_k(s) = \min\{k,s^+\} + \max\{-k,s^-\}$, for every $s \in \mathbb{R}$. We denote by $|\Omega|$ the Lebesgue measure of a measurable set Ω in $\mathbb{R}^{\mathbb{N}}$. For $1 \leqslant p \leqslant +\infty$, $\|u\|_p$ is the usual norm of a function $u \in L^p(\Omega)$. We equipped the standard Sobolev space $H_0^1(\Omega)$ with the norm $\|u\| = (\int_\Omega |\nabla u|^2)^{1/2}$. We denote by $\mathcal{S} = \sup\{\|u\|_{2^*}\colon \|u\| = 1\}$ the Sobolev embedding constant $(2^* = 2N/(N-2))$. By λ_1 (respectively, ϕ_1) we also denote the first positive eigenvalue (respectively, eigenfunction) of the Laplacian operator $-\Delta$ with zero Dirichlet boundary conditions.

2. A comparison principle

In this section we state a comparison result for the following quasilinear elliptic boundary value problem

$$\begin{cases} -\Delta u + H(x,u,\nabla u) = d(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $0 \leqslant d \in L^1(\Omega)$ and H is a nonnegative Carathédory function defined on $\Omega \times [0,+\infty) \times \mathbb{R}$ such that

$$H(x,s,t\xi) = t^2 H(x,s,\xi), \quad \forall s \geqslant 0,\ \forall t \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,\ \text{a.e. } x \in \Omega. \tag{2.2}$$

Moreover, we assume that there exists a continuously differentiable positive function $g : [0,+\infty) \to \mathbb{R}$ satisfying

$$0 \leqslant g(s)|\xi|^2 \leqslant H(x,s,\xi), \quad \forall s \geqslant 0,\ \forall t \in \mathbb{R},\ \forall \xi \in \mathbb{R}^N,\ \text{a.e. } x \in \Omega. \tag{2.3}$$

In the sequel, we denote by G the primitive of g given by $G(s) = \int_0^s g(t)\,dt$ for every $s \geqslant 0$.

We say that $0 \leqslant \bar{u} \in H_0^1(\Omega)$ is a supersolution for (2.1) if $H(x,\bar{u},\nabla\bar{u}) \in L^1(\Omega)$ and

$$\int_\Omega \nabla\bar{u}\nabla\phi + \int_\Omega H(x,\bar{u},\nabla\bar{u})\phi \geqslant \int_\Omega d(x)\phi,$$

for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Analogously we say that $0 \leqslant \underline{u} \in H_0^1(\Omega)$ is a subsolution for (2.1) if $H(x,\underline{u},\nabla\underline{u}) \in L^1(\Omega)$ and the reverse inequality holds. If $u \in H_0^1(\Omega)$ is a sub and a supersolution, then u is called a solution of (2.1).

Theorem 2.1. *Assume that H satisfies (2.2), (2.3) and that there exists $\theta \geqslant 0$ such that*

$$\theta\partial_s H(x,s,\xi) + \theta g(s) H(x,s,\xi) - \theta\big[g'(s) + g^2(s)\big]|\xi|^2 - \frac{1}{4}\big|2g(s)\xi - \partial_\xi H(x,s,\xi)\big|^2 \geqslant 0, \tag{2.4}$$

for a.e. $x \in \Omega$, every $s \geqslant 0$ and $\xi \in \mathbb{R}^N$. If v_1, v_2 are respectively a sub and a supersolution for (2.1) with $v_1 \in L^\infty(\Omega)$, then $v_1 \leqslant v_2$.

Remark 2.2. We remark explicitly that, in contrast with the comparison principle stated in [3] for a general quasilinear elliptic differential operators in divergence form, we do not require that H to be independent of $x \in \Omega$. Moreover, we do not impose the (increasing) monotonicity of H with respect to s as in [5,6]. In the particular case $H(x,s,\xi) = h(x,s)|\xi|^2$ then (2.2) is trivially satisfied and (2.3)–(2.4) reduces to the existence of g, θ such that for every $s \geqslant 0$, a.e. $x \in \Omega$, $0 < g(s) \leqslant h(x,s)$ and

$$0 \leqslant \theta\partial_s\big(h(x,s) - g(s)\big) + \big(h(x,s) - g(s)\big)\big((1+\theta)g(s) - h(x,s)\big). \tag{2.5}$$

Observe that if $g(s) - h(x,s)$ is decreasing in s, then (2.5) is satisfied.

Remark 2.3. Existence of solution for (2.1) is known from [9,10], the previous theorem shows that this solution is unique.

Proof of Theorem 2.1. We consider the C^3-function ψ defined in $[0,+\infty)$ by $\psi(s) = \int_0^s e^{-G(r)}\,dr$ for each $s \geqslant 0$. Observe that ψ is increasing with decreasing derivative ψ' and that $\psi(s) \leqslant s$, for every $s \geqslant 0$. Thus $w = \psi(v_1) - \psi(v_2) \in H^1(\Omega)$ and $w^+ \in H_0^1(\Omega)$. If n is the integer part of $\theta + 1$, we denote also $S(w) = w^n$. Taking $\psi'(v_1)S(w^+)$ (respectively, $\psi'(v_2)S(w^+)$) as test function in the inequality satisfied by v_1 (respectively, v_2), subtracting the resulting inequalities and taking into account that ψ' is nonincreasing, S is increasing with $S(0) = 0$, $d(x) \geqslant 0$ and (2.2), we obtain

$$\begin{aligned} 0 \geqslant{} & \int_\Omega \big[\psi''(v_1)|\nabla v_1|^2 + H(x,v_1,\nabla v_1)\psi'(v_1) - \psi''(v_2)|\nabla v_2|^2 - H(x,v_2,\nabla v_2)\psi'(v_2)\big]S\big(w^+\big) \\ & + \int_\Omega S'\big(w^+\big)\big[\psi'(v_1)\nabla v_1 - \psi'(v_2)\nabla v_2\big]\cdot\nabla w^+ \end{aligned}$$

$$= \int_\Omega \left[\frac{\psi''(v_1)|\nabla\psi(v_1)|^2 + H(x, v_1, \nabla\psi(v_1))\psi'(v_1)}{\psi'(v_1)^2} \right.$$
$$\left. - \frac{\psi''(v_2)|\nabla\psi(v_2)|^2 + H(x, v_2, \nabla\psi(v_2))\psi'(v_2)}{\psi'(v_2)^2} \right] S(w^+) + \int_\Omega S'(w^+)|\nabla w^+|^2$$
$$= \int_{\{w>0\}} S(w) \int_0^1 \frac{d}{dt}\left[\frac{\psi''(s)|\eta|^2 + H(x, s, \eta)\psi'(s)}{\psi'(s)^2} \right] dt + \int_{\{w>0\}} |\nabla w|^2 S'(w^+)$$

where $s = \psi^{-1}(t\psi(v_1) + (1-t)\psi(v_2))$ and $\eta = t\nabla\psi(v_1) + (1-t)\nabla\psi(v_2)$. After computing the above derivative, we derive that

$$\int_{\{w>0\}} wS(w) \int_0^1 \left(\frac{\psi'''(s)\psi'(s) - 2\psi''(s)^2}{\psi'(s)^4} |\eta|^2 \right) dt$$
$$+ \int_{\{w>0\}} wS(w) \int_0^1 \left(\frac{\partial_s H(x, s, \eta)\psi'(s)^2 - H(x, s, \eta)\psi''(s)\psi'(s)}{\psi'(s)^4} \right) dt + \int_{\{w>0\}} S(w) \int_0^1 \frac{\psi''(s)}{\psi'(s)^2} 2\eta \cdot \nabla w \, dt$$
$$+ \int_{\{w>0\}} S(w) \int_0^1 \frac{\psi'(s)}{\psi'(s)^2} \partial_\xi H(x, s, \eta) \cdot \nabla w \, dt + \int_{\{w>0\}} |\nabla w|^2 S'(w^+) \leqslant 0.$$

Multiplying by $\tilde\theta = \frac{\theta}{n}$ and using Young inequality in the set $\{w > 0\}$

$$\tilde\theta \left| S(w) \left(\frac{\psi''(s)}{\psi'(s)^2} 2\eta + \frac{\psi'(s)}{\psi'(s)^2} \partial_\xi H(x, s, \eta) \right) \cdot \nabla w \right|$$
$$\leqslant \tilde\theta^2 S'(w)|\nabla w|^2 + \frac{S^2(w^+)}{4S'(w^+)} \left| \frac{\psi''(s)}{\psi'(s)^2} 2\eta + \frac{\psi'(s)}{\psi'(s)^2} \partial_\xi H(x, s, \eta) \right|^2,$$

we get

$$\int_{\{w>0\}} \int_0^1 \tilde\theta(1-\tilde\theta) S'(w)|\nabla w|^2$$
$$+ \tilde\theta w S(w) \left(\frac{\partial_s H(x, s, \eta)\psi'(s)^2 - H(x, s, \eta)\psi''(s)\psi'(s)}{\psi'(s)^4} + \frac{\psi'''(s)\psi'(s) - 2\psi''(s)^2}{\psi'(s)^4} |\eta|^2 \right)$$
$$- \frac{S^2(w^+)}{S'(w^+)} \left| \frac{\psi''(s)}{\psi'(s)^2} 2\eta + \frac{\psi'(s)}{\psi'(s)^2} \partial_\xi H(x, s, \eta) \right|^2 dt \leqslant 0.$$

Taking into account the definition of ψ and S and using the variational characterization of λ_1, we have

$$\int_{\{w>0\}} \int_0^1 \tilde\theta(1-\tilde\theta) \frac{4\lambda_1 n}{(n+1)^2} w^{n+1} + \frac{w^{n+1}}{n\psi'(s)^2} \left[\theta\partial_s H(x, s, \eta) + \theta g(s) H(x, s, \eta) - \theta\left(g'(s) + g(s)^2\right)|\eta|^2 \right.$$
$$\left. - \frac{1}{4}\left|2g(s)\eta - \partial_\xi H(x, s, \eta)\right|^2 \right] dt \leqslant 0.$$

By (2.4), this implies that $w^+ \equiv 0$; i.e. $v_1 \leqslant v_2$ and the proof is finished. □

Remark 2.4. Observe that Theorem 2.1 assures the uniqueness of bounded solution $u \in H_0^1(\Omega)$ for (2.1). Moreover, the same conclusion holds, arguing as in the proof with $S(T_k(w^+))$ instead of $S(w^+)$, if we suppose that

$$\lim_{k\to\infty} \int_{\{w^+\geqslant k\}} k^{n-1}|\nabla w|^2 = 0.$$

This occurs, for example, in the case $n = 1$, which correspond to $0 \leqslant \theta < 1$ and $S(w) = w$ in the previous proof.

3. Minimal solutions and stability condition

Let H be a Carathéodory nonnegative function in $\Omega \times [0, +\infty) \times \mathbb{R}$ satisfying (2.2) and assume that there exist $M \geqslant 1$ and a C^1 nonnegative function g such that

$$g(s)|\xi|^2 \leqslant H(x, s, \xi) \leqslant Mg(s)|\xi|^2, \tag{3.1}$$

for every $x \in \Omega$, $s \geqslant 0$, $\xi \in \mathbb{R}^N$. Let also f be a continuous function in $[0, +\infty)$ with $f(0) > 0$.

We show the existence of a parameter λ^* such that problem (P_λ) has a solution if $\lambda < \lambda^*$ and no solution provided that $\lambda > \lambda^*$. Recall that $u \in H_0^1(\Omega)$ is a solution if $u \geqslant 0$ in Ω, $H(x, u, \nabla u) \in L^1(\Omega)$, $f(u) \in L^1(\Omega)$ and

$$\int_\Omega \nabla u \nabla \phi + \int_\Omega H(x, u, \nabla u)\phi = \lambda \int_\Omega f(u)\phi, \tag{3.2}$$

for every $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We say that a solution $u \in H_0^1(\Omega)$ of (P_λ) is regular if, in addition, $u \in L^\infty(\Omega)$.

Theorem 3.1. *Let H be a Carathéodory nonnegative function in $\Omega \times [0, +\infty) \times \mathbb{R}$ satisfying (2.2), (2.4) and (3.1). Let also f be a derivable, strictly increasing, nonnegative function in $[0, +\infty)$ with $f(0) > 0$ such that $\frac{1}{f(s)} \in L^1(0, +\infty)$. Assume that $f'(s)|\xi|^2 - H(x, s, \xi)f(s)$ is an increasing function in s for every $x \in \Omega$, and that there exists a positive constant c such that (recall that G is a primitive of g)*

$$\liminf_{s\to+\infty} \frac{f(s)e^{-MG(s)}}{\int_0^s e^{-MG(r)}\,dr} > 0, \tag{3.3}$$

and

$$\left|\frac{f'(s)}{f^2(s)}\right| \leqslant c\big(1 + \sqrt{g(s)}\big), \quad \forall s \geqslant 0. \tag{3.4}$$

Then there exists $\lambda^* \in (0, +\infty)$ such that (P_λ) admits a bounded minimal solution u_λ for every $\lambda \in (0, \lambda^*)$ and no solution for $\lambda > \lambda^*$.

Remark 3.2. In the particular model case $H(x, s, \xi) = h(x, s)|\xi|^2$ with $g(s) \leqslant h(x, s) \leqslant Mg(s)$ then H satisfies (2.2) and (3.1), while condition (2.4) is reduced to (2.5). Moreover, $f'(s)|\xi|^2 - H(x, s, \xi)f(s)$ is an increasing function in s for every $x \in \Omega$ whenever $f'(s) - h(x, s)f(s)$ is an increasing function in s for every $x \in \Omega$.

Remark 3.3. We point out the independence of hypotheses (3.3) and (3.4) in the above theorem. Indeed, the proof of the existence of solution for small λ only requires hypothesis (3.4) (i.e., it does not use (3.3)). On the other hand, the nonexistence of nontrivial solution for large λ only uses (3.3).

Remark 3.4. In the semilinear case ($H \equiv 0$), condition (3.3) is reduced to the standard condition $\liminf_{s\to+\infty} f(s)/s > 0$. This superlinearity hypothesis on f may also be a sufficient condition for (3.3) for some cases of a general H. For example, this is true if $H(x, s, \xi) = c|\xi|^2/(1 + s)$ with $0 < c < 1$. When $c \geqslant 1$ in the above example, the superlinear condition on f has to be strengthened. Specifically, (3.3) holds true provided that $\lim_{s\to+\infty} f(s)/s^c > 0$ (respectively, $\lim_{s\to+\infty} f(s)/(s \ln s) > 0$) if $c > 1$ (respectively, $c = 1$).

Remark 3.5. Observe that if $f(0) > 0$ then (3.4) is verified for small $s > 0$. Moreover, since f is increasing, $\liminf_{s\to\infty} f'(s)/f^2(s) = 0$ (note that f'/f^2 is integrable with primitive $-1/f$). Thus, if the function $\frac{f'(s)}{f^2(s)}$ has limit at infinity, then f verifies (3.4) at infinity.

Proof of Theorem 3.1. First, we show that (3.3) implies that (P_λ) has no positive solution for $\lambda > \lambda_1/c_1$ for some positive constant c_1. Even more, for these λ's we prove that (P_λ) does not have a supersolution $\bar{u} \in H_0^1(\Omega)$. Indeed, let $\bar{u} \in H_0^1(\Omega)$ be satisfying $f(\bar{u}), H(x,\bar{u},\nabla\bar{u}) \in L^1(\Omega)$ and

$$\int_\Omega \nabla\bar{u}\nabla\phi + \int_\Omega H(x,\bar{u},\nabla\bar{u})\phi \geqslant \lambda \int_\Omega f(\bar{u})\phi, \tag{3.5}$$

for every $\phi \in H_0^1(\Omega)\cap L^\infty(\Omega)$. The function $\phi = e^{-MG(T_k(\bar{u}))}\phi_1$ belongs to $H_0^1(\Omega)\cap L^\infty(\Omega)$ and we can take it as test function in (3.5) to get, using (3.1),

$$\int_\Omega \nabla\bar{u}\nabla\phi_1 e^{-MG(T_k(\bar{u}))} + \int_{\{\bar{u}\geqslant k\}} H(x,\bar{u},\nabla\bar{u})e^{-MG(k)}\phi_1 \geqslant \lambda \int_\Omega f(\bar{u})e^{-MG(T_k(\bar{u}))}\phi_1.$$

By the Lebesgue dominated convergence theorem, taking limits as k tends to ∞, we have

$$\int_\Omega \nabla\bar{u}\nabla\phi_1 e^{-MG(\bar{u})} \geqslant \lambda \int_\Omega f(\bar{u})e^{-MG(\bar{u})}\phi_1.$$

Using that $f \geqslant f(0) > 0$ and (3.3), there exists a positive constant c_1 such that $f(s)e^{-MG(s)} \geqslant c_1 \int_0^s e^{-MG(r)}\,dr$, for every $s > 0$. Consequently, if $z(x) = \int_0^{\bar{u}(x)} e^{-MG(r)}\,dr$, then $f(\bar{u})e^{-MG(\bar{u})} \geqslant c_1 z$. Observing that z belongs to $H_0^1(\Omega)$, this means that

$$\lambda_1 \int_\Omega z\phi_1 = \int_\Omega \nabla z\nabla\phi_1 \geqslant \lambda c_1 \int_\Omega z\phi_1,$$

i.e. $\lambda_1 \geqslant c_1\lambda$, as desired.

In particular, if we consider the set Λ of these $\lambda > 0$ for which (P_λ) has a solution, we have $\Lambda \subset [0, \lambda_1/c_1)$ and therefore Λ is bounded. In addition, Λ is an interval. In fact, we observe that if $0 < \mu \in \Lambda$, then (P_μ) has a solution $w \in H_0^1(\Omega)$. For every fixed $\lambda < \mu$, we claim that $\lambda \in \Lambda$, i.e. that problem (P_λ) has a bounded supersolution $\bar{u} \in H_0^1(\Omega)$. To verify it we follow closely [12] and define $h(s) = \int_0^s \frac{dt}{f(t)}$, which is continuous and strictly increasing. In addition, it is also bounded since $\frac{1}{f(s)} \in L^1(0,+\infty)$. We take $\Phi(s) = h^{-1}(\frac{\lambda}{\mu}h(s))$ and we will show that $\bar{u} = \Phi(w)$ is the desired bounded supersolution. To prove it, we first observe that $\bar{u} \in H_0^1(\Omega)\cap L^\infty(\Omega)$ (which clearly also implies that $f(\bar{u}), g(\bar{u})|\nabla\bar{u}|^2 \in L^1(\Omega)$ and thus, by (3.1), that $H(x,\bar{u},\nabla\bar{u}) \in L^1(\Omega)$). Indeed, by the boundedness of Φ, $\bar{u} \in L^\infty(\Omega)$. Since $\frac{1}{f}, \Phi \in L^\infty(0,+\infty)$, we deduce that $\frac{f(\Phi(w))}{f(w)} \in L^\infty(\Omega)$ and that

$$\nabla\bar{u} = \frac{\lambda}{\mu}\frac{f(\Phi(w))}{f(w)}\nabla w \in L^2(\Omega),$$

from which, using that $\Phi(0) = 0$, we obtain $\bar{u} \in H_0^1(\Omega)\cap L^\infty(\Omega)$.

Now, by (3.4), if $\phi \in H_0^1(\Omega)\cap L^\infty(\Omega)$, then $\frac{f(\Phi(w))}{f(w)}\phi \in H_0^1(\Omega)\cap L^\infty(\Omega)$ and

$$\begin{aligned}\int_\Omega \nabla\bar{u}\nabla\phi + \int_\Omega H(x,\bar{u},\nabla\bar{u})\phi &= \int_\Omega \Phi'(w)\nabla w\nabla\phi + \int_\Omega H\big(x,\Phi(w),\Phi'(w)\nabla w\big)\phi \\ &= \mu\int_\Omega f(w)\Phi'(w)\phi - \int_\Omega H(x,w,\nabla w)\Phi'(w)\phi \\ &\quad - \int_\Omega \Phi''(w)|\nabla w|^2\phi + \int_\Omega H\big(x,\Phi(w),\Phi'(w)\nabla w\big)\phi\end{aligned}$$

$$= \lambda \int_\Omega f(\bar{u})\phi + \int_\Omega \big[H\big(x, \Phi(w), \Phi'(w)\nabla w\big) - H(x, w, \nabla w)\Phi'(w) - \Phi''(w)|\nabla w|^2\big]\phi.$$

To conclude that $\bar{u}$ is a supersolution for (P_λ), it is enough to show that

$$H\big(x, \Phi(w), \Phi'(w)\nabla w\big) - H(x, w, \nabla w)\Phi'(w) - \Phi''(w)|\nabla w|^2 \geqslant 0.$$

By (2.2) this is equivalent to prove that

$$\frac{\lambda}{\mu}\big[f'\big(\Phi(w)\big)|\nabla u|^2 - H\big(x, \Phi(w), \nabla w\big) f\big(\Phi(w)\big)\big] \leqslant f'(w)|\nabla w|^2 - H(x, w, \nabla w) f(w).$$

Since $\frac{\lambda}{\mu} < 1$ and $\Phi(s) \leqslant s$, this is deduced by the hypothesis imposed on the monotony of $f'(s)|\xi|^2 - H(x, s, \xi) f(s)$.

On the other hand, since $f(0) > 0$, $u_0 \equiv 0$ is a (bounded) subsolution of (P_λ). For an integer $n \geqslant 1$, we define by induction u_n as the unique positive and bounded solution of

$$\begin{cases} -\Delta u_n + H(x, u_n, \nabla u_n) = \lambda f\big(u_{n-1}(x)\big), & x \in \Omega, \\ u_n = 0, & x \in \partial\Omega. \end{cases} \tag{3.6}$$

The existence of u_n is guaranteed in [9]. Observe that the uniqueness can be deduced from Theorem 2.1. Moreover, if $u_{n-1}(x) \leqslant \bar{u}(x)$ a.e. $x \in \Omega$, using that f is increasing, we deduce that $f(u_{n-1}(x)) \leqslant f(\bar{u}(x))$ a.e. $x \in \Omega$ and Theorem 2.1 allows to conclude that $u_n \leqslant \bar{u}$. Therefore, since $u_0 \leqslant \bar{u}$, we have inductively that $u_n \leqslant \bar{u}$ for every $n \geqslant 0$. Even more, if $u_{n-1}(x) \leqslant u_n(x)$ a.e. $x \in \Omega$ we deduce again from Theorem 2.1 that $u_n \leqslant u_{n+1}$. Thus, since $u_0(x) \leqslant u_1(x)$ a.e. $x \in \Omega$ we have that

$$0 < u_n \leqslant u_{n+1} \leqslant \cdots \leqslant \bar{u}.$$

In particular, u_n is converging almost everywhere in Ω to some u_λ satisfying $0 < u_n \leqslant u_\lambda \leqslant \bar{u}$ and $\|u_n\|_\infty \leqslant \|\bar{u}\|_\infty$ for every $n \in \mathbb{N}$.

Taking u_n as test function in (3.6), and using that H is positive, we obtain

$$\int_\Omega |\nabla u_n|^2 \leqslant \lambda f\big(\|\bar{u}\|_\infty\big)\|\bar{u}\|_\infty|\Omega|.$$

Therefore, u_n weakly converges to u_λ in $H_0^1(\Omega)$ and the convergence is strongly in $L^q(\Omega)$ for $1 \leqslant q < 2^*$. The compactness of u_n is a consequence of Lemma 4 in [10] which implies that u_λ is a bounded solution of (P_λ) and, consequently, Λ is an interval.

We point out that it has been proved that if $\mu \in \Lambda$ and $\lambda \in (0, \mu)$, then (P_λ) has a bounded solution $u_\lambda \in H_0^1(\Omega)$. In addition, observe that, by the previously cited comparison principle, every solution $w \in H_0^1(\Omega)$ of (P_λ) satisfies that $0 < u_n \leqslant w$ for every $n \geqslant 1$. This implies $u_\lambda \leqslant w$ and it proves that u_λ is the minimal solution of (P_λ).

To conclude the proof we only have to prove that Λ is not empty. Indeed, by Remark 2.3, let $z \in H_0^1(\Omega) \cap L^\infty(\Omega)$ be the unique solution of the problem

$$\begin{cases} -\Delta z + H(x, z, \nabla z) = 1, & x \in \Omega, \\ z = 0, & x \in \partial\Omega. \end{cases}$$

Clearly, there exists $\varepsilon > 0$ such that if $\lambda \in [0, \varepsilon)$, then $\lambda f(z(x)) \leqslant 1$ in Ω, so that z is a bounded supersolution of (P_λ). Using again the iterative scheme (3.6) with z playing the role of $\bar{u}$, we deduce the existence of a solution of (P_λ). Hence, the set Λ contains the interval $[0, \varepsilon)$ and it is not empty. □

Remark 3.6. In the previous proof we have proved that if f is a continuous and strictly increasing function with $f(0) > 0$ and $\bar{u}$ is a bounded supersolution of $(P_{\bar{\lambda}})$, then the limit of the sequence u_n given by (3.6) is the bounded minimal solution u_λ of (P_λ) for every $0 < \lambda \leqslant \bar{\lambda}$. As a consequence, $u_\lambda(x)$ is increasing in λ. Moreover, if $w \in H_0^1(\Omega)$ is a solution for (P_μ) then there exists such a bounded supersolution $\bar{u}$ for every $\bar{\lambda} < \mu$ and $\bar{u} < w$.

In the following lemma we prove a condition satisfied by minimal regular solutions.

Lemma 3.7. *Assume that H satisfies (2.2), (3.1) and there exists $0 < \tau \leqslant 1$ such that*

$$(1-\tau)\big[\partial_s H(x,s,\xi) + Mg(s)H(x,s,\xi) - \big(Mg'(s) + M^2 g(s)^2\big)|\xi|^2\big] + \frac{1}{4}\big|-2Mg(s)\xi + \partial_\xi H(x,s,\xi)\big|^2 \leqslant 0. \tag{3.7}$$

If u is a minimal regular solution of (P_λ), then

$$\frac{1}{\tau}\int_\Omega |\nabla\phi|^2 \geqslant \lambda \int_\Omega \big[f'(u) - Mg(u)f(u)\big]\phi^2, \tag{3.8}$$

for every $\phi \in H_0^1(\Omega)$.

Remark 3.8. In the particular case $H(x,s,\xi) = h(x,s)|\xi|^2$, (3.7) reduces to

$$0 \geqslant (1-\tau)\partial_s\big(h(x,s) - Mg(s)\big) + \big(h(x,s) - Mg(s)\big)\big(h(x,s) - \tau Mg(s)\big).$$

We observe that a sufficient condition is that $\partial_s(Mg(s) - h(x,s)) \geqslant 0$.

Remark 3.9. Observe that (3.8) plays the role of the stability condition of the semilinear case (see Theorem 4.6 below).

Remark 3.10. If $|\{x \in \Omega\colon\ f'(u(x)) - Mg(u(x))f(u(x)) > 0\}| > 0$, then inequality (3.8), with $\tau = 1$, means that $\lambda_1[f'(u) - Mg(u)f(u)] \geqslant \lambda$, where $\lambda_1[d(x)]$ denotes, for $d(x) \in L^q(\Omega)$ $(q > N/2)$, the first positive eigenvalue associated to the weighted eigenvalue problem:

$$\begin{cases} -\Delta u = \lambda d(x) u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

This occurs in particular, if $f' - Mgf$ is strictly increasing and nonnegative and $u = u_\lambda$ is a minimal bounded solution of (P_λ) with $\lambda \in (0, \lambda^*]$. Moreover, if $\lambda < \lambda^*$, then we can show that $\lambda_1[f'(u_\lambda) - Mg(u_\lambda)f(u_\lambda)] > \lambda$. Indeed, by Remark 3.6, we have $0 \leqslant u_\lambda \leqslant u_{\bar\lambda}$, for every $0 < \lambda < \bar\lambda < \lambda^*$. Therefore, using that $f' - Mgf$ is increasing, we get

$$\lambda_1\big[f'(u_\lambda) - Mg(u_\lambda)f(u_\lambda)\big] \geqslant \lambda_1\big[f'(u_{\bar\lambda}) - Mg(u_{\bar\lambda})f(u_{\bar\lambda})\big] \geqslant \bar\lambda > \lambda.$$

Proof of Lemma 3.7. Observe that $u = \lim_{n\to\infty} u_n$. We define $\psi(s) = \int_0^s e^{-MG(r)}\,dr$ for every $s \geqslant 0$. Choosing $\phi\psi'(u_n)$ as test function in the equation satisfied by u_n and $\phi\psi'(u)$ in the equation satisfied by u and subtracting them we obtain

$$\int_\Omega \nabla w \nabla\phi = \lambda \int_\Omega \big[f(u)\psi'(u) - f(u_{n-1})\psi'(u_n)\big]\phi - \int_\Omega \big[-Mg(u)|\nabla u|^2 + H(x,u,\nabla u)\big]\phi\psi'(u) + \int_\Omega \big[-Mg(u_n)|\nabla u_n|^2 + H(x,u_n,\nabla u_n)\big]\phi\psi'(u_n)$$

where $w = \psi(u) - \psi(u_n) \in H_0^1(\Omega)$. For any $\phi \in C_0^\infty(\Omega)$ and $\delta > 0$ we can take $\frac{\phi^2}{w+\delta}$ as test function to obtain, denoting $h_\delta(x) := \frac{f(u)\psi'(u) - f(u_{n-1})\psi'(u_n)}{w+\delta}$, that

$$\begin{aligned}
\lambda \int_\Omega h_\delta(x)\phi^2 &= 2\int_\Omega \frac{\phi}{w+\delta}\nabla w\nabla\phi - \int_\Omega \frac{\phi^2}{(w+\delta)^2}|\nabla w|^2 + \int_\Omega \frac{\psi''(u)|\nabla\psi(u)|^2 + H(x,u,\nabla\psi(u))\psi'(u)}{\psi'(u)^2}\,\frac{\phi^2}{w+\delta} \\
&\quad - \int_\Omega \frac{\psi''(u_n)|\nabla\psi(u_n)|^2 + H(x,u_n,\nabla\psi(u_n))\psi'(u_n)}{\psi'(u_n)^2}\,\frac{\phi^2}{w+\delta} \\
&= 2\int_\Omega \frac{\phi}{w+\delta}\nabla w\nabla\phi - \int_\Omega \frac{\phi^2}{(w+\delta)^2}|\nabla w|^2 + \int_\Omega \frac{\phi^2}{w+\delta}\int_0^1 \frac{d}{dt}\left(\frac{\psi''(s)|\eta|^2 + H(x,s,\eta)\psi'(s)}{\psi'(s)^2}\right) dt
\end{aligned}$$

where $s=\psi^{-1}(t\psi(u)+(1-t)\psi(u_n))$ and $\eta=t\nabla\psi(u)+(1-t)\nabla\psi(u_n)$. Observe that, from Young inequality we have $|2\frac{\phi}{w+\delta}\nabla w\nabla\phi|\leqslant\frac{1}{\tau}|\nabla\phi|^2+\tau\frac{\phi^2}{(w+\delta)^2}|\nabla w|^2$. Thus

$$\lambda\int_\Omega h_\delta(x)\phi^2\leqslant\frac{1}{\tau}\int_\Omega|\nabla\phi|^2+(\tau-1)\int_\Omega\frac{\phi^2}{(w+\delta)^2}|\nabla w|^2+\int_\Omega\frac{\phi^2}{w+\delta}\int_0^1\frac{d}{dt}\left(\frac{\psi''(s)|\eta|^2+H(x,s,\eta)\psi'(s)}{\psi'(s)^2}\right)dt.$$

By computing the above derivative, this inequality reduces to

$$\begin{aligned}\lambda\int_\Omega h_\delta(x)\phi^2\leqslant{}&\frac{1}{\tau}\int_\Omega|\nabla\phi|^2+(\tau-1)\int_\Omega\frac{\phi^2}{(w+\delta)^2}|\nabla w|^2\\&+\int_\Omega\frac{\phi^2 w}{w+\delta}\int_0^1\frac{-[Mg'(s)+M^2g^2(s)]|\eta|^2+\partial_sH(x,s,\eta)+Mg(s)H(x,s,\eta)}{\psi'(s)^2}\,dt\\&+\int_{\{w>0\}}\phi^2\int_0^1\frac{\nabla w}{w+\delta}\left(\frac{-2Mg(s)\eta+\partial_\xi H(x,s,\eta)}{\psi'(s)}\right)dt.\end{aligned}\tag{3.9}$$

In the case $\tau<1$, taking into account that

$$\left|\frac{\nabla w}{w+\delta}\left(\frac{-2Mg(s)\eta+\partial_\xi H(x,s,\eta)}{\psi'(s)}\right)\right|\leqslant(1-\tau)\frac{|\nabla w|^2}{(w+\delta)^2}+\frac{1}{4(1-\tau)}\left|\frac{-2Mg(s)\eta+\partial_\xi H(x,s,\eta)}{\psi'(s)}\right|^2,$$

we can assure that

$$\begin{aligned}\lambda\int_\Omega h_\delta(x)\phi^2\leqslant{}&\frac{1}{\tau}\int_\Omega|\nabla\phi|^2+\int_{\{w>0\}}\frac{\phi^2 w}{w+\delta}\int_0^1\frac{[-Mg'(s)-M^2g^2(s)]|\eta|^2+\partial_sH(x,s,\eta)+Mg(s)H(x,s,\eta)}{\psi'(s)^2}\,dt\\&+\int_{\{w>0\}}\phi^2\int_0^1\left(\frac{1}{4(1-\tau)}\left|\frac{2\psi''(s)\eta+\partial_\xi H(x,s,\eta)\psi'(s)}{\psi'(s)^2}\right|^2\right)dt.\end{aligned}\tag{3.10}$$

By (3.7) we have

$$-\big[Mg'(s)+M^2g^2(s)\big]|\eta|^2+\partial_sH(x,s,\eta)+Mg(s)H(x,s,\eta)\leqslant 0.$$

Thus, by Fatou lemma, taking limits as δ goes to zero in (3.10), we deduce that

$$\begin{aligned}\lambda\int_\Omega h(x)\phi^2\leqslant{}&\frac{1}{\tau}\int_\Omega|\nabla\phi|^2+\int_{\{w>0\}}\phi^2\int_0^1\frac{[-Mg'(s)-M^2g^2(s)]|\eta|^2+\partial_sH(x,s,\eta)+Mg(s)H(x,s,\eta)}{\psi'(s)^2}\,dt\\&+\int_{\{w>0\}}\phi^2\int_0^1\left(\frac{1}{4(1-\tau)}\left|\frac{2\psi''(s)\eta+\partial_\xi H(x,s,\eta)\psi'(s)}{\psi'(s)^2}\right|^2\right)dt,\end{aligned}$$

where $h(x):=\frac{f(u)\psi'(u)-f(u_{n-1})\psi'(u_n)}{w}$. In particular, using again hypothesis (3.7), we obtain

$$\lambda\int_\Omega h(x)\phi^2\leqslant\frac{1}{\tau}\int_\Omega|\nabla\phi|^2.$$

The above inequality is also deduced if $\tau=1$. Indeed, (3.7) and (2.2) imply that, in this case, $H(x,s,\xi)=Mg(s)|\xi|^2$ and (3.9) becomes

$$\lambda\int_\Omega h_\delta(x)\phi^2\leqslant\int_\Omega|\nabla\phi|^2.$$

Using that f is increasing and the generalized mean value theorem, we deduce that

$$h(x) \geqslant \frac{f(u)\psi'(u) - f(u_n)\psi'(u_n)}{\psi(u) - \psi(u_n)} = f'(\theta_n) - Mg(\theta_n)f(\theta_n),$$

for some $\theta_n \in [u_n, u]$. Therefore, we obtain

$$\frac{1}{\tau}\int_\Omega |\nabla\phi|^2 \geqslant \lambda \int_\Omega \big(f'(\theta_n) - Mg(\theta_n)f(\theta_n)\big)\phi^2,$$

which, by taking limits as $n \to \infty$, implies

$$\frac{1}{\tau}\int_\Omega |\nabla\phi|^2 \geqslant \lambda \int_\Omega \big[f'(u) - Mg(u)f(u)\big]\phi^2,$$

for every $\phi \in C_0^\infty(\Omega)$. Now, using that $f'(u) - Mg(u)f(u) \in L^\infty(\Omega)$ and the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, it yields (3.8). □

4. Extremal solution

In order to study the existence of solution of (P_{λ^*}), if u_λ denotes the minimal solution of (P_λ) given by Theorem 3.1 for $0 < \lambda < \lambda^*$, we first give sufficient conditions for the boundedness of u_λ in $H_0^1(\Omega)$.

Lemma 4.1. *Assume, in addition of hypotheses of Theorem 3.1 and Lemma 3.7, that*

$$\lim_{s\to+\infty} \frac{s^2(f'(s) - Mg(s)f(s))e^{G(s)}}{f(s)\int_0^s e^{G(t)}\,dt} = \rho > \frac{1}{\tau}. \tag{4.1}$$

Then the set $\{\|u_\lambda\|_{H_0^1(\Omega)}\colon \lambda \in (0, \lambda^*)\}$ *is bounded.*

Remark 4.2. In the semilinear case ($H \equiv 0$), condition (4.1) reduces to

$$\lim_{s\to+\infty} \frac{sf'(s)}{f(s)} = \rho > 1,$$

which is satisfied for example if $f(s) = e^s$ or $f(s) = (1+s)^p$ with $p > 1$. With respect to quasilinear equations, hypothesis (4.1) is also satisfied in the following cases:

- $f(s) = e^{\varpi s}$ and $H(x,s,\xi) = m(x)|\xi|^2$ with $0 < m_1 \leqslant m(x) \leqslant m_2 < \varpi$. In this case, $g(s) = m_1$, $M = \frac{m_2}{m_1}$, $\tau = \frac{m_1}{m_2}$.
- $f(s) = (1+s)^p$ and $H(x,s,\xi) = \frac{m(x)}{1+s}|\xi|^2$ with $1 < m_1 \leqslant m(x) \leqslant m_2 < p$ and $(p - m_2)(m_1 + 1) > \frac{m_2-1}{m_1-1}$. In this case, $g(s) = m_1/(1+s)$, $M = \frac{m_2}{m_1}$, $\tau = \frac{m_1-1}{m_2-1}$.

Proof of Lemma 4.1. Observe that $\varphi(s) = e^{-G(s)}\int_0^s e^{G(t)}\,dt$ satisfies $\varphi'(s) + g(s)\varphi(s) = 1$. Thus, if we take $\phi = \varphi(u_\lambda)$ as test function in (3.2) with $u = u_\lambda$, we obtain from (3.1) that

$$\lambda\int_\Omega f(u_\lambda)\varphi(u_\lambda) = \int_\Omega |\nabla u_\lambda|^2\varphi'(u_\lambda) + \int_\Omega H(x,u_\lambda,\nabla u_\lambda)\varphi(u_\lambda)$$
$$\geqslant \int_\Omega |\nabla u_\lambda|^2\big[\varphi'(u_\lambda) + g(u_\lambda)\varphi(u_\lambda)\big] = \int_\Omega |\nabla u_\lambda|^2.$$

Choosing $\phi = u_\lambda$ in the stability condition (3.8) satisfied by u_λ, we get

$$\lambda\int_\Omega f(u_\lambda)\varphi(u_\lambda) \geqslant \int_\Omega |\nabla u_\lambda|^2 \geqslant \tau\lambda\int_\Omega \big[f'(u_\lambda) - Mg(u_\lambda)f(u_\lambda)\big]u_\lambda^2.$$

By (4.1), there exists $C > 0$ such that $\frac{\frac{1}{\varepsilon}+\rho}{2}\varphi(s)f(s) \leqslant [f'(s) - Mg(s)f(s)]s^2 + C$, for every $s \geqslant 0$ and, consequently, we deduce that $\int_\Omega f(u_\lambda)\varphi(u_\lambda)$ (and hence $\int_\Omega |\nabla u_\lambda|^2$) is bounded for $\lambda \in (0, \lambda^*)$. □

Remark 4.3. We remark explicitly that in the above proof we have seen that $\int_\Omega f(u_\lambda)\varphi(u_\lambda)$ is bounded for $\lambda \in (0, \lambda^*)$.

Now, we give sufficient conditions to prove that there exists *extremal solution*.

Theorem 4.4. *If, in addition to the hypotheses of Lemma 4.1, we assume that g is bounded, then u_λ converges almost everywhere in Ω as $\lambda \to \lambda^*$ to a function $u^* \in H_0^1(\Omega)$ which is a (not necessarily bounded) solution of the quasilinear problem (P_{λ^*}).*

Proof. By Lemma 4.1, u_λ is bounded in $H_0^1(\Omega)$ and then there exists $u^* \in H_0^1(\Omega)$ such that u_λ weakly converges to u^*. Now, we prove that u^* is a solution of (P_{λ^*}) dividing the proof in two steps. In the first one, we show that $\{f(u_\lambda)\colon 0 < \lambda < \lambda^*\}$ is bounded in $L^1(\Omega)$. In the second step, we see that this boundedness implies that u^* solves (P_{λ^*}).

Step 1: *$\{f(u_\lambda)\colon 0 < \lambda < \lambda^*\}$ is bounded in $L^1(\Omega)$ and $f(u^*) \in L^1(\Omega)$.* Indeed, since g is bounded, if $a > 0$ is an upper bound of g, then the function $\varphi(s) := \int_0^s e^{-\int_t^s g(\tau)\,d\tau}\,dt \geqslant \frac{1}{a}(1 - e^{-as})$, for $s \geqslant 0$. In particular, for every fixed $s_0 > 0$ it follows that $\inf_{s\geqslant s_0}\varphi(s) > 0$ and we have

$$\int_\Omega f(u_\lambda) \leqslant f(s_0)|\Omega| + \frac{1}{\inf_{s\geqslant s_0}\varphi(s)}\int_\Omega f(u_\lambda)\varphi(u_\lambda),$$

which, by Remark 4.3, implies that $\int_\Omega f(u_\lambda)$ is bounded for $\lambda \in (0, \lambda^*)$. This implies, using the monotone convergence theorem (the function $\lambda \to u_\lambda$ is increasing, see Remark 3.6) that $f(u^*) \in L^1(\Omega)$.

Step 2: *u^* is a solution of (P_{λ^*}).* First, taking $T_\varepsilon(u_\lambda)/\varepsilon$ as test function in Eq. (3.2) satisfied by u_λ we have

$$\int_\Omega H(x, u_\lambda, \nabla u_\lambda)\frac{T_\varepsilon(u_\lambda)}{\varepsilon} \leqslant \lambda \int_\Omega f(u_\lambda)\frac{T_\varepsilon(u_\lambda)}{\varepsilon} \leqslant \lambda^* \int_\Omega f(u_\lambda).$$

Using Fatou lemma when ε tends to zero, we obtain by Step 1 the boundedness in $L^1(\Omega)$ of $H(x, u_\lambda, \nabla u_\lambda)$, more precisely, $\|H(x, u_\lambda, \nabla u_\lambda)\|_1 \leqslant \|\lambda^* f(u^*)\|_1$ for every $0 < \lambda < \lambda^*$. Hence $\|\lambda f(u_\lambda) - H(x, u_\lambda, \nabla u_\lambda)\|_1 \leqslant 2\|\lambda^* f(u^*)\|_1$. By [8, Theorem 2.1], this assures that $\nabla u_\lambda(x) \to \nabla u^*(x)$ $(\lambda \to \lambda^*)$ almost everywhere in Ω. Then we can use again Fatou lemma as λ goes to λ^* to get $H(x, u^*, \nabla u^*) \in L^1(\Omega)$.

Now, in order to verify that u^* is a solution of (P_{λ^*}), we closely follow [7]. Taking a nonnegative function $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function in Eq. (3.2) satisfied by u_λ, we can apply again Fatou lemma to obtain

$$\int_\Omega H\big(x, u^*, \nabla u^*\big)\phi \leqslant \lambda^* \int_\Omega f\big(u^*\big)\phi - \int_\Omega \nabla u^* \nabla\phi. \tag{4.2}$$

On the other hand, taking the function $e^{M[G(T_k(u^*))-G(u_\lambda)]}\phi$ as test function, it follows that

$$\begin{aligned}
&\int_\Omega \nabla u_\lambda \nabla\phi e^{M[G(T_k(u^*))-G(u_\lambda)]} + \int_\Omega Mg\big(T_k\big(u^*\big)\big)e^{M[G(T_k(u^*))-G(u_\lambda)]}\nabla T_k\big(u^*\big)\nabla u_\lambda\phi \\
&\quad= -\int_\Omega H(x, u_\lambda, \nabla u_\lambda)e^{M[G(T_k(u^*))-G(u_\lambda)]}\phi + \int_\Omega Mg(u_\lambda)e^{M[G(T_k(u^*))-G(u_\lambda)]}|\nabla u_\lambda|^2\phi \\
&\qquad+ \lambda \int_\Omega f(u_\lambda)e^{M[G(T_k(u^*))-G(u_\lambda)]}\phi.
\end{aligned}$$

By the weak convergence of u_λ to u^* (as $\lambda \to \lambda^*$) and the convergence of $e^{M[G(T_k(u^*))-G(u_\lambda)]}$ to $e^{M[G(T_k(u^*))-G(u^*)]}$ in $L^2(\Omega)$ we have the convergence of the left-hand side of the previous identity to

$$\int_\Omega \nabla u^* \nabla \phi e^{M[G(T_k(u^*))-G(u^*)]} + \int_\Omega M g\big(T_k(u^*)\big) e^{M[G(T_k(u^*))-G(u^*)]} \nabla T_k\big(u^*\big) \nabla u^* \phi.$$

Thus, using (3.1) and once again by Fatou lemma in the right-hand side of this identity, we get

$$\begin{aligned}
&\int_\Omega \nabla u^* \nabla \phi e^{M[G(T_k(u^*))-G(u^*)]} + \int_\Omega M g\big(T_k(u^*)\big) e^{M[G(T_k(u^*))-G(u^*)]} \nabla T_k\big(u^*\big) \nabla u^* \phi \\
&\quad \geqslant -\int_\Omega H\big(x, u^*, \nabla u^*\big) e^{M[G(T_k(u^*))-G(u^*)]} \phi + \int_\Omega M g\big(u^*\big) e^{M[G(T_k(u^*))-G(u^*)]} \big|\nabla u^*\big|^2 \phi \\
&\qquad + \lambda^* \int_\Omega f\big(u^*\big) e^{M[G(T_k(u^*))-G(u^*)]} \phi.
\end{aligned}$$

Using that $e^{M[G(T_k(u^*))-G(u^*)]} \leqslant 1$ and passing to the limit in the previous inequality as $k \to +\infty$, the Lebesgue dominated convergence theorem implies that

$$\int_\Omega \nabla u^* \nabla \phi + \int_\Omega H\big(x, u^*, \nabla u^*\big) \phi \geqslant \lambda^* \int_\Omega f\big(u^*\big) \phi,$$

which together to (4.2) implies

$$\int_\Omega \nabla u^* \nabla \phi + \int_\Omega H\big(x, u^*, \nabla u^*\big) \phi = \lambda^* \int_\Omega f\big(u^*\big) \phi,$$

for every $0 \leqslant \phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$. □

Remark 4.5. By taking u_λ as test function in (3.2) with $u = u_\lambda$, we have

$$\int_\Omega |\nabla u_\lambda|^2 + \int_\Omega H(x, u_\lambda, \nabla u_\lambda) u_\lambda = \lambda \int_\Omega f(u_\lambda) u_\lambda.$$

Thus, recalling that u_λ is bounded in $H_0^1(\Omega)$, in the case in which $g(s)s$ is a bounded function then $f(u^*)u^* \in L^1(\Omega)$.

We point out that since the constant M in (3.1) can be chosen arbitrarily large, then the condition (3.8) cannot be optimal in general.

Next result shows that if $M = 1$ in hypothesis (3.1); i.e., if $H(x, s, \xi) = g(s)|\xi|^2$, then we can extend the semilinear characterization of the minimal solution as the solutions satisfying (3.8) with $\tau = 1$ (see [13]). A solution $u \in H_0^1(\Omega)$ of (P_λ) with $H(x, s, \xi) = g(s)|\xi|^2$ is called stable if it satisfies condition (3.8).

Observe that, under the hypotheses of Theorem 4.4, the minimal solution $u_\lambda(x)$ is increasing in λ and converging to $u^*(x)$ almost everywhere in Ω. Since $f' - gf$ is increasing, the monotone convergence theorem implies that u^* satisfies also the condition (3.8); that is u^* is stable. We also prove that λ^* can be characterized as the unique possible value of λ for which problem (P_λ) admits a singular stable solution.

Theorem 4.6. *Assume that $H(x, s, \xi) = g(s)|\xi|^2$ with g a continuous nonnegative function and let f be a derivable function satisfying (3.3), (3.4) and such that f and $f' - gf$ are strictly increasing positive functions. If $u \in H_0^1(\Omega)$ is a stable solution of (P_λ), then u is the minimal solution of (P_λ). In particular, if additionally u is singular, then $\lambda = \lambda^*$. Moreover, if $\lambda = \lambda^*$ and $1/f(u)$, $f(u)u \in L^1(\Omega)$ then $u = u^* = \lim_{\lambda \to \lambda^*} u_\lambda$.*

Proof. Let $u \in H_0^1(\Omega)$ be a stable solution of (P_λ). We claim that $u \leqslant v$, for any other solution $v \in H_0^1(\Omega)$ of (P_λ). Indeed, given $\phi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, take $e^{-G(T_k(v))}\phi$ and $e^{-G(T_k(u))}\phi$ as test functions in the equations satisfied respectively by v and u. After passing to the limit as k tends to infinity we get

$$\int_\Omega \nabla v \nabla \phi e^{-G(v)} = \lambda \int_\Omega f(v) e^{-G(v)} \phi,$$

and

$$\int_\Omega \nabla u \nabla \phi e^{-G(u)} = \lambda \int_\Omega f(u) e^{-G(u)} \phi.$$

Subtracting both identities and choosing $\phi = w_k := T_k(\psi(v) - \psi(u))^-$ (with $\psi(s) = \int_0^s e^{-G(t)}\, dt$), it follows that

$$\int_\Omega |\nabla w_k|^2 = \lambda \int_\Omega \bigl(f(v) e^{-G(v)} - f(u) e^{-G(u)}\bigr) w_k.$$

Using that $w_k \leqslant (\psi(v) - \psi(u))^-$ and passing to the limit when k goes to infinity we obtain that $w := (\psi(v) - \psi(u))^-$ satisfies

$$\int_\Omega |\nabla w|^2 \leqslant \lambda \int_\Omega \bigl(f(v) e^{-G(v)} - f(u) e^{-G(u)}\bigr) w.$$

Since u is a stable solution, $\int_\Omega |\nabla w|^2 \geqslant \lambda \int_\Omega (f'(u) - g(u) f(u)) w^2$, and we derive that

$$\int_\Omega \bigl[f(v) e^{-G(v)} - f(u) e^{-G(u)} - \bigl(f'(u) - g(u) f(u)\bigr) w\bigr] w \geqslant 0.$$

Since $f' - gf$ is strictly increasing, we observe that the integrand in the preceding inequality is strictly positive. Indeed,

$$\begin{aligned}
& f(v) e^{-G(v)} - f(u) e^{-G(u)} - \bigl(f'(u) - g(u) f(u)\bigr) w \\
&= \int_u^v \bigl(f'(s) - g(s) f(s)\bigr) e^{-G(s)}\, ds - \int_u^v \frac{f'(u) - g(u) f(u)}{v - u} w\, ds \\
&= \int_u^v \biggl[\bigl(f'(s) - g(s) f(s)\bigr) e^{-G(s)} - \frac{f'(u) - g(u) f(u)}{v - u} w\biggr] ds \\
&> \int_u^v \bigl(f'(u) - g(u) f(u)\bigr) \biggl[e^{-G(s)} - \frac{w}{v - u}\biggr] ds \\
&= \bigl(f'(u) - g(u) f(u)\bigr) \bigl[\bigl(\psi(v) - \psi(u)\bigr) - w\bigr] = 0,
\end{aligned}$$

and, consequently, we deduce that $w = 0$, or equivalently, $\psi(v) \geqslant \psi(u)$. Using that ψ is an increasing function we obtain $v \geqslant u$, proving the claim and, therefore, u is the minimal solution for (P_λ).

Recalling that by Theorem 3.1, the minimal solution of (P_λ) for $\lambda < \lambda^*$ is bounded, we also obtain that the unique parameter λ for which (P_λ) may admit a singular stable solution u is $\lambda = \lambda^*$.

Finally, if $\lambda = \lambda^*$, since $u \in H_0^1(\Omega)$ is a (not necessarily singular) stable minimal solution for (P_{λ^*}), then, $u_\lambda \leqslant u$ for every $\lambda < \lambda^*$ (see Remark 3.6) and thus $u^* = \lim_{\lambda \to \lambda^*} u_\lambda \leqslant u$. On the other hand, taking u_λ as test function in the equation satisfied by u_λ and, using that $g \geqslant 0$ and that f is increasing, we get

$$\int_\Omega |\nabla u_\lambda|^2 \leqslant \int_\Omega |\nabla u_\lambda|^2 \bigl(1 + g(u_\lambda) u_\lambda\bigr) = \lambda \int_\Omega f(u_\lambda) u_\lambda \leqslant \lambda^* \int_\Omega f(u) u.$$

Therefore u_λ is bounded in $H_0^1(\Omega)$ and thus $u^* \in H_0^1(\Omega)$ and it is its weak limit as λ tends to λ^*. Taking into account that u is the minimal solution for (P_{λ^*}) then, in order to conclude that $u^* = u$, it is enough to show that u^* is actually a solution for (P_{λ^*}). This is a consequence of the integrability of $f(u^*)$ due to the inequality

$$0 \leqslant f(u_\lambda) \leqslant f\left(u^*\right) \leqslant f(u)$$

with $f(u) \in L^1(\Omega)$ (since u is a solution of (P_λ)). □

A natural question arises: *is the extremal solution bounded or not*? In [15] (see also [13,21]), for the semilinear case, sufficient conditions are given to assure that the extremal solution is bounded. We extend these sufficient conditions to the quasilinear case.

Theorem 4.7. *Assume that hypotheses of Lemma 4.1 hold true with g bounded. If, in addition,*

$$\lim_{s\to+\infty} g(s)\frac{f(s)}{f'(s)} = \alpha < \frac{1}{M}, \qquad \lim_{s\to+\infty} \frac{f(s)[f'(s)-Mg(s)f(s)]'}{f'(s)[f'(s)-Mg(s)f(s)]} = \mu \tag{4.3}$$

and

$$N < 4\tau(1-M\alpha) + 2\mu + 4\sqrt{\tau(1-M\alpha)\bigl[\tau(1-M\alpha)+\mu+\alpha-1\bigr]}, \tag{4.4}$$

then the extremal solution u^* given by Theorem 4.4 is bounded.

Proof. If we define $\nu := 2\tau(1-M\alpha) + \mu + 2\sqrt{\tau(1-M\alpha)[\tau(1-M\alpha)+\mu+\alpha-1]}$, by (4.4), we can fix $\beta \in (N/2, \nu)$. Let $\varphi(s)$ be a continuously differentiable function with $\varphi(0) = 0$ and

$$\varphi(s) = \left(\frac{f(s)^\beta}{f'(s) - Mg(s)f(s)}\right)^{\frac{1}{2}} \quad \text{for } s \geqslant 1.$$

For $\lambda < \lambda^*$ we choose $\phi = \varphi(u_\lambda)$ as test function in the stability condition (3.8) to get

$$\int_\Omega |\nabla u_\lambda|^2 \varphi'(u_\lambda)^2 \geqslant \tau\lambda \int_\Omega f(u_\lambda)^\beta + \tau\lambda \int_{\{u_\lambda \leqslant 1\}} \bigl[f'(u_\lambda) - Mg(u_\lambda)f(u_\lambda)\bigr]\varphi(u_\lambda)^2 - \tau\lambda \int_{\{u_\lambda \leqslant 1\}} f(u_\lambda)^\beta. \tag{4.5}$$

Now, we define $\psi(s) = \int_0^s \varphi'(t)^2 e^{G(t)-G(s)}\,dt$. Using L'Hôpital rule and (4.3) we have

$$\lim_{s\to+\infty} \frac{\psi(s)}{f(s)^{\beta-1}} = \lim_{s\to+\infty} \frac{\int_0^s \varphi'(t)^2 e^{\int_0^t g(\tau)\,d\tau}\,dt}{f(s)^{\beta-1} e^{\int_0^s g(\tau)\,d\tau}\,dt} = \frac{1}{4}\frac{(\beta-\mu)^2}{(\beta-1+\alpha)(1-M\alpha)}.$$

In particular, since $\beta < \nu$, we can choose γ in the interval $(\frac{(\beta-\mu)^2}{4(\beta-1+\alpha)(1-M\alpha)}, \tau)$ and $K > 0$ such that

$$\psi(s) \leqslant \gamma f(s)^{\beta-1} + K \quad \text{for } s \geqslant 0.$$

Thus, taking $\psi(u_\lambda)$ as test function in (3.2) with $u = u_\lambda$ and using (3.1), it follows

$$\begin{aligned}
\int_\Omega |\nabla u_\lambda|^2 \varphi'(u_\lambda)^2 &= \int_\Omega |\nabla u_\lambda|^2 \psi'(u_\lambda) + \int_\Omega g(u_\lambda)|\nabla u_\lambda|^2 \psi(u_\lambda) \\
&\leqslant \lambda \int_\Omega f(u_\lambda)\psi(u_\lambda) \\
&\leqslant \lambda\gamma \int_\Omega f(u_\lambda)^\beta + K\lambda \int_\Omega f(u_\lambda).
\end{aligned}$$

This, jointly with (4.5) gives

$$\begin{aligned}
(\tau-\gamma)\lambda \int_\Omega f(u_\lambda)^\beta &\leqslant K\lambda \int_\Omega f(u_\lambda) + \tau\lambda \int_{\{u_\lambda \leqslant 1\}} f(u_\lambda)^\beta \\
&\leqslant K\lambda^* \int_\Omega f\left(u^*\right) + \tau\lambda^* f(1)^\beta |\Omega|.
\end{aligned}$$

Thus, $\gamma < \tau$ and Fatou lemma imply that $f(u^*) \in L^\beta(\Omega)$. As a consequence, since $\beta > \frac{N}{2}$, the Stampacchia theorem (see Lemma 5.1 in [23]) assures that $u^* \in L^\infty(\Omega)$. $\square$

Remark 4.8. In some cases, condition (4.4) can be improved. For example, we can analyze the case that $g \geqslant 0$ and, for some $q, k > 1$, $f(s) \sim ks^q$ for $s \gg 0$. In this particular case, the boundedness of the extremal solution u^* in the above result is deduced from the L^β-integrability of the power $(u^*)^q$ for some $\beta > N/2$. However, using a bootstrap argument, it is also possible to deduce that $u^* \in L^\infty(\Omega)$ if we just have $\beta > \frac{N}{2}\frac{q-1}{q}$. Thus, condition (4.4) can be improved in this case to

$$N < \frac{q}{q-1}\Big(4\tau(1-M\alpha) + 2\mu + 4\sqrt{\tau(1-M\alpha)\big[\tau(1-M\alpha)+\mu+\alpha-1\big]}\Big).$$

We conclude this section and the paper by showing some applications of the preceding theorems to some particular cases of nonlinearities f and H. First we consider the case of problem (1.2) proving Theorem 1.1.

Proof of Theorem 1.1. Take $f(s) = (1+s)^p$ and $H(x,s,\xi) = \frac{m(x)}{1+s}|\xi|^2$, with $1 < m_1 \leqslant m(x) \leqslant m_2 < p$ and $1 < p$. Observe that (3.1) is satisfied with $g(s) = \frac{m_1}{1+s}$ and $M = \frac{m_2}{m_1}$. Moreover, f and g are nonnegative C^1-functions in $[0,\infty)$ with $f(0) > 0$, f strictly increasing, $f'(s) - \frac{m(x)}{1+s}f(s)$ is strictly increasing in s for every $x \in \Omega$ and g bounded. In addition, $1/f \in L^1(0,+\infty)$ and conditions (2.2), (2.4) for $\theta = \frac{m_2-m_1}{m_1-1}$, (3.3) and (3.4) are satisfied. Thus, by Theorem 3.1, there exists $\lambda^* > 0$ such that (1.2) has a minimal regular solution u_λ for every $\lambda < \lambda^*$ and no solution for every $\lambda > \lambda^*$. On the other hand, condition (3.7) holds true with $\tau = \frac{m_1-1}{m_2-1}$ and, if $(p-m_2)(m_1+1) > \frac{m_2-1}{m_1-1}$, then also condition (4.1) is satisfied (see Remark 4.2) and Theorem 4.4 assures in this case that $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is an *extremal solution*. Finally, condition (4.3) is satisfied with $\alpha = m_1/p$ and $\mu = (p-1)/p$ which implies, taking into account Theorem 4.7 and Remark 4.8, that u^* is regular provided that

$$3 \leqslant N < 4\frac{m_1-1}{m_2-1}\frac{p-m_2}{p-1} + 2 + 4\frac{m_1-1}{m_2-1}\sqrt{\frac{p-m_2}{p-1}}. \qquad \square$$

Remark 4.9. In the case $m_1 = m(x) = m_2$ Eq. (2.4) is trivially satisfied even for $m_1 \leqslant 1$.

Remark 4.10. Using similar arguments to these ones in [17,21], we can study (see [1]) the radial solutions for (1.2) with $m(x) = 1$ and $p = 2$, that is, the problem

$$\begin{cases} -\Delta w + \dfrac{|\nabla w|^2}{1+w} = \lambda(1+w)^2 & \text{in } B_1(0), \\ w = 0 & \text{on } \partial B_1(0). \end{cases} \qquad (Q_\lambda)$$

A phase plane technique proves, for every $\lambda > 0$, the existence of infinitely many negative radially increasing solutions with $w(0) = -1$. Moreover, in this case there exist infinitely many bounded sign-changing solutions. Even more, if we denote $\bar\lambda = 2(N-2)$ and $\lambda^* := \sup\{\lambda > 0\colon (Q_\lambda) \text{ admits positive radial solution}\}$ we have:

1. If $N \geqslant 10$ then $\lambda^* = \bar\lambda$ and (Q_λ) has a unique positive radial regular solution for every $\lambda \in (0,\lambda^*)$.
2. If $2 < N < 10$ then $\bar\lambda < \lambda^* < +\infty$ and $(Q_{\bar\lambda})$ has infinitely many positive regular radial solutions and a unique positive singular solution for $5 \leqslant N < 10$.

Similarly, it is possible to handle the case of exponential nonlinearities $f(s)$.

Theorem 4.11. *If $0 < m_1 \leqslant m(x) \leqslant m_2 < \varpi$, then there exists $\lambda^* > 0$ such that the problem*

$$\begin{cases} -\Delta u + m(x)|\nabla u|^2 = \lambda e^{\varpi u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a minimal regular solution u_λ for every $0 < \lambda < \lambda^*$ and no solution for every $\lambda > \lambda^*$. Moreover, $u^* = \lim_{\lambda\to\lambda^*} u_\lambda$ is a solution (the extremal solution) for $\lambda = \lambda^*$ and, if

$$3 \leqslant N < 4\frac{m_1}{m_2}\frac{\varpi - m_2}{\varpi} + 2 + 4\frac{m_1}{m_2}\sqrt{\frac{\varpi - m_2}{\varpi}},$$

then u^* is also regular.

Proof. Take $f(s) = e^{\varpi s}$ and $H(x, s, \xi) = m(x)|\xi|^2$ which satisfies (3.1) with $g(s) = m_1$ and $M = \frac{m_2}{m_1}$. Hypotheses of Theorems 3.1, 4.4 and 4.7 are satisfied. Indeed, see Remarks 3.2 and 4.2 and use $\alpha = 1/\varpi$ and $\mu = 1$ to verify condition (4.3). The proof is concluded applying these theorems. □

Remark 4.12. In the particular semilinear case, i.e., $M = 0$, we again obtain the sufficient condition $3 \leqslant N < 10$ for the regularity of the extremal solution.

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