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Ann. I. H. Poincaré – AN 31 (2014) 267–279

ANNALES DE L'INSTITUT HENRI POINCARÉ ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

On the Cauchy problem of a weakly dissipative μ-Hunter–Saxton equation

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Received 23 August 2011; received in revised form 9 February 2013; accepted 16 February 2013

Available online 14 March 2013

Abstract

In this paper, we study the Cauchy problem of a weakly dissipative μ-Hunter–Saxton equation. We first establish the local well-posedness for the weakly dissipative μ-Hunter–Saxton equation by Kato's semigroup theory. Then, we derive the precise blow-up scenario for strong solutions to the equation. Moreover, we present some blow-up results for strong solutions to the equation. Finally, we give two global existence results to the equation.

MSC: 35Q35; 35G25; 58D05

Keywords: A weakly dissipative μ-Hunter–Saxton; Blow-up scenario; Blow-up; Strong solutions; Global existence

1. Introduction

Recently, the μ-Hunter–Saxton (also called μ-Camassa–Holm) equation

$$\mu(u)_t - u_{txx} = -2\mu(u)u_x + 2u_x u_{xx} + u u_{xxx},$$

which is originally derived and studied in [21] attracts a lot of attention. Here $u(t,x)$ is a time-dependent function on the unit circle $\mathbb{S} = \mathbb{R}/\mathbb{Z}$ and $\mu(u) = \int_{\mathbb{S}} u\,dx$ denotes its mean. In [21], the authors show that if interactions of rotators and an external magnetic field is allowed, then the μ-Hunter–Saxton (μHS) equation can be viewed as a natural generalization of the rotator equation. Moreover, the μHS equation describes the geodesic flow on $\mathcal{D}^s(\mathbb{S})$ with the right-invariant metric given at the identity by the inner product [21]

$$(u, v) = \mu(u)\mu(v) + \int_{\mathbb{S}} u_x v_x\,dx.$$

In [21,25], the authors showed that the μHS equation admits both periodic one-peakon solution and the multi-peakons. Moreover, in [13,15], the authors also discussed the μHS equation.

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0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2013.02.008

One of the closest relatives of the μHS equation is the Camassa–Holm equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx},$$

which was introduced firstly by Fokas and Fuchssteiner in [12] as an abstract equation with a bihamiltonian structure. Meanwhile, it was derived by Camassa and Holm in [2] as a shallow water approximation independently. The Camassa–Holm equation is a model for shallow water waves [2,9,18] and a re-expression of the geodesic flow both on the diffeomorphism group of the circle [8] and on the Bott–Virasoro group [23]. The Camassa–Holm equation has a bi-Hamiltonian structure [12] and is completely integrable [3]. The possibility of the relevance of Camassa–Holm to the modeling of tsunamis was raised in [7]. It is worth to point out that a long-standing open problem in hydrodynamics was the derivation of a model equation that can capture breaking waves as well as peaked traveling waves, cf. the discussion in [30]. The quest for peaked traveling waves is motivated by the desire to find waves replicating a feature that is characteristic for the waves of great height-waves of largest amplitude that are exact traveling solutions of the governing equations for water waves, whether periodic or solitary, cf. [5]. Breaking waves are solutions that remain bounded but their slope becomes unbounded in finite time, cf. [6]. Both these aspects are modeled by the Camassa–Holm equation. Recently, the Camassa–Holm equation has been studied extensively, cf. [1,26,34,35]. The other closest relatives of the μHS equation is the Hunter–Saxton equation [16]

$$u_{txx} + 2u_x u_{xx} + uu_{xxx} = 0,$$

which is an asymptotic equation for rotators in liquid crystals and modeling the propagation of weakly nonlinear orientation waves in a massive nematic liquid crystal. The orientation of the molecules is described by the field of unit vectors $(\cos u(t,x), \sin u(t,x))$ [37]. The Hunter–Saxton equation also arises in a different physical context as the high-frequency limit [17] of the Camassa–Holm equation. Similar to the Camassa–Holm equation, the Hunter–Saxton equation also has a bi-Hamiltonian structure [18,27] and is completely integrable [17]. The initial value problem of the Hunter–Saxton equation also has been studied extensively, cf. [16,24,37].

In general, it is difficult to avoid energy dissipation mechanisms in a real world. So, it is reasonable to study the model with energy dissipation. In [14] and [28], the authors discussed the energy dissipative KdV equation from different aspects. Weakly dissipative CH equation and weakly dissipative DP equation have been studied in [33] and [11,31,32] respectively. Recently, some results for a weakly dissipative μDP equation are proved in [22]. It is worthy to note that the local well-posedness result in [22] is obtained by using a method based on a geometric argument.

In this paper, we will discuss the Cauchy problem of the following weakly dissipative μHS equation:

$$\begin{cases} y_t + uy_x + 2u_x y + \lambda y = 0, & t > 0,\ x \in \mathbb{R}, \\ y = \mu(u) - u_{xx}, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geqslant 0,\ x \in \mathbb{R}, \end{cases} \tag{1.1}$$

or in the equivalent form:

$$\begin{cases} \mu(u)_t - u_{txx} + 2\mu(u)u_x - 2u_x u_{xx} - uu_{xxx} + \lambda\big(\mu(u) - u_{xx}\big) = 0, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t \geqslant 0,\ x \in \mathbb{R}. \end{cases} \tag{1.2}$$

Here the constant λ is assumed to be positive and the term $\lambda y = \lambda(\mu(u) - u_{xx})$ models energy dissipation. For $\mu(u) = 0$, (1.2) becomes weakly dissipative Hunter–Saxton equation, which has been studied in [29].

The paper is organized as follows. In Section 2, we establish the local well-posedness of the initial value problem associated with Eq. (1.1). In Section 3, we derive the precise blow-up scenario. In Section 4, we present two explosion criteria of strong solutions to Eq. (1.1) with general initial data. In Section 5, we give two new global existence results of strong solutions to Eq. (1.1).

Notation. Given a Banach space Z, we denote its norm by $\|\cdot\|_Z$. Since all spaces of functions are over $\mathbb{S} = \mathbb{R}/\mathbb{Z}$, for simplicity, we drop $\mathbb{S}$ in our notations if there is no ambiguity. We let $[A, B]$ denote the commutator of linear operator A and B. For convenience, we let $(\cdot|\cdot)_{s\times r}$ and $(\cdot|\cdot)_s$ denote the inner products of $H^s \times H^r$, $s, r \in \mathbb{R}_+$ and H^s, $s \in \mathbb{R}_+$, respectively. Moreover, we denote by $G(X, M, \beta)$ the set of all linear operators A in X such that $-A$ generates a C_0-semigroup $\{e^{-tA}\}$ with $\|e^{-tA}\| \leqslant Me^{\beta t}$. In particular, A is quasi-m-accretive if $A \in G(X, 1, \beta)$.

2. Local well-posedness

In this section, we will establish the local well-posedness for the Cauchy problem of Eq. (1.1) in H^s, $s > \frac{3}{2}$, by applying Kato's theory.

For convenience, we state here Kato's theory in the form suitable for our purpose. Consider the abstract quasi-linear equation:

$$\frac{dv}{dt} + A(v)v = f(v), \quad t > 0, \ v(0) = v_0. \tag{2.1}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X and let $Q : Y \to X$ be a topological isomorphism. Let $L(Y, X)$ denote the space of all bounded linear operators from Y to X ($L(X)$, if $X = Y$). Assume that:

(i) $A(y) \in L(Y, X)$ for $y \in Y$ with

$$\big\|\big(A(y) - A(z)\big)w\big\|_X \leqslant \mu_1\|y - z\|_X\|w\|_Y, \quad y, z, w \in Y,$$

and $A(y) \in G(X, 1, \beta)$ (i.e. $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\big\|\big(B(y) - B(z)\big)w\big\|_X \leqslant \mu_2\|y - z\|_Y\|w\|_X, \quad y, z \in Y, \ w \in X.$$

(iii) $f : Y \to Y$ and extends also to a map from X to X. f is bounded on bounded sets in Y, and

$$\big\|f(y) - f(z)\big\|_Y \leqslant \mu_3\|y - z\|_Y, \quad y, z \in Y,$$
$$\big\|f(y) - f(z)\big\|_X \leqslant \mu_4\|y - z\|_X, \quad y, z \in Y.$$

Here μ_1, μ_2, μ_3 and μ_4 depend only on $\max\{\|y\|_Y, \|z\|_Y\}$.

Theorem 2.1. *(See [19].) Assume that* (i), (ii) *and* (iii) *hold. Given $v_0 \in Y$, there exist a maximal $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to Eq.* (2.1) *such that*

$$v = v(\cdot, v_0) \in C\big([0, T); Y\big) \cap C^1\big([0, T); X\big).$$

Moreover, the map $v_0 \to v(\cdot, v_0)$ is continuous from Y to

$$C\big([0, T); Y\big) \cap C^1\big([0, T); X\big).$$

On one hand, with $y = \mu(u) - u_{xx}$, the first equation in (1.2) takes the form of a quasi-linear evolution equation of hyperbolic type:

$$u_t + uu_x = -\partial_x A^{-1}\left(2\mu(u)u + \frac{1}{2}u_x^2\right) - \lambda u, \tag{2.2}$$

where $A = \mu - \partial_x^2$ is an isomorphism between H^s and H^{s-2} with the inverse $v = A^{-1}w$ given explicitly by [10,21]

$$\begin{aligned} v(x) = {} & \left(\frac{x^2}{2} - \frac{x}{2} + \frac{13}{12}\right)\mu(w) + \left(x - \frac{1}{2}\right)\int_0^1\int_0^y w(s)\,ds\,dy \\ & - \int_0^x\int_0^y w(s)\,ds\,dy + \int_0^1\int_0^y\int_0^s w(r)\,dr\,ds\,dy. \end{aligned} \tag{2.3}$$

Since A^{-1} and ∂_x commute, the following identities hold

$$A^{-1}\partial_x w(x) = \left(x - \frac{1}{2}\right)\int_0^1 w(x)\,dx - \int_0^x w(y)\,dy + \int_0^1\int_0^x w(y)\,dy\,dx, \tag{2.4}$$

and

$$A^{-1}\partial_x^2 w(x) = -w(x) + \int_0^1 w(x)\,dx. \tag{2.5}$$

On the other hand, integrating both sides of the first equation in (1.2) with respect to x on $\mathbb{S}$, we obtain

$$\frac{d}{dt}\mu(u) = -\lambda\mu(u).$$

Then it follows that

$$\mu(u) = \mu(u_0)e^{-\lambda t} := \mu_0 e^{-\lambda t}, \tag{2.6}$$

where

$$\mu_0 := \mu(u_0) = \int_{\mathbb{S}} u_0(x)\,dx.$$

Combining (2.2) and (2.6), Eq. (1.2) can be rewritten as

$$\begin{cases} u_t + uu_x = -\partial_x A^{-1}\left(2\mu_0 e^{-\lambda t}u + \frac{1}{2}u_x^2\right) - \lambda u, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t, x+1) = u(t,x), & t \geqslant 0,\ x \in \mathbb{R}. \end{cases} \tag{2.7}$$

The remainder of this section is devoted to the local well-posedness result. Firstly, we will give a useful lemma.

Lemma 2.1. *(See [19].) Let r, t be real numbers such that $-r < t \leqslant r$. Then*

$$\|fg\|_{H^t} \leqslant c\|f\|_{H^r}\|g\|_{H^t}, \quad \text{if } r > \frac{1}{2},$$

$$\|fg\|_{H^{t+r-\frac{1}{2}}} \leqslant c\|f\|_{H^r}\|g\|_{H^t}, \quad \text{if } r < \frac{1}{2},$$

where c is a positive constant depending on r, t.

Theorem 2.2. *Given $u_0 \in H^s$, $s > \frac{3}{2}$, then there exists a maximal $T = T(\lambda, u_0) > 0$, and a unique solution u to* (2.7) *(or* (1.1)*) such that*

$$u = u(\cdot, u_0) \in C\big([0,T); H^s\big) \cap C^1\big([0,T); H^{s-1}\big).$$

Moreover, the solution depends continuously on the initial data, i.e., the mapping

$$u_0 \to u(\cdot, u_0) : H^s \to C\big([0,T); H^s\big) \cap C^1\big([0,T); H^{s-1}\big)$$

is continuous.

Proof. For $u \in H^s$, $s > \frac{3}{2}$, we define the operator $A(u) = u\partial_x$. Similar to Lemma 2.6 in [36], we have that $A(u)$ belongs to $G(H^{s-1}, 1, \beta)$, that is, $-A(u)$ generates a C_0-semigroup $T(t)$ on H^{s-1} and $\|T(t)\|_{L(H^{s-1})} \leqslant e^{t\beta}$ for all $t \geqslant 0$. Analogous to Lemma 2.7 in [36], we get that $A(u) \in L(H^s, H^{s-1})$ and

$$\big\|\big(A(z) - A(y)\big)w\big\|_{H^{s-1}} \leqslant \mu_1\|z - y\|_{H^{s-1}}\|w\|_{H^s},$$

for all $z, y, w \in H^s$.

Let $Q = \Lambda = (1-\partial_x^2)^{\frac{1}{2}}$. Define $B(z) = QA(z)Q^{-1} - A(z)$ for $z \in H^s$, $s > \frac{3}{2}$. Similar to Lemma 2.8 in [36], we deduce that $B(z) \in L(H^{s-1})$ and

$$\left\|\big(B(z)-B(y)\big)w\right\|_{H^{s-1}} \leqslant \mu_2\|z-y\|_{H^s}\|w\|_{H^{s-1}},$$

for all $z, y \in H^s$ and $w \in H^{s-1}$. Where μ_1, μ_2 are positive constants.

Set

$$f(u) = -\partial_x\big(\mu-\partial_x^2\big)^{-1}\left(2\mu_0 e^{-\lambda t}u + \frac{1}{2}u_x^2\right) - \lambda u.$$

Let $y, z \in H^s$, $s > \frac{3}{2}$. Since H^{s-1} is a Banach algebra, it follows that

$$\begin{aligned}
\left\|f(y)-f(z)\right\|_{H^s} &\leqslant \left\|-\partial_x\big(\mu-\partial_x^2\big)^{-1}\left(2\mu_0 e^{-\lambda t}(y-z) + \frac{1}{2}\big(y_x^2-z_x^2\big)\right)\right\|_{H^s} + \lambda\|y-z\|_{H^s} \\
&\leqslant \left\|2\mu_0 e^{-\lambda t}(y-z) + \frac{1}{2}(y_x+z_x)(y_x-z_x)\right\|_{H^{s-1}} + \lambda\|y-z\|_{H^s} \\
&\leqslant 2|\mu_0|\|y-z\|_{H^{s-1}} + \frac{1}{2}\|y_x+z_x\|_{H^{s-1}}\|y_x-z_x\|_{H^{s-1}} + \lambda\|y-z\|_{H^s} \\
&\leqslant \big(2|\mu_0| + \|y\|_{H^s} + \|z\|_{H^s} + \lambda\big)\|y-z\|_{H^s}.
\end{aligned}$$

Furthermore, taking $z = 0$ in the above inequality, we obtain that f is bounded on bounded set in H^s. Moreover,

$$\begin{aligned}
\left\|f(y)-f(z)\right\|_{H^{s-1}} &\leqslant \left\|-\partial_x\big(\mu-\partial_x^2\big)^{-1}\left(2\mu_0 e^{-\lambda t}(y-z) + \frac{1}{2}\big(y_x^2-z_x^2\big)\right)\right\|_{H^{s-1}} + \lambda\|y-z\|_{H^{s-1}} \\
&\leqslant \left\|2\mu_0 e^{-\lambda t}(y-z) + \frac{1}{2}(y_x+z_x)(y_x-z_x)\right\|_{H^{s-2}} + \lambda\|y-z\|_{H^{s-1}} \\
&\leqslant 2|\mu_0|\|y-z\|_{H^{s-1}} + \frac{c}{2}\|y_x+z_x\|_{H^{s-1}}\|y_x-z_x\|_{H^{s-2}} + \lambda\|y-z\|_{H^{s-1}} \\
&\leqslant \big(2|\mu_0| + c\big(\|y\|_{H^s} + \|z\|_{H^s}\big) + \lambda\big)\|y-z\|_{H^{s-1}},
\end{aligned}$$

here we applied Lemma 2.1 with $r = s-1$, $t = s-2$. Set $Y = H^s$, $X = H^{s-1}$. It is obvious that Q is an isomorphism of Y onto X. Applying Theorem 2.1, we obtain the local well-posedness of Eq. (1.1) in H^s, $s > \frac{3}{2}$, and $u \in C([0,T]; H^s) \cap C^1([0,T]; H^{s-1})$. This completes the proof of Theorem 2.2. □

Remark 2.1. Similar to the proof of Theorem 2.3 in [36], we have that the maximal time of existence $T > 0$ in Theorem 2.2 is independent of the Sobolev index $s > \frac{3}{2}$.

3. The precise blow-up scenario

In this section, we present the precise blow-up scenario for strong solutions to Eq. (1.1). We first recall the following lemmas.

Lemma 3.1. *(See [20].) If $r > 0$, then $H^r \cap L^\infty$ is an algebra. Moreover*

$$\|fg\|_{H^r} \leqslant c\big(\|f\|_{L^\infty}\|g\|_{H^r} + \|f\|_{H^r}\|g\|_{L^\infty}\big),$$

where c is a constant depending only on r.

Lemma 3.2. *(See [20].) If $r > 0$, then*

$$\left\|\left[\Lambda^r, f\right]g\right\|_{L^2} \leqslant c\big(\|\partial_x f\|_{L^\infty}\left\|\Lambda^{r-1}g\right\|_{L^2} + \left\|\Lambda^r f\right\|_{L^2}\|g\|_{L^\infty}\big),$$

where c is a constant depending only on r.

Lemma 3.3. *(See [4,13].) If $f \in H^1(\mathbb{S})$ is such that $\int_{\mathbb{S}} f(x)\,dx = 0$, then we have*

$$\max_{x\in\mathbb{S}} f^2(x) \leqslant \frac{1}{12}\int_{\mathbb{S}} f_x^2(x)\,dx.$$

Next we prove the following useful result on global existence of solutions to (1.1).

Theorem 3.1. *Let $u_0 \in H^s$, $s > \frac{3}{2}$, be given and assume that T is the maximal existence time of the corresponding solution u to* (2.7) *with the initial data u_0. If there exists $M > 0$ such that*

$$\left\|u_x(t,\cdot)\right\|_{L^\infty} \leqslant M, \quad t \in [0,T),$$

then the H^s-norm of $u(t,\cdot)$ does not blow up on $[0,T)$.

Proof. Let u be the solution to (2.7) with the initial data $u_0 \in H^s$, $s > \frac{3}{2}$, and let T be the maximal existence time of the corresponding solution u, which is guaranteed by Theorem 2.2. Throughout this proof, $c > 0$ stands for a generic constant depending only on s.

Applying the operator Λ^s to the first equation in (2.7), multiplying by $\Lambda^s u$, and integrating over $\mathbb{S}$, we obtain

$$\frac{d}{dt}\|u\|_{H^s}^2 = -2(uu_x,u)_s - 2\left(u, \partial_x\left(\mu-\partial_x^2\right)^{-1}\left(2\mu_0 e^{-\lambda t}u + \frac{1}{2}u_x^2\right)\right)_s - 2\lambda(u,u)_s. \tag{3.1}$$

Let us estimate the first term of the right hand side of (3.1).

$$\begin{aligned}
\left|(uu_x,u)_s\right| &= \left|\left(\Lambda^s(u\partial_x u), \Lambda^s u\right)_0\right| \\
&= \left|\left(\left[\Lambda^s,u\right]\partial_x u, \Lambda^s u\right)_0 + \left(u\Lambda^s\partial_x u, \Lambda^s u\right)_0\right| \\
&\leqslant \left\|\left[\Lambda^s,u\right]\partial_x u\right\|_{L^2}\left\|\Lambda^s u\right\|_{L^2} + \frac{1}{2}\left|\left(u_x\Lambda^s u, \Lambda^s u\right)_0\right| \\
&\leqslant \left(c\|u_x\|_{L^\infty} + \frac{1}{2}\|u_x\|_{L^\infty}\right)\|u\|_{H^s}^2 \\
&\leqslant c\|u_x\|_{L^\infty}\|u\|_{H^s}^2,
\end{aligned} \tag{3.2}$$

where we used Lemma 3.2 with $r = s$. Furthermore, we estimate the second term of the right hand side of (3.1) in the following way:

$$\begin{aligned}
\left|\left(u, \partial_x\left(\mu-\partial_x^2\right)^{-1}\left(2\mu_0 e^{-\lambda t}u + \frac{1}{2}u_x^2\right)\right)_s\right| &\leqslant \left\|\partial_x\left(\mu-\partial_x^2\right)^{-1}\left(2\mu_0 e^{-\lambda t}u + \frac{1}{2}u_x^2\right)\right\|_{H^s}\|u\|_{H^s} \\
&\leqslant \left\|2\mu_0 e^{-\lambda t}u + \frac{1}{2}u_x^2\right\|_{H^{s-1}}\|u\|_{H^s} \\
&\leqslant c\left(|\mu_0|\|u\|_{H^s} + \|u_x\|_{L^\infty}\|u_x\|_{H^{s-1}}\right)\|u\|_{H^s} \\
&\leqslant c\left(|\mu_0| + \|u_x\|_{L^\infty}\right)\|u\|_{H^s}^2,
\end{aligned} \tag{3.3}$$

where we used Lemma 3.1 with $r = s-1$. Combining (3.2) and (3.3) with (3.1), we get

$$\frac{d}{dt}\|u\|_{H^s}^2 \leqslant c\left(|\mu_0| + \|u_x\|_{L^\infty} + 2\lambda\right)\|u\|_{H^s}^2.$$

An application of Gronwall's inequality and the assumption of the theorem yield

$$\|u\|_{H^s}^2 \leqslant e^{c(|\mu_0|+M+2\lambda)t}\|u_0\|_{H^s}^2.$$

This completes the proof of the theorem. □

The following result describes the precise blow-up scenario for sufficiently regular solutions to Eq. (1.1).

Theorem 3.2. *Let $u_0 \in H^s$, $s > \frac{3}{2}$ be given and let T be the maximal existence time of the corresponding solution u to* (2.7) *with the initial data u_0. Then the corresponding solution blows up in finite time if and only if*

$$\liminf_{t\to T}\Big\{\min_{x\in\mathbb{S}} u_x(t,x)\Big\} = -\infty.$$

Proof. Applying a simple density argument, Remark 2.1 implies that we only need to consider the case $s = 3$. Multiplying the first equation in (1.1) by y and integrating over $\mathbb{S}$ with respect to x yield

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} y^2\,dx &= 2\int_{\mathbb{S}} y(-uy_x - 2u_x y - \lambda y)\,dx \\ &= -2\int_{\mathbb{S}} uyy_x\,dx - 4\int_{\mathbb{S}} u_x y^2\,dx - 2\lambda\int_{\mathbb{S}} y^2\,dx \\ &= -3\int_{\mathbb{S}} u_x y^2\,dx - 2\lambda\int_{\mathbb{S}} y^2\,dx.\end{aligned}$$

So, if there is a constant $M > \lambda > 0$ such that

$$u_x(t,x) \geqslant -M, \quad \forall (t,x) \in [0,T)\times\mathbb{S},$$

then

$$\frac{d}{dt}\int_{\mathbb{S}} y^2\,dx \leqslant (3M - 2\lambda)\int_{\mathbb{S}} y^2\,dx.$$

Gronwall's inequality implies that

$$\int_{\mathbb{S}} y^2\,dx \leqslant e^{(3M-2\lambda)t}\int_{\mathbb{S}} y^2(0,x)\,dx.$$

Note that

$$\int_{\mathbb{S}} y^2\,dx = \mu(u)^2 + \int_{\mathbb{S}} u_{xx}^2\,dx \geqslant \|u_{xx}\|_{L^2}^2.$$

Since $u_x \in H^2 \subset H^1$ and $\int_{\mathbb{S}} u_x\,dx = 0$, Lemma 3.3 implies that

$$\|u_x\|_{L^\infty} \leqslant \frac{1}{2\sqrt{3}}\|u_{xx}\|_{L^2} \leqslant e^{\frac{(3M-2\lambda)t}{2}}\big\|y(0,x)\big\|_{L^2}.$$

Theorem 3.1 ensures that the solution u does not blow up in finite time.

On the other hand, by Sobolev's embedding theorem it is clear that if

$$\liminf_{t\to T}\Big\{\min_{x\in\mathbb{S}} u_x(t,x)\Big\} = -\infty,$$

then $T < \infty$. This completes the proof of the theorem. □

4. Blow-up

In this section, we discuss the blow-up phenomena of Eq. (1.1) and prove that there exist strong solutions to (1.1) which do not exist globally in time.

Firstly, for $u_0 \in H^s$, $s > \frac{3}{2}$, we will give some useful estimates for the corresponding solution u. By the first equation of (1.2) and (2.6), a direct computation implies that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{S}} u_x^2\,dx &= 2\int_{\mathbb{S}} u(-u_{txx})\,dx \\ &= 2\int_{\mathbb{S}} u\big(-\mu(u)_t - 2\mu(u)u_x + 2u_xu_{xx} + uu_{xxx} - \lambda\mu(u) + \lambda u_{xx}\big)\,dx \\ &= -2\mu(u)_t\mu(u) - 2\lambda\big(\mu(u)\big)^2 - 2\lambda\int_{\mathbb{S}} u_x^2\,dx \\ &= -2\lambda\int_{\mathbb{S}} u_x^2\,dx.\end{aligned}$$

It follows that

$$\int_{\mathbb{S}} u_x^2\,dx = \int_{\mathbb{S}} u_{0,x}^2\,dx\cdot e^{-2\lambda t} := \mu_1^2 e^{-2\lambda t}, \tag{4.1}$$

where $\mu_1 = (\int_{\mathbb{S}} u_{0,x}^2\,dx)^{\frac{1}{2}}$. Note that $\int_{\mathbb{S}}(u(t,x)-\mu(u))\,dx = \mu(u)-\mu(u) = 0$. By Lemma 3.3, we find that

$$\max_{x\in\mathbb{S}}\big[u(t,x)-\mu(u)\big]^2 \leqslant \frac{1}{12}\int_{\mathbb{S}} u_x^2(t,x)\,dx \leqslant \frac{1}{12}\mu_1^2.$$

So we have

$$\big\|u(t,\cdot)\big\|_{L^\infty} \leqslant |\mu_0| + \frac{\sqrt{3}}{6}\mu_1. \tag{4.2}$$

Lemma 4.1. *(See [6].) Let $t_0 > 0$ and $v \in C^1([0,t_0); H^2(\mathbb{R}))$. Then for every $t \in [0,t_0)$ there exists at least one point $\xi(t) \in \mathbb{R}$ with*

$$m(t) := \inf_{x\in\mathbb{R}}\big\{v_x(t,x)\big\} = v_x\big(t,\xi(t)\big),$$

and the function m is almost everywhere differentiable on $(0,t_0)$ with

$$\frac{d}{dt}m(t) = v_{tx}\big(t,\xi(t)\big) \quad \text{a.e. on } (0,t_0).$$

Theorem 4.1. *Let $u_0 \in H^s$, $s > \frac{3}{2}$, $u_0 \not\equiv c$ for $\forall c \in \mathbb{R}$ and T be the maximal time of the solution u to (1.1) with the initial data u_0. If u_0 satisfies the following condition*

$$\int_{\mathbb{S}} u_{0,x}^3\,dx < -3\lambda\mu_1^2 - \mu_1\sqrt{9\lambda^2\mu_1^2 + 2K},$$

where $K = 6|\mu_0|\mu_1^2(|\mu_0| + \frac{\sqrt{3}}{6}\mu_1)$, then the corresponding solution to (1.1) blows up in finite time.

Proof. As mentioned earlier, here we only need to show that the above theorem holds for $s = 3$. Differentiating the first equation of Eq. (2.7) with respect to x, we have

$$u_{tx} = -\frac{1}{2}u_x^2 - uu_{xx} + 2\mu_0 e^{-\lambda t}u - \lambda u_x - 2\mu_0^2 e^{-2\lambda t} - \frac{1}{2}\mu_1^2 e^{-2\lambda t}. \tag{4.3}$$

Then, it follows that

$$
\begin{aligned}
\frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx &= \int_{\mathbb{S}} 3u_x^2 u_{xt}\,dx \\
&= 3\int_{\mathbb{S}} u_x^2\left(-\frac{1}{2}u_x^2 - uu_{xx} + 2\mu_0 e^{-\lambda t}u - \lambda u_x - 2\mu_0^2 e^{-2\lambda t} - \frac{1}{2}\mu_1^2 e^{-2\lambda t}\right)dx \\
&\leqslant -\frac{3}{2}\int_{\mathbb{S}} u_x^4\,dx - 3\int_{\mathbb{S}} uu_x^2 u_{xx}\,dx + 6\mu_0 e^{-\lambda t}\int_{\mathbb{S}} uu_x^2\,dx - 3\lambda\int_{\mathbb{S}} u_x^3\,dx \\
&= -\frac{1}{2}\int_{\mathbb{S}} u_x^4\,dx + 6\mu_0 e^{-\lambda t}\int_{\mathbb{S}} uu_x^2\,dx - 3\lambda\int_{\mathbb{S}} u_x^3\,dx \\
&\leqslant -\frac{1}{2}\int_{\mathbb{S}} u_x^4\,dx - 3\lambda\int_{\mathbb{S}} u_x^3\,dx + 6|\mu_0|\mu_1^2\left(|\mu_0| + \frac{\sqrt{3}}{6}\mu_1\right) \\
&:= -\frac{1}{2}\int_{\mathbb{S}} u_x^4\,dx - 3\lambda\int_{\mathbb{S}} u_x^3\,dx + K.
\end{aligned}
$$

Using the following inequality

$$
\left|\int_{\mathbb{S}} u_x^3\,dx\right| \leqslant \left(\int_{\mathbb{S}} u_x^4\,dx\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}} u_x^2\,dx\right)^{\frac{1}{2}} \leqslant \left(\int_{\mathbb{S}} u_x^4\,dx\right)^{\frac{1}{2}}\mu_1,
$$

and letting

$$
m(t) = \int_{\mathbb{S}} u_x^3\,dx,
$$

we have

$$
\begin{aligned}
\frac{d}{dt}m(t) &\leqslant -\frac{1}{2\mu_1^2}m^2(t) - 3\lambda m(t) + K \\
&= -\frac{1}{2\mu_1^2}\left(m(t) + 3\lambda\mu_1^2 + \mu_1\sqrt{9\lambda^2\mu_1^2 + 2K}\right)\left(m(t) + 3\lambda\mu_1^2 - \mu_1\sqrt{9\lambda^2\mu_1^2 + 2K}\right).
\end{aligned}
$$

Taking $A = 3\lambda\mu_1^2$, $B = \mu_1\sqrt{9\lambda^2\mu_1^2 + 2K}$. Then, we have

$$
\frac{d}{dt}m(t) \leqslant -\frac{1}{2\mu_1^2}\big(m(t) + A + B\big)\big(m(t) + A - B\big). \tag{4.4}
$$

Note that if $m(0) < -A - B$ then $m(t) < -A - B$ for all $t \in [0, T)$. In fact, if not, since $m(t)$ is continuous on $[0, T)$, there exists a point $t_0 \in (0, T)$ such that $m(t_0) = -A - B$ and $m(t) < -A - B$, a.e. on $(0, t_0)$. By (4.4), we have

$$
\frac{dm(t)}{dt} < 0, \quad \text{a.e. } (0, t_0).
$$

Integrating this inequality, we have

$$
m(t_0) \leqslant m(0) < -A - B.
$$

This is a contradiction. From the inequality (4.4), we obtain

$$
\frac{m(0) + A + B}{m(0) + A - B}e^{\frac{B}{\mu_1^2}t} - 1 \leqslant \frac{2B}{m(t) + A - B} \leqslant 0.
$$

Since $0<\frac{m(0)+A+B}{m(0)+A-B}<1$, there exists

$$0<T\leqslant \frac{\mu_1}{\sqrt{9\lambda^2\mu_1^2+2K}}\ln\frac{m(0)+3\lambda\mu_1^2-\mu_1\sqrt{9\lambda^2\mu_1^2+2K}}{m(0)+3\lambda\mu_1^2+\mu_1\sqrt{9\lambda^2\mu_1^2+2K}},$$

such that $\lim_{t\to T}m(t)=-\infty$. On the other hand,

$$\int_{\mathbb{S}} u_x^3\,dx\geqslant \min_{x\in\mathbb{S}}u_x(t,x)\int_{\mathbb{S}} u_x^2\,dx=\min_{x\in\mathbb{S}}u_x(t,x)\cdot\mu_1^2e^{-2\lambda t}.$$

Applying Theorem 3.2, the solution u blows up in finite time. □

Theorem 4.2. *Let $u_0\in H^s$, $s>\frac{3}{2}$, and T be the maximal time of the solution u to* (1.1) *with the initial data u_0. If*

$$\min_{x\in\mathbb{S}}u_0'(x)<-\lambda-\sqrt{\lambda^2+2K},$$

with $K=2|\mu_0|(|\mu_0|+\frac{\sqrt{3}}{6}\mu_1)$, then the corresponding solution to (1.1) *blows up in finite time.*

Proof. As mentioned earlier, here we only need to show that the above theorem holds for $s=3$. Define now

$$m(t):=\min_{x\in\mathbb{S}}\big[u_x(t,x)\big],\quad t\in[0,T)$$

and let $\xi(t)\in\mathbb{S}$ be a point where this minimum is attained by using Lemma 4.1. It follows that

$$m(t)=u_x\big(t,\xi(t)\big).$$

Clearly $u_{xx}(t,\xi(t))=0$ since $u(t,\cdot)\in H^3(\mathbb{S})\subset C^2(\mathbb{S})$. Evaluating (4.3) at $(t,\xi(t))$, we obtain

$$\begin{aligned}\frac{dm(t)}{dt}&=-\frac{1}{2}m^2(t)+2\mu_0e^{-\lambda t}u\big(t,\xi(t)\big)-\lambda m(t)-2\mu_0^2e^{-2\lambda t}-\frac{1}{2}\mu_1^2e^{-2\lambda t}\\&\leqslant -\frac{1}{2}m^2(t)-\lambda m(t)+2|\mu_0|\left(|\mu_0|+\frac{\sqrt{3}}{6}\mu_1\right)\\&:=-\frac{1}{2}m^2(t)-\lambda m(t)+K\\&=-\frac{1}{2}\big(m(t)+\lambda+\sqrt{\lambda^2+2K}\big)\big(m(t)+\lambda-\sqrt{\lambda^2+2K}\big).\end{aligned}$$

Note that if $m(0)<-\lambda-\sqrt{\lambda^2+2K}$ then $m(t)<-\lambda-\sqrt{\lambda^2+2K}$ for all $t\in[0,T)$. From the above inequality we obtain

$$\frac{m(0)+\lambda+\sqrt{\lambda^2+2K}}{m(0)+\lambda-\sqrt{\lambda^2+2K}}e^{\sqrt{\lambda^2+2K}t}-1\leqslant\frac{2\sqrt{\lambda^2+2K}}{m(t)+\lambda-\sqrt{\lambda^2+2K}}\leqslant 0.$$

Since $0<\frac{m(0)+\lambda+\sqrt{\lambda^2+2K}}{m(0)+\lambda-\sqrt{\lambda^2+2K}}<1$, then there exists

$$0<T\leqslant\frac{1}{\sqrt{\lambda^2+2K}}\ln\frac{m(0)+\lambda-\sqrt{\lambda^2+2K}}{m(0)+\lambda+\sqrt{\lambda^2+2K}},$$

such that $\lim_{t\to T}m(t)=-\infty$. Theorem 3.2 implies the solution u blows up in finite time. □

5. Global existence

In this section, we will present some global existence results. Firstly, we give a useful lemma.

Given $u_0 \in H^s$ with $s > \frac{3}{2}$. Theorem 2.2 ensures the existence of a maximal $T > 0$ and a solution u to (2.7) such that

$$u = u(\cdot, u_0) \in C\big([0,T); H^s\big) \cap C^1\big([0,T); H^{s-1}\big).$$

Consider now the following initial value problem

$$\begin{cases} q_t = u(t,q), & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{5.1}$$

Lemma 5.1. *Let $u_0 \in H^s$ with $s > \frac{3}{2}$, $T > 0$ be the maximal existence time. Then Eq.* (5.1) *has a unique solution $q \in C^1([0,T) \times \mathbb{R}; \mathbb{R})$ and the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with*

$$q_x(t,x) = exp\left(\int_0^t u_x\big(s, q(s,x)\big)\,ds\right) > 0, \quad (t,x) \in [0,T) \times \mathbb{R}.$$

Moreover, with $y = \mu(u) - u_{xx}$, we have

$$y\big(t, q(t,x)\big) q_x^2 = y_0(x) e^{-\lambda t}.$$

Proof. The proof of the first conclusion is similar to the proof of Lemma 4.1 in [38], so we omit it here. By the first equation in (1.1) and Eq. (5.1), we have

$$\begin{aligned} \frac{d}{dt} y\big(t, q(t,x)\big) q_x^2 &= (y_t + y_x q_t) q_x^2 + y \cdot 2 q_x q_{xt} \\ &= (y_t + u y_x) q_x^2 + 2 y u_x q_x^2 \\ &= (y_t + u y_x + 2 y u_x) q_x^2 = -\lambda y q_x^2. \end{aligned}$$

It follows that $y(t, q(t,x)) q_x^2 = y_0(x) e^{-\lambda t}$. □

Theorem 5.1. *If $y_0(x) = \mu_0 - u_{0,xx}(x) \in H^1$ does not change sign, then the corresponding solution u of the initial value u_0 exists globally in time.*

Proof. Note that given $t \in [0,T)$, there is a $\xi(t) \in \mathbb{S}$ such that $u_x(t, \xi(t)) = 0$ by the periodicity of u to x-variable. If $y_0(x) \geqslant 0$, then Lemma 5.1 implies that $y(t,x) \geqslant 0$. For $x \in [\xi(t), \xi(t)+1]$, we have

$$\begin{aligned} -u_x(t,x) &= -\int_{\xi(t)}^x \partial_x^2 u(t,x)\,dx = \int_{\xi(t)}^x \big(y - \mu(u)\big)\,dx = \int_{\xi(t)}^x y\,dx - \mu(u)\big(x - \xi(t)\big) \\ &\leqslant \int_{\mathbb{S}} y\,dx - \mu(u)\big(x - \xi(t)\big) = \mu(u)\big(1 - x + \xi(t)\big) \leqslant |\mu_0|. \end{aligned}$$

It follows that $u_x(t,x) \geqslant -|\mu_0|$. On the other hand, if $y_0(x) \leqslant 0$, then Lemma 5.1 implies that $y(t,x) \leqslant 0$. Therefore, for $x \in [\xi(t), \xi(t)+1]$, we have

$$\begin{aligned} -u_x(t,x) &= -\int_{\xi(t)}^x \partial_x^2 u(t,x)\,dx = \int_{\xi(t)}^x \big(y - \mu(u)\big)\,dx = \int_{\xi(t)}^x y\,dx - \mu(u)\big(x - \xi(t)\big) \\ &\leqslant -\mu(u)\big(x - \xi(t)\big) \leqslant |\mu_0|. \end{aligned}$$

It follows that $u_x(t,x) \geqslant -|\mu_0|$. This completes the proof by using Theorem 3.3. □

Corollary 5.1. *If the initial value $u_0 \in H^3$ such that*

$$\|u_{0,xxx}\|_{L^2} \leqslant 2\sqrt{3}|\mu_0|,$$

then the corresponding solution u of u_0 exists globally in time.

Proof. Since u is periodic to x-variable, $\int_{\mathbb{S}} u_{0,xx}\,dx = u_{0,x}|_0^1 = 0$. Lemma 3.3 implies that

$$\|u_{0,xx}\|_{L^\infty} \leqslant \frac{\sqrt{3}}{6}\|u_{0,xxx}\|_{L^2}.$$

If $\mu_0 \geqslant 0$, then

$$y_0(x) = \mu_0 - u_{0,xx}(x) \geqslant \mu_0 - \frac{\sqrt{3}}{6}\|u_{0,xxx}\|_{L^2} \geqslant \mu_0 - |\mu_0| = 0.$$

If $\mu_0 \leqslant 0$, then

$$y_0(x) = \mu_0 - u_{0,xx}(x) \leqslant \mu_0 + \|u_{0,xx}\|_{L^\infty} \leqslant \mu_0 + \frac{\sqrt{3}}{6}\|u_{0,xxx}\|_{L^2} \leqslant \mu_0 + |\mu_0| = 0.$$

This completes the proof by using Theorem 5.1. □

Acknowledgements

This work was partially supported by NNSFC (No. 11271382), RFDP (No. 20120171110014), the key project of Sun Yat-sen University (No. c1185) and the Doctoral Scientific Research Foundation of Zhengzhou University of Light Industry. The authors would like to thank the referees for useful comments and suggestions.

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