Available online at www.sciencedirect.com
SciVerse ScienceDirect
Ann. I. H. Poincaré – AN 31 (2014) 315–338
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Global regularity for the energy-critical NLS on $\mathbb{S}^3$ ☆

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Received 23 October 2012; received in revised form 18 March 2013; accepted 21 March 2013

Available online 17 April 2013

Abstract

We establish global existence for the energy-critical nonlinear Schrödinger equation on $\mathbb{S}^3$. This follows similar lines to the work on $\mathbb{T}^3$ but requires new extinction results for linear solutions and bounds on the interaction of a Euclidean profile and a linear wave of much higher frequency that are adapted to the new geometry.

1. Introduction

We consider the question of global well-posedness for the defocusing energy-critical nonlinear Schrödinger equation on $\mathbb{S}^3$, namely

$$(i\partial_t - \Delta_{\mathbb{S}^3})u + |u|^4 u = 0. \tag{1.1}$$

The goal of this work is to apply the method introduced by Ionescu and the first author in [32,33] to the energy critical NLS on the three dimensional sphere. We therefore follow the same general lines. The main novelty in this paper is the proof of the extinction lemma for the linear flow and the bound on the interaction between a high-frequency linear wave and a low frequency profile which in the case of the sphere requires new arguments related to the different geometry.

The study of the Schrödinger equation on compact manifolds was initiated by Bourgain [11,12] for torii and systematically developed by Burq–Gérard–Tzvetkov for arbitrary compact manifolds, where the sphere appeared as a natural challenging problem, somewhat complementary to the case of the torus. More precisely, on the torus, the spectrum is badly localized, but still regular and with low multiplicity and there is a nice basis of eigenfunctions coming from the product structure; on the sphere, the spectrum is as simple as it can be, but has very high multiplicity, with eigenfunctions of different character which are in some sense as bad as can be. Informally speaking, on the sphere,

☆ B.P. was partially supported by NSF grant DMS-1142293. N.T. is partially supported by the ERC grant Dispeq.
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0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2013.03.006

the oscillations in time and in space appear as rather decoupled and have to be treated differently. We also refer to [5,6,8,11,23,26,27,30,31,37,46,47] for other works on the nonlinear Schrödinger equation in different geometries.

On the torus $\mathbb{T}^3$, Bourgain [11] proved global existence for subquintic nonlinearities. Local existence for the energy-critical problem was obtained in [29] and extended to global existence in [33]. Global existence for the defocusing problem on $\mathbb{S}^3$ was obtained for subquintic nonlinearities in [15], local existence for the quintic problem was established in [28]. In this paper, we prove global existence for the energy-critical problem, namely

Theorem 1.1. *For any $u_0 \in H^1(\mathbb{S}^3)$, there exists a unique global strong solution of (1.1) satisfying $u(0) = u_0$. In addition, if $u_0 \in H^s$ for some $s \geqslant 1$, then $u \in C(\mathbb{R} : H^s)$.*

Since the Cauchy problem is ill-posed in H^1 for superquintic nonlinearities (see [16]), this completes the local and global analysis of well-posedness in H^1. With the results in [21,31], this establishes global existence for the energy-critical problem in $\mathbb{R}^3$, $\mathbb{H}^3$ and $\mathbb{S}^3$.

For supercritical nonlinearities, classical compactness results yield global existence of weak solutions for (1.1), see e.g. [19]. Their uniqueness (and regularity) is, however an open problem. In particular, the results in [1] suggest the possibility of H^s loss of regularity for weak solutions for some $1 \leqslant s \leqslant 3/2$.

The proof of Theorem 1.1 brings together the different contributions developed in [15–18,28,29,33] which address (among other things) the subcritical nonlinear Schrödinger equation, the analysis of products of eigenfunctions, boundedness of the first iterate in the energy-critical case, global existence for large data for the energy-critical problem and global analysis of the corresponding problem in the case of the torus $\mathbb{T}^3$.

In the study of the nonlinear Schrödinger equation on a manifold, the (difficult) study of the linear flow is very important and is presumably specific to each particular setting. This is one of the major ingredients that limit the generality of the present work and we do not add in new information on that aspect (and we do rely heavily on the analysis developed in [12,16,28,29]). A "good" understanding of the linear flow should automatically yield global existence for the defocusing energy-subcritical problem, and local existence and stability for the energy-critical problem.

In addressing the global existence for the energy-critical problem on the sphere, we need to revisit the main nonlinear ingredients in [33] and reinterpret them. While we are not yet able to give a general result, even conditionally on a good linear theory, several aspects start to emerge for the key ingredients.

The first one concerns the application of the profile decomposition, which seems to hold in a very general context. To properly work, it requires an extinction argument which is provided here by Lemma 4.4. Since one already has sufficiently good Strichartz estimates, one only needs an improvement on the Sobolev inequality for the linear flow. This comes from two aspects. On the one hand, by purely elliptic considerations, one can track down when the Sobolev inequality is inefficient. Very precise estimate are available to quantify this (see e.g. [43,45]). Here, since we need to beat this inequality by a fixed but large constant, we rely on the explicit formula for the eigenprojectors, but in general, such information might follow from estimate of the Green function away from the diagonal.

Once this has been taken into account, we are left with a part of the solution that has more structure and we need to use the fact that, under the linear flow, it cannot remain concentrated for all times, which, for the moment, we can only do using some argument coming from the Euclidean Fourier transform, or from Weyl bounds, which are quite sensitive to fine properties of the spectrum. This is done here in Lemma A.1.

The second main ingredient is an understanding of the linearization of the equation around an arbitrary profile for certain initial data (the remainder in the profile decomposition). In general, we expect solutions to essentially follow the linear flow. In the Euclidean $\mathbb{R}^3$ case, this would follow from local smoothing estimates. In a compact manifold, this might follow for short time if one can get quantitative bounds on the concentration of eigenfunctions of $\Delta_{\mathbb{S}^3}$ to points, i.e. the absence of semi-classical measure concentrating on points (see e.g. [2]). Such information is provided by Lemma 2.3 which is valid for an arbitrary smooth manifold. This is then used in Lemma 5.3 to control the first iterate of the above mentioned linearization, but this latter result uses the particular localization of the spectrum on the sphere in an essential way (see also [33] for a similar arguments relying more on the "Euclidean-like" localization of the spectrum – in the sense that it forms a 3 dimensional lattice).

While the analysis in [32,33] can probably be combined with the new estimate on the linear flow in [14] to yield global existence for the defocusing energy-critical Schrödinger equation on $\mathbb{T}^4$, let us mention several other open problems with increasing (in our opinion) level of difficulty. 1) The analysis developed here might extend to the case of Zoll manifolds provided one obtains the appropriate bounds on eigenprojectors, possibly from arguments in the spirit of Lemma 2.2. 2) The analysis of the same problem in the space $\mathbb{S}^2 \times \mathbb{S}^1$ seems to require nontrivial adaptations from the arguments given in [28,33] and here, even for small initial data. This is partly due to the failure of good L^4 bilinear estimates for eigenprojectors. 3) The case of $\mathbb{S}^4$ remains a challenging open problem where new ideas seem needed due to the failure of the $L^4_{x,t}$-Strichartz estimates which implies that the second iterate is unbounded, see [15].

Another interesting case that can be addressed with a similar analysis is the energy-critical problem in the unit ball $B(0,1) \subset \mathbb{R}^3$ with Dirichlet boundary condition and *radial* data.[1] In Appendix A.2, we shall give the main modifications required to prove

Theorem 1.2. *Let $s \geqslant 1$. For any $u_0 \in H^s \cap H^1_D(B(0,1))$ radial,*[2] *there exists a unique strong solution of* (1.1)*, $u \in C(\mathbb{R} : H^s)$.*

Global existence for finite-energy solutions to the nonlinear Schrödinger equation on two dimensional domains was already obtained by Anton [3]. We also refer to [4,9,10,42] for other results in three dimensions and to [22,34,37,40] for global existence and scattering results in the exterior of the unit ball.

In Section 2, we review some notation and introduce our main spaces. In Section 3, we review the local well-posedness theory. In Section 4, we present the profile decomposition on $\mathbb{S}^3$. In Section 5, we prove Theorem 1.1. Finally in Appendix A, we prove some additional results needed in the course of the proof and give the ingredients for the proof of Theorem 1.2.

2. Notations and preliminaries

In this section we summarize our notations and collect several lemmas that are used in the rest of the paper.

Given two quantities A and B, the notation $A \lesssim B$ means that $A \leqslant CB$, with C uniform with respect to the set where A and B varies. We write $A \simeq B$ when $A \lesssim B \lesssim A$. If the constant C involved has some explicit dependency, we emphasize it by a subscript. Thus $A \lesssim_u B$ means that $A \leqslant C(u)B$ for some constant $C(u)$ depending on u.

We write $F(z) = z|z|^4$ the nonlinearity in (1.1). For $p \in \mathbb{N}^n$ a vector, we denote by $\mathfrak{O}_{p_1,\dots,p_n}(a_1,\dots,a_n)$ a $|p|$-linear expression which is a product of p_1 terms which are either equal to a_1 or its complex conjugate $\bar{a}_1$ and similarly for $p_j, a_j, 2 \leqslant j \leqslant n$.

2.1. The three sphere

We can view $\mathbb{S}^3$ as the unit sphere in the quaternion field and this endows $\mathbb{S}^3$ with a group structure with the north pole $O = (1,0,0,0)$ as the unit element. This also endows $\mathbb{S}^3 \subset \mathbb{R}^4$ with the structure of a Riemannian manifold with distance d_g which is also given by

$$d_g(P,Q) = \angle(P,Q),$$

where $\angle(P,Q)$ denotes the angle between the rays starting at the origin and passing through P and Q. For $Q \in \mathbb{S}^3$, we define R_Q to be the right multiplication by Q^{-1}. This defines an isometry of $\mathbb{S}^3$.

We can parameterize $\mathbb{S}^3$ in exponential radial coordinates $P \mapsto (\theta,\omega)$ where $\theta = d_g(O,P)$ and $\omega \in \mathbb{S}^2$. In fact we have the global mapping[3]

$$[0,\pi] \times [0,\pi] \times \mathbb{S}^1 \ni (\theta,\psi,\varphi) \mapsto (\cos\theta, \sin\theta\cos\psi, \sin\theta\sin\psi\cos\varphi, \sin\theta\sin\psi\sin\varphi).$$

In these coordinates, we have that

$$\begin{aligned}\Delta_{\mathbb{S}^3} &= \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\sin^2\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\Delta_{\mathbb{S}^2} = \frac{\partial^2}{\partial\theta^2} + \frac{2\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\Delta_{\mathbb{S}^2}\\ &= \frac{1}{\sin^2\theta}\frac{\partial}{\partial\theta}\sin^2\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta\sin\psi}\frac{\partial}{\partial\psi}\sin\psi\frac{\partial}{\partial\psi} + \frac{1}{\sin^2\theta\sin^2\psi}\frac{\partial^2}{\partial\varphi^2}.\end{aligned} \tag{2.1}$$

[1] The case of arbitrary data remains an outstanding open problem, where even the linear flow is still not satisfactorily understood [3,4,9,10,42].

[2] Here H^1_D is the completion for the H^1-norm of the smooth functions compactly supported in $B(0,1)$.

[3] Here by $\mathbb{S}^1$ we mean $[0,2\pi]$ with the endpoints identified.

In these coordinates, we also have the explicit formula for the Haar measure

$$d\nu_g = (\sin\theta)^2 \sin\psi \, d\theta d\psi d\varphi.$$

2.2. Spherical harmonics

We will consider the operator $L = -\Delta_{\mathbb{S}^3} + 1$. For $k \in \mathbb{N}^*$, we define $\mathcal{E}_k$ to be the space of $k-1$-th spherical harmonics. We have an L^2-orthonormal decomposition

$$L^2(\mathbb{S}^3) = \bigoplus_{k\in\mathbb{N}^*} \mathcal{E}_k$$

and π_k defined above is the orthogonal projection on $\mathcal{E}_k$. These satisfy that for any $\varphi \in \mathcal{E}_k$, $L\varphi = k^2\varphi$. We recall the following bounds from Sogge [44]

$$\|\pi_q f\|_{L^p(\mathbb{S}^3)} \lesssim q^{1-3/p} \|f\|_{L^2(\mathbb{S}^3)}, \quad 4 \leqslant p \leqslant \infty. \tag{2.2}$$

We then define projectors on $I \subset \mathbb{R}$ by

$$P_I = \sum_{k\in I} \pi_k, \qquad P_{\leqslant N} = \sum_{k\in\mathbb{N}} \eta\left(\frac{k}{N}\right)\pi_k, \qquad P_N = P_{\leqslant N} - P_{\leqslant N/2} = \sum_{k\in\mathbb{N}} \eta_N(k)\pi_k, \tag{2.3}$$

for $\eta \in C_c^\infty(\mathbb{R})$ such that $\eta(x) = 1$ when $|x| \leqslant 1$ and $\eta(x) = 0$ when $|x| \geqslant 2$ and where $\eta_N(x) = \eta(x/N) - \eta(2x/N)$. In particular all the sums over N below are implicitly taken to be over all dyadic integers, $N = 2^k$ for some $k \in \mathbb{N}$.

In fact, we can be more precise about the spectral projectors. We define the Zonal function of order k, $\mathbf{Z}_k$ as

$$Z_k(\theta) = k\frac{\sin(k\theta)}{\sin\theta}, \qquad \mathbf{Z}_k(P) = Z_k\big(\angle(P, O)\big), \tag{2.4}$$

where O denotes the north pole. One may directly check that these are eigenfunctions of the Laplace–Beltrami operator on $\mathbb{S}^3$ defined in (2.1). These allow to get the following classical result:

Lemma 2.1. *The spectral projection on the $k-1$-th eigenspace can be written as*

$$[\pi_k f](P) = \frac{1}{2\pi^2} \int_{\mathbb{S}^3} \mathbf{Z}_k(R_P Q) f(Q)\, d\nu_g(Q). \tag{2.5}$$

Proof. Denote, for this proof only Π_k as the operator defined by the right-hand side of (2.5). Using the symmetry

$$\mathbf{Z}_k(R_P Q) = Z_k\big(\angle(P, Q)\big) = \mathbf{Z}_k(R_Q P) = \overline{\mathbf{Z}_k(R_Q P)},$$

and the fact that since R_Q is an isometry, Π_k commutes with $\Delta_{\mathbb{S}^3}$ and we see that $L\Pi_k f = k^2 \Pi_k f$. This also shows that Π_k is self-adjoint. Therefore it is sufficient to prove that for any $g \in C^{10}(\mathbb{S}^3)$, there holds that

$$g = \sum_{k\geqslant 1} \Pi_k g. \tag{2.6}$$

Since Π_k commutes with rotations, it suffices to prove that this equality holds at the north pole O. We switch to exponential coordinates. Using Fourier analysis on $[0, \pi]$, we see that

$$\sin\theta \cdot g(\theta, \omega) = \sum_{k\geqslant 1} c_k(\omega) \sin(k\theta), \quad c_k(\omega) = \frac{2}{\pi} \int_0^\pi g(\theta, \omega) \sin(\theta) \sin(k\theta)\, d\theta.$$

In other words,

$$g(\theta, \omega) = \sum_{k\geqslant 1} c_k(\omega) \frac{\sin(k\theta)}{\sin\theta}.$$

Integrating this over $\omega \in \mathbb{S}^2$ and letting $\theta \to 0$, we find (since $c_k(\omega) \in l^1_k(k^2)$ uniformly[4] in ω) that

$$\begin{aligned} g(O) &= \lim_{\theta\to 0}\frac{1}{4\pi}\int_{\mathbb{S}^2} g(\theta,\omega)\,d\omega = \lim_{\theta\to 0}\sum_{k\geqslant 1}\frac{1}{4\pi}\int_{\mathbb{S}^2} c_k(\omega)\frac{\sin(k\theta)}{\sin\theta}\,d\omega \\ &= \sum_{k\geqslant 1}\frac{1}{4\pi}\int_{\mathbb{S}^2} kc_k(\omega)\,d\omega = \sum_{k\geqslant 1}\frac{1}{2\pi^2}\int_0^{\pi}\int_{\mathbb{S}^2} g(\theta,\omega)k\frac{\sin(k\theta)}{\sin\theta}\sin^2\theta\,d\theta\,d\omega \\ &= \sum_{k\geqslant 1}\Pi_k g(O). \end{aligned}$$

This shows (2.6) and finishes the proof. □

The spectral projectors π_q satisfy a convenient reproducing formula highlighted in [16]: for $\chi \in \mathcal{S}(\mathbb{R})$ such that $\chi(0) = 1$ and $\hat{\chi}$ supported on $[\varepsilon, 2\varepsilon]$,

$$\chi_q\pi_q = \pi_q\chi_q = \pi_q, \qquad \chi_q = \chi(\sqrt{L} - q). \tag{2.7}$$

The interest of this comes from the following description of χ_q:

Lemma 2.2. *(See [16, Lemma 2.3].) There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$, we can decompose*

$$\chi_q = qT_q + R_q, \qquad \|R_q\|_{L^2\to H^{10}} \lesssim q^{-10} \tag{2.8}$$

and there exists $\delta > 0$ such that for any $x_0 \in \mathbb{S}^3$, there exists a system of coordinates centered at x_0 such that for any $|x| \leqslant \delta$,

$$T_q f(x) = \int_{\mathbb{R}^3} e^{-iq\,d_g(x,y)} a(x,y,q) f(y)\,dy,$$

where $a(x, y, q)$ is a polynomial in $1/q$ with smooth coefficients supported on the set

$$\big\{(x,y) \in V \times V:\ |x| \leqslant \delta \ll \varepsilon/C \leqslant |y| \leqslant C\varepsilon\big\}.$$

In the study of the linearization of (2.10) at a profile, we will need the following quantitative version of the fact that quantum measures do not concentrate on points.

Lemma 2.3. *Let $N \geqslant 1$ be a dyadic number and fix $P \in \mathbb{S}^3$, then there holds that*

$$\|\mathbb{1}_{B(P,N^{-1})}\pi_q\|_{L^2\to L^2} = \big\|\pi_q[\mathbb{1}_{B(P,N^{-1})}\cdot]\big\|_{L^2\to L^2} \lesssim N^{-1/2} + q^{-2}. \tag{2.9}$$

Remark 2.4. Note that this estimate is sharp when testing against zonal harmonics of degree $p \geqslant N$. In addition, the proof holds on any compact smooth Riemannian manifold.

Proof of Lemma 2.3. This claim essentially follows from [16]. We give here the modification necessary to obtain it. It suffices to prove the second bound as the first follows by duality. Also, we may assume that $N \gg 1$.

Using (2.7) and (2.8), remarking that

$$\pi_q\mathbb{1}_{B(P,N^{-1})} = q\pi_q T_q\mathbb{1}_{B(P,N^{-1})} + \pi_q R_q\mathbb{1}_{B(P,N^{-1})}, \qquad \|\pi_q\|_{L^2\to L^2} \leqslant 1,$$

we see that it suffices to show that

$$\|T_q\mathbb{1}_{B(P,N^{-1})}\|_{L^2\to L^2} \lesssim \big(q^2N\big)^{-1/2}.$$

[4] Here we denote $l^1_k(k^2)$ the set of sequences which are summable in k for the measure $k^2\,dk$. We also denote $d\omega$ the Haar measure on $\mathbb{S}^2$.

Now, using the notation of [16, page 12], we can decompose

$$T_q = \int_{r=\delta_1}^{\delta_2} T_q^r \, dr$$

where for a finite number of charts covering $\mathbb{S}^3$ and centered at points x_k, there holds that

$$\big[1_{B(x_k,\delta)} T_q^r f\big](Q) = \int_{\mathbb{S}^2} e^{-iq\, d_g(Q, \exp_{x_k}(r\omega))} a\big(Q, \exp_{x_k}(r\omega), q\big) \kappa(r,\omega) f_r(\omega)\, d\omega,$$

$$f_r(\omega) = f\big(\exp_{x_k}(r\omega)\big),$$

where κ is a new smooth function. Applying Hölder's inequality in r, we obtain for any $Q \in B(x_k, \delta)$

$$\big|T_q(1_{B(P,N^{-1})} f)(Q)\big|^2 \lesssim N^{-1} \int_{r=\delta_1}^{\delta_2} \big|T_q^r f(Q)\big|^2 \, dr$$

since by the triangle inequality, for any x_k, we have that

$$Q \in B\big(P, N^{-1}\big), \quad d_g(x_k, Q) = r \quad \Rightarrow \quad d_g(x_k, P) - N^{-1} \leqslant r \leqslant d_g(x_k, P) + N^{-1}.$$

The result then follows from [16, Lemma 2.14] which implies that

$$q \big\| T_q^r f_r \big\|_{L^2} \lesssim \| f_r \|_{L^2}. \qquad \square$$

2.3. Linear analysis

In fact, for simplicity of notations, we will replace Eq. (1.1) by

$$(i\partial_t + L)u + |u|^4 u = 0. \tag{2.10}$$

This is completely equivalent since a solution $u(x,t)$ solves (2.10) if and only if $v(x,t) = e^{-it} u(x,t)$ solves (1.1).

For solutions of (2.10), we recall the conservation laws

$$E(u) = \frac{1}{2} \int_{\mathbb{S}^3} \left[\big|\nabla u(x)\big|^2 + \frac{1}{3} \big|u(x)\big|^6 \right] dx, \qquad M(u) = \int_{\mathbb{S}^3} \big|u(x)\big|^2 \, dx. \tag{2.11}$$

Here and below dx refers to the Haar measure on $\mathbb{S}^3$. These conserved quantities provide a uniform in time control on the H^1 norm and motivate our choice of function spaces.

Function spaces. The strong spaces are similar to the one used by Herr [28], adapting previous ideas from Herr–Tataru–Tzvetkov [29,30]. Namely

$$\|u\|_{\widetilde{X}^s(\mathbb{R})} := \left(\sum_{k \in \mathbb{N}^*} k^{2s} \big\| e^{itk^2} \pi_k u(t) \big\|_{U_t^2(L^2)}^2 \right)^{\frac{1}{2}},$$

$$\|u\|_{\widetilde{Y}^s(\mathbb{R})} := \left(\sum_{k \in \mathbb{N}^*} k^{2s} \big\| e^{itk^2} \pi_k u(t) \big\|_{V_t^2(L^2)}^2 \right)^{\frac{1}{2}}, \tag{2.12}$$

where we refer to [25,28–30,39] for a description of the spaces $U^p(L^2)$, $V^p(L^2)$ and of their properties. Note in particular that

$$\widetilde{X}^1(\mathbb{R}) \hookrightarrow \widetilde{Y}^1(\mathbb{R}) \hookrightarrow L^\infty\big(\mathbb{R}, H^1\big).$$

We denote by $U_L^p(L^2)$ the space $e^{itL} U^p(L^2)$.

For intervals $I \subset \mathbb{R}$, we define $X^s(I)$, $s \in \mathbb{R}$, in the usual way as restriction norms, thus

$$X^1(I) := \Big\{ u \in C\big(I : H^1\big)\colon \|u\|_{X^s(I)} := \sup_{J \subseteq I, |J| \leqslant 1} \Big[\inf_{v \cdot \mathbf{1}_J(t) = u \cdot \mathbf{1}_J(t)} \|v\|_{\widetilde{X}^s} \Big] < \infty \Big\}.$$

The spaces $Y^s(I)$ are defined in a similar way. The norm controlling the inhomogeneous term on an interval $I = (a, b)$ is then defined as

$$\|h\|_{N(I)} := \Bigg\| \int_a^t e^{i(t-s)L} h(s)\, ds \Bigg\|_{X^1(I)}. \tag{2.13}$$

We also need a weaker critical norm

$$\|u\|_{Z(I)} := \sum_{p \in \{p_0, p_1\}} \sup_{J \subseteq I, |J| \leqslant 1} \Big(\sum_{N = 2^k, k \in \mathbb{N}} N^{5-p/2} \big\| P_N u(t) \big\|^p_{L^p_{x,t}(\mathbb{S}^3 \times J)} \Big)^{1/p},$$
$$p_0 = 4 + 1/10, \qquad p_1 = 100. \tag{2.14}$$

This definition, in particular the choice of the exponents p_0, p_1, is motivated by the Strichartz estimates from Theorem 2.5 below. This norm is divisible and, thanks to sufficiently strong multilinear Strichartz estimates, still controls the global evolution, as will be manifest from the local theory in Section 3. Moreover, as a consequence of Corollary 2.6 below,

$$\|u\|_{Z(I)} \lesssim \|u\|_{X^1(I)},$$

thus Z is indeed a weaker norm.

Definition of solutions. Given an interval $I \subseteq \mathbb{R}$, we call $u \in C(I : H^1(\mathbb{S}^3))$ a strong solution of (2.10) if $u \in X^1(I)$ and u satisfies that for all $t, s \in I$,

$$u(t) = e^{i(t-s)L} u(s) + i \int_s^t e^{i(t-t')L} \big(u(t') \big| u(t') \big|^4 \big)\, dt'.$$

Dispersive estimates. We recall the following result from [28, Lemma 3.5].

Theorem 2.5. *If $p > 4$ then*

$$\big\| P_N e^{itL} f \big\|_{L^p_{x,t}(\mathbb{S}^3 \times [-1,1])} \lesssim_p N^{\frac{3}{2} - \frac{5}{p}} \|P_N f\|_{L^2(\mathbb{S}^3)}.$$

As a consequence of the properties of the U^p_L spaces, we have:

Corollary 2.6. *If $p > 4$ then for any dyadic integer N and any time interval I, $|I| \leqslant 1$,*

$$\|P_N u\|_{L^p_{x,t}(\mathbb{S}^3 \times I)} \lesssim N^{\frac{3}{2} - \frac{5}{p}} \|u\|_{U^p_L(I, L^2)}. \tag{2.15}$$

We will also use the following results from Herr [28].

Proposition 2.7. *(See [28, Lemma 2.5].) If $f \in L^1_t(I, H^1(\mathbb{S}^3))$ then*

$$\|f\|_{N(I)} \lesssim \sup_{\{\|v\|_{Y^{-1}(I)} \leqslant 1\}} \int_{\mathbb{S}^3 \times I} f(x,t) \overline{v(x,t)}\, dx\, dt. \tag{2.16}$$

In particular, there holds for any smooth function g that

$$\|g\|_{X^1([0,1])} \lesssim \big\| g(0) \big\|_{H^1} + \Big(\sum_N \big\| P_N (i\partial_t + L) g \big\|^2_{L^1_t([0,1], H^1)} \Big)^{\frac{1}{2}}. \tag{2.17}$$

3. Local well-posedness and stability theory

In this section we present large-data local well-posedness and stability results that allow us to connect nearby intervals of nonlinear evolution. This is essentially a modification of the results in [28]. We need the following notation

$$\|u\|_{Z'(I)} = \|u\|_{Z(I)}^{\frac{1}{2}} \|u\|_{X^1(I)}^{\frac{1}{2}}. \tag{3.1}$$

We start with the following nonlinear estimate:

Lemma 3.1. *There exists $\delta > 0$ such that if u_1, u_2, u_3 satisfy $P_{N_i} u_i = u_i$ with $N_1 \geqslant N_2 \geqslant N_3 \geqslant 1$ and $|I| \leqslant 1$, then*

$$\|u_1 u_2 u_3\|_{L^2_{x,t}(\mathbb{S}^3 \times I)} \lesssim \left(\frac{N_3}{N_1} + \frac{1}{N_2}\right)^{\delta} \|u_1\|_{Y^0(I)} \|u_2\|_{Z'(I)} \|u_3\|_{Z'(I)} \tag{3.2}$$

and, with $p_0 = 4 + 1/10$ as in (2.14),

$$\|u_1 u_2 u_3\|_{L^2_{x,t}(\mathbb{S}^3 \times I)} \lesssim N_1^{1/2-5/p_0} N_2^{1/2-5/p_0} N_3^{10/p_0-2} \|u_1\|_{Z(I)} \|u_2\|_{Z(I)} \|u_3\|_{Z(I)}. \tag{3.3}$$

Proof. Inequality (3.2) follows from interpolation between the two estimates

$$\|u_1 u_2 u_3\|_{L^2_{x,t}(\mathbb{S}^3 \times I)} \lesssim \left(\frac{N_3}{N_1} + \frac{1}{N_2}\right)^{\delta} N_2 N_3 \|u_1\|_{V^2_L(I)} \|u_2\|_{V^2_L(I)} \|u_3\|_{V^2_L(I)},$$

$$\|u_1 u_2 u_3\|_{L^2_{x,t}(\mathbb{S}^3 \times I)} \lesssim \|u_1\|_{V^2_L(I)} \big(\|u_2\|_{Z(I)} \|u_3\|_{Z(I)}\big)^{\frac{3}{5}} \big(\|u_2\|_{X^1(I)} \|u_3\|_{X^1(I)}\big)^{\frac{2}{5}}.$$

The first is taken directly from [28, Corollary 3.7], while the second follows from the following modifications of its proof. We start with the estimate

$$\|u_1 u_2 u_3\|_{L^2_{x,t}} \lesssim \big[\max\big(N_2^2/N_1, 1\big)\big]^{1/2-2/p_1} N_2^{1+\varepsilon-2/p_2} N_3^{\frac{3}{2}-\varepsilon-\frac{2}{p_3}} \|u_1\|_{U^2_L} \|u_2\|_{U^2_L} \|u_3\|_{U^2_L} \tag{3.4}$$

valid for $\varepsilon > 0$ and $4 < p_1, p_2, p_3 < +\infty$ satisfying $1/p_1 + 1/p_2 + 1/p_3 = 1/2$ which we borrow from the proof of [28, Proposition 3.6]. Independently, using Theorem 2.5 and Hölder's inequality, we obtain

$$\begin{aligned} \|u_1 u_2 u_3\|_{L^2_{x,t}} &\lesssim \|u_1\|_{L^{q_1}_{x,t}} \|u_2\|_{L^{q_2}_{x,t}} \|u_3\|_{L^{q_3}_{x,t}} \\ &\lesssim N_1^{\frac{3}{2}-\frac{5}{q_1}} N_2^{\frac{1}{2}-\frac{5}{q_2}} N_3^{\frac{1}{2}-\frac{5}{q_3}} \|u_1\|_{U^{q_1}_L} \big(N_2^{\frac{5}{q_2}-\frac{1}{2}} \|u_2\|_{L^{q_2}_{x,t}}\big) \big(N_3^{\frac{5}{q_3}-\frac{1}{2}} \|u_3\|_{L^{q_3}_{x,t}}\big) \end{aligned} \tag{3.5}$$

where $4 < q_1, q_2, q_3 < +\infty$ satisfy $1/q_1 + 1/q_2 + 1/q_3 = 1/2$.

In the case $N_1 \leqslant N_2^2$, we may choose

$$p_1 = 40, \qquad q_1 = q_2 = p_2 = 25/6, \qquad p_3 = 200/47, \qquad q_3 = 50, \qquad \varepsilon = 1/100$$

and apply [28, Lemma 2.4].

In the case $N_2^2 \leqslant N_1$, we use (3.4) with the same exponents, while (3.5) is replaced by

$$\begin{aligned} \|u_1 u_2 u_3\|_{L^2_{x,t}} &\lesssim \|u_1\|_{L^\infty_t L^2_x} \|u_2\|_{L^{p_2}_t L^\infty_x} \|u_3\|_{L^{p_2}_t L^\infty_x} \\ &\lesssim (N_2 N_3)^{\frac{3}{p_2}} \|u_1\|_{L^\infty_t L^2_x} \|u_2\|_{L^{p_2}_{x,t}} \|u_3\|_{L^{p_2}_{x,t}} \\ &\lesssim (N_2 N_3)^{\frac{1}{2}-\frac{2}{p_2}} \|u_1\|_{U^4_L} \|u_2\|_Z \|u_3\|_Z \end{aligned}$$

and we apply again [28, Lemma 2.4].

Finally, (3.3) follows from (3.5) with $q_1 = q_2 = p_0$ and $q_3 = 20 p_0$. □

From here on, we have an estimate formally identical to the nonlinear estimate in [33, Lemma 3.1] and the following lemma and propositions are proved using straightforward adaptation from [33, Section 3] (see also [32]).

Lemma 3.2. *For $u_k \in X^1(I)$, $k = 1\dots5$, $|I| \leqslant 1$, the estimate*

$$\left\| \prod_{i=1}^{5} \tilde{u}_k \right\|_{N(I)} \lesssim \sum_{\sigma \in \mathfrak{S}_5} \|u_{\sigma(1)}\|_{X^1(I)} \prod_{j \geqslant 2} \|u_{\sigma(j)}\|_{Z'(I)}$$

holds true, where $\tilde{u}_k \in \{u_k, \overline{u_k}\}$. In fact, we have that

$$\left\| \sum_{B \geqslant 1} P_B \tilde{u}_1 \prod_{j=2}^{5} P_{\leqslant DB} \tilde{u}_j \right\|_{N(I)} \lesssim_D \|u_1\|_{X^1(I)} \prod_{j=2}^{5} \|u_j\|_{Z'(I)}. \tag{3.6}$$

We have a local existence result:

Proposition 3.3 *(Local well-posedness).*

(i) *Given $E > 0$, there exists $\delta_0 = \delta_0(E) > 0$ such that if $\|\phi\|_{H^1(\mathbb{S}^3)} \leqslant E$ and*

$$\|e^{itL}\phi\|_{Z(I)} \leqslant \delta_0$$

on some interval $I \ni 0$, $|I| \leqslant 1$, then there exists a unique solution $u \in X^1(I)$ of (2.10) satisfying $u(0) = \phi$. Besides

$$\|u - e^{itL}\phi\|_{X^1(I)} \lesssim_E \|e^{itL}\phi\|_{Z(I)}^{3/2}.$$

The quantities $E(u)$ and $M(u)$ defined in (2.11) are conserved on I.

(ii) *If $u \in X^1(I)$ is a solution of (2.10) on some open interval I and*

$$\|u\|_{Z(I)} < +\infty$$

then u can be extended as a nonlinear solution to a neighborhood of $\bar{I}$ and

$$\|u\|_{X^1(I)} \leqslant C\big(E(u), \|u\|_{Z(I)}\big)$$

for some constant C depending on $E(u)$ and $\|u\|_{Z(I)}$.

The main result in this section is the following:

Proposition 3.4 *(Stability). Assume I is an open bounded interval, $\rho \in [-1, 1]$, and $\widetilde{u} \in X^1(I)$ satisfies the approximate Schrödinger equation*

$$(i\partial_t + L)\widetilde{u} + \rho\widetilde{u}|\widetilde{u}|^4 = e \quad \text{on } \mathbb{S}^3 \times I. \tag{3.7}$$

Assume in addition that

$$\|\widetilde{u}\|_{Z(I)} + \|\widetilde{u}\|_{L_t^\infty(I, H^1(\mathbb{S}^3))} \leqslant M, \tag{3.8}$$

for some $M \in [1, \infty)$. Assume $t_0 \in I$ and $u_0 \in H^1(\mathbb{S}^3)$ is such that the smallness condition

$$\|u_0 - \widetilde{u}(t_0)\|_{H^1(\mathbb{S}^3)} + \|e\|_{N(I)} \leqslant \epsilon \tag{3.9}$$

holds for some $0 < \epsilon < \epsilon_1$, where $\epsilon_1 \leqslant 1$ is a small constant $\epsilon_1 = \epsilon_1(M) > 0$.

Then there exists a strong solution $u \in X^1(I)$ of the Schrödinger equation

$$(i\partial_t + L)u + \rho u|u|^4 = 0 \tag{3.10}$$

such that $u(t_0) = u_0$ and

$$\begin{aligned} &\|u\|_{X^1(I)} + \|\widetilde{u}\|_{X^1(I)} \leqslant C(M), \\ &\|u - \widetilde{u}\|_{X^1(I)} \leqslant C(M)\epsilon. \end{aligned} \tag{3.11}$$

4. Profiles

4.1. Analysis of Euclidean profiles

In this section we prove precise estimates showing how to compare Euclidean and spherical solutions of both linear and nonlinear Schrödinger equations. Of course, such a comparison is only meaningful in the case of rescaled data that concentrate at a point. We follow closely the arguments in [32,31], the main novelty being in Lemma 4.4.

Recall η defined[5] in (2.3). Given $\phi \in \dot{H}^1(\mathbb{R}^3)$ and a real number $N \geqslant 1$ we define

$$T_N\phi = f_N \in H^1(\mathbb{S}^3), \qquad f_N(y) = N^{\frac{1}{2}}\eta\big(N^{1/2}d_g(O,y)\big)\phi\big(N\exp_O^{-1}(y)\big) \tag{4.1}$$

and observe that

$$T_N : \dot{H}^1(\mathbb{R}^3) \to H^1(\mathbb{S}^3) \text{ is a linear operator with } \|T_N\phi\|_{H^1(\mathbb{S}^3)} \lesssim \|\phi\|_{\dot{H}^1(\mathbb{R}^3)}$$

and that

$$\|T_N\phi\|_{L^1} \lesssim N^{-\frac{5}{2}}\|\phi\|_{L^1}, \qquad \|T_N\phi\|_{L^2} \lesssim N^{-1}\|\phi\|_{L^2}.$$

We define also

$$E_{\mathbb{R}^3}(\phi) = \frac{1}{2}\int_{\mathbb{R}^3}\left[|\nabla_{\mathbb{R}^3}\phi|^2 + \frac{1}{3}|\phi|^6\right]dx.$$

We will use the main theorem of [21] (see also [36] and [13,24,38] for previous results), in the following form.

Theorem 4.1. *Assume $\psi \in \dot{H}^1(\mathbb{R}^3)$. Then there is a unique global solution $v \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^3))$ of the initial-value problem*

$$(i\partial_t - \Delta_{\mathbb{R}^3})v + v|v|^4 = 0, \qquad v(0) = \psi, \tag{4.2}$$

and

$$\|v\|_{L_t^4L_x^\infty(\mathbb{R}^3\times\mathbb{R}))} + \|\nabla_{\mathbb{R}^3}v\|_{(L_t^\infty L_x^2 \cap L_t^2L_x^6)(\mathbb{R}^3\times\mathbb{R})} \leqslant \widetilde{C}\big(E_{\mathbb{R}^3}(\psi)\big). \tag{4.3}$$

Moreover this solution scatters in the sense that there exists $\psi^{\pm\infty} \in \dot{H}^1(\mathbb{R}^3)$ such that

$$\big\|v(t) - e^{-it\Delta}\psi^{\pm\infty}\big\|_{\dot{H}^1(\mathbb{R}^3)} \to 0 \tag{4.4}$$

as $t \to \pm\infty$. Besides, if $\psi \in H^5(\mathbb{R}^3)$ then $v \in C(\mathbb{R} : H^5(\mathbb{R}^3))$ and

$$\sup_{t\in\mathbb{R}}\big\|v(t)\big\|_{H^5(\mathbb{R}^3)} \lesssim_{\|\psi\|_{H^5(\mathbb{R}^3)}} 1.$$

Again, we emphasize that this extends readily to the case when $-\Delta_{\mathbb{R}^3}$ is replaced by $1 - \Delta_{\mathbb{R}^3}$.

Our first result in this section is the following lemma:

Lemma 4.2. *Assume $\phi \in \dot{H}^1(\mathbb{R}^3)$, $T_0 \in (0,\infty)$, and $\rho \in \{0,1\}$ are given, and define f_N as in (4.1). Then the following conclusions hold:*

(i) *There is $N_0 = N_0(\phi, T_0)$ sufficiently large such that for any $N \geqslant N_0$ there is a unique solution $U_N \in C((-T_0N^{-2}, T_0N^{-2}) : H^1(\mathbb{S}^3))$ of the initial-value problem*

$$(i\partial_t - \Delta + 1)U_N = \rho U_N|U_N|^4, \qquad U_N(0) = f_N. \tag{4.5}$$

[5] The role of η is to avoid "tail" effects coming from the fact that ϕ might not vanish outside of $B(0,R)$ for any R.

(ii) *Assume* $\varepsilon_1 \in (0,1]$ *is sufficiently small (depending only on* $E_{\mathbb{R}^3}(\phi)$*),* $\phi' \in H^5(\mathbb{R}^3)$*, and* $\|\phi - \phi'\|_{\dot H^1(\mathbb{R}^3)} \leqslant \varepsilon_1$*. Let* $v' \in C(\mathbb{R} : H^5(\mathbb{R}^3))$ *denote the solution of the initial-value problem*

$$(i\partial_t - \Delta_{\mathbb{R}^3} + 1)v' = \rho v' \big|v'\big|^4, \qquad v'(0) = \phi'.$$

For $R, N \geqslant 1$ *we define*

$$\begin{aligned} &v'_R(x,t) = \eta\big(|x|/R\big)v'(x,t), \quad (x,t) \in \mathbb{R}^3 \times (-T_0, T_0),\\ &v'_{R,N}(x,t) = N^{\frac12} v'_R\big(Nx, N^2 t\big), \quad (x,t) \in \mathbb{R}^3 \times \big(-T_0N^{-2}, T_0N^{-2}\big),\\ &V_{R,N}(y,t) = v'_{R,N}\big(\exp_O^{-1}(y), t\big) \quad (y,t) \in \mathbb{S}^3 \times \big(-T_0N^{-2}, T_0N^{-2}\big). \end{aligned} \tag{4.6}$$

Then there is $R_0 \geqslant 1$ *(depending on* T_0 *and* ϕ' *and* ε_1*) such that, for any* $R \geqslant R_0$*,*

$$\limsup_{N\to\infty} \|U_N - V_{R,N}\|_{X^1(-T_0N^{-2}, T_0N^{-2})} \lesssim_{E_{\mathbb{R}^3}(\phi)} \varepsilon_1. \tag{4.7}$$

In particular, for any $N \geqslant N_0$*,*

$$\|U_N\|_{X^1(-T_0N^{-2}, T_0N^{-2})} \lesssim_{E_{\mathbb{R}^3}(\phi)} 1. \tag{4.8}$$

Remark 4.3. As is shown in [16, Appendix A] (see also [20]), for times $0 \leqslant t \ll N^{-2}$, the effect of the dispersion is weak and a good approximation for (2.10) is the simple *ODE*

$$i\partial_t u = u|u|^4 - u.$$

This lemma shows how to take into account the effect of the dispersion on the interval $[N^{-2}, TN^{-2}]$ for T large, so as to complement the conclusion of Lemma 4.4 below.

Proof of Lemma 4.2. In fact, we show that $V_{R,N}$ in (ii) gives such a good ansatz that we can apply the stability Proposition 3.4 and obtain (4.7), which in particular implies (i). All of the constants in this proof are allowed to depend on $E_{\mathbb{R}^3}(\phi)$. Using Theorem 4.1

$$\begin{aligned} &\big\|v'\big\|_{L^4_tL^\infty_x(\mathbb{R}\times\mathbb{R}^3)} + \big\|\nabla_{\mathbb{R}^3} v'\big\|_{(L^\infty_t L^2_x \cap L^2_t L^6_x)(\mathbb{R}^3\times\mathbb{R})} \lesssim 1,\\ &\sup_{t\in\mathbb{R}} \big\|v'(t)\big\|_{H^5(\mathbb{R}^3)} \lesssim_{\|\phi'\|_{H^5(\mathbb{R}^3)}} 1. \end{aligned} \tag{4.9}$$

Let

$$\begin{aligned} e_R(x,t) := {}&\big[(i\partial_t - \Delta_{\mathbb{R}^3} + 1)v'_R - \rho v'_R \big|v'_R\big|^4\big](x,t) = \rho\big(\eta\big(|x|/R\big) - \eta\big(|x|/R\big)^5\big)v'(x,t)\big|v'(x,t)\big|^4\\ &- R^{-2}v'(x,t)\eta''\big(|x|/R\big) - 2R^{-1}|x|^{-1}v'(x,t)\eta'\big(|x|/R\big) - 2R^{-1}\sum_{j=1}^4 \partial_r v'(x,t)\eta'\big(|x|/R\big). \end{aligned}$$

Since $|v'(x,t)| \lesssim_{\|\phi'\|_{H^5(\mathbb{R}^3)}} 1$, see (4.9), it follows that

$$\big|e_R(x,t)\big| + \sum_{k=1}^3 \big|\partial_k e_R(x,t)\big| \lesssim_{\|\phi'\|_{H^5(\mathbb{R}^3)}} \mathbf{1}_{[R,2R]}\big(|x|\big)\cdot\Big[\big|v'(x,t)\big| + \sum_{k=1}^3 \big|\partial_k v'(x,t)\big| + \sum_{k,j=1}^3 \big|\partial_k\partial_j v'(x,t)\big|\Big].$$

Therefore

$$\lim_{R\to\infty} \big\||e_R| + |\nabla_{\mathbb{R}^3} e_R|\big\|_{L^\infty_t L^2_x(\mathbb{R}^3\times(-T_0,T_0))} = 0. \tag{4.10}$$

Letting

$$e_{R,N}(x,t) := \big[(i\partial_t - \Delta_{\mathbb{R}^3} + 1)v'_{R,N} - \rho v'_{R,N}\big|v'_{R,N}\big|^4\big](x,t) = N^{\frac52} e_R\big(Nx, N^2t\big),$$

it follows from (4.10) that there is $R_0 \geqslant 1$ such that, for any $R \geqslant R_0$ and $N \geqslant 1$,

$$\big\| |e_{R,N}| + |\nabla_{\mathbb{R}^3} e_{R,N}| \big\|_{L^1_t L^2_x(\mathbb{R}^3\times(-T_0N^{-2},T_0N^{-2}))} \leqslant \varepsilon_1. \tag{4.11}$$

With $V_{R,N}(y,t) = v'_{R,N}(\exp_O^{-1}(y),t)$ as in (4.6) and $N \geqslant 10R$, let

$$\begin{aligned} E_{R,N}(y,t) &:= \big[(i\partial_t + L)V_{R,N} - \rho V_{R,N}|V_{R,N}|^4\big](y,t) \\ &= e_{R,N}\big(\exp_O^{-1}(y),t\big) + 2(1/\phi - 1/\sin\phi)\big(\partial_\phi v'_{R,N}\big)\big(\exp_O^{-1}(y),t\big) \\ &\quad + \big(1/\phi^2 - 1/\sin^2\phi\big)\big(\Delta_{\mathbb{S}^2} v'_{R,N}\big)\big(\exp_O^{-1}(y),t\big) \end{aligned} \tag{4.12}$$

where we have used the formula in (2.1). We remark that

$$\begin{aligned} &\big\|\phi\partial_\phi v'_{R,N}\big(\exp_O^{-1}(y),t\big)\big\|_{L^1_tL^2_x} + \big\|\phi\nabla\big(\partial_\phi v'_{R,N}\big)\big(\exp_O^{-1}(y),t\big)\big\|_{L^1_tL^2_x} \lesssim_{R,T} N^{-2}, \\ &\big\|\Delta_{\mathbb{S}^2} v'_{R,N}\big(\exp_O^{-1}(y),t\big)\big\|_{L^1_tL^2_x} + \big\|\nabla\big(\Delta_{\mathbb{S}^2} v'_{R,N}\big)\big(\exp_O^{-1}(y),t\big)\big\|_{L^1_tL^2_x} \lesssim_{R,T} N^{-2}. \end{aligned}$$

Using (4.11), it follows that for any R_0 sufficiently large there is N_0 such that for any $N \geqslant N_0$

$$\big\| |\nabla^1 E_{R_0,N}| \big\|_{L^1_tL^2_x(\mathbb{S}^3\times(-T_0N^{-2},T_0N^{-2}))} \leqslant 2\varepsilon_1. \tag{4.13}$$

To verify the hypothesis (3.8) of Proposition 3.4, we estimate for N large enough, using (4.9)

$$\sup_{t\in(-T_0N^{-2},T_0N^{-2})} \big\|V_{R_0,N}(t)\big\|_{H^1(\mathbb{S}^3)} \leqslant \sup_{t\in(-T_0N^{-2},T_0N^{-2})} \big\|v'_{R_0,N}(t)\big\|_{H^1(\mathbb{R}^3)} \lesssim 1 \tag{4.14}$$

and using (2.17), (4.13) and

$$\big\|V_{R,N}|V_{R,N}|^4\big\|_{L^1_tH^1} \lesssim \|v'\|^4_{L^4L^\infty_x}\|v'\|_{L^\infty_tH^1_x} \lesssim 1$$

we obtain that

$$\|V_{R,N}\|_{X^1} \lesssim 1.$$

Finally, to verify the inequality on the first term in (3.9) we estimate, for R_0, N large enough,

$$\begin{aligned} \big\|f_N - V_{R_0,N}(0)\big\|_{H^1(\mathbb{S}^3)} &\lesssim \big\|\phi_N - v'_{R_0,N}(0)\big\|_{\dot H^1(\mathbb{R}^3)} \lesssim \big\|\eta\big(N^{\frac12}\cdot\big)\phi - v'_{R_0}(0)\big\|_{\dot H^1(\mathbb{R}^3)} \\ &\lesssim \big\|\big(1-\eta\big(N^{\frac12}\cdot\big)\big)\phi\big\|_{\dot H^1(\mathbb{R}^3)} + \|\phi-\phi'\|_{\dot H^1(\mathbb{R}^3)} + \big\|\phi' - v'_{R_0}(0)\big\|_{\dot H^1(\mathbb{R}^3)} \\ &\lesssim \varepsilon_1. \end{aligned} \tag{4.15}$$

The conclusion of the lemma follows from Proposition 3.4, provided that ε_1 is fixed sufficiently small depending on $E_{\mathbb{R}^3}(\phi)$. □

To understand linear and nonlinear evolutions beyond the Euclidean window we need an additional extinction lemma:

Lemma 4.4. *Let $\phi \in \dot H^1(\mathbb{R}^3)$ and define f_N as in (4.1). For any $\varepsilon > 0$, there exist $T = T(\phi,\varepsilon)$ and $N_0(\phi,\varepsilon)$ such that for all $N \geqslant N_0$, there holds that*

$$\big\|e^{itL}f_N\big\|_{Z(TN^{-2},T^{-1})} \lesssim \varepsilon. \tag{4.16}$$

Remark 4.5. Note that the analysis in [15] already gives the result on an interval of time of the form $[TN^{-2}, N^{-1}]$. However for our application, it is important to obtain an upper bound independent of N.

Proof of Lemma 4.4. Using Strichartz estimates and interpolation, we see that it suffices to obtain this for $p = \infty$ in the definition of Z, i.e.

$$\sup_M M^{-\frac12}\big\|P_Me^{itL}f_N\big\|_{L^\infty_{x,t}(\mathbb{S}^3\times[TN^{-2},T^{-1}])} \lesssim \varepsilon.$$

Fix $\varphi \in C_c^\infty(\mathbb{R}^3)$ such that

$$\|\phi - \varphi\|_{\dot{H}^1(\mathbb{R}^3)} \leqslant \varepsilon^2.$$

From the boundedness of T_N in (4.1), we deduce that it suffices to prove that

$$\sup_M M^{-1/2} \| P_M e^{itL} \varphi_N \|_{L^\infty_{x,t}} \leqslant \varepsilon, \quad \varphi_N = T_N \varphi.$$

Let $Q = R^2 + \varepsilon^{-2}$, where R is the diameter of the support of φ. Using Bernstein estimate, we observe that

$$M^{-\frac{1}{2}} \| P_M e^{itL} \varphi_N \|_{L^\infty_{x,t}} \lesssim M \| P_M e^{itL} \varphi_N \|_{L^\infty_t L^2_x} \lesssim \min\left(\frac{M}{N}, \left(\frac{N}{M}\right)^{10}\right). \tag{4.17}$$

Thus, if $(M/N) \notin (Q^{-1}, Q)$, (4.16) holds. From now on, we assume that

$$Q^{-1} \leqslant M/N \leqslant Q.$$

We define

$$c_p(x) = [\pi_p \varphi_N](x).$$

This decouples the oscillations in time and the variations in space as follows:

$$P_M e^{itL} \varphi_N(x) = \sum_{p \leqslant 2M} \eta_M(p) e^{itp^2} c_p(x). \tag{4.18}$$

We consider two cases.

Case 1. When $d_g(O, x) \geqslant Q^6/N$. In this case, we can use the explicit formula (2.4) to get that the function is far from saturating Sobolev inequality

$$\sum_{M \leqslant p \leqslant 2M} \left|\pi_p(\varphi_N)(x)\right| \lesssim \varepsilon N^{\frac{1}{2}}. \tag{4.19}$$

From the formula (2.5) and the fact that in our case, for any Y in the support of φ_N, $\angle(Y, x) \geqslant Q^5/N$ we obtain that

$$\left|\pi_p(\varphi_N)(x)\right| \lesssim \|\varphi_N\|_{L^1} p\big(N/Q^5\big) \lesssim \varepsilon^2 Q^{-4} N^{-\frac{3}{2}} p.$$

Summing crudely over all $p \leqslant 2M$, we obtain that

$$\left| \sum_{p \leqslant 2M} \eta_M(p) e^{-itp^2} c_p(x) \right| \lesssim N^{-\frac{3}{2}} Q^{-4} \sum_{p \leqslant 2M} \varepsilon^2 p \leqslant \varepsilon N^{\frac{1}{2}},$$

which gives (4.16) in this case.

Case 2. When $d_g(O, x) \leqslant 2Q^6/N$. In this case, we claim that, uniformly in p, $d_g(O, x)$, there holds that

$$\begin{aligned} &\left|c_p(x)\right| \lesssim_\varphi Q^{10} N^{-\frac{1}{2}}, \\ &\left|c_p(x) - c_{p-1}(x)\right| \lesssim_\varphi Q^{10} N^{-\frac{3}{2}}, \\ &\left|c_p(x) - 2c_{p-1}(x) + c_{p-2}(x)\right| \lesssim_\varphi Q^{10} N^{-\frac{5}{2}}. \end{aligned} \tag{4.20}$$

This follows from the explicit formulas

$$c_p(Q) = \int_{\mathbb{S}^3} \mathbf{Z}_p(R_Q P)\varphi_N(P)\,d\nu_g(P),$$

$$c_p(Q) - c_{p-1}(Q) = \int_{\mathbb{S}^3} \mathbf{Z}_p^d(R_Q P)\varphi_N(P)\,d\nu_g(P),$$

$$c_p(Q) - 2c_{p-1}(Q) + c_{p-2}(Q) = \int_{\mathbb{S}^3} \mathbf{Z}_p^{dd}(R_Q P)\varphi_N(P)\,d\nu_g(P),$$

where

$$\begin{aligned}
\left|Z_p(\theta)\right| &= p\frac{|\sin(p\theta)|}{\sin\theta} \lesssim p^2,\\
\left|Z_p^d(\theta)\right| &= p\left|\sin(p\theta)\frac{1-\cos\theta}{\sin\theta} + \cos(p\theta) + \frac{\sin((p-1)\theta)}{p\sin\theta}\right| \lesssim p(1+p\theta),\\
\left|Z_p^{dd}(\theta)\right| &= (p-1)\left|\frac{\sin(p\theta)}{\sin\theta}\big[1-2\cos\theta+\cos(2\theta)\big]\right.\\
&\quad \left. + \cos(p\theta)\left[2 - \frac{\sin(2\theta)}{\sin\theta}\right] + \frac{2}{p-1}\frac{\cos\theta - \cos(2\theta)}{\sin\theta} + \frac{\sin p\theta}{p\sin\theta}\big[1-\cos(2\theta)\big]\right|\\
&\lesssim p^2\theta^2 + \theta.
\end{aligned}$$

We may now use (4.18), (4.20) together with Lemma A.1 (with $K = Q^{10}N^{-\frac{1}{2}}$) to find an acceptable T as in (4.16). More precisely, we fix $T_0 \geqslant \varepsilon^{-3}$, which forces either $(a,q) = (0,1)$ or $q \geqslant \varepsilon^{-2}$ and then choose $T \geqslant T_0$ in such a way as to satisfy (4.16). □

In the process, we have seen from (4.17), (4.19) and the end of the proof above that if $\varphi \in C_c^\infty(\mathbb{R}^3)$, then, for any ε, there exist $T_0 > 0$ and N_0 such that, whenever $T \geqslant T_0$ and $N \geqslant N_0$, there holds that

$$\sum_{M\geqslant 1} M^{-1/2}\left\|e^{itL}P_M(T_N\varphi)\right\|_{L^\infty(\mathbb{S}^3\times(TN^{-2},T^{-1}))} \lesssim \varepsilon. \tag{4.21}$$

We conclude this section with a proposition describing nonlinear solutions of the initial-value problem (2.10) corresponding to data concentrating at a point. In view of the profile analysis in the next section, we need to consider slightly more general data. Given $f \in L^2(\mathbb{S}^3)$, $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{S}^3$ we define

$$(\Pi_{t_0,x_0})f(x) = \big(e^{-it_0L}\tau_{x_0}f\big)(x),$$

where $\tau_{x_0}f(x) = f(R_{x_0}x)$.

Let $\widetilde{\mathcal{F}}_e$ denote the set of renormalized Euclidean frames[6]

$$\begin{aligned}
\widetilde{\mathcal{F}}_e := \Big\{&(N_k,t_k,x_k)_{k\geqslant 1}\colon\ N_k \in [1,+\infty),\ t_k \to 0,\ x_k \in \mathbb{S}^3,\ N_k \to +\infty,\\
&\text{and either } t_k = 0 \text{ for any } k \geqslant 1 \text{ or } \lim_{k\to\infty} N_k^2|t_k| = +\infty\Big\}.
\end{aligned} \tag{4.22}$$

Proposition 4.6. *Assume that $\mathcal{O} = (N_k,t_k,x_k)_k \in \widetilde{\mathcal{F}}_e$, $\phi \in \dot{H}^1(\mathbb{S}^3)$, and let $U_k(0) = \Pi_{t_k,x_k}(T_{N_k}\phi)$.*

(i) *There exists $\tau = \tau(\phi)$ such that for k large enough (depending only on ϕ and $\mathcal{O}$) there is a nonlinear solution $U_k \in X^1(-\tau,\tau)$ of Eq. (2.10) with initial data $U_k(0)$, and*

$$\|U_k\|_{X^1(-\tau,\tau)} \lesssim_{E_{\mathbb{R}^3}(\phi)} 1. \tag{4.23}$$

[6] We will later consider a slightly more general class of frames, called *Euclidean frames*, see Definition 4.7. For our later application, it suffices to prove Proposition 4.6 under the stronger assumption that $\mathcal{O}$ is a renormalized Euclidean frame.

(ii) *There exists a Euclidean solution* $u \in C(\mathbb{R} : \dot{H}^1(\mathbb{R}^3))$ *of*

$$(i\partial_t - \Delta_{\mathbb{R}^3} + 1)u + u|u|^4 = 0 \tag{4.24}$$

with scattering data $\phi^{\pm\infty}$ *defined as in* (4.4) *such that the following holds, up to a subsequence: for any* $\varepsilon > 0$, *there exists* $T(\phi, \varepsilon)$ *such that for all* $T \geqslant T(\phi, \varepsilon)$ *there exists* $R(\phi, \varepsilon, T)$ *such that for all* $R \geqslant R(\phi, \varepsilon, T)$, *there holds that*

$$\|U_k - \tilde{u}_k\|_{X^1(\{|t-t_k| \leqslant TN_k^{-2}\} \cap \{|t| \leqslant T^{-1}\})} \leqslant \varepsilon, \tag{4.25}$$

for k large enough, where

$$\tilde{u}_k(x,t) = N_k^{\frac{1}{2}} \eta\big(N_k d_g(x_k, x)/R\big) u\big(N_k \exp_{x_k}^{-1}(x), N_k^2(t - t_k)\big).$$

In addition, up to a subsequence,[7]

$$\big\|U_k(t) - \Pi_{t_k - t, x_k} T_{N_k} \phi^{\pm\infty}\big\|_{X^1(\{\pm(t-t_k) \geqslant TN_k^{-2}\} \cap \{|t| \leqslant T^{-1}\})} \leqslant \varepsilon, \tag{4.26}$$

for k large enough (*depending on* ϕ, ε, T, R).

Proof. This follows from minor adaptation of the proof in [33, Proposition 4.4]. Here Lemma 4.4 is used in an essential way. □

4.2. Profile decomposition

In this section we show that given a bounded sequence of functions $f_k \in H^1(\mathbb{S}^3)$ we can construct suitable *profiles* and express the sequence in terms of these profiles. The statements and the arguments in this section are very similar to those in [33, Section 5]. See also [32,31,35] for the original proofs of Keraani in the Euclidean geometry and [7,41] for earlier results.

The following is our main definition.

Definition 4.7.

(1) We define a Euclidean frame to be a sequence $\mathcal{F}_e = (N_k, t_k, x_k)_k$ with $N_k \geqslant 1$, $N_k \to +\infty$, $t_k \in \mathbb{R}$, $t_k \to 0$, $x_k \in \mathbb{S}^3$. We say that two frames $(N_k, t_k, x_k)_k$ and $(M_k, s_k, y_k)_k$ are orthogonal if

$$\lim_{k \to +\infty} \left(\left| \ln \frac{N_k}{M_k} \right| + N_k^2 |t_k - s_k| + N_k d_g(x_k, y_k) \right) = +\infty.$$

Two frames that are not orthogonal are called equivalent.

(2) If $\mathcal{O} = (N_k, t_k, x_k)_k$ is a Euclidean frame and if $\phi \in \dot{H}^1(\mathbb{R}^3)$, we define the Euclidean profile associated to $(\phi, \mathcal{O})$ as the sequence $\widetilde{\phi}_{\mathcal{O}_k}$

$$\widetilde{\phi}_{\mathcal{O}_k}(x) := \Pi_{t_k, x_k}(T_{N_k} \phi).$$

The following lemma summarizes some of the basic properties of profiles associated to equivalent/orthogonal frames. Its proof uses Lemma 4.2 with $\rho = 0$ to control linear evolutions inside the Euclidean window and Lemma 4.4 to control these evolutions outside such a window. Given these ingredients, the proof of Lemma 4.8 is very similar to the proof of Lemma 5.4 in [31], and is omitted.

[7] The definition of T_N is given in (4.1).

Lemma 4.8 *(Equivalence of frames).*

(i) *If $\mathcal{O}$ and $\mathcal{O}'$ are equivalent Euclidean profiles, then there exists an isometry $T : \dot{H}^1(\mathbb{R}^3) \to \dot{H}^1(\mathbb{R}^3)$ such that for any profile $\widetilde{\psi}_{\mathcal{O}'_k}$, up to a subsequence there holds that*

$$\limsup_{k\to+\infty} \|\widetilde{T\psi}_{\mathcal{O}_k} - \widetilde{\psi}_{\mathcal{O}'_k}\|_{H^1(\mathbb{S}^3)} = 0. \tag{4.27}$$

(ii) *If $\mathcal{O}$ and $\mathcal{O}'$ are orthogonal frames and $\widetilde{\psi}_{\mathcal{O}_k}$, $\widetilde{\varphi}_{\mathcal{O}'_k}$ are corresponding profiles, then, up to a subsequence,*

$$\lim_{k\to+\infty} \langle \widetilde{\psi}_{\mathcal{O}_k}, \widetilde{\varphi}_{\mathcal{O}'_k}\rangle_{H^1\times H^1(\mathbb{S}^3)} = 0,$$
$$\lim_{k\to+\infty} \big\langle |\widetilde{\psi}_{\mathcal{O}_k}|^3, |\widetilde{\varphi}_{\mathcal{O}'_k}|^3\big\rangle_{L^2\times L^2(\mathbb{S}^3)} = 0.$$

(iii) *If $\mathcal{O}$ is a Euclidean frame and $\widetilde{\psi}_{\mathcal{O}_k}$, $\widetilde{\varphi}_{\mathcal{O}_k}$ are two profiles corresponding to $\mathcal{O}$, then*

$$\lim_{k\to+\infty} \big(\|\widetilde{\psi}_{\mathcal{O}_k}\|_{L^2(\mathbb{S}^3)} + \|\widetilde{\varphi}_{\mathcal{O}_k}\|_{L^2(\mathbb{S}^3)}\big) = 0,$$
$$\lim_{k\to+\infty} \langle \widetilde{\psi}_{\mathcal{O}_k}, \widetilde{\varphi}_{\mathcal{O}_k}\rangle_{H^1\times H^1(\mathbb{S}^3)} = \langle \psi, \varphi\rangle_{\dot{H}^1\times\dot{H}^1(\mathbb{R}^3)}.$$

Definition 4.9. We say that a sequence of functions $\{f_k\}_k \subseteq H^1(\mathbb{S}^3)$ is absent from a frame $\mathcal{O}$ if for every profile $\psi_{\mathcal{O}_k}$ associated to $\mathcal{O}$,

$$\int_{\mathbb{S}^3} (f_k \overline{\widetilde{\psi}}_{\mathcal{O}_k} + \nabla f_k \nabla \overline{\widetilde{\psi}}_{\mathcal{O}_k})\, dx \to 0$$

as $k \to +\infty$.

Note in particular that a profile associated to a frame $\mathcal{O}$ is absent from any frame orthogonal to $\mathcal{O}$.

The following proposition is the core of this section. Its proof is similar to the proof of [32, Proposition 5.5], and is omitted.

Proposition 4.10. *Consider $\{f_k\}_k$ a sequence of functions in $H^1(\mathbb{S}^3)$ satisfying*

$$\limsup_{k\to+\infty} \|f_k\|_{H^1(\mathbb{S}^3)} \lesssim E \tag{4.28}$$

and a sequence of intervals $I_k = (-T_k, T^k)$ such that $|I_k| \to 0$ as $k \to +\infty$. Up to passing to a subsequence, assume that $f_k \rightharpoonup g \in H^1(\mathbb{S}^3)$. There exists a sequence of profiles $\widetilde{\psi}^\alpha_{\mathcal{O}^\alpha_k}$ associated to pairwise orthogonal Euclidean frames $\mathcal{O}^\alpha$ such that, after extracting a subsequence, for every $J \geqslant 0$

$$f_k = g + \sum_{1\leqslant\alpha\leqslant J} \widetilde{\psi}^\alpha_{\mathcal{O}^\alpha_k} + R^J_k \tag{4.29}$$

where R^J_k is absent from the frames $\mathcal{O}^\alpha$, $\alpha \leqslant J$, and is small in the sense that

$$\limsup_{J\to+\infty} \limsup_{k\to+\infty} \Big[\sup_{N\geqslant 1, t\in I_k, x\in\mathbb{S}^3} N^{-\frac{1}{2}} \big|\big(e^{itL} P_N R^J_k\big)(x)\big|\Big] = 0. \tag{4.30}$$

Besides, we also have the following orthogonality relations

$$\|f_k\|^2_{L^2} = \|g\|^2_{L^2} + \|R^J_k\|^2_{L^2} + o_k(1),$$
$$\|\nabla f_k\|^2_{L^2} = \|\nabla g\|^2_{L^2} + \sum_{\alpha\leqslant J} \|\nabla_{\mathbb{R}^3}\psi^\alpha\|^2_{L^2(\mathbb{R}^3)} + \|\nabla R^J_k\|^2_{L^2} + o_k(1),$$
$$\lim_{J\to+\infty} \limsup_{k\to+\infty} \Big| \|f_k\|^6_{L^6} - \|g\|^6_{L^6} - \sum_{\alpha\leqslant J} \|\widetilde{\varphi}^\alpha_{\mathcal{O}^\alpha_k}\|^6_{L^6}\Big| = 0, \tag{4.31}$$

where $o_k(1) \to 0$ as $k \to +\infty$, possibly depending on J.

The proof of the last bound in (4.31) relies on the estimate

$$\limsup_{J\to+\infty}\limsup_{k\to+\infty}\|R_k^J\|_{L^6(\mathbb{S}^3)}=0.$$

This is a consequence of (4.30) and the bound

$$\|f\|_{L^6(\mathbb{S}^3)}^6\lesssim\|f\|_{H^1(\mathbb{S}^3)}^2\Big(\sup_{N\geqslant 1}N^{-1/2}\|P_Nf\|_{L^\infty(\mathbb{S}^3)}\Big)^4,\tag{4.32}$$

for any $f\in H^1(\mathbb{S}^3)$, see for example [32, Lemma 2.3] for a similar proof.

5. Global existence

5.1. Induction on energy

We follow a strategy derived from [38]. From Proposition 3.3, we see that to prove Theorem 1.1, it suffices to prove that solutions remain bounded in Z on intervals of length at most 1. To obtain this, we induct on the energy $E(u)$.

Define

$$\Lambda_*(L)=\limsup_{\tau\to 0}\sup\big\{\|u\|_{Z(I)}^2,\ E(u)\leqslant L,\ |I|\leqslant\tau\big\}$$

where the supremum is taken over all strong solutions of (2.10) of energy less than or equal to L and all intervals I of length $|I|\leqslant\tau$. In addition, define

$$E_{max}=\sup\big\{L:\ \Lambda_*(L)<+\infty\big\}.\tag{5.1}$$

We see that Theorem 1.1 is equivalent to the following statement.

Theorem 5.1. $E_{max}=+\infty$. *In particular every solution of* (2.10) *is global.*

Proof. Suppose for contradiction that $E_{max}<+\infty$. From now on, all our constants are allowed to depend on E_{max}. By definition, there exist a sequence of intervals I_k and a sequence of solutions u_k such that

$$E(u_k)\to E_{max},\qquad |I_k|\to 0,\qquad \|u_k\|_{Z(I_k)}\to+\infty\tag{5.2}$$

and $0\in I_k$. We now apply Proposition 4.10 to the sequence $\{u_k(0)\}_k$ with I_k. This gives a sequence of profiles $\widetilde{\psi}^\alpha_{\mathcal{O}_k^\alpha}$, $\alpha,k=1,2,\ldots,$ and a decomposition

$$u_k(0)=g+\sum_{1\leqslant\alpha\leqslant J}\widetilde{\psi}^\alpha_{\mathcal{O}_k^\alpha}+R_k^J.$$

Using Lemma 4.8 and passing to a subsequence, we may renormalize every Euclidean profile, that is, up to passing to an equivalent profile, we may assume that for every Euclidean frame $\mathcal{O}^\alpha$, $\mathcal{O}^\alpha\in\widetilde{\mathcal{F}}_e$, see definition (4.22). Besides, using Lemma 4.8 and passing to a subsequence once again, we may assume that for every $\alpha\neq\beta$, either $N_k^\alpha/N_k^\beta+N_k^\beta/N_k^\alpha\to+\infty$ as $k\to+\infty$ or $N_k^\alpha=N_k^\beta$ for all k and in this case, either $t_k^\alpha=t_k^\beta$ as $k\to+\infty$ or $(N_k^\alpha)^2|t_k^\alpha-t_k^\beta|\to+\infty$ as $k\to+\infty$.

From (4.31) and Lemma 4.8(iii) we see that, after extracting a subsequence,

$$E(\alpha):=\lim_{k\to+\infty}E\big(\widetilde{\psi}^\alpha_{\mathcal{O}_k^\alpha}\big)\in(0,E_{max}],$$

$$\lim_{J\to+\infty}\Big[\sum_{1\leqslant\alpha\leqslant J}E(\alpha)+\lim_{k\to+\infty}E\big(R_k^J\big)\Big]\leqslant E_{max}-E(g).\tag{5.3}$$

We consider also the remainder and note that, for $p \in \{p_0, p_1\}$ and $q = (p_0 + 4)/2 > 4$,

$$\begin{aligned}
&\sum_N N^{5-p/2} \| P_N e^{itL} R_k^J \|_{L^p_{x,t}(\mathbb{S}^3 \times I_k)}^p \\
&\quad\lesssim \Big[\sup_N N^{-\frac{1}{2}} \| e^{itL} P_N R_k^J \|_{L^\infty_{x,t}(\mathbb{S}^3 \times I_k)} \Big]^{p-q} \sum_N \big[N^{5/q-1/2} \| P_N e^{itL} R_k^J \|_{L^q_{x,t}(\mathbb{S}^3 \times I_k)} \big]^q \\
&\quad\lesssim \Big[\sup_N N^{-\frac{1}{2}} \| e^{itL} P_N R_k^J \|_{L^\infty_{x,t}(\mathbb{S}^3 \times I_k)} \Big]^{p-q} \sum_N N^q \| P_N R_k^J \|_{L^2_x(\mathbb{S}^3)}^q \\
&\quad\lesssim \Big[\sup_N N^{-\frac{1}{2}} \| e^{itL} P_N R_k^J \|_{L^\infty_{x,t}(\mathbb{S}^3 \times I_k)} \Big]^{p-q}.
\end{aligned}$$

Therefore

$$\limsup_{J \to +\infty} \limsup_{k \to +\infty} \| e^{itL} R_k^J \|_{Z(I_k)} = 0. \tag{5.4}$$

Case I. $\{u_k(0)\}_k$ converges strongly in $H^1(\mathbb{S}^3)$ to its limit g which satisfies $E(g) = E_{max}$. Then, by Strichartz estimates, there exists $\eta > 0$ such that, for k large enough

$$\| e^{itL} u_k(0) \|_{Z(I_k)} \leqslant \| e^{itL} g \|_{Z(-\eta,\eta)} + o_k(1) \leqslant \delta_0,$$

where δ_0 is given by the local theory in Proposition 3.3. In this case, we conclude that $\|u_k\|_{Z(I_k)} \lesssim 2\delta_0$ which contradicts (5.2).

Case IIa. $g = 0$ and there are no profiles. Then, taking J sufficiently large, we get that, for k large enough,

$$\| e^{itL} u_k(0) \|_{Z(I_k)} \leqslant \delta_0,$$

where δ_0 is as above. Once again, this contradicts (5.2).

Case IIb. $g = 0$ and there is only one Euclidean profile, such that

$$u_k(0) = \widetilde{\psi}_{\mathcal{O}_k} + o_k(1)$$

in $H^1(\mathbb{S}^3)$ (see (5.3)), where $\mathcal{O}$ is a Euclidean frame. In this case, we let U_k be the solution of (2.10) with initial data $U_k(0) = \widetilde{\psi}_{\mathcal{O}_k}$ and we use (4.23) to get, for k large enough

$$\|U_k\|_{Z(I_k)} \leqslant \|U_k\|_{Z(-\delta,\delta)} \lesssim 1 \quad \text{and} \quad \lim_{k \to +\infty} \| U_k(0) - u_k(0) \|_{H^1} \to 0.$$

We may use Proposition 3.4 to deduce that

$$\|u_k\|_{Z(I_k)} \lesssim \|u_k\|_{X^1(I_k)} \lesssim 1$$

which contradicts (5.2).

Case III. $E(g) < E_{max}$ and $E(\alpha) < E_{max}$ for any $\alpha = 1, 2, \ldots$. Up to relabeling the profiles, we can assume that for all α, $E(\alpha) \leqslant E(1) < E_{max} - \eta$, $E(g) < E_{max} - \eta$ for some $\eta > 0$. Now for every linear profile $\widetilde{\psi}^\alpha_{\mathcal{O}^\alpha_k}$, we define the associated nonlinear profile U_k^α as the maximal solution of (2.10) with initial data $U_k^\alpha(0) = \widetilde{\psi}^\alpha_{\mathcal{O}^\alpha_k}$. A more precise description of each nonlinear profile is given by Proposition 4.6. Similarly, we define W to be the nonlinear solution of (2.10) with initial data g. In view of the induction hypothesis

$$\|W\|_{Z(-1,1)} + \| U_k^\alpha \|_{Z(-1,1)} \leqslant 3\Lambda(E_{max} - \eta/2, 2) \lesssim 1,$$

where from now on all the implicit constants are allowed to depend on $\Lambda(E_{max} - \eta/2, 2)$. Using Proposition 3.4 it follows that for any α and any $k > k_0(\alpha)$ sufficiently large,

$$\|W\|_{X^1(-1,1)} + \| U_k^\alpha \|_{X^1(-1,1)} \lesssim 1. \tag{5.5}$$

For $J, k \geqslant 1$ we define

$$U^J_{prof,k} := W + \sum_{\alpha=1}^{J} U^\alpha_k.$$

We show first that there is a constant Q such that

$$\|U^J_{prof,k}\|^2_{X^1(-1,1)} + \|W\|^2_{X^1(-1,1)} + \sum_{\alpha=1}^{J} \|U^\alpha_k\|^2_{X^1(-1,1)} + \sum_{\alpha=1}^{J} \|U^\alpha_k - e^{itL}\widetilde{\psi}^\alpha_{\mathcal{O}^\alpha_k}\|_{X^1(-1,1)} \leqslant Q^2, \tag{5.6}$$

uniformly in J, for all $k \geqslant k_0(J)$ sufficiently large. Indeed, a simple fixed point argument as in Section 3 shows that there exists $\delta_0 > 0$ such that if

$$\|\phi\|_{H^1(\mathbb{S}^3)} = \delta \leqslant \delta_0$$

then the unique strong solution of (2.10) with initial data ϕ is global and satisfies

$$\|u\|_{X^1(-2,2)} \leqslant 2\delta \quad \text{and} \quad \|u - e^{itL}\phi\|_{X^1(-2,2)} \lesssim \delta^2. \tag{5.7}$$

From (5.3), we know that there are only finitely many profiles such that $E(\alpha) \geqslant \delta_0/2$. Without loss of generality, we may assume that for all $\alpha \geqslant A$, $E(\alpha) \leqslant \delta_0$. Using (4.31), (5.5), and (5.7) we then see that

$$\begin{aligned}
\|U^J_{prof,k}\|_{X^1(-1,1)} &= \Big\| W + \sum_{1\leqslant\alpha\leqslant J} U^\alpha_k \Big\|_{X^1(-1,1)} \\
&\leqslant \|W\|_{X^1(-1,1)} + \sum_{1\leqslant\alpha\leqslant A} \|U^\alpha_k\|_{X^1(-1,1)} + \Big\| \sum_{A\leqslant\alpha\leqslant J} \big(U^\alpha_k - e^{itL}U^\alpha_k(0)\big) \Big\|_{X^1(-1,1)} \\
&\quad + \Big\| e^{itL} \sum_{A\leqslant\alpha\leqslant J} U^\alpha_k(0) \Big\|_{X^1(-1,1)} \\
&\lesssim 1 + A + \sum_{A\leqslant\alpha\leqslant J} E(\alpha) + \Big\| \sum_{A\leqslant\alpha\leqslant J} U^\alpha_k(0) \Big\|_{H^1} \lesssim 1.
\end{aligned}$$

The bound on $\sum_{\alpha=1}^{J} \|U^\alpha_k\|^2_{X^1(-1,1)}$ is similar (in fact easier), which gives (5.6).

We now claim that

$$U^J_{app,k} = W + \sum_{1\leqslant\alpha\leqslant J} U^\alpha_k + e^{itL}R^J_k$$

is an approximate solution for all $J \geqslant J_0$ and all $k \geqslant k_0(J)$ sufficiently large. We saw in (5.6) that $U^J_{app,k}$ has bounded X^1-norm. Let $\varepsilon = \varepsilon(2Q^2)$ be the constant given in Proposition 3.4. We compute, with $F(z) = z|z|^4$,

$$\begin{aligned}
e &= (i\partial_t + L)U^J_{app,k} - F\big(U^J_{app,k}\big) = F\big(U^J_{app,k}\big) - F(W) - \sum_{1\leqslant\alpha\leqslant J} F\big(U^\alpha_k\big) \\
&= F\big(U^J_{prof,k} + e^{itL}R^J_k\big) - F\big(U^J_{prof,k}\big) + F\big(U^J_{prof,k}\big) - F(W) - \sum_{1\leqslant\alpha\leqslant J} F\big(U^\alpha_k\big)
\end{aligned}$$

and appealing to Lemma 5.2 below, we obtain that

$$\limsup_{k\to+\infty} \|e\|_{N(I_k)} \leqslant \varepsilon/2$$

for $J \geqslant J_0(\varepsilon)$. In this case, we may use Proposition 3.4 to conclude that u_k satisfies

$$\|u_k\|_{X^1(I_k)} \lesssim \|U^J_{app,k}\|_{X^1(I_k)} \leqslant \|U^J_{prof,k}\|_{X^1(-1,1)} + \|e^{itL}R^J_k\|_{X^1(-1,1)} \lesssim 1,$$

which contradicts (5.2). This finishes the proof. □

We have now proved our main theorem, except for the following important assertion.

Lemma 5.2. *With the notations in Case* III *of the proof of Theorem* 5.1*, we have that, for fixed* J*,*

$$\limsup_{k\to+\infty}\left\|F\left(U^J_{prof,k}\right)-F(W)-\sum_{1\leqslant\alpha\leqslant J}F\left(U^\alpha_k\right)\right\|_{N(I_k)}=0. \tag{5.8}$$

Besides, we also have that

$$\limsup_{J\to+\infty}\limsup_{k\to+\infty}\left\|F\left(U^J_{prof,k}+e^{itL}R^J_k\right)-F\left(U^J_{prof,k}\right)\right\|_{N(I_k)}=0. \tag{5.9}$$

The proof of this lemma is identical to the proof in [33, Section 7], with [33, Lemma 7.1] replaced by Lemma 5.3 below.

Recall from Section 2 that $\mathfrak{O}_{4,1}(a,b)$ denotes a quantity which is quartic in $\{a,\bar a\}$ and linear in $\{b,\bar b\}$.

Lemma 5.3. *Let* $O\in\mathbb{S}^3$ *and assume that* $B,N\geqslant 2$ *are dyadic numbers and* $\omega:\mathbb{S}^3\times(-1,1)\to\mathbb{C}$ *is a function satisfying* $|\nabla^j\omega|\leqslant N^{j+1/2}\mathbf{1}_{\{d_g(x,O)\leqslant N^{-1},|t|\leqslant N^{-2}\}}$*,* $j=0,1$*. Then*

$$\left\|\mathfrak{O}_{4,1}\left(\omega,e^{itL}P_{>BN}f\right)\right\|_{L^1((-1,1),H^1)}\lesssim B^{-1}\|f\|_{H^1(\mathbb{S}^3)}.$$

Proof. The general strategy of the proof is similar to the one in [33] on $\mathbb{T}^3$. We may assume that $\|f\|_{H^1(\mathbb{S}^3)}=1$ and $f=P_{>BN}f$. We notice that

$$\begin{aligned}\left\|\mathfrak{O}_{4,1}\left(\omega,e^{itL}P_{>BN}f\right)\right\|_{L^1((-1,1),H^1)}&\lesssim\left\|\mathfrak{O}_{4,1}\left(\omega,\nabla e^{itL}f\right)\right\|_{L^1((-1,1),L^2)}\\&\quad+\left\|e^{itL}f\right\|_{L^\infty_tL^2_x}\|\omega\|^3_{L^4_tL^\infty_x}\left\||\nabla\omega|+|\omega|\right\|_{L^4_tL^\infty_x}\\&\lesssim\left\|\mathfrak{O}_{4,1}\left(\omega,\nabla e^{itL}f\right)\right\|_{L^1((-1,1),L^2)}+B^{-1}.\end{aligned}$$

Let $\chi_N=\mathbf{1}_{B(O,2N^{-1})}$ and $W(x,t):=N^4\chi_N(x)\eta(N^2t)$ and write

$$\begin{aligned}\left\|\mathfrak{O}_{4,1}\left(\omega,\nabla e^{itL}f\right)\right\|^2_{L^1((-1,1),L^2)}&\lesssim N^{-2}\left\|W^{\frac12}\nabla e^{itL}f\right\|^2_{L^2(\mathbb{S}^3\times(-1,1))}\\&\lesssim N^{-2}\sum_{j=1}^3\int_{-1}^1\left\langle e^{itL}\partial_jf,We^{itL}\partial_jf\right\rangle_{L^2\times L^2(\mathbb{S}^3)}dt\\&\lesssim N^{-2}\sum_{j=1}^3\left\langle\partial_jf,\left[\int_{-1}^1e^{-itL}We^{itL}\,dt\right]\partial_jf\right\rangle_{L^2\times L^2(\mathbb{S}^3)}.\end{aligned}$$

Therefore, it remains to prove that

$$\|K\|_{L^2(\mathbb{S}^3)\to L^2(\mathbb{S}^3)}\lesssim N^2B^{-1}\quad\text{where }K=P_{>BN}\int_{\mathbb{R}}e^{-itL}We^{itL}P_{>BN}\,dt. \tag{5.10}$$

We look at the Fourier coefficients

$$\begin{aligned}K_{p,q}&=\pi_pK\pi_q\\&=N^4\big(1-\eta(p/BN)\big)\big(1-\eta(q/BN)\big)\int_{\mathbb{R}}e^{-it[p^2-q^2]}\eta\left(N^2t\right)dt\cdot[\pi_p\chi_N\pi_q]\\&=N^2\big(1-\eta(p/BN)\big)\big(1-\eta(q/BN)\big)\hat\eta\left(N^{-2}\left(p^2-q^2\right)\right)\cdot[\pi_p\chi_N\pi_q].\end{aligned}$$

Using Schur's lemma, it suffices to prove that

$$\sup_{p\geqslant BN}\sum_{q\in\mathbb{Z}}\big(1-\eta(q/BN)\big)\left|\hat\eta\left(N^{-2}\left(p^2-q^2\right)\right)\right|\|\pi_p\chi_N\pi_q\|_{L^2\to L^2}\lesssim B^{-1}.$$

The new ingredient we need is the following

$$\|\pi_p \chi_N \pi_q\|_{L^2 \to L^2} \lesssim N^{-1} + \min(p,q)^{-2} \tag{5.11}$$

which is a consequence[8] of (2.9). Assuming (5.11), we finish the proof as follows: for any $p \geqslant BN$,

$$\begin{aligned}\sum_{q\in\mathbb{Z}}\big(1-\eta(q/BN)\big)\big|\hat{\eta}\big(N^{-2}\big(p^2-q^2\big)\big)\big|\|\pi_p\chi_N\pi_q\|_{L^2\to L^2} &\lesssim \sum_{q\geqslant BN} N^{-1}\cdot\big[1+N^{-2}\big|p^2-q^2\big|\big]^{-10}\\ &\lesssim \sum_{q\geqslant BN} N^{-1}\cdot\big[1+B|p-q|/N\big]^{-10}\\ &\lesssim B^{-1}\end{aligned}$$

which finishes the proof. □

Appendix A

A.1. Weyl sum estimate

For a sequence $c=(c_p)_p$, we define the linear difference operator δ by

$$(\delta c)_p = c_p - c_{p-1}$$

and for $j \geqslant 1$, $\delta^{j+1}c = \delta(\delta^j c)$. The following lemma is essentially from [11] in a slightly different formulation.

Lemma A.1. *Assume that $(c_p)_p$ satisfies*

$$\big|\delta^j c\big| \lesssim KN^{-j}, \quad 0 \leqslant j \leqslant 2,$$

and that

$$\{p\colon\ c_p \neq 0\} \subset [-QN, QN].$$

For $t \in [-\pi,\pi]$ let $t/\pi = a/q + \beta$, $0 \leqslant |a| \leqslant q \leqslant N$ and $|\beta| \leqslant 1/(Nq)$ be its Dirichlet approximation. Define

$$S(t) = \sum_p c_p e^{it|p|^2},$$

then there holds that

$$\big|S(t)\big| \lesssim KQ^{\frac{3}{2}}\frac{N}{\sqrt{q(1+N^2|\beta|)}}. \tag{A.1}$$

Proof. We may assume that $K=1$. We first compute

$$|S|^2 = \sum_{a,b}\bar{c}_a c_b e^{it[|b|^2-|a|^2]} = \sum_m e^{it|m|^2}\sigma_m,$$

$$\sigma_m = \sum_p \bar{c}_p c_{p+m} e^{it2mp}.$$

We shall not use the oscillations that might be present in the above sum beyond the following claim:

$$|\sigma_m| \lesssim \frac{NQ}{[1+N\,\mathrm{dist}(mt/\pi,\mathbb{Z})]^2}. \tag{A.2}$$

If $N\,\mathrm{dist}(mt/\pi,\mathbb{Z}) < 1$, the bound is clear. Otherwise, we simply observe that, letting $z=e^{i2mt}$ and $C_p = \bar{c}_p c_{p+m}$, there holds, uniformly in m,

[8] Note that we use both bounds in (2.9).

$$(1-z)\sum_p C_p z^p = \sum_p (\delta C)_p z^p,$$
$$(1-z)^2\sum_p C_p z^p = \sum_p \big(\delta^2 C\big)_p z^p,$$
$$\big|\delta^j C_p\big| \lesssim N^{-j}, \quad 0 \leqslant j \leqslant 2.$$

This gives (A.2).

Now, we can finish the proof. We may assume that $a \geqslant 0$ and $|\beta| \neq 0$. For any $m \in \mathbb{Z}$, we define

$$b(m) = am \mod q, \qquad b(m) \in \mathbb{Z}_q = \{0, 1, \dots, q-1\}.$$

Since $(a, q) = 1$, a is invertible in $\mathbb{Z}_q$ and the mapping $r \mapsto b(r)$ is a bijection $\mathbb{Z}_q \to \mathbb{Z}_q$. We now distinguish two cases.

The nonresonant case[9]: $b(r) \notin \mathcal{R} = \{0, 1, \dots, 3Q, q-3Q, \dots, q-2, q-1\}$. In this case, since $|m| \leqslant 2QN$ and

$$mt/\pi = \frac{ma}{q} + m\beta \in \mathbb{Z} + \frac{b(m)}{q} + \left[-\frac{2Q}{q}, \frac{2Q}{q}\right],$$

we may use the oscillations in $b(m)$ since

$$\operatorname{dist}(mt/\pi, \mathbb{Z}) = \frac{b(m)}{q} + m\beta \geqslant \frac{3}{5}\min\left\{\frac{b(m)}{q}, \frac{q-b(m)}{q}\right\}$$

so that, we can estimate the corresponding contribution by

$$\sum_{m:\ b(m)\notin\mathcal{R}} |\sigma_m| \lesssim \frac{Q}{N}\sum_{m:\ b(m)\notin\mathcal{R}} \frac{q^2}{[b(m)]^2} \lesssim \frac{Qq^2}{N}\sum_{k\geqslant 2}\sum_{m:\ b(m)=k}\frac{1}{k^2} \lesssim \frac{Q^2q^2}{N}\frac{N}{q} \lesssim Q^2 q$$

which is acceptable.

The resonant case. In this case, we are left with a worse bound in (A.2), but fortunately, there are only $6Q$ of them and we can estimate them one by one. Thus, from now on, we assume that $b(m)$ is fixed. Then, clearly,

$$\big\{\operatorname{dist}(mt/\pi, \mathbb{Z}):\ b(m) = k\big\}$$

is contained in at most $1 + Q/q$ arithmetic sequences of length $O(N)$ and increment $2q|\beta|$. Hence its contribution can be estimated by

$$Q\min\left(\frac{N}{q}QN, \sum_{k\geqslant 0}\frac{NQ}{(1+N2kq|\beta|)^2}\right) \lesssim Q^2\min\left(\frac{N^2}{q}, \sum_{2kq|\beta|N\leqslant 1} N + \sum_{2kq|\beta|N\geqslant 1}\frac{1}{N(kq|\beta|)^2}\right)$$
$$\lesssim Q^2\min\left(\frac{N^2}{q}, \frac{1}{q|\beta|}\right).$$

Again, this is acceptable. □

A.2. The case of the ball with Dirichlet boundary conditions and radial data

Here we give the main ingredients to prove Theorem 1.2. The analysis of the Dirichlet problem on $B(0,\pi)$ is not so different from the analysis on $\mathbb{S}^3$ due to the relation

$$(1-\Delta_{\mathbb{S}^3})f = \frac{\theta^2}{\sin^2\theta}\Delta_{\mathbb{R}^3}\left[\frac{\sin^2\theta}{\theta^2}f\right]$$

where θ denotes the distance to the origin.[10] This leads to the relation

[9] This case is of course vacuous if $q \leqslant 10Q$.

[10] Here we identify functions on $\mathbb{S}^3$ with functions on $B(0,\pi)$ through the relation $f(x) \simeq f(\exp_O x)$, where O denotes the north pole.

$$\left(e^{-it\Delta_{B^3_D}}\varphi\right) = g \cdot e^{itL}\left(\frac{\varphi}{g}\right), \qquad g(\theta,\omega) = \frac{\sin(\theta)}{\theta}. \tag{A.3}$$

Since we also have that

$$\left\|\frac{\varphi}{g}\right\|_{L^2(\mathbb{S}^3)} = \|\varphi\|_{L^2(B^3)},$$

we can directly transfer the linear estimates on $\mathbb{S}^3$ to estimates on the ball with Dirichlet condition. In particular, we recover all the results of Section 3. In Section 4, we also see that Lemma 4.4 holds directly, while the other lemmas do not depend on the geometry and hence trivially hold. Note in particular that the radial Sobolev inequality

$$\left(\frac{|x|}{\pi - |x|}\right)^{\frac{1}{2}}\left|u(x)\right| \lesssim \|\nabla u\|_{L^2(B)},$$

valid for all functions vanishing at π forces all the Euclidean profiles to only concentrate at the origin. In Section 5, the main novelty is in the linear Lemma 5.3, which again holds equally, thanks to (A.3). The other parts of the proof need only minor modifications.

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