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Duality methods for a class of quasilinear systems

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Abstract

Duality methods are used to generate explicit solutions to nonlinear Hodge systems, demonstrate the well-posedness of boundary value problems, and reveal, via the Hodge–Bäcklund transformation, underlying symmetries among superficially different forms of the equations.

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1. Introduction

After well over a half-century, the equations of Hodge and Kodaira remain a fruitful approach to the theory of irrotational fields, which they endow with the rich topological structure of de Rham cohomology. See, *e.g.*, Ch. 7 of [12], or [16], for introductions. A solution to the Hodge–Kodaira equations is a k-form ω which is *closed* ($d\omega = 0$) and *co-closed* ($\delta\omega = 0$) under the exterior derivative d, where δ is its formal adjoint.

Most of the interesting classical fields are quasilinear. The nonlinear Hodge theory conjectured by Bers and realized by Sibner and Sibner [17] introduces Hodge-like equations which model irrotational velocity fields associated with steady, ideal compressible flow. In that extension, the requirement of classical Hodge theory that the solution ω be co-closed under exterior differentiation is weakened to the requirement that only the product of ω and a possibly nonlinear term ρ must have this property.

Classical fields are frequently characterized by vortices. So although most conservative field theories are quasilinear, most quasilinear field theories are not conservative (even locally), and it is worthwhile to study the analytic properties of equations in which the requirement that the solution be closed under exterior differentiation is also weakened. Thus in a recent paper [9] we studied the invariantly defined system ([13, Sec. VI], [14, Sec. 4])

$$\begin{cases} \delta\big(\rho(Q)\omega\big) = 0, \\ d\omega = \Gamma \wedge \omega \end{cases} \tag{1.1}$$

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for unknown $\omega \in \Lambda^k(\Omega)$, $k \in \mathbb{Z}^+$, with Ω a smooth open domain in $\mathbb{R}^n$, and continuously differentiable $\Gamma \in \Lambda^1(\Omega)$. Here $Q = |\omega|^2 = *(\omega \wedge *\omega)$, with $*$ denoting the Hodge duality operator $* : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$; ρ is a positive, Hölder-continuously differentiable function of Q, which is generally given by the physical or geometric context. We call (1.1) the *nonlinear Hodge–Frobenius equations*, as they generalize the nonlinear Hodge equations

$$\begin{cases} \delta(\rho(Q)\omega) = 0, \\ d\omega = 0 \end{cases} \tag{1.2}$$

introduced in [17]. In this paper we study (1.1) and also variants in which the term Γ in the second equation – the *Frobenius condition* – may depend on ω, or in which the co-differential equation assumes a special inhomogeneous form and $\rho = \rho(\mathbf{x}, Q)$ may depend explicitly on $\mathbf{x} \in \Omega$.

The Frobenius condition represents a weakening of the local conservation hypothesis $d\omega = 0$ in system (1.2). The resulting field is no longer locally conservative, but generates a closed ideal. For this reason, it is completely integrable (in the sense of Frobenius) for forms of degree or co-degree equal to 1, or for general k under the additional hypothesis that Γ be exact; see, *e.g.*, [4, Sec. 4-2]. The hypothesis that Γ be exact is automatically satisfied in the case $k = 1$ or $k = n - 1$. If Γ is exact, say $\Gamma = d\eta$ for $\eta \in \Lambda^0(\Omega)$, solutions to Eqs. (1.1) are locally exact when multiplied by an integrating factor; that is, they have the local form

$$e^\eta \omega = d\Psi$$

for $\Psi \in \Lambda^{k-1}(\Omega)$; see the discussions in Secs. 2.1 and 2.2 of [9] and in Sec. 1 of [10]. For many applications the weaker condition

$$\omega \wedge d\omega = 0 \tag{1.3}$$

suffices in place of the Frobenius condition; see, *e.g.*, Sec. 1.2 of [9]. In cases for which the Frobenius condition is used only to imply (1.3), or for cases in which it is interpreted as a condition for an integrating factor, the 1-form Γ need not be prescribed: any nonsingular Γ will do.

Diverse choices of the mass density ρ arise in models of classical fields. These models are reviewed in [15, Sec. 2.7 and Chs. 5 and 6]. Most classical fields which satisfy quasilinear partial differential equations are vectorial, and these vectorial solutions correspond via isomorphism to 1-form solutions of (1.1) or (1.2). But occasionally there are matrix-valued solutions of quasilinear field equations, and some of these correspond to 2-form-valued solutions of the nonlinear Hodge or Hodge–Frobenius equations. Examples of equations having 1-form solutions include the continuity equations for the velocity field of a steady, compressible fluid flow [17] and for certain models of shallow hydrodynamic flow [15]. Examples of equations having 2-form solutions include nonlinear Maxwell's equations for electromagnetic fields [11], Born–Infeld fields [20], and certain twisted variants of these [13,19]. The variety of applications discussed in [15] and the references cited therein suggest that Eqs. (1.1) and (1.2) are rather generic: they apply, under various additional hypotheses, to a wide variety of models. For this reason, it is worthwhile to study their analytic properties, as we do here and in [9], without focusing on any particular application.

1.1. Organization of the paper

In Section 2 we derive the existence of solutions to a Hodge–Frobenius system, in which the solution is co-closed and the Frobenius condition is nonlinear, from the existence of an appropriate class of A-harmonic forms.

In Section 3 we give an algebraic criterion for inverting the operator A. That criterion can be applied also to the hyperbolic range of the corresponding nonlinear Hodge–Frobenius system. We use this inversion to write an explicit formula for the solutions to the system and generate a concrete example.

In Section 4 we show that certain superficially different models for classical fields can be shown to be Hodge–Bäcklund transforms of each other. In that section we transform different types of nonlinear Hodge–Frobenius systems, including a variational form of these systems, into nonlinear Hodge systems (1.2) of particular type.

In Section 5 we prove the existence and uniqueness of solutions to boundary value problems of Dirichlet and Neumann type in the elliptic regime, for inhomogeneous nonlinear Hodge–Frobenius systems in which the 1-form Γ in the Frobenius condition is exact. We do so for both linear and nonlinear Frobenius conditions. The results of Section 5 are an application of the results obtained in Section 4 and of known results for the conventional nonlinear Hodge equations (1.2) in the elliptic range.

2. Relation to A-harmonic forms

It was observed in Section 1 that the Frobenius condition emerges as a natural weakening of the conservation hypothesis $d\omega = 0$. But the Hodge–Frobenius equations also arise naturally from the nonlinear Hodge equations (1.2) in a completely different way, as a dual, or conjugate form of the equations. The use of conjugate forms in nonlinear Hodge theory goes back at least to [17]. Dirichlet and Neumann problems for Eqs. (1.2) were introduced in [18].

If $u \in \Lambda^{k-1}$ and $v \in \Lambda^{k+1}$, then the Cauchy–Riemann equations can be written in the form $du = \delta v$. More generally, we may consider A-harmonic extensions. We call the differential forms $u \in \Lambda^{k-1}$ and $v \in \Lambda^{k+1}$ *conjugate A-harmonic forms* if they satisfy the equation

$$A(x, du) = \delta v, \tag{2.1}$$

where $A : \Omega \times \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ is a differential operator of order 0 and Ω is a domain of $\mathbb{R}^n$; see, *e.g.*, [1] for an exposition and [5] for analytic properties.

We specify A to be given by

$$A(x, \omega) = A(\omega) = \rho\big(|\omega|^2\big)\omega, \quad \omega \in \Lambda^k(\Omega), \tag{2.2}$$

and impose further conditions on A or Ω as we require them. Our immediate goal is to define Hodge–Frobenius fields in terms of conjugate A-harmonic k-forms. We say that A is *invertible* if there exists an operator $B : \Omega \times \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ such that

$$B\big(x, A(x, \omega)\big) = \tilde{\omega},$$
$$A\big(x, B(x, \tilde{\omega})\big) = \omega \quad \forall \omega, \tilde{\omega} \in \Lambda^k(\Omega).$$

Associated to A is the differential operator $\tilde{A}$ of order 1, $\tilde{A} : \Omega \times \Lambda^{k-1}(\Omega) \to \Lambda^k(\Omega)$, $(x, u) \to A(x, du)$, and the second order differential equation $\delta \tilde{A}(x, u) = 0$ (with its inhomogeneous variants), of which the co-differential equation in (1.1), in the special case of ω exact and $\rho(Q) = Q^{p/2}$, is the p-harmonic equation.

Proposition 2.1. *Let A be given by (2.2), with ρ sufficiently smooth and positive. Assume A to be invertible. Let $u \in \Lambda^{k-1}(\Omega)$ and $v \in \Lambda^{k+1}(\Omega)$ be sufficiently smooth, conjugate A-harmonic forms. Then $\tilde{\omega} \equiv \delta v = A(du) \in \Lambda^k(\Omega)$ is a solution to the Hodge–Frobenius equations in the form*

$$\begin{cases} \delta\tilde{\omega} = 0, \\ d\tilde{\omega} = \Gamma \wedge \tilde{\omega} \end{cases} \tag{2.3}$$

with $\Gamma \equiv d \ln \rho(|B(\tilde{\omega})|^2)$, where $B \equiv A^{-1}$. Conversely, for any given $\tilde{\omega} \in \Lambda^k(\Omega)$ satisfying (2.3), with $\Gamma \equiv d \ln \rho(|B(\tilde{\omega})|^2)$, the k-form $\omega \equiv B(\tilde{\omega})$ satisfies Eqs. (1.2). If in addition Ω is contractible, then $\tilde{\omega} = \delta v$, $\omega = du$ for some conjugate A-harmonic forms $u \in \Lambda^{k-1}(\Omega)$, $v \in \Lambda^{k+1}(\Omega)$.

Proof. To prove the first assertion we proceed as follows. The co-closedness of $\tilde{\omega}$ comes directly from the fact that the generalized Cauchy–Riemann equations (2.1) are satisfied and that $\delta^2 = 0$ on differential forms of class $\mathcal{C}^2$. Furthermore,

$$du = \frac{1}{\rho(|du|^2)} A(du) = \eta\big(|\delta v|^2\big)\delta v, \tag{2.4}$$

with $\eta(|\delta v|^2)$ (well) defined by the formula $\eta(|\delta v|^2)\rho(|B(\delta v)|^2) = 1$. We conclude that $\eta(|\delta v|^2) > 0$, as $\rho(|du|^2) > 0$ by hypothesis. Having set $\tilde{\omega} \equiv \delta v$, (2.4) implies $0 = d^2 u = d(\eta(|\tilde{\omega}|^2)\tilde{\omega})$. This yields the nonlinear Frobenius condition in (2.3) with $\tilde{\eta}(|\tilde{\omega}|^2) \equiv -\ln \eta(|\tilde{\omega}|^2) = \ln \rho(|B(\tilde{\omega})|^2)$.

Conversely, substituting $A(\omega) = \rho(|\omega|^2)\omega$ for $\tilde{\omega}$ in the first equation in (2.3), one obtains the first equation in (1.2). Likewise, the second equation in (2.3) can be multiplied by $e^{-\eta}$ and rewritten as

$$0 = d\big(\tilde{\omega} e^{-\tilde{\eta}(|\tilde{\omega}|^2)}\big) = d\left(A(\omega)\frac{1}{\rho(|\omega|^2)}\right) = d\omega.$$

If Ω is a contractible domain, the application of the Poincaré Lemma and its adjoint version to ω and $\tilde{\omega}$, respectively, completes the proof. □

The following proposition gives a partial converse to Proposition 2.1. A systematic approach to the study of the invertibility of the operator A, leading to a method to construct explicit solutions to Eqs. (1.1), is postponed until Section 3.

Proposition 2.2. *Let $\tilde{\eta} : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ be a prescribed smooth function and I be an interval such that the function $f : \tilde{t} \to t \equiv \tilde{t} \exp[-2\tilde{\eta}(\tilde{t})]$ restricted to I is 1:1. Let $\rho : f(I) \to \mathbb{R}^+$ be defined by $\rho(t) = \exp[\tilde{\eta}(f^{-1}(t))]$. Then for each $\tilde{\omega} \in \Lambda^k(\tilde{\Omega})$ satisfying (2.3) with $\Gamma = d[\tilde{\eta}(|\tilde{\omega}|^2)]$ there exists a unique $\omega \in \Lambda^k(\Omega)$ satisfying $A(\omega) = \tilde{\omega}$, with A defined as in (2.2). Such ω also satisfies (1.2) with ρ as prescribed. Conversely, if $\omega \in \Lambda^k(\Omega)$ satisfies system (1.2) with ρ as prescribed, then the differential form $\tilde{\omega} \equiv A(\omega)$ satisfies (2.3) with $\Gamma = d\tilde{\eta}$, $\tilde{\eta}$ as prescribed. If the domains Ω and $\tilde{\Omega}$ are contractible, our assertions are true with ω replaced by an exact form du and $\tilde{\omega}$ replaced by a co-exact form δv, yielding conjugate A-harmonic forms u, v. Moreover, ω satisfies homogeneous Dirichlet or Neumann conditions on Ω if and only if $\tilde{\omega}$ does as well.*

Proof. Let $\tilde{\omega}$ satisfy (2.3) with $\Gamma = d[\tilde{\eta}(|\tilde{\omega}|^2)]$. Then the differential form $\omega \equiv \exp[-\tilde{\eta}(|\tilde{\omega}|^2)]\tilde{\omega}$ satisfies

$$A(\omega) \equiv \rho\big(|\omega|^2\big)\omega \equiv e^{\tilde{\eta}(f^{-1}(|\omega|^2))}\omega = \tilde{\omega}.$$

For ρ as prescribed, suppose that the differential k-forms ω_1, ω_2 satisfy

$$\rho\big(|\omega_1|^2\big)\omega_1 = \tilde{\omega} = \rho\big(|\omega_2|^2\big)\omega_2, \quad |\omega_j|^2 \in f(I).$$

From this we see that $\omega_1 = \omega_2$ if and only if $|\omega_1| = |\omega_2|$. By taking absolute values and squaring that formula, we also conclude that $|\omega_1| = |\omega_2|$ is the unique inverse image under f of $|\tilde{\omega}|^2$. Thus $\omega_1 = \omega_2$.

As $\tilde{\omega} = \rho(|\omega|^2)\omega$, homogeneous Dirichlet or Neumann boundary conditions for ω become homogeneous Dirichlet or Neumann boundary conditions for $\tilde{\omega}$.

The remainder of the proof is contained in the proof of Proposition 2.1. □

Remark 2.3. Proposition 2.2 gives a precise correspondence between solutions of the Hodge–Frobenius equations (2.3) with nonlinear constraint and solutions of (1.2). Such a correspondence provides the basis to obtain existence and uniqueness theorems for Dirichlet or Neumann problems from analogous theorems for the conventional nonlinear Hodge theory; see [18]. In Section 4 this correspondence is extended to systems of the form (1.1) under conditions on Γ and on the density function in (1.1) sufficient to guarantee the ellipticity condition

$$0 < \rho^2(Q) + 2Q\rho'(Q)\rho(Q) \tag{2.5}$$

for the transformed system. It is necessary to assume appropriate smoothness of the boundary of the domain, of the coefficients of the equation and of ω in order to guarantee the well-posedness of the Dirichlet and Neumann problems; *cf.* Section 5, Theorems 5.2, 5.3, and Theorems 1 and 2 of [18].

3. The construction of solutions

We now want to use the operator A defined in (2.2) to prove results which are independent of equation type. In Proposition 2.1 we assumed the existence of an inverse for the quasilinear coefficient A. In this section we define conditions under which that hypothesis is satisfied.

Theorem 3.1. *Let A be defined via the formula (2.2). Assume that ρ is such that the function*

$$\phi_\rho(t) \equiv t\rho^2(t), \tag{3.1}$$

when restricted to the connected interval (t_1, t_2), satisfies

$$\frac{d\phi_\rho}{dt} > 0 \quad or \quad \frac{d\phi_\rho}{dt} < 0. \tag{3.2}$$

Let $\Lambda^k(\Omega)_{t_1,t_2}$ denote the set of differential k-forms ω such that $t_1 \leqslant |\omega|^2 \leqslant t_2$, and let (r_1, r_2) be the image under ϕ_ρ of the interval (t_1, t_2). Then

$$A_{|\Lambda^k(\Omega)_{t_1,t_2}} : \Lambda^k(\Omega)_{t_1,t_2} \to \Lambda^k(\Omega)_{r_1,r_2}, \tag{3.3}$$

and its restriction to $\Lambda^k(\Omega)_{t_1,t_2}$ is invertible with inverse

$$B : \Lambda^k(\Omega)_{r_1,r_2} \to \Lambda^k(\Omega)_{t_1,t_2}, \quad \tilde{\omega} \to \tilde{\omega}/\rho\big(\psi\big(|\tilde{\omega}|^2\big)\big),$$
$$\text{with } \psi \equiv \phi_{\rho_{|(t_1,t_2)}}{}^{-1} : (r_1, r_2) \to (t_1, t_2). \tag{3.4}$$

Proof. Condition (3.2) implies by monotonicity that there exists an inverse $\psi : (r_1, r_2) \to (t_1, t_2)$ of the map ϕ_ρ defined in (3.1) on the interval (t_1, t_2). Condition (3.3) is satisfied because

$$\big|A(\omega)\big|^2 \equiv \big|\rho\big(|\omega|^2\big)\omega\big|^2 = \rho^2\big(|\omega|^2\big)|\omega|^2 \equiv \phi_\rho\big(|\omega|^2\big),$$

with $t_1 \leqslant |\omega|^2 \leqslant t_2$. Similarly, for k-forms $\tilde{\omega} \in \Lambda^k(\Omega)_{\tau_1,\tau_2}$ and B defined by the formula in (3.4) one has

$$\big|B(\tilde{\omega})\big|^2 = \frac{|\tilde{\omega}|^2}{\rho^2(\psi(|\tilde{\omega}|^2))} = \frac{|\tilde{\omega}|^2\psi(|\tilde{\omega}|^2)}{\rho^2(\psi(|\tilde{\omega}|^2))\psi(|\tilde{\omega}|^2)}. \tag{3.5}$$

The denominator in (3.5) can be rewritten as

$$\psi\big(|\tilde{\omega}|^2\big)\rho^2\big(\psi\big(|\tilde{\omega}|^2\big)\big) = \phi_\rho\big(\psi\big(|\tilde{\omega}|^2\big)\big) = |\tilde{\omega}|^2, \tag{3.6}$$

yielding $|B(\tilde{\omega})|^2 = \psi(|\tilde{\omega}|^2) \in (t_1, t_2)$. That is, $B : \Lambda^k(\Omega)_{r_1,r_2} \to \Lambda^k(\Omega)_{t_1,t_2}$. For k-forms $\omega \in \Lambda^k(\Omega)_{t_1,t_2}$ we have

$$B\big(A(\omega)\big) = \frac{A(\omega)}{\rho(\psi(|A(\omega)|^2))} = \frac{\rho(|\omega|^2)\omega}{\rho(\psi(\rho^2(|\omega|^2)|\omega|^2))} = \frac{\rho(|\omega|^2)\omega}{\rho(\psi(\phi_\rho(|\omega|^2)))} = \omega.$$

Likewise, for k-forms $\tilde{\omega} \in \Lambda^k(\Omega)_{r_1,r_2}$ we have

$$A\big(B(\tilde{\omega})\big) = \rho\big(\big|B(\tilde{\omega})\big|^2\big)B(\tilde{\omega}) = \rho\bigg(\frac{|\tilde{\omega}|^2}{\rho^2(\psi(|\tilde{\omega}|^2))}\bigg)\frac{\tilde{\omega}}{\rho(\psi(|\tilde{\omega}|^2))} = \tilde{\omega},$$

in which, for the last equality, we have divided (3.6) by $\rho^2(\psi(|\tilde{\omega}|^2))$ and substituted the result into this equation. This concludes the proof. □

Note that the conditions in (3.2) are precisely the conditions that make the system (1.2), and also (1.1) with linear Frobenius condition, either elliptic (if $d\phi_\rho/dt > 0$) or hyperbolic (if $d\phi_\rho/dt < 0$); *cf.* (2.5).

Remark 3.2. We have divided by ρ at various steps of the proof of Theorem 3.1. Clearly this can be done if $\rho = \rho(t) > 0\ \forall t \in \mathbb{R}^+ \cup \{0\}$. Nonetheless, the milder assumption $\rho(\psi(|\tilde{\omega}|^2)) \neq 0$ is sufficient for the purpose of finding smooth solutions to the equation $A(\omega) = \tilde{\omega}$ with prescribed $\tilde{\omega}$. In some applications, this assumption can be weakened furthermore; *cf.* [10].

Theorem 3.1 can be used to construct explicit k-form-valued solutions to the nonlinear co-differential equation $\delta(\rho(|\omega|^2)\omega) = 0$ in (1.1). For a detailed exposition of the method and the construction of various examples, see [10]. Briefly, one argues by the Poincaré Lemma that a solution ω on a contractible domain of $\mathbb{R}^n$ always admits a "stream $(n-k-1)$-form" f, that is, a form f satisfying $\rho(Q)\omega = *df$. Theorem 3.1 can then be applied directly to obtain the solution formula

$$\omega = \frac{*df}{\rho(\psi(|df|^2))}, \tag{3.7}$$

where ψ denotes the inverse(s) of the function ϕ_ρ given by (3.1). The classical solutions ω are well defined except possibly at the *sonic hypersurface* dividing the elliptic from the hyperbolic regime. The singular set will depend on f, ρ and ψ. Sometimes it is possible to define ω with continuity, or even higher regularity, across the sonic hypersurface;

cf. [10, Sec. 5.1.1]. In general such a property is not achieved (see Ch. 6 of [15] and references cited therein). On non-contractible domains, one can still write (3.7) and produce examples of solutions to the co-differential equation in (1.1). More generally, one can replace the exact forms df in (3.7) by prescribed closed $(n-k)$-forms. Satisfaction of the Frobenius condition for some Γ can be shown and is equivalent to the existence of an integrating factor in the cases $k=1, n-1$; *cf.* [10].

As an example, let us consider system (1.1) with prescribed density $\rho(Q) = |Q-1|^{-1/2}$, $Q \neq 1$, for a differential form of degree 2 in 4 dimensions. This choice of ρ corresponds to the Euclidean Born–Infeld model if $Q < 1$ and to the Lorentzian Born–Infeld model if $Q > 1$. All non-cavitating classical solutions ω can be expressed by (3.7) on contractible domains Ω. Cavitating solutions may be expressed by (3.7) as a limit. In this example, the function ϕ_ρ appearing in (3.1) is

$$\phi_\rho(Q) = \frac{Q}{|Q-1|}, \quad Q \neq 1,$$

with inverses $\psi_+ \equiv [\phi_{\rho_{|[0,1)}}]^{-1} : [0,\infty) \to [0,1)$, $\psi_- \equiv \phi_{\rho_{|(1,\infty)}}^{-1} : (1,\infty) \to (1,\infty)$, given by

$$\psi_\pm : \xi \to \frac{\xi}{\xi \pm 1}.$$

Corresponding to these inverses of ϕ_ρ, one obtains the families of solutions

$$\mathcal{W}_\pm = \left\{ \omega_\pm = \frac{*df}{\sqrt{|df|^2 \pm 1}} \text{ with } f \in \Lambda^1(\Omega) \right\}, \tag{3.8}$$

with the solutions in $\mathcal{W}_+$ being defined (and uniformly bounded) for smoothly prescribed generalized stream 1-forms f, and the solutions in $\mathcal{W}_-$ requiring the additional condition $|df| > 1$. The family $\mathcal{W}_-$ contains unbounded solutions corresponding to choices of generalized stream forms which satisfy $|df| = 1$ at points of the domain Ω. One may also prescribe generalized stream forms f such that $|df| \to \infty$ when approaching a smooth manifold, say γ_∞, contained in Ω. As $|\omega_\pm| \to 1$ when approaching γ_∞, one may in some cases patch together the two types of solutions with some regularity. But the co-differential equation in (1.1) would not be satisfied on γ_∞ (as ρ would blow up). Differential forms $\omega_+ \in \mathcal{W}^+$ and $\omega_- \in \mathcal{W}^-$ that satisfy a linear Frobenius condition would then solve (1.1) in the elliptic and hyperbolic regime, respectively.

4. Hodge–Bäcklund transformations of solutions

One finds in the literature a bewildering redundancy of choices for the mass density ρ; see Sec. 1 of [7], Sec. 2 of [8], and the pairs of densities discussed in [15, Sec. 2.7 and Ch. 6], in connection with the Born–Infeld and extremal surface equations. It is natural to wonder whether there is a mathematical operation underlying the varieties of density. In this section we extend Theorem 6.1 of [9], which related two particular densities by an application of the Hodge–Bäcklund transformation; see also the special cases studied in [2,3,6,19,20]. A different motivation for seeking a relation between pairs of densities comes from the fact that when we introduce Hodge–Bäcklund transformations we acquire an inhomogeneous right-hand side which has a natural variational interpretation. In fact, the Euler–Lagrange equation for the nonlinear Hodge energy

$$E_{NH} = \frac{1}{2} \int_M \int_0^Q \rho(s)\, ds\, dM, \tag{4.1}$$

where M is an n-dimensional Riemannian manifold and Γ is prescribed, is the inhomogeneous equation (*cf.* [9, Sec. 5.1])

$$\delta\big[\rho(Q)\omega\big] = (-1)^{n(k+1)} * \big(\Gamma \wedge *\rho(Q)\omega\big).$$

Definition 4.1. We define the pair of continuously differentiable densities $(\rho, \hat\rho)$ to be a *dual pair* if $\rho : I \subset \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$, $\hat\rho : \hat I \equiv \phi_\rho(I) \to \mathbb{R}^+$, with $\phi_\rho : t \in I \to \hat t \equiv t\rho^2(t)$, and the pair $(\rho, \hat\rho)$ satisfies the identity

$$\rho(t)\hat\rho(\hat t) \equiv 1. \tag{4.2}$$

Definition 4.1 implies that the functions ϕ_ρ and $\hat{\phi}_{\hat{\rho}}$, defined analogously, are inverses of one another; thus both are 1:1. In fact, by squaring and multiplying by t throughout, one obtains $t = t\rho^2(t)\hat{\rho}^2(\hat{t}) = \hat{t}\hat{\rho}^2(\hat{t}) = \hat{\phi}_{\hat{\rho}}(\hat{t})$. Therefore, the relation of duality defined above is symmetric. For the same reason, ellipticity of the system (1.1) or (1.2) is preserved under the transformation $\rho \to \hat{\rho}$. Moreover, the relation (4.2) defines $\hat{\rho}$ in terms of ρ and vice versa.

An example of a dual pair of densities is the pair $(\rho, \hat{\rho})$, with $\rho(t) = 1/\sqrt{1+t}$ – associated in the applications with the Born–Infeld model and with the minimal surface equation – and $\hat{\rho}(t) = 1/\sqrt{1-t}$ with $t < 1$, associated with the maximal surface equation. The density $\rho(t) = 1/\sqrt{t-1}$ with $t > 1$ is self-dual and is associated with extremal surfaces in Minkowski space.

We find in the following proposition that systems having the form (4.3) can be related to each other by Hodge–Bäcklund transformations.

Proposition 4.2. *Let Σ, Γ be given, continuous differential* 1*-forms, $(\rho, \hat{\rho})$ be a prescribed dual pair of densities. Then the k-form ω satisfies the nonlinear Hodge–Frobenius system*

$$\begin{cases} d * \big(\rho\big(|\omega|^2\big)\omega\big) = \Sigma \wedge *\big(\rho(Q)\omega\big), \\ d\omega = \Gamma \wedge \omega \end{cases} \tag{4.3}$$

if and only if the $(n-k)$-form $\xi \equiv *(\rho(|\omega|^2)\omega)$ satisfies the dual system

$$\begin{cases} d * \big(\hat{\rho}\big(|\xi|^2\big)\xi\big) = \Gamma \wedge *\big(\hat{\rho}\big(|\xi|^2\big)\xi\big), \\ d\xi = \Sigma \wedge \xi. \end{cases} \tag{4.4}$$

Proof. Multiplying the definition $\xi \equiv *(\rho(|\omega|^2)\omega)$ by $\hat{\rho}(|\xi|^2)$ and using (4.2), we obtain $*\hat{\rho}(|\xi|^2)\xi = *_{n-k} *_k \omega \equiv \sigma_k\omega$, where the value of $\sigma_k = \pm 1$ depends on the order k of the differential form ω and on the dimension n of the domain Ω. By the second equation in (4.3) this yields

$$d\big(*\hat{\rho}\big(|\xi|^2\big)\xi\big) = d(\sigma_k\omega) = \sigma_k\, d\omega = (\sigma_k)^2\Gamma \wedge *\big(\hat{\rho}\big(|\xi|^2\big)\xi\big) = \Gamma \wedge *\big(\hat{\rho}\big(|\xi|^2\big)\xi\big),$$

which is the first equation in the system (4.4). The second equation in (4.4) is the first equation in the system (4.3) with a change in notation. □

If $\Gamma = \Sigma \equiv 0$, Proposition 4.2 yields the standard Hodge duality result for the conventional nonlinear Hodge equations (1.2); see [5,17].

Theorem 4.3. *Let $\eta, \zeta : I \subset \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ be prescribed continuously differentiable functions, with the additional hypothesis on η that the function $f_\eta : t \in I \to t\exp[-2\eta(t)] \in \mathbb{R}^+ \cup \{0\}$ be invertible with inverse g_η. Let the terms Σ, Γ and the mass density ρ be prescribed by $\Sigma = d[\zeta(|\omega|^2)]$, $\Gamma = d[\eta(|\omega|^2)]$, $\rho = \rho_1(x, |\omega|^2)$, for $|\omega|^2 \in I$ in (4.3). Then for every classical solution ω_1 of system (4.3) there is a classical solution ω_0 of the conventional nonlinear Hodge equations (1.2) with mass density $\rho_0(x, |\omega_0|^2)$, where ρ_0 depends on ρ_1, η and ζ and ω_0 is related to ω_1 by C^1 conformal transformations. The ellipticity condition for system (1.2) holds if and only if g'_η and $\partial_t\phi_{\rho_1 e^{-\zeta}}$ have the same sign. The converse also holds.*

Proof. Let ω_1 be a differential form satisfying $|\omega_1|^2 \in I$ and $\rho = \rho_1(x, |\omega_1|^2)$ be a prescribed density function. Define

$$\omega_0 = e^{-\eta(|\omega_1|^2)}\omega_1. \tag{4.5}$$

Then $|\omega_0|^2 \in f(I)$ and $|\omega_1|^2 = g_\eta(|\omega_0|^2)$. This enables us to define a density function

$$\rho_0\big(x, |\omega_0|^2\big) = e^{\eta(g_\eta(|\omega_0|^2)) - \zeta(g_\eta(|\omega_0|^2))}\rho_1\big(x, g_\eta\big(|\omega_0|^2\big)\big). \tag{4.6}$$

Conversely, given a differential form ω_0 satisfying $|\omega_0|^2 \in f(I)$, a prescribed density function $\rho = \rho_0(x, |\omega_0|^2)$, and functions η and ζ as defined in the hypotheses of this theorem, one can rewrite definition (4.5) as

$$\omega_1 = e^{\eta(g_\eta(|\omega_0|^2))}\omega_0,$$

and define ω_1 in terms of ω_0. Likewise, formula (4.6) can be rewritten as

$$\rho_1\big(x,|\omega_1|^2\big)=e^{\zeta(|\omega_1|^2)-\eta(|\omega_1|^2)}\rho_0\big(x,f_\eta\big(|\omega_1|^2\big)\big),$$

defining ρ_1 in terms of ρ_0.

It is easily seen that the differential form ω_0 satisfies system (1.2) with density function $\rho_0(x,|\omega_0|^2)$ if and only if ω_1 satisfies system (4.3) with density function $\rho_1(x,|\omega_1|^2)$ and coefficients η and ζ as in the hypotheses of the theorem. In fact,

$$e^\eta\, d\omega_0=e^\eta\, d\big(e^{-\eta}\omega_1\big)=-d\eta\wedge\omega_1+d\omega_1,\quad \text{with } \eta=\eta\big(|\omega_1|^2\big)=\eta\big(g_\eta\big(|\omega_0|^2\big)\big),$$

and

$$e^\zeta\, d*(\rho_0\omega_0)=e^\zeta\, d*\big(e^{-\zeta}\rho_1\omega_1\big)=-d\zeta\wedge*(\rho_1\omega_1)+d*(\rho_1\omega_1),\quad \text{with } \zeta=\zeta\big(|\omega_1|^2\big).$$

Finally, the ellipticity condition for system (1.2) with $\rho=\rho_0$ is

$$\frac{\partial\phi_{\rho_0}(x,\hat t)}{\partial\hat t}>0,\quad \text{with } \phi_{\rho_0}(x,\hat t)=\hat t\rho_0^2(x,\hat t),\ \hat t\in f_\eta(I).$$

Squaring both sides of (4.6) and multiplying by $\hat t$, one obtains

$$\begin{aligned}\phi_{\rho_0}(x,\hat t)&=\hat t\rho_0^2(x,\hat t)=\hat t e^{2\eta(g_\eta(\hat t))}\big(\rho_1\big(x,g_\eta(\hat t)\big)e^{-\zeta(g_\eta(\hat t))}\big)^2\\&=\hat t e^{2\eta(t)}\big(\rho_1(x,t)e^{-\zeta(t)}\big)^2=t\big(\rho_1(x,t)e^{-\zeta(t)}\big)^2=\phi_{\rho_1 e^{-\zeta}}(x,t),\quad \text{with } t=g_\eta(\hat t).\end{aligned}$$

Thus,

$$\frac{\partial\phi_{\rho_0}(x,\hat t)}{\partial\hat t}=\frac{\partial\phi_{\rho_1 e^{-\zeta}}(x,t)}{\partial t}g'_\eta(\hat t),\quad \text{with } t=g_\eta(\hat t).\qquad\Box$$

Proposition 4.4. *Let $\eta,\zeta:\Omega\to\mathbb{R}$ be prescribed continuously differentiable functions. Then for every classical solution ω_1 of (4.3) with mass density ρ_1, coefficients $\Sigma=d\zeta$ and $\Gamma=d\eta$, there is a classical solution ω_0 of the nonlinear Hodge equations (1.2) with density ρ_0. Here ρ_0 depends on ρ_1, η and ζ; ω_0 is related to ω_1 by C^1 conformal transformations. The converse also holds. Ellipticity is preserved by this correspondence.*

Proof. Given a k-form ω_1 and a density function $\rho_1(|\omega_1|^2)$, define

$$\omega_0=e^{-\eta(x)}\omega_1\quad \text{and}\quad \rho_0\big(x,|\omega_0|^2\big)=e^{\eta(x)-\zeta(x)}\rho_1\big(e^{2\eta(x)}|\omega_0|^2\big).$$

If ω_1 satisfies (4.3) with mass density ρ_1 and coefficients Σ and Γ as in the hypotheses of the proposition, then ω_0 satisfies (1.2) with density ρ_0. In fact,

$$\begin{aligned}&d\omega_0=d\big(e^{-\eta}\omega_1\big)=e^{-\eta}(-d\eta\wedge\omega_1+d\omega_1)=0,\\&d*(\rho_0\omega_0)=d\big(e^{-\zeta}*\rho_1\omega_1\big)=e^{-\zeta}\big(-d\zeta\wedge*(\rho_1\omega_1)+d*(\rho_1\omega_1)\big)=0.\end{aligned}$$

The converse, for prescribed ω_0 and ρ_0 holds with ω_1 and ρ_1 defined by

$$\omega_1=e^{\eta(x)}\omega_0\quad \text{and}\quad \rho_1\big(x,|\omega_1|^2\big)=e^{\zeta(x)-\eta(x)}\rho_0\big(e^{-2\eta(x)}|\omega_1|^2\big).\qquad\Box$$

The prescription $\eta(x)=\zeta(x)$ in Proposition 4.4, yielding the simpler relation between densities $\rho_0=\rho_1(\exp[2\eta(x)]|\omega_0|^2)$, corresponds to the variational equations for the nonlinear Hodge–Frobenius theory for gradient-recursive k-forms (that is, with prescribed exact Γ).

5. Boundary value problems

Theorem 4.3 – with nonlinear Frobenius condition – and Proposition 4.4 allow us to extend the existence and uniqueness theorem for the Dirichlet and Neumann problems established in [18] for the conventional nonlinear Hodge theory (proven in their strongest formulation for 1-forms) to system (4.3) for gradient-recursive forms. For this application we use the results in [18] in their general formulation for density functions which may depend explicitly on x. Here M denotes an oriented, finite Riemannian manifold of dimension n with $\mathcal{C}^\infty$ boundary [5]. The following theorems correspond to Theorems 1, 2 in [18]. We establish the following definition.

Definition 5.1. The triplet of functions (ρ, ζ, η) is said to be an *admissible system* if the following conditions hold: (a) $f_\eta : t \in \mathbb{R}^+ \cup \{0\} \to \hat{t} \equiv t \exp[-2\eta(t)] \in \mathbb{R}^+ \cup \{0\}$ is 1:1 and onto; (b) $\rho_0 \equiv \rho(x,t)\exp[\eta(t) - \zeta(x,t)] \in [k, 1/k]$ for some constant $k > 0$, $\forall(x,t)$; (c) there exists $T > 0$ s.t. $(\partial_t \phi_{\rho e^{-\zeta}}) g'_\eta > 0$, $\forall x$, $\forall t \in (0,T)$; here g_η denotes the inverse of f_η. The *sonic speed* associated with an admissible system (ρ, ζ, η) is $Q_s \equiv \sup\{T\}$ such that (c) is satisfied. A k-form ω is said to be *subsonic* if $\max_{x\in M} |\omega|^2 < Q_s$.

Following [18], the inhomogeneous Dirichlet boundary data are given by an element of the space $\mathcal{D} = \ker d \oplus \mathcal{C}^{1+\alpha}(\bar{M})$, while inhomogeneous Neumann data are given by an element of the space $\mathcal{N} = \ker d \oplus \mathcal{N}_2$, with $\mathcal{N}_2 = \ker \delta$ if $n \leqslant 3$, $\mathcal{N}_2 = 0$ if $n > 3$. We denote by $T\omega$, $N\omega$ respectively, the restriction to the boundary of the tangential component, normal component respectively, of ω.

Theorem 5.2. *Let (ρ, ζ, η) be an admissible system of class $\mathcal{C}^{2+\alpha}$ in x and $\mathcal{C}^{1+\alpha}$ in t, with sonic speed Q_s. There is an open connected set $\mathcal{O} \in \mathcal{D}$ containing the origin such that for each pair of 1-forms $(\gamma, \sigma) \in \mathcal{O}$, there is a unique subsonic 1-form $\omega \in \mathcal{C}^{1+\alpha}(\bar{M})$ having the same relative periods as γ, satisfying*

$$\begin{cases} d * \big(\rho\big(|\omega|^2\big)\omega\big) = d\zeta \wedge *\big(\rho(Q)\omega\big) + d * \sigma, \\ d\omega = d\eta \wedge \omega, \\ T\big(e^{-\eta(|\omega|^2)}\omega\big) = T\gamma \quad \text{on } \partial M. \end{cases} \tag{5.1}$$

Moreover, for any given continuous path $(\gamma(\tau), \sigma(\tau))$ on $\mathcal{D}$, the solution $\omega(\tau)$ will also depend continuously on τ in the uniform norm and, either is subsonic $\forall\tau$ or there exists a number τ_s such that $\sup_{x\in M} |\omega|^2(\tau) \to Q_s$ as $\tau \to \tau_s$.

Proof. By Theorem 4.3, system (5.1) is transformed into

$$\begin{cases} d * \big(\rho_0\big(x, |\omega_0|^2\big)\omega_0\big) = d * \sigma, \\ d\omega_0 = 0, \\ T\omega_0 = T\gamma \quad \text{on } \partial M, \end{cases} \tag{5.2}$$

with $\rho_0(x, \hat{t}) = \exp[\eta(g(\hat{t})) - \zeta(x, g(\hat{t}))]\rho(x, g(\hat{t}))$. By Theorem 4.3, ρ_0 is *admissible* as defined in [18]; that is, $\rho_0(x,t) \in [k, 1/k]$ and $\partial_{\hat{t}}\phi_{\rho_0} > 0$ $\forall \hat{t} \in (0, f_\eta(T))$. Therefore, the conclusions in Theorem 1 of [18] extend to (5.1). By Theorem 4.3 the 1-form $\omega = \exp[\eta(g(|\omega_0|^2))]\omega_0$ is the unique solution to (5.1) as required. □

Theorem 5.3. *Let (ρ, ζ, η) be an admissible system of class $\mathcal{C}^{2+\alpha}$ in x and $\mathcal{C}^{1+\alpha}$ in t, with sonic speed Q_s. There is an open connected set $\mathcal{O} \in \mathcal{N}$ containing the origin such that for each pair of 1-forms $(\gamma, \nu) \in \mathcal{O}$, there is a unique subsonic 1-form $\omega \in \mathcal{C}^{1+\alpha}(\bar{M})$ having the same absolute periods as γ, satisfying*

$$\begin{cases} d * \big(\rho\big(x, |\omega|^2\big)\omega\big) = d\zeta \wedge *\big(\rho(Q)\omega\big), \\ d\omega = d\eta \wedge \omega, \\ N\big(\rho e^{-\zeta}\omega\big) = N\nu \quad \text{on } \partial M. \end{cases} \tag{5.3}$$

Moreover, for any given continuous path $(\gamma(\tau), \nu(\tau))$ on $\mathcal{O}$, the same conclusions as in Theorem 5.2 hold for the path of solutions $\omega(\tau)$.

Proof. By Theorem 4.3, system (5.3) is transformed into

$$\begin{cases} d * \big(\rho_0\big(x, |\omega_0|^2\big)\omega_0\big) = 0, \\ d\omega_0 = 0, \\ N(\rho_0\omega_0) = N\nu \quad \text{on } \partial M, \end{cases} \tag{5.4}$$

with $\rho_0(x, \hat{t}) = \exp[\eta(g(\hat{t})) - \zeta(x, g(\hat{t}))]\rho(x, g(\hat{t}))$, satisfying the hypotheses of Theorem 2 of [18]. Again, by Theorem 4.3 $\omega = \exp[\eta(g(|\omega_0|^2))]\omega_0$ is the unique solution to (5.3) as required. □

Remark 5.4. For a linear Frobenius condition, that is, if $\eta = \eta(x)$, simpler versions of Theorems 5.2 and 5.3 hold and their proofs are a direct application of Proposition 4.4 to Theorems 1, 2 in [18]. For simplicity, we have not addressed the question on whether the surjectivity hypothesis on f_η can be removed in Theorems 5.2 and 5.3.

Remark 5.5. It is natural to expect that, at least in the case of the variational equation (4.1), Theorem 4.3 and Proposition 4.4 would lead to decomposition theorems for gradient-recursive differential forms, mirroring the conventional nonlinear Hodge decomposition theorems. Furthermore, the duality result of Proposition 4.2 has potential importance in extending nonlinear Hodge decomposition theorems to include differential forms satisfying the nonlinear Hodge–Frobenius equations that are not necessarily gradient-recursive. Because all recursive forms of degree or co-degree 1 are gradient-recursive, this investigation would be of special interest for applications to forms of degree $k \neq 1, n-1$. In this regard, we observe that the Frobenius theorem for 1-forms, stating that 1-forms that generate a closed ideal are integrable, does not extend to k-forms with $k \neq 1, n-1$.

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