

# Localized minimizers of flat rotating gravitational systems

Jean Dolbeault<sup>a,\*</sup>, Javier Fernández<sup>b</sup>

<sup>a</sup> Ceremade, Université Paris Dauphine, Place de Latre de Tassigny, F-75775 Paris cedex 16, France

<sup>b</sup> Departamento Automática y Computación, Universidad Pública de Navarra, Campus Arrosadía s/n, 31.006 Pamplona, Spain

Received 9 November 2006; accepted 2 January 2007

Available online 17 July 2007

---

## Abstract

We study a two-dimensional system in solid rotation at constant angular velocity driven by a self-consistent three-dimensional gravitational field. We prove the existence of stationary solutions of such a flat system in the rotating frame as long as the angular velocity does not exceed some critical value which depends on the mass. The solutions can be seen as stationary solutions of a kinetic equation with a relaxation-time collision kernel forcing the convergence to the polytropic gas solutions, or as stationary solutions of an extremely simplified drift-diffusion model, which is derived from the kinetic equation by formally taking a diffusion limit. In both cases, the solutions are critical points of a free energy functional, and can be seen as *localized minimizers* in an appropriate sense.

© 2007 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

MSC: primary 35Q72; secondary 45E10, 76S05, 35B30

Keywords: Stellar dynamics; Gravitation; Mass; Rotation; Angular velocity; Entropy; Diffusion limit; Drift-diffusion; Hardy–Littlewood–Sobolev inequality; Bounded solutions; Minimization; Solutions with compact support; Radial solutions; Localized minimizers

---

## 0. Introduction

In this paper, we look for some special solutions of flat systems in rotation made of gravitating particles. We adopt the point of view of continuum mechanics, at two different levels of description. At the *kinetic level*, the system is described by a distribution function on the phase space. The microscopic structure of the system is revealed by the fact that at some point in the physical space, the dispersion of velocities is taken into account. At the *macroscopic level*, we consider only the spatial density of mass in the physical space. Although the distribution of velocities is observable in some cases, the macroscopic density of mass is by far a more interesting and fundamental object.

Many solutions of the equations of mechanics for gravitational systems have been identified over the last two centuries, from a numerical point of view as well as by explicit calculations, which reflect well the diversity of observational data, see, *e.g.*, [9]. In celestial mechanics, the key question consists less in finding solutions than in understanding their stability, since in most of practical cases, time scales are such that the observation of an unstable solution is highly improbable. Stability issues for finite dimensional dynamical systems of particles have been studied for about a century but the corresponding investigations in continuum mechanics are much more recent. We

---

\* Corresponding author.

E-mail addresses: [dolbeaul@ceremade.dauphine.fr](mailto:dolbeaul@ceremade.dauphine.fr) (J. Dolbeault), [fcojavier.fernandez@unavarra.es](mailto:fcojavier.fernandez@unavarra.es) (J. Fernández).

may for instance quote the use of Casimir functionals in fluid mechanics in [1,2] as one of the earliest contributions, and also [61,59]. At the level of kinetic equations for gravitating systems, the topic has been developed mostly by G. Wolansky, G. Rein and Y. Guo, see for instance [49,60,35,36], and the review papers [38,53]. Only very simple configurations have been tackled up to now, see [37,51,52,27,55,10,54,25,41] and references therein for some other recent contributions. Roughly speaking, stability properties have been established only for radially symmetric stationary solutions (in the reference frame of the center of mass) characterized as minima of a free energy functional, after taking into account the Galilean invariance. Notice here that we are going to restrict the topic to classical mechanics, hence discarding any further consideration involving relativity or quantum mechanics. See [40] and [46,24] for some contributions in this direction.

Our goal is to make a first step towards more complex geometries than the ones which have been considered up to now, and to open a new field of research in kinetic models. Typical examples one would like to study are double systems, or systems with a spiral geometry. For the moment, nobody knows how to deal with such systems at the kinetic level. We will expose later some results obtained with fluid models of gravitating particles. In this paper, we will use a dramatic simplification, which consists in reducing the problem to the study of nonlinear diffusion equations by taking an appropriate diffusion limit and consider solutions at the macroscopic level. Such a limit makes a lot of sense when one is interested in quantities like the mass density. A rigorous derivation in the case of a given potential has recently been done in [26], but the diffusion limit is still formal in presence of a self-consistent potential as it is the case here. One could argue that the solutions found in this paper are solutions both at the kinetic and macroscopic levels, but this would be somewhat excessive since, after formally deriving the diffusion limit, we will systematically adopt the macroscopic level point of view.

Let us insist on several important features of the model we shall consider. First of all, one is interested in *compactly supported solutions*. This can be mathematically enforced by imposing boundary conditions and many papers in physics and astrophysics journals do it, but it is in most of the cases rather unphysical. We will use artificial boundary conditions as a technical step in our proofs, but we will be able to remove them in the end. Another important issue is the type of *statistical distribution* one wants to take into consideration. Although Maxwellian (Gaussian) distributions are very popular in statistical physics, such distributions turn out not to make much sense in astrophysics since they give rise to equilibrium solutions with infinite mass, while one is clearly interested in systems with finite mass, even if it can be huge when measured in standard units. One of the proposed remedies is to consider *Gibbs states*, *i.e.*, equilibrium solutions, which are polytropic type distribution functions at the kinetic level. By polytropic, we simply mean that the distribution function is assumed to be a power law of the microscopic energy. In astrophysics, such models are known as the model of *polytropic gases*, or *polytropes*. These Gibbs states turn out to give porous media equations in the diffusion limit. Such equations are well known in mathematics and various areas of physics, and have always raised lots of interest. Coupled with a gravitational Poisson interaction, they have the nice property of having stationary solutions with finite mass and compact support (which is also the case of King's model). Another feature that we want to include is that the solutions under consideration can be characterized as *critical points*, and eventually *localized minimizers* if some judicious constraints are imposed on the system, of some functionals which can be interpreted from a thermodynamical point of view as *free energies*. In astrophysics and in the case of the porous media equations, such functionals are sometimes called Tsallis' entropies [58]. They are perfectly consistent with the free energies involving the so-called Casimir functionals at the kinetic level (in the polytropic gases case), and they provide useful estimates for diffusion limits, see [26]. How to select one such functional has to be determined by physics and should involve a collision mechanism whose discussion goes far beyond the scope of this introduction. We refer again to [26] for more detailed comments.

Many gravitating systems are rotating and the total angular momentum determines a global axis of rotation. Since the goal of this paper is only to make a first mathematical step into a whole world of models, we are going to assume several simplifying hypotheses. First of all, we will consider a system in a *solid motion of rotation*: in the rotating reference frame, we look for stationary solutions, and the only novelty is that a centrifugal force term has to be added. We will see however that even such a small change produces tremendous differences at the level of the solutions. For instance, solutions with a given mass may exist only as long as the angular velocity is not too big. We will further restrict the model to flat systems, so that *two-dimensional coordinates* can be used, but we will keep the gravitational interaction in its three-dimensional version. Although highly simplified, such a model could for instance be quite relevant for some physical situations like accretion disks of self-gravitating dust. Many features could be added to the model, like the presence of a given central force field created by a dense core or a massive object, say a star. To

keep our analysis as elementary as possible, we shall only consider self-consistent gravitational interactions. From the mathematical point of view, scalings are crucial. We will therefore consider mostly *power law* dependences, which are easier to handle. That is the case of polytropic gases, or polytropes, which have already been mentioned above.

Strictly speaking, we are not going to prove any stability result neither at the kinetic level nor at the macroscopic level of nonlinear diffusions. As far as we know, such questions for systems in a solid motion of rotation are completely unknown. Our purpose is to characterize some special critical points of the free energy in the rotating reference frame, the localized minimizers, which are good candidates for a local stability analysis, that is still to be done.

This paper is organized as follows. The first section is devoted to a precise description of the model at the kinetic level and to the formal diffusion limit. The main results are then stated. For the convenience of the reader who is interested only in the mathematical results, all definitions and notations needed for the understanding of these results have been summarized in Appendix A. In the second section, the results of G. Rein [50] in the nonrotational case are adapted to our setting. See [30] for a more recent presentation of essentially the same results, which appeared as a preprint during the completion of this paper, and [29] for a related paper. Although mostly not original, the sketches of the proofs are given, since we will reuse them in the case of a nonzero angular velocity. The third section is devoted to the proof of the main results and some additional properties of the solutions, and the last one to some additional properties of the solutions.

As one can infer from this rather long introduction, our results are connected with many topics in modeling. Establishing an exhaustive list of relevant references would be a difficult task. Let us mention only a few papers in which the interested reader will be able to find a more complete list of references.

For an introduction to statistical physics of gravitating systems and issues about bounded domains models, we refer to [48]. An interesting list of issues concerning Maxwellian statistics and gravitational Vlasov–Poisson systems has been developed in [3]. Concerning power laws in astrophysics and Gibbs states, see [9], and [38] for some mathematical properties of such equilibrium states. From a physics point of view, see [15] for a recent justification of diffusion models and related thermodynamical functionals in astrophysics and two-dimensional turbulence, and references therein for earlier papers. Lyapunov functionals corresponding to porous media in an astrophysical context are usually called Tsallis’ entropies, referring to [58], and the evolution equation is often called the generalized Smoluchowski–Poisson equation. See [6–8] for some related papers. A whole series of papers concerning Lyapunov functionals, diffusion equations and formal diffusion limits has been written by Chavanis et al., see [12–19]. For a justification of the diffusion limits from a more mathematical point of view, let us quote [5,34], and also [26] and references therein. We will not insist on the following point in the paper, but it deserves to be mentioned that the energy profile of the Gibbs state is related with the entropy generating function by some convex duality, see [5,57,20,14,17]. This last paper and [5] contain formal Chapman–Enskog expansions of the type we are going to use in Section 1. As a last observation, we quote two recent papers, [40,39], which are devoted to *local minimizers*, in the sense that they are local in the function space. We will consider a different notion of minimizers, which is based on the localization of the support of the solutions, and for this reason, we shall call them *localized minimizers*.

Further references will be quoted throughout this paper whenever appropriate.

## 1. A kinetic model and its diffusion limit

Our goal is to describe steady state solutions of a flat rotating gravitational system. We start with a dynamical kinetic model and show at a formal level how solutions converge in a certain diffusion limit to solutions of a nonlinear drift-diffusion equation.

### 1.1. A kinetic model

At the kinetic level, we consider a *flat gravitational system* in three dimensions. In our model, the support of the mass distribution is contained in a two-dimensional plane and we assume that the system rotates around an axis of symmetry which is perpendicular to the plane. This amounts to consider a nonnegative solution of the following gravitational three-dimensional Vlasov–Poisson–Boltzmann system

$$\begin{cases} \partial_t F + v \cdot \nabla_x F - \nabla_x \psi \cdot \nabla_v F = Q_\omega(F), \\ \Delta \psi = \int_{\mathbb{R}^2 \times \mathbb{R}} F dv dw, \end{cases}$$

where the *distribution function*  $F$  is a measure concentrated on the manifold

$$\{(x, z), (v, w)\} \in (\mathbb{R}^2 \times \mathbb{R}) \times (\mathbb{R}^2 \times \mathbb{R}): z = 0, w = 0\}$$

and  $\mathcal{Q}_\omega(F)$  is a collision kernel which depends on the angular velocity  $\omega$ , to be specified later. We will consider the nonnegative distribution function  $F$  as a function of  $t \in \mathbb{R}$ ,  $x \in \mathbb{R}^2$  and  $v \in \mathbb{R}^2$ , and impose  $\psi$  to be given as a solution of the three-dimensional gravitational Poisson equation, corresponding to a measure valued distribution of mass whose support is constrained to the plane

$$\psi(t, x) = -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} F(t, x, v) dv.$$

Relevant position and velocity variables, respectively  $x$  and  $v$ , being in  $\mathbb{R}^2$ , we shall identify  $\mathbb{R}^2$  with  $\mathbb{C}$ . Instead of  $x = (x_1, x_2)$  and  $v = (v_1, v_2)$ , we shall write  $x = x_1 + ix_2$  and  $v = v_1 + iv_2$ , with the convention that  $v \cdot \nabla_x F$  means  $\text{Re}[(v_1 + iv_2)(\partial f/\partial x_1 - i\partial f/\partial x_2)] = \text{Re}[v \overline{\nabla_x f}]$ , and so on. We are interested in the effects of a rotation with constant angular velocity, and as a first step in this direction, we investigate the simple case where the solution has a global solid motion of rotation. For our purpose, it is therefore convenient to rewrite the equations in a rotating frame with constant angular velocity  $\omega$ . Using complex notations, we make the change of variables

$$(x, v) \mapsto (xe^{i\omega t}, (v + i\omega x)e^{i\omega t}) =: \mathcal{R}_{\omega,t}(x, v)$$

and define a new distribution function  $f$  by

$$F(t, x, v) = f(t, xe^{i\omega t}, (v + i\omega x)e^{i\omega t}) = (f \circ \mathcal{R}_{\omega,t})(x, v).$$

The equation satisfied by  $f$  can be written as

$$\partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2 \text{Re}(i\omega v \overline{\nabla_v f}) - \nabla_x \phi \cdot \nabla_v f = Q(f)$$

where the collision kernel  $Q$  is defined by  $Q(f) := \mathcal{Q}_\omega(F) \circ \mathcal{R}_{\omega,t}^{-1}$  and the potential  $\phi$  is given by

$$\phi(t, x) = -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f(t, x, v) dv.$$

Written in Cartesian coordinates, the equation satisfied by  $f$  is

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2\omega v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f &= Q(f), \\ \phi &= -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f dv, \end{aligned}$$

where  $a \wedge b := a^\perp \cdot b = (-a_2, a_1) \cdot (b_1, b_2) = a_1 b_2 - a_2 b_1 = \text{Re}[i(a_1 + ia_2)(b_1 - ib_2)]$ .

We are interested in stationary solutions and in the relaxation towards stationary solutions. For that purpose, we introduce a collision kernel which relaxes the distribution function towards a local Gibbs state in the rotating frame. By a local Gibbs state, we mean a function

$$G_f(t, x, v) = \gamma \left( \frac{1}{2}|v|^2 + \phi(t, x) - \frac{1}{2}\omega^2|x|^2 + \mu_f(t, x) \right),$$

where  $\gamma$  is a given energy profile and  $\mu_f$  a local Lagrange multiplier associated to the constraint

$$\int_{\mathbb{R}^2} G_f(t, x, v) dv = \int_{\mathbb{R}^2} f(t, x, v) dv. \tag{1}$$

Several kernels can be chosen, see for instance [17,26]. Here we simply consider the *relaxation time approximation* kernel which relaxes to the local Gibbs state in the rotating frame

$$\mathcal{Q}_\omega(F) = G_f \circ \mathcal{R}_{\omega,t} - F$$

or, equivalently,

$$Q(f) = G_f - f.$$

At this point we want to stress that such a kernel does not have any deep physical signification, but should be considered as a sound simplification of more realistic collision kernels. From a mathematical point of view, it is only a kind of a projection operator onto the local Gibbs state. See [26] for more comments. Notice that the local Gibbs state  $G_f$  has no mean velocity in the rotating reference frame. This justifies why  $Q$  does not depend on  $\omega$  while  $Q_\omega$  explicitly depends on the angular velocity. Taking  $Q$  independent of  $\omega$  simply means that the collision kernel tends to relax the solution towards a state in solid motion rotating at constant angular velocity  $\omega$ .

Condition (1) can be solved as follows: Define  $\bar{\mu}$  implicitly by the condition

$$\int_{\mathbb{R}^2} \gamma \left( \frac{1}{2} |v|^2 + \bar{\mu}(\rho) \right) dv = \rho,$$

which means that

$$\bar{\mu}(\rho) = \Gamma^{-1}(\rho) \tag{2}$$

where

$$\Gamma(s) := 2\pi \int_s^\infty \gamma(\sigma) d\sigma.$$

It is straightforward to check that the local Lagrange multiplier associated to (1) is such that

$$\mu_f(t, x) = \bar{\mu}(\rho(t, x)) - \phi(t, x).$$

If  $\Gamma$  is not invertible, an appropriate notion of generalized inverse has to be defined.

Collecting all these remarks, the problem written in the rotating frame solves the system

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2\omega v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f &= G_f - f, \\ \phi &= -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f dv. \end{aligned} \tag{3}$$

The angular velocity parametrizes the set of the solutions. For simplicity, we will restrict our statements to the case of *polytropic gases*, or *polytropes*, which corresponds to an energy profile given by a power law:

$$\gamma(s) := \left( \frac{-s}{k+1} \right)_+^k \quad \text{and} \quad \bar{\mu}(\rho) = -(k+1) \left( \frac{\rho}{2\pi} \right)^{\frac{1}{k+1}}$$

for some parameter  $k \in \mathbb{R}^+$ . The generalization to other nonincreasing energy profiles  $\gamma$  is easy and we leave it to the reader, but in any case, for a reason that will be made clear later on, we shall assume that  $\gamma \equiv 0$  on  $[0, +\infty)$ . In the polytropes case, we get

$$\Gamma(s) := 2\pi \left( \frac{-s}{k+1} \right)_+^{k+1} \quad \forall s \in \mathbb{R}.$$

We are going to focus on stationary solutions with fixed mass

$$M = \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f dx dv > 0. \tag{4}$$

For that purpose, let

$$\beta(s) := \int_s^0 \gamma^{-1}(\sigma) d\sigma$$

and define the free energy functional

$$\mathcal{F}_\omega[f] := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ f \left( \frac{1}{2} |v|^2 - \frac{1}{2} \omega^2 |x|^2 + \frac{1}{2} \phi \right) + \beta(f) \right] dx dv.$$

At a formal level, if  $f$  is a solution of (3), then

$$\frac{d}{dt} \mathcal{F}_\omega[f(t, \cdot, \cdot)] := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (G_f - f) \left( \frac{1}{2} |v|^2 + \beta'(f) \right) dx dv,$$

so that by writing  $\frac{1}{2} |v|^2 - \frac{1}{2} \omega^2 |x|^2 + \phi + \mu_f = -\beta'(G_f)$ , and using (1) and  $\beta'(f) = -\gamma^{-1}(f)$ , the entropy production term takes the form

$$\frac{d}{dt} \mathcal{F}_\omega[f(t, \cdot, \cdot)] := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} (G_f - f) (\gamma^{-1}(G_f) - \gamma^{-1}(f)) dx dv, \tag{5}$$

which has a negative sign if  $\gamma$  is nonincreasing. Moreover, if  $\gamma$  is decreasing on its support, it vanishes if and only if  $G_f = f$  almost everywhere on the support of  $f$ . As a consequence,  $f$  is in the kernel of the collision operator. Any stationary state  $f$  has therefore to be a global Gibbs state in the sense that

$$\mu_f(x) = \phi(x) - \frac{1}{2} \omega^2 |x|^2 - \mu^*$$

on the support of  $\rho := \int_{\mathbb{R}^2} f(\cdot, v) dv$ , for some constant  $\mu^* \in \mathbb{R}$ . To be precise, there is one such constant for each connected component of the support of  $\rho$ . This is easily deduced by reinjecting the local Gibbs state in the left-hand side of (3). As a consequence, such a stationary distribution function is a critical point of the free energy functional under the mass constraint  $\iint_{\mathbb{R}^2 \times \mathbb{R}^2} f dx dv = M > 0$ , fixed, and  $\mu^*$  can be identified as the Lagrange multiplier associated to this constraint. Note that because of the centrifugal force term  $-\omega^2 x$ , which gives rise to the potential energy term  $-\frac{1}{2} \omega |x|^2$ , the free energy does not have any global minimizer and is actually not even bounded from below. It is however easy to recover that a global Gibbs state taking the form

$$f^\infty(x, v) := \gamma \left( \frac{1}{2} |v|^2 + \phi^\infty(x) - \frac{1}{2} \omega^2 |x|^2 - \mu^* \right)$$

with  $\phi^\infty(x) := -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f^\infty(x, v) dv$  and  $\mu^*$  such that  $\iint_{\mathbb{R}^2 \times \mathbb{R}^2} f^\infty(x, v) dx dv = M$  is a critical point of  $\mathcal{F}_\omega$  under constraint (4).

In the special case of the polytropes, we can choose

$$\beta(f) = \frac{f^q}{q-1} \quad \text{with } k = \frac{1}{q-1} \iff q = 1 + \frac{1}{k}.$$

Note that  $\beta$  is defined up to a constant. The functional  $f \mapsto \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \beta(f) dx dv$  is convex as soon as  $\gamma$  is nonincreasing, which holds true in the case of the polytropes. Looking for stationary solutions with compact support requires  $q > 1$  or equivalently  $k > 0$ , and the problem is reduced to solve the fixed point equation

$$\phi = -\frac{1}{4\pi|x|} * \rho \quad \text{with } \rho = \Gamma \left( \phi - \frac{1}{2} \omega^2 |x|^2 - \mu^* \right), \tag{6}$$

where the last identity has to be satisfied only on the support of  $\rho$ , and  $\mu^*$  is implicitly fixed by the condition  $\int_{\mathbb{R}^2} \rho dx = M$ . Since

$$\Gamma(s) = 2\pi \left( -\frac{q-1}{q} s \right)^{\frac{q}{q-1}},$$

this amounts to write

$$-\frac{q}{q-1} \rho^{q-1} + \phi_{\text{eff}} - \mu^* = 0$$

on the support of  $\rho$ , where the *effective potential* in the rotating frame is defined by

$$\phi_{\text{eff}}(x) := -\frac{1}{4\pi|x|} * \rho - \frac{1}{2} \omega^2 |x|^2.$$

**1.2. Reduced variational problem**

Because of (5), if  $\gamma$  is decreasing on its support, stationary solutions of (3) are Gibbs states and therefore critical points of the free energy functional. But they are not global minimizers of the free energy functional, which, as said before, turns out to be unbounded from below for any nonzero angular velocity. However, since  $\rho = \int_{\mathbb{R}^2} G_f dv$ , we can write

$$\mathcal{F}_\omega[f] - \mathcal{F}_\omega[G_f] = \int_{\mathbb{R}^2} dx \left\{ \int_{\mathbb{R}^2} [\beta(f) - \beta(G_f) - \beta'(G_f)(f - G_f)] dv \right\},$$

so that to any critical point of

$$\rho \mapsto \mathcal{F}_\omega[\bar{G}_\rho] =: \mathcal{G}_\omega[\rho] \quad \text{with } \bar{G}_\rho(x, v) := \gamma\left(\frac{1}{2}|v|^2 + \bar{\mu}(\rho)\right)$$

and  $\phi(x) := -\frac{1}{4\pi|x|} * \rho$ , we may associate a critical point,  $\bar{G}_\rho$ , of  $\mathcal{F}_\omega$  under constraint (4). This reduced variational problem takes the form

$$\mathcal{G}_\omega[\rho] = \int_{\mathbb{R}^2} \left[ h(\rho) + \left(\frac{1}{2}\phi(x) - \frac{1}{2}\omega^2|x|^2\right)\rho \right] dx$$

with

$$\begin{aligned} h(\rho) &:= \int_{\mathbb{R}^2} \left[ (\beta \circ \gamma)\left(\frac{1}{2}|v|^2 + \bar{\mu}(\rho)\right) + \frac{1}{2}|v|^2\gamma\left(\frac{1}{2}|v|^2 + \bar{\mu}(\rho)\right) \right] dv \\ &= 2\pi \int_0^\infty [(\beta \circ \gamma)(s + \bar{\mu}(\rho)) + s\gamma(s + \bar{\mu}(\rho))] ds \\ &= 2\pi \int_{\bar{\mu}(\rho)}^\infty [(\beta \circ \gamma)(s) + s\gamma(s)] ds - \rho\bar{\mu}(\rho) \\ &= H(\bar{\mu}(\rho)) - \rho\bar{\mu}(\rho) \end{aligned}$$

with  $H(s) := \int_0^s \Gamma(\sigma) d\sigma$ . Here we used the fact that  $G(\bar{\mu}(\rho)) = \rho$ . Notice that  $h'(\rho) = -\bar{\mu}(\rho)$ . In the special case of the polytropes, we obtain

$$h(\rho) = \frac{\kappa}{m-1} \rho^m \quad \text{with } m = 2 - \frac{1}{q} = 1 + \frac{1}{k+1}, \quad \kappa = \frac{1}{m} (2\pi)^{1-m}.$$

Hence finding stationary solutions to (3) is equivalent to find critical points of  $\mathcal{G}_\omega$  on the set  $\{\rho \in L^1(\mathbb{R}^2): \rho \geq 0 \text{ a.e., } \int_{\mathbb{R}^2} \rho dx = M\}$ .

**1.3. Diffusion limit**

The stationary states of the kinetic equation are also the stationary states of a nonlinear drift-diffusion problem, whose associated free energy is the functional  $\mathcal{G}_\omega$ . This can be seen at a formal level by taking the diffusion limit in Eq. (3). By diffusion limit, we mean that the physical parameters of the problem have to be adjusted in order that in the corresponding regime the dynamics is dominated by the collision kernel and studied on a large time scale. After an appropriate adimensionalization, this means that we consider the equations

$$\begin{aligned} \varepsilon \partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2\omega v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f &= \frac{1}{\varepsilon} (G_f - f), \\ \phi &= -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f dv \end{aligned}$$

in the singular limit  $\varepsilon \rightarrow 0$ . Multiply the first equation by 1 and  $v$ , and integrate with respect to  $v$ . The corresponding system is

$$\partial_t \rho + \nabla_x \cdot j = 0, \tag{7}$$

$$\varepsilon^2 \partial_t j + \nabla_x \cdot \int_{\mathbb{R}^2} v \otimes v f \, dv - \omega^2 x \rho + \rho \nabla_x \phi = -j, \tag{8}$$

where

$$\rho = \int_{\mathbb{R}^2} f \, dv \quad \text{and} \quad j := \frac{1}{\varepsilon} \int_{\mathbb{R}^2} v f \, dv.$$

At a heuristic level, it is quite easy to identify the limit by an appropriate Chapman–Enskog expansion, see for instance [4,5,17]. For a rigorous approach, but in the case where there is no self-consistent potential, we refer to [26] and references therein. Let  $f = f_0 + \varepsilon f_1 + O(\varepsilon^2)$  and formally identify the limits order by order. At order  $O(\varepsilon^{-1})$ , we get

$$G f_0 - f_0 = O(\varepsilon).$$

By passing to the limit  $\varepsilon \rightarrow 0$ , this means

$$f_0 = \gamma \left( \frac{1}{2} |v|^2 + \bar{\mu}(\rho) \right)$$

where  $\rho$  is the solution of (7). To complete the description of the diffusion limit, we only need to identify  $j$ . Consider therefore (8). Again at a formal level, by identifying the first order term in  $\varepsilon$ , we get

$$j = -\nabla_x \cdot \int_{\mathbb{R}^2} v \otimes v f_0 \, dv + \omega^2 x \rho - \rho \nabla_x \phi + O(\varepsilon).$$

Passing formally to the limit  $\varepsilon \rightarrow 0$ , we get

$$j = -\nabla_x (v(\rho)) + \omega^2 x \rho - \rho \nabla_x \phi,$$

where  $v$  is given in terms of  $\rho$  by

$$v(\rho) := \frac{1}{2} \int_{\mathbb{R}^2} |v|^2 \gamma \left( \frac{1}{2} |v|^2 + \bar{\mu}(\rho) \right) \, dv. \tag{9}$$

Notice that

$$v'(\rho) = 2\pi \int_0^\infty s \gamma'(s + \bar{\mu}(\rho)) \, ds \cdot \bar{\mu}'(\rho) = -2\pi \int_0^\infty \gamma(s + \bar{\mu}(\rho)) \, ds \cdot \bar{\mu}'(\rho) = -\rho \cdot \bar{\mu}'(\rho),$$

using again the fact that  $\Gamma(\bar{\mu}(\rho)) = \rho$ , so that

$$v(\rho) = - \int_0^\rho \sigma \bar{\mu}'(\sigma) \, d\sigma.$$

Summarizing, we have obtained that, in the limit  $\varepsilon \rightarrow 0$ ,  $\rho = \rho(t, x)$  is a solution of

$$\begin{aligned} \partial_t \rho &= \nabla \cdot [\nabla v(\rho) - \omega^2 x \rho + \rho \nabla_x \phi], \\ \phi &= -\frac{1}{4\pi |x|} * \rho \end{aligned} \tag{10}$$

with  $\bar{\mu}$  and  $v$  given by (2) and (9). In the case of the polytropes, we have

$$v(\rho) = \kappa \rho^m \quad \text{with} \quad \kappa = \frac{1}{m} (2\pi)^{1-m}.$$



1.4. Ranges of validity

We will refer to [4,5,17,13] for further discussions on formal asymptotics and applications in physics, and to [26] and references therein for rigorous proofs of diffusion limits providing nonlinear diffusion equations.

As quoted in [5], the functional  $\mathcal{G}_\omega$  is a Lyapunov functional for (10). For a solution, using the fact that  $h'(\rho) = -\bar{\mu}(\rho)$ , we get

$$\frac{d}{dt} \mathcal{G}_\omega[\rho(t, \cdot)] = - \int_{\mathbb{R}^2} \rho |\nabla \bar{\mu}(\rho) - (\nabla_x \phi - \omega^2 x)|^2 dx.$$

This expression can be recovered by taking the limit as  $\varepsilon \rightarrow 0$  of (5). See [4,5] for more details. As a consequence, we recover that for the nonlinear diffusion equation (10) as for the kinetic model (3), stationary states are given by (6).

The assumption  $\gamma \equiv 0$  on  $[0, +\infty)$  guarantees the existence of solutions with finite mass. At the kinetic level, see [38]. This can also be seen at the level of the diffusion equation. With porous media type diffusion, which is the case for all polytropes, the diffusion coefficient  $v'(\rho)$  becomes 0 as the density decays to 0 and stationary solutions have compact support. Otherwise, all mass would diffuse and runaway because of the centrifugal force. Such a runaway phenomenon occurs for instance with linear diffusions, or more elaborate nonlinear diffusions like the one based on the Fermi–Dirac distribution, see for instance [13]. For completeness, let us mention a related work in three dimensions, [47], and references therein. We will come back to this paper in the conclusion.

Up to now, no restrictions have been imposed on the power  $k \in \mathbb{R}^+$ . Since

$$m = 2 - \frac{1}{q} = \frac{k + 2}{k + 1},$$

the range of  $m$  is therefore  $(1, 2)$ , which corresponds to  $q \in (1, \infty)$ . However, in the case  $\omega = 0$ , the existence of a minimizer is achieved if and only if  $q$  is big enough to prevent concentration of minimizing sequences. The reason goes as follows. By a standard interpolation method, one can prove that for  $\rho = \int_{\mathbb{R}^2} f dv$ ,  $f \geq 0$  a.e.,

$$\|\rho\|_{L^m(\mathbb{R}^2)} \leq C_{\text{Interp}} \|f\|_{L^q(\mathbb{R}^2 \times \mathbb{R}^2)}^{\frac{q}{2q-1}} \left( \iint_{\mathbb{R}^2 \times \mathbb{R}^2} |v|^2 f dx dv \right)^{\frac{q-1}{2q-1}} \quad \text{with } m = 2 - \frac{1}{q}$$

for some positive constant  $C_{\text{Interp}}$ , see for instance [23]. Then

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x - y|} dx dy \leq C_{\text{HLS}} \|\rho\|_{L^{4/3}(\mathbb{R}^2)}^2$$

for some positive constant  $C_{\text{HLS}}$ , by the Hardy–Littlewood–Sobolev inequality, see, e.g., [43]. The free energies  $\mathcal{F}_\omega$  and  $\mathcal{G}_\omega$  are therefore bounded from below if  $4/3 \in (1, m)$ , which amounts to

$$q \in (3/2, \infty) \iff m \in (4/3, 2) \iff k \in (0, 2).$$

We refer to [50] for more details. The critical case corresponds to  $q = \frac{3}{2}$ ,  $m = \frac{4}{3}$ ,  $k = 2$  and will not be dealt with here since it requires a specific analysis which is out of the scope of this paper.

In [50], Rein considers flat distribution functions  $f$  as in Section 1.1, which solve the Vlasov–Poisson system without collision kernel, in the case with no angular motion, i.e.,  $\omega = 0$ , and proves the existence of a radially symmetric and compactly supported minimizer of the free energy functional  $\mathcal{F}_\omega$  on the set

$$\left\{ f \in L^1(\mathbb{R}^4): f \geq 0, f \text{ is radially symmetric, } \iint_{\mathbb{R}^2 \times \mathbb{R}^2} f dx dv = M \right\}.$$

Here by *radially symmetric* we mean that  $f$  depends only on  $|x|$ ,  $|v|$  and  $(x \cdot v)$ , and  $\beta$  is assumed to be a convex function verifying appropriate conditions both near the origin and at infinity. In Section 2 we expose the results of [50,30] without the radial symmetry assumption, but in the simpler setting of nonlinear diffusions. These results can also be seen as relevant for the Euler–Poisson equation, see [30]. The more general case of a nonzero angular velocity  $\omega$  is considered in Section 3.

Before giving more details, let us mention again that a major difficulty appears when  $\omega$  is turned on, since there are no more global minimizers of the free energy. The difficulty can be overcome by solving the problem with positions in a fixed ball of radius  $R$ , which from a physical point of view prevents the runaway of the particles when the centrifugal force dominates the gravitational attraction. In this compactly supported case, the results for the nonrotational case can be adapted, see Section 3 for details, and as  $\omega$  tends to 0, minimizers converge, up to a subsequence, to a minimizer corresponding to  $\omega = 0$ . The minimizers in the rotational case are however not known to be radial, not even for very small angular velocities, but their support is close to the one of solutions with  $\omega = 0$ . The nonradiality of the minimizers, at least for sufficiently large angular velocities, would not be a complete surprise. Indeed the results of [33,32] and their extensions do not apply because the centrifugal force term has the wrong monotonicity, and a phenomenon of symmetry breaking has been observed for instance in the case of Caffarelli–Kohn–Nirenberg inequalities, see [11,28], and of the Hénon problem, see [56], which share some qualitative features with our minimization problem. If  $R$  has been taken large enough initially, solutions are global stationary solutions not only in the ball of radius  $R$ , but also in the whole euclidean space. By a continuation argument, one can prove that such an  $R$  can be found for larger and larger values of  $\omega$ , until some critical value is reached. This critical value depends on the mass  $M$  of the solution.

### 1.5. Main results

From now on, we assume that the setting is the one of the polytropes:

$$\beta(f) = \frac{f^q}{q-1}, \quad h(\rho) = \frac{\kappa}{m-1} \rho^m,$$

with  $q \in (3/2, \infty)$ ,  $m \in (4/3, 2)$  or  $k \in (0, 2)$ . More general settings are not very difficult to handle, but require case by case technical assumptions, and will therefore not be considered here. Unless it is explicitly specified, variables are all in  $\mathbb{R}^2$ . Integrals are also supposed to be extended to the whole space unless otherwise stated.

The above derivation of the model is quite lengthy, so for the convenience of the reader, we have summarized all useful equations and notations in an appendix. Recall that

$$\mathcal{G}_\omega[\rho] := \frac{\kappa}{m-1} \int_{\mathbb{R}^2} \rho^m dx - \frac{\omega^2}{2} \int_{\mathbb{R}^2} |x|^2 \rho dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x-y|} dx dy$$

is the so called *reduced free energy functional*, and consider the set

$$\mathcal{X}_M := \left\{ \rho \in \mathcal{L}(\mathbb{R}^2): \exists R > 0 \text{ such that } \text{supp}(\rho) \subset B(0, R) \text{ and } \int_{\mathbb{R}^2} \rho dx = M \right\},$$

where  $\mathcal{L}(\mathbb{R}^2)$  is the set of nonnegative functions  $\rho$  in  $L^1(\mathbb{R}^2)$  satisfying the assumption

$$\rho(x) = \rho(-x) \quad \forall x \in \mathbb{R}^2 \text{ a.e.} \tag{11}$$

**Definition.** A *localized minimizer* is a critical point  $\rho$  of  $\mathcal{G}_\omega$  which is compactly supported in a ball  $B(0, R - \varepsilon)$  for some  $R > 0$  and  $\varepsilon \in (0, R)$ , and which is a minimizer of  $\mathcal{G}_\omega$  restricted to the set  $\{\rho \in L^1_+(\mathbb{R}^2): \text{supp}(\rho) \subset B(0, R) \text{ and } \int_{\mathbb{R}^2} \rho dx = M\}$ .

We will also consider *radial localized minimizers* which are radial critical points as above, with the minimization set further restricted to radial functions.

**Theorem 1.** For any  $M > 0$ , there exists  $\omega_*(M) = \omega_* > 0$  and  $\omega^*(M) = \omega^* > 0$  with  $\omega_* \leq \omega^*$  such that

- (i) If  $\omega \in [-\omega_*, \omega_*]$ ,  $\mathcal{G}_\omega[\rho]$  admits a localized minimizer  $\rho_\infty^\omega \in \mathcal{X}_M$ .
- (ii) If  $|\omega| > \omega^*$ ,  $\mathcal{G}_\omega[\rho]$  admits no localized minimizer.

Our next goal is to give a characterization of  $\omega_*$ . The idea is simpler to explain in the case of radial functions. Define therefore  $\mathcal{S}_{M,\text{rad}}^\omega$  as the set of radial localized minimizers of mass  $M$ . Each  $\rho_\infty^\omega \in \mathcal{S}_{M,\text{rad}}^\omega$  can be explicitly written in terms of its own potential

$$\rho_\infty^\omega(x) = A_m \left( \lambda[\rho_\infty^\omega] + \frac{\omega^2}{2} |x|^2 - \phi_\infty^\omega(x) \right)_+^{\frac{1}{m-1}} \quad \text{where } \phi_\infty^\omega = -\frac{1}{4\pi|\cdot|} * \rho_\infty^\omega,$$

with  $A_m := [\frac{m-1}{\kappa m}]^{1/(m-1)}$ , and  $\lambda[\rho_\infty^\omega] < 0$  a parameter which is determined by  $M$ , but eventually depending on  $\rho_\infty^\omega$ . We define

$$R_1[\rho_\infty^\omega] := \min\{r > 0: \text{supp}(\rho_\infty^\omega) \subset B(0, r)\}$$

and notice that for any  $x \in \partial B(0, R_1[\rho_\infty^\omega])$ ,  $\lambda[\rho_\infty^\omega] = \phi_\infty^\omega(x) - \frac{\omega^2}{2} |x|^2$ . Let

$$h[\rho_\infty^\omega] := \sup\left\{ \phi_\infty^\omega(x) - \frac{\omega^2}{2} |x|^2: |x| > R_1[\rho_\infty^\omega] \right\}.$$

We observe that  $\lambda[\rho_\infty^0] < h[\rho_\infty^0] = 0$ . This suggests to consider

$$\sup\left\{ \omega \geq 0: \sup_{\rho_\infty^\omega \in \mathcal{S}_{M,\text{rad}}^\omega} (\lambda[\rho_\infty^\omega] - h[\rho_\infty^\omega]) < 0 \right\}.$$

Since, even for radial functions, uniqueness of localized minimizers is not known if  $\omega \neq 0$ , further precautions are therefore needed. Let

$$R_2[\rho_\infty^\omega] = \max\left\{ r > R_1[\rho_\infty^\omega]: \exists x \in \partial B(0, r) \text{ such that } \phi_\infty^\omega(x) - \frac{\omega^2}{2} |x|^2 = h[\rho_\infty^\omega] \right\}.$$

In fact, in the radial case with  $\omega \neq 0$ , the effective potential  $\phi_\infty^\omega(x) - \frac{\omega^2}{2} |x|^2$  attains its maximum at only one point.

We shall say that *Property*  $\mathcal{P}_\omega^{\text{rad}}$  holds true if and only if there exists  $R(\omega)$  such that  $R_1[\rho_\infty^\omega] < R(\omega) < R_2[\rho_\infty^\omega]$  for any  $\rho_\infty^\omega \in \mathcal{S}_{M,\text{rad}}^\omega$ .

**Theorem 2.** *For any  $M > 0$ , the maximal interval in  $\omega$  containing  $\omega = 0$  for which *Property*  $\mathcal{P}_\omega^{\text{rad}}$  holds true is an open interval.*

Let  $\omega_*^{\text{rad}} := \sup\{\omega \geq 0: \mathcal{P}_\omega^{\text{rad}} \text{ holds true}\}$ . Observe that if there is at most one localized minimizer  $\rho_\omega$  for any given value of  $\omega$ , then  $\mathcal{P}_\omega^{\text{rad}}$  holds true as long as  $\lambda[\rho_\infty^\omega] < h[\rho_\infty^\omega]$ . On the other hand, for any  $M > 0$  and for any  $\omega \in (-\omega_*^{\text{rad}}, \omega_*^{\text{rad}})$ , there exists a radial localized minimizer and  $\lambda[\rho_\infty^\omega] < h[\rho_\infty^\omega]$  for any  $\rho_\infty^\omega \in \mathcal{S}_{M,\text{rad}}^\omega$ . The proof is based on a continuation argument: we shall prove that for any  $\omega_0 \geq 0$ , if  $\mathcal{P}_{\omega_0}^{\text{rad}}$  holds true, then one can construct radial localized minimizers for any  $\omega > \omega_0$ ,  $\omega - \omega_0$  small enough, and *Property*  $\mathcal{P}_\omega^{\text{rad}}$  still holds true, see Section 3.5 for more details. If  $\mathcal{G}_{\omega_*^{\text{rad}}}$  has a unique radial compactly supported critical point, then

$$\lambda[\rho_\infty^{\omega_*^{\text{rad}}}] = h[\rho_\infty^{\omega_*^{\text{rad}}}] .$$

If localized minimizers are not radial, one can also define  $\lambda[\rho_\infty^\omega]$  and  $h[\rho_\infty^\omega]$  for each  $\rho_\infty^\omega$  using a set of paths connecting  $\text{supp}(\rho_\infty^\omega)$  to infinity. Under an additional technical assumption, we will be able to extend the statement of Theorem 2 to this situation. See Theorem 25 in Section 3 for more details. Notice that as soon as symmetry is broken, uniqueness is also lost due to the invariance under rotation of the equations.

Because of the technical assumption in the nonradial case, we will only give a lower bound for  $\omega_*$ . It is an open question to decide whether  $\omega_* = \omega^*$  or not, but one can reasonably conjecture that localized minimizers exist for any  $\omega \in (0, \omega^*)$ .

Notice that with the above definition of localized minimizers, an endpoint of the family  $(\rho_\infty^\omega)_{\omega \in (0, \omega_*)}$  cannot be considered as a localized minimizer. On the opposite, if  $\omega \rightarrow 0$ , the localized minimizers converge, up to a subsequence, to a global radial minimizer of  $\mathcal{G}_0$  verifying condition (11), as we shall see later.

## 2. Preliminary results: the case $\omega = 0$

We start by fixing  $\omega = 0$  and study the existence of minimizers of the energy for the following nonrotational problem. On the set

$$\mathcal{X}_M = \left\{ \rho \in L^1_+(\mathbb{R}^2) : \int_{\mathbb{R}^2} \rho(x) dx = M \right\},$$

where  $L^1_+(\mathbb{R}^2)$  denotes the set of functions  $\rho \in L^1(\mathbb{R}^2)$  that are positive almost everywhere, and with  $\phi_\rho$  given in terms of  $\rho$  by

$$\phi_\rho = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{\rho(y)}{|x-y|} dy,$$

consider the reduced free energy

$$\mathcal{G}_0[\rho] = \frac{\kappa}{m-1} \int_{\mathbb{R}^2} \rho^m dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi_\rho \rho dx = \frac{\kappa}{m-1} \int_{\mathbb{R}^2} \rho^m dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x-y|} dx dy.$$

**Problem 1.** Fix  $M > 0$  and  $m \in (3/2, 2)$  and find all minimizers of

$$\mathcal{I}_M^0 = \inf\{\mathcal{G}_0[\rho] : \rho \in \mathcal{X}_M\}.$$

This problem has been considered by Rein in [50] in view of stability results for kinetic equations, and more recently in [30] using a reduced free energy functional and tools of the concentration-compactness method. We will therefore give only the sketches of the proofs, for the completeness of this paper, and also because we are going to reuse several intermediate results and techniques in the rotational case,  $\omega \neq 0$ . Our presentation differs from the ones of [50,30] only by minor details (see below) and we do not claim any originality here.

**Theorem 3.** Fix  $M > 0$  and  $m \in (3/2, 2)$ . The infimum  $\mathcal{I}_M^0$  is achieved by at least one minimizer  $\rho_\infty^0 \in \mathcal{X}_M$  of  $\mathcal{G}_0$ . All minimizers are compactly supported, radially symmetric, monotone nonincreasing and take the form

$$\rho_\infty^0(x) = \left( \frac{m-1}{\kappa m} \right)^{\frac{1}{m-1}} (\lambda^0 - \phi_\infty^0(x))_+^{\frac{1}{m-1}} \quad \forall x \in \mathbb{R}^2,$$

for some  $\lambda^0 < 0$  which is the same for all of them. Here  $\phi_\infty^0 := \phi_{\rho_\infty^0}$  denotes the potential associated to the density  $\rho_\infty^0$ .

To prove this result, one has to consider a minimizing sequence  $(\rho_n)_{n \in \mathbb{N}}$  for Problem 1 and it is sufficient to prove that there exists  $\rho_\infty^0 \in \mathcal{X}_M$  such that, up to the extraction of a subsequence,  $\rho_n$  weakly converges  $\rho_\infty^0$  in  $L^1 \cap L^m(\mathbb{R}^2)$  and  $\lim_{n \rightarrow \infty} \mathcal{G}_0[\rho_n] \geq \mathcal{G}_0[\rho_\infty^0]$ . The proof of Theorem 3 relies on a series of lemmata. Observe that in the setting of Theorem 3,  $m > 3/2 > 4/3$  so that  $L^{4/3}(\mathbb{R}^2) \subset L^1 \cap L^m(\mathbb{R}^2)$ .

**Lemma 4.** For any  $\rho \in L^1(\mathbb{R}^2) \cap L^m(\mathbb{R}^2)$ ,  $m \in (4/3, 2)$ ,  $\phi_\rho$  is in  $L^q(\mathbb{R}^2)$  for  $\frac{1}{q} = \frac{1}{m} - \frac{1}{2}$  and there exists a positive constant  $C$  such that

$$\frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x-y|} dx dy \leq C \|\rho\|_{L^1(\mathbb{R}^2)}^{2\theta} \|\rho\|_{L^m(\mathbb{R}^2)}^{2(1-\theta)}$$

where  $\theta = \frac{3m-4}{4(m-1)}$ . As a consequence,

$$\mathcal{G}_0[\rho] \geq \frac{\kappa}{m-1} \|\rho\|_{L^m(\mathbb{R}^2)}^m - \frac{C}{2} \|\rho\|_{L^1(\mathbb{R}^2)}^{2\theta} \|\rho\|_{L^m(\mathbb{R}^2)}^{2(1-\theta)} \tag{12}$$

is bounded from below for any  $m \in (3/2, 2)$ .

**Proof.** By the usual Hardy–Littlewood–Sobolev inequality,  $\phi_\rho$  belongs to  $L^q(\mathbb{R}^2)$ . Using Hölder’s inequality, we get

$$\frac{1}{4\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x-y|} dx dy \leq C \|\rho\|_{L^{4/3}(\mathbb{R}^2)}^2,$$

and  $\|\rho\|_{L^{4/3}(\mathbb{R}^2)}^2$  can be interpolated between  $L^1$  and  $L^m$  again by Hölder’s inequality. Concerning the reduced free energy, observe that

$$x \mapsto \frac{\kappa}{m-1} x^m - \frac{C}{2} M^{2\theta} x^{2(1-\theta)}$$

is bounded from below as long as  $2(1-\theta) < m$ , that is for  $m \in (3/2, 2)$ .  $\square$

**Lemma 5.** *With the above notations,  $-\infty < \mathcal{I}_M^0 < 0$  if  $m \in (3/2, 2)$ . As a consequence,  $\mathcal{I}_M^0$  is achieved on the set  $\{\rho \in \mathcal{X}_M: \|\rho\|_{L^m(\mathbb{R}^2)} \leq A\}$  for some positive constant  $A$  which only depends on  $M$  and  $m$ .*

**Proof.** This follows from a scaling argument. Consider  $\rho \in \mathcal{X}_M$  and for any  $\lambda > 0$ , define  $\rho^\lambda \in \mathcal{X}_M$  by

$$\rho^\lambda(x) := \lambda^2 \rho(\lambda x) \quad \forall x \in \mathbb{R}^2.$$

Then

$$\mathcal{G}_0[\rho^\lambda] = \lambda^{2(m-1)} \left( \frac{\kappa}{m-1} \int_{\mathbb{R}^2} \rho^m dx - \frac{\lambda^{3-2m}}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x-y|} dx dy \right)$$

is negative for  $\lambda$  small enough, and so is  $\mathcal{I}_M^0$ . Then by (12),  $x = \|\rho\|_{L^m(\mathbb{R}^2)}$  is such that  $\frac{\kappa}{m-1} x^m - \frac{C}{2} M^{2\theta} x^{2(1-\theta)} \leq 0$ , which determines an explicit expression of

$$A := \left( \frac{(m-1)C}{2\kappa} \right)^{\frac{2(m-1)}{m(2m-3)}} M^{\frac{3m-4}{m(2m-3)}}. \quad \square$$

**Lemma 6.** *Let  $0 < M_1 < M_2$ . Then*

$$\mathcal{I}_{M_1}^0 \geq \left( \frac{M_1}{M_2} \right)^{\frac{3m-4}{2m-3}} \mathcal{I}_{M_2}^0.$$

**Proof.** This also follows from a scaling argument. For  $\rho \in \mathcal{X}_{M_2}$ , take  $\mu, \lambda > 0$  and define  $\rho^{\lambda,\mu}$  by

$$\rho^{\lambda,\mu}(x) = \mu \rho(\lambda x) \quad \forall x \in \mathbb{R}^2.$$

Then  $\rho^{\lambda,\mu} \in \mathcal{X}_{M_1}$  if and only if  $M_1 = \mu \lambda^{-2} M_2$ . Since

$$\mathcal{G}_0[\rho^{\lambda,\mu}] = \frac{\mu^m \lambda^{-2\kappa}}{m-1} \int_{\mathbb{R}^2} \rho^m dx - \frac{1}{8\pi} \mu^2 \lambda^{-3} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x-y|} dx dy,$$

by taking  $\mu^m \lambda^{-2} = \mu^2 \lambda^{-3}$ , i.e.,  $\lambda = \mu^{2-m}$ , we get  $\mathcal{G}_0[\rho^{\lambda,\mu}] = \mu^{3m-4} \mathcal{G}_0[\rho]$ . Hence

$$\mu = \left( \frac{M_1}{M_2} \right)^{\frac{1}{2m-3}} \quad \text{and} \quad \mathcal{G}_0[\rho^{\lambda,\mu}] = \left( \frac{M_1}{M_2} \right)^{\frac{3m-4}{2m-3}} \mathcal{G}_0[\rho],$$

which completes the proof.  $\square$

Observe that by Lemmata 5 and 6, we can relax the mass constraint in Problem 1 and simply require that  $\int_{\mathbb{R}^2} \rho dx \leq M$ .

Denote by  $\rho^*$  the radially symmetric decreasing rearrangement associated to a given function  $\rho$ .

**Lemma 7.** For any  $\rho \in \mathcal{X}_M$ ,

$$\mathcal{G}_0[\rho^*] \leq \mathcal{G}_0[\rho].$$

Moreover, the inequality is strict unless  $\rho$  and  $\rho^*$  are equal up to a translation.

**Proof.** It is an immediate consequence of the preservation of the  $L^p$  norm, for any  $p \geq 1$ , by the symmetric rearrangement, and of Riesz’ inequality, see [44] for more details.  $\square$

Define

$$M(R) := \int_{|x|>R} \rho(x) dx.$$

**Lemma 8.** (See [50].) There exists a constant  $K > 0$  such that, for any radially symmetric  $\rho \in L^1_+ \cap L^{4/3}(\mathbb{R}^2)$  and any  $R > 0$

$$- \int_{|x|>R} \phi_\rho \rho dx \leq \frac{K M(R)}{\sqrt{R}} \|\rho\|_{L^{4/3}(\mathbb{R}^2)}.$$

For any  $\alpha \geq 2$ , there exists a positive constant  $C(\alpha)$  such that

$$1 - (1 - x)^\alpha - x^\alpha \geq C(\alpha)x(1 - x) \quad \forall x \in (0, 1).$$

With  $m \in (3/2, 2)$ , take  $\alpha = \frac{3m-4}{2m-3} > 2$ . Let  $C_m := C(\frac{3m-4}{2m-3})$  and  $A$  be the constant in Lemma 5. If  $\|\rho\|_{L^m(\mathbb{R}^2)} \leq A$ , then  $\|\rho\|_{L^{4/3}(\mathbb{R}^2)} \leq M^\theta A^{1-\theta}$ , with  $\theta = \frac{3m-4}{4(m-1)}$ .

**Lemma 9.** (See [50].) For any radially symmetric  $\rho \in \mathcal{X}_M$  such that  $\|\rho\|_{L^m(\mathbb{R}^2)} \leq A$  and any  $R > 0$ ,

$$\mathcal{G}_0[\rho] - \mathcal{I}_M^0 \geq - \left( \frac{C_m \mathcal{I}_M^0}{M^2} (M - M(R)) + \frac{K M^\theta A^{1-\theta}}{\sqrt{R}} \right) M(R).$$

**Proof.** Let  $\rho_1 := \rho \mathbb{I}_{B(0,R)}$  and  $\rho_2 := \rho - \rho_1$  and  $\phi_i = \phi_{\rho_i}$ . By Lemma 8

$$\mathcal{G}_0[\rho] = \frac{\kappa}{m-1} \left( \int_{\mathbb{R}^2} \rho_1^m dx + \int_{\mathbb{R}^2} \rho_2^m dx \right) + \frac{1}{2} \int_{\mathbb{R}^2} \phi_1 \rho_1 dx + \frac{1}{2} \int_{\mathbb{R}^2} \phi_2 \rho_2 dx + \int_{\mathbb{R}^2} \phi_1 \rho_2 dx$$

is bounded from below by

$$\mathcal{I}_{M-M(R)}^0 + \mathcal{I}_{M(R)}^0 - \frac{K M^\theta A^{1-\theta} M(R)}{\sqrt{R}}.$$

By Lemma 6 and since  $\mathcal{I}_M^0 < 0$ , with  $x = \frac{M}{M(R)}$ , we get

$$\mathcal{G}_0[\rho] - \mathcal{I}_M^0 + \frac{K M^\theta A^{1-\theta} M(R)}{\sqrt{R}} \geq [(1-x)^\alpha + x^\alpha - 1] \mathcal{I}_M^0 \geq -C_m x(1-x) \mathcal{I}_M^0,$$

so the proof is complete.  $\square$

**Proof of Theorem 3.** Consider a minimizing sequence  $(\rho_n)_{n \in \mathbb{N}}$  for Problem 1. We can further assume that all  $\rho_n$  are radial and such that  $\|\rho_n\|_{L^m(\mathbb{R}^2)} \leq A$ . By Lemma 9, we can also require that the functions  $\rho_n$  are supported in a fixed ball of radius

$$R \leq \left( \frac{K M^{1+\theta} A^{1-\theta}}{C_m |\mathcal{I}_M^0|} \right)^2.$$

Hence, up to the extraction of a subsequence,  $\rho_n$  converges as  $n \rightarrow \infty$  to some limit  $\rho_\infty^0 \in \mathcal{X}_M$  weakly in  $L^1 \cap L^m(\mathbb{R}^2)$ . By the Fréchet–Kolmogorov criterion,  $\phi_{\rho_n}$  strongly converges to  $\phi_{\rho_\infty^0}$  in  $L^4(\mathbb{R}^2)$ , which shows that  $\mathcal{G}_0[\rho_\infty^0] = \mathcal{I}_M^0$ . See [50] for more details.

The expression of  $\rho_\infty^0$  follows from the Euler–Lagrange equations, where the Lagrange multiplier  $\lambda^0$  associated to the mass constraint comes out to be negative. For the assertion that the Lagrange multiplier is the same for all minimizers, for any  $\rho_\infty^0$  a minimizer consider the scaling  $\rho^\mu(x) := \mu^2 \rho_\infty^0(\mu x)$ . Then the function  $F(\mu) = \mathcal{G}_0(\rho^\mu)$  attains a minimum at  $\mu = 1$ . This means that

$$2\kappa \int_{\mathbb{R}^2} (\rho_\infty^0)^m dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho_\infty^0(x)\rho_\infty^0(y)}{|x - y|} dx dy = 0.$$

So

$$\mathcal{I}_M^0 = \frac{3 - 2m}{m - 1} \kappa \int_{\mathbb{R}^2} (\rho_\infty^0)^m dx,$$

so the  $L^m$  norm and the potential energy of all the minimizers is the same. Since

$$\lambda_0[\rho_\infty^0]M = \frac{\kappa m}{m - 1} \int_{\mathbb{R}^2} (\rho_\infty^0)^m dx + \int_{\mathbb{R}^2} \phi_\infty^0 \rho_\infty^0 dx,$$

the result follows.  $\square$

Up to our knowledge, uniqueness of minimizers for a given mass has not been proved yet, although it seems quite probable.

As a conclusion to this section, we observe that the inequality  $\mathcal{G}_0[\rho] \geq \mathcal{I}_M^0$  for all  $\rho \in \mathcal{X}_M$  can be rewritten as

$$\frac{\kappa}{m - 1} \|\rho\|_{L^m(\mathbb{R}^2)}^m dx + |\mathcal{I}_0^1| \|\rho\|_{L^1(\mathbb{R}^2)}^{\frac{3m-4}{2m-3}} \geq \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x - y|} dx dy \quad \forall \rho \in \mathcal{X}_M.$$

Written for  $\rho^\lambda(x) := \lambda^{3/2} \rho(\lambda x)$ , the right-hand side of the above inequality is independent of  $\lambda$ . By optimizing with respect to  $\lambda > 0$ , we get

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x - y|} dx dy \leq C_m \|\rho\|_{L^1(\mathbb{R}^2)}^{\frac{3m-4}{2(m-1)}} \|\rho\|_{L^m(\mathbb{R}^2)}^{\frac{m}{2(m-1)}} \quad \forall \rho \in \mathcal{X}_M,$$

with  $C_m := 16\pi \kappa^{\frac{1}{2(m-1)}} \left(\frac{m-1}{2m-3} |\mathcal{I}_0^1|\right)^{\frac{2m-3}{2(m-1)}}$ . Using  $\kappa = \frac{1}{m} (2\pi)^{1-m}$ , we get

$$C_m := 8\sqrt{2\pi} m^{-\frac{1}{2(m-1)}} \left(\frac{m-1}{2m-3} |\mathcal{I}_0^1|\right)^{\frac{2m-3}{2(m-1)}}.$$

By homogeneity, we observe that the constraint  $\|\rho\|_{L^1(\mathbb{R}^2)} = M$  does not play any role anymore and that the inequality is invariant under scalings.

**Corollary 10.** *Let  $m \in (\frac{3}{2}, 2)$ . For any  $\rho \in L^1_+ \cap L^m(\mathbb{R}^2)$ ,*

$$\iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x - y|} dx dy \leq C_m \|\rho\|_{L^1(\mathbb{R}^2)}^{\frac{3m-4}{2(m-1)}} \|\rho\|_{L^m(\mathbb{R}^2)}^{\frac{m}{2(m-1)}},$$

*the constant  $C_m$  is optimal and the equality case is achieved by*

$$\rho(x) = \left(-1 - \phi_\rho(x)\right)_+^{\frac{1}{m-1}} \quad \forall x \in \mathbb{R}^2$$

*where  $\rho$  is radially symmetric and  $\phi_\rho = \frac{1}{4\pi|\cdot|} * \rho$ .*

### 3. The case $\omega \neq 0$

#### 3.1. The variational problem

Consider now the case  $\omega \neq 0$  and recall that for a density  $\rho \in L^1_+(\mathbb{R}^2)$ , the reduced free energy is

$$\mathcal{G}_\omega[\rho] = \kappa \int_{\mathbb{R}^2} \frac{\rho^m}{m-1} dx - \frac{\omega^2}{2} \int_{\mathbb{R}^2} |x|^2 \rho dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho(x)\rho(y)}{|x-y|} dx dy.$$

For a given  $M > 0$ , the set  $\mathcal{X}_M$  is the set of nonnegative functions  $\rho$  with compact support in  $L^1(\mathbb{R}^2)$  satisfying condition (11) and such that  $\int_{\mathbb{R}^2} \rho dx = M$ .

**Problem 2.** Fix  $M > 0$ ,  $m \in (3/2, 2)$ , and find a critical point of  $\mathcal{G}_\omega$  in  $\mathcal{X}_M$ .

Because of the centrifugal force term,  $\mathcal{G}_\omega$  is not bounded from below if  $\omega \neq 0$  and there is therefore no global minimizer. Our strategy is to find *localized minimizers*, i.e., functions  $\rho$  in  $\mathcal{X}_M$  with compact support and such that for some  $R > 0$ ,  $\text{supp}(\rho) \subset B(0, r)$  with  $r < R$  and  $\rho$  minimizes  $\mathcal{G}_\omega$  on the set of functions in  $\mathcal{X}_M$  with support in  $B(0, R)$ . From the proof of existence of minimizers in the nonrotational case it can be concluded that there exists a radius  $R_0 = R_0(M)$  such that the supports of all the minimizers verifying the symmetry condition (11) are contained in  $B(0, R_0)$ .

#### 3.2. Minimization for densities supported in a fixed ball

As a first step, we fix  $R > 0$  and prove the existence of minimizers with densities supported in the ball  $B(0, R)$ . The scheme is basically the one of Section 2, so we shall skip most of the details.

**Lemma 11.** Consider  $\rho \in L^m(\mathbb{R}^2)$ ,  $m \in (4/3, 2)$ , with compact support in  $B(0, R)$  for some  $R > 0$ . Then

$$\mathcal{G}_\omega[\rho] \geq \mathcal{G}_0[\rho] - \frac{\omega^2}{2} R^2 \|\rho\|_{L^1(\mathbb{R}^2)},$$

which is bounded from below by  $\mathcal{I}_M^0 - \omega^2 R^2 M/2$  with  $M = \|\rho\|_{L^1(\mathbb{R}^2)}$ .

**Proof.** The proof is a straightforward consequence of Lemma 4 after noticing that functions in  $L^m(\mathbb{R}^2)$  with compact support are also in  $L^1(\mathbb{R}^2)$ .  $\square$

**Lemma 12.** For any  $M > 0$  and any  $R > R_0(M)$ ,

$$-\infty < \mathcal{I}_M^\omega(R) := \inf\{\mathcal{G}_\omega[\rho] : \rho \in \mathcal{X}_M, \text{supp}(\rho) \subset B(0, R)\} < 0.$$

**Proof.** The minimizer of the nonrotational problem is an admissible function in  $\mathcal{X}_M$ , and its energy is negative.  $\square$

**Proposition 13.** For any  $M > 0$  and any  $R > R_0(M)$ , there exists a function  $\rho_\infty^\omega \in \mathcal{X}_M$  with  $\text{supp}(\rho) \subset \overline{B(0, R)}$  such that  $\mathcal{G}_\omega[\rho_\infty^\omega] = \mathcal{I}_M^\omega(R)$ . Moreover, there exists  $\lambda^\omega \in \mathbb{R}$  such that

$$\rho_\infty^\omega = A_m \left( \lambda^\omega + \frac{\omega^2}{2} |x|^2 - \phi_\infty^\omega(x) \right)_+^{\frac{1}{m-1}} \quad \forall x \in \overline{B(0, R)}.$$

Here  $A_m := [\frac{m-1}{\kappa m}]^{1/(m-1)}$  and  $\phi_\infty^\omega$  denotes the potential associated to the density  $\rho_\infty^\omega$ .

**Proof.** Let  $(\rho_n^\omega)_{n \in \mathbb{N}}$  be a minimizing sequence. There exists  $\rho_\infty^\omega \in L^1 \cap L^m(\mathbb{R}^2)$  such that

$$\rho_n^\omega \rightharpoonup \rho_\infty^\omega \quad \text{in } L^1 \cap L^m(\mathbb{R}^2).$$



As in Theorem 3,  $\|\rho_\infty^\omega\|_{L^1(\mathbb{R}^2)} = M$  and  $\lim_{n \rightarrow \infty} \mathcal{G}_\omega[\rho_n^\omega] = \mathcal{G}_\omega[\rho_\infty^\omega]$ . On  $B(0, R)$ ,  $x \mapsto |x|^2$  is bounded, so the convergence of  $\int_{\mathbb{R}^2} |x|^2 \rho_n^\omega dx$  to  $\int_{\mathbb{R}^2} |x|^2 \rho_\infty^\omega dx$  is straightforward.  $\square$

Observe that we have not proven yet that  $\text{supp}(\rho_\infty^\omega) \subset B(0, r)$  for some  $r < R$  with  $R$  well chosen, *i.e.*, that  $\rho_\infty^\omega$  is a localized minimizer. This is the purpose of the next section, at least when  $\omega$  is small.

### 3.3. Existence of a localized minimizer for small values of $\omega$

For a given  $R > 0$ , the family  $(\rho_\infty^\omega)_\omega$  is uniformly bounded in  $L^m(\mathbb{R}^2)$  as  $\omega \rightarrow 0$ . This follows from (12) written for  $\rho = \rho_\infty^\omega$ , namely

$$\frac{\kappa}{m-1} \|\rho\|_{L^m(\mathbb{R}^2)}^m - \frac{C}{2} \|\rho\|_{L^1(\mathbb{R}^2)}^{2\theta} \|\rho\|_{L^m(\mathbb{R}^2)}^{2(1-\theta)} \leq \mathcal{G}_0[\rho] = \mathcal{G}_\omega[\rho_\infty^\omega] + \frac{\omega^2}{2} \int_{\mathbb{R}^2} |x|^2 \rho_\infty^\omega dx < \frac{\omega^2}{2} MR^2$$

using  $\mathcal{G}_\omega[\rho_\infty^\omega] = \mathcal{I}_M^\omega(R) < 0$ . Hence,  $(\rho_\infty^\omega)_\omega$  is weakly relatively compact in  $L^1 \cap L^m(\mathbb{R}^2)$  as  $\omega \rightarrow 0$  and, up to a subsequence, the limit can be identified with a minimizer  $\rho_\infty^0$  for the nonrotational case.

**Theorem 14.** *Let  $\rho_\infty^\omega$  be a minimizer in the sense of Proposition 13, for some  $R > R_0(M)$ . Then, up to a subsequence,  $(\rho_\infty^\omega)_\omega$  strongly converges in  $L^1 \cap L^m(\mathbb{R}^2)$  as  $\omega \rightarrow 0$  to a radial minimizer  $\rho_\infty^0$  of Problem 1 verifying condition (11).*

**Proof.** Let  $\rho_\infty$  be the weak limit of a convergent subsequence  $(\rho_n)_{n \in \mathbb{N}} = (\rho_\infty^{\omega_n})_{n \in \mathbb{N}}$ , with  $\lim_{n \rightarrow \infty} \omega_n = 0$ . Since  $\rho_n$  is a minimizer of  $\mathcal{I}_M^{\omega_n}(R)$  and any limit  $\rho_\infty^0$  is in the corresponding set of admissible functions, it follows that

$$\mathcal{G}_{\omega_n}[\rho_\infty^0] - \frac{\omega_n^2}{2} \int_{\mathbb{R}^2} |x|^2 \rho_\infty^0 dx \geq \mathcal{G}_{\omega_n}[\rho_n] \geq \mathcal{G}_0[\rho_n] - \frac{\omega_n^2}{2} MR^2 \geq \mathcal{G}_0[\rho_\infty^0] - \frac{\omega_n^2}{2} MR^2,$$

thus showing that  $\mathcal{G}_0[\rho_\infty] = \lim_{n \rightarrow \infty} \mathcal{G}_{\omega_n}[\rho_n] = \mathcal{I}_M^0$  using arguments similar to those of Theorem 3:  $(\rho_n)_{n \in \mathbb{N}}$  is a minimizing sequence for Problem 1. If the convergence in  $L^m(\mathbb{R}^2)$  was not strong, then we would get  $\mathcal{G}_0[\rho_\infty] < \mathcal{I}_M^0$ . Up to the extraction of a further subsequence, the convergence also holds almost everywhere and in  $L^1(\mathbb{R}^2)$ . Using condition (11), the proof is complete.  $\square$

**Corollary 15.** *Let  $\lambda^\omega$  be defined as in Proposition 13. Then  $\lim_{\omega \rightarrow 0} \lambda^\omega = \lambda^0$  where  $\lambda^0$  is defined in Theorem 3.*

**Proof.** Since up to a subsequence  $\phi_\infty^\omega$  strongly converges to some  $\phi_\infty^0$  in  $L^4(\mathbb{R}^2)$ , the result follows from the observation that on  $B(0, R_0(M))$ ,

$$\lambda^\omega = \left( \frac{1}{A_m} \rho_\infty^\omega \right)^{m-1} - \frac{\omega^2}{2} |x|^2 + \phi_\infty^\omega \rightarrow \left( \frac{1}{A_m} \rho_\infty^0 \right)^{m-1} - \frac{\omega^2}{2} |x|^2 + \phi_{\rho^0} = \lambda^0 \quad \text{a.e.}$$

and the fact that the Lagrange multiplier is the same for all the minimizers of the nonrotational case.  $\square$

**Lemma 16.** *Let  $\omega > 0$  and assume that Problem 2 admits a localized minimizer,  $\rho_\infty^\omega$ . Then there exists  $\epsilon > 0$  such that  $\rho_\infty^\omega \in L^{2+\epsilon}(\mathbb{R}^2)$ .*

**Proof.** Take  $\mu > \lambda^\omega + \frac{\omega^2}{2} R^2$ , and observe that by Proposition 13,  $\int_{\mathbb{R}^2} (\rho_\infty^\omega)^{2+\epsilon} dx = A_m^{2+\epsilon} (I_1 + I_2)$  with

$$I_1 := \int_{|\phi_\infty^\omega| < \mu} \left( \lambda^\omega + \frac{\omega^2}{2} |x|^2 - \phi_\infty^\omega \right)_+^{\frac{2+\epsilon}{m-1}} dx \leq M \left( \mu - |\lambda^\omega| + \frac{\omega^2}{2} R^2 \right)^{\frac{2+\epsilon}{m-1} - 1},$$

$$I_2 := \int_{|\phi_\infty^\omega| \geq \mu} \left( \lambda^\omega + \frac{\omega^2}{2} |x|^2 - \phi_\infty^\omega \right)_+^{\frac{2+\epsilon}{m-1}} dx \leq \frac{2}{\mu^s} \int_{|\phi_\infty^\omega| \geq \mu} (\phi_\infty^\omega)^{\frac{2+\epsilon}{m-1}} dx,$$

with  $s = q - \frac{2+\epsilon}{m-1} > 0$ ,  $1/q = 1/m - 1/2$  as given by Lemma 4, and with both integrals  $I_1$  and  $I_2$  restricted to the support of  $\rho_\infty^\omega$ . Now, also from the first part of Lemma 4, the result will follow as long as  $\epsilon$  is small.  $\square$

**Lemma 17.** *The potential  $\phi_\infty^\omega$  is a continuous function.*

**Proof.** By Hardy’s inequality, for any  $x, y \in \mathbb{R}^2$ ,

$$|\phi_\infty^\omega(x) - \phi_\infty^\omega(y)| \leq \|\rho_\infty^\omega\|_{L^{2+\epsilon}(\mathbb{R}^2)} \left( \int_{\mathcal{K}} \left| \frac{1}{|z-x|} - \frac{1}{|z-y|} \right|^{\frac{2+\epsilon}{1+\epsilon}} dz \right)^{\frac{1+\epsilon}{2+\epsilon}}$$

with  $\mathcal{K} := B(x, R) \cup B(y, R)$ .  $\square$

**Corollary 18.** *With the above notations, there exists  $\phi_\infty^0$  such that, up to a subsequence,*

$$\lim_{\omega \rightarrow 0} \|\phi_\infty^\omega - \phi_\infty^0\|_{L^\infty(\mathbb{R}^2)} = 0.$$

**Proof.** This is an easy consequence of Theorem 14 and Lemma 17.  $\square$

**Corollary 19.** *For any  $\epsilon \in (0, 1)$ , there exists  $\omega_\epsilon$  such that  $\text{supp}(\rho_\infty^\omega) \subset B(0, (1 + \epsilon)R_0)$  for any  $\omega \in (0, \omega_\epsilon)$ .*

As a consequence, for any  $R > R_0$ , if  $\omega$  is small enough,  $\rho_\infty^\omega$  is a localized minimizer for Problem 2. This proves Theorem 1(i).

**Proof.** Since a minimizer  $\rho_\infty^0$  of Problem 1 is radial, it solves the following fixed point identity

$$\phi_\infty^0(r) = -\frac{1}{2\pi} \int_0^\infty \frac{s}{r} f\left(\frac{s}{r}\right) \rho_\infty^0(s) ds, \quad \rho_\infty^0(r) = (\lambda^0 - \phi_\infty^0(r))_+^{\frac{1}{m-1}} \mathbb{1}_{[0,R]}(r),$$

where  $\lambda^0$  is implicitly fixed by the condition:  $2\pi \int_0^\infty \rho_\infty^0(r)r dr = M$ . By writing the kernel  $-\frac{1}{4\pi|x|}$  in radial coordinates and by integrating with respect to the angle, we obtain

$$f(t) = \int_0^\pi \frac{d\theta}{\sqrt{1+t^2-2t \cos \theta}}.$$

See [50] for more details. A simple calculation shows that

$$\frac{d\phi_\infty^0}{dr}(r) = \frac{1}{2\pi r^2} \int_0^\infty \int_0^\pi \frac{1 - (s/r) \cos \theta}{(1 + s^2/r^2 - 2(s/r) \cos \theta)^{3/2}} d\theta \rho_\infty^0(s) s ds,$$

so that the potential  $\phi_\infty^0$  is an increasing function at least for  $|x| \geq R_0$ .

Take  $R = (1 + \epsilon_0)R_0$  and consider minimizers  $\rho_\infty^\omega$  supported in the ball  $B(0, R)$ . For any  $\epsilon \in (0, \epsilon_0)$ , we define  $\mathcal{A}_\epsilon := B(0, R) \setminus B(0, (1 + \epsilon)R_0)$ . Because of the monotonicity property of  $\phi_\infty^0$ , there exists  $\delta = \delta(\epsilon)$  such that

$$\lambda^0 - \phi_\infty^0 < -2\delta \quad \forall x \in \mathcal{A}_\epsilon.$$

By Corollaries 15 and 17, for  $\omega$  small enough,

$$\lambda^\omega - \phi_\infty^\omega + \frac{\omega^2}{2}|x|^2 < -\delta \quad \forall x \in \mathcal{A}_\epsilon,$$

which proves that  $\text{supp}(\phi_\infty^\omega) \subset B(0, (1 + \epsilon)R_0)$ .  $\square$

3.4. There are no solutions for large values of  $\omega$

**Lemma 20.** Let  $\omega_1 > \omega_2 \geq 0$  and consider two minimizers  $\rho_\infty^{\omega_1}$  and  $\rho_\infty^{\omega_2}$  in the sense of Proposition 13. Then

- (i)  $\mathcal{G}_\omega[\rho_\infty^{\omega_1}] \leq \mathcal{G}_\omega[\rho_\infty^{\omega_2}]$ ,
- (ii)  $\int_{\mathbb{R}^2} |x|^2 \rho_\infty^{\omega_1} dx \geq \int_{\mathbb{R}^2} |x|^2 \rho_\infty^{\omega_2} dx$ .

**Proof.** Since  $\rho_\infty^{\omega_2}$  is a minimizer for angular velocity  $\omega_2$ , we observe that

$$\begin{aligned} & \frac{\kappa}{m-1} \int_{\mathbb{R}^2} (\rho_\infty^{\omega_2})^m dx - \frac{1}{2} \omega_2^2 \int_{\mathbb{R}^2} |x|^2 \rho_\infty^{\omega_2} dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho_\infty^{\omega_2}(x) \rho_\infty^{\omega_2}(y)}{|x-y|} dx dy \\ & \leq \frac{\kappa}{m-1} \int_{\mathbb{R}^2} (\rho_\infty^{\omega_1})^m dx - \frac{1}{2} \omega_2^2 \int_{\mathbb{R}^2} |x|^2 \rho_\infty^{\omega_1} dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho_\infty^{\omega_1}(x) \rho_\infty^{\omega_1}(y)}{|x-y|} dx dy \\ & \leq \frac{\kappa}{m-1} \int_{\mathbb{R}^2} (\rho_\infty^{\omega_1})^m dx - \frac{1}{2} \omega_1^2 \int_{\mathbb{R}^2} |x|^2 \rho_\infty^{\omega_1} dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho_\infty^{\omega_1}(x) \rho_\infty^{\omega_1}(y)}{|x-y|} dx dy \end{aligned}$$

which is the statement of (i). By adding the following two inequalities

$$\begin{aligned} & \frac{\kappa}{m-1} \int_{\mathbb{R}^2} (\rho_\infty^{\omega_1})^m dx - \frac{1}{2} \omega_2^2 \int_{\mathbb{R}^2} |x|^2 \rho_\infty^{\omega_1} dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho_\infty^{\omega_1}(x) \rho_\infty^{\omega_1}(y)}{|x-y|} dx dy \\ & \geq \frac{\kappa}{m-1} \int_{\mathbb{R}^2} (\rho_\infty^{\omega_2})^m dx - \frac{1}{2} \omega_2^2 \int_{\mathbb{R}^2} |x|^2 \rho_\infty^{\omega_2} dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho_\infty^{\omega_2}(x) \rho_\infty^{\omega_2}(y)}{|x-y|} dx dy \end{aligned}$$

and

$$\begin{aligned} & \frac{\kappa}{m-1} \int_{\mathbb{R}^2} (\rho_\infty^{\omega_2})^m dx - \frac{1}{2} \omega_1^2 \int_{\mathbb{R}^2} |x|^2 \rho_\infty^{\omega_2} dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho_\infty^{\omega_2}(x) \rho_\infty^{\omega_2}(y)}{|x-y|} dx dy \\ & \geq \frac{\kappa}{m-1} \int_{\mathbb{R}^2} (\rho_\infty^{\omega_1})^m dx - \frac{1}{2} \omega_1^2 \int_{\mathbb{R}^2} |x|^2 \rho_\infty^{\omega_1} dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho_\infty^{\omega_1}(x) \rho_\infty^{\omega_1}(y)}{|x-y|} dx dy, \end{aligned}$$

we obtain (ii).  $\square$

**Lemma 21.** Let  $\rho_\infty^\omega$  be a critical point for Problem 2. There exists a positive constant  $C$  such that, if  $\rho_\infty^0$  is a radial minimizer for Problem 1, then

$$\omega^2 \leq \frac{C}{\int_{\mathbb{R}^2} |x|^2 \rho_\infty^0(x) dx}.$$

**Proof.** Observe that if  $\rho_\infty^\omega \in \mathcal{X}_M$ , the rescaled functions  $x \mapsto \rho_\infty^{\omega, \lambda}(x) := \lambda^2 \rho_\infty^\omega(\lambda x)$  are also in  $\mathcal{X}_M$  for any  $\lambda > 0$ . Hence the function  $F_\omega(\lambda) := \mathcal{G}_\omega[\rho_\infty^{\omega, \lambda}]$  attains a minimum at  $\lambda = 1$ , which means that

$$\begin{aligned} & 2\kappa \int_{\mathbb{R}^2} (\rho_\infty^\omega)^m dx + \omega^2 \int_{\mathbb{R}^2} |x|^2 \rho_\infty^\omega dx - \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho_\infty^\omega(x) \rho_\infty^\omega(y)}{|x-y|} dx dy = 0, \\ & \omega^2 = \frac{-2\kappa \int_{\mathbb{R}^2} (\rho_\infty^\omega)^m dx + \frac{1}{8\pi} \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{\rho_\infty^\omega(x) \rho_\infty^\omega(y)}{|x-y|} dx dy}{\int_{\mathbb{R}^2} |x|^2 \rho_\infty^\omega dx}. \end{aligned}$$

On the one hand, by Lemma 20(ii),

$$\omega^2 \leq -\frac{\mathcal{G}_0[\rho_\infty^\omega]}{(m-1) \int_{\mathbb{R}^2} |x|^2 \rho_\infty^\omega dx} \leq \frac{|\mathcal{G}_0[\rho_\infty^\omega]|}{(m-1) \int_{\mathbb{R}^2} |x|^2 \rho_\infty^0 dx}.$$

On the other hand, by (12),

$$|\mathcal{G}_0[\rho_\infty^\omega]| \leq \frac{C}{2} M^{2\theta} \|\rho_\infty^\omega\|_{L^m(\mathbb{R}^2)}^{2(1-\theta)} - \frac{\kappa}{m-1} \|\rho_\infty^\omega\|_{L^m(\mathbb{R}^2)}^m$$

is bounded because the function  $t \mapsto \frac{C}{2} M^{2\theta} t^{2(1-\theta)} - \frac{\kappa}{m-1} t^m$  is bounded on  $\mathbb{R}^+$  for any  $m \in (3/2, 2)$ .  $\square$

Since all the minimizers for the nonrotational setting have the same  $L^m$  norm, by interpolation,  $\int_{\mathbb{R}^2} |x|^2 \rho_\infty^0(x) dx$  can be uniformly bounded by a constant which depends only on the mass and  $\mathcal{I}_M^0$ , and we obtain the statement (ii) of Theorem 1.

### 3.5. A continuation method for radial minimizers

**Lemma 22.** *Let  $\rho$  be a nonnegative radial function and assume that  $\text{supp}(\rho) \subset B(0, R)$  for some  $R > 0$ . If for some  $x_0 \in \mathbb{R}^2 \setminus B(0, R)$   $x_0 \cdot (\nabla \phi_\rho(x_0) - \omega^2 x_0) \leq 0$  holds, where  $\phi_\rho = -\frac{1}{4\pi|\cdot|} * \rho$ , then  $x_0 \cdot (\nabla \phi_\rho(\mu x_0) - \omega^2(\mu x_0)) < 0$  for any  $\mu > 1$ .*

**Proof.** If  $\rho$  is a radial function,

$$\phi_\rho(x) = -\frac{1}{4\pi} \int_{\mathbb{R}^2} \frac{\rho(y)}{|x-y|} dy$$

can be rewritten as

$$\phi_\rho(x) = -\frac{1}{4\pi} \int_0^\infty s \rho(s) \int_{-\pi}^\pi \frac{d\theta}{\sqrt{r^2 + s^2 - 2rs \cos \theta}} = -\frac{1}{2\pi} \int_0^\infty \rho(s) \frac{s}{r} f\left(\frac{s}{r}\right) ds$$

with  $r = |x|$  and

$$f(t) := \int_0^\pi \frac{d\theta}{\sqrt{1+t^2-2t \cos \theta}}.$$

Hence

$$\frac{x}{|x|} \cdot \nabla \phi_\rho(x) = \frac{1}{2\pi} \int_0^\infty \rho(s) \frac{s}{r^2} \left[ f\left(\frac{s}{r}\right) + \frac{s}{r} f'\left(\frac{s}{r}\right) \right] ds.$$

We observe that

$$f(t) + t f'(t) = \int_0^\pi \frac{1-t \cos \theta}{(1+t^2-2t \cos \theta)^{3/2}} d\theta$$

is nonnegative for  $t < 1$ , that is  $s < r$ . If  $\rho$  is radial,  $\text{supp}(\rho) \subset \overline{B(0, R)}$  and  $x \in \mathbb{R}^2 \setminus B(0, R)$ , then

$$\frac{x}{|x|} \cdot \nabla \phi_\rho(x) > 0.$$

Moreover, for any  $t \in (0, 1)$ ,

$$\begin{aligned} \frac{d}{dt} [t^2(f(t) + t f'(t))] &= \int_0^\pi t \frac{(3 \cos(2\theta) + 1)t^2 - 8 \cos \theta t + 4}{2(t^2 - 2 \cos \theta t + 1)^{5/2}} d\theta \\ &= t \frac{(t-1)^2 K(4t/(t+1)^2) + (3-t^2) E(4t/(t+1)^2)}{(t-1)^2 \sqrt{(t+1)^2}}, \end{aligned}$$

where  $K$  and  $E$  are respectively the complete elliptic integrals of the first and second kind, and satisfy the relations

$$K(0) = \frac{\pi}{2} \quad \text{and} \quad 0 \leq \frac{dK}{dk} = \int_0^1 \frac{dt}{\sqrt{(1-t^2)[1-(1-k^2)t^2]}} = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k}.$$

Since  $K$  and  $E$  take positive values for  $t \in (0, 1)$ , it follows that  $K(4t/(t+1)^2)$  and  $E(4t/(t+1)^2)$  take positive values for  $t \in (0, 1)$ . Hence

$$\frac{d}{dt} [t^2(f(t) + tf'(t))] \geq 0$$

and we conclude that  $\mu \mapsto \frac{x}{|x|} \cdot \nabla \phi_\rho(\mu x)$  is decreasing for  $\mu > 1$ .  $\square$

Consider a critical point  $\rho$  of  $\mathcal{G}_\omega$  and assume that  $\rho$  is radial and compactly supported. Then  $\rho$  can be explicitly written in terms of its own potential, which amounts to consider the integral equation

$$\rho(x) = A_m \left( \lambda[\rho] + \frac{\omega^2}{2} |x|^2 - \phi_\infty^\omega(x) \right)_+^{\frac{1}{m-1}} \quad \text{where} \quad \phi_\infty^\omega = -\frac{1}{4\pi|\cdot|} * \rho.$$

We can also define  $R_1[\rho] := \min\{r > 0: \text{supp}(\rho) \subset B(0, r)\}$  and notice that for any  $x \in \partial B(0, R_1[\rho])$ ,  $\lambda[\rho] = \phi_\infty^\omega(x) - \frac{\omega^2}{2} |x|^2$ . Let

$$h[\rho] := \sup \left\{ \phi_\infty^\omega(x) - \frac{\omega^2}{2} |x|^2: |x| > R_1[\rho] \right\}.$$

From Lemma 22, we deduce the

**Corollary 23.** *Let  $\rho$  be a nonnegative radial compactly supported function in  $\mathcal{L}(\mathbb{R}^2)$ . If  $\rho$  is a critical point of  $\mathcal{G}_\omega$  such that  $\int_{\mathbb{R}^2} \rho \, dx = M$  and  $h[\rho] > \lambda[\rho]$ , then  $x \cdot (\nabla \phi_\rho - \omega^2 x) > 0$  for any  $x \in \partial B(0, R_1[\rho])$ .*

As in Section 1.5, define  $S_{M,\text{rad}}^\omega$  as the set of radial localized minimizers with angular velocity  $\omega$ . We recall that by definition of a radial localized minimizer, for any  $\rho_\infty^\omega \in S_{M,\text{rad}}^\omega$ , there exists  $\varepsilon[\rho_\infty^\omega] > 0$  such that

$$\mathcal{G}_\omega[\rho_\infty^\omega] = \min \left\{ \mathcal{G}_\omega[\rho]: \rho \in \mathcal{L}_{\text{rad}}(\mathbb{R}^2), \text{supp}(\rho) \subset B(0, R_1[\rho_\infty^\omega] + \varepsilon[\rho_\infty^\omega]) \text{ and } \int_{\mathbb{R}^2} \rho \, dx = M \right\}$$

where  $\mathcal{L}_{\text{rad}}(\mathbb{R}^2)$  is the set of nonnegative radial functions  $\rho$  in  $L^1(\mathbb{R}^2)$ . Notice that the assumption (11) is automatically satisfied.

**Proof of Theorem 2.** We have already seen in Section 3.3 that  $\mathcal{P}_0^{\text{rad}}$  is true. Now fix  $\omega_0 > 0$  such that  $\mathcal{P}_{\omega_0}^{\text{rad}}$  is true in  $[0, \omega_0]$ . First of all observe that, up to a subsequence,  $\rho_\infty^\omega$  converges strongly in  $L^1 \cap L^m(\mathbb{R}^2)$  as  $\omega \rightarrow \omega_0$  to a radial localized minimizer  $\rho_\infty^{\omega_0}$ . It is also true that  $\lambda(\rho_\infty^\omega) \rightarrow \lambda(\rho_\infty^{\omega_0})$ . Moreover, it is easy to see that Lemmata 16 and 17, and Corollary 18 also hold for  $\omega \rightarrow \omega_0$ . So, in order to complete the proof we only need to get an analog of Corollary 19. But to do so, only the monotonicity property of the potential in a small neighborhood of the support of  $\rho_\infty^{\omega_0}$  is needed. In our case this is ensured by Corollary 23.  $\square$

### 3.6. Strict localized minimizers

In the nonrotational case the minimizer is radial. Here by radial we mean radial in the plane, or axially symmetric if we consider the three-dimensional flat system, with mass concentrated on the plane. The key point is the fact that the potential energy term is decreased under symmetric rearrangements whereas the other term appearing in the energy remains unchanged. On the other hand, in the rotational case the term  $\int_{\mathbb{R}^2} |x|^2 \rho \, dx$  behaves in the opposite direction, and the resulting behaviour of the energy is unclear. Is the minimizer of the rotational problem radial? We have no answer yet for this question, although it seems reasonable to expect that minimizers are not radial at least for angular velocities large enough.

The strategy of proof for radial localized minimizers (Lemma 22) cannot be extended to the nonradial case, as it is shown by the following result.

**Proposition 24.** *Assume that  $\text{supp}(\rho) \subset \overline{B(0, R)}$  for some  $R > 0$ . There exists  $t_0 \in (1, \sqrt{3/2})$  such that, if for some  $x_0 \in B(0, t_0 R)^c$ ,*

$$x_0 \cdot (\nabla \phi_\rho(x_0) - \omega^2 x_0) \leq 0, \quad (13)$$

and if we define

$$t \mapsto f(t) := x_0 \cdot (\nabla \phi_\rho(tx_0) - \omega^2(tx_0)),$$

then  $f'$  takes negative values for any  $t > t_0$ . Alternatively, if  $\text{supp}(\rho) \subset \overline{B(0, 3R/5)}$ , the same result holds with  $t_0 = 1$ .

On the contrary, there exists a nonnegative function  $\rho \in L^1(\mathbb{R}^2)$  with  $\text{supp}(\rho) \subset \overline{B(0, R)}$  for some  $R > 0$  such that  $f$  defined as above takes negative (respectively, positive) values in an interval  $(1, 1 + \varepsilon)$  (respectively  $(1 + \varepsilon, 1 + 2\varepsilon)$ ) for some  $\varepsilon > 0$ .

**Proof.** This proof is based on a computation. We start with the following observations:

(1)  $x_0 \cdot y \leq |x_0|^2$  for any  $y \in \text{supp}(\rho)$ , so that

$$-\frac{x_0 \cdot (x_0 - y)}{|x_0 - y|^3} \leq 0.$$

(2) Using the expression of  $\phi_\rho$ , we get

$$f(t) = \int_{\mathbb{R}^2} \frac{x_0 \cdot (tx_0 - y)}{|tx_0 - y|^3} \frac{\rho(y)}{4\pi} dy - t\omega^2|x_0|^2.$$

(3) Assumption (13) means  $f(1) \leq 0$ , that is

$$\omega^2|x_0|^2 \geq \int_{\mathbb{R}^2} \frac{x_0 \cdot (x_0 - y)}{|x_0 - y|^3} \frac{\rho(y)}{4\pi} dy.$$

Hence

$$f(t) \leq \int_{\mathbb{R}^2} \left[ \frac{x_0 \cdot (tx_0 - y)}{|tx_0 - y|^3} - t \frac{x_0 \cdot (x_0 - y)}{|x_0 - y|^3} \right] \frac{\rho(y)}{4\pi} dy.$$

With  $r := |y|/|x_0| \in [0, 1]$  and  $\alpha := (x_0 \cdot y)(|x_0||y|)^{-1} \in [-1, 1]$ , we observe that

$$|tx_0 - y|^2 - |x_0 - y|^2 = |x_0|^2(t^2 - 1 - 2tr(\alpha - 1)) \geq |x_0|^2(t^2 - 1) \geq 0$$

for any  $t \geq 1$ . For any given  $(r, \alpha) \in [0, 1] \times [-1, 1]$ , define

$$g(t) := |x_0| \left[ \frac{x_0 \cdot (tx_0 - y)}{|tx_0 - y|^3} - t \frac{x_0 \cdot (x_0 - y)}{|x_0 - y|^3} \right].$$

We compute

$$g'(t) = \frac{|x_0|^5}{|tx_0 - y|^5} (t^2 - 2tr\alpha + r^2 - 3(t - r\alpha)^2) - |x_0| \frac{x_0 \cdot (x_0 - y)}{|x_0 - y|^3}$$

and either  $(t^2 - 2tr\alpha + r^2 - 3(t - r\alpha)^2) \leq 0$ , or not. In the first case,  $g'(t) \leq 0$ . Assume from now on that the second case holds. With  $s := r\alpha \in [-1, 1]$ , this means

$$0 \leq t^2 - 2st + r^2 - 3(t - s)^2 = -2t^2 + 4st + r^2 - 3s^2,$$

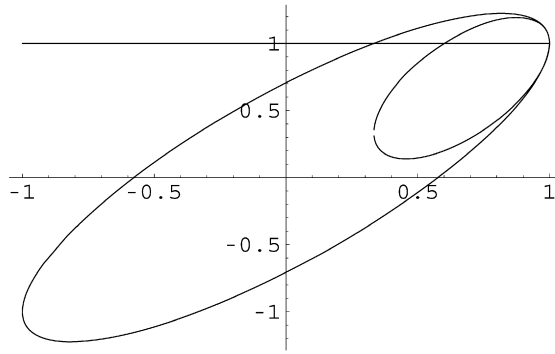


Fig. 1.

which defines a family of domains whose boundary are ellipses  $\mathcal{E}_r$  parametrized by  $r \in [0, 1]$ , all of them contained in  $\mathcal{E}_1$  corresponding to  $r = 1$ , which is itself included in the half-plane corresponding to  $s > 0$  and has a nonempty intersection with the half-plane corresponding to  $t > 1$ . Using  $|tx_0 - y|^5 \geq |x_0 - y|^5$ , we get

$$g'(t) \leq \frac{|x_0|^5}{|x_0 - y|^5} (-2t^2 + 4st + r^2 - 3s^2) - |x_0| \frac{x_0 \cdot (x_0 - y)}{|x_0 - y|^3} =: \frac{h(t)}{|x_0 - y|^5},$$

where

$$h(t) = -2t^2 + 4st - 1 + 3s + r^2s - 5s^2.$$

For  $s > 0$ , the function  $h$  is nonnegative inside a family of ellipses parametrized by  $r \in [0, 1]$ , all of them contained in the one corresponding to  $r = 1$ , which itself contains  $\mathcal{E}_1$ . See Fig. 1. However,  $h(t) < 0$  in the half-plane corresponding to  $t > \sqrt{3/2} \approx 1.22474 \dots$

Notice that the condition  $h(1) > 0$  means  $s \in (3/5, 1)$ .

For more accurate estimates, one has to use

$$g'(t) = \frac{r^2 - 3s^2 - 2t^2 + 4st}{(r^2 + t^2 - 2st)^{5/2}} - \frac{1 - s}{(r^2 - 2s + 1)^{3/2}}.$$

Numerically, one finds that the optimal value for  $t_0$  is  $t_0 \approx 1.17355 \dots$ , which, for  $r = 1$  and  $s \approx 0.898872 \dots$ , gives  $g'(t_0) = 0$  and  $g'(t) > 0$  for any  $t \in (0, t_0)$ . Explicit but ugly algebraic computations can be done, which show that  $g'(t)$  takes negative values in a neighborhood of  $t = 1_+$  and changes sign for higher values of  $t$ . Details are left to the reader.  $\square$

In Proposition 24, we only used the Poisson equation but did not take into account the fact that we are interested in the sign of  $f$  only for critical points of  $\mathcal{G}_\omega$ . It is an open question to decide whether in such a case  $f$  is automatically positive or not.

Because of the above considerations and in order to extend the method which has been proposed for the radial case to the nonradial one, we have to impose an additional restriction on the notion of localized minimizers.

**Definition.** A *strict localized minimizer*  $\rho$  is a localized minimizer of  $\mathcal{G}_\omega$  for which there exists an open simply connected set  $\Omega$  with the two properties:

- (1)  $\text{supp}(\rho) \subset \Omega$ ,
- (2)  $\{x \in \mathbb{R}^2: \rho(x) - \frac{\omega^2}{2}|x|^2 > \lambda[\rho]\} = \Omega \setminus \text{supp}(\rho)$ .

3.7. A continuation method for nonradial but strict localized minimizers

Define  $\mathcal{S}_M^\omega$  as the set of strict localized minimizers. Exactly as for radial minimizers, each  $\rho_\infty^\omega \in \mathcal{S}_M^\omega$  can be explicitly written in terms of its own potential, which amounts to consider the integral equation

$$\rho_\infty^\omega(x) = A_m \left( \lambda[\rho_\infty^\omega] + \frac{\omega^2}{2} |x|^2 - \phi_\infty^\omega(x) \right)_+^{\frac{1}{m-1}} \quad \text{where } \phi_\infty^\omega = -\frac{1}{4\pi|\cdot|} * \rho_\infty^\omega,$$

$A_m := [\frac{m-1}{km}]^{1/(m-1)}$  and  $\lambda[\rho_\infty^\omega] < 0$  is a parameter which is determined by  $M$ , but eventually depends on  $\rho_\infty^\omega$ . Let  $\mu[\rho_\infty^\omega] := \min_{x \in \text{supp}(\rho_\infty^\omega)} (\phi_\infty^\omega(x) - \frac{\omega^2}{2} |x|^2)$  and consider the set of paths

$$\Gamma[\rho_\infty^\omega] := \left\{ \gamma \in C^0([0, \infty); \mathbb{R}^2) : \gamma(0) \in \text{supp}(\rho_\infty^\omega), (\phi_{\text{eff}}^\omega)_\infty(\gamma(0)) = \mu[\rho_\infty^\omega], \lim_{s \rightarrow \infty} |\gamma(s)| = \infty \right\}$$

where  $(\phi_{\text{eff}}^\omega)_\infty(x) := \phi_\infty^\omega(x) + \frac{\omega^2}{2} |x|^2$ . Along a path  $\gamma \in \Gamma[\rho_\infty^\omega]$ , define the highest altitude of the effective potential  $(\phi_{\text{eff}}^\omega)_\infty$  as

$$h[\gamma, \omega, \rho_\infty^\omega] := \sup_{s \in [0, \infty)} \left( \phi_\infty^\omega(\gamma(s)) - \frac{\omega^2}{2} |\gamma(s)|^2 \right).$$

We shall say that *Property  $\mathcal{P}_\omega$*  holds true if and only if there exists an open set  $\mathcal{D}(\omega)$  such that for any  $\rho_\infty^\omega \in \mathcal{S}_M^\omega$ , there exists a rotation  $\mathcal{R}$  around the origin such that

- (1)  $\text{supp}(\rho_\infty^\omega) \subset \mathcal{R}^{-1}(\mathcal{D}(\omega))$ ,
- (2) If  $\phi_\infty^\omega(x) - \frac{\omega^2}{2} |x|^2 = \inf_{\gamma \in \Gamma[\rho_\infty^\omega]} h[\gamma, \omega, \rho_\infty^\omega]$ , then  $\mathcal{R}x \in \mathbb{R}^2 \setminus \mathcal{D}(\omega)$ .

Observe that, since  $\lambda[\rho_\infty^\omega] - h[\gamma, \omega, \rho_\infty^\omega] \leq 0$ , Property  $\mathcal{P}_\omega$  true implies that every minimizer  $\rho_\infty^\omega \in \mathcal{S}_M^\omega$  is a strict localized minimizer. We could also consider only the case  $\mathcal{R} = \text{Id}$  but this would probably be by far more restrictive.

**Theorem 25.** *For any  $M > 0$ , the maximal interval in  $\omega$  containing  $\omega = 0$  for which Property  $\mathcal{P}_\omega$  holds true is an open interval.*

**Proof.** First of all observe that, up to a subsequence,  $(\rho_\infty^\omega)_\omega$  converges strongly to some minimizer  $\rho_\infty^{\omega_0}$  as  $\omega \rightarrow \omega_0$  in  $L^1 \cap L^m(\mathbb{R}^2)$ . The analogues of Lemmata 16 and 17, and Corollaries 15 and 18 also hold for  $\omega \rightarrow \omega_0$ . Suppose that  $\mathcal{P}_{\omega_0}$  holds. It follows that for any  $\rho_\infty^{\omega_0} \in \mathcal{S}_M^{\omega_0}$

$$h[\gamma, \omega_0, \rho_\infty^{\omega_0}] > \lambda[\rho_\infty^{\omega_0}].$$

By continuity, the same will be true for any  $\rho_\infty^\omega \in \mathcal{S}_M^\omega$  as long as  $|\omega - \omega_0|$  is small enough. Since Property  $\mathcal{P}_\omega$  holds true for  $\omega \rightarrow 0$ , the proof is complete.  $\square$

Observe that Theorem 1(i) is a consequence of Theorem 25.

As in the radial case, if we knew that localized minimizers are all strict, then we could characterize the supremum of  $\omega > 0$  such that  $\mathcal{P}_\omega$  holds as

$$\sup \left\{ \omega_0 > 0 : \forall \omega \in (0, \omega_0), \sup_{\rho_\infty^\omega \in \mathcal{S}_M^\omega} \left( \inf_{\gamma \in \Gamma[\rho_\infty^\omega]} h[\gamma, \omega, \rho_\infty^\omega] - \lambda[\rho_\infty^\omega] \right) > 0 \right\}.$$

4. Concluding remarks

4.1. Ground states and critical points

In the porous medium context, *ground states*, or to be precise, nonnegative solutions converging to 0 at infinity, have been studied already long ago, see for instance [22,21]. It has been proved that such solutions may have several connected components. It has also to be noted that such ground states are not minimizers of an energy, but only critical



points, which can be somewhat misleading from the point of view of physics. There is usually no external potential to interact with.

From a variational point of view, after the seminal paper [31], a very extended literature has been devoted to the construction of critical points concentrated at specific locations determined by an external potential, and it is out of the scope of this paper to review even the latest contributions. Finding such critical points has however been done mostly in the context of the Schrödinger operator and we are not aware of any contribution which applies to porous media equations coupled to the gravitational Poisson equations.

#### 4.2. Fluid models and axial symmetry

Various stationary solutions of fluid models have been studied. In [42], the existence of steady, compactly supported, axisymmetric solutions of the Euler–Poisson system in the three-dimensional space is considered, for prescribed mass and angular momentum. In this paper, the author proves that stationary solutions, which are seen as critical points of a free energy functional, exist as long as the angular velocity is not too big, whereas no solution exists for angular velocities bigger than some critical value. This result is quite similar to Theorem 1, but in a three-dimensional setting and in the case of axisymmetric densities. Let us emphasize that a function  $f$  is *axisymmetric* if, in a cylindrical coordinates system  $(r, \theta, z) \in [0, \infty) \times [0, 2\pi) \times \mathbb{R}$ ,  $f$  only depends on  $r$  and  $z$ . We will use below a weaker notion of *partial symmetry*.

Steady, spherically symmetric solutions of the three-dimensional Euler–Poisson system are also considered in [45], but instead of fixing the mass as a constraint, the authors prescribe the value of the central density, *i.e.*, the density at the origin. They prove the existence of steady solutions as long as the central density is bigger than some critical value which depends on the angular velocity. Moreover, for a fixed central density, the radius of the support increases with angular velocity.

Concerning the existence of solutions with fixed mass and large angular momenta, but without symmetry assumption, McCann in [47] builds critical points of the free energy functional made of solutions with support splitted into two disjoint components, rotating at constant angular velocity. To each component is associated a mass, and the mass ratio is specified. Such critical points are local minimizers, where *local* has to be understood with respect to the Wasserstein metric, see [47] for more details.

The results of our paper are concerned with the simplified situation of *flat systems*, but such a simplified framework allows us to give more detailed results, for instance on the continuation of the solutions or on how to characterize the critical angular velocity.

#### 4.3. Scalings

The centrifugal force term in the free energy introduces a length scale. Scalings are therefore of no use to study the problem corresponding to  $\omega \neq 0$ , fixed, but there is a scaling invariance exactly as in the case  $\omega = 0$  if one also varies  $\omega$  in terms of the scaling parameter.

**Proposition 26.** *For any  $M > 0$ , the critical angular velocities  $\omega_*(M)$  and  $\omega^*(M)$  in the sense of Theorems 1 and 25 satisfy*

$$\omega_*(M) = M^{\frac{1}{2} \frac{m-3}{2m-3}} \omega_*(1) \quad \text{and} \quad \omega^*(M) = M^{\frac{1}{2} \frac{m-3}{2m-3}} \omega^*(1).$$

Moreover,  $\rho_\infty^\omega \in \mathcal{X}_M$  is a localized minimizer with angular velocity  $\omega$  if and only if  $(\rho_\infty^\omega)^\lambda \in \mathcal{X}_1$  with  $\rho^\lambda(x) := \lambda \rho(\lambda^{2-m}x)$ ,  $\lambda = M^{-1/(2m-3)}$ , is also a localized minimizer, with angular velocity  $\lambda^{(m-3)/2}\omega$ .

For any given  $\omega > 0$ ,

- (1) *there is a localized minimizer in  $\mathcal{X}_M$  if  $M > (\omega/\omega_*(1))^{2(2m-3)/(m-3)}$ ,*
- (2) *there is no localized minimizer in  $\mathcal{X}_M$  if  $M < (\omega/\omega^*(1))^{2(2m-3)/(m-3)}$ .*

**Proof.** For any  $\lambda > 0$ , let  $\rho^\lambda(x) := \lambda \rho(\lambda^{2-m}x)$  and observe that  $\text{supp}(\rho^\lambda) = \lambda^{m-2} \text{supp}(\rho)$ ,

$$\|\rho^\lambda\|_{L^1(\mathbb{R}^2)} = \lambda^{2m-3} \|\rho\|_{L^1(\mathbb{R}^2)} \quad \text{and} \quad \mathcal{G}_{\omega_2}[\rho^\lambda] = \lambda^{3m-4} \mathcal{G}_{\lambda^{(m-3)/2}\omega_1}[\rho].$$

With  $\lambda = (M_2/M_1)^{1/(2m-3)}$  and  $\lambda^{(m-3)/2}\omega_2 = \omega_1$ , it is clear that  $\rho^\lambda$  is a localized minimizer of  $\mathcal{G}_{\omega_2}$  if  $\rho$  is a localized minimizer of  $\mathcal{G}_{\omega_1}$ .  $\square$

4.4. *Dynamical stability*

As far as we know, the notion of localized minimizers is new and seems particularly well adapted to gravitational problems. Such a notion is motivated by the study of the dynamical stability, which is still to be understood. However, at a formal level, we can state a result and formulate an open problem which should attract interest.

**Proposition 27.** *Fix  $M > 0$  and  $\omega \in (0, \omega_*(M))$  such that  $\mathcal{P}_\omega$  holds and consider a strict localized minimizer  $\rho_\infty^\omega \in \mathcal{X}_M$  with associated Lagrange multiplier  $\lambda^\omega$  such that, a.e. on the support of  $\rho_\infty^\omega$ ,*

$$\lambda^\omega = \left(\frac{1}{A_m} \rho_\infty^\omega\right)^{m-1} - \frac{\omega^2}{2}|x|^2 + \phi_\infty^\omega.$$

*For any  $\varepsilon > 0$ , sufficiently small, there exists  $\eta > 0$  such that the following property holds. Let  $\rho$  be a solution of the evolution equation (10), that is*

$$\begin{aligned} \partial_t \rho &= \nabla \cdot [\nabla v(\rho) - \omega^2 x \rho + \rho \nabla_x \phi], \\ \phi &= -\frac{1}{4\pi|x|} * \rho \end{aligned}$$

*with initial datum  $\rho(x, t = 0) = \rho_0 \in \mathcal{X}_M$  and assume that*

$$\mathcal{G}_\omega[\rho_0] < \mathcal{I}_M^\omega + \varepsilon.$$

*Then there exists  $\eta > 0$  such that, if for any  $t > 0$ ,*

$$\left(\frac{1}{A_m} \rho(\cdot, t)\right)^{m-1} - \frac{\omega^2}{2}|x|^2 + \phi_\rho(\cdot, t) < \lambda^\omega + f(\varepsilon), \quad x \in \text{supp}(\rho(\cdot, t)) \text{ a.e.}, \tag{14}$$

*for some continuous function  $f$  with  $f(0) = 0$ ,  $f(s) > 0$  for any  $s > 0$ , then*

$$\sup_{\rho_\infty^\omega \in \mathcal{S}_M^\omega} \|\rho(\cdot, t) - \rho_\infty^\omega\|_{L^m(\mathbb{R}^2)} < \eta \quad \forall t > 0.$$

Such a stability result is a rather easy consequence of the localized minimization method developed in this paper, but proving that a bound like (14) holds for any  $t > 0$  if it holds at  $t = 0$  is an open question.

An alternative approach based on McCann’s approach in [47] can also be used. Since Eq. (10) is formally the gradient flow of the reduced free energy with respect to the Wasserstein distance, a dynamical stability result should hold in the corresponding sense.

4.5. *Partial symmetry*

One of the features that we have not established in this paper is the symmetry breaking, which is known to be a difficult question. See for instance [11,28] for related issues in the simpler case of the Caffarelli–Kohn–Nirenberg inequality. Using the *Schwarz foliated symmetrization*, see for instance [56], the symmetry assumption (11) results in a partial symmetry property of the localized minimizers which can be described as follows.

**Proposition 28.** *If  $\rho$  is a localized minimizers of  $\mathcal{G}_\omega$  in the sense of Theorem 1, up to a rotation, in radial coordinates  $(r, \theta) \in (0, \infty) \times [0, 2\pi)$ ,*

$$\rho(r, \theta + \pi) = \rho(r, \theta) \quad \text{and} \quad \rho(r, 2\pi - \theta) = \rho(r, \theta),$$

*and for any  $r > 0$ ,  $\theta \mapsto \rho(r, \theta)$  is monotone nonincreasing on  $(0, \pi/2)$ .*

The proof is left to the reader.

**Acknowledgements**

J.F. was supported by a Basque government postdoctoral grant for the formation of young researchers. The two authors thank I. Flores, O. Sánchez and J. Soler for stimulating discussions.

© 2006 by the authors. This paper may be reproduced, in its entirety, for non-commercial purposes.

**Appendix A. Summary of the notations**

To a given energy profile  $\gamma$ , we associate a local Gibbs state

$$G_f(t, x, v) := \gamma\left(\frac{1}{2}|v|^2 + \phi(t, x) - \frac{1}{2}\omega^2|x|^2 + \mu_f(t, x)\right).$$

Here  $G_f$  is defined with respect to a given distribution function  $f$  and  $\mu_f$  is the local Lagrange multiplier associated to the constraint

$$\int_{\mathbb{R}^2} G_f(t, x, v) dv = \int_{\mathbb{R}^2} f(t, x, v) dv =: \rho_f. \tag{1}$$

With  $\bar{\mu}$  implicitly by the condition  $\int_{\mathbb{R}^2} \gamma(\frac{1}{2}|v|^2 + \bar{\mu}(\rho)) dv = \rho$ , which means that

$$\bar{\mu}(\rho) = \Gamma^{-1}(\rho) \tag{2}$$

where  $\Gamma(s) := 2\pi \int_s^\infty \gamma(\sigma) d\sigma$ , then  $\mu_f$  is explicitly given by  $\mu_f = \bar{\mu} \circ \rho_f - \phi$ . At the kinetic level, the main purpose of the paper is to find a stationary solution  $(f, \phi)$  of

$$\begin{aligned} \partial_t f + v \cdot \nabla_x f + \omega^2 x \cdot \nabla_v f + 2\omega v \wedge \nabla_v f - \nabla_x \phi \cdot \nabla_v f &= G_f - f =: Q(f), \\ \phi &= -\frac{1}{4\pi|x|} * \int_{\mathbb{R}^2} f dv, \end{aligned} \tag{3}$$

where  $Q$  is a *relaxation time approximation* kernel. The free energy functional is defined as

$$\mathcal{F}_\omega[f] := \iint_{\mathbb{R}^2 \times \mathbb{R}^2} \left[ f\left(\frac{1}{2}|v|^2 - \frac{1}{2}\omega^2|x|^2 + \frac{1}{2}\phi\right) + \beta(f) \right] dx dv$$

with  $\beta(s) := \int_s^0 \gamma^{-1}(\sigma) d\sigma$ .

The *reduced free energy functional*, is

$$\mathcal{G}_\omega[\rho] := \int_{\mathbb{R}^2} \left[ h(\rho) + \left(\frac{1}{2}\phi - \frac{1}{2}\omega^2|x|^2\right)\rho \right] dx$$

with  $h(\rho) = H(\bar{\mu}(\rho)) - \rho \bar{\mu}(\rho)$  and  $H(s) := \int_0^s \Gamma(\sigma) d\sigma$ . Notice that  $\Gamma(\bar{\mu}(\rho)) = \rho$ ,  $h'(\rho) = -\bar{\mu}(\rho)$  and  $\mathcal{G}_\omega[\rho]$  is such that  $\mathcal{G}_\omega[\rho] = \mathcal{F}_\omega[\bar{G}_\rho]$  with  $\bar{G}_\rho(x, v) := \gamma(\frac{1}{2}|v|^2 + \bar{\mu}(\rho))$ . In the diffusion limit, with  $\nu(\rho) := \frac{1}{2} \int_{\mathbb{R}^2} |v|^2 \gamma(\frac{1}{2}|v|^2 + \bar{\mu}(\rho)) dv$ ,  $\rho_f$  converges to a solution of the system

$$\begin{aligned} \partial_t \rho &= \nabla \cdot [\nabla \nu(\rho) - \omega^2 x \rho + \rho \nabla_x \phi], \\ \phi &= -\frac{1}{4\pi|x|} * \rho, \end{aligned} \tag{10}$$

and  $\mathcal{G}_\omega$  is the associated free energy functional (Lyapunov functional). Hence finding stationary solutions to (3) or (10) is equivalent to find critical points of  $\mathcal{G}_\omega$  on the set  $\{\rho \in L^1(\mathbb{R}^2) : \rho \geq 0 \text{ a.e.}, \int_{\mathbb{R}^2} \rho dx = M\}$ .

The case of *polytropic gases*, or *polytropes*, corresponds to

$$\gamma(s) := \left(\frac{-s}{k+1}\right)_+^k \quad \text{and} \quad \bar{\mu}(\rho) = -(k+1) \left(\frac{\rho}{2\pi}\right)^{\frac{1}{k+1}}$$

for some parameter  $k \in \mathbb{R}^+$ . As a consequence, we have

$$\Gamma(s) = 2\pi \left( \frac{-s}{k+1} \right)_+^{k+1} \quad \text{and} \quad \beta(f) = \frac{f^q}{q-1} \quad \text{with} \quad k = \frac{1}{q-1} \quad \iff \quad q = 1 + \frac{1}{k},$$

$$h(\rho) = \frac{\kappa}{m-1} \rho^m \quad \text{with} \quad m = 2 - \frac{1}{q} = 1 + \frac{1}{k+1}, \quad \kappa = \frac{1}{m} (2\pi)^{1-m},$$

and the nonlinear diffusion equation holds with  $v(\rho) = \kappa \rho^m$ .

## References

- [1] V.I. Arnol'd, On conditions for non-linear stability of plane stationary curvilinear flows of an ideal fluid, Dokl. Akad. Nauk SSSR 162 (1965) 975–978.
- [2] V.I. Arnol'd, An a priori estimate in the theory of hydrodynamic stability, Izv. Vyssh. Uchebn. Zaved. Mat. 1966 (1966) 3–5.
- [3] F. Bavaud, Equilibrium properties of the Vlasov functional: the generalized Poisson–Boltzmann–Emden equation, Rev. Modern Phys. 63 (1991) 129–148.
- [4] N. Ben Abdallah, J. Dolbeault, Relative entropies for the Vlasov–Poisson system in bounded domains, C. R. Acad. Sci. Paris Sér. I Math. 330 (2000) 867–872.
- [5] N. Ben Abdallah, J. Dolbeault, Relative entropies for kinetic equations in bounded domains (irreversibility, stationary solutions, uniqueness), Arch. Ration. Mech. Anal. 168 (2003) 253–298.
- [6] P. Biler, P. Laurençot, T. Nadzieja, On an evolution system describing self-gravitating Fermi–Dirac particles, Adv. Differential Equations 9 (2004) 563–586.
- [7] P. Biler, T. Nadzieja, R. Stańczy, Nonisothermal systems of self-attracting Fermi–Dirac particles, in: Nonlocal Elliptic and Parabolic Problems, in: Banach Center Publ., vol. 66, Polish Acad. Sci., Warsaw, 2004, pp. 61–78.
- [8] P. Biler, R. Stańczy, Parabolic-elliptic systems with general density–pressure relations, Tech. rep., Mathematical Institute of the University of Wrocław, 2004.
- [9] J. Binney, S. Tremaine, Galactic Dynamics, Princeton University Press, Princeton, 1987.
- [10] A. Burchard, Y. Guo, Compactness via symmetrization, J. Funct. Anal. 214 (2004) 40–73.
- [11] F. Catrina, Z.-Q. Wang, On the Caffarelli–Kohn–Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions, Comm. Pure Appl. Math. 54 (2001) 229–258.
- [12] P.-H. Chavanis, Generalized thermodynamics and Fokker–Planck equations: applications to stellar dynamics and two-dimensional turbulence, Phys. Rev. E (2003) 036108.
- [13] P.-H. Chavanis, Generalized Fokker–Planck equations and effective thermodynamics, Phys. A 340 (2004) 57–65. News and expectations in thermostatics.
- [14] P.-H. Chavanis, Generalized kinetic equations and effective thermodynamics, Banach Center Publ. 66 (2004) 79–102.
- [15] P.-H. Chavanis, Generalized thermodynamics and kinetic equations: Boltzmann, Landau, Kramers and Smoluchowski, Physica A 332 (2004) 89.
- [16] P.-H. Chavanis, Hamiltonian and Brownian systems with long-range interactions, Physica A 361 (2006) 55–80.
- [17] P.-H. Chavanis, P. Laurençot, M. Lemou, Chapman–Enskog derivation of the generalized Smoluchowski equation, Physica A 341 (2004) 145–164.
- [18] P.-H. Chavanis, M. Ribot, C. Rosier, C. Sire, On the analogy between self-gravitating Brownian particles and bacterial populations, in: Banach Center Publ., vol. 66, 2004, p. 103.
- [19] P.-H. Chavanis, C. Sire, Anomalous diffusion and collapse of self-gravitating Langevin particles in  $d$  dimensions, Phys. Rev. E 69 (2004) 016116.
- [20] J.F. Collet, Extensive Lyapounov functionals for moment-preserving evolution equations, C. R. Math. Acad. Sci. Paris, Ser. I 334 (2002) 429–434.
- [21] C. Cortázar, M. Elgueta, P. Felmer, Existence of signed solutions for a semilinear elliptic boundary value problem, Differential Integral Equations 7 (1994) 293–299.
- [22] C. Cortázar, M. Elgueta, P. Felmer, On a semilinear elliptic problem in  $\mathbb{R}^N$  with a non-Lipschitzian nonlinearity, Adv. Differential Equations 1 (1996) 199–218.
- [23] J. Dolbeault, Monokinetic charged particle beams: qualitative behavior of the solutions of the Cauchy problem and 2d time-periodic solutions of the Vlasov–Poisson system, Comm. Partial Differential Equations 25 (2000) 1567–1647.
- [24] J. Dolbeault, P. Felmer, J. Mayorga-Zambrano, Compactness properties for trace-class operators and applications to quantum mechanics, Monatshefte für Mathematik (2008), in press.
- [25] J. Dolbeault, J. Fernández, O. Sánchez, Stability for the gravitational Vlasov–Poisson system in dimension two, Comm. Partial Differential Equations 31 (2006) 1425–1449.
- [26] J. Dolbeault, P. Markowich, D. Oelz, C. Schmeiser, Nonlinear diffusions as limit of kinetic equations with relaxation collision kernels, Arch. Rational Mech. Anal. 186 (2007) 133–158.
- [27] J. Dolbeault, O. Sánchez, J. Soler, Asymptotic behaviour for the Vlasov–Poisson system in the stellar-dynamics case, Arch. Ration. Mech. Anal. 171 (2004) 301–327.

- [28] V. Felli, M. Schneider, Perturbation results of critical elliptic equations of Caffarelli–Kohn–Nirenberg type, *J. Differential Equations* 191 (2003) 121–142.
- [29] R. Firt, Stability of disk-like galaxies. II: The Kuzmin disk, *Analysis (Munich)* 27 (2007) 405–424.
- [30] R. Firt, G. Rein, Stability of disk-like galaxies. I: Stability via reduction, *Analysis (Munich)* 26 (2006) 507–525.
- [31] A. Floer, A. Weinstein, Nonspreading wave packets for the cubic Schrödinger equation with a bounded potential, *J. Funct. Anal.* 69 (1986) 397–408.
- [32] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, *Commun. Math. Phys.* 68 (1979) 209–243.
- [33] B. Gidas, W.M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in  $\mathbf{R}^n$ , in: *Mathematical Analysis and Applications, Part A*, in: *Adv. in Math. Suppl. Stud.*, vol. 7, Academic Press, New York, 1981, pp. 369–402.
- [34] T. Goudon, F. Poupaud, Approximation by homogenization and diffusion of kinetic equations, *Comm. Partial Differential Equations* 26 (2001) 537–569.
- [35] Y. Guo, G. Rein, Existence and stability of Camm type steady states in galactic dynamics, *Indiana Univ. Math. J.* 48 (1999) 1237–1255.
- [36] Y. Guo, G. Rein, Stable steady states in stellar dynamics, *Arch. Ration. Mech. Anal.* 147 (1999) 225–243.
- [37] Y. Guo, G. Rein, Isotropic steady states in galactic dynamics, *Commun. Math. Phys.* 219 (2001) 607–629.
- [38] Y. Guo, G. Rein, Stable models of elliptical galaxies, *Mon. Not. R. Astronom.* (2003).
- [39] Y. Guo, G. Rein, A non-variational approach to nonlinear stability in stellar dynamics applied to the King model, *Commun. Math. Phys.* 271 (2007) 489–509.
- [40] M. Hadžić, G. Rein, Global existence and nonlinear stability for the relativistic Vlasov–Poisson system in the gravitational case, *Indiana Univ. Math. J.* 56 (2007) 2453–2488.
- [41] M. Lemou, F. Méhats, P. Raphael, Orbital stability and singularity formation for Vlasov–Poisson systems, *C. R. Math. Acad. Sci. Paris, Ser. I* 341 (2005) 269–274.
- [42] Y.Y. Li, On uniformly rotating stars, *Arch. Ration. Mech. Anal.* 115 (1991) 367–393.
- [43] E.H. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, *Ann. of Math. (2)* 118 (1983) 349–374.
- [44] E.H. Lieb, M. Loss, *Analysis*, second ed., *Graduate Studies in Mathematics*, vol. 14, American Mathematical Society, Providence, RI, 2001.
- [45] T. Luo, J. Smoller, Rotating fluids with self-gravitation in bounded domains, *Arch. Ration. Mech. Anal.* 173 (2004) 345–377.
- [46] P. Markowich, G. Rein, G. Wolansky, Existence and nonlinear stability of stationary states of the Schrödinger–Poisson system, *J. Statist. Phys.* 106 (2002) 1221–1239.
- [47] R.J. McCann, Stable rotating binary stars and fluid in a tube, *Houston J. Math.* 32 (2006) 603–631 (electronic).
- [48] T. Padmanabhan, Statistical mechanics of gravitating systems, *Phys. Rep.* 188 (1990) 285–362.
- [49] G. Rein, Non-linear stability for the Vlasov–Poisson system—the energy-Casimir method, *Math. Methods Appl. Sci.* 17 (1994) 1129–1140.
- [50] G. Rein, Flat steady states in stellar dynamics – existence and stability, *Commun. Math. Phys.* 205 (1999) 229–247.
- [51] G. Rein, Reduction and a concentration-compactness principle for energy-Casimir functionals, *SIAM J. Math. Anal.* 33 (2001) 896–912 (electronic).
- [52] G. Rein, Non-linear stability of gaseous stars, *Arch. Ration. Mech. Anal.* 168 (2003) 115–130.
- [53] G. Rein, Nonlinear stability of Newtonian galaxies and stars from a mathematical perspective, *Ann. New York Acad. Sci.* 1045 (2005) 103–119.
- [54] O. Sánchez, J. Soler, Orbital stability for polytropic galaxies, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 23 (2006) 781–802.
- [55] J. Schaeffer, Steady states in galactic dynamics, *Arch. Ration. Mech. Anal.* 172 (2004) 1–19.
- [56] D. Smets, M. Willem, Partial symmetry and asymptotic behavior for some elliptic variational problems, *Calc. Var. Partial Differential Equations* 18 (2003) 57–75.
- [57] G. Toscani, Remarks on entropy and equilibrium states, *Appl. Math. Lett.* 12 (1999) 19–25.
- [58] C. Tsallis, Possible generalization of Boltzmann–Gibbs statistics, *J. Statist. Phys.* 52 (1988) 479–487.
- [59] V.A. Vladimirov, K.I. Il'in, On Arnol'd's variational principles in fluid mechanics, in: *The Arnol'dfest*, Toronto, ON, 1997, in: *Fields Inst. Commun.*, vol. 24, Amer. Math. Soc., Providence, RI, 1999, pp. 471–495.
- [60] G. Wolansky, On nonlinear stability of polytropic galaxies, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 16 (1999) 15–48.
- [61] G. Wolansky, M. Ghil, An extension of Arnol'd's second stability theorem for the Euler equations, *Physica D* 94 (1996) 161–167.