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A general class of phase transition models with weighted interface energy

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Abstract

We study a family of singular perturbation problems of the kind

$$\inf\left\{\frac{1}{\varepsilon}\int_\Omega f(u,\varepsilon\nabla u,\varepsilon\rho)\,dx\colon \int_\Omega u=m_0,\ \int_\Omega \rho=m_1\right\},$$

where u represents a fluid density and the non-negative energy density f vanishes only for $u=\alpha$ or $u=\beta$. The novelty of the model is the additional variable $\rho\geqslant 0$ which is also unknown and interplays with the gradient of u in the formation of interfaces. Under mild assumptions on f, we characterize the limit energy as $\varepsilon\to 0$ and find for each f a transition energy (well defined when $u\in BV(\Omega;\{\alpha,\beta\})$ and ρ is a measure) which depends on the $n-1$ dimensional density of the measure ρ on the jump set of u. An explicit formula is also given.
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1. Introduction and main results

Bubbles, foams and the role of surfactants have been the object of focused study in the last years; a surfactant is, roughly speaking, a substance which may be added to a mixture of two phases (gas/fluid, fluid/fluid, even metallic foams) to help with the formation of interfaces, by locally lowering surface tension.

Several Ginzburg–Landau-type models exist in the literature (see e.g. [3–6,8–10,12]), based on the study of the molecular interaction energies of the two phases and the surfactant. Due to the amphiphilic character of tensioactives, the energy should take into account the gradient of the orientation of surfactant molecules, but as this gives rise to

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highly complex numerical problems, in many instances (see [5,8–10,12]) only the average concentration of surfactant is considered, thus leading to a bulk energy of the form

$$\int f(u,\nabla u,\rho)\,dx$$

where u is the phase parameter (i.e. a scalar function which takes two different prescribed values, say α and β, in the two phases) and ρ is the density of the surfactant.

As an example, a recent paper [5] by Fonseca, Morini and Slastikov uses a particular model by Perkins, Sekerka, Warren and Langer which is a modification of the standard van der Waals–Cahn–Hilliard model for fluid phase transition, namely their energy is given by

$$\int_\Omega \frac{1}{\varepsilon}W(u)+\varepsilon|\nabla u|^2+\varepsilon\big(\rho-|\nabla u|\big)^2\,dx, \tag{1.1}$$

where W is a double-well potential and ε is the scaling parameter that is commonly used to drive the system towards phase separation. Fonseca, Morini and Slastikov deduce from the model the limit energy at the interface between two phases.

In this paper we deal with a very general class of energy functionals, and under mild assumptions we derive the limit energy on the interface, depending on the bulk energy density f; to be more precise, let $\alpha<\beta$ be two real numbers and let $f:\mathbb{R}\times\mathbb{R}^n\times[0,+\infty[\,\to[0,+\infty[$ be a continuous function satisfying

(H1) $f(s,0,0)=0 \Longleftrightarrow s\in\{\alpha,\beta\}$;
(H2) for all $s\in\mathbb{R}$, $f(s,\cdot,\cdot)$ is convex;
(H3) $\begin{cases} \text{for all } s\in\mathbb{R},\ f(s,z,\gamma)\geqslant f(s,0,0),\\ \text{for all } s\in\mathbb{R},\ z\mapsto f(s,z,0) \text{ attains a strict minimum at } z=0,\\ \text{for all } s\in\mathbb{R},\ \frac{\partial f}{\partial z}(s,0,0)=0,\\ \text{if } s\in\{\alpha,\beta\},\ \frac{\partial f}{\partial \gamma}(s,0,0)=0;\end{cases}$
(H4) there exist $C,C'>0$ such that $f(s,0,0)\geqslant\frac{1}{C}|s|-C'$.

If Ω is a bounded open set with Lipschitz boundary, the total energy we attach to the system reads

$$F_\varepsilon(u,\rho)=\begin{cases}\frac{1}{\varepsilon}\int_\Omega f(u,\varepsilon\nabla u,\varepsilon\rho)\,dx & \text{if } (u,\rho)\in W^{1,1}(\Omega)\times L^1_+(\Omega),\\ +\infty & \text{otherwise.}\end{cases} \tag{1.2}$$

Due to the scaling and to conditions (H1)–(H3) we expect that if a sequence $(u_\varepsilon,\rho_\varepsilon)$ is uniformly bounded in energy (that is $\sup_\varepsilon F_\varepsilon(u_\varepsilon,\rho_\varepsilon)<+\infty$), then up to a subsequence we will have $u_\varepsilon\rightharpoonup u$ and $\rho_\varepsilon\rightharpoonup\rho$ where $u(x)\in\{\alpha,\beta\}$ and $\rho\in\mathcal{M}_+(\overline{\Omega})$ is a non-negative Radon measure supported by $\overline{\Omega}$. We will show that in fact the convergence of u_ε is strong in $L^1(\Omega)$ and the limit u belongs to the space $BV(\Omega;\{\alpha,\beta\})$: thus u is of the form $u=\alpha\mathbb{1}_A+\beta\mathbb{1}_{\Omega\setminus A}$ where A is a set with finite perimeter. Then the total limit energy will be concentrated on the interface ∂A (*essential boundary of* A). The physical interpretation of the above is that the two phases actually separate, and all the energy resides at the interface.

The interest now lies in describing this energy; for this task we introduce the *conical envelope* of f defined by:

$$f_c(s,z,\gamma):=\inf_{\lambda>0}\frac{1}{\lambda}f(s,\lambda z,\lambda\gamma), \tag{1.3}$$

and then, for all $(z,\Theta)\in\mathbb{R}^n\times[0,+\infty[$, we set

$$\sigma(z,\Theta):=\inf\left\{\int_\alpha^\beta f_c\big(s,z,\gamma(s)\big)\,ds\colon\ \gamma\in L^1(\alpha,\beta;R_+),\ \int_\alpha^\beta\gamma(s)\,ds\leqslant\Theta\right\}. \tag{1.4}$$

The properties of f_c and σ are summarized in Section 3, see Lemmas 3.2 and 3.4. It turns out that the value $\sigma(z,\Theta)$ given in (1.4) is the reduced expression for the minimum of a one-dimensional problem in which u varies between α and β and the density ρ has a prescribed integral. The direction z figures out the normal to the transition layer. As observed in many other scalar models (see the forefather [11] and many of the references in [5]) the transition profile will be unidimensional. If for a two-valued BV function $\alpha\mathbb{1}_A + \beta\mathbb{1}_{\Omega\setminus A}$ we denote by S_u the jump set of u, that is the reduced boundary of A inside Ω, and by ν_u the normal to S_u pointing towards higher values of u, for any such function and for any non-negative Radon measure ρ on $\overline{\Omega}$ we define

$$F(u,\rho) := \int_{S_u} \sigma\big(\nu_u(x), \rho^0(x)\big)\, d\mathcal{H}^{n-1}, \tag{1.5}$$

where

$$\rho^0(x) = \frac{d\rho}{d(Ju)}(x), \qquad Ju := \mathcal{H}^{n-1} \llcorner S_u. \tag{1.6}$$

We remark that according to the previous formulas the only portion of surfactant which plays a role in the total energy concentrates at the interface, that is it has a density with respect to the *singular* surface measure.

After some preliminary material in Section 3, we will prove in Section 4 the Γ-convergence of F_ε to F and as an immediate corollary we obtain the following result:

Theorem 1.1. *Assume that $m_1 > 0$ and that $\alpha|\Omega| < m_0 < \beta|\Omega|$. Let $(u_\varepsilon, \rho_\varepsilon)$ be an optimal pair (possibly up to a small error vanishing as $\varepsilon \to 0$) for the minimum problem*

$$\inf\left\{ \frac{1}{\varepsilon}\int_\Omega f(u, \varepsilon\nabla u, \varepsilon\rho)\, dx:\ \int_\Omega u = m_0,\ \int_\Omega \rho = m_1 \right\}.$$

Then the sequence $(u_\varepsilon, \rho_\varepsilon)$ is relatively compact for the normed topology of $L^1(\Omega)$ times the weak convergence in $\mathcal{M}_+(\overline{\Omega})$, and any cluster point $(\bar{u}, \bar{\rho})$ belongs to $BV(\Omega; \{\alpha, \beta\}) \times \mathcal{M}_+(\overline{\Omega})$ and solves the problem

$$\inf\left\{ F(u,\rho):\ (u,\rho) \in BV\big(\Omega; \{\alpha,\beta\}\big) \times \mathcal{M}_+(\overline{\Omega}),\ \int_\Omega u\, dx = m_0,\ \int_{\overline{\Omega}} \rho = m_1 \right\}.$$

To prove one of the inequalities involved in our main result (Theorem 4.1), we will introduce an integral representation formula for the relaxed functional of $\int_\Omega f_c(u, \nabla u, \rho)\, dx$ on $BV \times \mathcal{M}_+$. Instead, the other inequality requires lengthy technical approximation arguments to find and implement an optimal transition profile.

We remark that the definition (1.4) of σ may seem too difficult to handle to be of any practical use, but in Section 2 we provide an explicit formula which helps with computing the function σ without solving the variational problem in (1.4), and we apply it to the study of some particular cases and a discussion on the underlying physical models including the one proposed in [5]. Due to the interest of these considerations for the applications, we placed this section right after the introduction, even though we will have to refer to some material covered later on.

To conclude this introduction, we emphasize that in this paper we give a mathematical foundation to a large class of surfactant-driven energies and that the approximation procedure we propose may be usefully applied for the numerical treatment of interface problems in the presence of a surfactant.

2. A representation formula, examples and comments

In this section we will analyze in deeper detail the properties of the integrand $\sigma(z,\Theta)$ which describes the interface energy arising from our model. As will be proved in Lemma 3.4, the function $\sigma(z,\Theta)$ given by (1.4) is convex, lower semicontinuous and 1-homogeneous on $\mathbb{R}^n \times \mathbb{R}^+$. Moreover $\sigma(z,\cdot)$ is monotone non-increasing. In the following we extend the definition of σ to all of $\mathbb{R}^{n+1}$ by setting

$$\sigma(z,\Theta) = +\infty \quad \text{if } \Theta < 0,$$

which obviously preserves the previous properties of σ.

Explicit computations are not an easy task. However we found a quite simple formula for the *partial* Legendre–Fenchel conjugate of σ with respect to the variable Θ, that is the function $\sigma^*(z,\Theta^*)$ given by

$$\sigma^*(z,\Theta^*) = \sup_{\Theta\in\mathbb{R}} \big[\Theta^*\Theta - \sigma(z,\Theta)\big].$$

This formula will be helpful in order to study the differentiability of $\sigma(z,\cdot)$. Notice that, as a convex and l.s.c. function on the real line, $\sigma(z,\cdot)$ can be recovered from $\sigma^*(z,\cdot)$ through the biconjugate formula $\sigma(z,\Theta)=\sup\{\Theta\Theta^*-\sigma^*(z,\Theta^*)\colon \Theta^*\in\mathbb{R}\}$. Analogously, we denote by $f_c^*(s,z,\cdot)$ the (partial) Legendre–Fenchel conjugate of $f_c(s,z,\cdot)$. We also denote by $\partial\sigma(z,\Theta)$ the partial subdifferential defined by

$$\partial\sigma(z,\Theta)=\big\{\Theta^*\colon \sigma(z,\Theta)+\sigma^*(z,\Theta^*)=\Theta\Theta^*\big\}.$$

Since $\sigma=+\infty$ for $\Theta<0$, we have that $\partial\sigma(z,0)=\left]-\infty,\tau\right]$ where $\tau=\sigma'(z,0^+)$ is the slope at 0 of $\sigma(z,\cdot)$.

Lemma 2.1. *For any $z\in\mathbb{R}^n$*

$$\sigma^*(z,\Theta^*)=\begin{cases}\int_\alpha^\beta f_c^*(s,z,\Theta^*)\,ds & \text{if } \Theta^*\leqslant 0,\\ +\infty & \text{if } \Theta^*>0.\end{cases}\tag{2.1}$$

Proof. As σ is bounded, see (3.14), we immediately infer that $\sigma^*(z,\Theta^*)=+\infty$ whenever $\Theta^*>0$. Thus from now on we concentrate on the case $\Theta^*\leqslant 0$. We get from definition (1.4)

$$\begin{aligned}\sigma^*(z,\Theta^*)&=\sup_{\Theta\geqslant 0}\big[\Theta^*\Theta-\sigma(z,\Theta)\big]\\&=\sup_{\Theta\geqslant 0}\left[\Theta^*\Theta-\inf\left\{\int_\alpha^\beta f_c\big(s,z,\gamma(s)\big)\,ds\colon \gamma\in L^1_+,\ \int_\alpha^\beta\gamma(s)\,ds\leqslant\Theta\right\}\right]\\&=\sup_{\Theta\geqslant 0}\left[\sup_{\gamma\in L^1([\alpha,\beta];R_+),\int_\alpha^\beta\gamma(s)\,ds\leqslant\Theta}\left(\Theta^*\Theta-\int_\alpha^\beta f_c\big(s,z,\gamma(s)\big)\,ds\right)\right]\\&=\sup_{\gamma\in L^1([\alpha,\beta];R_+)}\left[\sup_{\Theta\geqslant\int_\alpha^\beta\gamma(s)\,ds}\left(\Theta^*\Theta-\int_\alpha^\beta f_c\big(s,z,\gamma(s)\big)\,ds\right)\right].\end{aligned}$$

Since $\Theta^*\leqslant 0$, it is never convenient to take $\Theta>\int_\alpha^\beta\gamma(s)\,ds$. Thus we obtain

$$\begin{aligned}\sigma^*(z,\Theta^*)&=\sup_{\gamma\in L^1([\alpha,\beta];R_+)}\left[\Theta^*\int_\alpha^\beta\gamma(s)\,ds-\int_\alpha^\beta f_c\big(s,z,\gamma(s)\big)\,ds\right]\\&=\sup_{\gamma\in L^1([\alpha,\beta];R_+)}\int_\alpha^\beta\big[\Theta^*\gamma(s)-f_c\big(s,z,\gamma(s)\big)\big]\,ds\\&=\int_\alpha^\beta\sup_{\gamma(s)\geqslant 0}\big[\Theta^*\gamma(s)-f_c\big(s,z,\gamma(s)\big)\big]\,ds\\&=\int_\alpha^\beta f_c^*(s,z,\Theta^*)\,ds,\end{aligned}$$

where in the third line we interchange "sup" and "integral" (see [2], Theorem VII.7). □

We remark that, despite the function f_c^* may very well be finite also for some positive values of Θ^* (see Example 2.5), the domain of $\sigma(z,\cdot)$, and thus of $\partial\sigma(z,\cdot)$, lies in $\left]-\infty,0\right]$. As a consequence of Lemma 2.1 we have:

Lemma 2.2. *Let $z \in \mathbb{R}^n$, $\Theta \geqslant 0$ and $\Theta^* < 0$. Then $\Theta^* \in \partial\sigma(z,\Theta)$ if and only if there exists $\gamma \in L^1([\alpha,\beta];\mathbb{R}^+)$ such that*

$$\int_\alpha^\beta \gamma(s)\,ds \leqslant \Theta, \quad \Theta^* \in \partial f_c\big(s,z,\gamma(s)\big) \text{ for a.e. } s \in [\alpha,\beta]. \tag{2.2}$$

In particular

$$\partial\sigma(z,0) = \bigcap_{s\in[\alpha,\beta]} \operatorname{arg\,min} f_c^*(s,z,\cdot) = \big]-\infty,\tau(z)\big],$$

where $\tau(z) = \inf_{s\in[\alpha,\beta]} f_c'(s,z,0^+)$ and $f_c'(s,z,0^+)$ is the slope at 0 of $f_c(s,z,\cdot)$.

Notice that this lemma ensures the existence of an optimal γ associated with Θ provided $\partial\sigma(z,\Theta)$ contains a negative element Θ^*. The special case $\Theta^* = 0$ is discussed later in Remark 2.4.

Proof. If γ satisfies (2.2), then

$$\sigma(z,\Theta) \leqslant \int_\alpha^\beta f_c\big(s,z,\gamma(s)\big)\,ds = \int_\alpha^\beta \big[\Theta^*\gamma(s) - f_c^*(s,z,\Theta^*)\big]\,ds \\ \leqslant \Theta\Theta^* - \sigma^*(z,\Theta^*),$$

showing that $\Theta^* \in \partial\sigma(z,\Theta)$. Remark that this implication holds also if $\Theta^* = 0$.

Conversely, assume that $\Theta^* \in \partial\sigma(z,\Theta)$ with $\Theta^* < 0$; we begin with the case $\Theta > 0$. As we will see in Lemma 3.5, for every $\varepsilon > 0$ there exists $\gamma_\varepsilon(s)$ such that

$$\int_\alpha^\beta f_c\big(s,z,\gamma_\varepsilon(s)\big)\,ds \leqslant \sigma(z,\Theta) + \varepsilon, \qquad \int_\alpha^\beta \gamma_\varepsilon(s)\,ds = \Theta.$$

Set $h_\varepsilon(s) := f_c(s,z,\gamma_\varepsilon(s)) + f_c^*(s,z,\Theta^*) - \Theta^*\gamma_\varepsilon(s)$. Then by Lemma 2.1 the function $f_c^*(\cdot,z,\Theta^*)$ is integrable and

$$\int_\alpha^\beta h_\varepsilon(s)\,ds \leqslant \varepsilon + \sigma(z,\Theta) + \sigma^*(z,\Theta^*) - \Theta\Theta^* = \varepsilon.$$

As in addition h_ε and f_c are nonnegative, we infer that $h_\varepsilon \to 0$ in $L^1([\alpha,\beta])$. Recalling that $\Theta^* < 0$, we are led to

$$0 \leqslant \gamma_\varepsilon \leqslant \frac{1}{-\Theta^*}\big[h_\varepsilon - f_c^*(\cdot,z,\Theta^*)\big].$$

Thus the sequence γ_ε is equiintegrable and therefore weakly relatively compact in $L^1([\alpha,\beta)]$. Let γ be a weak cluster point. Then $\lim_{\varepsilon\to0}\int_\alpha^\beta \gamma_\varepsilon(s)\,ds = \int_\alpha^\beta \gamma(s)\,ds$ and, since $f_c(s,z,\cdot)$ is convex continuous, we have by the classical weak lower semicontinuity property of convex integral functionals

$$\liminf_{\varepsilon\to0}\int_\alpha^\beta f_c\big(s,z,\gamma_\varepsilon(s)\big)\,ds \geqslant \int_\alpha^\beta f_c\big(s,z,\gamma(s)\big)\,ds.$$

Therefore

$$0 = \lim_{\varepsilon\to0}\int_\alpha^\beta h_\varepsilon(s)\,ds \geqslant \int_\alpha^\beta \big[f_c\big(s,z,\gamma(s)\big) + f_c^*(s,z,\Theta^*) - \Theta^*\gamma(s)\big]\,ds \geqslant 0,$$

and we conclude that $\gamma(s)$ satisfies (2.2).

Now let $\Theta = 0$: in this case we obviously have $\gamma \equiv 0$ and condition (2.2) reduces to $\Theta^* \in \partial f_c(s,z,0)$ for a.e. $s \in [\alpha,\beta]$. By Lemma 2.3, we have $\partial f_c(s,z,0) = \arg\min f_c^*(s,z,\cdot) =]-\infty, \tau(s)]$ where

$$\tau(s) := f_c'(s,z,0^+) = \inf_{t>0} \frac{f_c(s,z,t)}{t}.$$

The continuity of f_c, proved in Lemma 3.2, implies that $\tau(s)$ is upper semicontinuous. Thus the condition on Θ^* satisfied a.e. reduces to $\Theta^* \leqslant \tau$ where $\tau = \inf\{\tau(s)\colon s \in [\alpha,\beta]\}$. □

Lemma 2.3. *Let g be a convex lower semicontinuous function on $\mathbb{R}$ such that $g(x) = +\infty$ for $x < 0$; the following three conditions are then equivalent*:

$$\text{(i): } g(x) \geqslant g(0) + \tau x \ \forall x \geqslant 0, \quad \text{(ii): } g^*(\tau) = -g(0), \quad \text{(iii): } g^*(x^*) = g^*(\tau) \ \forall x^* \leqslant \tau.$$

Therefore $\partial g(0) = \arg\min g^* =]-\infty, g'(0^+)]$.

Proof. Since (i) may be rewritten $\tau x - g(x) \leqslant -g(0)$ it is clearly equivalent to (ii). Now we observe that $g^*(x^*) = \sup_{x\geqslant 0}[x^*x - g(x)]$ is monotone non decreasing as a function of x^*. Therefore condition (iii) is equivalent to $g^*(\tau) = \min g^*$. Since g is convex l.s.c. we have $\inf g^* = -g^{**}(0) = -g(0)$ and the last condition reduces to (ii). □

Remark 2.4. In view of Lemma 2.3, one has $0 \in \partial\sigma(z,\overline{\Theta})$ if and only if $\sigma(z,\cdot)$ reaches its minimum at $\overline{\Theta}$ and remains constant on $[\overline{\Theta}, +\infty[$. This means that $\overline{\Theta}$ is a saturation constant beyond which any further addition of surfactant has no effect on the surface tension. If there exists an integrable selection $\gamma(s)$ of the minimum set of $f_c(s,z,\cdot)$, then this selection satisfies (2.2) with $\Theta^* = 0$, and as already mentioned in the proof of Lemma 2.2 this implies that $0 \in \partial\sigma(z,\Theta)$ where $\Theta = \int_\alpha^\beta \gamma(s)\,ds$. Therefore we obtain:

$$\overline{\Theta} = \inf\left\{\int_\alpha^\beta \gamma(s)\,ds\colon\ f_c\bigl(s,z,\gamma(s)\bigr) = \min f_c(s,z,\cdot) = -f_c^*(s,z,0) \text{ a.e.}\right\}.$$

Notice also that such a finite $\overline{\Theta}$ exists if and only if the slope at 0 of $\sigma^*(z,\cdot)$ is finite, which, in view of (2.1), is equivalent to the condition $\int_\alpha^\beta (f_c^*)'(s,z,0^+)\,ds < +\infty$.

In the ensuing examples we will endeavour to describe the behaviour of the surface energy σ, by pushing its computation as far as possible. In addition to Lemma 2.3 we will use the fact that, for any function f and constants $C, K \neq 0$, there holds

$$g(x) = Cf(x/K) \quad\Rightarrow\quad g^*(x^*) = Cf^*(Kx^*/C). \tag{2.3}$$

For a wide class of energies, including those considered in Examples 2.5 and 2.6 below, the specific form of the energy density f allows for easy computation of f_c. Indeed assume

$$f(s,z,\gamma) = W(s) + g^2(s,z,\gamma)$$

with $W \geqslant 0$ and with $g \geqslant 0$ positively homogeneous of degree 1 with respect to the pair (z,γ), so that g^2 is positively homogeneous of degree 2: then one directly obtains

$$f_c(s,z,\gamma) = 2\sqrt{W(s)}\,g(s,z,\gamma).$$

In view of (2.1), we need to compute f_c^*: this is simpler when, in addition, f depends on z only through $|z|$. Then writing

$$g(s,z,\gamma) = |z|\psi\bigl(s,\gamma/|z|\bigr)$$

(with $\psi(s,t) = +\infty$ for $t < 0$), and leaving to the reader the easy changes for the cases $z = 0$ or $W(s) = 0$, one has by (2.3)

$$
\begin{aligned}
f_c^*(s,z,\Theta^*) &= \sup_{\gamma>0}\big[\Theta^*\gamma - 2\sqrt{W(s)}|z|\psi\big(s,\gamma/|z|\big)\big]\\
&= |z|\sup_{t>0}\big[\Theta^* t - 2\sqrt{W(s)}\psi(s,t)\big]\\
&= 2\sqrt{W(s)}|z|\psi^*\left(s, \frac{\Theta^*}{2\sqrt{W(s)}}\right).
\end{aligned}
$$

Eventually one has only to compute the conjugate of the function (of one real variable) $\psi(s,\cdot)$.

Example 2.5. The functional (1.1) may now be dealt with quite easily: indeed we apply to the function $f(s,z,\gamma) = W(s) + |z|^2 + (|z|-\gamma)^2$ the considerations above to get

$$f_c(s,z,\gamma) = 2\sqrt{W(s)}\,|z|\psi\big(\gamma/|z|\big)$$

with $\psi(t) = \sqrt{1+(1-t)^2}$ for $t>0$. Since in (1.5) we only use $|z|=1$, with a little abuse of notation we have

$$f_c(s,1,\gamma) = 2\sqrt{W(s)}\sqrt{1+(1-\gamma)^2};$$

it is straightforward to get

$$
\psi^*(t^*) = \begin{cases} +\infty & \text{if } t^* > 1,\\ t^* - \sqrt{1-(t^*)^2} & \text{if } -\frac{1}{\sqrt{2}} \leqslant t^* \leqslant 1,\\ -\sqrt{2} & \text{if } t^* \leqslant -\frac{1}{\sqrt{2}} \end{cases}
$$

(we remark that this function is finite also for some positive values of its argument), so that by (2.3)

$$
f_c^*(s,1,\Theta^*) = \begin{cases} \Theta^* - \sqrt{4W(s) - (\Theta^*)^2} & \text{if } -\sqrt{2}\sqrt{W(s)} \leqslant \Theta^* \leqslant 0,\\ -2\sqrt{2}\sqrt{W(s)} & \text{if } \Theta^* \leqslant -\sqrt{2}\sqrt{W(s)}. \end{cases}
$$

Although going further would require to know W in explicit form, we can draw some physically interesting features of the limit interface $\sigma(z,\Theta)(= |z|\sigma(1,\Theta))$.

(1) By (2.1) the function $\sigma^*(1,\cdot)$ is constant exactly on $]-\infty, -\sqrt{2\max W}]$; by Lemmas 2.2 or 2.3 this implies that the slope at $\Theta = 0$ of $\sigma(1,\Theta)$ is $-\sqrt{2\max W}$: the value of this constant accounts the influence of small additions of surfactant to a surfactant-free mixture. In particular, we have for small Θ

$$\sigma(1,\Theta) \approx \sigma(1,0) - \sqrt{2\max W}\,\Theta,$$

where

$$\sigma(1,0) = \int_\alpha^\beta f_c(s,1,0)\,ds = 2\int_\alpha^\beta \sqrt{W(s)}\,ds.$$

(2) Employing the prime to denote differentiation with respect to Θ^*, since $(f_c^*)'(s,1,0^-) \equiv 1$, by (2.1) we have $(\sigma^*)'(1,0^-) = \beta - \alpha$, which by Lemma 2.3 implies that $\sigma(1,\Theta)$ is constant for $\Theta \geqslant \beta - \alpha$: this shows that this model exhibits saturation, that is, if the amount of surfactant is larger than $\beta - \alpha$ the exceeding part has no influence on the interface energy. One may easily see that in this case if $\Theta = \beta - \alpha$ the minimizing density given by Lemma 2.2 is the constant 1, whereas if $\Theta > \beta - \alpha$ the minimizing sequence concentrates the exceeding mass around points where $W(s) = 0$, that is at $s = \alpha$ or $s = \beta$. Physically this means that the extra surfactant concentrates inside the pure phases.

Example 2.6. We may modify Example 2.5 above to obtain an uneven distribution of surfactant across the interface, or to obtain cases where saturation is neither achieved: fix a positive function $\bar\gamma(s)$ and define

$$\psi(s,t) = \sqrt{1+\big(1 - t/\bar\gamma(s)\big)^2}$$

so that the minimum point of ψ is now at $t=\bar\gamma(s)$ instead of $t\equiv 1$. Leaving out the computations, one may check that there is saturation if and only if $\bar\gamma\in L^1$, because

$$(\sigma^*)'(1,0^-)=\int_\alpha^\beta \bar\gamma(s)\,ds=:\Theta_0,$$

the value at which (if finite) the surfactant reaches saturation by Lemma 2.3.

Example 2.7. Another interesting energy is given by

$$f(s,z,\gamma)=W(s)+\frac{|z|^p}{(\delta+\gamma)^{p-1}}$$

where $\delta>0$ and $p>1$. Setting

$$\mu_\delta(s)=\left(\frac{(p-1)\delta}{W(s)}\right)^{1/p}$$

one has

$$f_c(s,z,\gamma)=\begin{cases}\frac{p|z|}{\mu_\delta^{p-1}}-\frac{p-1}{\mu_\delta^p}\gamma & \text{if } 0\leqslant\gamma\leqslant\mu_\delta(s)|z|,\\ \frac{|z|^p}{\gamma^{p-1}} & \text{if } \gamma\geqslant\mu_\delta(s)|z|.\end{cases}$$

The left part of the graph of f_c with respect to γ is the tangent line drawn from $(0,f_c(s,z,0))$ to the graph of $|z|^p/\gamma^{p-1}$; here again some comments may be made.

(1) There is no saturation: this seems clear since f_c is decreasing. It can be proved by computing f_c^*; indeed for all $s\neq\alpha,\beta$ one has for a suitable $t(s)>0$ and for all $\Theta^*\in\,]-t(s),0[$:

$$f_c^*(s,1,\Theta^*)=-p(p-1)^{(1-p)/p}(-\Theta^*)^{(p-1)/p}.$$

In particular $(f_c^*)'(s,1,0)\equiv+\infty$, which implies $(\sigma^*)'(1,0)=+\infty$.

(2) For $\Theta^*\leqslant\Theta_0^*:=-\frac{\max W(s)}{\delta}$, one has

$$f_c^*(s,1,\Theta^*)=-\frac{p}{(\mu_\delta(s))^{p-1}}$$

so that $\sigma^*(1,\cdot)$ is constant on $]-\infty,\Theta_0^*]$. From Lemma 2.3 we deduce the approximate behaviour of the surface energy for small amounts of surfactant:

$$\sigma(1,\Theta)\approx\sigma(1,0)+\Theta_0^*\Theta,$$

where

$$\sigma(1,0)=\frac{p}{((p-1)\delta)^{(p-1)/p}}\int_\alpha^\beta W^{(p-1)/p}(s)\,ds.$$

(3) Pushing δ to zero leads to the following explicit expression for the limit surface tension coefficient

$$\sigma(1,\Theta)=\frac{(\beta-\alpha)^p}{\Theta^{p-1}}.$$

This can be easily checked by plugging $f_c(s,z,\gamma)=|z|^p/\gamma^{p-1}$ in (1.4) and by choosing a constant profile $\gamma(s)$. In the underlying physical model, the phase transition may thus only occur at those interfaces on which the surfactant has a positive density.

3. Preliminary results

The first part of this section is devoted to a precise study of the properties of the conical envelope f_c defined in (1.3) and of the interface energy density σ introduced in (1.4) in relation with a one-dimensional variational problem. In the second part of the section, we recall some useful features of convex functionals on measures. Then in the last part, we establish some approximation properties for the limit functional introduced in (1.5).

3.1. Interfacial integrands

Lemma 3.1. *Under assumptions* (H1)–(H3)

$$f_c(s,z,0)=0 \text{ for some } z\neq 0 \quad\Longleftrightarrow\quad s\in\{\alpha,\beta\}, \tag{3.1}$$

$$f_c(\alpha,z,0)=f_c(\beta,z,0)=0, \tag{3.2}$$

$$f_c(\alpha,0,\gamma)=f_c(\beta,0,\gamma)=0. \tag{3.3}$$

Proof. Assume that $f_c(s,z,0)=0$ for some $z\neq 0$ and $s\notin\{\alpha,\beta\}$: this means that for a suitable sequence λ_n of positive numbers

$$\frac{1}{\lambda_n}f(s,\lambda_n z,0)\to 0; \tag{3.4}$$

by the minimum property $(\mathrm{H3})_2$ we have

$$\frac{1}{\lambda_n}f(s,\lambda_n z,0)\geqslant\frac{f(s,0,0)}{\lambda_n},$$

and if $\lambda_n\not\to+\infty$ we deduce as $n\to+\infty$

$$f(s,0,0)=0,$$

which implies by (H1) that $s\in\{\alpha,\beta\}$. If otherwise $\lambda_n\to+\infty$, by the convexity of f we have as soon as $\lambda_n\geqslant 1$

$$f(s,\lambda_n z,0)\geqslant f(s,0,0)+\lambda_n\big(f(s,z,0)-f(s,0,0)\big),$$

thus in particular

$$\frac{1}{\lambda_n}f(s,\lambda_n z,0)\geqslant f(s,z,0)-f(s,0,0)$$

which by (3.4) would give $f(s,z,0)\leqslant f(s,0,0)$, thus contradicting the strict minimality in $(\mathrm{H3})_2$ because f is convex and $z\neq 0$, which proves (3.1).

Confining ourselves to $s=\alpha$, by (H1) we may write

$$\frac{f(\alpha,\lambda z,0)}{\lambda}=\frac{f(\alpha,\lambda z,0)-f(\alpha,0,0)}{\lambda}\to z\frac{\partial f}{\partial z}(\alpha,0,0)=0$$

by $(\mathrm{H3})_3$, thus $f_c(\alpha,0,0)\leqslant 0$ by (1.3), and also (3.2) is proved.

The proof of (3.3) is the same (switch z and γ), just remark that we did not use the strict minimality given by $(\mathrm{H3})_3$ but simply minimality—which in the case of $(s,0,\cdot)$ is provided by $(\mathrm{H3})_1$—and we need to compute the partial derivative only at $s=\alpha$, which is allowed by $(\mathrm{H3})_4$. □

In addition to the function f_c defined in (1.3) we will use the function defined for all $M\geqslant 1$ as

$$f_c^M(s,z,\gamma)=\inf_{M^{-1}\leqslant\lambda\leqslant M}\frac{1}{\lambda}f(s,\lambda z,\lambda\gamma). \tag{3.5}$$

Moreover it is useful to introduce an auxiliary lower-bound function: we set

$$k(s)=\inf\big\{\gamma+f_c(s,z,\gamma):\ \gamma\geqslant 0,\ |z|^2+\gamma^2=1\big\}. \tag{3.6}$$

We then have:

Lemma 3.2. *Under the assumptions* (H1)–(H3) *the function* $f_c(s,z,\gamma)$ *is continuous*; *for every* s *it is convex,* 1*-homogeneous and subadditive with respect to* (z,γ) *and*

$$f_c(\alpha,\cdot,\cdot)=f_c(\beta,\cdot,\cdot)=0. \tag{3.7}$$

In particular there exists

$$K=\max\bigl\{f_c(s,z,\gamma)\colon\ s\in[\alpha,\beta],\ \gamma\geqslant 0,\ |z|+\gamma=1\bigr\}$$

and

$$f_c(s,z,\gamma)\leqslant K\bigl(|z|+\gamma\bigr)\quad \forall s\in[\alpha,\beta],\ \gamma\geqslant 0,\ z\in\mathbb{R}^n. \tag{3.8}$$

The function f_c^M *is also continuous, and*

$$f_c^M\searrow f_c\quad \textit{as } M\to+\infty; \tag{3.9}$$

in particular it satisfies the inequalities

$$f_c(s,z,\gamma)\leqslant f_c^M(s,z,\gamma)\leqslant f_c^1(s,z,\gamma)=f(s,z,\gamma). \tag{3.10}$$

Finally the function $k(s)$ *is continuous and*

$$k(s)>0\quad \textit{if } s\notin\{\alpha,\beta\}. \tag{3.11}$$

Proof. Some of the properties of f_c might be deduced from [1], proof of Lemma 3.1 in Appendix A, but since our assumptions on (z,γ) are a little less restrictive we prefer to carry on the adapted proofs here.

To prove that f_c is convex, fix s, take two couples (z_1,γ_1) and (z_2,γ_2) and choose a number $\vartheta\in\,]0,1[$; now take any two positive numbers λ_1,λ_2, and for the sake of convenience set

$$\frac{1}{K}=\frac{\vartheta}{\lambda_1}+\frac{1-\vartheta}{\lambda_2}.$$

We have

$$\begin{aligned}
&\frac{\vartheta}{\lambda_1}f(s,\lambda_1z_1,\lambda_1\gamma_1)+\frac{1-\vartheta}{\lambda_2}f(s,\lambda_2z_2,\lambda_2\gamma_2)\\
&\quad=\frac{1}{K}\biggl[\frac{\vartheta/\lambda_1}{\vartheta/\lambda_1+(1-\vartheta)/\lambda_2}f(s,\lambda_1z_1,\lambda_1\gamma_1)+\frac{(1-\vartheta)/\lambda_2}{\vartheta/\lambda_1+(1-\vartheta)/\lambda_2}f(s,\lambda_2z_2,\lambda_2\gamma_2)\biggr]\\
&\quad\geqslant\frac{1}{K}f\bigl(s,K\bigl(\vartheta z_1+(1-\vartheta)z_2\bigr),K\bigl(\vartheta\gamma_1+(1-\vartheta)\gamma_2\bigr)\bigr)\\
&\quad\geqslant f_c\bigl(s,\vartheta z_1+(1-\vartheta)z_2,\vartheta\gamma_1+(1-\vartheta)\gamma_2\bigr),
\end{aligned}$$

where we used the convexity of f in the second-last inequality. Taking the infimum with respect to λ_1,λ_2 gives the convexity inequality.

The fact that f_c is 1-homogeneous is obvious from the definition; since f_c is convex and 1-homogeneous it is also subadditive. Convexity and (3.2), (3.3) together with the fact that $f\geqslant 0$ imply (3.7).

To prove continuity we remark that as f_c is the infimum of a family of continuous functions, it is upper semicontinuous, and we have only to prove that it is also lower semicontinuous. Assume

$$s_n\to s,\quad z_n\to z,\quad \gamma_n\to\gamma$$

and set

$$\ell=\liminf_{n\to+\infty}f_c(s_n,z_n,\gamma_n).$$

Upon passing to a subsequence it is not restrictive to assume that indeed

$$\ell=\lim_{n\to+\infty}f_c(s_n,z_n,\gamma_n),$$

and we have to prove that $f_c(s,z,\gamma) \leqslant \ell$, which of course we must do only if $\ell < +\infty$. For every n, by (1.3) we may select λ_n such that

$$\left| f_c(s_n, z_n, \gamma_n) - \frac{1}{\lambda_n} f(s_n, \lambda_n z_n, \lambda_n \gamma_n) \right| < \frac{1}{n},$$

thus

$$\frac{1}{\lambda_n} f(s_n, \lambda_n z_n, \lambda_n \gamma_n) \to \ell.$$

It is not restrictive to assume also $\lambda_n \to \lambda \in [0, +\infty]$; now we have three cases depending on the limit λ.
Case $\lambda = 0$. We remark that by $(\mathrm{H3})_1$

$$\frac{1}{\lambda_n} f(s_n, 0, 0) \leqslant \frac{1}{\lambda_n} f(s_n, \lambda_n z_n, \lambda_n \gamma_n) \to \ell < +\infty,$$

thus $f(s_n, 0, 0) \to 0$, but the continuity of f implies $f(s, 0, 0) = 0$, which in turn by (H1) implies that $s \in \{\alpha, \beta\}$. But then (3.7) clearly implies $0 = f_c(s, z, \gamma) \leqslant \ell$.
Case $0 < \lambda < +\infty$. This is the easiest, because by the continuity of f and (1.3) we get

$$f_c(s,z,\gamma) \leqslant \frac{1}{\lambda} f(s, \lambda z, \lambda \gamma) = \lim_{n\to+\infty} \frac{1}{\lambda_n} f(s_n, \lambda_n z_n, \lambda_n \gamma_n) = \ell.$$

Case $\lambda = +\infty$. Fix $\mu > 0$: by the convexity of f we have as soon as $\lambda_n \geqslant \mu$

$$\frac{1}{\mu}\big(f(s_n, \mu z_n, \mu\gamma_n) - f(s_n, 0, 0)\big) \leqslant \frac{1}{\lambda_n}\big(f(s_n, \lambda_n z_n, \lambda_n \gamma_n) - f(s_n, 0, 0)\big);$$

letting $n \to +\infty$ by the continuity of f we get

$$\frac{1}{\mu} f(s, \mu z, \mu\gamma) \leqslant \frac{1}{\mu} f(s, 0, 0) + \ell$$

and the result follows by (1.3). The remaining statements about f_c follow immediately from continuity and homogeneity.

The proof of the continuity of f_c^M is the same, and there is no need of the last case; as for (3.9), (3.10) we remark that definitions (1.3) and (3.5) immediately imply that the family of functions $M \mapsto f_c^M$ decreases pointwise to f_c as $M \to \infty$.

We now turn to k: as f_c is continuous, the function $(s,z,\gamma) \mapsto \gamma + f_c(s,z,\gamma)$ is uniformly continuous on every set

$$[s_1, s_2] \times (z,\gamma)\colon\ \gamma \geqslant 0,\ |z|^2 + \gamma^2 = 1,$$

which implies continuity with respect to s of its minimum value with respect to the other two variables, which is k by (3.6).

To prove that $k(s)$ is positive if $s \notin \{\alpha, \beta\}$, assume that for some s we have $k(s) = 0$: by the continuity of f_c there exist (z,γ) such that

$$\gamma \geqslant 0, \quad |z|^2 + \gamma^2 = 1, \quad \gamma + f_c(s,z,\gamma) = 0. \tag{3.12}$$

Since $f_c \geqslant 0$, we have immediately $\gamma = 0$, thus (3.12) reads $f_c(s,z,0) = 0$ for some $z \neq 0$, and (3.1) ends the proof. □

For the sake of completeness we recall a selection lemma, see [2] Theorem 3.6 or [7] Theorem 3.1, stated here in the special case we will need:

Lemma 3.3. *Let Λ be a measurable multifunction defined on $[0, R]$ and whose values are nonempty closed subsets of $\mathbb{R}$; if for every $c \geqslant 0$ the set $\{t\colon\ \Lambda(t) \cap [c, +\infty[\neq \emptyset\}$ is measurable, there exists a measurable function $\lambda : [0, R] \to \mathbb{R}$ such that $\lambda(t) \in \Lambda(t)$.*

We now come to a relevant variational interpretation of the function σ introduced in (1.4): define for all $R > 0$

$$\sigma_R(z,\Theta) = \inf\Biggl\{ \int_0^R f\bigl(w(t), zw'(t), \rho(t)\bigr)\,dt\colon\ w(0)=\alpha,\ w(R)=\beta,$$
$$w \in \mathrm{Lip}(0,R),\ w'(t) > 0 \text{ when } w(t) < \beta,\ \int_0^R \rho(t)\,dt \leqslant \Theta \Biggr\} \tag{3.13}$$

(here and elsewhere, measurability is understood whenever an integral appears); then:

Lemma 3.4. *For every (z,Θ) we have $\sigma_R(z,\Theta) \searrow \sigma(z,\Theta)$ when $R \to +\infty$. Moreover the function σ is convex, continuous, 1-homogeneous, subadditive, and it is non-increasing with respect to Θ. Accordingly, there is a constant $C > 0$ such that*

$$\sigma(z,\Theta) \leqslant \sigma(z,0) \leqslant C \quad \text{on } \bigl\{|z| = 1\bigr\}. \tag{3.14}$$

Proof. The fact that $R \mapsto \sigma_R$ is decreasing is obvious. To prove that $\sigma_R \geqslant \sigma$, take any couple (w,ρ) as in (3.13); we first clean away the part where $w = \beta$. Assume

$$\{w < \beta\} = [0, R_0[;$$

then $w : [0, R_0] \to [\alpha, \beta]$ is invertible and since $f \geqslant 0$ we have

$$\int_0^R f\bigl(w(t), zw'(t), \rho(t)\bigr)\,dt \geqslant \int_0^{R_0} f\bigl(w(t), zw'(t), \rho(t)\bigr)\,dt$$
$$= \int_0^{R_0} \frac{1}{w'(t)} f\biggl(w(t), zw'(t), \frac{\rho(t)}{w'(t)} w'(t)\biggr) w'(t)\,dt$$
$$\geqslant \int_0^{R_0} f_c\biggl(w(t), z, \frac{\rho(t)}{w'(t)}\biggr) w'(t)\,dt \tag{3.15}$$

where we used (1.3). We change to the variable $s = w(t) \in [\alpha, \beta]$, so $t = w^{-1}(s)$, and we define an important new function (the density ρ reparametrized by the value of w) by

$$\gamma(s) = \frac{\rho(w^{-1}(s))}{w'(w^{-1}(s))};$$

remark that

$$\int_0^{R_0} f_c\biggl(w(t), z, \frac{\rho(t)}{w'(t)}\biggr) w'(t)\,dt = \int_\alpha^\beta f_c\bigl(s, z, \gamma(s)\bigr)\,ds \tag{3.16}$$

and that

$$\Theta \geqslant \int_0^R \rho(t)\,dt \geqslant \int_0^{R_0} \rho(t)\,dt = \int_\alpha^\beta \gamma(s)\,ds,$$

so (1.4), (3.15), (3.16) give

$$\int_0^R f\bigl(w(t), zw'(t), \rho(t)\bigr)\,dt \geqslant \sigma(z,\Theta)$$

whence

$$\sigma_R \geqslant \sigma. \tag{3.17}$$

Proving that $\sigma = \inf \sigma_R$ requires more care. Set

$$\mathcal{I}(z,\Theta) = \inf_{R>0} \sigma_R(z,\Theta), \tag{3.18}$$

fix any R, w, ρ such that

$$w(0)=\alpha, \quad w(R)=\beta, \quad w \in \mathrm{Lip}, \quad w' > 0 \text{ on }]0,R[, \quad \int_0^R \rho(t)\,dt \leqslant \Theta \tag{3.19}$$

and, for the time being, also

$$\rho \text{ bounded}, \tag{3.20}$$

and take any $R_0 > 0$ and any 1–1 Lipschitz function $\varphi : [0,R_0] \to [0,R]$ with Lipschitz inverse and $\varphi' > 0$. We define on $[0,R_0]$ two functions w_0, ρ_0 by

$$w_0 = w \circ \varphi, \qquad \rho_0 = (\rho \circ \varphi)\varphi';$$

then

$$w_0(0)=\alpha, \quad w_0(R_0)=\beta, \quad w_0' > 0 \quad \text{on }]0,R_0[$$

and

$$\int_0^{R_0} \rho_0(s)\,ds = \int_0^{R_0} \rho\big(\varphi(s)\big)\varphi'(s)\,ds = \int_0^R \rho(t)\,dt \leqslant \Theta,$$

therefore by (3.13)

$$\begin{aligned}\mathcal{I}(z,\Theta) \leqslant \sigma_{R_0}(z,\Theta) &\leqslant \int_0^{R_0} f\big(w_0(s), z w_0'(s), \rho_0(s)\big)\,ds \\ &= \int_0^R \frac{1}{\varphi'(\varphi^{-1}(t))} f\big(w(t), z w'(t)\varphi'\big(\varphi^{-1}(t)\big), \rho(t)\varphi'\big(\varphi^{-1}(t)\big)\big)\,dt.\end{aligned}$$

If we set

$$\lambda(t) = \varphi'\big(\varphi^{-1}(t)\big) = 1/(\varphi^{-1})'(t)$$

the choices made up to this point impose on λ only the following restrictions:

$$\lambda \text{ measurable on } [0,R], \qquad M^{-1} \leqslant \lambda \leqslant M \quad \text{for some } M \tag{3.21}$$

(the bounds are due to Lipschitz continuity of φ and φ^{-1}, and there is no restriction concerning the integral of λ because R_0 may still be any positive number). We have proved that

$$\mathcal{I}(z,\Theta) \leqslant \int_0^R \frac{1}{\lambda(t)} f\big(w(t), z w'(t)\lambda(t), \rho(t)\lambda(t)\big)\,dt \tag{3.22}$$

for any (R, w, ρ) satisfying (3.19), (3.20) and any λ satisfying (3.21).

Fix $M > 1$ and choose $\varepsilon > 0$: we check that Lemma 3.3 may be applied to

$$\Lambda(t) = \left\{\lambda \in [M^{-1}, M]:\ \frac{1}{\lambda} f\big(w(t), z w'(t)\lambda, \rho(t)\lambda\big) \leqslant f_c^M\big(w(t), z w'(t), \rho(t)\big) + \frac{\varepsilon}{R}\right\}.$$

Indeed closedness is easy, non-emptyness follows from the definition of f_c^M, and we must check that

$$S_c = \{t\colon \exists\lambda \geqslant c\colon \lambda \in \Lambda(t)\}$$

is measurable if $c > 0$. Assume for a while that $w \in \mathcal{C}^1$ and $\rho \in \mathcal{C}^0$: by Lemma 3.2 the function

$$(t,\lambda) \mapsto \frac{1}{\lambda} f\big(w(t), zw'(t)\lambda, \rho(t)\lambda\big) - f_c^M\big(w(t), zw'(t), \rho(t)\big)$$

is continuous on $[0,R] \times [c,M]$, so S_c is compact; the general case follows by approximation.

By Lemma 3.3 there exists a measurable function λ_M defined on $[0,R]$ such that

$$M^{-1} \leqslant \lambda_M(t) \leqslant M$$

and for a.e. t

$$\frac{1}{\lambda_M(t)} f\big(w(t), zw'(t)\lambda_M(t), \rho(t)\lambda_M(t)\big) \leqslant f_c^M\big(w(t), zw'(t), \rho(t)\big) + \frac{\varepsilon}{R}.$$

Picking exactly this function λ_M in (3.22) we deduce

$$\mathcal{I}(z,\Theta) \leqslant \varepsilon + \int_0^R f_c^M\big(w(t), zw'(t), \rho(t)\big)\,dt;$$

since ε was arbitrary we may drop it to get

$$\mathcal{I}(z,\Theta) \leqslant \int_0^R f_c^M\big(w(t), zw'(t), \rho(t)\big)\,dt. \tag{3.23}$$

Now, the argument of f_c^M lies in a compact subset of $[\alpha,\beta] \times \mathbb{R}^{n+1}$ by (3.19), (3.20), so by the continuity of f and (3.9), (3.10) we may apply the dominated convergence theorem in (3.23) and get as $M \to +\infty$

$$\mathcal{I}(z,\Theta) \leqslant \int_0^R f_c\big(w(t), zw'(t), \rho(t)\big)\,dt. \tag{3.24}$$

Condition (3.20) may now be lifted by approximation, using (3.8): we may thus in any case write

$$\mathcal{I}(z,\Theta) \leqslant \int_0^R f_c\left(w(t), z, \frac{\rho(t)}{w'(t)}\right) w'(t)\,dt \overset{[w(t)=s]}{=} \int_\alpha^\beta f_c\big(s, z, \gamma(s)\big)\,ds, \tag{3.25}$$

where we put

$$\gamma(s) = \frac{\rho(w^{-1}(s))}{w'(w^{-1}(s))}.$$

Since w and ρ need only satisfy the mild conditions (3.19), the function γ need only satisfy

$$\gamma \geqslant 0, \qquad \int_\alpha^\beta \gamma(s)\,ds \leqslant \Theta,$$

so by (1.4), (3.25) we get

$$\mathcal{I}(z,\Theta) \leqslant \sigma(z,\Theta),$$

which together with (3.17), (3.18) concludes the proof.

As for the remaining statements, monotonicity, convexity and 1-homogeneity are directly verified from the definition, and subadditivity follows from the latter two. Since convexity implies continuity at all interior points of the domain, and upper semicontinuity at the boundary, we only need to prove lower semicontinuity at $(z,0)$: take

$(z_n, \Theta_n) \to (z, 0)$ and, assuming without loss of generality that $\sigma(z_n, \Theta_n)$ has a limit, by (1.4) let $\gamma_n \in L^1([\alpha, \beta])$ be such that

$$\lim_{n\to+\infty} \int_\alpha^\beta f_c\big(s, z_n, \gamma_n(s)\big)\, ds = \lim_{n\to+\infty} \sigma(z_n, \Theta_n), \qquad \int_\alpha^\beta \gamma_n(s)\, ds \leqslant \Theta_n.$$

Since $\Theta_n \to 0$ the sequence γ_n converges to zero in L^1, but f_c is continuous by Lemma 3.2 and it is positive, so Fatou's lemma implies

$$\sigma(z, 0) = \int_\alpha^\beta f_c(s, z, 0)\, ds \leqslant \lim_{n\to+\infty} \int_\alpha^\beta f_c\big(s, z_n, \gamma_n(s)\big)\, ds = \lim_{n\to+\infty} \sigma(z_n, \Theta_n).$$

The bound (3.14) follows immediately from continuity. □

Two remarks are in order: first, the lemma we just finished shows that to approach the value $\sigma(z, \Theta)$ one must carefully tune the distribution $\rho(s)$ of the amount Θ of surfactant with the transition profile $w(s)$, so the two are coupled. Second, it appears that the amount of surfactant needed to reach the transition cost $\sigma(z, \Theta)$ can be lower than the given maximum Θ. In fact it is not worth to take care of the unused portion as is shown in the following lemma, where the inequality constraint $\int_0^R \rho(t)\, dt \leqslant \Theta$ is substituted with an equality.

Lemma 3.5. *Let $\tilde\sigma_R$ be defined by*

$$\tilde\sigma_R(z, \Theta) = \inf\Bigg\{ \int_0^R f\big(w(t), zw'(t), \rho(t)\big)\, dt\colon\ w(0) = \alpha,\ w(R) = \beta,$$
$$w \in \mathrm{Lip}(0, R),\ w'(t) > 0 \text{ when } w(t) < \beta,\ \int_0^R w(t)\, dt = \Theta \Bigg\}, \tag{3.26}$$

then $\sigma(z, \Theta) = \liminf_{R\to\infty} \tilde\sigma_R(z, \Theta)$.

Proof. Since $\tilde\sigma_R \geqslant \sigma_R$, by Lemma 3.4 we already have $\sigma \leqslant \liminf_{R\to\infty} \tilde\sigma_R$. Fix z, Θ and $R > 0$ and take $\varepsilon > 0$: we will find a number $R' \geqslant R$ such that $\tilde\sigma_{R'}(z, \Theta) \leqslant \sigma_R(z, \Theta) + \varepsilon$, which will conclude the proof. Let w and ρ be as in (3.13) and such that

$$\int_0^R f\big(w(t), zw'(t), \rho(t)\big)\, dt < \sigma(z, \Theta) + \frac{\varepsilon}{2}$$

and assume

$$\int_0^R \rho(t)\, dt = \Theta - C < \Theta;$$

we will extend w and ρ beyond R by setting

$$w(t) = \beta, \quad \rho(t) = \frac{C}{M} \quad \text{on } [R, R + M]$$

for some M to be determined, then we define $R' = R + M$. Indeed by $(\mathrm{H3})_4$ we have

$$f(\beta, 0, \rho) \leqslant \rho\, \eta(\rho) \tag{3.27}$$

for some nondecreasing function η such that $\eta(\rho) \to 0$ as $\rho \to 0$, and

$$\int_R^{R'} f\big(w(t), zw'(t), \rho(t)\big)\, dt = \int_R^{R+M} f(\beta, 0, C/M)\, dt \leqslant C\eta\left(\frac{C}{M}\right),$$

and if M is so large that $C\eta(C/M) < \varepsilon/2$, we have $\int_0^{R'} \rho(t)\,dt = \Theta$ whereas

$$\tilde{\sigma}_{R'}(z,\Theta) \leqslant \int_0^{R'} f\big(w(t), zw'(t), \rho(t)\big)\,dt < \int_0^{R} f\big(w(t), zw'(t), \rho(t)\big)\,dt + \frac{\varepsilon}{2} < \sigma_R(z,\Theta) + \varepsilon. \qquad \square$$

3.2. Convex functionals on measures

Let X a locally compact metric space and let $\varphi(x,p)$ be a Borel function on $X \times \mathbb{R}^d$ to $\mathbb{R}_+$ which is positively 1-homogeneous in p. Then we consider the functional on the space $\mathcal{M}(X;\mathbb{R}^d)$ of R^d-valued vector measures on X defined by

$$\Phi : \lambda \in \mathcal{M}(X;\mathbb{R}^d) \to \int_X \varphi\left(x, \frac{d\lambda}{d\theta}\right) d\theta,$$

where θ is a positive Radon measure on X such that $\lambda \ll \theta$. By the homogeneity of φ, is easy to check that the integral above does not depend on the choice of θ and therefore we may rewrite functional Φ in a more intrinsic way as

$$\int_X \varphi(x,\lambda) = \int_X \varphi\left(x, \frac{d\lambda}{d\theta}\right) d\theta \quad \text{for every } \theta \text{ such that } \lambda \ll \theta. \tag{3.28}$$

Furthermore, the following additivity property holds

$$\int_X \varphi(x, \lambda_1 + \lambda_2) = \int_X \varphi(x,\lambda_1) + \int_X \varphi(x,\lambda_2) \quad \text{whenever } \lambda_1 \perp \lambda_2$$

(where $\lambda_1 \perp \lambda_2$ means that λ_1 and λ_2 are mutually singular).

Now we turn to the continuity (resp. lower semicontinuity) properties of functional Φ. We will say that a sequence $\{\lambda_n\}$ converges weakly to λ (denoted $\lambda_n \overset{*}{\rightharpoonup} \lambda$) in $\mathcal{M}(X;\mathbb{R}^d)$ if we have $\int_X u\,d\lambda_n \to \int_X u\,d\lambda_n$ for every continuous test function compactly supported in X. A straightforward variant of an important result due to Reshetnyak is the following

Theorem 3.6.

(i) (Lower semicontinuity) *Assume that $\varphi(x,\cdot)$ is convex, nonnegative and that φ is l.s.c. on $X \times \mathbb{R}^d$. Then*

$$\lambda_n \overset{*}{\rightharpoonup} \lambda \quad \Longrightarrow \quad \liminf_n \int_X \varphi(x,\lambda_n) \geqslant \int_X \varphi(x,\lambda).$$

(ii) (Continuity) *Assume that X is compact and that φ is continuous on $X \times \mathbb{R}^d$. Let $\{\lambda_n\}$ a sequence in $\mathcal{M}(X;\mathbb{R}^d)$ such that*

$$\lambda_n \overset{*}{\rightharpoonup} \lambda, \quad \int_X \varphi_0(\lambda_n) \to \int_X \varphi_0(\lambda),$$

where $\varphi_0 : \mathbb{R}^d \to \mathbb{R}_+$ is a suitable strictly convex positively 1*-homogeneous function* (*i.e.* $\varphi_0(z_1+z_2) < \varphi_0(z_1) + \varphi_0(z_2)$ *whenever* $|z_1| = |z_2| = 1$ *and* $z_1 \neq |z_2|$). *Then we have*

$$\lim_n \int_X \varphi(x,\lambda_n) = \int_X \varphi(x,\lambda).$$

Assertion (i) of this theorem will be used in Section 4 with $X = \Omega \times]\alpha,\beta[$ and $d = n+1$, whereas assertion (ii) will be used with $X = \overline{\Omega}$ and $d = n$.

3.3. Approximation properties for the limit energy

In order to simplify the construction of recovery sequences for the limit energy $F(u,\rho)$, it will be very useful to reduce to the case where (u,ρ) is suitably regular: this will be possible using Proposition 3.10 below. Before that, we need some lemmas (two Lipschitz approximation results for measures and an approximation result for sets of finite perimeter) and we introduce a distance on the set of Radon measures.

Lemma 3.7. *Let Ω be a bounded open set and let $\mu \in \mathcal{M}_+(\overline{\Omega})$ be a non-negative Radon measure such that $\mu(\partial\Omega) = 0$. If $\theta \in L^1_\mu$, there exists a sequence θ_h of Lipschitz functions defined on $\mathbb{R}^n$ and with compact support in Ω such that $\int |\theta - \theta_h|\,d\mu \to 0$. Moreover one may take $\int \theta_h\,d\mu = \int \theta\,d\mu$.*

Proof. By Hahn–Banach Theorem, proving that the subspace V of Lipschitz functions compactly supported in Ω is dense in L^1_μ is equivalent to showing that an element $w \in L^\infty_\mu$ (dual space) vanishes whenever it satisfies $\int w\,\theta\,d\mu = 0$ for all $\varphi \in V$. Let w be such a function and let $B \Subset \Omega$ be a closed ball. Then there exists a sequence of smooth functions $\varphi_h \in V$ ranging in $[0,1]$ such that $\varphi_h \to \mathbb{1}_B$ and whose support lies in a slightly bigger ball $B' \Subset \Omega$. By using dominated convergence, we obtain

$$\int_B w\,d\mu = \lim_h \int \varphi_h w\,d\mu = 0$$

yielding that $w = 0$ at every Lebesgue's point of w in Ω, that is μ-a.e. over $\mathbb{R}^n$ since $\mu(\partial\Omega) = 0$. So we have proved that there exists a sequence $\tilde{\theta}_h$ in V such that $\tilde{\theta}_h \to \theta$ in L^1_μ. In particular $r_h := \int \tilde{\theta}_h\,d\mu$ converges to $r := \int \theta\,d\mu$. Pick an element $\varphi_0 \in V$ such that $\int \varphi_0\,d\mu = 1$. Then $\theta_h := \tilde{\theta}_h + (r - r_h)\varphi_0$ satisfies all requirements. □

Lemma 3.8. *Let Ω be a bounded open set and let $\omega \in \mathcal{M}_+(\overline{\Omega})$ be a non-negative Radon measure. There exists a sequence ω_h of non-negative Lipschitz functions defined on $\mathbb{R}^n$ and with compact support in Ω such that $\omega_h\,d\mathcal{L}^n \rightharpoonup \omega$ weakly in the sense of measures. Moreover one may take $\int_\Omega \omega_h\,dx = \omega(\overline{\Omega})$.*

Proof. We prove that the weak closure of the set of Radon measures of the form $f\,d\mathcal{L}^n$ with f Lipschitz, non-negative and compactly supported in Ω is the space of all non-negative Radon measures in $\overline{\Omega}$: this density result clearly follows from the fact that if $\phi \in C^0(\overline{\Omega})$ satisfies $\int \phi f\,dx = 0$ for all such f then $\phi = 0$. The last condition is dealt with as in Lemma 3.7. □

The following is a variant of an approximation result for sets with finite perimeter to be found in [11] (see also [1]).

Lemma 3.9. *Let A be a set of finite perimeter in Ω. Denote by ∂A the essential boundary of A and by ν_A its generalized outward normal. Then there exists a sequence $\{A_h\}$ of bounded subsets of $\mathbb{R}^n$ with C^2 boundary satisfying*

$$|A_h \cap \Omega| = |A|, \quad \left|(A_h \cap \Omega)\Delta A\right| \to 0, \quad \mathcal{H}^{n-1}(\partial A_h \cap \partial\Omega) = 0$$

and such that for every non-negative convex continuous function σ on $\mathbb{R}^n \times [0,+\infty[$ and every non-negative Lipschitz function ρ on $\mathbb{R}^n$

$$\lim_h \int_{\Omega\cap\partial A_h} \sigma\left(\nu_{A_h}, \rho(x)\right) d\mathcal{H}^{n-1} = \int_{\partial A} \sigma\left(\nu_A, \rho(x)\right) d\mathcal{H}^{n-1}.$$

Proof. For the construction of A_h we refer to Lemma 4.3 in [1] where it is noticed that the vector measure $\lambda_h := D\mathbb{1}_{A_h}$ converges tightly to $\lambda := D\mathbb{1}_A$, that is

$$\mathcal{H}^{n-1}(\partial A_h \cap \Omega) = \int_{\overline{\Omega}} |\lambda_h| \to \int_{\overline{\Omega}} |\lambda| = \mathcal{H}^{n-1}(\partial A \cap \Omega). \tag{3.29}$$

The last assertion follows by using property (ii) in Theorem 3.6 with $\varphi_0(z) = |z|$ and $\varphi(x,z) = \sigma(z,\rho(x))$. □

We introduce a distance on the set of positive Radon measures $\mathcal{M}_+(\overline{\Omega})$ by

$$\mathrm{d}^*(\mu,\nu)=\sum_{n=1}^{\infty}2^{-n}\left|\langle\mu-\nu,\phi_n\rangle\right|,\tag{3.30}$$

where $\{\phi_n\}_n$ is a dense subset of $C^0(\overline{\Omega};[0,1])$. It is clear that this distance satisfies

$$\mathrm{d}^*(t\mu_1,t\mu_2)=t\,\mathrm{d}^*(\mu_1,\mu_2)\quad\forall t\geqslant 0,\tag{3.31}$$

$$\mathrm{d}^*(\mu_1+\mu_2,\mu_3+\mu_4)\leqslant\mathrm{d}^*(\mu_1,\mu_3)+\mathrm{d}^*(\mu_2,\mu_4).\tag{3.32}$$

Also, the topology induced by this distance on the space of positive Radon measures with total mass not exceeding a given value is equivalent to weak convergence, so in particular on the set $\{\mu\in\mathcal{M}_+(\overline{\Omega}):\ \mu(\overline{\Omega})\leqslant m_1\}$ we have

$$\mu_h\rightharpoonup\mu\quad\Longleftrightarrow\quad\mathrm{d}^*(\mu_h,\mu)\to 0.$$

We may state and prove the main result of this subsection.

Proposition 3.10. *Let $u\in BV(\Omega;\{\alpha,\beta\})$, with $u=\alpha\mathbb{1}_A+\beta\mathbb{1}_{\Omega\setminus A}$ for some $A\subset\Omega$, and let $\rho\in\mathcal{M}_+(\overline{\Omega})$ be a non-negative Radon measure. There exist a sequence $\{A_h\}$ of bounded subsets of $\mathbb{R}^n$ and two sequences θ_h,ω_h of non-negative Lipschitz functions compactly supported in Ω, such that, setting*

$$u_h:=\alpha\mathbb{1}_{A_h\cap\Omega}+\beta\mathbb{1}_{\Omega\setminus A_h},\qquad\rho_h:=\theta_h\mathcal{H}^{n-1}\llcorner\partial A_h+\omega_h\mathcal{L}^n\llcorner\Omega$$

the following properties hold:

(i) $\partial A_h\in C^2$, $\mathcal{H}^{n-1}(\partial A_h\cap\partial\Omega)=0$, $\int_\Omega|u_h-u|\,dx\to 0$, $\int_\Omega u_h\,dx=\int_\Omega u\,dx$;
(ii) $\rho_h\rightharpoonup\rho$ *and* $\rho_h(\overline{\Omega})=\rho(\overline{\Omega})$ *for every* h;
(iii) $\lim_{h\to+\infty}F(u_h,\rho_h)=F(u,\rho)$.

Proof. We first remark that if θ,ω are Lipschitz and $v\in BV(\Omega;\{\alpha,\beta\})$ then setting $\bar\rho=\theta\mathcal{H}^{n-1}\llcorner S_v+\omega\mathcal{L}^n\llcorner\Omega$ we have by (1.5), (1.6)

$$F(v,\bar\rho)=\int_{S_v}\sigma\big(\nu_v,\theta(x)\big)\,d\mathcal{H}^{n-1}.$$

Setting as in (1.6)

$$\rho^0=\frac{d\rho}{dJu},$$

what we have to show, in order to prove all three properties, is that

$$\forall h\in\mathbb{N},\ \exists A_h,\tilde\theta_h,\omega_h:\quad\|u_h-u\|_1+\mathrm{d}^*(\rho_h,\rho)<\frac{C}{h},\tag{3.33}$$

$$\int_{S_{u_h}}\sigma\big(\nu_{u_h},\tilde\theta_h(x)\big)\,d\mathcal{H}^{n-1}<\int_{S_u}\sigma\big(\nu_u,\rho^0(x)\big)\,d\mathcal{H}^{n-1}+\frac{C}{h}\tag{3.34}$$

for some constant C, with $A_h,\tilde\theta_h,\omega_h$ as prescribed and the condition on $\int u_h$ and $\rho_h(\overline{\Omega})$ satisfied.

Fix h. We begin by applying Lemma 3.7 to

$$\mu:=Ju=\mathcal{H}^{n-1}\llcorner S_u,\qquad\theta:=\rho^0=\frac{d\rho}{d\mu}$$

(recall that $S_u\subset\Omega$) and Lemma 3.8 to

$$\omega=\rho-\theta\mathcal{H}^{n-1}\llcorner S_u$$

to obtain two sequences θ_j, ω_j. By the continuity of σ and the convergence of θ_j to ρ^0 we have, using also (3.14),

$$\int_{S_u} \left|\sigma(\nu_u, \theta_j) - \sigma(\nu_u, \rho^0)\right| d\mathcal{H}^{n-1} \to 0$$

as $j \to +\infty$. By the convergence results in the two lemmas, we may select an index (which we label h) such that

$$\int_{S_u} \left|\sigma(\nu_u, \theta_h) - \sigma(\nu_u, \rho^0)\right| d\mathcal{H}^{n-1} < \frac{1}{h}, \qquad \mathrm{d}^*(\omega_h + \theta_h J u, \rho) < \frac{1}{h}. \tag{3.35}$$

Now we apply Lemma 3.9 to get a sequence A_j and define u_j accordingly. We remark that by the convergence of $D\mathbb{1}_{A_j}$ to $D\mathbb{1}_A$ we have from Theorem 3.6 that for every continuous function ϕ

$$\int_{S_{u_j}} \phi\theta_h \, d\mathcal{H}^{n-1} \to \int_{S_u} \phi\theta_h \, d\mathcal{H}^{n-1}$$

and in particular

$$p_j := \int_{S_{u_j}} \theta_h \, d\mathcal{H}^{n-1} \to p := \int_{S_u} \theta_h \, d\mathcal{H}^{n-1} = \int_{S_u} \rho^0 \, d\mathcal{H}^{n-1}.$$

These properties together say that

$$\frac{p}{p_j}\theta_h \mathcal{H}^{n-1} \llcorner S_{u_j} \rightharpoonup \theta_h \mathcal{H}^{n-1} \llcorner S_u, \qquad \int_{S_{u_j}} \frac{p}{p_j}\theta_h \, d\mathcal{H}^{n-1} = \int_{S_u} \rho^0 \, d\mathcal{H}^{n-1}.$$

Since θ_h is bounded, by the continuity of σ we deduce

$$\sup_{|z|=1,\ x\in\mathbb{R}^n} \left|\sigma\left(z, \frac{p}{p_j}\theta_h(x)\right) - \sigma\big(z, \theta_h(x)\big)\right| \to 0$$

as $j \to +\infty$, so that by (3.29)

$$\int_{S_{u_j}} \left|\sigma\left(\nu_{u_j}, \frac{p}{p_j}\theta_h\right) - \sigma(\nu_{u_j}, \theta_h)\right| d\mathcal{H}^{n-1} \to 0.$$

Finally, Lemma 3.9 implies that

$$\int_{S_{u_j}} \sigma(\nu_{u_j}, \theta_h) \, d\mathcal{H}^{n-1} \to \int_{S_u} \sigma(\nu_u, \theta_h) \, d\mathcal{H}^{n-1},$$

so that by the previous formula

$$\int_{S_{u_j}} \sigma\left(\nu_{u_j}, \frac{p}{p_j}\theta_h\right) d\mathcal{H}^{n-1} \to \int_{S_u} \sigma(\nu_u, \theta_h) \, d\mathcal{H}^{n-1}$$

as $j \to +\infty$. We may therefore select an index j (which we relabel h) such that defining

$$\tilde\theta_h = \frac{p}{p_h}\theta_h, \qquad \rho_h = \tilde\theta_h \mathcal{H}^{n-1} \llcorner S_{u_h} + \omega_h \mathcal{L}^n \llcorner \Omega$$

and using also (3.35) all conditions in (3.33), (3.34) are satisfied. □

4. Main result

We will use two sets of mass constraints:

$$\int_\Omega u_\varepsilon\,dx = m_0, \qquad \int_\Omega \rho_\varepsilon\,dx = m_1 \tag{4.1}$$

and

$$\int_\Omega u\,dx = m_0, \qquad \rho(\overline{\Omega}) = m_1. \tag{4.2}$$

Theorem 4.1. *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary, and let f satisfy* (H1), …, (H4). *If F_ε, F are defined as in* (1.2), (1.5) *respectively, then*:

(i) (compactness) *if* $\{(u_\varepsilon,\rho_\varepsilon)\}_\varepsilon \subset W^{1,1}(\Omega)\times L^1_+(\Omega)$ *satisfies* (4.1) *and*

$$\sup F_\varepsilon(u_\varepsilon,\rho_\varepsilon) < +\infty$$

then there exists $(u,\rho)\in BV(\Omega;\{\alpha,\beta\})\times\mathcal{M}_+(\overline{\Omega})$ *satisfying* (4.2) *such that at least for a subsequence*

$$u_\varepsilon \to u \quad \textit{strongly in } L^1,$$
$$\rho_\varepsilon \rightharpoonup \rho \quad \textit{weakly in the sense of Radon measures};$$

(ii) (lower bound) *if* $\{(u_\varepsilon,\rho_\varepsilon)\}_\varepsilon \subset W^{1,1}(\Omega)\times L^1_+(\Omega)$ *satisfies* (4.1) *and*

$$u_\varepsilon \to u, \qquad \rho_\varepsilon \rightharpoonup \rho$$

for some $(u,\rho)\in BV(\Omega;\{\alpha,\beta\})\times\mathcal{M}_+(\overline{\Omega})$ *then* (u,ρ) *satisfies* (4.2) *and*

$$F(u,\rho) \leqslant \liminf_{\varepsilon\to0} F_\varepsilon(u_\varepsilon,\rho_\varepsilon);$$

(iii) (upper bound) *if* $(u,\rho)\in BV(\Omega;\{\alpha,\beta\})\times\mathcal{M}_+(\overline{\Omega})$ *satisfies* (4.2) *there exists* $\{(u_\varepsilon,\rho_\varepsilon)\}_\varepsilon \subset W^{1,1}(\Omega)\times L^1_+(\Omega)$ *satisfying* (4.1) *such that*

$$u_\varepsilon \to u, \quad \rho_\varepsilon \rightharpoonup \rho, \quad \limsup_{\varepsilon\to0} F_\varepsilon(u_\varepsilon,\rho_\varepsilon) \leqslant F(u,\rho). \tag{4.3}$$

Conditions (ii) and (iii) express the fact that the sequence F_ε is Γ-converging to F in the space $BV\times\mathcal{M}_+$ endowed with the strong L^1 topology times the weak topology of measures. The remainder of this section is devoted to the proof of the three parts of the theorem.

4.1. Compactness

We truncate the function k defined in (3.6) by setting

$$\tilde{k}(s) = \min\{1, k(s)\},$$

so that

$$0 \leqslant \tilde{k}(s) \leqslant 1, \qquad \tilde{k}(s) = 0 \Leftrightarrow s \in \{\alpha,\beta\} \tag{4.4}$$

by (3.11); we also define

$$K(s) = \int_0^s \tilde{k}(t)\,dt,$$

thus K is a $\mathcal{C}^1$ function, strictly increasing and satisfying

$$\bigl|K(s)\bigr| \leqslant |s|. \tag{4.5}$$

Assume

$$F_\varepsilon(u_\varepsilon,\rho_\varepsilon)\leqslant C; \tag{4.6}$$

since $\int\rho_\varepsilon\,dx=m_1$, we have (writing C in place of $C+m_1$: throughout the proof we will denote by the same letter C any harmless constant)

$$\begin{aligned}C&\geqslant\int_\Omega \rho_\varepsilon+\frac{1}{\varepsilon}f(u_\varepsilon,\varepsilon\nabla u_\varepsilon,\varepsilon\rho_\varepsilon)\,dx\geqslant\int_\Omega \rho_\varepsilon+f_c(u_\varepsilon,\nabla u_\varepsilon,\rho_\varepsilon)\,dx\\&\geqslant\int_\Omega\sqrt{|\nabla u_\varepsilon|^2+\rho_\varepsilon^2}k(u_\varepsilon)\,dx\geqslant\int_\Omega k(u_\varepsilon)|\nabla u_\varepsilon|\,dx\\&\geqslant\int_\Omega\tilde{k}(u_\varepsilon)|\nabla u_\varepsilon|\,dx=\int_\Omega\big|\nabla\big(K(u_\varepsilon)\big)\big|\,dx,\end{aligned}$$

thus

$$\int_\Omega k(u_\varepsilon)|\nabla u_\varepsilon|\,dx\leqslant C,\qquad \big\|\nabla\big(K(u_\varepsilon)\big)\big\|_{L^1}\leqslant C. \tag{4.7}$$

By (4.5), (4.6) and by using the definitions of f_c and F_ε we deduce

$$\int_\Omega\big|K(u_\varepsilon)\big|\,dx\leqslant\int_\Omega|u_\varepsilon|\,dx\leqslant CC'+C\int_\Omega f(u_\varepsilon,\varepsilon\nabla u_\varepsilon,\varepsilon\rho_\varepsilon)\,dx\leqslant CC'+C\varepsilon F_\varepsilon(u_\varepsilon,\rho_\varepsilon)\leqslant C,$$

whence

$$\big\|K(u_\varepsilon)\big\|_{L^1}\leqslant C. \tag{4.8}$$

We may also write by (4.6)

$$\int_\Omega\frac{1}{\varepsilon}f(u_\varepsilon,0,0)\,dx\leqslant F_\varepsilon(u_\varepsilon,\rho_\varepsilon)\leqslant C$$

so that

$$\int_\Omega f(u_\varepsilon,0,0)\,dx\to 0. \tag{4.9}$$

We begin to switch back and forth between u_ε and $K(u_\varepsilon)$: by (4.7), (4.8) we deduce that the sequence $K(u_\varepsilon)$ is bounded in $W^{1,1}(\Omega)$, thus strongly compact in BV; in particular, up to a subsequence, there exists a function ψ such that

$$\psi\in BV(\Omega),\quad K(u_\varepsilon)\to\psi\quad\text{weakly in } BV \text{ and strongly in } L^1.$$

We deduce also from this that

$$K(u_\varepsilon)\to\psi\quad\text{a.e. in }\Omega.$$

Since K is 1–1 by (4.4) we have that

$$u_\varepsilon=K^{-1}\big(K(u_\varepsilon)\big)\to u:=K^{-1}(\psi)\quad\text{a.e. in }\Omega. \tag{4.10}$$

By (H4) we have

$$|u_\varepsilon|\leqslant CC'+Cf(u_\varepsilon,0,0)\,dx$$

but (4.9) implies that the non-negative sequence $f(u_\varepsilon,0,0)$ converges in L^1, thus the previous two formulas imply that

$$u_\varepsilon\to u\quad\text{strongly in } L^1(\Omega).$$

Now the continuity of f implies that

$$f(u_\varepsilon,0,0)\to f(u,0,0)\quad \text{a.e.}$$

but since $f(u_\varepsilon,0,0)\to 0$ we obtain

$$f\big(u(x),0,0\big)=0\quad \text{a.e.},$$

which by (H1) gives $u(x)\in\{\alpha,\beta\}$ a.e.; call A the set where $u(x)=\alpha$: then $\psi(x)=K(\alpha)$ on A and $\psi(x)=K(\beta)$ otherwise, so

$$\psi=K(\alpha)\mathbb{1}_A+K(\beta)\mathbb{1}_{\Omega\setminus A}$$

but as $\psi\in BV$ the set A is (equivalent to one) of finite perimeter in Ω and thus

$$u=\alpha\mathbb{1}_A+\beta\mathbb{1}_{\Omega\setminus A}\in BV(\Omega).$$

As for the convergence of ρ_ε we have that (4.1) implies that for a subsequence $\rho_\varepsilon\rightharpoonup\rho$, but $\rho(\overline{\Omega})\geqslant m_1$ because $\overline{\Omega}$ is compact, and $\rho(\overline{\Omega})\leqslant\rho(\mathbb{R}^n)\leqslant m_1$ because $\mathbb{R}^n$ is open.

4.2. Lower bound inequality

Assume $u=\alpha\mathbb{1}_A+\beta\mathbb{1}_{\Omega\setminus A}$ is a BV function and let $\{(u_\varepsilon,\rho_\varepsilon)\}_\varepsilon$ be as in the statement; we may assume

$$\liminf_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon,\rho_\varepsilon)=\lim_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon,\rho_\varepsilon)<+\infty.$$

We first get rid of the areas where $u_\varepsilon\notin[\alpha,\beta]$: indeed, if we consider the truncation operator $T(t)=(t\vee\alpha)\wedge\beta$ we have

$$T(u_\varepsilon)\rightharpoonup T(u)=u\quad \text{weakly in } BV$$

and

$$F_\varepsilon(u_\varepsilon,\rho_\varepsilon)=\int_\Omega\frac{1}{\varepsilon}f(u_\varepsilon,\varepsilon\nabla u_\varepsilon,\varepsilon\rho_\varepsilon)\,dx\overset{(1.3)}{\geqslant}\int_\Omega f_c(u_\varepsilon,\nabla u_\varepsilon,\rho_\varepsilon)\,dx\geqslant\int_\Omega f_c\big(T(u_\varepsilon),\nabla T(u_\varepsilon),\rho_\varepsilon\big)\,dx$$

because $f_c(u_\varepsilon,\cdot,\cdot)\geqslant 0$ whereas $f_c(T(u_\varepsilon),\nabla T(u_\varepsilon),\cdot)=0$ where $u_\varepsilon\neq T(u_\varepsilon)$ due to (3.7). To avoid the weight of the notation, we may thus with no loss of generality assume that

$$\alpha\leqslant u_\varepsilon\leqslant\beta$$

and prove that

$$\liminf_{\varepsilon\to 0}\int_\Omega f_c(u_\varepsilon,\nabla u_\varepsilon,\rho_\varepsilon)\,dx\geqslant F(u,\rho).\tag{4.11}$$

We want to rewrite the integral at the left-hand side in a way that will let us apply Reshetnyak's theorem 3.6, so we transform it into an integral (in one more dimension) on the graph of u_ε. We begin by remarking that there is no contribution to the integral from the parts where $u_\varepsilon=\alpha$ or $u_\varepsilon=\beta$, because $\nabla u_\varepsilon=0$ a.e. on these sets and due to (3.7). Now we associate with each $(u_\varepsilon,\rho_\varepsilon)$ three bounded Radon measures which we define on $\Omega\times{]\alpha,\beta[}$ as

$$\begin{aligned}
\langle\zeta_\varepsilon,\varphi\rangle&=\int_\Omega\varphi(x,u_\varepsilon)\nabla u_\varepsilon\,dx=\int_{\Omega\times]\alpha,\beta[}\varphi(x,t)\frac{\nabla u_\varepsilon}{\sqrt{1+|\nabla u_\varepsilon|^2}}\,d\mathcal{H}^n\llcorner G_{u_\varepsilon},\\
\langle m_\varepsilon,\varphi\rangle&=\int_\Omega\varphi(x,u_\varepsilon)\rho_\varepsilon\,dx=\int_{\Omega\times]\alpha,\beta[}\varphi(x,t)\frac{\rho_\varepsilon}{\sqrt{1+|\nabla u_\varepsilon|^2}}\,d\mathcal{H}^n\llcorner G_{u_\varepsilon},\\
\langle\mu_\varepsilon,\varphi\rangle&=\int_\Omega\varphi(x,u_\varepsilon)\,dx=\int_{\Omega\times]\alpha,\beta[}\frac{\varphi(x,t)}{\sqrt{1+|\nabla u_\varepsilon|^2}}\,d\mathcal{H}^n\llcorner G_{u_\varepsilon};
\end{aligned}\tag{4.12}$$

the first one is vector-valued, all three are supported by the graph G_{u_ε} of u_ε and clearly

$$\zeta_\varepsilon \ll \mu_\varepsilon, \quad \frac{d\zeta_\varepsilon}{d\mu_\varepsilon} = \nabla u_\varepsilon, \qquad m_\varepsilon \ll \mu_\varepsilon, \quad \frac{dm_\varepsilon}{d\mu_\varepsilon} = \rho_\varepsilon.$$

Recalling (3.7), by the area formula we may write

$$\begin{aligned}\int_\Omega f_c(u_\varepsilon, \nabla u_\varepsilon, \rho_\varepsilon)\,dx &= \int_{\Omega\times]\alpha,\beta[} f_c\big(t, \nabla u_\varepsilon(x), \rho_\varepsilon(x)\big)\frac{1}{\sqrt{1+|\nabla u_\varepsilon|^2}}\,d\mathcal{H}^n \llcorner G_{u_\varepsilon}\\ &= \int_{\Omega\times]\alpha,\beta[} f_c\big(t, \nabla u_\varepsilon(x), \rho_\varepsilon(x)\big)\,d\mu_\varepsilon\\ &= \int_{\Omega\times]\alpha,\beta[} f_c\left(t, \frac{d\zeta_\varepsilon}{d\mu_\varepsilon}, \frac{dm_\varepsilon}{d\mu_\varepsilon}\right)d\mu_\varepsilon\\ &= \int_{\Omega\times]\alpha,\beta[} f_c(t, \zeta_\varepsilon, m_\varepsilon), \end{aligned}\tag{4.13}$$

by (3.28). We will prove that up to subsequences we have

$$\zeta_\varepsilon \rightharpoonup \zeta, \qquad m_\varepsilon \rightharpoonup m \tag{4.14}$$

weakly in the sense of measures, therefore by Reshetnyak's theorem 3.6

$$\liminf_{\varepsilon\to 0} \int_{\Omega\times]\alpha,\beta[} f_c(t, \zeta_\varepsilon, m_\varepsilon) \geqslant \int_{\Omega\times]\alpha,\beta[} f_c(t, \zeta, m). \tag{4.15}$$

The measures m_ε are uniformly bounded, and on the other hand (4.7), by (3.11) and the continuity of k, implies

$$\int_{\{x\in\Omega:\ \alpha+\delta<u_\varepsilon(x)<\beta-\delta\}} \big|\nabla u_\varepsilon(x)\big|\,dx \leqslant c_\delta,$$

hence also ζ_ε is compact and (4.14) is proved: we will now identify the limits of the measures ζ_ε as $\varepsilon \to 0$; to this aim it is enough to take as test functions those which are products of a function of x times a function of t, that is we take

$$\varphi(x,t) = \psi(x)a(t):$$

we begin with ζ_ε. We define

$$A(t) = \int_\alpha^t a(s)\,ds$$

and we have

$$\langle \zeta_\varepsilon, \varphi\rangle = \int_\Omega \psi(x)a(u_\varepsilon)\nabla u_\varepsilon\,dx = \int_\Omega \psi(x)\nabla\big(A(u_\varepsilon)\big)\,dx = -\int_\Omega \operatorname{div}\psi(x)A(u_\varepsilon)\,dx. \tag{4.16}$$

The support of $a(t)$ is contained in an interval $]\alpha', \beta'[$, and (4.10) implies that the bounded functions u_ε converge pointwise a.e. to u, thus $A(u_\varepsilon)$ converges in L^1 and we deduce that the last integral converges as $\varepsilon \to 0$ to

$$-\int_\Omega \operatorname{div}\psi(x)A(u)\,dx = -\int_{\{x\in\Omega:\ u(x)=\beta\}} \operatorname{div}\psi(x)A(\beta)\,dx \tag{4.17}$$

by the form of u and the fact that $A(\alpha) = 0$. Since u is in BV, recalling the definitions of S_u, Ju and ν_u given in Section 1 we may write

$$-\int_{\{x\in\Omega:\ u(x)=\beta\}} \operatorname{div}\psi(x)A(\beta)\,dx = \int_{S_u} \psi(x)\Bigg(\int_\alpha^\beta a(t)\,dt\Bigg)\nu_u(x)\,d\mathcal{H}^{n-1}(x)$$
$$= \int_\Omega \Bigg(\int_\alpha^\beta \varphi(x,t)\,dt\Bigg)\nu_u(x)\,dJu(x),$$

and joining this to (4.16), (4.17) we get

$$\langle \zeta_\varepsilon, \varphi\rangle \to \int_\Omega \Bigg(\int_\alpha^\beta \varphi(x,t)\,dt\Bigg)\nu_u(x)\,dJu(x).$$

What we proved for product functions extends by density, and we may thus write

$$\zeta_\varepsilon \rightharpoonup \zeta := \big(\nu_u(x)\,Ju\big)\otimes\big(\mathcal{L}^1(t)\llcorner\,]\alpha,\beta[\big) \tag{4.18}$$

weakly in the sense of measures.

It is not easy to characterize the limit of m_ε; to better understand the outcome it is useful, in addition to (4.12), to write also

$$\langle m_\varepsilon, \varphi\rangle = \int_{\Omega\times]\alpha,\beta[} \varphi(x,t)\big(\rho_\varepsilon(x)\,d\mathcal{L}^n(x)\otimes\delta_{u_\varepsilon(x)}(t)\big). \tag{4.19}$$

Thus m_ε is a product measure which (as G_{u_ε} has no vertical part) on the vertical side is concentrated on a single real value for each x. The projection of m_ε on Ω is just $\rho_\varepsilon\,d\mathcal{L}^n$, and what we know of the measure m in (4.14) is that, due to the assumption $\rho_\varepsilon \rightharpoonup \rho$, the projection of m on Ω is ρ. Define

$$\mu = Ju\otimes\mathcal{L}^1\llcorner\,]\alpha,\beta[;$$

we may split m into its absolutely continuous and singular parts with respect to μ, as $m = m_a\,d\mu + m_s$, and due to our considerations for $\mathcal{H}^{n-1}$-a.e. $x\in S_u$ there exists a probability measure p^x on $]\alpha,\beta[$ such that, denoting by $p_a^x(t)\,dt$ its absolutely continuous part with respect to $\mathcal{L}^1\llcorner\,]\alpha,\beta[$,

$$m_a\,d\mu = \frac{d\rho}{dJu}(x)\,Ju\otimes p_a^x(t)\big(\mathcal{L}^1(t)\llcorner\,]\alpha,\beta[\big).$$

We remark that as p^x was a probability, now

$$\int_\alpha^\beta p_a^x(t)\,dt \leqslant 1. \tag{4.20}$$

Going back to (4.15) and recalling (4.18) we may write

$$\int_{\Omega\times]\alpha,\beta[} f_c(t,\zeta,m) = \int_{\Omega\times]\alpha,\beta[} f_c\bigg(t,\frac{d\zeta}{d\mu},\frac{dm}{d\mu}\bigg)\,d\mu + \int_{\Omega\times]\alpha,\beta[} f_c(t,0,m_s)$$
$$\geqslant \int_{\Omega\times]\alpha,\beta[} f_c\bigg(t,\frac{d\zeta}{d\mu},\frac{dm}{d\mu}\bigg)\,d\mu$$
$$= \int_{S_u}\Bigg(\int_\alpha^\beta f_c\bigg(t,\nu_u,\frac{d\rho}{dJu}(x)p_a^x(t)\bigg)\,dt\Bigg)\,dJu$$
$$\geqslant \int_{S_u} \sigma\bigg(\nu_u,\frac{d\rho}{dJu}(x)\bigg)\,dJu \tag{4.21}$$

by (4.20) and (1.4). The proof of (4.11) follows by collecting (4.13), (4.15), (4.21) and the definition (1.2) of $F(u,\rho)$.

Going back to (4.19), we see that now that the graph of u has vertical parts, the measure m, besides eventually possessing a singular part with respect to $Ju\otimes\mathcal{L}^1$, spreads its pointwise projection $\frac{d\rho}{dJu}(x)$ on the vertical line from α to β in the fashion that turns out to be most favorable in terms of $f_c(t,\nu_u,\cdot)$-energy. A word about the singular part of m, which plays no role in the final energy: it is easy to remark that if by chance the approximating functions u_ε take the value α (or β) on a set S of positive measure, and if we take ρ_ε to be any (even huge) constant on S, then by (H1)–(H3)

$$\frac{1}{\varepsilon}\int_S f(u_\varepsilon,\varepsilon\nabla u_\varepsilon,\varepsilon\rho_\varepsilon)\,dx\to 0$$

and the energy contribution of this part of the limit m is indeed zero.

4.3. Upper bound inequality

We use the distance introduced in (3.30); using a simplified notation (we omit some spaces and conditions) it is readily seen that (4.3) is equivalent to

$$\begin{aligned}&\forall j\in\mathbb{N},\ \exists\{u_\varepsilon^j,\rho_\varepsilon^j\}_{\varepsilon>0},\ \varepsilon_j:\quad \forall\varepsilon<\varepsilon_j\\ &\|u_\varepsilon^j-u\|_1+\mathrm{d}^*(\rho_\varepsilon^j,\rho)<\frac{C}{j},\qquad F_\varepsilon(u_\varepsilon^j,\rho_\varepsilon^j)<F(u,\rho)+\frac{C}{j}\end{aligned}\tag{4.22}$$

for some fixed constant C. Indeed it is trivial that (4.3) implies (4.22); for the converse, one may first make sure that $\varepsilon_j\searrow 0$ by setting

$$\tilde\varepsilon_1=\varepsilon_1,\qquad \tilde\varepsilon_{j+1}=\min\{\varepsilon_{j+1},\tilde\varepsilon_j/2\},$$

so that in particular $\tilde\varepsilon_j\leqslant\varepsilon_j$, then one defines for all $\varepsilon\leqslant\varepsilon_1$

$$j(\varepsilon)=\bar j\quad\Longleftrightarrow\quad \tilde\varepsilon_{\bar j}\geqslant\varepsilon>\tilde\varepsilon_{\bar j+1};$$

as $\tilde\varepsilon_j\searrow 0$, also $j(\varepsilon)\to\infty$ when $\varepsilon\to 0$, and

$$\|u_\varepsilon^{j(\varepsilon)}-u\|_1+\mathrm{d}^*(\rho_\varepsilon^{j(\varepsilon)},\rho)\to 0,\qquad \limsup_{\varepsilon\to 0}F_\varepsilon(u_\varepsilon^{j(\varepsilon)},\rho_\varepsilon^{j(\varepsilon)})\leqslant F(u,\rho).$$

We will therefore concentrate on (4.22). We have several tasks ahead, and we accordingly subdivide the proof into several steps: at the beginning we deal with the case when everything is smooth, then we will use the approximation result of Lemma 3.10.

Step 1: smooth interface, preliminaries and first approximation of ρ on S_u. Let $u\in BV(\Omega;\{\alpha,\beta\})$ be a function with a nice interface: precisely we assume that there exists a bounded open set $U\subset\mathbb{R}^n$ with smooth boundary such that

$$\mathcal{H}^{n-1}(\partial U\cap\partial\Omega)=0\tag{4.23}$$

and that if we set

$$A=\Omega\cap U,\quad B=\Omega\setminus\overline{U},\quad \Omega^0=A\cup B\tag{4.24}$$

then

$$u=\alpha\mathbb{1}_A+\beta\mathbb{1}_B,\qquad S_u=\Omega\cap\partial U.$$

We remark that inside Ω the vector $\nu_u(x)$ coincides with the outward normal vector $n_U(x)$ to U; to avoid doubling the notation, we define $\nu_u(x)$ on all of ∂U as $n_U(x)$.

Since U is smooth, denoting by $\mathrm{dist}^\pm$ the signed distance to ∂U, i.e.

$$\mathrm{dist}^\pm(x)=\mathrm{dist}(x,\overline{U})-\mathrm{dist}(x,\mathbb{R}^n\setminus U),$$

if we define for all $r>0$

$$U^r=\{x:\ |\mathrm{dist}^\pm(x)|\leqslant r\}\tag{4.25}$$

there exists a number $T > 0$ such that the projection $\Pi(x)$ of x onto ∂U is well defined on the strip U^T, and the mapping

$$\Phi(x) = \big(\Pi(x), \mathrm{dist}^{\pm}(x)\big), \qquad \Phi : U^T \to \partial U \times [-T, T] \tag{4.26}$$

is a smooth diffeomorphism whose Jacobian satisfies

$$J\Phi \leqslant c \quad \text{on } U^T, \qquad J\Phi^{-1} \leqslant c \quad \text{on } \Phi(U^T). \tag{4.27}$$

We remark for further reference that

$$\max_{\Phi(U^r)} |J\Phi^{-1} - 1| \to 0 \quad \text{as } r \to 0. \tag{4.28}$$

We also define for all $r \in \mathbb{R}$

$$\Omega_r = \begin{cases} \{x \in \Omega\colon \mathrm{dist}(x, \partial\Omega) > r\} & \text{if } r \geqslant 0, \\ \{x \in \mathbb{R}^n\colon \mathrm{dist}(x, \Omega) < -r\} & \text{if } r < 0. \end{cases}$$

Let ρ be a nice function: precisely we assume that there exist two non-negative Lipschitz functions ρ^0, ρ^1 compactly supported in Ω such that

$$\rho = \rho^0(x)\mathcal{H}^{n-1} \llcorner S_u + \rho^1 \mathcal{L}^n \llcorner \Omega, \qquad \rho(\overline{\Omega}) = m_1 \tag{4.29}$$

and we set

$$m_1' = \int_{S_u} \rho^0 d\mathcal{H}^{n-1}, \qquad m_2' = \int_{\Omega} \rho^1\, dx = m_1 - m_1'; \tag{4.30}$$

if $m_1' = 0$ much of the trouble we are going to be into would be spared, because $\sigma(z, 0) = \max \sigma(z, \cdot)$, thus we assume that, alas, this is not the case. Eventually reducing the value of the number T above, we also assume that

$$\mathrm{spt}\, \rho^0 \cup \mathrm{spt}\, \rho^1 \Subset \Omega_T.$$

The lives of the two measures split for a while: steps $1, \dots, 4$ are devoted to the approximation of ρ^0 and step 5 to ρ^1; the two proofs are entirely independent, and may be read in any order. We warn that to improve readability (and since confusion is unlikely) we will frequently employ the same symbol to denote a measure and its density with respect to a base measure which is clear from the context: thus for example we will denote by ρ^0 both the measure $\rho^0(x)\mathcal{H}^{n-1} \llcorner S_u$ and its density $\rho^0(x)$. Also, when dealing with subsets of ∂U we will use terms as *open* and *boundary* referring to the relative topology, without further mentioning it.

To help the reader not to get lost, we include a map of what lies forth: for simplicity imagine x is the coordinate on the jump set, and t orthogonal to it; for every x we want to approximate $\sigma(\nu_u(x), \rho^0(x))$ with a smooth transition from α to β. To do this we must rely on Lemma 3.4 and (3.13) which link $\sigma(\nu_u(x), \rho^0(x))$ with something containing the original integrand f; unfortunately this gives rise to a family of "near best" transitions, one for each x, given in the notation of (3.13) by the couples $(w_x(t), \rho_x(t))$, where we stress the dependence on the point x.

This family has no regularity properties at all with respect to x, thus we are forced to replace it by a piecewise constant (with respect to x) choice, later to be smoothed out by a partition of unity method. We will therefore select small intervals, and pick points x where $\int f(w_x, \nu_u(x) w_x', \rho_x)\, dt$ nearly matches $\sigma(\nu_u(x), \rho^0(x))$. This process may ruin the condition of constant total mass on ρ, and we remark that as this mass might be entirely concentrated on the jump set, we cannot risk to exceed m_1': we therefore must voluntarily reduce ρ^0 so that its mass is less than m_1', which will leave us with some room for the piecewise constant approximation without mass problems; we will later put the extra mass remaining in a dustbin far from the jump set. One last problem will come from the volume constraint on u: to deal with this we will move the transition strip a little around the jump set, and as we do not want this translation to ruin all we did to preserve the mass of ρ by letting some of it fall outside Ω, all future movements will have amplitude not exceeding T.

The first brick is the dustbin: take, say, a ball $P \subset \Omega$ far from S_u and $\partial\Omega$: we may assume its distance from both is at least T. Fix $j \in \mathbb{N}$; as this will remain the same throughout the proof, dependence on j is harmless, and will be stressed only very sparingly. We measure how much mass we may afford to put in the dustbin: let $\Delta m > 0$ be such that

$$\mathrm{d}^*\left(\frac{2\Delta m}{|P|} \mathbb{1}_P, 0\right) < \frac{1}{j}$$

(this Δm is one of the j-dependent quantities). By (3.31) this implies that

$$0 \leqslant c \leqslant \Delta m \quad \Rightarrow \quad \mathrm{d}^*\left(\frac{1}{|P|} c \mathbb{1}_P, 0\right) < \frac{1}{j}. \tag{4.31}$$

We reserve half the dustbin's capacity for each of ρ^0 and ρ^1: choose m_1'' such that

$$0 < m_1'' < m_1', \qquad m_1'' > m_1' - \Delta m.$$

In step 3 we will have to venture on ∂U a bit beyond Ω: thus we define the function ρ^0 on all of ∂U by extending it as zero outside Ω. We thus remark that the function $\sigma(\nu_u(x), \rho^0(x))$ is well defined on ∂U. For simplicity of notation, it is also convenient to restrict the function ρ^0 to ∂U, so that $\operatorname{spt} \rho^0 \Subset \Omega \cap \partial U$. Now remark that by (4.23)

$$\lim_{r \to 0^+} \mathcal{H}^{n-1}\big(\partial U \cap (\Omega_{-r} \setminus \Omega_r)\big) = 0,$$

so we may take $0 < r_j < T/3$, and a constant $\vartheta_j < 1$ so close to 1 that if we define a smaller function

$$\rho_{sm} = \vartheta_j \rho^0$$

we have

$$\mathcal{H}^{n-1}(\partial U \cap \Omega_{-3r_j}) \leqslant 2\mathcal{H}^{n-1}(S_u)$$

and

$$\mathrm{d}^*(\rho^0, \rho_{sm}) < \frac{1}{j}, \qquad m_1'' < \int_{S_u} \rho_{sm} \, d\mathcal{H}^{n-1} < m_1'; \tag{4.32}$$

the introduction of the number $\vartheta_j < 1$ was needed only to get that the very last inequality is strict, so from now on we will be able to slightly change the function without worrying whether this increases or decreases the norm.

We also impose on r_j the following restriction:

$$\mathrm{d}^*\big(\rho^1, \rho^1 \llcorner (\Omega \setminus U^{r_j})\big) < \frac{1}{j}, \qquad \rho^1(\Omega \cap U^{r_j}) < \Delta m. \tag{4.33}$$

We define an enlarged relative of S_u:

$$\Sigma_u = \partial U \cap \overline{\Omega_{-2r_j}}.$$

By the bound (3.14), we may also assume that r_j was so small that

$$\int_{\Sigma_u} \sigma\big(\nu(x), \rho_{sm}(x)\big) \, d\mathcal{H}^{n-1} < \int_{S_u} \sigma\big(\nu(x), \rho^0(x)\big) \, d\mathcal{H}^{n-1} + \frac{1}{j}. \tag{4.34}$$

Step 2: near best slope and piecewise constant approximation. We are not finished with ρ^0 yet; pick a small number $\vartheta > 0$ such that both functions

$$\rho_{sm}^- = (\rho_{sm} - \vartheta)^+, \qquad \rho_{sm}^+ = \rho_{sm} + \vartheta$$

satisfy (4.32), (4.34) when substituted in place of ρ_{sm}: this is possible since all inequalities are strict, and this will give us some freedom to change ρ once more. We may also suppose the number ϑ to be so small that for any non-negative continuous function $\bar\rho \in L^1(\partial U)$

$$\rho_{sm}^- \leqslant \bar\rho \leqslant \rho_{sm}^+ \quad \Rightarrow \quad \mathrm{d}^*(\rho^0, \bar\rho) < \frac{1}{j}.$$

For brevity we define

$$\xi(x) = \sigma\big(\nu_u(x), \rho_{sm}(x)\big) + \frac{1}{j\mathcal{H}^{n-1}(\Sigma_u)},$$

so that in particular ξ is a continuous function satisfying

$$\xi(x) > \sigma\big(\nu_u(x), \rho_{sm}(x)\big), \qquad \int_{\Sigma_u} \xi(x) \, d\mathcal{H}^{n-1} < \int_{S_u} \sigma\big(\nu_u(x), \rho^0(x)\big) \, d\mathcal{H}^{n-1} + \frac{2}{j} \tag{4.35}$$

by (4.34). We apply Lemma 3.5 to obtain for each point $x \in \partial U$ a number r_x and functions w_x, ρ_x as in (3.26) such that

$$\int_0^{r_x} \rho_x(t)\,dt = \rho_{sm}(x), \qquad \int_0^{r_x} f\big(w_x(t), \nu_u(x)w_x'(t), \rho_x(t)\big)\,dt < \xi(x)$$

(due to the first condition we have $\rho_x \equiv 0$ outside $\operatorname{spt}\rho^0$). In particular for $Y = x$ we have

$$\rho_{sm}(Y) - \vartheta < \int_0^{r_x} \rho_x(t)\,dt < \rho_{sm}^+(Y), \qquad \int_0^{r_x} f\big(w_x(t), \nu_u(Y)w_x'(t), \rho_x(t)\big)\,dt < \xi(Y).$$

Since ρ_{sm}, f, ν_u, ξ are continuous functions, these inequalities hold also for all Y in a neighbourhood I_x of x in ∂U, and in particular since $\rho_x \geqslant 0$ we have for all $Y \in I_x$

$$\rho_{sm}^-(Y) \leqslant \int_0^{r_x} \rho_x(t)\,dt < \rho_{sm}^+(Y), \qquad \int_0^{r_x} f\big(w_x(t), \nu_u(Y)w_x'(t), \rho_x(t)\big)\,dt < \xi(Y)$$

(we had to use the possibly negative function $\rho_{sm} - \vartheta$ only to have a clean strict inequality at boundary points of the support of ρ_{sm}). We impose another restriction: if $x \in \Omega_{2r_j}$ then $I_x \subset \Omega_{2r_j}$, and for all $x \in \partial U \setminus \operatorname{spt}\rho^0$ we assume that I_x does not intersect $\operatorname{spt}\rho^0$: these points will thus never contribute in terms of ρ, since $\rho_{sm} = 0$ on all of I_x; instead, the sets I_x for all points in $\operatorname{spt}\rho^0$ will never go beyond Ω_{2r_j}. Finally we assume these sets are nice, precisely that I_x is the intersection of ∂U with an open ball B_x centered at x and with radius less than the number T defined in (4.25), and we define W_x to be the intersection of Σ_u with the ball of *half* the radius of B_x, thus W_x is relatively open in Σ_u.

We cover Σ_u with a finite number of these sets, $W_1, \dots, W_k$ with $W_m \equiv W_{x_m}$, and let I_m, r_m, w_m, ρ_m be the sets, numbers and functions $I_{x_m}, r_{x_m}, w_{x_m}, \rho_{x_m}$ respectively. To make the sets disjoint, we replace each W_m by

$$W_m \setminus \bigcup_{i=1}^{m-1} \overline{W_i},$$

which we relabel W_m not to add to the already exuberant notation, and we remark that I_m is still an open neighbourhood of W_m. We get rid of the numbers r_m by calling $r = \max\{r_m\}$ and by extending the functions w_m and ρ_m as β and 0 respectively on $]r_m, r]$: this does not change the situation, as $f(\beta, 0, 0) = 0$. For our future convenience we may go one step further, extending w_m, ρ_m on all of $\mathbb{R}$ as α, 0 for $t < 0$ and β, 0 for $t > r$: still nothing changes.

We summarize the situation: we have

$$\Sigma_u = W_1 \cup \dots \cup W_k \cup N$$

with $\mathcal{H}^{n-1}(N) = 0$ and all sets disjoint; we also have functions w_m, ρ_m such that for all points $x \in I_m$

$$\rho_{sm}^-(x) \leqslant \int_0^{r} \rho_m(t)\,dt < \rho_{sm}^+(x), \qquad \int_0^{r} f\big(w_m, \nu_u(x)w_m'(t), \rho_m(t)\big)\,dt < \xi(x). \tag{4.36}$$

Step 3: glueing together the pieces in a neighbourhood of Σ_u. In this step we use ideas from the proof of [1]; basically we will use w_m and ρ_m in W_m to define a function in a small "cylinder" sticking out from W_m into Ω, but although this might be fine for ρ, the resulting function w is irregular, so we will interpolate between neighbouring patches.

Take a number $\delta > 0$, whose value we will be able to fix later, smaller than half the smallest radius of the balls B_m and smaller than r_j, and let

$$W_m^\delta = \big\{x \in W_m\colon \operatorname{dist}(x, \partial U \setminus W_m) \geqslant \delta\big\}, \qquad \widetilde{W}_m^\delta = \big\{x \in \partial U\colon \operatorname{dist}(x, W_m) < \delta\big\}:$$

since $\widetilde{W}_1^\delta \cup \dots \cup \widetilde{W}_k^\delta$ is a neighbourhood of Σ_u and each W_m^δ is compact in $\widetilde{W}_m^\delta$, we may take a partition of unity $\psi_1, \dots, \psi_k$ on Σ_u such that

$$\psi_m = 1 \quad \text{in } W_m^\delta, \qquad \operatorname{spt}\psi_m \subset \widetilde{W}_m^\delta \subset I_m, \qquad |D\psi_m| \leqslant \frac{C}{\delta} \tag{4.37}$$

for some C independent of δ.

Recalling (4.26), we remark that for $\vartheta > r/T$ the function

$$\sum_{m=1}^{k} \psi_m\big(\Pi(x)\big) w_m\big(\vartheta \operatorname{dist}^{\pm}(x)\big)$$

is regular and it is well defined for all $x \in \mathbb{R}^n$ such that

$$\big|\operatorname{dist}^{\pm}(x)\big| \leqslant T, \tag{4.38}$$

which contains an n-dimensional neighbourhood of ∂U; moreover where

$$(\psi_1 + \cdots + \psi_k)\big(\Pi(x)\big) = 1 \tag{4.39}$$

its value on one face of the neighbourhood is α, its value on the other is β. This may seem a good candidate for the transition, just extending the resulting function as α or β elsewhere, but there are two (minor) problems: for one thing, there may be points in Ω satisfying (4.38) but not (4.39), and the proposed extension would then be discontinuous; also, recalling (4.24), the proposed extension would have value α in all of A, but β only in a portion of B, thus missing the integral constraint. Thanks to the introduction of Σ_u we will show that it is easy to deal with the first problem, while to tackle the second we will slightly shift the centre of the neighbourhood in the normal direction to ∂U.

Recall that $T > 3r_j$ and that $\operatorname{dist}(\partial U \setminus \Sigma_u, \Omega) \geqslant 2r_j$, so that

$$x \in \Omega,\ \big|\operatorname{dist}^{\pm}(x)\big| < 2r_j \quad\Rightarrow\quad \Pi(x) \in \Sigma_u \quad\Rightarrow\quad \sum_{m=1}^{k} \psi_m\big(\Pi(x)\big) = 1. \tag{4.40}$$

Now set (the reason for each entry at the right-hand side will be clear in the sequel)

$$R_j = \min\big\{r_j/2,\ 1/\big(2cj\mathcal{H}^{n-1}(\Sigma_u)\big)\big\} < T \tag{4.41}$$

where c was defined in (4.27); we are about to set a first j-dependent restriction on ε, which will be used to determine the number ε_j of (4.22): recalling (4.24), (4.25), we define in Ω for all $\varepsilon < R_j/r$

$$u_\varepsilon^\delta(x) = \sum_{m=1}^{k} \psi_m\big(\Pi(x)\big) w_m\left(\frac{\operatorname{dist}^{\pm}(x)}{\varepsilon} + \tau_\varepsilon\right) \quad \text{if } x \in \Omega \cap U^{R_j}, \tag{4.42}$$

$$u_\varepsilon^\delta(x) = \alpha \quad \text{if } x \in A \setminus U^{R_j}, \qquad u_\varepsilon^\delta(x) = \beta \quad \text{if } x \in B \setminus U^{R_j},$$

where $\tau_\varepsilon \in [0, r]$ will be chosen in a few lines. By (4.40), if $x \in \Omega$ is such that $\operatorname{dist}^{\pm}(x) = R_j$ then $|\operatorname{dist}^{\pm}(x)| < r_j$ and

$$\frac{\operatorname{dist}^{\pm}(x)}{\varepsilon} + \tau_\varepsilon \geqslant r$$

no matter what $\tau_\varepsilon \geqslant 0$ is, so $w_m(\cdot) = \beta$ for all m and therefore $u_\varepsilon^\delta(x) = \beta$; analogous considerations hold on the other side of U^{R_j}, where $\operatorname{dist}^{\pm}(x) = -R_j$ and the value turns out to be α for any $\tau_\varepsilon \leqslant r$. Thus the resulting function u_ε^δ is in $W^{1,1}(\Omega)$; moreover it is always between α and β because so do the functions w_m of which it is a convex combination, and it agrees with the target function u except on a set whose measure by (4.41) does not exceed $1/j$, so that

$$\|u_\varepsilon^\delta - u\|_1 \leqslant \frac{\beta - \alpha}{j}. \tag{4.43}$$

Finally, for $\tau_\varepsilon = 0$ we have $u_\varepsilon^\delta = \alpha$ in A, so $\int u_\varepsilon^\delta\,dx \leqslant \int u\,dx$, whereas for $\tau_\varepsilon = r$ we have $u_\varepsilon^\delta = \beta$ in B, so $\int u_\varepsilon^\delta\,dx \geqslant \int u\,dx$. Therefore there exists a value of τ_ε for which the volume constraint

$$\int_\Omega u_\varepsilon^\delta\,dx = m_0 = \int_\Omega u\,dx$$

is satisfied. The family $\{u_\varepsilon^\delta\}_\varepsilon$ thus satisfies its share of (4.22).

We go back to ρ: define for $m = 1, \dots, k$

$$\bar\rho_m = \int_0^r \rho_m(t)\,dt$$

and define on ∂U

$$\rho_p^\delta(x) = \sum_{m=1}^k \psi_m(x)\bar\rho_m.$$

By our choices and recalling (4.37) we have that

$$\rho_p^\delta(x) = \bar\rho_m \text{ in } W_m^\delta, \quad \bar\rho_m = 0 \text{ if } I_m \cap \operatorname{spt}\rho^0 = \emptyset, \quad \bar\rho_m \neq 0 \quad \Rightarrow \quad \widetilde W_m^\delta \subset I_m \subset \Omega_{2r_j}$$

and in particular

$$\operatorname{spt}\rho_p^\delta \subset S_u \cap \Omega_{2r_j} \tag{4.44}$$

and also, since at each point the value $\rho_p^\delta(x)$ is a convex combination of only those $\bar\rho_m$ for which $x \in I_m$, by the first part of (4.36) we have

$$\rho_{sm}^-(x) \leqslant \rho_p^\delta(x) < \rho_{sm}^+(x) \quad \text{on } \partial U.$$

In particular we deduce by (4.34), the monotonicity of σ and the choice of ρ_{sm}^-, ρ_{sm}^+ made at the beginning of step 2

$$\int_{\Sigma_u} \sigma\big(\nu_u(x), \rho_p^\delta(x)\big)\,d\mathcal{H}^{n-1} < \int_{S_u} \sigma\big(\nu_u(x), \rho^0(x)\big)\,d\mathcal{H}^{n-1} + \frac{1}{j}.$$

We define

$$\rho_\varepsilon^\delta(x) = \frac{1}{\varepsilon}\sum_{m=1}^k \psi_m\big(\Pi(x)\big)\rho_m\left(\frac{\operatorname{dist}^\pm(x)}{\varepsilon} + \tau_\varepsilon\right) \quad \text{if } x \in \Omega \cap U^{R_j}$$

and zero elsewhere. It is clear that as $\varepsilon \to 0$ this family converges weakly to the measure ρ_p^δ, and also that $\int_\Omega \rho_\varepsilon^\delta\,dx \to \int_{S_u} \rho_p^\delta\,d\mathcal{H}^{n-1}$, so we deduce by (4.32) and the choice of ρ_{sm}^-, ρ_{sm}^+ that for ε small (this is the second condition to build ε_j)

$$\mathrm{d}^*(\rho^0, \rho_\varepsilon^\delta) < \frac{1}{j}, \qquad m_1'' < \int_\Omega \rho_\varepsilon^\delta(x)\,dx < m_1'; \tag{4.45}$$

since the support of each ρ_m is in $[0, r]$, the support of ρ_ε^δ is contained in

$$\big\{y \in \Omega\colon\ y = x + t\nu_u(x),\ x \in \operatorname{spt}\rho_p^\delta,\ |t| \leqslant \varepsilon r\big\},$$

but as we chose

$$\varepsilon r \leqslant R_j \leqslant r_j$$

we finally deduce by (4.44)

$$\operatorname{spt}\rho_\varepsilon^\delta \subset \Omega_{r_j} \cap U^{r_j}.$$

The remark extends to u_ε^δ, for which we may say

$$\big\{x \in \Omega\colon\ u_\varepsilon^\delta(x) \neq \alpha, \beta\big\} \subset \Omega \cap U^{\varepsilon r} \subset \Omega \cap U^{r_j}.$$

We summarize: concentrating on ρ^0 and the jump, we found a candidate family of transitions u_ε^δ which converges in L^1 to u and satisfies the integral constraint, and a candidate family of L^1 functions ρ_ε^δ whose support is a narrow strip close to S_u and far from $\partial\Omega$, whose integral is not too far from $\int_{S_u} \rho^0$ and which are sufficiently close to ρ^0 in the d^* metric.

Step 4: energy estimate and conclusion for ρ^0. We restrict ourselves to the strip $\Omega \cap U^{\varepsilon r}$, outside which $f(u_\varepsilon^\delta, \varepsilon\nabla u_\varepsilon^\delta, \varepsilon\rho_\varepsilon^\delta) \equiv 0$. We add another restriction about ε_j: we will assume that

$$\varepsilon \leqslant \delta$$

(δ has still to be chosen, but we will do so depending only on j). Since the functions $w_m(t)$ we got from Lemma 3.5 were Lipschitz, from (4.37), (4.42) and the regularity of ∂U we deduce

$$\left|\nabla u_\varepsilon^\delta(x)\right| \leqslant \frac{C}{\varepsilon} \quad \text{if } \Pi(x) \in W_m^\delta, \qquad \left|\nabla u_\varepsilon^\delta(x)\right| \leqslant \frac{C}{\varepsilon} + \frac{C}{\delta} \leqslant \frac{2C}{\varepsilon} \quad \text{otherwise,}$$

where C is a constant independent of ε, δ, but which takes into account all sorts of j-dependent quantities. We also have

$$\rho_\varepsilon^\delta(x) \leqslant \frac{C}{\varepsilon}.$$

We divide Σ_u into the constancy patches W_m^δ and the patch interpolation set

$$G^\delta = \Sigma_u \setminus \bigcup_{m=1}^{k} W_m^\delta;$$

recalling (4.40), (4.41), and since $\alpha \leqslant u_\varepsilon^\delta \leqslant \beta$, we have

$$\Pi(x) \in G^\delta \quad \Rightarrow \quad f(u_\varepsilon^\delta, \varepsilon\nabla u_\varepsilon^\delta, \varepsilon\rho_\varepsilon^\delta) \leqslant M,$$

where by the continuity of f we set

$$M = \max\bigl\{f(s,z,\gamma):\ \alpha \leqslant s \leqslant \beta,\ |z| \leqslant 2C,\ 0 \leqslant \gamma \leqslant C\bigr\}.$$

As $\varepsilon \to 0$, we have

$$\left|\bigl\{x \in \Omega \cap U^{\varepsilon r}:\ \Pi(x) \in G^\delta\bigr\}\right| \approx 2\varepsilon r \mathcal{H}^{n-1}(G^\delta \cap \overline{\Omega}),$$

and we may now specify the value of δ: precisely, remarking that

$$\mathcal{H}^{n-1}(G^\delta \cap \overline{\Omega}) \to 0 \qquad \text{as } \delta \to 0,$$

we take δ so that

$$\mathcal{H}^{n-1}(G^\delta \cap \overline{\Omega}) \leqslant \frac{1}{4rMj}.$$

From this moment δ is fixed (depending on j), so referring to (4.22) we may define

$$u_\varepsilon^j := u_\varepsilon^\delta \quad \text{in } \Omega, \qquad \rho_\varepsilon^j := \rho_\varepsilon^\delta \quad \text{in } \Omega \cap U^{r_j},$$

and we obtain

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_{\{x\in\Omega\cap U^{r_j}:\ \Pi(x)\in G^\delta\}} f(u_\varepsilon^j, \varepsilon\nabla u_\varepsilon^j, \varepsilon\rho_\varepsilon^j)\,dx \leqslant \frac{1}{2j},$$

therefore for all sufficiently small ε, which is a further restriction on ε_j,

$$\frac{1}{\varepsilon} \int_{\{x\in\Omega\cap U^{r_j}:\ \Pi(x)\in G^\delta\}} f(u_\varepsilon^j, \varepsilon\nabla u_\varepsilon^j, \varepsilon\rho_\varepsilon^j)\,dx < \frac{1}{j}. \tag{4.46}$$

Now take a patch W_m^δ and remark that on $\{x \in \Omega \cap U^{\varepsilon r}:\ \Pi(x) \in W_m^\delta\}$ we have

$$u_\varepsilon^j(x) = w_m\left(\frac{\mathrm{dist}^\pm(x)}{\varepsilon} + \tau_\varepsilon\right), \qquad \rho_\varepsilon^j(x) = \frac{1}{\varepsilon}\rho_m\left(\frac{\mathrm{dist}^\pm(x)}{\varepsilon} + \tau_\varepsilon\right),$$

thus by the change of variables

$$y = x + \varepsilon(t - \tau_\varepsilon)\nu_u(x)$$

we have

$$\frac{1}{\varepsilon}\int\limits_{\{y\in\Omega\cap U^{\varepsilon r}:\ \Pi(y)\in W_m^\delta\}} f(u_\varepsilon^j,\varepsilon\nabla u_\varepsilon^j,\varepsilon\rho_\varepsilon^j)\,dy=\int\limits_{W_m^\delta} d\mathcal{H}^{n-1}(x)\int\limits_0^r f\big(w_m(t),\nu_u(x)w_m'(t),\rho_m(t)\big)J\Phi^{-1}(y)\,dt,$$

where $y\in\Omega\cap U^{\varepsilon r}$ is the only place where ε hides. Recalling (4.28) and later using (4.36) we have

$$\begin{aligned}\limsup_{\varepsilon\to0}\frac{1}{\varepsilon}\int\limits_{\{y\in\Omega\cap U^{\varepsilon r}:\ \Pi(y)\in W_m^\delta\}} f(u_\varepsilon^j,\varepsilon\nabla u_\varepsilon^j,\varepsilon\rho_\varepsilon^j)\,dy&=\int\limits_{W_m^\delta} d\mathcal{H}^{n-1}(x)\int\limits_0^r f\big(w_m(t),\nu_u(x)w_m'(t),\rho_m(t)\big)\,dt\\&<\int\limits_{W_m^\delta}\xi(x)\,d\mathcal{H}^{n-1},\end{aligned}$$

so that for ε sufficiently small (this is the last restriction on ε_j as far as ρ^0 is concerned)

$$\frac{1}{\varepsilon}\int\limits_{\{y\in\Omega\cap U^{r_j}:\ \Pi(y)\in W_m^\delta\}} f(u_\varepsilon^j,\varepsilon\nabla u_\varepsilon^j,\varepsilon\rho_\varepsilon^j)\,dy<\int\limits_{W_m^\delta}\xi(x)\,d\mathcal{H}^{n-1},\tag{4.47}$$

where we used again (H1). Summing with respect to m and collecting (4.35), (4.46) and (4.47) we get at last

$$\frac{1}{\varepsilon}\int\limits_{\Omega\cap U^{r_j}} f\big(u_\varepsilon^j(x),\varepsilon\nabla u_\varepsilon^j(x),\varepsilon\rho_\varepsilon^j(x)\big)\,dx<\int\limits_{\Sigma_u}\xi(x)\,d\mathcal{H}^{n-1}+\frac{1}{j}<\int\limits_{S_u}\sigma\big(\nu_u(x),\rho^0(x)\big)\,d\mathcal{H}^{n-1}+\frac{3}{j}\tag{4.48}$$

for all ε sufficiently small.

Step 5: reduction to a simpler problem and approximation of ρ^1. Recalling the subadditivity property (3.32) of d^* and comparing (4.43), (4.45), (4.48) with our goal (4.22), we see that we still have to define ρ_ε^j also outside the strip U^{r_j} in such a way that using the hybrid notation $[\alpha;\beta]$ to denote α in A and β in B,

$$\int\limits_\Omega\rho_\varepsilon^j\,dx=m_1,\qquad \mathrm{d}^*\big(\rho_\varepsilon^j\llcorner(\Omega\setminus U^{r_j}),\rho^1\big)<\frac{C}{j},\qquad \frac{1}{\varepsilon}\int\limits_{\Omega\setminus U^{r_j}} f\big([\alpha;\beta],0,\varepsilon\rho_\varepsilon^j\big)\,dx<\frac{C}{j},$$

where ρ^1 was defined in (4.29). Indeed the last condition and (4.48), given (1.2) and (1.5), immediately imply the last inequality in (4.22). Recall the definition (4.30) of m_2'; if we are in the lucky case $m_2'=0$ then $\rho_j=0$ outside U^{r_j} and that's the end. Else, the task ahead is not too hard: we remark that the function $\rho^j=\rho^1\llcorner(\Omega\setminus U^{r_j})$ satisfies

$$\rho^j\in L^\infty,\quad \operatorname{spt}\rho^j\subset\Omega\setminus U^{r_j},\quad \mathrm{d}^*(\rho^j,\rho^1)<\frac{1}{j},\quad m_2'-\Delta m<\int\limits_\Omega\rho^j\,dx\leqslant m_2'\tag{4.49}$$

by (4.33), and we are done: call

$$c_\varepsilon=m_1-\int\limits_\Omega\rho_\varepsilon^j\,dx$$

and remembering (4.31) define on $\Omega\setminus U^{r_j}$

$$\rho_\varepsilon^j=\rho^j+\frac{1}{|P|}c_\varepsilon\mathbb{1}_P$$

(recall that ρ_ε^j is already defined on $\Omega\cap U^{r_j}$). This family satisfies the integral constraint $\int\rho_\varepsilon^j=m_1$, also $\mathrm{d}^*(\rho_\varepsilon^j,\rho)<C/j$ and by the L^∞ bound, since $c_\varepsilon\leqslant m_1$, we deduce by (3.27)

$$\rho_\varepsilon^j\leqslant M\quad\text{on }\Omega\setminus U^{r_j}\quad\Rightarrow\quad f\big([\alpha;\beta],0,\varepsilon\rho_\varepsilon^j\big)\leqslant\varepsilon M\eta(\varepsilon M)\quad\text{on }\Omega\setminus U^{r_j}$$

so that

$$\frac{1}{\varepsilon}\int_{\Omega\setminus U_j^r} f\big([\alpha;\beta],0,\varepsilon\rho_\varepsilon^j\big)\,dx \leqslant M\eta(\varepsilon M)\to 0$$

as $\varepsilon\to 0$, and we may suppose that

$$\frac{1}{\varepsilon}\int_{\Omega\setminus U_j^r} f\big([\alpha;\beta],0,\varepsilon\rho_\varepsilon^j\big)\,dx \leqslant \frac{1}{j}$$

for $\varepsilon\leqslant\varepsilon_j$ suitably small (last restriction on ε_j).

Step 6: general case. By the metric nature of both L^1 convergence and weak convergence of Radon measures with equibounded mass, (4.22) in the general case now follows easily from Proposition 3.10: let u_h,ρ_h be the sequence given by this proposition and fix j; there exists $\bar h$ such that

$$\|u_{\bar h}-u\|_1+\mathrm{d}^*(\rho_h,\rho)<\frac{1}{j},\qquad F(u_{\bar h},\rho_h)<F(u,\rho)+\frac{1}{j}.$$

In steps $1,\dots,5$ we proved that (4.22) holds if u,ρ are nice, so in particular if $(u_\varepsilon^j,\rho_\varepsilon^j)$ is the sequence given by (4.22) with $u_{\bar h},\rho_{\bar h}$ in place of u,ρ we have for all $\varepsilon<\varepsilon_j$

$$\|u_\varepsilon^j-u\|_1+\mathrm{d}^*(\rho_\varepsilon^j,\rho)<\frac{C+1}{j},\qquad F_\varepsilon(u_\varepsilon^j,\rho_\varepsilon^j)<F(u,\rho)+\frac{C+1}{j}.$$

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