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# Regularity of Hölder continuous solutions of the supercritical quasi-geostrophic equation

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**Abstract**

We present a regularity result for weak solutions of the 2D quasi-geostrophic equation with supercritical ($\alpha < 1/2$) dissipation $(-\Delta)^\alpha$: If a Leray–Hopf weak solution is Hölder continuous $\theta \in C^\delta(\mathbb{R}^2)$ with $\delta > 1 - 2\alpha$ on the time interval $[t_0, t]$, then it is actually a classical solution on $(t_0, t]$.


*MSC:* 76D03; 35Q35

*Keywords:* 2D quasi-geostrophic equation; Supercritical dissipation; Regularity; Weak solutions

## 1. Introduction

We discuss the surface 2D quasi-geostrophic (QG) equation

$$\partial_t \theta + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \quad x \in \mathbb{R}^2,\ t > 0, \tag{1.1}$$

where $\alpha > 0$ and $\kappa \geqslant 0$ are parameters, and the 2D velocity field $u = (u_1, u_2)$ is determined from $\theta$ by the stream function $\psi$ via the auxiliary relations

$$(u_1, u_2) = (-\partial_{x_2}\psi, \partial_{x_1}\psi), \qquad (-\Delta)^{1/2}\psi = -\theta. \tag{1.2}$$

Using the notation $\Lambda \equiv (-\Delta)^{1/2}$ and $\nabla^\perp \equiv (\partial_{x_2}, -\partial_{x_1})$, the relations in (1.2) can be combined into

$$u = \nabla^\perp \Lambda^{-1}\theta = (-\mathcal{R}_2\theta, \mathcal{R}_1\theta), \tag{1.3}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the usual Riesz transforms in $\mathbb{R}^2$. The 2D QG equation with $\kappa > 0$ and $\alpha = \frac{1}{2}$ arises in geophysical studies of strongly rotating fluids (see [5,16] and references therein) while the inviscid QG equation ((1.1) with $\kappa = 0$) was derived to model frontogenesis in meteorology, a formation of sharp fronts between masses of hot and cold air (see [7,10,16]).

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The problem at the center of the mathematical theory concerning the 2D QG equation is whether or not it has a global in time smooth solution for any prescribed smooth initial data. In the subcritical case $\alpha > \frac{1}{2}$, the dissipative QG equation has been shown to possess a unique global smooth solution for every sufficiently smooth initial data (see [8,17]). In contrast, when $\alpha \leqslant \frac{1}{2}$, the issue of global existence and uniqueness is more difficult and has still unanswered aspects. Recently this problem has attracted a significant amount of research ([1–6,9,11–15,18–24]). In Constantin, Córdoba and Wu [6], we proved in the critical case ($\alpha = \frac{1}{2}$) the global existence and uniqueness of classical solutions corresponding to any initial data with $L^\infty$-norm comparable to or less than the diffusion coefficient $\kappa$. In a recently posted preprint in arXiv [14], Kiselev, Nazarov and Volberg proved that smooth global solutions exist for any $C^\infty$ periodic initial data, by removing the $L^\infty$-smallness assumption on the initial data of [6]. Caffarelli and Vasseur (arXiv reference [1]) establish the global regularity of the Leray–Hopf type weak solutions (in $L^\infty((0,\infty); L^2) \cap L^2((0,\infty); \mathring{H}^{1/2})$) of the critical 2D QG equation with $\alpha = \frac{1}{2}$ in general $\mathbb{R}^n$.

In this paper we present a regularity result of weak solutions of the dissipative QG equation with $\alpha < \frac{1}{2}$ (the supercritical case). The result asserts that if a Leray–Hopf weak solution $\theta$ of (1.1) is in the Hölder class $C^\delta$ with $\delta > 1 - 2\alpha$ on the time interval $[t_0, t]$, then it is actually a classical solution on $(t_0, t]$. The proof involves representing the functions in Hölder space in terms of the Littlewood–Paley decomposition and using Besov space techniques. When $\theta$ is in $C^\delta$, it also belongs to the Besov space $\mathring{B}^{\delta(1-2/p)}_{p,\infty}$ for any $p \geqslant 2$. By taking $p$ sufficiently large, we have $\theta \in C^{\delta_1} \cap \mathring{B}^{\delta_1}_{p,\infty}$ for $\delta_1 > 1 - 2\alpha$. The idea is to show that $\theta \in C^{\delta_2} \cap \mathring{B}^{\delta_2}_{p,\infty}$ with $\delta_2 > \delta_1$. Through iteration, we establish that $\theta \in C^\gamma$ with $\gamma > 1$. Then $\theta$ becomes a classical solution.

The results of this paper can be easily extended to a more general form of the quasi-geostrophic equation in which $x \in \mathbb{R}^n$ and $u$ is a divergence-free vector field determined by $\theta$ through a singular integral operator.

The rest of this paper is divided into two sections. Section 2 provides the definition of Besov spaces and necessary tools. Section 3 states and proves the main result.

## 2. Besov spaces and related tools

This section provides the definition of Besov spaces and several related tools. We start with a some notation. Denote by $\mathcal{S}(\mathbb{R}^n)$ the usual Schwarz class and $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions. $\hat{f}$ denotes the Fourier transform of $f$, namely

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx.$$

The fractional Laplacian $(-\Delta)^\alpha$ can be defined through the Fourier transform

$$\widehat{(-\Delta)^\alpha f} = |\xi|^{2\alpha} \hat{f}(\xi).$$

Let

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}, \int_{\mathbb{R}^n} \phi(x) x^\gamma\,dx = 0,\ |\gamma| = 0, 1, 2, \ldots \right\}.$$

Its dual $\mathcal{S}_0'$ is given by

$$\mathcal{S}_0' = \mathcal{S}'/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P},$$

where $\mathcal{P}$ is the space of polynomials. In other words, two distributions in $\mathcal{S}'$ are identified as the same in $\mathcal{S}_0'$ if their difference is a polynomial.

It is a classical result that there exists a dyadic decomposition of $\mathbb{R}^n$, namely a sequence $\{\Phi_j\} \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\operatorname{supp} \hat{\Phi}_j \subset A_j, \quad \hat{\Phi}_j(\xi) = \hat{\Phi}_0(2^{-j}\xi) \quad \text{or} \quad \Phi_j(x) = 2^{jn}\Phi_0(2^j x)$$

and

$$\sum_{k=-\infty}^{\infty} \hat{\Phi}_k(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } \xi = 0, \end{cases}$$

where

$$A_j = \{\xi \in \mathbb{R}^n \colon\ 2^{j-1} < |\xi| < 2^{j+1}\}.$$

As a consequence, for any $f \in \mathcal{S}_0'$,

$$\sum_{k=-\infty}^{\infty} \Phi_k * f = f. \tag{2.1}$$

For notational convenience, set

$$\Delta_j f = \Phi_j * f, \quad j = 0, \pm 1, \pm 2, \ldots. \tag{2.2}$$

**Definition 2.1.** For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the homogeneous Besov space $\mathring{B}^s_{p,q}$ is defined by

$$\mathring{B}^s_{p,q} = \{f \in \mathcal{S}_0' \colon\ \|f\|_{\mathring{B}^s_{p,q}} < \infty\},$$

where

$$\|f\|_{\mathring{B}^s_{p,q}} = \begin{cases} \left(\sum_j \left(2^{js}\|\Delta_j f\|_{L^p}\right)^q\right)^{1/q} & \text{for } q < \infty, \\ \sup_j 2^{js}\|\Delta_j f\|_{L^p} & \text{for } q = \infty. \end{cases}$$

For $\Delta_j$ defined in (2.2) and $S_j \equiv \sum_{k<j} \Delta_k$,

$$\Delta_j \Delta_k = 0 \quad \text{if } |j-k| \geqslant 2 \quad \text{and} \quad \Delta_j(S_{k-1} f \Delta_k f) = 0 \quad \text{if } |j-k| \geqslant 3.$$

The following proposition lists a few simple facts that we will use in the subsequent section.

**Proposition 2.2.** *Assume that $s \in \mathbb{R}$ and $p, q \in [1, \infty]$.*

(1) *If $1 \leqslant q_1 \leqslant q_2 \leqslant \infty$, then $\mathring{B}^s_{p,q_1} \subset \mathring{B}^s_{p,q_2}$.*
(2) (Besov embedding) *If $1 \leqslant p_1 \leqslant p_2 \leqslant \infty$ and $s_1 = s_2 + n(\frac{1}{p_1} - \frac{1}{p_2})$, then $\mathring{B}^{s_1}_{p_1,q}(\mathbb{R}^n) \subset \mathring{B}^{s_2}_{p_2,q}(\mathbb{R}^n)$.*
(3) *If $1 < p < \infty$, then*

$$\mathring{B}^s_{p,\min(p,2)} \subset \mathring{W}^{s,p} \subset \mathring{B}^s_{p,\max(p,2)},$$

*where $\mathring{W}^{s,p}$ denotes a standard homogeneous Sobolev space.*

We will need a Bernstein type inequality for fractional derivatives.

**Proposition 2.3.** *Let $\alpha \geqslant 0$. Let $1 \leqslant p \leqslant q \leqslant \infty$.*

(1) *If $f$ satisfies*

$$\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^n \colon\ |\xi| \leqslant K 2^j\},$$

*for some integer $j$ and a constant $K > 0$, then*

$$\|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^n)} \leqslant C_1 2^{2\alpha j + jn(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)}.$$

(2) *If $f$ satisfies*

$$\operatorname{supp} \hat{f} \subset \{\xi \in \mathbb{R}^n \colon\ K_1 2^j \leqslant |\xi| \leqslant K_2 2^j\} \tag{2.3}$$

*for some integer $j$ and constants $0 < K_1 \leqslant K_2$, then*

$$C_1 2^{2\alpha j} \|f\|_{L^q(\mathbb{R}^n)} \leqslant \|(-\Delta)^\alpha f\|_{L^q(\mathbb{R}^n)} \leqslant C_2 2^{2\alpha j + jn(\frac{1}{p} - \frac{1}{q})} \|f\|_{L^p(\mathbb{R}^n)},$$

*where $C_1$ and $C_2$ are constants depending on $\alpha$, $p$ and $q$ only.*

The following proposition provides a lower bound for an integral that originates from the dissipative term in the process of $L^p$ estimates (see [21,4]).

**Proposition 2.4.** *Assume either $\alpha \geqslant 0$ and $p = 2$ or $0 \leqslant \alpha \leqslant 1$ and $2 < p < \infty$. Let $j$ be an integer and $f \in \mathcal{S}'$. Then*

$$\int_{\mathbb{R}^n} |\Delta_j f|^{p-2} \Delta_j f \Lambda^{2\alpha} \Delta_j f \, dx \geqslant C 2^{2\alpha j} \|\Delta_j f\|_{L^p}^p$$

*for some constant $C$ depending on $n$, $\alpha$ and $p$.*

## 3. The main theorem and its proof

**Theorem 3.1.** *Let $\theta$ be a Leray–Hopf weak solution of* (1.1)*, namely*

$$\theta \in L^\infty\big([0,\infty); L^2(\mathbb{R}^2)\big) \cap L^2\big([0,\infty); \mathring{H}^\alpha(\mathbb{R}^2)\big). \tag{3.1}$$

*Let $\delta > 1 - 2\alpha$ and let $0 < t_0 < t < \infty$. If*

$$\theta \in L^\infty\big([t_0, t]; C^\delta(\mathbb{R}^2)\big), \tag{3.2}$$

*then*

$$\theta \in C^\infty\big((t_0, t] \times \mathbb{R}^2\big).$$

**Proof.** First, we notice that (3.1) and (3.2) imply that

$$\theta \in L^\infty\big([t_0, t]; \mathring{B}^{\delta_1}_{p,\infty}(\mathbb{R}^2)\big),$$

for any $p \geqslant 2$ and $\delta_1 = \delta(1 - \frac{2}{p})$. In fact, for any $\tau \in [t_0, t]$,

$$\begin{aligned}\big\|\theta(\cdot,\tau)\big\|_{\mathring{B}^{\delta_1}_{p,\infty}} &= \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^p} \\ &\leqslant \sup_j 2^{\delta_1 j} \|\Delta_j \theta\|_{L^\infty}^{1-\frac{2}{p}} \|\Delta_j \theta\|_{L^2}^{\frac{2}{p}} \\ &\leqslant \big\|\theta(\cdot,\tau)\big\|_{C^\delta}^{1-\frac{2}{p}} \big\|\theta(\cdot,\tau)\big\|_{L^2}^{\frac{2}{p}}.\end{aligned}$$

Since $\delta > 1 - 2\alpha$, we have $\delta_1 > 1 - 2\alpha$ when

$$p > p_0 \equiv \frac{2\delta}{\delta - (1 - 2\alpha)}.$$

Next, we show that

$$\theta \in L^\infty\big([t_0, t]; \mathring{B}^{\delta_1}_{p,\infty} \cap C^{\delta_1}\big)$$

implies

$$\theta(\cdot, t) \in \mathring{B}^{\delta_2}_{p,\infty} \cap C^{\delta_2}$$

for some $\delta_2 > \delta_1$ to be specified. Let $j$ be an integer. Applying $\Delta_j$ to (1.1), we get

$$\partial_t \Delta_j \theta + \kappa \Lambda^{2\alpha} \Delta_j \theta = -\Delta_j (u \cdot \nabla \theta). \tag{3.3}$$

By Bony's notion of paraproduct,

$$\begin{aligned}\Delta_j (u \cdot \nabla \theta) = &\sum_{|j-k|\leqslant 2} \Delta_j (S_{k-1} u \cdot \nabla \Delta_k \theta) + \sum_{|j-k|\leqslant 2} \Delta_j (\Delta_k u \cdot \nabla S_{k-1} \theta) \\ &+ \sum_{k \geqslant j-1} \sum_{|k-l|\leqslant 1} \Delta_j (\Delta_k u \cdot \nabla \Delta_l \theta).\end{aligned} \tag{3.4}$$

Multiplying (3.3) by $p|\Delta_j\theta|^{p-2}\Delta_j\theta$, integrating with respect to $x$, and applying the lower bound

$$\int_{\mathbb{R}^d} |\Delta_j f|^{p-2}\Delta_j f \Lambda^{2\alpha}\Delta_j f\,dx \geqslant C2^{2\alpha j}\|\Delta_j f\|_{L^p}^p$$

of Proposition 2.4, we obtain

$$\frac{d}{dt}\|\Delta_j\theta\|_{L^p}^p + C\kappa 2^{2\alpha j}\|\Delta_j\theta\|_{L^p}^p \leqslant I_1 + I_2 + I_3, \tag{3.5}$$

where $I_1$, $I_2$ and $I_3$ are given by

$$\begin{aligned}
I_1 &= -p\sum_{|j-k|\leqslant 2}\int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot\Delta_j(S_{k-1}u\cdot\nabla\Delta_k\theta)\,dx,\\
I_2 &= -p\sum_{|j-k|\leqslant 2}\int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot\Delta_j(\Delta_k u\cdot\nabla S_{k-1}\theta)\,dx,\\
I_3 &= -p\sum_{k\geqslant j-1}\int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot\sum_{|k-l|\leqslant 1}\Delta_j(\Delta_k u\cdot\nabla\Delta_l\theta)\,dx.
\end{aligned}$$

We first bound $I_2$. By Hölder's inequality

$$I_2 \leqslant C\|\Delta_j\theta\|_{L^p}^{p-1}\sum_{|j-k|\leqslant 2}\|\Delta_k u\|_{L^p}\|\nabla S_{k-1}\theta\|_{L^\infty}.$$

Applying Bernstein's inequality, we obtain

$$\begin{aligned}
I_2 &\leqslant C\|\Delta_j\theta\|_{L^p}^{p-1}\sum_{|j-k|\leqslant 2}\|\Delta_k u\|_{L^p}\sum_{m\leqslant k-1}2^m\|\Delta_m\theta\|_{L^\infty}\\
&\leqslant C\|\Delta_j\theta\|_{L^p}^{p-1}\sum_{|j-k|\leqslant 2}\|\Delta_k u\|_{L^p}2^{(1-\delta_1)k}\sum_{m\leqslant k-1}2^{(m-k)(1-\delta_1)}2^{m\delta_1}\|\Delta_m\theta\|_{L^\infty}.
\end{aligned}$$

Thus, for $1-\delta_1>0$, we have

$$I_2 \leqslant C\|\Delta_j\theta\|_{L^p}^{p-1}\|\theta\|_{C^{\delta_1}}\sum_{|j-k|\leqslant 2}\|\Delta_k u\|_{L^p}2^{(1-\delta_1)k}.$$

We now estimate $I_1$. The standard idea is to decompose it into three terms: one with commutator, one that becomes zero due to the divergence-free condition and the rest. That is, we rewrite $I_1$ as

$$\begin{aligned}
I_1 &= -p\sum_{|j-k|\leqslant 2}\int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta\,dx - p\int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot(S_j u\cdot\nabla\Delta_j\theta)\,dx\\
&\quad - p\sum_{|j-k|\leqslant 2}\int |\Delta_j\theta|^{p-2}\Delta_j\theta\cdot(S_{k-1}u - S_j u)\cdot\nabla\Delta_j\Delta_k\theta\,dx\\
&= I_{11}+I_{12}+I_{13},
\end{aligned}$$

where we have used the simple fact that $\sum_{|k-j|\leqslant 2}\Delta_k\Delta_j\theta = \Delta_j\theta$, and the brackets [] represent the commutator, namely

$$[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta = \Delta_j(S_{k-1}u\cdot\nabla\Delta_k\theta) - S_{k-1}u\cdot\nabla\Delta_j\Delta_k\theta.$$

Since $u$ is divergence free, $I_{12}$ becomes zero. $I_{12}$ can also be handled without resort to the divergence-free condition. In fact, integrating by parts in $I_{12}$ yields

$$I_{12} = \int |\Delta_j\theta|^p\nabla\cdot S_j u\,dx \leqslant \|\Delta_j\theta\|_{L^p}^p\|\nabla\cdot S_j u\|_{L^\infty}.$$

By Bernstein's inequality,

$$
\begin{aligned}
|I_{12}| &\leqslant \|\Delta_j\theta\|_{L^p}^p \sum_{m\leqslant j-1} 2^m \|\Delta_m u\|_{L^\infty} \\
&= \|\Delta_j\theta\|_{L^p}^p 2^{(1-\delta_1)j} \sum_{m\leqslant j-1} 2^{(1-\delta_1)(m-j)} 2^{m\delta_1} \|\Delta_m u\|_{L^\infty}.
\end{aligned}
$$

For $1-\delta_1>0$,

$$
|I_{12}| \leqslant C\|\Delta_j\theta\|_{L^p}^p 2^{(1-\delta_1)j}\|u\|_{C^{\delta_1}} \leqslant C\|\Delta_j\theta\|_{L^p}^{p-1} 2^{(1-2\delta_1)j}\|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}}\|u\|_{C^{\delta_1}}.
$$

We now bound $I_{11}$ and $I_{13}$. By Hölder's inequality,

$$
|I_{11}| \leqslant p\|\Delta_j\theta\|_{L^p}^{p-1} \sum_{|j-k|\leqslant 2} \big\|[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta\big\|_{L^p}.
$$

To bound the commutator, we have by the definition of $\Delta_j$

$$
[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta = \int \Phi_j(x-y)\big(S_{k-1}(u)(x)-S_{k-1}(u)(y)\big)\cdot\nabla\Delta_k\theta(y)\,dy.
$$

Using the fact that $\theta\in C^{\delta_1}$ and thus

$$
\big\|S_{k-1}(u)(x)-S_{k-1}(u)(y)\big\|_{L^\infty} \leqslant \|u\|_{C^{\delta_1}}|x-y|^{\delta_1},
$$

we obtain

$$
\big\|[\Delta_j, S_{k-1}u\cdot\nabla]\Delta_k\theta\big\|_{L^p} \leqslant 2^{-\delta_1 j}\|u\|_{C^{\delta_1}} 2^k\|\Delta_k\theta\|_{L^p}.
$$

Therefore,

$$
|I_{11}| \leqslant Cp\|\Delta_j\theta\|_{L^p}^{p-1} 2^{-\delta_1 j}\|u\|_{C^{\delta_1}} \sum_{|j-k|\leqslant 2} 2^k\|\Delta_k\theta\|_{L^p}.
$$

The estimate for $I_{13}$ is straightforward. By Hölder's inequality,

$$
\begin{aligned}
|I_{13}| &\leqslant p\|\Delta_j\theta\|_{L^p}^{p-1} \sum_{|j-k|\leqslant 2} \|S_{k-1}u - S_j u\|_{L^p}\|\nabla\Delta_j\theta\|_{L^\infty} \\
&\leqslant Cp\|\Delta_j\theta\|_{L^p}^{p-1} 2^{(1-\delta_1)j}\|\theta\|_{C^{\delta_1}} \sum_{|j-k|\leqslant 2} \|\Delta_k u\|_{L^p}.
\end{aligned}
$$

We now bound $I_3$. By Hölder's inequality and Bernstein's inequality,

$$
\begin{aligned}
|I_3| &\leqslant p\|\Delta_j\theta\|_{L^p}^{p-1} \bigg\|\Delta_j\nabla\cdot\bigg(\sum_{k\geqslant j-1}\sum_{|l-k|\leqslant 1}\Delta_l u\Delta_k\theta\bigg)\bigg\|_{L^p} \\
&\leqslant p\|\Delta_j\theta\|_{L^p}^{p-1} 2^j\|u\|_{C^{\delta_1}} \sum_{k\geqslant j-1} 2^{-\delta_1 k}\|\Delta_k\theta\|_{L^p}.
\end{aligned} \tag{3.6}
$$

Inserting the estimates for $I_1$, $I_2$ and $I_3$ in (3.5) and eliminating $p\|\Delta_j\theta\|_{L^p}^{p-1}$ from both sides, we get

$$
\begin{aligned}
\frac{d}{dt}\|\Delta_j\theta\|_{L^p} + C\kappa 2^{2\alpha j}\|\Delta_j\theta\|_{L^p} &\leqslant C2^{(1-2\delta_1)j}\|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}}\|u\|_{C^{\delta_1}} + C2^{-\delta_1 j}\|u\|_{C^{\delta_1}} \sum_{|j-k|\leqslant 2} 2^k\|\Delta_k\theta\|_{L^p} \\
&\quad + C\|\theta\|_{C^{\delta_1}} \sum_{|j-k|\leqslant 2} \|\Delta_k u\|_{L^p} 2^{(1-\delta_1)k} + C2^{(1-\delta_1)j}\|\theta\|_{C^{\delta_1}} \sum_{|j-k|\leqslant 2} \|\Delta_k u\|_{L^p} \\
&\quad + C2^j\|u\|_{C^{\delta_1}} \sum_{k\geqslant j-1} 2^{-\delta_1 k}\|\Delta_k\theta\|_{L^p}.
\end{aligned} \tag{3.7}
$$

The terms on the right can be further bounded as follows.

$$C2^{-\delta_1 j}\|u\|_{C^{\delta_1}}\sum_{|j-k|\leqslant 2}2^k\|\Delta_k\theta\|_{L^p}=C2^{(1-2\delta_1)j}\|u\|_{C^{\delta_1}}\sum_{|j-k|\leqslant 2}2^{\delta_1 k}\|\Delta_k\theta\|_{L^p}2^{(k-j)(1-\delta_1)}$$
$$\leqslant C2^{(1-2\delta_1)j}\|u\|_{C^{\delta_1}}\|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}},$$

$$C\|\theta\|_{C^{\delta_1}}\sum_{|j-k|\leqslant 2}\|\Delta_k u\|_{L^p}2^{(1-\delta_1)k}=C2^{(1-2\delta_1)j}\|\theta\|_{C^{\delta_1}}\sum_{|j-k|\leqslant 2}2^{\delta_1 k}\|\Delta_k u\|_{L^p}2^{(k-j)(1-2\delta_1)}$$
$$\leqslant C2^{(1-2\delta_1)j}\|\theta\|_{C^{\delta_1}}\|u\|_{\mathring{B}^{\delta_1}_{p,\infty}},$$

$$C2^{(1-\delta_1)j}\|\theta\|_{C^{\delta_1}}\sum_{|j-k|\leqslant 2}\|\Delta_k u\|_{L^p}=C2^{(1-2\delta_1)j}\|\theta\|_{C^{\delta_1}}\sum_{|j-k|\leqslant 2}2^{\delta_1 k}\|\Delta_k u\|_{L^p}2^{(j-k)\delta_1}$$
$$\leqslant C2^{(1-2\delta_1)j}\|\theta\|_{C^{\delta_1}}\|u\|_{\mathring{B}^{\delta_1}_{p,\infty}}$$

and

$$C2^{j}\|u\|_{C^{\delta_1}}\sum_{k\geqslant j-1}2^{-\delta_1 k}\|\Delta_k\theta\|_{L^p}=C2^{(1-2\delta_1)j}\|u\|_{C^{\delta_1}}\sum_{k\geqslant j-1}2^{-2\delta_1(k-j)}2^{\delta_1 k}\|\Delta_k\theta\|_{L^p}$$
$$\leqslant C2^{(1-2\delta_1)j}\|u\|_{C^{\delta_1}}\|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}}.$$

We can write (3.7) in the following integral form

$$\left\|\Delta_j\theta(t)\right\|_{L^p}\leqslant e^{-C\kappa 2^{2\alpha j}(t-t_0)}\left\|\Delta_j\theta(t_0)\right\|_{L^p}$$
$$+C\int_{t_0}^{t}e^{-C\kappa 2^{2\alpha j}(t-s)}2^{(1-2\delta_1)j}\left(\|\theta\|_{C^{\delta_1}}\|u\|_{\mathring{B}^{\delta_1}_{p,\infty}}+\|u\|_{C^{\delta_1}}\|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}}\right)ds.$$

Multiplying both sides by $2^{(2\alpha+2\delta_1-1)j}$ and taking the supremum with respect to $j$, we get

$$\left\|\theta(t)\right\|_{\mathring{B}^{2\delta_1+2\alpha-1}_{p,\infty}}\leqslant\sup_j\left\{e^{-C\kappa 2^{2\alpha j}(t-t_0)}2^{(\delta_1+2\alpha-1)j}\right\}\left\|\theta(t_0)\right\|_{\mathring{B}^{\delta_1}_{p,\infty}}$$
$$+C\kappa^{-1}\sup_j\left\{\left(1-e^{-C\kappa 2^{2\alpha j}(t-t_0)}\right)\right\}\max_{s\in[t_0,t]}\left\|\theta(s)\right\|_{\mathring{B}^{\delta_1}_{p,\infty}}\left\|\theta(s)\right\|_{C^{\delta_1}}.$$

Here we have used the fact that

$$\|u\|_{C^{\delta_1}}\leqslant\|\theta\|_{C^{\delta_1}}\quad\text{and}\quad\|u\|_{\mathring{B}^{\delta_1}_{p,\infty}}\leqslant\|\theta\|_{\mathring{B}^{\delta_1}_{p,\infty}}.$$

Therefore, we conclude that if

$$\theta\in L^\infty\left([t_0,t];\mathring{B}^{\delta_1}_{p,\infty}\cap C^{\delta_1}\right),$$

then

$$\theta(\cdot,t)\in\mathring{B}^{2\delta_1+2\alpha-1}_{p,\infty}. \tag{3.8}$$

Since $\delta_1>1-2\alpha$, we have $2\delta_1+2\alpha-1>\delta_1$ and thus gain regularity. In addition, according to the Besov embedding of Proposition 2.2,

$$\mathring{B}^{2\delta_1+2\alpha-1}_{p,\infty}\subset\mathring{B}^{\delta_2}_{\infty,\infty},$$

where

$$\delta_2=2\delta_1+2\alpha-1-\frac{2}{p}=\delta_1+\left(\delta_1-\left(1-2\alpha+\frac{2}{p}\right)\right).$$

We have $\delta_2>\delta_1$ when

$$p>p_1\equiv\frac{2}{\delta_1-(1-2\alpha)}.$$

Noting that

$$\mathring{B}^{\delta_2}_{\infty,\infty} \cap L^\infty = C^{\delta_2},$$

we conclude that, for $p > \max\{p_0, p_1\}$,

$$\theta(\cdot, t) \in \mathring{B}^{\delta_2}_{p,\infty} \cap C^{\delta_2}$$

for some $\delta_2 > \delta_1$. The above process can then be iterated with $\delta_1$ replaced by $\delta_2$. A finite number of iterations allow us to obtain that

$$\theta(\cdot, t) \in C^\gamma$$

for some $\gamma > 1$. The regularity in the spatial variable can then be converted into regularity in time. We have thus established that $\theta$ is a classical solution to the supercritical QG equation. Higher regularity can be proved by well-known methods. □

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