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Suboptimal boundary controls for elliptic equation in critically perforated domain

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Abstract

In this paper we study the asymptotic behaviour, as ε tends to zero, of a class of boundary optimal control problems $\mathbb{P}_\varepsilon$, set in ε-periodically perforated domain. The holes have a critical size with respect to ε-sized mesh of periodicity. The support of controls is contained in the set of boundaries of the holes. This set is divided into two parts, on one part the controls are of Dirichlet type; on the other one the controls are of Neumann type. We show that the optimal controls of the homogenized problem can be used as suboptimal ones for the problems $\mathbb{P}_\varepsilon$.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, be a bounded open domain, and let ε be a small positive parameter. To define a perforated domain Ω_ε, we introduce the following sets: $Y = [-1/2, +1/2)^n$; Q and K are compact subsets of Y such that $0 \in \operatorname{int} K \cap \partial Q$,

$$\Theta_\varepsilon = \left\{\mathbf{k} = (k_1, k_2, \dots, k_n) \in \mathbb{Z}^n\colon\ (\varepsilon Y + \varepsilon\mathbf{k}) \cap \Omega \neq \emptyset\right\}; \tag{1.1}$$

$$Y_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \left\{\varepsilon(Y + \mathbf{k})\right\}; \qquad T_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} \{\varepsilon^{n/(n-1)} Q + \varepsilon\mathbf{k}\}; \tag{1.2}$$

$$S_\varepsilon = \begin{cases} \varepsilon^{n/(n-2)} K \cap \partial(\varepsilon^{n/(n-1)} Q), & n \geqslant 3, \\ \exp(-1/\varepsilon^2) K \cap \partial(\varepsilon^{n/(n-1)} Q), & n = 2, \end{cases} \tag{1.3}$$

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$$\Gamma_\varepsilon^D = \left[\bigcup_{\mathbf{k}\in\Theta_\varepsilon} \{S_\varepsilon + \varepsilon\mathbf{k}\} \right] \cap \overline{\Omega}, \qquad \Gamma_\varepsilon^N = [\partial T_\varepsilon \setminus \Gamma_\varepsilon^D] \cap \overline{\Omega}. \tag{1.4}$$

Then we set $\Omega_\varepsilon = \Omega \setminus T_\varepsilon$. The principal feature of the perforated domain Ω_ε is the fact that the size of the holes $Q_\varepsilon + \varepsilon\mathbf{k}$ and their boundaries Γ_ε^D, Γ_ε^N are not proportional to the size of the periodicity cell εY. In Ω_ε we consider the following boundary value problem:

$$\left.\begin{array}{ll} -\Delta y_\varepsilon + y_\varepsilon = f_\varepsilon, & \text{in } \Omega_\varepsilon, \\ \partial_\nu y_\varepsilon = -k_0 y_\varepsilon + p_\varepsilon, & \text{on } \Gamma_\varepsilon^N, \\ y_\varepsilon = u_\varepsilon, & \text{on } \Gamma_\varepsilon^D, \\ y_\varepsilon = 0, & \text{on } \Sigma_\varepsilon = \partial\Omega \cap \partial\Omega_\varepsilon, \end{array}\right\} \tag{1.5}$$

where $f_\varepsilon \in L^2(\Omega)$ is a given function, k_0 is a positive constant, $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative.

In (1.5) u_ε and p_ε are the control functions which act on the system through the set of boundaries of the holes. We say that the control functions u_ε and p_ε are admissible if the following conditions hold:

$$p_\varepsilon \in L^2(\Gamma_\varepsilon^N), \quad u_\varepsilon \in U_\varepsilon = \left\{ a|_{\Gamma_\varepsilon^D} \colon a \in H_0^1(\Omega) \cap H^2(\Omega), \|a\|_{H^2(\Omega)} \leqslant C_0 \right\}. \tag{1.6}$$

Then the optimal control problem $\mathbb{P}_\varepsilon$ can be formulated as follows: Given $z_\varepsilon \in L^2(\Omega)$, $C_0 > 0$, find a triple $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ such that

$$I_\varepsilon(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) = \inf_{(u_\varepsilon, p_\varepsilon, y_\varepsilon)\in\Xi_\varepsilon} I_\varepsilon(u_\varepsilon, p_\varepsilon, y_\varepsilon), \tag{1.7}$$

where the cost functional I_ε and the set of admissible triplets Ξ_ε are defined as

$$I_\varepsilon = \int_{\Omega_\varepsilon} |\nabla y_\varepsilon|^2\, dx + \int_{\Omega_\varepsilon} |y_\varepsilon - z_\varepsilon|^2 dx + \int_{\Gamma_\varepsilon^N} p_\varepsilon^2 d\mathcal{H}^{n-1} + \beta(\varepsilon) \int_{\Gamma_\varepsilon^D} u_\varepsilon^2 d\mathcal{H}^{n-1}, \tag{1.8}$$

$$\Xi_\varepsilon = \left\{ (u_\varepsilon, p_\varepsilon, y_\varepsilon) \in H^1(\Gamma_\varepsilon^D) \times L^2(\Gamma_\varepsilon^N) \times H^1(\Omega_\varepsilon; \Sigma_\varepsilon) \colon (u_\varepsilon, p_\varepsilon, y_\varepsilon) \text{ satisfies (1.5)–(1.6)} \right\}. \tag{1.9}$$

Here $H^1(\Omega_\varepsilon; \Sigma_\varepsilon) = \{y_\varepsilon \in H^1(\Omega_\varepsilon)\colon y_\varepsilon = 0 \text{ on } \Sigma_\varepsilon\}$, $\beta(\varepsilon) = \varepsilon^{n/(2-n)}$ if $n \geqslant 3$, and $\beta(\varepsilon) = \varepsilon^2 \exp(\varepsilon^{-2})$ if $n = 2$.

The asymptotic analysis of the boundary value problems in perforated domain with small holes (without controls) has been widely studied by many authors. We mainly could mention Cioranescu and Donato and Murat and Zuazua [7], Cioranescu and Saint Jean Paulin [10], Cioranescu and Murat [9], Dal Maso and Murat [14], Marchenko and Khruslov [23], Zhikov and Kozlov and Oleinik [29], Scrypnik [27]. It is well known the interesting effect of homogenization of the Poisson equation with (zero) Dirichlet conditions on the boundary of the holes, when a "strange term" appears in the limit equation (see [9,23]). Another effect of homogenization of the same equation with a critical size of the holes, when nonhomogeneous Neumann conditions on the boundary of the holes are assumed, was studied by Conca and Donato [11]. In this case some constant that is proportional to the limit of the total flux of the solution through the boundary of the holes, appears in the limit equation. Cardone and D'Apice and De Maio in [5] and Corbo Esposito and D'Apice and Gaudiello in [12] examined the same equation with mixed boundary conditions on the holes. As proved in [12], in the context of perforated domains with a rather simple geometry of the holes, an interference phenomenon in the homogenization of such boundary value problems is present.

Optimal control problems in perforated domains have been the object of intensive research in the past years [8,18, 24,26]. The numerical computation of such problems is very complicated through thick perforations of Ω_ε. Therefore, the asymptotic analysis is one of the main approaches to the study of optimization problems in perforated domains. The goal of this paper is to obtain an appropriate approximation for the optimal solutions to the problem $\mathbb{P}_\varepsilon$ for small enough values of ε. Using the ideas of the Γ-convergence theory and the concept of the variational convergence of constrained minimization problems (see [2,3,19,20]), we show that the homogenized problem for the original one can be recovered in the following analytical form:

$$\begin{aligned} &\int_\Omega (\nabla y \cdot \nabla\varphi)\, dx + \rho^* \int_\Omega (y - a)\varphi\, d\mu^* + \left(1 + k_0|\partial Q|_H\right) \int_\Omega y\varphi\, dx \\ &\quad = \int_\Omega f\varphi\, dx + |\partial Q|_H \int_\Omega p\varphi\, dx, \quad \forall\varphi \in H_0^1(\Omega) \cap L^2(\Omega, d\mu^*). \end{aligned} \tag{1.10}$$

$$y \in H_0^1(\Omega), \quad p \in L^2(\Omega), \quad a \in H^2(\Omega) \cap H_0^1(\Omega), \tag{1.11}$$

$$y - a \in L^2(\Omega, d\mu^*), \qquad \|a\|_{H^2(\Omega)} \leqslant C_0, \tag{1.12}$$

$$I_0(a, p, y) = \int_\Omega |\nabla y|^2\,dx + \int_\Omega |y - z^\partial|^2\,dx + \rho^* \int_\Omega (y-a)^2\,d\mu^* + |\partial Q|_H \int_\Omega p^2\,dx + |K \cap \partial \Lambda|_H \int_\Omega a^2\,dx \longrightarrow \inf. \tag{1.13}$$

Here the parameters $\rho^*, |\partial Q|_H, |K \cap \partial \Lambda|_H \in \mathbb{R}$ and the Borel measure μ^* are coming from the geometry of control zones. In contrast to $\mathbb{P}_\varepsilon$, the limit control problem (1.10)–(1.13) contains two independent distributed control functions. We show that this problem has a unique optimal solution, derive the corresponding optimality conditions, and establish that the optimal solution for the homogenized problem can be used as a suboptimal control for the original one.

2. Preliminaries and notation

Throughout the paper we suppose that Ω is a measurable set in the sense of Jordan; the small parameter ε varies in a strictly decreasing sequence of positive numbers which converges to 0; Q and K are compact subsets of Y such that $0 \in \operatorname{int} K \cap \partial Q$; the set Q has Lipschitz boundary ∂Q, $\operatorname{int} Q$ is a strongly connected set, $Q \subset \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n\colon x_1 \geqslant 0\}$, and its boundary ∂Q contains the origin; $A = B(\mathbf{0}, r_0)$ is an open ball centered at the origin with a radius $r_0 < 1/2$, so that $A \Subset Y$ and $K \Subset A$ (see Fig. 1 for 2-d example); $C_0 > 0$ is a constant independent of ε; the functions $f_\varepsilon \in L^2(\Omega)$, $z_\varepsilon \in L^2(\Omega)$ are such that $f_\varepsilon \rightharpoonup f$ and $z_\varepsilon \to z^\partial$ in $L^2(\Omega)$ as $\varepsilon \to 0$. For any subset $E \subset \Omega$ we denote by $|E|$ its n-dimensional Lebesgue measure $\mathcal{L}^n(E)$, whereas $|\partial E|_H$ denotes the $(n-1)$-dimensional Hausdorff measure of manifold ∂E on $\mathbb{R}^n$. We suppose that the sets $K \cap \partial Q^\varsigma$ and $\partial Q \setminus (K \cap \partial Q^\varsigma)$ have nonzero capacity for any $\varsigma > 1$, where $Q^\varsigma = \{\varsigma x, \forall x = (x_1, \ldots, x_n) \in Q\}$ is the homothetic stretching of Q by a factor of ς. Hence $|K \cap \partial Q^\varsigma|_H \neq 0$ for all $\varsigma > 1$.

Let $\mathcal{M}_b(\Omega)$ be the space of bounded Borel measures on Ω with values in $[0, +\infty]$. Let $\mathcal{M}_0^+(\Omega)$ be the cone of all nonnegative Borel measures μ on Ω such that $\mu(B) = 0$ for every set $B \subseteq \Omega$ with $\operatorname{cap}(B, \Omega) = 0$, and $\mu(B) = \inf\{\mu(U)\colon U \text{ quasi open}, B \subseteq U\}$ for every Borel set $B \subseteq \Omega$. Note that if $\mu \in \mathcal{M}_0^+(\Omega)$, then the functions of $H^1(\Omega)$ are defined μ-almost everywhere and are μ-measurable in Ω, hence the space $H^1(\Omega) \cap L^2(\Omega, d\mu)$ is well defined.

In view of [21] (see Theorems 1.1 and 2.2, Chapter IV), the following result can be easily proved: *for every fixed ε and for any control functions $u_\varepsilon \in H^1(\Gamma_\varepsilon^D)$ and $p_\varepsilon \in L^2(\Gamma_\varepsilon^N)$ there exists a unique function $y_\varepsilon = y_\varepsilon(u_\varepsilon, p_\varepsilon)$ such that*

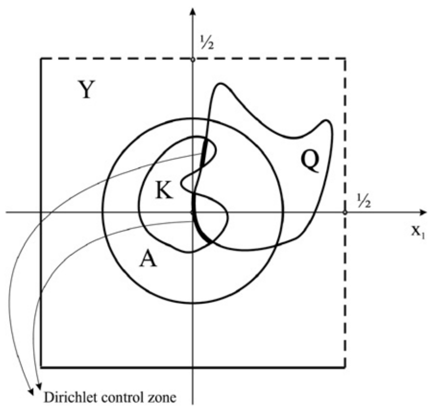

Fig. 1. Example of perforation scheme.

$$
\begin{aligned}
&y_\varepsilon - a_\varepsilon \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon),\\
&\int_{\Omega_\varepsilon} (\nabla y_\varepsilon \cdot \nabla\varphi)\, dx + \int_{\Omega_\varepsilon} y_\varepsilon \varphi\, dx + k_0 \int_{\Gamma_\varepsilon^N} y_\varepsilon \varphi\, d\mathcal{H}^{n-1}\\
&\quad = \int_{\Omega_\varepsilon} f_\varepsilon \varphi\, dx + \int_{\Gamma_\varepsilon^N} p_\varepsilon \varphi\, d\mathcal{H}^{n-1} \quad \forall \varphi \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon),
\end{aligned}
\tag{2.1}
$$

$$
\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \leqslant C\big[\|f_\varepsilon\|_{L^2(\Omega)} + D_1(\varepsilon)\|u_\varepsilon\|_{H^1(\Gamma_\varepsilon^D)} + D_2(\varepsilon)\|p_\varepsilon\|_{L^2(\Gamma_\varepsilon^N)}\big],
\tag{2.2}
$$

where C, $D_1(\varepsilon)$, and $D_2(\varepsilon)$ are some positive constants, C is independent of ε, and the function $a_\varepsilon \in H_0^1(\Omega) \cap H^2(\Omega)$ is such that $\|a_\varepsilon\|_{H^2(\Omega)} \leqslant C_0$ and $a_\varepsilon|_{\Gamma_\varepsilon^D} = u_\varepsilon$. In the sequel we call the function y_ε weak solution to the problem (1.5) and identify y_ε with its quasi continuous representative [14].

Let $\{(u_\varepsilon^n, p_\varepsilon^n, y_\varepsilon^n)\} \subset \Xi_\varepsilon$ be a minimizing sequence for the problem $\mathbb{P}_\varepsilon$. Then, using the compact embedding result $H^{3/2}(\Gamma_\varepsilon^D) \hookrightarrow H^1(\Gamma_\varepsilon^D)$ which implies $u_\varepsilon^n \to u_\varepsilon^0 = a_\varepsilon^0|_{\Gamma_\varepsilon^D}$ strongly in $H^1(\Gamma_\varepsilon^D)$ and the direct method of Calculus of variation, we come to the following conclusion (for details see [16], [13]):

Theorem 2.1. *For every ε there exists a unique solution $(u_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \Xi_\varepsilon$ of the optimal control problem $\mathbb{P}_\varepsilon$.*

3. On formulation of the homogenization problem

We begin this section with the description of the geometry of the perforated domain Ω_ε. We describe the class of admissible solutions to problem $\mathbb{P}_\varepsilon$ in the terms of singular periodic Borel measures on $\mathbb{R}^n$. To do so, we use the approach of Zhikov, Bouchitté and Fragala (see [1,28]).

Let us denote by K^λ and Q^h the homothetic contractions of the sets K and Q at λ^{-1} and h^{-1} times. In what follows it is assumed that $0 < \lambda \ll h < 1$. Let the sets $\Gamma^{\lambda,h}$ and $\Lambda^{\lambda,h}$ be defined as follows:

$$
\Gamma^{\lambda,h} = K^\lambda \cap \partial Q^h, \qquad \Lambda^{\lambda,h} = \partial Q^h \setminus \Gamma^{\lambda,h}.
\tag{3.1}
$$

Let $\mu^{\lambda,h}$ and $\nu^{\lambda,h}$ be the normalized periodic Borel measures on $\mathbb{R}^n$ with the periodicity cell Y such that $\mu^{\lambda,h}$ is concentrated on $\Gamma^{\lambda,h}$, $\nu^{\lambda,h}$ is concentrated on $\Lambda^{\lambda,h}$, and both these measures are proportional to the $(n-1)$-dimensional Hausdorff measure. Since these measures are concentrated and uniformly distributed on the corresponding sets, it follows that $\mu^{\lambda,h}(Y \setminus \Gamma^{\lambda,h}) = 0$.

For any function $\varphi \in C^\infty(\mathbb{R}^n)$ we have

$$
\int_{\Gamma^{\lambda,h}} \varphi\, d\mathcal{H}^{n-1} = |\Gamma^{\lambda,h}|_H \int_Y \varphi\, d\mu^{\lambda,h} = \lambda^{n-1}|K \cap \partial Q^{h/\lambda}|_H \int_Y \varphi\, d\mu^{\lambda,h},
\tag{3.2}
$$

$$
\begin{aligned}
\int_{\Lambda^{\lambda,h}} \varphi\, d\mathcal{H}^{n-1} &= |\Lambda^{\lambda,h}|_H \int_Y \varphi\, d\nu^{\lambda,h} = \big(|\partial Q^h|_H - |\Gamma^{\lambda,h}|_H\big) \int_Y \varphi\, d\nu^{\lambda,h}\\
&= \big(h^{n-1}|\partial Q|_H - \lambda^{n-1}|K \cap \partial Q^{h/\lambda}|_H\big) \int_Y \varphi\, d\nu^{\lambda,h}.
\end{aligned}
\tag{3.3}
$$

We introduce also the scaling measures $\mu_\varepsilon^{\lambda,h}$ and $\nu_\varepsilon^{\lambda,h}$ by setting $\mu_\varepsilon^{\lambda,h}(B) = \varepsilon^n \mu^{\lambda,h}(\varepsilon^{-1}B)$, $\nu_\varepsilon^{\lambda,h}(B) = \varepsilon^n \nu^{\lambda,h}(\varepsilon^{-1}B)$ for every Borel set $B \subset \mathbb{R}^n$, and relate the parameters λ, h, and ε by the rule

$$
h(\varepsilon) = \varepsilon^{n/(n-1)}, \quad \lambda(\varepsilon) = \varepsilon^{n/(n-2)} \quad \text{if } n \geqslant 3, \quad \text{and} \quad \lambda(\varepsilon) = \exp(-\varepsilon^{-2}) \quad \text{if } n = 2.
\tag{3.4}
$$

Then $\int_{\varepsilon Y} d\mu_\varepsilon^{\lambda,h} = \varepsilon^n \int_Y d\mu^{\lambda,h} = \varepsilon^n$, $\int_{\varepsilon Y} d\nu_\varepsilon^{\lambda,h} = \varepsilon^n \int_Y d\nu^{\lambda,h} = \varepsilon^n$. It means that the measures $\mu_\varepsilon^{\lambda,h}$ and $\nu_\varepsilon^{\lambda,h}$ weakly converge to the Lebesgue measure: $d\mu_\varepsilon^{\lambda,h} \rightharpoonup dx$, $d\nu_\varepsilon^{\lambda,h} \rightharpoonup dx$, that is for every $\varphi \in C_0^\infty(\mathbb{R}^n)$ we have

$$
\lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \varphi\, d\mu_\varepsilon^{\lambda,h} = \int_{\mathbb{R}^n} \varphi\, dx, \qquad \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n} \varphi\, d\nu_\varepsilon^{\lambda,h} = \int_{\mathbb{R}^n} \varphi\, dx.
\tag{3.5}
$$

Remark 3.1. It is easy to see that the scaling measures $\mu_\varepsilon^{\lambda,h}$ and $\nu_\varepsilon^{\lambda,h}$ belong to the class $\mathcal{M}_0^+(\Omega)$. Hence the spaces $H_0^1(\Omega)\cap H^2(\Omega)\cap L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$ and $H^1(\Omega;\Sigma_\varepsilon)\cap L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$ are well defined [14].

Now we turn back to the definition of the set of admissible solutions of the problem $\mathbb{P}_\varepsilon$ (see (1.9)). We see that

$$\Gamma_\varepsilon^D = \bigcup_{\mathbf{k}\in\Theta_\varepsilon}\big[K^{\lambda(\varepsilon)}\cap\partial Q^{h(\varepsilon)}+\varepsilon\mathbf{k}\big], \qquad \Gamma_\varepsilon^N = \bigcup_{\mathbf{k}\in\Theta_\varepsilon}\big[\partial Q^{h(\varepsilon)}\setminus\big(K^{\lambda(\varepsilon)}\cap\partial Q^{h(\varepsilon)}\big)+\varepsilon\mathbf{k}\big].$$

Then, using properties (3.2)–(3.3) and setting

$$\sigma(\varepsilon)=\big(h(\varepsilon)^{n-1}|\partial Q|_H-\lambda(\varepsilon)^{n-1}|K\cap\partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H\big), \tag{3.6}$$

the term $\int_{\Gamma_\varepsilon^N} p_\varepsilon\varphi\,d\mathcal{H}^{n-1}$ of the integral identity (2.1) can be rewritten in the form

$$\begin{aligned}\int\limits_{\Gamma_\varepsilon^N} p_\varepsilon\varphi\,d\mathcal{H}^{n-1} &= \sum_{\mathbf{k}\in\Theta_\varepsilon}\ \int\limits_{\partial Q^{h(\varepsilon)}\setminus\Gamma^{\lambda(\varepsilon),h(\varepsilon)}+\varepsilon\mathbf{k}} p_\varepsilon\varphi\,d\mathcal{H}^{n-1}\\ &=\sigma(\varepsilon)\sum_{\mathbf{k}\in\Theta_\varepsilon}\ \int\limits_{\varepsilon(Y+\mathbf{k})}\hat p_\varepsilon\varphi\,d\nu^{\lambda(\varepsilon),h(\varepsilon)}(x/\varepsilon)=\varepsilon^{-n}\sigma(\varepsilon)\int\limits_\Omega \hat p_\varepsilon\varphi\,d\nu_\varepsilon^{\lambda,h}\end{aligned} \tag{3.7}$$

for every function $\varphi\in C_0^\infty(\mathbb{R}^n;\Sigma_\varepsilon\cup\Gamma_\varepsilon^D)=\{\psi\in C_0^\infty(\mathbb{R}^n)\colon\ \psi=0 \text{ on } \Sigma_\varepsilon\cup\Gamma_\varepsilon^D\}$.

Here $\hat p_\varepsilon$ is a function of $L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$ taking the same values as $p_\varepsilon\in L^2(\Gamma_\varepsilon^N)$ on Γ_ε^N. It is clear that for every boundary control $p_\varepsilon\in L^2(\Gamma_\varepsilon^N)$, one can find a function $\hat p_\varepsilon\in L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$ such that $p_\varepsilon=\hat p_\varepsilon$ on Γ_ε^N. Hence

$$\int\limits_{\Gamma_\varepsilon^N} p_\varepsilon^2\,d\mathcal{H}^{n-1}=\varepsilon^{-n}\sigma(\varepsilon)\int\limits_\Omega \hat p_\varepsilon^2\,d\nu_\varepsilon^{\lambda,h}, \tag{3.8}$$

where

$$\varepsilon^{-n}\sigma(\varepsilon)=\begin{cases}|\partial Q|_H-\varepsilon^{n/(n-2)}|K\cap\partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H, & \text{if } n\geqslant 3,\\ |\partial Q|_H-\frac{1}{\varepsilon^2}\exp(-\frac{1}{\varepsilon^2})|K\cap\partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H, & \text{if } n=2.\end{cases} \tag{3.9}$$

By analogy we obtain

$$k_0\int\limits_{\Gamma_\varepsilon^N} y_\varepsilon\varphi\,d\mathcal{H}^{n-1}=\varepsilon^{-n}\sigma(\varepsilon)k_0\int\limits_\Omega \breve y_\varepsilon\varphi\,d\nu_\varepsilon^{\lambda,h}, \tag{3.10}$$

$$\int\limits_{\Gamma_\varepsilon^D} u_\varepsilon^2\,d\mathcal{H}^{n-1}=\varepsilon^{-n}\lambda(\varepsilon)^{n-1}|K\cap\partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H\int\limits_\Omega a_\varepsilon^2\,d\mu_\varepsilon^{\lambda,h}. \tag{3.11}$$

Here $\breve y_\varepsilon\in H^1(\Omega;\Sigma_\varepsilon)$ is an extension of the weak solution y_ε to the problem (1.5) to the whole of domain Ω, and the function $a_\varepsilon\in H_0^1(\Omega)\cap H^2(\Omega)\cap L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$ is a prototype of the Dirichlet control $u_\varepsilon\in U_\varepsilon$ (see (1.6)).

Remark 3.2. In view of our initial suppositions, the measure $\nu_\varepsilon^{\lambda,h}$ is supported on the set with nonzero capacity for every $\varepsilon>0$. Since every element v of the space $H^1(\Omega;\Sigma_\varepsilon)\cap L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$ can be interpreted as a quasi continuous function, it is reasonable to suppose that for every element $v\in H^1(\Omega)$, one can find a sequence $\{v_k\in C(\overline{\Omega})\}_{k\in\mathbb{N}}$ such that

$$\sup_{k\in\mathbb{N}}\limsup_{\varepsilon\to 0}\int\limits_{\mathcal{V}_k}(v-v_k)^2\,d\nu_\varepsilon^{\lambda,h}<+\infty \quad\text{and}\quad \lim_{k\to\infty}\operatorname{cap}(\mathcal{V}_k,\Omega)=0,$$

where $\mathcal{V}_k=\{x\in\Omega\colon\ v\neq v_k\}$. We assume that the same property is valid for the elements of the space $H_0^1(\Omega)\cap L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$.

As a result, we can reformulate the original optimal control problem $\mathbb{P}_\varepsilon$ (1.7)–(1.9) as follows: Find some $(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \mathbb{X}_\varepsilon$ such that

$$\hat{I}_\varepsilon(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) = \inf_{(a_\varepsilon, p_\varepsilon, y_\varepsilon)\in\hat{\Xi}_\varepsilon} \hat{I}_\varepsilon(a_\varepsilon, p_\varepsilon, y_\varepsilon), \tag{3.12}$$

$$\mathbb{X}_\varepsilon = \big[H_0^1(\Omega)\cap H^2(\Omega)\cap L^2(\Omega, d\mu_\varepsilon^{\lambda,h})\big] \times L^2(\Omega, d\nu_\varepsilon^{\lambda,h}) \times \big[H^1(\Omega;\Sigma_\varepsilon)\cap L^2(\Omega, d\nu_\varepsilon^{\lambda,h})\big],$$

where

$$\hat{I}_\varepsilon(a_\varepsilon, p_\varepsilon, y_\varepsilon) = \int_\Omega \chi_\varepsilon|\nabla\breve{y}_\varepsilon|^2\,dx + \int_\Omega |\chi_\varepsilon\breve{y}_\varepsilon - z_\varepsilon^\partial|^2\,dx + \varepsilon^{-n}\sigma(\varepsilon)\int_\Omega p_\varepsilon^2\,d\nu_\varepsilon^{\lambda,h} + |K\cap\partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H \int_\Omega a_\varepsilon^2\,d\mu_\varepsilon^{\lambda,h}, \tag{3.13}$$

$$\hat{\Xi}_\varepsilon = \left\{ (a_\varepsilon, p_\varepsilon, y_\varepsilon) \;\middle|\; \begin{array}{l} \breve{y}_\varepsilon - a_\varepsilon \in H^1(\Omega;\Gamma_\varepsilon^D\cup\Sigma_\varepsilon),\ \|a\|_{H^2(\Omega)} \leqslant C_0, \\ \int_\Omega \chi_\varepsilon(\nabla\breve{y}_\varepsilon\cdot\nabla\varphi)\,dx + \int_\Omega \chi_\varepsilon\breve{y}_\varepsilon\varphi\,dx \\ \quad + k_0\varepsilon^{-n}\sigma(\varepsilon)\int_\Omega \breve{y}_\varepsilon\varphi d\nu_\varepsilon^{\lambda,h} - \int_\Omega \chi_\varepsilon f_\varepsilon\varphi\,dx \\ \quad = \varepsilon^{-n}\sigma(\varepsilon)\int_\Omega p_\varepsilon\varphi\,d\nu_\varepsilon^{\lambda,h},\ \forall\varphi\in H^1(\Omega;\Gamma_\varepsilon^D\cup\Sigma_\varepsilon). \end{array} \right\} \tag{3.14}$$

We denote with $\hat{\mathbb{P}}_\varepsilon$ the optimal control problem (3.12)–(3.14). It is clear that $\hat{\mathbb{P}}_\varepsilon$ has a unique solution $(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)$ for every ε [16,22]. This solution can be viewed as a prototype of the optimal triplet to $\mathbb{P}_\varepsilon$-problem. Moreover, in this case a priori norm estimate (2.2) takes the form

$$\|\breve{y}_\varepsilon\|_{H^1(\Omega,\chi_\varepsilon dx)} + \sqrt{\varepsilon^{-n}\sigma(\varepsilon)}\|y_\varepsilon\|_{L^2(\Omega,d\nu_\varepsilon^{\lambda,h})} \leqslant \hat{C}\Big[\sqrt{\varepsilon^{-n}\sigma(\varepsilon)}\|p_\varepsilon\|_{L^2(\Omega,d\nu_\varepsilon^{\lambda,h})} + \|f_\varepsilon\|_{L^2(\Omega)} + \sqrt{|K\cap\partial Q^{h(\varepsilon)/\lambda(\varepsilon)}|_H}\,\|a_\varepsilon\|_{H_0^1(\Omega)\cap L^2(\Omega,d\mu_\varepsilon^{\lambda,h})}\Big]. \tag{3.15}$$

To end this section, we list some auxiliary results that will be useful in the sequel. Let

$$\varsigma(\varepsilon) = h(\varepsilon)/\lambda(\varepsilon) = \left\{ \begin{array}{ll} \exp(\frac{-n}{n^2-3n+2}\ln\varepsilon) & \text{if } n \geqslant 3, \\ \varepsilon^2\exp(\frac{1}{\varepsilon^2}) & \text{if } n = 2. \end{array} \right\} \tag{3.16}$$

Then $\varsigma(\varepsilon) \in (1, +\infty)$ $\forall\varepsilon$ and $\lim_{\varepsilon\to 0}\varsigma(\varepsilon) = +\infty$. We are interested in the limit behaviour of the sequence $\{|K\cap\partial Q^{\varsigma(\varepsilon)}|_H\}$ as $\varepsilon\to 0$. We recall that the set $Q^{\varsigma(\varepsilon)} = \{\varsigma(\varepsilon)x, \forall x = (x_1,\dots,x_n)\in Q\}$ is the homothetic stretching of Q by a factor of $\varsigma(\varepsilon)$.

Proposition 3.3. *There exists an open cone $\Lambda \subset \{x\in\mathbb{R}^n\colon x_1 > 0\}$ such that*

$$\lim_{\varepsilon\to 0}\big|K\cap\partial Q^{\varsigma(\varepsilon)}\big|_H = |K\cap\partial\Lambda|_H. \tag{3.17}$$

Proof. Indeed, by the initial assumptions, the origin is a Lipschitz point of the boundary ∂Q and $\operatorname{int} Q$ is a strongly connected set in the classical sense. Hence, there is a neighbourhood $\mathcal{U}(0)$ such that $\mathcal{U}(0)\cap\operatorname{int} Q$ is a convex set [4,15]. Then $\Lambda = \{x\in t[\mathcal{U}(0)\cap\operatorname{int} Q]\ \forall t\in(0,+\infty)\}$ is a nonempty open cone.

Assume that the origin does not belong to a smooth part of the boundary ∂Q. Then the inclusion $K\cap\Lambda\subset K\cap Q^{\varsigma(\varepsilon)}$ holds true for ε small enough, and it immediately implies the existence of a value $\varepsilon_0 > 0$ such that $|K\cap\partial\Lambda|_H = |K\cap\partial Q^{\varsigma}(\varepsilon)|_H$ $\forall\varepsilon < \varepsilon_0$. If a part of the boundary ∂Q containing the origin is smooth, then it follows that there is a neighbourhood $\mathcal{U}(0)$ of the origin such that $\mathcal{U}(0)\cap\partial Q$ is the graph of a smooth function whose epigraph contains $\mathcal{U}(0)\cap Q$. So that, we may always suppose that there is a function $\Psi\colon\mathbb{R}^{n-1}\to\mathbb{R}_{\geqslant}$ satisfying $\Psi\in C_0^\infty(\mathbb{R}^{n-1})$ and $x_1 = \Psi(x_2,\dots,x_n)$ for every $x = (x_1,x_2,\dots,x_n)\in\mathcal{U}(0)\cap\partial Q$.

Let $\Lambda = \{x\in\mathbb{R}^n\colon x_1 > 0\}$. Then $\partial\Lambda = \{x\in\mathbb{R}^n\colon x_1 = 0\}$ and we deduce: $x = (x_1,x_2,\dots,x_n)\in K\cap\varsigma(\varepsilon)\partial Q$ for small enough ε if and only if $x\in\varsigma(\varepsilon)(\mathcal{U}(0)\cap\partial Q)$. Hence

$$x_1 = \frac{1}{\varsigma(\varepsilon)}\Psi\big(\varsigma(\varepsilon)x_2,\dots,\varsigma(\varepsilon)x_n\big) \quad \text{for every } x\in K\cap\varsigma(\varepsilon)\partial Q.$$

Since $\lim_{\varepsilon\to 0} \varsigma^{-1}(\varepsilon)\Psi(\varsigma(\varepsilon)x_2,\dots,\varsigma(\varepsilon)x_n)=0$ by the definition of the Hausdorff measure $\mathcal{H}^{n-1}$, we immediately obtain the required result. $\square$

Remark 3.4. As follows from Proposition 3.3 and its proof, the cone Λ can be recovered in an explicit form in the case when the origin belongs to a smooth part of the boundary ∂Q. Moreover, in view of (3.6) we have $\lim_{\varepsilon\to 0}\sigma(\varepsilon)/\varepsilon^n = |\partial Q|_H$.

In a similar way the following statement can be proved:

Proposition 3.5. *Let $\{\rho'_\varepsilon \in \mathbb{R}\}_{\varepsilon>0}$ be a sequence of numbers such that*

$$\rho'_\varepsilon = |A \setminus Q^{\varsigma(\varepsilon)}|/|A| \quad \forall\varepsilon > 0. \tag{3.18}$$

Then the sequence $\{\rho'_\varepsilon\}_{\varepsilon>0}$ is monotone and there exists a value $\rho^* \in [1/2, 1)$ such that $\lim_{\varepsilon\to 0}\rho'_\varepsilon = \rho^*$.

4. Convergence in the variable space $\mathbb{X}_\varepsilon$

Let us recall the main types of convergence in variable spaces occurring in the homogenization theory (see [28]). We cite them with respect to the family of the periodic Borel measure $\mu_\varepsilon^{\lambda,h}$. Here the parameters $\lambda=\lambda(\varepsilon)$ and $h=h(\varepsilon)$ are defined by (3.4). Let $\{v_\varepsilon^{\lambda,h}\in L^2(\Omega, d\mu_\varepsilon^{\lambda,h})\}$ be a bounded sequence, i.e. $\limsup_{\varepsilon\to 0}\int_\Omega (v_\varepsilon^{\lambda,h})^2\, d\mu_\varepsilon^{\lambda,h} < +\infty$.

1. The weak convergence $v_\varepsilon^{\lambda,h} \rightharpoonup v$ in $L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$ means that $v\in L^2(\Omega)$ and $\lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon^{\lambda,h}\varphi\, d\mu_\varepsilon^{\lambda,h} = \int_\Omega v\varphi\, dx$ for any $\varphi\in C_0^\infty(\Omega)$;
2. The strong convergence $v_\varepsilon^{\lambda,h}\to v$ in $L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$ means that $v\in L^2(\Omega)$ and $\lim_{\varepsilon\to 0}\int_\Omega v_\varepsilon^{\lambda,h} z_\varepsilon^{\lambda,h}\, d\mu_\varepsilon^{\lambda,h} = \int_\Omega vz\, dx$ if $z_\varepsilon^{\lambda,h}\rightharpoonup z$ in $L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$.

The following properties of the convergence in variable spaces hold:

(a) *Compactness criterium*: if a sequence is bounded in $L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$, then this sequence is compact in the sense of the weak convergence;
(b) *Property of lower semicontinuity*: if $v_\varepsilon^{\lambda,h}\rightharpoonup v$ in $L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$, then

$$\liminf_{\varepsilon\to 0}\int_\Omega (v_\varepsilon^{\lambda,h})^2\, d\mu_\varepsilon^{\lambda,h} \geqslant \int_\Omega v^2\, dx;$$

(c) *Criterium of strong convergence*: $v_\varepsilon^{\lambda,h}\to v$ if and only if $v_\varepsilon^{\lambda,h}\rightharpoonup v$ and $\lim_{\varepsilon\to 0}\int_\Omega (v_\varepsilon^{\lambda,h})^2 d\mu_\varepsilon^{\lambda,h} = \int_\Omega v^2\, dx$;
(d) Since $\mu_\varepsilon^{\lambda,h}\rightharpoonup dx$, it follows that $\lim_{\varepsilon\to 0}\int_\Omega \varphi\, d\mu_\varepsilon^{\lambda,h} = \int_\Omega\varphi\, dx$ $\forall\varphi\in C(\overline{\Omega})$ and $\limsup_{\varepsilon\to 0}\mu_\varepsilon^{\lambda,h}(F)\leqslant |F|$ for any compact set $F\subset\Omega$.

We begin with the following concept:

Definition 4.1. Let $\{v_\varepsilon^{\lambda,h}\in H_0^1(\Omega)\cap L^2(\Omega, d\mu_\varepsilon^{\lambda,h})\}$ be a bounded sequence. We say that this sequence converges weakly in $H_0^1(\Omega)\cap L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$ to $v\in H^1(\Omega)$ if

$$v_\varepsilon^{\lambda,h}\rightharpoonup v \quad \text{in } H_0^1(\Omega) \quad \text{and} \quad v_\varepsilon^{\lambda,h}\rightharpoonup v \quad \text{in } L^2(\Omega, d\mu_\varepsilon^{\lambda,h}). \tag{4.1}$$

In order to check the correctness of this definition we make use of the following auxiliary statements:

Lemma 4.2. *If $v\in H_0^1(\Omega)$, then $\lim_{\varepsilon\to 0}\int_\Omega v\varphi\, d\mu_\varepsilon^{\lambda,h} = \int_\Omega v\varphi\, dx$ $\forall\varphi\in C_0^\infty(\Omega)$.*

Proof. If $v\in C_0(\Omega)$, then the weak convergence $\mu_\varepsilon^{\lambda,h}\rightharpoonup dx$ of the measures immediately implies this relation. Let v be an arbitrary element of $H_0^1(\Omega)$. Then for every $\delta>0$ there exist a set $\mathcal{A}^\delta\subset\Omega$ and a function $v^\delta\in C_0(\Omega)$ such that $\mathrm{cap}(\mathcal{A}^\delta,\Omega)<\delta$, and $v^\delta=v$ on $\Omega\setminus\mathcal{A}^\delta$. In what follows, we suppose that the set $\mathcal{A}^\delta$ is closed. We now consider the following estimate:

$$\left|\int_\Omega v\varphi\,d\mu_\varepsilon^{\lambda,h}-\int_\Omega v\varphi\,dx\right|\leqslant\left|\int_\Omega (v-v^\delta)\varphi\,d\mu_\varepsilon^{\lambda,h}\right|$$
$$+\left|\int_\Omega v^\delta\varphi\,d\mu_\varepsilon^{\lambda,h}-\int_\Omega v^\delta\varphi\,dx\right|+\left|\int_\Omega (v-v^\delta)\varphi\,dx\right|=J_1+J_2+J_3.$$

Owing to the weak convergence $\mu_\varepsilon^{\lambda,h}\rightharpoonup dx$, we have $J_2\to 0$ as $\varepsilon\to 0$. By Lusin's Theorem we may suppose that there is a constant $d_1>0$ such that $\|v-v^\delta\|_{L^2(\Omega)}\leqslant d_1\delta$. Hence $J_3\leqslant d_1\|\varphi\|_{L^2(\Omega)}\delta$. As for the value J_1, we note that each of the measure $\mu_\varepsilon^{\lambda,h}$ is supported on a set with nonzero capacity. So, there is a constant $d_2>0$ such that $\|v-v^\delta\|_{L^2(\Omega,d\mu_\varepsilon^{\lambda,h})}\leqslant d_2$ for any $\delta>0$ small enough (see Remark 3.2). It implies the following estimate:

$$J_3=\left|\int_\Omega (v-v^\delta)\varphi\,d\mu_\varepsilon^{\lambda,h}\right|\leqslant d_2\|\varphi\|_{C(\Omega)}\left[\mu_\varepsilon^{\lambda,h}(\mathcal{A}^\delta)\right]^{1/2}.$$

As a result, we have:

(i) $\limsup_{\varepsilon\to 0}\mu_\varepsilon^{\lambda,h}(\mathcal{A}^\delta)\leqslant|\mathcal{A}^\delta|$ by property (d); (ii) $|\mathcal{A}^\delta|\leqslant d_3\mathrm{cap}^{n/(n-2)}(\mathcal{A}^\delta)$ by the properties of capacity (see [15]); (iii) $\mathrm{cap}(\mathcal{A}^\delta)<\delta$ by the initial assumption. Hence, summing up all estimates that were obtained before, we conclude $|\int_\Omega v\varphi\,d\mu_\varepsilon^{\lambda,h}-\int_\Omega v\varphi\,dx|\leqslant d\delta$ for any $\delta>0$ small enough (here the constant d does not depend on δ). This completes the proof. □

We now consider a more delicate situation.

Lemma 4.3. *Let* $\{v_\varepsilon^{\lambda,h}\in H_0^1(\Omega)\cap L^2(\Omega,d\mu_\varepsilon^{\lambda,h})\}$ *and* $v\in L^2(\Omega)$ *be such that* $v_\varepsilon^{\lambda,h}\rightharpoonup v$ *in* $H_0^1(\Omega)$ *and hence* $v_\varepsilon^{\lambda,h}\to v$ *in* $L^2(\Omega)$. *Then*

$$\lim_{\varepsilon\to 0}\left[\int_\Omega v_\varepsilon^{\lambda,h}\,d\mu_\varepsilon^{\lambda,h}-\int_\Omega v_\varepsilon^{\lambda,h}\,dx\right]=0. \tag{4.2}$$

Proof. As in the previous lemma, we introduce two functions $\tilde v_\varepsilon^{\lambda,h}\in C(\Omega)$ and $\tilde v\in C(\Omega)$ such that $\tilde v_\varepsilon^{\lambda,h}=v_\varepsilon^{\lambda,h}$ and $\tilde v=v$ quasi everywhere. Let us partition the set Ω into cubes εY with edges ε and denote these cubes with εY^j. Then there are points $x_j^{\lambda,h}\in\varepsilon Y^j$ such that

$$\int_\Omega \tilde v_\varepsilon^{\lambda,h}\,d\mu_\varepsilon^{\lambda,h}=\sum\int_{\varepsilon Y^j}\tilde v_\varepsilon^{\lambda,h}(x)\,d\mu_\varepsilon^{\lambda,h}+\sum\int_{\Omega\cap\varepsilon Y^j}\tilde v_\varepsilon^{\lambda,h}(x)\,d\mu_\varepsilon^{\lambda,h}$$
$$=\sum\tilde v_\varepsilon^{\lambda,h}(x_j^{\lambda,h})\int_{\varepsilon Y^j}d\mu_\varepsilon^{\lambda,h}+\sum\int_{\Omega\cap\varepsilon Y^j}\tilde v_\varepsilon^{\lambda,h}(x)\,d\mu_\varepsilon^{\lambda,h},$$

where the second sum is calculated over the set of the "boundary" cubes. By the definition of the measure $\mu_\varepsilon^{\lambda,h}$, we have $\int_{\varepsilon Y^j}d\mu_\varepsilon^{\lambda,h}=\varepsilon^n\int_Y d\mu_\varepsilon^{\lambda,h}=\varepsilon^n$. Hence

$$\int_\Omega \tilde v_\varepsilon^{\lambda,h}\,d\mu_\varepsilon^{\lambda,h}=\sum\tilde v_\varepsilon^{\lambda,h}(x_j^{\lambda,h})\varepsilon^n+\sum\int_{\Omega\cap\varepsilon Y^j}\tilde v_\varepsilon^{\lambda,h}(x)\,d\mu_\varepsilon^{\lambda,h}. \tag{4.3}$$

It is clear that an analogous representation takes place for the second term in (4.2), namely,

$$\int_\Omega \tilde v_\varepsilon^{\lambda,h}\,dx=\sum\tilde v_\varepsilon^{\lambda,h}(x_j)\int_{\varepsilon Y^j}dx+\sum\int_{\Omega\cap\varepsilon Y^j}\tilde v_\varepsilon^{\lambda,h}(x)\,dx$$
$$=\sum\tilde v_\varepsilon^{\lambda,h}(x_j)\varepsilon^n+\sum\int_{\Omega\cap\varepsilon Y^j}\tilde v_\varepsilon^{\lambda,h}(x)\,dx, \tag{4.4}$$

for some $x_j \in \varepsilon Y^j$. Note that

$$\left|\sum \int_{\Omega \cap \varepsilon Y^j} \tilde{v}_\varepsilon^{\lambda,h}(x)\, d\mu_\varepsilon^{\lambda,h}\right| \leqslant \sup_{j \in D(\varepsilon)} \Big(\sup_{x \in \Omega \cap \varepsilon Y^j} \big|\tilde{v}_\varepsilon^{\lambda,h}(x)\big|\Big) \varepsilon^n \cdot D(\varepsilon),$$

$$\left|\sum \int_{\Omega \cap \varepsilon Y^j} \tilde{v}(x)\, dx\right| \leqslant \sup_{j \in D(\varepsilon)} \Big(\sup_{x \in \Omega \cap \varepsilon Y^j} \big|\tilde{v}(x)\big|\Big) \varepsilon^n \cdot D(\varepsilon)$$

where $D(\varepsilon)$ is the quantity of the "boundary" cubes, and $\varepsilon^n D(\varepsilon) \to 0$ by Jordan's measurability property of the set $\partial\Omega$. Moreover, since $v_\varepsilon^{\lambda,h}, v \in H_0^1(\Omega)$, it follows that

$$\sup_{j \in D(\varepsilon)} \Big(\sup_{x \in \Omega \cap \varepsilon Y^j} \big|\tilde{v}(x)\big|\Big) < +\infty \quad \text{and} \quad \sup_{j \in D(\varepsilon)} \Big(\sup_{x \in \Omega \cap \varepsilon Y^j} \big|\tilde{v}_\varepsilon^{\lambda,h}(x)\big|\Big) < +\infty$$

for ε small enough.

Then, substituting (4.3) and (4.4) in (4.2), we come to the following relation:

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \left(\int_\Omega v_\varepsilon^{\lambda,h}\, d\mu_\varepsilon^{\lambda,h} - \int_\Omega v_\varepsilon^{\lambda,h}\, dx \right) \leqslant \lim_{\varepsilon \to 0} \sum \big(\tilde{v}_\varepsilon^{\lambda,h}(x_j^{\lambda,h}) - \tilde{v}_\varepsilon^{\lambda,h}(x_j)\big)\varepsilon^n \\
&\quad + \max\Big[\sup_{j \in D(\varepsilon)} \Big(\sup_{x \in \Omega \cap \varepsilon Y^j} \big|\tilde{v}_\varepsilon^{\lambda,h}(x)\big|\Big), \sup_{j \in D(\varepsilon)} \Big(\sup_{x \in \Omega \cap \varepsilon Y^j} \big|\tilde{v}(x)\big|\Big)\Big] \limsup_{\varepsilon \to 0} \big(\varepsilon^n \cdot D(\varepsilon)\big) \\
&\quad + \lim_{\varepsilon \to 0} (J_1 + J_2) = \lim_{\varepsilon \to 0} \Big(\sum \big(\tilde{v}_\varepsilon^{\lambda,h}(x_j^{\lambda,h}) - \tilde{v}_\varepsilon^{\lambda,h}(x_j)\big)\varepsilon^n \Big),
\end{aligned} \tag{4.5}$$

where $J_1 = \int_\Omega (v_\varepsilon^{\lambda,h} - \tilde{v}_\varepsilon^{\lambda,h})\, d\mu_\varepsilon^{\lambda,h}$, $J_2 = \int_\Omega (v_\varepsilon^{\lambda,h} - \tilde{v}_\varepsilon^{\lambda,h})\, dx$ and by the arguments of the previous lemma and Remark 3.2, we may suppose that $\lim_{\varepsilon \to 0}(J_1 + J_2) = 0$.

We now use the fact that $v_\varepsilon^{\lambda,h} \to v$ in $L^2(\Omega)$. One has

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_\Omega (v_\varepsilon^{\lambda,h} - v)^2\, dx &\leqslant 2 \lim_{\varepsilon \to 0} \int_\Omega (\tilde{v}_\varepsilon^{\lambda,h} - \tilde{v})^2\, dx + 2 \lim_{\varepsilon \to 0} J_0(\varepsilon) \\
&= 2 \lim_{\varepsilon \to 0} \sum \big(\tilde{v}_\varepsilon^{\lambda,h}(x_j^*) - \tilde{v}(x_j^*)\big)^2 \varepsilon^n = 0,
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
\lim_{\varepsilon \to 0} \int_\Omega (v_\varepsilon^{\lambda,h})^2\, dx - \int_\Omega v^2\, dx &= \lim_{\varepsilon \to 0} \int_\Omega (\tilde{v}_\varepsilon^{\lambda,h})^2\, dx - \int_\Omega \tilde{v}^2\, dx + \lim_{\varepsilon \to 0} J(\varepsilon) \\
&= \lim_{\varepsilon \to 0} \sum \big(\tilde{v}_\varepsilon^{\lambda,h}(x_j^*)\big)^2 \varepsilon^n - \lim_{\varepsilon \to 0} \sum \big(\tilde{v}(x_j)\big)^2 \varepsilon^n \\
&= \lim_{\varepsilon \to 0} \sum \big[\big(\tilde{v}_\varepsilon^{\lambda,h}(x_j^*)\big)^2 - (\tilde{v}(x_j))^2\big] \varepsilon^n,
\end{aligned} \tag{4.7}$$

where, as usual, we suppose that the values $J_0(\varepsilon) = \int_\Omega (\tilde{v}_\varepsilon^{\lambda,h} - v_\varepsilon^{\lambda,h} + v - \tilde{v})^2\, dx$ and $J(\varepsilon) = \int_\Omega [(v_\varepsilon^{\lambda,h})^2 - (\tilde{v}_\varepsilon^{\lambda,h})^2]\, dx + \int_\Omega [v^2 - \tilde{v}^2]\, dx$ are arbitrarily small.

Hence, by (4.5), the construction of Riemann sum, and the fact that $v \in H_0^1(\Omega)$, we conclude

$$\begin{aligned}
&\lim_{\varepsilon \to 0} \left| \int_\Omega \tilde{v}_\varepsilon^{\lambda,h}\, d\mu_\varepsilon^{\lambda,h} - \int_\Omega \tilde{v}_\varepsilon^{\lambda,h}\, dx \right| \\
&\quad \leqslant \lim_{\varepsilon \to 0} \left| \sum \big(\big[\tilde{v}_\varepsilon^{\lambda,h}(x_j^{\lambda,h}) - \tilde{v}(x_j^{\lambda,h})\big] + \big[\tilde{v}(x_j^{\lambda,h}) - \tilde{v}(x_j)\big] + \big[\tilde{v}(x_j) - \tilde{v}_\varepsilon^{\lambda,h}(x_j)\big]\big)\varepsilon^n \right| \\
&\quad \leqslant 2\sqrt{|\Omega|} \lim_{\varepsilon \to 0} \|v_\varepsilon^{\lambda,h} - v\|_{L^2(\Omega)} + \limsup_{\varepsilon \to 0} \left| \sum \big[\tilde{v}(x_j^{\lambda,h}) - \tilde{v}(x_j)\big]\varepsilon^n \right| \\
&\quad \leqslant \left| \limsup_{\varepsilon \to 0} \Big(\sum \tilde{v}(x_j^{\lambda,h})\varepsilon^n - \sum \tilde{v}(x_j)\varepsilon^n \Big) \right| = \left| \int_\Omega \tilde{v}\, dx - \int_\Omega \tilde{v}\, dx \right| = 0. \qquad \square
\end{aligned}$$

Taking into account the proof of the previous lemmas and relations (4.6)–(4.7), the following statement is readily ascertained:

Lemma 4.4. *Let $\{v_\varepsilon^{\lambda,h} \in H_0^1(\Omega) \cap L^2(\Omega, d\mu_\varepsilon^{\lambda,h})\}$ and $v \in H_0^1(\Omega)$ be such that $v_\varepsilon^{\lambda,h} \rightharpoonup v$ in $H_0^1(\Omega)$. Then*

$$\lim_{\varepsilon\to 0}\left[\int_\Omega (v_\varepsilon^{\lambda,h})^2\, d\mu_\varepsilon^{\lambda,h} - \int_\Omega (v_\varepsilon^{\lambda,h})^2\, dx\right] = 0; \tag{4.8}$$

$$\lim_{\varepsilon\to 0}\int_\Omega v^2\, d\mu_\varepsilon^{\lambda,h} = \int_\Omega v^2\, dx \quad \forall v \in H_0^1(\Omega). \tag{4.9}$$

Remark 4.5. Since the set Ω is bounded and $|\partial\Omega \setminus \Sigma_\varepsilon|_H \sim \varepsilon^{1-n} h^{n-1}(\varepsilon) = \varepsilon$, it follows that $|\Sigma_\varepsilon|_H \to |\partial\Omega|_H$ as $\varepsilon \to 0$. Hence, by property (d), the statements of the Lemmas 4.2–4.4 remain valid if the space $H_0^1(\Omega)$ is changed to $H^1(\Omega, \Sigma_\varepsilon)$.

Theorem 4.6. *Every bounded sequence $\{v_\varepsilon^{\lambda,h} \in H_0^1(\Omega) \cap L^2(\Omega, d\mu_\varepsilon^{\lambda,h})\}$ is relatively compact with respect to the weak convergence in the variable space $H_0^1(\Omega) \cap L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$.*

Proof. Since the sequence $\{v_\varepsilon^{\lambda,h}\}$ is bounded in $H_0^1(\Omega)$, we may suppose that there is an element $v \in H_0^1(\Omega)$ such that $v_\varepsilon^{\lambda,h} \to v$ weakly in $H_0^1(\Omega)$. Then the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ implies the strong convergence $v_\varepsilon^{\lambda,h} \to v$ in $L^2(\Omega)$. Hence, for every $\varphi \in C_0^\infty(\Omega)$, we have

$$\left|\int_\Omega v_\varepsilon^{\lambda,h}\varphi\, d\mu_\varepsilon^{\lambda,h} - \int_\Omega v\varphi\, dx\right| \leqslant \left|\int_\Omega v_\varepsilon^{\lambda,h}\varphi\, dx - \int_\Omega v_\varepsilon^{\lambda,h}\varphi\, d\mu_\varepsilon^{\lambda,h}\right| + \left|\int_\Omega v_\varepsilon^{\lambda,h}\varphi\, dx - \int_\Omega v\varphi\, dx\right|.$$

Passing to the limit in the right-hand part of this inequality as $\varepsilon \to 0$, we obtain $|\int_\Omega v_\varepsilon^{\lambda,h}\varphi\, d\mu_\varepsilon^{\lambda,h} - \int_\Omega v_\varepsilon^{\lambda,h}\varphi\, dx| \to 0$ (by Lemma 4.3), and $|\int_\Omega (v_\varepsilon^{\lambda,h} - v)\varphi\, dx| \to 0$ as a weak limit in $L^2(\Omega)$. The proof is complete. □

Using the above results, we introduce the concept of the weak convergence for the following sequences $\{y_\varepsilon \in H^1(\Omega_\varepsilon; \Sigma_\varepsilon)$: $\breve{y}_\varepsilon \in H^1(\Omega; \Sigma_\varepsilon) \cap L^2(\Omega, d\nu_\varepsilon^{\lambda,h})\}_{\varepsilon>0}$. Here $\breve{y}_\varepsilon$ is some extension of the function y_ε on the whole of Ω. Let us recall that the perforated domain Ω_ε considered here, satisfies the so-called "condition of strong connectedness" (see [23]). It means that there exist a family $\{P_\varepsilon\}_{\varepsilon>0}$ of extension operators $P_\varepsilon : H^1(\Omega_\varepsilon; \Sigma_\varepsilon) \to H^1(\Omega; \Sigma_\varepsilon)$ and a constant C independent of ε, such that $\|\nabla(P_\varepsilon y_\varepsilon)\|_{L^2(\Omega)} \leqslant C\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}$ for every $y_\varepsilon \in H^1(\Omega_\varepsilon; \Sigma_\varepsilon)$. So, we can assume that $\breve{y}_\varepsilon := P_\varepsilon y_\varepsilon$ for some extension operator with the above properties.

Definition 4.7. We say that a sequence $\{y_\varepsilon \in H^1(\Omega_\varepsilon; \Sigma_\varepsilon)\}_{\varepsilon>0}$ is weakly convergent in $H^1(\Omega; \Sigma_\varepsilon) \cap L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$ if there exists an element $y \in H_0^1(\Omega)$ such that $\breve{y}_\varepsilon \rightharpoonup y$ in$H^1(\Omega)$, and $y_\varepsilon \rightharpoonup y$ in $L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$.

We are now in a position to verify the correctness of this definition.

Theorem 4.8. *Every bounded sequence $\{y_\varepsilon \in H^1(\Omega_\varepsilon; \Sigma_\varepsilon)\}_{\varepsilon>0}$ is relatively compact with respect to the weak convergence in the variable space $H^1(\Omega; \Sigma_\varepsilon) \cap L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$.*

Proof. Taking into account Remark 4.5 and the fact that

$$\int_\Omega \breve{y}_\varepsilon\varphi\chi_{\Omega_\varepsilon}\, dx \xrightarrow{\varepsilon\to 0} \int_\Omega y\varphi\, dx \quad \text{and} \quad \int_\Omega y_\varepsilon\varphi\, d\nu_\varepsilon^{\lambda,h} = \int_\Omega \breve{y}_\varepsilon\varphi\, d\nu_\varepsilon^{\lambda,h} \quad \forall\varphi \in C_0^\infty(\Omega),$$

this theorem can be established in complete analogy with the proof of Theorem 4.6. The main difference is the addition of the property $v \in H_0^1(\Omega)$. However, $|\Sigma_\varepsilon|_H \to |\partial\Omega|_H$ as $\varepsilon \to 0$, and we obtain the required result. □

In fact, we can prove a more precise result.

Theorem 4.9. *Let* $\{y_\varepsilon^{\lambda,h} \in H^1(\Omega_\varepsilon;\Sigma_\varepsilon) \cap L^2(\Omega, d\nu_\varepsilon^{\lambda,h})\}_{\varepsilon>0}$ *be a bounded sequence such that* $\breve{y}_\varepsilon^{\lambda,h} \rightharpoonup y$ *in* $H^1(\Omega;\Sigma_\varepsilon) \cap L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$. *Then* $y \in H_0^1(\Omega)$ *and* $y_\varepsilon^{\lambda,h} \to y$ *strongly in* $L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$.

Proof. By the criterium of strong convergence in $L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$, to establish the convergence $y_\varepsilon^{\lambda,h} \to y$ in $L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$ it is enough to show that

$$y_\varepsilon^{\lambda,h} \rightharpoonup y \quad \text{in } L^2(\Omega, d\nu_\varepsilon^{\lambda,h}), \quad \text{and} \quad \lim_{\varepsilon\to 0}\int_\Omega (y_\varepsilon^{\lambda,h})^2\, d\nu_\varepsilon^{\lambda,h} = \int_\Omega y^2\, dx. \tag{4.10}$$

The first statement in (4.10) is valid by Definition 4.7. In order to prove the second one, we apply the following estimate:

$$\left|\int_\Omega (y_\varepsilon^{\lambda,h})^2\, d\nu_\varepsilon^{\lambda,h} - \int_\Omega y^2\, dx\right| \leqslant \left|\int_\Omega (y_\varepsilon^{\lambda,h})^2\, d\nu_\varepsilon^{\lambda,h} - \int_\Omega (y_\varepsilon^{\lambda,h})^2\, dx\right| + \left|\int_\Omega (\breve{y}_\varepsilon^{\lambda,h})^2\, dx - \int_\Omega y^2 dx\right|. \tag{4.11}$$

The second term in right-hand side of (4.11) tends to zero as $\varepsilon \to 0$ by the strong convergence of $\breve{y}_\varepsilon^{\lambda,h}$ to y in $L^2(\Omega)$. The first one is equal to zero as $\varepsilon \to 0$ by applying Lemma 4.4, and this concludes the proof. □

Let $\{(a_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ be a sequence of admissible solutions for the original problem. We assume that this sequence is bounded. Then, summing up the above given reasonings, we may introduce the following concept of the weak convergence in the variable space $\mathbb{X}_\varepsilon$.

Definition 4.10. We say that a bounded sequence $\{(a_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ is w-convergent to a triplet $(a, p, y) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega) \times H_0^1(\Omega)$ in the variable space $\mathbb{X}_\varepsilon$ as ε tends to zero (in symbols, $(a_\varepsilon, p_\varepsilon, y_\varepsilon) \stackrel{w}{\rightharpoonup} (a, p, y)$), if: (i) $a_\varepsilon \rightharpoonup a$ in $H^2(\Omega)$ and $a_\varepsilon \rightharpoonup a$ in $L^2(\Omega, d\mu_\varepsilon^{\lambda,h})$; (ii) $p_\varepsilon \rightharpoonup p$ in $L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$; (iii) $\breve{y}_\varepsilon \rightharpoonup y$ in $H^1(\Omega)$ and $y_\varepsilon \rightharpoonup y$ in $L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$.

In view of Theorems 4.6, 4.8 we come to the following conclusion:

Theorem 4.11. *Every bounded sequence of admissible solutions* $\{(a_\varepsilon, p_\varepsilon, y_\varepsilon)\}_{\varepsilon>0}$ *to problems* $\hat{\mathbb{P}}_\varepsilon$ *is relatively compact with respect to the* w*-convergence in* $\mathbb{X}_\varepsilon$.

We observe also that for the characteristic function χ_ε of the perforated domain Ω_ε, the following result is obvious [12].

Lemma 4.12. χ_ε *converges strongly to* 1 *both in* $L^2(\Omega)$ *and in the variable space* $L^2(\Omega, \chi_\varepsilon dx)$ *as* $\varepsilon \to 0$.

To conclude this section, we present some new results which will be useful in the sequel and which we feel to be interesting per se.

Proposition 4.13 *(Property of homothetic mean value). Let* $g : \mathbb{R}^n \to \mathbb{R}$ *be a* Y*-periodic function such that* $g \in L^2(\partial Q, d\mathcal{H}^{n-1})$. *Then*

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon h(\varepsilon)}\right) d\nu_\varepsilon^{\lambda,h} = \left(\frac{1}{|\partial Q|_H}\int_{\partial Q} g\, d\mathcal{H}^{n-1}\right)\int_\Omega \varphi\, dx \tag{4.12}$$

for any $\varphi \in C(\overline{\Omega})$. *In particular,* $\lim_{\varepsilon\to 0}\int_\Omega g(\frac{x}{\varepsilon h(\varepsilon)})\, d\nu_\varepsilon^{\lambda,h} = |\Omega|(\frac{1}{|\partial Q|_H}\int_{\partial Q} g\, d\mathcal{H}^{n-1})$.

Proof. It is evident that we can restrict our attention to the case when $g \geqslant 0$. Let us partition the set Ω into cubes εY with edges ε and denote these cubes by the symbols εY^j. Then

$$\int_{\Omega} \varphi(x) g\left(\frac{x}{\varepsilon h(\varepsilon)}\right) d\nu_{\varepsilon}^{\lambda,h} = \sum \int_{\varepsilon Y^j} \varphi(x) g\left(\frac{x}{\varepsilon h(\varepsilon)}\right) d\nu_{\varepsilon}^{\lambda,h} + \sum \int_{\Omega\cap\varepsilon Y^j} \varphi(x) g\left(\frac{x}{\varepsilon h(\varepsilon)}\right) d\nu_{\varepsilon}^{\lambda,h}$$
$$= \sum \varphi(x_j^{\lambda,h}) \int_{\varepsilon Y^j} g\left(\frac{x}{\varepsilon h(\varepsilon)}\right) d\nu_{\varepsilon}^{\lambda,h} + \sum \int_{\Omega\cap\varepsilon Y^j} \varphi(x) g\left(\frac{x}{\varepsilon h(\varepsilon)}\right) d\nu_{\varepsilon}^{\lambda,h}, \tag{4.13}$$

where $x_j^{\lambda,h} \in \varepsilon Y^j$ and the second sum is calculated over the set of the "boundary" cubes. By the definition of the scaling measure $\nu_{\varepsilon}^{\lambda,h}$ and due to the Y-periodicity of g, we have

$$\int_{\varepsilon Y^j} g\left(\frac{x}{\varepsilon h(\varepsilon)}\right) d\nu_{\varepsilon}^{\lambda,h} = \frac{\varepsilon^n}{\sigma(\varepsilon)} \int_{\Lambda^{\lambda(\varepsilon),h(\varepsilon)}} g\left(\frac{x}{h(\varepsilon)}\right) d\mathcal{H}^{n-1}, \tag{4.14}$$

where the set $\Lambda^{\lambda(\varepsilon),h(\varepsilon)}$ is defined by (3.1). Since $\Lambda^{\lambda,h} = \partial Q^h \setminus \Gamma^{\lambda,h}$, it follows that

$$\int_{\Lambda^{\lambda(\varepsilon),h(\varepsilon)}} g\, d\mathcal{H}^{n-1} = \int_{\partial Q^{h(\varepsilon)}} g\, d\mathcal{H}^{n-1} - \int_{\Gamma^{\lambda(\varepsilon),h(\varepsilon)}} g\, d\mathcal{H}^{n-1}.$$

Then, due to the definition of the homothetic contraction and using formula (3.4), we have

$$\int_{\partial Q^{h(\varepsilon)}} g\left(\frac{x}{h(\varepsilon)}\right) d\mathcal{H}^{n-1} = h^{n-1}(\varepsilon) \int_{\partial Q} g\, d\mathcal{H}^{n-1} = \varepsilon^n \int_{\partial Q} g\, d\mathcal{H}^{n-1}, \tag{4.15}$$

$$\int_{\Gamma^{\lambda(\varepsilon),h(\varepsilon)}} g\left(\frac{x}{h(\varepsilon)}\right) d\mathcal{H}^{n-1} = h^{n-1}(\varepsilon) \int_{h^{-1}(\varepsilon)\Gamma^{\lambda(\varepsilon),h(\varepsilon)}} g\, d\mathcal{H}^{n-1}. \tag{4.16}$$

From the definition of the set $\Gamma^{\lambda(\varepsilon),h(\varepsilon)}$ (see (3.1)), we obtain $h^{-1}(\varepsilon)\Gamma^{\lambda(\varepsilon),h(\varepsilon)} = K^{\lambda(\varepsilon)/h(\varepsilon)} \cap \partial Q = \lambda(\varepsilon)(K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)})$. Hence, by Proposition 3.3, we have

$$\lim_{\varepsilon\to 0} \left|\lambda(\varepsilon)(K \cap \partial Q^{h(\varepsilon)/\lambda(\varepsilon)})\right|_H = |K \cap \partial \Lambda|_H \lim_{\varepsilon\to 0} \lambda^{n-1}(\varepsilon) = 0.$$

Thus, combining relations (4.14)–(4.16), we conclude

$$\int_{\varepsilon Y^j} g\left(\frac{x}{\varepsilon h(\varepsilon)}\right) d\nu_{\varepsilon}^{\lambda,h} = \frac{\varepsilon^n}{\sigma(\varepsilon)} \varepsilon^n \left(\int_{\partial Q} g\, d\mathcal{H}^{n-1} + J(\varepsilon)\right), \tag{4.17}$$

$$J(\varepsilon) \leqslant \|g\|_{L^2(\partial Q, d\mathcal{H}^{n-1})} |K^{\lambda(\varepsilon)/h(\varepsilon)} \cap \partial Q|_H^{1/2} \overset{\varepsilon\to 0}{\longrightarrow} 0. \tag{4.18}$$

As a result, substituting (4.17) and (4.18) into (4.13), we have

$$\left|\int_{\Omega} \varphi(x) g\left(\frac{x}{\varepsilon h(\varepsilon)}\right) d\nu_{\varepsilon}^{\lambda,h} - \frac{\varepsilon^n}{\sigma(\varepsilon)} \left(\int_{\partial Q} g\, d\mathcal{H}^{n-1}\right) \sum \varphi(x_j^{\lambda,h}) \varepsilon^n\right|$$
$$\leqslant \frac{\varepsilon^n}{\sigma(\varepsilon)} J(\varepsilon) \sum \varphi(x_j^{\lambda,h}) \varepsilon^n + \frac{\varepsilon^n}{\sigma(\varepsilon)} \left(\int_{\partial Q} g\, d\mathcal{H}^{n-1} + J(\varepsilon)\right) \sup_{x\in\Omega} |\varphi| \varepsilon^n D(\varepsilon), \tag{4.19}$$

where $D(\varepsilon)$ is the quantity of the "boundary" cubes.

Since $\lim_{\varepsilon\to 0} \sum \varphi(x_j^{\lambda,h}) \varepsilon^n = \int_{\Omega} \varphi\, dx$ by construction of the Riemann sum, $\lim_{\varepsilon\to 0} J(\varepsilon) = 0$ by (4.18),

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^n}{\sigma(\varepsilon)} = |\partial Q|_H^{-1}$$

by (3.4), $\lim_{\varepsilon\to 0} \varepsilon^n D(\varepsilon) = 0$ by Jordan's measurability property of $\partial\Omega$, it follows that estimate (4.19) immediately leads to the required result. □

In a similar way we can prove the following result:

Proposition 4.14. *Let Λ be a cone which is defined in Proposition* 3.3*, and let $g:\mathbb{R}^n\to\mathbb{R}$ be a Y-periodic function such that $g\in H^1(K)$. Then*

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi(x) g\left(\frac{x}{\varepsilon\lambda(\varepsilon)}\right) d\mu_\varepsilon^{\lambda,h} = \left(\frac{1}{|K\cap\partial\Lambda|_H}\int_{K\cap\partial\Lambda} g\, d\mathcal{H}^{n-1}\right)\int_\Omega \varphi\, dx \tag{4.20}$$

for any $\varphi\in C(\overline{\Omega})$. In particular,

$$\lim_{\varepsilon\to 0}\int_\Omega g\left(\frac{x}{\varepsilon\lambda(\varepsilon)}\right) d\mu_\varepsilon^{\lambda,h} = |\Omega|\left(\frac{1}{|K\cap\partial\Lambda|_H}\int_{K\cap\partial\Lambda} g\, d\mathcal{H}^{n-1}\right).$$

Remark 4.15. The results of Propositions 4.13–4.14 are examples of bounded sequences in variable spaces whose weak limits can be recovered in an explicit form.

5. Definition of a homogenized problem, and its property

We begin this section with the following notion:

Definition 5.1. We say that the space $L^2(\Omega)$ possesses the weak approximation property with respect to the family of the Borel measures $\{\eta_\varepsilon^{\lambda,h}\}_{\varepsilon>0}$, if for every $\delta>0$ and any $p\in L^2(\Omega)$ there exist an element $q\in L^2(\Omega)$ and a sequence $\{q_\varepsilon^{\lambda,h}\in L^2(\Omega,d\eta_\varepsilon^{\lambda,h})\}_{\varepsilon>0}$, such that $\|p-q\|_{L^2(\Omega)}\leqslant\delta$ and $q_\varepsilon^{\lambda,h}\to q$ in $L^2(\Omega,d\eta_\varepsilon^{\lambda,h})$. In this case, the sequence $\{q_\varepsilon^{\lambda,h}\in L^2(\Omega,d\eta_\varepsilon^{\lambda,h})\}_{\varepsilon>0}$ is called δ-realizing sequence.

Lemma 5.2. *The weak approximation property for the $L^2(\Omega)$ with respect to the family of the Borel measures $\{\nu_\varepsilon^{\lambda,h}\}_{\varepsilon>0}$ is valid.*

Proof. Let p be any element of $L^2(\Omega)$. Since the inclusion $H_0^1(\Omega)\subset L^2(\Omega)$ is dense with respect to the strong topology for $L^2(\Omega)$, it follows that for a given value $\delta>0$ there is an element $q\in H_0^1(\Omega)$ such that $\|p-q\|_{L^2(\Omega)}\leqslant\delta$. Let us construct the δ-realizing sequence as follows: $q_\varepsilon^{\lambda,h}=q$ for every $\varepsilon>0$. In accordance with Lemmas 4.2, 4.4 and Theorem 4.9, we have $\lim_{\varepsilon\to0}\int_\Omega q\varphi\, d\nu_\varepsilon^{\lambda,h}=\int_\Omega q\varphi\, dx$ $\forall\varphi\in C_0^\infty(\Omega)$ and $\lim_{\varepsilon\to0}\int_\Omega q^2\, d\nu_\varepsilon^{\lambda,h}=\int_\Omega q^2\, dx$. Hence, by the criterium of strong convergence in $L^2(\Omega,d\nu_\varepsilon^{\lambda,h})$, we obtain the required result. □

In view of the main question of this paper, our next intention is to study the asymptotic behaviour of the problem $\hat{\mathbb{P}}_\varepsilon$ as $\varepsilon\to0$. To do so, we represent $\hat{\mathbb{P}}_\varepsilon$-problem for various values of ε, in the form of the following sequence:

$$\left\{\left\langle \inf_{(a_\varepsilon,p_\varepsilon,y_\varepsilon)\in\hat{\Xi}_\varepsilon} \hat{I}_\varepsilon(a_\varepsilon,p_\varepsilon,y_\varepsilon)\right\rangle;\ \varepsilon>0\right\}. \tag{5.1}$$

Then the definition of an appropriate homogenized optimal control problem to the family (3.12), can be reduced to the analysis of the limit properties of the sequence (5.1) as $\varepsilon\to0$. To get this limit in the form of some constrained minimization problem, we apply the scheme of the direct homogenization which was developed in [19,20]. However, in contrast to the usual concept of variational convergence (see for instance [2,3,19]), we introduce another one. The main reason for this, is the specific construction of the solution space $\mathbb{X}_\varepsilon$ and the absence of the strong approximation property for the "w-limit space" $\mathbb{Y}=[H^2(\Omega)\cap H_0^1(\Omega)]\times L^2(\Omega)\times H_0^1(\Omega)$ [25]. This means that perhaps not for every triplet $(a,p,y)\in[H^2(\Omega)\cap H_0^1(\Omega)]\times L^2(\Omega)\times H_0^1(\Omega)$ one can find a sequence $\{(a_\varepsilon,p_\varepsilon,y_\varepsilon)\colon (a_\varepsilon,p_\varepsilon,\check{y}_\varepsilon)\in\mathbb{X}_\varepsilon\}_{\varepsilon>0}$ such that $(a_\varepsilon,p_\varepsilon,y_\varepsilon)\overset{w}{\to}(a,p,y)$.

Definition 5.3. We say that a minimization problem

$$\left\langle \inf_{(a,p,y)\in\Xi_0} I_0(a,p,y)\right\rangle \tag{5.2}$$

is the weak variational limit of the sequence (5.1) with respect to the w-convergence in the variable space $\mathbb{X}_\varepsilon$ (or variational w-limit), if the following conditions are satisfied:

(1) if a sequence $\{(a_k, p_k, \breve{y}_k) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ w-converges to a triplet (a, p, y), and there exists a subsequence $\{\varepsilon_k\}$ of $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and $(a_k, p_k, y_k) \in \hat{\Xi}_{\varepsilon_k}$ for all k, then

$$(a, p, y) \in \Xi_0; \quad I_0(a, p, y) \leqslant \liminf_{k\to\infty} \hat{I}_{\varepsilon_k}(a_k, p_k, y_k); \tag{5.3}$$

(2) for every triplet $(a, p, y) \in \Xi_0$ and any value $\delta > 0$, there exists a δ-realizing sequence $\{(\hat{a}_\varepsilon^{\lambda,h}, \hat{p}_\varepsilon^{\lambda,h}, \breve{\hat{y}}_\varepsilon^{\lambda,h}) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ such that

$$(\hat{a}_\varepsilon^{\lambda,h}, \hat{p}_\varepsilon^{\lambda,h}, \hat{y}_\varepsilon^{\lambda,h}) \in \hat{\Xi}_\varepsilon \quad \forall\varepsilon > 0, \quad (\hat{a}_\varepsilon^{\lambda,h}, \hat{p}_\varepsilon^{\lambda,h}, \hat{y}_\varepsilon^{\lambda,h}) \xrightarrow{w} (\hat{a}, \hat{p}, \hat{y}), \tag{5.4}$$

$$\|(a, p, y) - (\hat{a}, \hat{p}, \hat{y})\|_{\mathbb{Y}} \leqslant \delta, \quad \text{and} \quad I_0(a, p, y) \geqslant \limsup_{\varepsilon\to0} \hat{I}_\varepsilon(\hat{a}_\varepsilon^{\lambda,h}, \hat{p}_\varepsilon^{\lambda,h}, \hat{y}_\varepsilon^{\lambda,h}) - \delta. \tag{5.5}$$

Theorem 5.4. *Assume that the constrained minimization problem* (5.2) *is the weak variational limit of the sequence* (5.1)*, and has a unique solution* $(a^0, p^0, y^0) \in \Xi_0$. *Let* $\{(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \hat{\Xi}_\varepsilon\}_{\varepsilon>0}$ *be the sequence of optimal triplets for* $\hat{\mathbb{P}}_\varepsilon$*-problems. Then*

$$(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \xrightarrow{w} (a^0, p^0, y^0), \tag{5.6}$$

$$\inf_{(a,p,y)\in\Xi_0} I_0(a, p, y) = I_0(a^0, p^0, y^0) = \lim_{\varepsilon\to0} \inf_{(a_\varepsilon,p_\varepsilon,y_\varepsilon)\in\hat{\Xi}_\varepsilon} \hat{I}_\varepsilon(a_\varepsilon, p_\varepsilon, y_\varepsilon). \tag{5.7}$$

Proof. First, observe that in view of Theorem 4.11, the w-compactness property holds true for the sequence of optimal solutions $\{(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \hat{\Xi}_\varepsilon\}_{\varepsilon>0}$. So, we may suppose that there exist a subsequence $\{(a_{\varepsilon_k}^0, p_{\varepsilon_k}^0, y_{\varepsilon_k}^0)\}_{k\in\mathbb{N}}$ of the sequence of optimal solutions and a triplet (a^*, p^*, y^*), such that $(a_{\varepsilon_k}^0, p_{\varepsilon_k}^0, y_{\varepsilon_k}^0) \xrightarrow{w} (a^*, p^*, y^*)$ as $\varepsilon_k \to 0$. Hence, property (5.3) leads to the following conclusion: $(a^*, p^*, y^*) \in \Xi_0$, and

$$\begin{aligned}\liminf_{k\to\infty} \min_{(a,p,y)\in\hat{\Xi}_{\varepsilon_k}} \hat{I}_{\varepsilon_k}(a, p, y) &= \liminf_{k\to\infty} \hat{I}_{\varepsilon_k}(a_{\varepsilon_k}^0, p_{\varepsilon_k}^0, y_{\varepsilon_k}^0)\\ &\geqslant I_0(a^*, p^*, y^*) \geqslant \min_{(a,p,y)\in\Xi_0} I_0(a, p, y) = I_0(u^0, p^0, y^0).\end{aligned} \tag{5.8}$$

Let us fix a value $\delta > 0$. Then, by property (ii) of Definition 5.3 there exists a δ-realizing sequence $\{(\hat{a}_\varepsilon, \hat{p}_\varepsilon, \hat{y}_\varepsilon) \in \hat{\Xi}_\varepsilon\}_{\varepsilon>0}$ such that $(\hat{a}_\varepsilon, \hat{p}_\varepsilon, \hat{y}_\varepsilon) \xrightarrow{w} (\hat{a}, \hat{p}, \hat{y})$,

$$\left\|(a^0, p^0, y^0) - (\hat{a}, \hat{p}, \hat{y})\right\|_{\mathbb{Y}} \leqslant \delta, \quad \text{and} \quad I_0(a^0, p^0, y^0) \geqslant \limsup_{\varepsilon\to0} \hat{I}_\varepsilon(\hat{a}_\varepsilon, \hat{p}_\varepsilon, \hat{y}_\varepsilon) - \delta.$$

Using this fact, we have

$$\begin{aligned}\min_{(a,p,y)\in\Xi_0} I_0(a, p, y) + \delta &= I_0(a^0, p^0, y^0) + \delta \geqslant \limsup_{\varepsilon\to0} I_\varepsilon(\hat{a}_\varepsilon, \hat{p}_\varepsilon, \hat{y}_\varepsilon)\\ &\geqslant \limsup_{\varepsilon\to0} \min_{(a,p,y)\in\hat{\Xi}_\varepsilon} \hat{I}_\varepsilon(a, p, y) \geqslant \limsup_{k\to\infty} \min_{(a,p,y)\in\hat{\Xi}_{\varepsilon_k}} \hat{I}_{\varepsilon_k}(a, p, y)\\ &= \limsup_{k\to\infty} \hat{I}_{\varepsilon_k}(a_{\varepsilon_k}^0, p_{\varepsilon_k}^0, y_{\varepsilon_k}^0).\end{aligned} \tag{5.9}$$

From this and (5.8) we deduce that

$$\liminf_{k\to\infty} \hat{I}_{\varepsilon_k}(a_{\varepsilon_k}^0, p_{\varepsilon_k}^0, y_{\varepsilon_k}^0) \geqslant \limsup_{k\to\infty} \hat{I}_{\varepsilon_k}(a_{\varepsilon_k}^0, p_{\varepsilon_k}^0, y_{\varepsilon_k}^0) - \delta.$$

Since this inequality holds true for sufficiently small $\delta > 0$, after combining (5.8) and (5.9) we get

$$I_0(a^*, p^*, y^*) = I_0(a^0, p^0, y^0) = \min_{(a,p,y)\in\Xi_0} I_0(a, p, y) = \lim_{k\to\infty} \min_{(a,p,y)\in\hat{\Xi}_{\varepsilon_k}} \hat{I}_{\varepsilon_k}(a, p, y).$$

Using these relations and the fact that an optimal triplet for the problem (5.2) is unique, we obtain $(a^*, p^*, y^*) = (a^0, p^0, y^0)$. Since this equality holds for the w-limits of all subsequences of $\{(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon>0}$, it follows that these limits coincide and therefore, (a^0, p^0, y^0) is the w-limit of the whole sequence $\{(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0)\}_{\varepsilon>0}$. Then, using the same argument for the sequence of minimizers as for the subsequence $\{(a^0_{\varepsilon_k}, p^0_{\varepsilon_k}, y^0_{\varepsilon_k})\}_{k\in\mathbb{N}}$, we have

$$\begin{aligned}\liminf_{\varepsilon\to 0} \min_{(a,p,y)\in\hat{\Xi}_\varepsilon} \hat{I}_\varepsilon(a,p,y) &= \liminf_{\varepsilon\to 0} \hat{I}_\varepsilon(a^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) \geqslant I_0(a^0, p^0, y^0)\\ &= \min_{(a,p,y)\in\Xi_0} I_0(a,p,y) \geqslant \limsup_{\varepsilon\to 0} \hat{I}_\varepsilon(a_\varepsilon, p_\varepsilon, y_\varepsilon) - \delta\\ &\geqslant \limsup_{\varepsilon\to 0} \min_{(a,p,y)\in\hat{\Xi}_\varepsilon} \hat{I}_\varepsilon(a,p,y) - \delta = \limsup_{\varepsilon\to 0} \hat{I}_\varepsilon(a^0_\varepsilon, p^0_\varepsilon, y^0_\varepsilon) - \delta \quad \forall\delta > 0\end{aligned}$$

and this concludes the proof. □

Definition 5.5. We say that the optimal control problem (1.5) admits homogenization as ε tends to zero with respect to the w-convergence in the variable space $\mathbb{X}_\varepsilon$, if for the corresponding sequence of the constrained minimization problems (5.1), there exists a weak variational limit which can be recovered in the form of some optimal control problem.

6. Convergence theorem and correctors

The main question of this section is the homogenization of the boundary value problem (1.5). Let $H^1_{\mathrm{per}}(Y)$ be the Sobolev space of Y-periodic functions. We begin with the following result:

Lemma 6.1. *There exists a sequence of functions $\{w^{\lambda,h}\}_{h>\lambda>0}$ satisfying*

(H1) $w^{\lambda,h} \in H^1_{\mathrm{per}}(Y)$, $w^{\lambda,h} = 0$ *on* $K^\lambda \cap \partial Q^h$, $0 \leqslant w^{\lambda,h} \leqslant 1$;
(H2) $w^{\lambda,h} = 1$ *in* $Y \setminus A^h$;
(H3) $w^{\lambda,h}(x_1, x_2, \ldots, x_n) = w^{\lambda,h}(-x_1, x_2, \ldots, x_n)$ $\forall x \in A^h$, $\forall h > \lambda > 0$;
(H4) $w^{\lambda,h} \rightharpoonup 1$ *weakly in* $H^1_{\mathrm{per}}(Y)$ *and strongly in* $L^2_{\mathrm{per}}(Y)$.

Proof. Let us define the following objects

$$\begin{aligned}\mathcal{A} = \big\{&\{v^{\lambda,h}\}\colon\ v^{\lambda,h} = 0 \text{ on } K^\lambda \cap \partial Q^h, v^{\lambda,h}(x_1, x_2, \ldots, x_n) = v^{\lambda,h}(-x_1, x_2, \ldots, x_n)\\ &\forall x \in A^h, \forall h > \lambda > 0, v^{\lambda,h} \rightharpoonup 1 \text{ in } H^1_{\mathrm{per}}(Y), v^{\lambda,h} = 1 \text{ in } Y \setminus A^h\big\},\\ \alpha = \inf&\left\{\liminf_{(h>\lambda)\to 0} \int_Y |\nabla v^{\lambda,h}|^2\,dx\colon \{v^{\lambda,h}\} \in \mathcal{A}\right\}.\end{aligned}$$

Note that the set $\mathcal{A}$ is not empty. Indeed, if we define the functions $v^{\lambda,h}$ as follows

$$v^{\lambda,h} \in H^1_{\mathrm{per}}(Y), \quad \Delta v^{\lambda,h} = 0 \quad \text{in } A^h \setminus A^\lambda, \quad v^{\lambda,h} = 0 \quad \text{in } A^\lambda, v^{\lambda,h} = 1 \quad \text{in } Y \setminus A^h$$

one has immediately $\{v^{\lambda,h}\} \in \mathcal{A}$. For any $k \in \mathbb{N}$, we consider a sequence $\{v^{\lambda,h}_k\} \in \mathcal{A}$ such that

$$\liminf_{(h>\lambda)\to 0} \int_Y |\nabla v^{\lambda,h}_k|^2\,dx < \alpha + \frac{1}{k}.$$

Let $\hat{v}^{\lambda,h}_k = T(v^{\lambda,h}_k)$, where $T(s) = |s|$ if $-1 \leqslant s \leqslant 1$, and $T(s) = 1$ otherwise. Then $\{\hat{v}^{\lambda,h}_k\} \in \mathcal{A}$, $0 \leqslant \hat{v}^{\lambda,h}_k \leqslant 1$, and

$$\liminf_{(h>\lambda)\to 0} \int_Y |\nabla \hat{v}^{\lambda,h}_k|^2\,dx \leqslant \liminf_{(h>\lambda)\to 0} \int_Y |\nabla v^{\lambda,h}_k|^2\,dx < \alpha + \frac{1}{k}.$$

By Rellich–Kondrashov's compactness and Lebesgue's dominated convergence theorems, we conclude that the embedding $H^1_{\mathrm{per}}(Y) \cap L^\infty(Y) \hookrightarrow L^q(Y)$ $(1 \leqslant q < +\infty)$ is compact. As a result, the sequence $\{\hat{v}^{\lambda,h}_k\}$ converges strongly to 1 in $L^2(Y)$ as $(h > \lambda) \to 0$ for every fixed k. Then it is possible to define a subsequence (λ_k, h_k) of (λ, h) which is decreasing and tends to 0, such that

$$\int_Y |\nabla \hat{v}_k^{\lambda_k, h_k}|^2 \, dx < \alpha + 2/k, \qquad \|\hat{v}_k^{\lambda_k, h_k} - 1\|_{L^2(Y)} < 1/k.$$

Then the desired sequence $\{w^{\lambda,h}\}_{h>\lambda>0}$ is defined by $w^{\lambda,h} = \hat{v}_k^{\lambda_k,h_k}$. $\square$

From now on, we suppose that each of the functions $w^{\lambda,h}$ satisfying conditions (H1)–(H4), is extended by Y-periodicity onto $\mathbb{R}^n$. We set

$$w_\varepsilon(x) = w^{\lambda(\varepsilon),h(\varepsilon)}(x/\varepsilon) \quad \forall x \in \Omega, \ \forall \varepsilon > 0.$$

From Lemma 6.1, we have

(P1) $w_\varepsilon \in H^1(\Omega)$, $0 \leqslant w_\varepsilon \leqslant 1$;
(P2) $w_\varepsilon = 0$ on $\Gamma_\varepsilon^D = \bigcup_{\mathbf{k} \in \Theta_\varepsilon} [K^{\lambda(\varepsilon)} \cap \partial Q^{h(\varepsilon)} + \varepsilon \mathbf{k}]$;
(P3) $w_\varepsilon = 1$ in $\Omega \setminus \bigcup_{\mathbf{k} \in \Theta_\varepsilon} (A^{h(\varepsilon)} + \varepsilon \mathbf{k})$;
(P4) $w_\varepsilon(x_1, x_2, \dots, x_n) = w_\varepsilon(-x_1, x_2, \dots, x_n)$ $\forall x \in A^{\lambda(\varepsilon)}$, $\forall \mathbf{k} \in \Theta_\varepsilon$ and $\varepsilon > 0$;
(P5) $w_\varepsilon \rightharpoonup 1$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$ as $\varepsilon \to 0$.

Note that the sequence $\{|\nabla w_\varepsilon|^2\}$ is bounded in $L^1(\Omega)$. So that, extracting if necessary, a subsequence, we can suppose the existence of a bounded nonnegative Radon measure μ^* such that $|\nabla w_\varepsilon|^2$ converges to μ^* in the weak sense of the space $\mathcal{M}_b(\Omega)$. Following in many aspects Casado-Díaz ([6], Theorem 2.1), the following quite similar result can be proved:

Theorem 6.2. *Let $\{w_\varepsilon \in H^1(\Omega)\}$ be a sequence satisfying the properties* (P1)–(P5). *Then*

(L1) *$|\nabla w_\varepsilon|^2 \to \mu^*$ weakly in $\mathcal{M}_b(\Omega)$, where $\mu^* \in \mathcal{M}_0^+$, i.e. $\int_\Omega \varphi |\nabla w_\varepsilon|^2 \, dx \to \int_\Omega \varphi \, d\mu^*$ for any $\varphi \in C_0^\infty(\Omega)$;*
(L2) *for any $v_\varepsilon \in H^1(\Omega; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$, and for any $v \in H_0^1(\Omega)$ such that $v_\varepsilon \rightharpoonup v$ in $H^1(\Omega)$, we have*

$$v \in L^2(\Omega, d\mu^*), \qquad \int_\Omega \varphi(\nabla v_\varepsilon \cdot \nabla w_\varepsilon) \, dx \to \int_\Omega \varphi v \, d\mu^* \quad \forall \varphi \in C_0^\infty(\Omega). \tag{6.1}$$

In fact, the measure $\mu^* \in \mathcal{M}_0^+$ that appeared as the weak limit of $|\nabla w_\varepsilon|^2$ in the space $\mathcal{M}_b(\Omega)$, can be recovered in an explicit form. For this, we recall some properties of capacity (see Theorem 2 of Section 4.7.1 of [15]).

Lemma 6.3. *Let D be an open subset of $\mathbb{R}^n$ and B be a compact subset of D. Then*

(i) *if $\{D_i\}_{i \in \mathbb{N}}$ is an increasing sequence of open sets such that $\bigcup_{i \in \mathbb{N}} D_i = D$, then* $\lim_{i \to \infty} \operatorname{cap}(B, D_i) = \operatorname{cap}(B, D)$;
(ii) *if $\{D_i \subset E\}_{i \in \mathbb{N}}$ is a decreasing sequence of compact sets such that $\cap_{i \in \mathbb{N}} D_i = \operatorname{cl} D$, then* $\lim_{i \to \infty} \operatorname{cap}(D_i, E) = \operatorname{cap}(D, E)$;
(iii) *if $D_1 \subset D_2$, then* $\operatorname{cap}(D_1, E) \leqslant \operatorname{cap}(D_2, E)$;
(iv) *if $t > 0$, then* $\operatorname{cap}(tB, tD) = t^{n-2} \operatorname{cap}(B, D)$.

We now give the recovery result of the measure μ^*.

Lemma 6.4. *Assume that the origin belongs to a smooth part of the boundary ∂Q ($\partial Q(0) \in C^\infty$). Then for a sequence $\{w_\varepsilon \in H^1(\Omega)\}$ which satisfies properties* (P1)–(P5), *we have $|\nabla w_\varepsilon|^2 \rightharpoonup \mu^*$ weakly in $\mathcal{M}_b(\Omega)$, where*

$$\mu^* = \operatorname{cap}\bigl(K \cap \{x \in \mathbb{R}^n \colon x_1 = 0\}\bigr) \quad \textit{if } n \geqslant 3, \qquad \mu^* = 2\pi \quad \textit{if } n = 2. \tag{6.2}$$

Proof. The proof follows standard techniques in such situations (see [15]) and, in some aspects, it is similar to the one given in [12]. First of all, we note that for any function $\varphi \in C_0^\infty(\Omega)$, $\varepsilon > 0$, and every $\mathbf{k} \in \Theta_\varepsilon$ we have the following inequality:

$$\varphi(x_{\mathbf{k}}^\varepsilon) \int\limits_{\varepsilon Y+\varepsilon\mathbf{k}} |\nabla w_\varepsilon|^2\,dx \leqslant \int\limits_{\varepsilon Y+\varepsilon\mathbf{k}} |\nabla w_\varepsilon|^2 \varphi\,dx \leqslant \varphi(y_{\mathbf{k}}^\varepsilon) \int\limits_{\varepsilon Y+\varepsilon\mathbf{k}} |\nabla w_\varepsilon|^2\,dx, \tag{6.3}$$

where $x_{\mathbf{k}}^\varepsilon, y_{\mathbf{k}}^\varepsilon \in \varepsilon Y + \varepsilon\mathbf{k}$.

Let us begin with the case $n \geqslant 3$. From the definition of the capacity and Theorem 6.2, it readily follows that $\int_{\varepsilon Y+\varepsilon\mathbf{k}} |\nabla w_\varepsilon|^2\,dx = \operatorname{cap}(K^{\lambda(\varepsilon)} \cap \partial Q^{h(\varepsilon)}, A^{h(\varepsilon)})$. Then, taking into account property (iv) of Lemma 6.3 and relation (3.4), we have

$$\begin{aligned} \int\limits_{\varepsilon Y+\varepsilon\mathbf{k}} |\nabla w_\varepsilon|^2\,dx &= \operatorname{cap}\big(\lambda(\varepsilon)[K \cap \partial Q^{\varsigma(\varepsilon)}], A^{h(\varepsilon)}\big) \\ &= \lambda^{n-2}(\varepsilon)\operatorname{cap}\big([K \cap \partial Q^{\varsigma(\varepsilon)}], A^{\varsigma(\varepsilon)}\big) = \varepsilon^n \operatorname{cap}\big([K \cap \partial Q^{\varsigma(\varepsilon)}], A^{\varsigma(\varepsilon)}\big), \end{aligned} \tag{6.4}$$

where $\varsigma(\varepsilon) = h(\varepsilon)/\lambda(\varepsilon) = \exp(-n\ln\varepsilon/(n^2 - 3n + 2))$ for $n \geqslant 3$.

Now we interpret the sequence $\{\operatorname{cap}([K \cap \partial Q^{\varsigma(\varepsilon)}], A^{\varsigma(\varepsilon)})\}_{\varepsilon>0}$ as a two parametric one: $\{\Lambda_{\delta,\varepsilon} = \operatorname{cap}([K \cap \partial Q^{\varsigma(\delta)}], A^{\varsigma(\varepsilon)})\}_{\delta,\varepsilon>0}$. Since this sequence is monotone with respect to the parameter ε, it follows that $\lim_{\delta,\varepsilon\to 0} \Lambda_{\delta,\varepsilon} = \lim_{\varepsilon\to 0} \Lambda_{\delta(\varepsilon),\varepsilon}$ for every sequence $\{\delta(\varepsilon)\}$ converging to zero. Then due to the following inequality:

$$\begin{aligned} &\big|\operatorname{cap}\big([K \cap \partial Q^{\varsigma(\delta)}], A^{\varsigma(\varepsilon)}\big) - \operatorname{cap}(K \cap D)\big| \leqslant \big|\operatorname{cap}\big([K \cap \partial Q^{\varsigma(\delta)}]\big) - \operatorname{cap}(K \cap D)\big| \\ &\quad + \big|\operatorname{cap}\big([K \cap \partial Q^{\varsigma(\delta)}], A^{\varsigma(\varepsilon)}\big) - \operatorname{cap}\big([K \cap \partial Q^{\varsigma(\delta)}]\big)\big| = J'(\delta) + J''(\delta,\varepsilon) \end{aligned} \tag{6.5}$$

and using property (i) of Lemma 6.3, we have $\lim_{\varepsilon\to 0} J''(\delta,\varepsilon) = 0$ for every $\delta > 0$.

To examine the limit properties of the sequence $\{J'(\delta)\}_{\delta>0}$, we have to perform its analysis in a more precise form. Namely, since a part of boundary ∂Q containing the origin is smooth, it follows that there is a neighbourhood $\mathcal{U}(0)$ of the origin such that $\mathcal{U}(0) \cap \partial Q$ is a graph of a smooth function whose epigraph contains $\mathcal{U}(0) \cap Q$. So, we may suppose that there is a function $\Psi : \mathbb{R}^{n-1} \to \mathbb{R}_\geqslant$ such that $\Psi \in C_0^\infty(\mathbb{R}^{n-1})$ and $x_1 = \Psi(x_2, \ldots, x_n)$ for every $x = (x_1, x_2, \ldots, x_n) \in \mathcal{U}(0) \cap \partial Q$.

Then the following conclusion is valid: $x = (x_1, x_2, \ldots, x_n) \in K \cap \varepsilon^{-1}\partial Q$ for ε small enough if and only if $x \in \varepsilon^{-1}(\mathcal{U}(0) \cap \partial Q)$ and hence $x_1 = \varepsilon\Psi(x_2/\varepsilon, \ldots, x_n/\varepsilon)$. As a result, for any sufficiently small ε_0, there exists a constant $C' > 0$ such that

$$K \cap \varepsilon^{-1}\partial Q \subset K \cap \Pi_r \quad \forall\varepsilon \leqslant \varepsilon_0, \text{ with } r = C'\varepsilon_0 \|\Psi\|_{C(\mathbb{R}^{n-1} \cap \mathcal{U}(0))}$$

where $\Pi_r = \{x \in \mathbb{R}^n\colon 0 \leqslant x_1 \leqslant r\}$. Then by properties (ii)–(iii) of Lemma 6.3, we have the following implication:

$$\lim_{r\to 0} \operatorname{cap}(K \cap \Pi_r) = \operatorname{cap}(K \cap D) \quad \text{and} \quad K \cap \varepsilon^{-1}\partial Q \subset K \cap \Pi_{\varepsilon C'\|\Psi\|_{C(\mathbb{R}^{n-1} \cap \mathcal{U}(0))}} \quad \forall\varepsilon > 0$$

implies that $\lim_{\varepsilon\to 0} \operatorname{cap}(K \cap \varepsilon^{-1}\partial Q) = \operatorname{cap}(K \cap D)$. Hence

$$J'(\delta) = \big|\operatorname{cap}\big([K \cap \partial Q^{\varsigma(\delta)}]\big) - \operatorname{cap}(K \cap D)\big| \leqslant C'\|\Psi\|_{C(\mathbb{R}^{n-1}) \cap \mathcal{U}(0)}\, \varsigma(\delta) \tag{6.6}$$

for δ small enough.

Summing up relations (6.3) for every $\mathbf{k} \in \Theta_\varepsilon$, and taking into account (6.4)–(6.6), we come to

$$\begin{aligned} &\big[\operatorname{cap}(K \cap D) - C'\varsigma(\delta)\|\Psi\|_{C(\mathbb{R}^{n-1})} - J''(\delta,\varepsilon)\big] \sum_{\mathbf{k}\in\Theta_\varepsilon} \varepsilon^n \varphi(x_{\mathbf{k}}^\varepsilon) \leqslant \sum_{\mathbf{k}\in\Theta_\varepsilon} \int\limits_{\varepsilon Y+\varepsilon\mathbf{k}} |\nabla w_\varepsilon|^2 \varphi\,dx \\ &\quad \leqslant \big[\operatorname{cap}(K \cap D) + C'\varsigma(\delta)\|\Psi\|_{C(\mathbb{R}^{n-1})} + J''(\delta,\varepsilon)\big] \sum_{\mathbf{k}\in\Theta_\varepsilon} \varepsilon^n \varphi(y_{\mathbf{k}}^\varepsilon). \end{aligned} \tag{6.7}$$

Therefore, if we consider the construction of the Riemann sum for $\int_\Omega \varphi\,dx$, setting $\delta = \varepsilon$, and passing to the limit in (6.8) as $\varepsilon \to 0$, we immediately obtain the required result $\lim_{\varepsilon\to 0} \int_\Omega |\nabla w_\varepsilon|^2 \varphi\,dx = \operatorname{cap}(K \cap D) \int_\Omega \varphi\,dx$.

If $n = 2$, then we have a similar situation to the previous one. The only difference concerns the following obvious equality

$$\int\limits_{\varepsilon Y+\varepsilon\mathbf{k}} |\nabla w_\varepsilon|^2 dx = \operatorname{cap}\bigl(\exp(-1/\varepsilon^2)K \cap \partial Q, A\bigr) = \operatorname{cap}\bigl(K \cap \exp(1/\varepsilon^2)\partial Q, \exp(1/\varepsilon^2)A\bigr).$$

For the sequence $\{\Lambda_{\delta,\varepsilon} = \operatorname{cap}(K \cap \exp(1/\delta^2)\partial Q, \exp(1/\varepsilon^2)A)\}_{\delta>0,\varepsilon>0}$, we can apply the above arguments. Therefore, there is a constant $C'' > 0$ such that for ε small enough

$$\bigl|\Lambda_{\delta,\varepsilon} - \operatorname{cap}\bigl(K \cap D, \exp(1/\varepsilon^2)A\bigr)\bigr| < C'' \exp(-1/\delta^2). \tag{6.8}$$

However, as follows from [12] (see Lemma 3.3), we have

$$\operatorname{cap}\bigl(K \cap D, \exp(1/\varepsilon^2)A\bigr) = 2\pi\varepsilon^2(1 + c_\varepsilon), \quad \text{where } \lim_{\varepsilon\to 0} c_\varepsilon = 0. \tag{6.9}$$

Then, summing up relations (6.4) for all $\mathbf{k} \in \Theta_\varepsilon$ and taking into account (6.8) and (6.9), we obtain

$$\begin{aligned}\bigl[2\pi(1 + c_\varepsilon) - C''\varepsilon^{-2}\exp(-1/\delta^2)\bigr] \sum_{\mathbf{k}\in\Theta_\varepsilon} \varepsilon^2\varphi(x^\varepsilon_{\mathbf{k}}) &\leqslant \sum_{\mathbf{k}\in\Theta_\varepsilon} \int\limits_{\varepsilon Y+\varepsilon\mathbf{k}} |\nabla w_\varepsilon|^2\varphi\, dx \\ &\leqslant \bigl[2\pi(1 + c_\varepsilon) - C''\varepsilon^{-2}\exp(-1/\delta^2)\bigr] \sum_{\mathbf{k}\in\Theta_\varepsilon} \varepsilon^2\varphi(y^\varepsilon_{\mathbf{k}}).\end{aligned} \tag{6.10}$$

Setting $\delta = \varepsilon$ and passing to the limit as $\varepsilon \to 0$, we get $\lim_{\varepsilon\to 0} \int_\Omega |\nabla w_\varepsilon|^2\varphi dx = 2\pi$. $\square$

Corollary 6.5. *Under the assumptions of Lemma* 6.4 *concerning the local smoothness property of the boundary ∂Q, item* (L2) *of Theorem* 6.2 *can be made more precise in the following way*: *for any $v_\varepsilon \in H^1(\Omega; \Gamma^D_\varepsilon \cup \Sigma_\varepsilon)$, and for any $v \in H^1_0(\Omega)$ such that $v_\varepsilon \rightharpoonup v$ in $H^1(\Omega)$, we have*

$$\int\limits_\Omega \varphi(\nabla v_\varepsilon \cdot \nabla w_\varepsilon)\, dx \to \mu^* \int\limits_\Omega \varphi v\, dx \quad \forall\varphi \in C^\infty_0(\Omega), \tag{6.11}$$

where the multiplier μ^* is defined by (6.2). *Moreover, in this case we have* (*see Proposition* 3.3) $|K \cap \partial\Lambda|_H = |K \cap D|_H$.

The following result is crucial in this section:

Theorem 6.6. *Let $\{v_\varepsilon \in H^1(\Omega_\varepsilon; \Gamma^D_\varepsilon \cup \Sigma_\varepsilon)\}$ be a bounded sequence such that $v_\varepsilon \rightharpoonup v$ in the variable space $H^1(\Omega_\varepsilon; \Gamma^D_\varepsilon \cup \Sigma_\varepsilon) \cap L^2(\Omega, d\nu^{\lambda,h}_\varepsilon)$. Let $\{\rho'_\varepsilon\}_{\varepsilon>0}$ be the sequence of numbers that was defined in Proposition* 3.5 *and ρ^* be its limit. Then for the sequence $\{w_\varepsilon \in H^1(\Omega)\}$ with properties* (P1)–(P5) *we have*

$$v \in L^2(\Omega, d\mu^*), \quad \int\limits_{\Omega_\varepsilon} \varphi(\nabla v_\varepsilon \cdot \nabla w_\varepsilon)\, dx \to \rho^* \int\limits_\Omega \varphi v\, d\mu^* \quad \forall\varphi \in C^\infty_0(\Omega). \tag{6.12}$$

Proof. Denote by $\breve{v}_\varepsilon \in H^1(\Omega; \Gamma^D_\varepsilon \cup \Sigma_\varepsilon)$ some extensions of the functions v_ε, and define the following sets:

$$J_\varepsilon = \{\mathbf{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n\colon\ (\varepsilon Y + \varepsilon\mathbf{k}) \subset \Omega\},$$

$$A_\varepsilon = \bigcup_{\mathbf{k}\in\Theta_\varepsilon} [A^{h(\varepsilon)} + \varepsilon\mathbf{k}] = \biggl(\bigcup_{\mathbf{k}\in J_\varepsilon} [A^{h(\varepsilon)} + \varepsilon\mathbf{k}]\biggr) \cup \biggl(\bigcup_{\mathbf{k}\in\Theta_\varepsilon\setminus J_\varepsilon} [A^{h(\varepsilon)} + \varepsilon\mathbf{k}]\biggr) = A'_\varepsilon \cup A''_\varepsilon.$$

It is clear that for any bounded sequence $\{z_\varepsilon \in H^1(\Omega)\}$, we have

$$\begin{aligned}\int_{\Omega} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx &= \int_{\Omega\cap A_\varepsilon} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx \\ &= \int_{(\Omega\setminus\Omega_\varepsilon)\cap A_\varepsilon} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx + \int_{\Omega_\varepsilon\cap A_\varepsilon} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx \\ &= \int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx + \int_{\Omega_\varepsilon\cap A'_\varepsilon} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx \\ &\quad + \int_{(\Omega\setminus\Omega_\varepsilon)\cap A''_\varepsilon} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx + \int_{\Omega_\varepsilon\cap A''_\varepsilon} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx. \end{aligned} \tag{6.13}$$

Since each of the sets $(\Omega \setminus \Omega_\varepsilon)\cap A''_\varepsilon$ and $\Omega_\varepsilon \cap A''_\varepsilon$ is located along the boundary $\partial\Omega$, it follows that $|(\Omega \setminus \Omega_\varepsilon)\cap A''_\varepsilon| \to 0$ and $|\Omega_\varepsilon \cap A''_\varepsilon| \to 0$ as $\varepsilon \to 0$. Hence, in view of the boundedness of $\{z_\varepsilon \in H^1(\Omega)\}$ and $\{w_\varepsilon \in H^1(\Omega)\}$ we conclude

$$\int_{(\Omega\setminus\Omega_\varepsilon)\cap A''_\varepsilon} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx \xrightarrow{\varepsilon\to 0} 0, \qquad \int_{\Omega_\varepsilon\cap A''_\varepsilon} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx \xrightarrow{\varepsilon\to 0} 0, \tag{6.14}$$

$$\lim_{\varepsilon\to 0}\left[\int_{\Omega} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx - \int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx - \int_{\Omega_\varepsilon\cap A'_\varepsilon} \varphi(\nabla z_\varepsilon \cdot \nabla w_\varepsilon)\,dx\right] = 0. \tag{6.15}$$

For $\{v_\varepsilon \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)\}$ and $\varphi \in C_0^\infty(\Omega)$, let us define a function $\psi \in C_0^\infty(\Omega)$ in the following way (always possible by the property (P4) of w_ε and some freedom of choosing of the extension operators $\breve{v}_\varepsilon = P_\varepsilon(v_\varepsilon)$, [5,12])

$$\rho''_\varepsilon \int_{\Omega_\varepsilon\cap A'_\varepsilon} \varphi(\nabla v_\varepsilon \cdot \nabla w_\varepsilon)\,dx = \rho'_\varepsilon \int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon} \psi(\nabla \breve{v}_\varepsilon \cdot \nabla w_\varepsilon)\,dx, \forall\varepsilon > 0.$$

Here

$$\rho'_\varepsilon = \frac{|A^{\lambda(\varepsilon)} \setminus Q^{h(\varepsilon)}|}{|A^{\lambda(\varepsilon)}|} = \frac{|A \setminus Q^{h(\varepsilon)/\lambda(\varepsilon)}|}{|A|}, \qquad \rho''_\varepsilon = \frac{|A^{\lambda(\varepsilon)} \cap Q^{h(\varepsilon)}|}{|A^{\lambda(\varepsilon)}|} = \frac{|A \cap Q^{h(\varepsilon)/\lambda(\varepsilon)}|}{|A|}.$$

It is clear that $\rho'_\varepsilon + \rho''_\varepsilon = 1$ for every $\varepsilon > 0$. Then

$$\begin{aligned}\int_{\Omega_\varepsilon\cap A'_\varepsilon} \varphi(\nabla v_\varepsilon \cdot \nabla w_\varepsilon)\,dx &= \rho'_\varepsilon\left[\int_{\Omega_\varepsilon\cap A'_\varepsilon} \varphi(\nabla v_\varepsilon \cdot \nabla w_\varepsilon)\,dx + \int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon} \psi(\nabla \breve{v}_\varepsilon \cdot \nabla w_\varepsilon)\,dx\right] \\ &= \rho'_\varepsilon\left[\int_{\Omega_\varepsilon\cap A'_\varepsilon} \varphi(\nabla \breve{v}_\varepsilon \cdot \nabla w_\varepsilon)\,dx + \int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon} \psi(\nabla \breve{v}_\varepsilon \cdot \nabla w_\varepsilon)\,dx\right] \\ &= \rho'_\varepsilon\left[\int_{\Omega_\varepsilon\cap A'_\varepsilon} \varphi(\nabla \breve{v}_\varepsilon \cdot \nabla w_\varepsilon)\,dx + \int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon} \varphi(\nabla \breve{v}_\varepsilon \cdot \nabla w_\varepsilon)\,dx\right] \\ &\quad + \rho'_\varepsilon\left(\int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon} (\psi - \varphi)(\nabla \breve{v}_\varepsilon \cdot \nabla w_\varepsilon)\,dx\right). \end{aligned} \tag{6.16}$$

Since the sequences $\{\breve{v}_\varepsilon\}$ and $\{w_\varepsilon\}$ are equibounded in $H^1(\Omega)$, ρ'_ε tends to ρ^* as $\varepsilon \to 0$, and $d_\vartheta = \max\{|\psi(x) - \varphi(y)|\colon |x - y| < \vartheta\}$ tends to zero as $\vartheta \to 0$, we easily obtain

$$\rho'_\varepsilon\left|\int_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon} (\psi - \varphi)(\nabla \breve{v}_\varepsilon \cdot \nabla w_\varepsilon)\,dx\right| \leqslant \|\nabla \breve{v}_\varepsilon\|_{L^2(\Omega)} \|\nabla w_\varepsilon\|_{L^2(\Omega)} d_{2\lambda(\varepsilon)r_0} \xrightarrow{\varepsilon\to 0} 0. \tag{6.17}$$

As a result, taking properties (6.14)–(6.17) into account, we come to the following relation:

$$\begin{aligned}\lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon}\varphi(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,dx &= \lim_{\varepsilon\to 0}\Bigg[\int\limits_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,dx+\int\limits_{\Omega_\varepsilon\cap A''_\varepsilon}\varphi(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,dx\Bigg]\\ &= \lim_{\varepsilon\to 0}\int\limits_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,dx\\ &= \lim_{\varepsilon\to 0}\rho'_\varepsilon\cdot\lim_{\varepsilon\to 0}\Bigg[\int\limits_{\Omega_\varepsilon\cap A'_\varepsilon}\varphi(\nabla\breve v_\varepsilon\cdot\nabla w_\varepsilon)\,dx+\int\limits_{(\Omega\setminus\Omega_\varepsilon)\cap A'_\varepsilon}\varphi(\nabla\breve v_\varepsilon\cdot\nabla w_\varepsilon)\,dx\Bigg]\\ &= \rho^*\lim_{\varepsilon\to 0}\int\limits_{\Omega}\varphi(\nabla\breve v_\varepsilon\cdot\nabla w_\varepsilon)\,dx. \end{aligned}\tag{6.18}$$

In order to complete this proof it only remains to apply property (6.1) of Theorem 6.2 and the fact that $v\in H_0^1(\Omega)$ due to Theorem 4.8. □

Remark 6.7. As follows from Proposition 3.5, the condition $\rho^*\in[1/2,1)$ is always valid. In particular, using the suppositions of Lemma 6.4 concerning the smoothness of the boundary ∂Q in a neighbourhood of the origin, we have $\rho'_\varepsilon=|A\setminus Q^{h(\varepsilon)/\lambda(\varepsilon)}|/|A|\xrightarrow{\varepsilon\to 0}\rho^*=1/2$. So, the main result of Theorem 6.6 can be viewed as follows:

$$\int\limits_{\Omega_\varepsilon}\varphi(\nabla v_\varepsilon\cdot\nabla w_\varepsilon)\,dx\to(1/2)\mu^*\int\limits_{\Omega}\varphi v\,dx\quad\forall\varphi\in C_0^\infty(\Omega),$$

where μ^* is defined by (6.2).

We are now in a position to prove the main result of this section concerning the passage to the limit as $\varepsilon\to 0$ in the following integral identity:

$$\begin{aligned}&\int\limits_{\Omega}\chi_\varepsilon(\nabla\breve y_\varepsilon\cdot\nabla\varphi)\,dx+\int\limits_{\Omega}\chi_\varepsilon\breve y_\varepsilon\varphi\,dx+k_0\varepsilon^{-n}\sigma(\varepsilon)\int\limits_{\Omega}\breve y_\varepsilon\varphi\,d\nu_\varepsilon^{\lambda,h}\\ &\quad=\int\limits_{\Omega}\chi_\varepsilon f_\varepsilon\varphi\,dx+\varepsilon^{-n}\sigma(\varepsilon)\int\limits_{\Omega}p_\varepsilon\varphi\,d\nu_\varepsilon^{\lambda,h},\quad\forall\varphi\in H^1(\Omega;\Gamma_\varepsilon^D\cup\Sigma_\varepsilon).\end{aligned}\tag{6.19}$$

Here $\{(a_\varepsilon,p_\varepsilon,\breve y_\varepsilon)\in\mathbb{X}_\varepsilon\}_{\varepsilon>0}$ is an equibounded sequence of admissible triplets, and $\sigma(\varepsilon)$ is defined by (3.9).

By Theorem 4.8, this sequence is relatively compact with respect to the weak convergence in the variable space $\mathbb{X}_\varepsilon$. So, we may suppose that there exists a triplet $(a,p,y)\in[H^2(\Omega)\cap H_0^1(\Omega)]\times L^2(\Omega)\times H_0^1(\Omega)$ such that $(a_\varepsilon,p_\varepsilon,y_\varepsilon)\xrightarrow{w}(a,p,y)$.

Theorem 6.8. *Let ρ^* be a limit of the sequence* (3.18) *as $\varepsilon\to 0$, and let*

$$\{a_\varepsilon\in H_0^1(\Omega)\cap H^2(\Omega)\cap L^2(\Omega,d\mu_\varepsilon^{\lambda,h})\}\quad\text{and}\quad\{p_\varepsilon\in L^2(\Omega,d\nu_\varepsilon^{\lambda,h})\}\tag{6.20}$$

be any bounded sequences of admissible controls for $\hat{\mathbb{P}}_\varepsilon$-problems such that

$$a_\varepsilon\rightharpoonup a\quad\text{in } H^2(\Omega)\cap H_0^1(\Omega)\cap L^2(\Omega,d\mu_\varepsilon^{\lambda,h});\tag{6.21}$$

$$p_\varepsilon\rightharpoonup p\quad\text{in } L^2(\Omega,d\nu_\varepsilon^{\lambda,h}).\tag{6.22}$$

Let $\{y_\varepsilon=y_\varepsilon(a_\varepsilon,p_\varepsilon)\in H^1(\Omega,\Sigma_\varepsilon)\cap L^2(\Omega,d\nu_\varepsilon^{\lambda,h})\}_{\varepsilon>0}$ be the corresponding solutions to problem (1.5). *Then $(a_\varepsilon,p_\varepsilon,y_\varepsilon)\xrightarrow{w}(a,p,y)$ as $\varepsilon\to 0$, $y-a\in L^2(\Omega,d\mu^*)$, and y is the unique function in $H_0^1(\Omega)$ satisfying the following integral identity*:

$$\int\limits_{\Omega}(\nabla y\cdot\nabla\varphi)\,dx+\big(1+k_0|\partial Q|_H\big)\int\limits_{\Omega}y\varphi\,dx+\rho^*\int\limits_{\Omega}(y-a)\varphi\,d\mu^*$$

$$= \int_\Omega f\varphi\,dx + |\partial Q|_H \int_\Omega p\varphi\,dx, \quad \forall\varphi \in H_0^1(\Omega) \cap L^2(\Omega, d\mu^*). \tag{6.23}$$

Proof. Let $\{w_\varepsilon \in H^1(\Omega)\}_{\varepsilon>0}$ be a sequence defined by Theorem 6.2. Let $\varphi \in C_0^\infty(\Omega)$ be a fixed function. It is clear that $w_\varepsilon\varphi \in H^1(\Omega; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$ for every $\varepsilon > 0$. Take $w_\varepsilon\varphi$ as test functions in (6.19). Then the following integral identity holds true for every $\varepsilon > 0$

$$\int_\Omega \chi_\varepsilon\big(\nabla(\breve{y}_\varepsilon - a_\varepsilon) \cdot \nabla(w_\varepsilon\varphi)\big)\,dx + \int_\Omega \chi_\varepsilon\big(\nabla a_\varepsilon \cdot \nabla(w_\varepsilon\varphi)\big)\,dx + \int_\Omega \chi_\varepsilon \breve{y}_\varepsilon w_\varepsilon\varphi\,dx + k_0\varepsilon^{-n}\sigma(\varepsilon)\int_\Omega \breve{y}_\varepsilon w_\varepsilon\varphi\,d\nu_\varepsilon^{\lambda,h}$$
$$= \int_\Omega \chi_\varepsilon f_\varepsilon w_\varepsilon\varphi\,dx + \varepsilon^{-n}\sigma(\varepsilon)\int_\Omega p_\varepsilon w_\varepsilon\varphi\,d\nu_\varepsilon^{\lambda,h}, \quad \forall\varphi \in H^1(\Omega; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon). \tag{6.24}$$

Observe that in view of the boundedness of $\{f_\varepsilon \in L^2(\Omega)\}$, by using estimate (3.15) and Theorem 4.11, we may suppose that there is a function $y \in H_0^1(\Omega)$ such that $(a_\varepsilon, p_\varepsilon, y_\varepsilon) \overset{w}{\rightharpoonup} (a, p, y)$ as $\varepsilon \to 0$. We now pass to the limit in (6.24) as $\varepsilon \to 0$. We do it for each term of (6.24) separately. Observe first that

$$\int_\Omega \chi_\varepsilon\big(\nabla(\breve{y}_\varepsilon - a_\varepsilon) \cdot \nabla(w_\varepsilon\varphi)\big)\,dx = \int_\Omega \chi_\varepsilon w_\varepsilon\big(\nabla(\breve{y}_\varepsilon - a_\varepsilon) \cdot \nabla\varphi\big)\,dx + \int_\Omega \chi_\varepsilon\varphi\big(\nabla(\breve{y}_\varepsilon - a_\varepsilon) \cdot \nabla w_\varepsilon\big)\,dx.$$

Take into account the following facts: $\chi_\varepsilon w_\varepsilon \to 1$ strongly in $L^2(\Omega)$ (see Lemma 4.12), $\breve{y}_\varepsilon - a_\varepsilon \rightharpoonup v = y - a$ in $H_0^1(\Omega)$, $v_\varepsilon = \breve{y}_\varepsilon - a_\varepsilon \in H^1(\Omega_\varepsilon; \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$ for every $\varepsilon > 0$. Then, by Theorem 6.6 we have

$$\int_\Omega \chi_\varepsilon w_\varepsilon\big(\nabla(\breve{y}_\varepsilon - a_\varepsilon) \cdot \nabla\varphi\big)\,dx \xrightarrow{\varepsilon\to 0} \int_\Omega \big(\nabla(y-a) \cdot \nabla\varphi\big)\,dx, \tag{6.25}$$

$$\int_\Omega \chi_\varepsilon\varphi\big(\nabla(\breve{y}_\varepsilon - a_\varepsilon) \cdot \nabla w_\varepsilon\big)\,dx \xrightarrow{\varepsilon\to 0} \rho^*\int_\Omega \varphi(y-a)\,d\mu^*, \quad y - a \in L^2(\Omega, d\mu^*). \tag{6.26}$$

By (3.14) it follows that $\{a_\varepsilon\}$ is bounded in $H^2(\Omega) \cap H_0^1(\Omega)$, so

$$a_\varepsilon \to a \quad \text{weakly in } H^2(\Omega) \quad \text{and hence} \quad \nabla a_\varepsilon \to \nabla a \quad \text{strongly in } \big[L^2(\Omega)\big]^n.$$

Then, due to (6.21)–(6.22) and since $\nabla w_\varepsilon \rightharpoonup 0$ in $[L^2(\Omega)]^n$, we obtain

$$\int_\Omega \chi_\varepsilon\big(\nabla a_\varepsilon \cdot \nabla(w_\varepsilon\varphi)\big)\,dx \xrightarrow{\varepsilon\to 0} 0, \qquad \int_\Omega \chi_\varepsilon\breve{y}_\varepsilon w_\varepsilon\varphi\,dx \xrightarrow{\varepsilon\to 0} \int_\Omega y\varphi\,dx, \tag{6.27}$$

$$k_0\varepsilon^{-n}\sigma(\varepsilon) \xrightarrow{\varepsilon\to 0} k_0|\partial Q|_H \quad \text{by (3.4)}, \qquad \int_\Omega \chi_\varepsilon f_\varepsilon w_\varepsilon\varphi\,dx \xrightarrow{\varepsilon\to 0} \int_\Omega f\varphi\,dx, \tag{6.28}$$

$$\int_\Omega \breve{y}_\varepsilon w_\varepsilon\varphi\,d\nu_\varepsilon^{\lambda(\varepsilon),h(\varepsilon)} \xrightarrow{\varepsilon\to 0} \int_\Omega y\varphi\,dx \quad \text{as weak limit in } L^2(\Omega, d\nu_\varepsilon^{\lambda,h}), \tag{6.29}$$

$$\varepsilon^{-n}\sigma(\varepsilon)\int_\Omega p_\varepsilon w_\varepsilon\varphi\,d\nu_\varepsilon^{\lambda(\varepsilon),h(\varepsilon)} \xrightarrow{\varepsilon\to 0} |\partial Q|_H \int_\Omega p\varphi\,dx. \tag{6.30}$$

Thus the required relation (6.23) is established for any function $\varphi \in C_0^\infty(\Omega)$. Moreover, from the fact that $\breve{y}_\varepsilon - a_\varepsilon \in H^1(\Omega, \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$ and $a_\varepsilon \to a$ in $H_0^1(\Omega)$, we conclude that $(\breve{y}_\varepsilon - a_\varepsilon) \rightharpoonup (y - a)$ in $H^1(\Omega)$, and hence $y \in H_0^1(\Omega)$.

To conclude, we note that the integral identity (6.23) can always be interpreted as the variational formulation of the problem

$$\left.\begin{aligned} &-\Delta y + (1 + k_0|\partial Q|_H)y + \rho^*(y-a)\mu^* = f + p|\partial Q|_H, \\ &y \in H_0^1(\Omega), \quad y - a \in L^2(\Omega, d\mu^*), \end{aligned}\right\} \tag{6.31}$$

with respect to which the following result is well known: for every $a \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^2(\Omega, d\mu^*)$, $p \in L^2(\Omega)$, and $f \in L^2(\Omega)$ there exists a unique solution of (6.31) (see [14]). This completes the proof. □

The following statement is a direct consequence of well known results of the theory of boundary value problems [21].

Corollary 6.9. *Let* $(a_1, p_1, y_1), (a_2, p_2, y_2) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega) \times H_0^1(\Omega)$ *be any triplets satisfying the relation* (6.31). *Then there exists a constant* $\hat{C} > 0$ $(\hat{C} = \hat{C}(\Omega, |\partial Q|, \rho^*))$ *such that*

$$\|y_1 - y_2\|_{H_0^1(\Omega)\cap L^2(\Omega,d\mu^*)} \leqslant \hat{C}\big[\|a_1 - a_2\|_{H^2(\Omega)\cap H_0^1(\Omega)} + \|p_1 - p_2\|_{L^2(\Omega)} + \|a_1 - a_2\|_{L^2(\Omega,d\mu^*)}\big]. \tag{6.32}$$

7. Identification of the homogenized optimal control problem

In this section we show that for the sequence (5.1), there exists a weak variational limit with respect to the w-convergence, and it can be recovered in an explicit form. We begin with the following result:

Lemma 7.1. *Let* $\{(a_\varepsilon, p_\varepsilon, y_\varepsilon) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ *be a bounded sequence of admissible solutions, assumed to be* w*-convergent to a triplet* $(a, p, y) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega) \times H_0^1(\Omega)$. *Then*

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} |\nabla y_\varepsilon|^2 dx = \int_\Omega |\nabla y|^2\, dx + \rho^* \int_\Omega (y - a)^2\, d\mu^*, \tag{7.1}$$

where the measure μ^* *and value* ρ^* *are defined in Theorems* 6.2 *and* 6.6, *respectively.*

Proof. We first observe that

$$\int_{\Omega_\varepsilon} |\nabla y_\varepsilon|^2\, dx = \int_\Omega \chi_\varepsilon |\nabla \breve{y}_\varepsilon - \nabla a_\varepsilon|^2\, dx + 2\int_\Omega \chi_\varepsilon (\nabla \breve{y}_\varepsilon \cdot \nabla a_\varepsilon)\, dx - \int_\Omega \chi_\varepsilon |\nabla a_\varepsilon|^2\, dx. \tag{7.2}$$

Then, taking into account the facts that $\nabla a_\varepsilon \to \nabla a$ in $[L^2(\Omega)]^n$, $\breve{y}_\varepsilon \rightharpoonup y$ in $H^1(\Omega)$, and $\chi_\varepsilon \to 1$ in $L^2(\Omega)$, we have

$$\int_\Omega \chi_\varepsilon (\nabla \breve{y}_\varepsilon \cdot \nabla a_\varepsilon)\, dx \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega \nabla y \cdot \nabla a dx, \qquad \int_\Omega \chi_\varepsilon |\nabla a_\varepsilon|^2\, dx \overset{\varepsilon\to 0}{\longrightarrow} \int_\Omega |\nabla a|^2\, dx. \tag{7.3}$$

Since $(\breve{y}_\varepsilon - a_\varepsilon) \in H^1(\Omega, \Gamma_\varepsilon^D \cup \Sigma_\varepsilon)$ for every $\varepsilon > 0$, it follows that we can take $\breve{y}_\varepsilon - a_\varepsilon$ as a test function φ in (6.19). Then the following equality is ensured:

$$\begin{aligned}\int_\Omega \chi_\varepsilon |\nabla \breve{y}_\varepsilon - \nabla a_\varepsilon|^2\, dx = &- \int_\Omega \chi_\varepsilon \nabla a_\varepsilon \cdot (\nabla \breve{y}_\varepsilon - \nabla a_\varepsilon)\, dx - \int_\Omega \chi_\varepsilon \breve{y}_\varepsilon (\breve{y}_\varepsilon - a_\varepsilon)\, dx \\ &- k_0 \varepsilon^{-n} \sigma(\varepsilon) \int_\Omega \breve{y}_\varepsilon (\breve{y}_\varepsilon - a_\varepsilon)\, d\nu_\varepsilon^{\lambda,h} + \int_\Omega \chi_\varepsilon f_\varepsilon (\breve{y}_\varepsilon - a_\varepsilon)\, dx \\ &+ \varepsilon^{-n} \sigma(\varepsilon) \int_\Omega p_\varepsilon (\breve{y}_\varepsilon - a_\varepsilon)\, d\nu_\varepsilon^{\lambda,h}.\end{aligned} \tag{7.4}$$

By properties (6.25)–(6.30), we obtain

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_\Omega \chi_\varepsilon |\nabla \breve{y}_\varepsilon - \nabla a_\varepsilon|^2\, dx = &- \int_\Omega \nabla a \cdot (\nabla y - \nabla a)\, dx - \int_\Omega y(y - a)\, dx \\ -k_0 |\partial Q|_H \lim_{\varepsilon\to 0} \int_\Omega \breve{y}_\varepsilon (\breve{y}_\varepsilon - a_\varepsilon)\, d\nu_\varepsilon^{\lambda,h} &+ \int_\Omega f(y - a)\, dx + |\partial Q|_H \lim_{\varepsilon\to 0} \int_\Omega p_\varepsilon (\breve{y}_\varepsilon - a_\varepsilon)\, d\nu_\varepsilon^{\lambda,h}.\end{aligned}$$

Since $(\breve{y}_\varepsilon - a_\varepsilon) \in H^1(\Omega, \Gamma_\varepsilon^D \cup \Sigma_\varepsilon) \cap L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$, we have: $\breve{y}_\varepsilon - a_\varepsilon \to y - a$ strongly in $L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$ (by Theorem 4.9), $\breve{y}_\varepsilon \rightharpoonup y$ in $L^2(\Omega, d\nu_\varepsilon^{\lambda,h})$ (by Theorem 4.8). Hence, in view of the definition of the strong convergence in variable spaces, we conclude

$$\lim_{\varepsilon\to 0}\int_\Omega \breve{y}_\varepsilon(\breve{y}_\varepsilon - a_\varepsilon)\, d\nu_\varepsilon^{\lambda,h} = \int_\Omega y(y-a)\, dx, \qquad \lim_{\varepsilon\to 0}\int_\Omega p_\varepsilon(\breve{y}_\varepsilon - a_\varepsilon)\, d\nu_\varepsilon^{\lambda,h} = \int_\Omega p(y-a)\, dx.$$

As a result, we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_\Omega \chi_\varepsilon|\nabla\breve{y}_\varepsilon - \nabla a_\varepsilon|^2\, dx &= -\int_\Omega \nabla a\cdot(\nabla y - \nabla a)\, dx - \int_\Omega y(y-a)\, dx \\ &\quad - k_0|\partial Q|_H\int_\Omega y(y-a)\, dx + \int_\Omega f(y-a)\, dx + |\partial Q|_H\int_\Omega p(y-a)\, dx. \end{aligned}\tag{7.5}$$

Let us consider the integral identity (6.23) with the test function $\varphi = y - a$. By a rearrangement, we have

$$\begin{aligned}&\int_\Omega |\nabla y - \nabla a|^2\, dx + \rho^*\int_\Omega y(y-a)\, d\mu^* - \rho^*\int_\Omega a(y-a)\, d\mu^* \\ &\quad = -\int_\Omega \nabla a\cdot(\nabla y - \nabla a)\, dx - (1 + k_0|\partial Q|_H)\int_\Omega y(y-a)\, dx \\ &\qquad + \int_\Omega f(y-a)\, dx + |\partial Q|_H\int_\Omega p(y-a)\, dx. \end{aligned}\tag{7.6}$$

The comparison of (7.5) with (7.6) leads to the following equality:

$$\lim_{\varepsilon\to 0}\int_\Omega \chi_\varepsilon|\nabla\breve{y}_\varepsilon - \nabla a_\varepsilon|^2\, dx = \int_\Omega |\nabla y - \nabla a|^2\, dx + \rho^*\int_\Omega (y-a)^2\, d\mu^*.$$

which, together with (7.2)–(7.3), concludes the proof. □

We are now in position to establish the identification result of the weak variational limit for the sequence of constrained minimization problems (5.1).

Theorem 7.2. *For the sequence* (5.1) *there exists a unique weak variational limit with respect to the w-convergence which can be represented in the form* (5.2)*, where the cost functional I_0 and the set of admissible solutions Ξ_0 are defined as follows*:

$$\begin{aligned} I_0(a,p,y) &= \int_\Omega |\nabla y|^2\, dx + \int_\Omega |y - z^\partial|^2\, dx + \rho^*\int_\Omega (y-a)^2\, d\mu^* \\ &\quad + |\partial Q|_H\int_\Omega p^2\, dx + |K\cap\partial\Lambda|_H\int_\Omega a^2\, dx, \end{aligned}\tag{7.7}$$

$$\Xi_0 = \left\{ (a,p,y)\;\middle|\; \begin{array}{l} y\in H_0^1(\Omega),\quad p\in L^2(\Omega), \\ a\in H^2(\Omega)\cap H_0^1(\Omega),\quad y-a\in L^2(\Omega, d\mu^*), \\ \|a\|_{H^2(\Omega)} \leqslant C_0, \\ \int_\Omega(\nabla y\cdot\nabla\varphi)\, dx + \rho^*\int_\Omega (y-a)\varphi\, d\mu^* \\ \quad +(1+k_0|\partial Q|_H)\int_\Omega y\varphi\, dx = \int_\Omega f\varphi\, dx + |\partial Q|_H\int_\Omega p\varphi\, dx, \\ \forall\varphi\in H_0^1(\Omega)\cap L^2(\Omega, d\mu^*) \end{array} \right\}\tag{7.8}$$

Here Λ is a cone in $\mathbb{R}^n$ (*see Proposition* 3.3).

Proof. The proof of this theorem is divided into two steps each of them concerns the verification of the corresponding item of Definition 5.3.

STEP 1: *Statement (1) of Definition 5.3 is valid.*

Let $\{(a_k, p_k, \breve{y}_k) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ be a bounded sequence which is w-convergent to a triplet $(u, p, y) \in [H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega) \times H_0^1(\Omega)$. Let $\{\varepsilon_k\}$ be a subsequence of $\{\varepsilon\}$ such that $\varepsilon_k \to 0$ as $k \to \infty$ and $(a_k, p_k, y_k) \in \hat{\Xi}_{\varepsilon_k}$ for all $k \in \mathbb{N}$. Then, due to Theorem 6.8, we have that the w-limit triplet (u, p, y) satisfies integral identity (6.23), and moreover

$$C_0 \geqslant \liminf_{k\to\infty} \|a_k\|_{H^2(\Omega)} \geqslant \|a\|_{H^2(\Omega)}$$

by the lower semicontinuity of $\|\cdot\|_{H^2(\Omega)}$ with respect to the weak convergence in $H^2(\Omega)$. So, inclusion (5.3) holds true.

We now turn back to the test inequality (5.3). By the property of the lower semicontinuity of the weak convergence in variable spaces, Proposition 3.3, and relation (3.4), we have

$$\lim_{k\to\infty}\left[\varepsilon_k^{-n}\sigma(\varepsilon_k)\int_\Omega p_{\varepsilon_k}^2\, d\nu_{\varepsilon_k}^{\lambda(\varepsilon_k),h(\varepsilon_k)} + |K\cap\partial Q^{\varsigma(\varepsilon_k)}|_H \int_\Omega a_{\varepsilon_k}^2\, d\mu_{\varepsilon_k}^{\lambda(\varepsilon_k),h(\varepsilon_k)}\right]$$
$$\geqslant |\partial Q|_H \int_\Omega p^2\, dx + |K\cap\partial\Lambda|_H\int_\Omega a^2\, dx.$$

To conclude it remains only to apply Lemma 7.1.

STEP 2: *Statement (2) of Definition 5.3 holds true*.

Let $(a, p, y) \in \Xi_0$ be an admissible triplet for the minimization problem (5.2). As readily follows from (7.7), for any triplet $(a, \hat{p}, \hat{y}) \in \Xi_0$ there exists a constant $\gamma > 0$ depending on Ω, z^∂, ρ^*, p, y, and $|\partial Q|_H$, such that

$$\left|I_0(a,p,y) - I_0(a,\hat{p},\hat{y})\right| \leqslant \gamma\left(\|y-\hat{y}\|^2_{H_0^1(\Omega)\cap L^2(\Omega,d\mu^*)} + \|p-\hat{p}\|^2_{L^2(\Omega)}\right). \tag{7.9}$$

Let $1 > \delta > 0$ be a given value. Using the density of the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$, we can choose an element $\hat{p} \in H^1(\Omega)$ such that $\|p - \hat{p}\|_{L^2(\Omega)} < \varrho$, where

$$\varrho < \max\left\{\delta/\sqrt{\gamma(1+\hat{C})}, \delta/\hat{C}\right\}. \tag{7.10}$$

Let $\hat{y} = \hat{y}(a, \hat{p})$ be the corresponding solution of the boundary value problem (6.31). Then due to estimates (6.32) and (7.9), we have

$$\left|I_0(a,p,y) - I_0(a,\hat{p},\hat{y})\right| \leqslant \gamma(1+\hat{C})\|p-\hat{p}\|^2_{L^2(\Omega)} < \delta, \tag{7.11}$$
$$\|y-\hat{y}\|_{H_0^1(\Omega)\cap L^2(\Omega,d\mu^*)} \leq \hat{C}\|p-\hat{p}\|_{L^2(\Omega)} < \delta.$$

We now construct the δ-realizing sequence $\{(\hat{a}_\varepsilon^{\lambda,h}, \hat{p}_\varepsilon^{\lambda,h}, \hat{y}_\varepsilon^{\lambda,h}) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$ as follows: $\hat{a}_\varepsilon^{\lambda,h} = a$, $\hat{p}_\varepsilon^{\lambda,h} = \hat{p}$, and we take $\hat{y}_\varepsilon^{\lambda,h}$ as the corresponding solution of the original boundary value problem (1.5). Hence $(\hat{a}_\varepsilon^{\lambda,h}, \hat{p}_\varepsilon^{\lambda,h}, \hat{y}_\varepsilon^{\lambda,h}) \in \hat{\Xi}_\varepsilon$ for every $\varepsilon > 0$. It is clear that this sequence in equibounded in $\mathbb{X}_\varepsilon$. Moreover, by Lemma 4.2 and Theorems 4.8–4.9, we have

$$\hat{a}_\varepsilon^{\lambda,h} \rightharpoonup a \quad \text{in } L^2(\Omega, d\mu_\varepsilon^{\lambda,h}) \quad \text{and} \quad \int_\Omega (\hat{a}_\varepsilon^{\lambda,h})^2\, d\mu_\varepsilon^{\lambda,h} \stackrel{\varepsilon\to 0}{\longrightarrow} \int_\Omega a^2\, dx; \tag{7.12}$$

$$\hat{p}_\varepsilon^{\lambda,h} \rightharpoonup \hat{p} \quad \text{in } L^2(\Omega, d\nu_\varepsilon^{\lambda,h}) \quad \text{and} \quad \int_\Omega (\hat{p}_\varepsilon^{\lambda,h})^2\, d\nu_\varepsilon^{\lambda,h} \stackrel{\varepsilon\to 0}{\longrightarrow} \int_\Omega \hat{p}^2\, dx. \tag{7.13}$$

Then, applying Theorem 4.11, we conclude that the sequence $\{(\hat{a}_\varepsilon^{\lambda,h}, \hat{p}_\varepsilon^{\lambda,h}, \hat{y}_\varepsilon^{\lambda,h})\}$ is compact with respect to the w-convergence. Let $(a, \hat{p}, \hat{y}^*)$ be its w-limit. Due to Theorem 6.8 we have $(a, \hat{p}, \hat{y}^*) \in \Xi_0$. Since the boundary value problem (6.31) has a unique solution for every fixed a and $\hat{p}$, it follows that $\hat{y}^* = \hat{y}$, and hence $(\hat{a}_\varepsilon^{\lambda,h}, \hat{p}_\varepsilon^{\lambda,h}, \hat{y}_\varepsilon^{\lambda,h}) \stackrel{w}{\rightharpoonup} (a, \hat{p}, \hat{y})$ as $\varepsilon \to 0$.

Consequently, properties (5.4) are fulfilled. It remains only to verify inequality (5.5). To do so, we use properties (7.12)–(7.13) and Lemma 7.1. Then $\lim_{\varepsilon\to 0}\hat{I}_\varepsilon(\hat{a}_\varepsilon^{\lambda,h},\hat{p}_\varepsilon^{\lambda,h},\hat{y}_\varepsilon^{\lambda,h}) = I_0(a,\hat{p},\hat{y})$. To conclude we apply inequality (7.11). This yields the required result

$$I_0(a,p,y) \geqslant \lim_{\varepsilon\to 0}\hat{I}_\varepsilon(\hat{a}_\varepsilon^{\lambda,h},\hat{p}_\varepsilon^{\lambda,h},\hat{y}_\varepsilon^{\lambda,h}) - \delta$$

end this ends the proof. □

It is now clear that the constrained minimization problem (5.2) can be interpreted as an optimal control problem. So, in accordance with Definition 5.5, we can give the following deduction: for the optimal control problem (1.5)–(1.9) (so-called $\mathbb{P}_\varepsilon$-problem) there exists a unique homogenized one with respect to w-convergence as $\varepsilon \to 0$ and it can be represented in the form (1.10)–(1.13).

Proposition 7.3. *The limit optimal control problem* (1.10)–(1.13) *has a unique solution.*

Proof. The proof is quite similar to that given in Theorem 2.1. The main difference is the choice of the topology for the space of admissible solutions $[H^2(\Omega)\cap H_0^1(\Omega)]\times L^2(\Omega)\times H_0^1(\Omega)$ with respect to which the set Ξ_0 and the cost functional I_0 possess the required topological properties, one of which has to guarantee the inclusion (1.12). It is clear that this topology can be taken as $\tau = (w_{H^2(\Omega)\cap H_0^1(\Omega)})\times(w_{L^2(\Omega)})\times(w_{H_0^1(\Omega)})$, where $w_{(\cdot)}$ denotes the weak topology of the corresponding Banach space. Indeed, due to the fact that $\mu^*\in\mathcal{M}_0^+(\Omega)$, the space $H_0^1(\Omega)\cap L^2(\Omega,d\mu^*)$ is well defined (see Remark 3.1). Hence, if $y_n \rightharpoonup y$ in $H_0^1(\Omega)$ and $a_n \rightharpoonup a$ in $H^2(\Omega)\cap H_0^1(\Omega)$, then $(y_n - a_n)\rightharpoonup (y-a)$ in $L^2(\Omega,d\mu^*)$. Moreover, it can be easily checked (by passing to the limit in (1.10), (1.12)) that the set Ξ_0 is τ-closed and the cost functional I_0 is τ-lower semicontinuous. In another aspects we do not modify the proof of Theorem 2.1 is then valid with any more modifications. □

Thus, combining the results of Theorems 2.1, 5.4, and Proposition 7.3, we come to the following conclusion concerning the variational properties of the homogenized optimal control problem (1.10)–(1.13):

Theorem 7.4. *Let* $\{(a_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0)\in\hat{\Xi}_\varepsilon\}_{\varepsilon>0}$ *be the optimal solutions of the problems* $\hat{\mathbb{P}}_\varepsilon$. *Then*

$$\lim_{\varepsilon\to 0}\hat{I}_\varepsilon(a_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0) = \inf_{(a,p,y)\in\Xi_0} I_0(a,p,y) = I_0(a^0,p^0,y^0), \tag{7.14}$$

and

$$(a_\varepsilon^0,p_\varepsilon^0,y_\varepsilon^0)\xrightarrow{w}(a^0,p^0,y^0)\quad \textit{in the variable space } \mathbb{X}_\varepsilon. \tag{7.15}$$

8. Optimality conditions for the homogenized problem and suboptimal controls for $\mathbb{P}_\varepsilon$-problem

In this section we derive the optimality conditions for the problem (1.10)–(1.13) from which an optimal triplet may be determined. For this, we use the Lagrange multiplier principle. We obtain the weak form of the optimality system equations that an optimal triplet (a^0,p^0,y^0) and Lagrange multipliers must satisfy. This optimality system can serve as a basis for the construction of suboptimal solutions to the original problem in perforated domains.

We recall some central tenet of the Lagrange multiplier principle. Let $\mathbb{Y}$, $\mathbb{U}$, and $\mathbb{V}$ be the Banach spaces. Let $I:\mathbb{Y}\times\mathbb{U}\to\overline{\mathbb{R}}$ be a cost functional, and let $F(y,u):\mathbb{Y}\times\mathbb{U}\to\mathbb{V}$ be a mapping. Let U_∂ be a closed subset of $\mathbb{U}$ with a nonempty interior. We have the following minimization problem:

$$I(y,u)\longrightarrow \inf,\quad F(y,u)=0,\quad u\in U_\partial. \tag{8.1}$$

The Lagrange functional for the problem (8.1) is defined by

$$\mathcal{L}(y,u,\lambda,\psi) = \lambda I(y,u) + \langle F(y,u),\psi\rangle,\quad \text{where } \lambda\in\mathbb{R}_+,\ \psi\in\mathbb{V}'. \tag{8.2}$$

Theorem 8.1. *(Ioffe and Tikhomirov [17].) Let* $(y^0,u^0)\in\mathbb{Y}\times\mathbb{U}$ *be a solution of* (8.1). *Assume that the mappings* $y\to I(y,u)$ *and* $y\to F(y,u)$ *are continuously differentiable at* $y\in\mathcal{O}(y^0)$ *and* $\operatorname{Im}F_y'(y^0,u^0)=\mathbb{V}$. *Assume that the* *mapping $u \to I(y,u)$ is convex, I is differentiable at (y^0,u^0), and that the mapping $u \to F(y,u)$ is continuous and affine. Then λ can be taken as* 1 *and there exists a $\psi \in \mathbb{V}'$ such that*

$$\big\langle \mathcal{L}'_y(y^0,u^0,1,\psi),h\big\rangle = 0 \quad \forall h \in \mathbb{V} \quad \text{and} \quad \big\langle \mathcal{L}'_u(y^0,u^0,1,\psi),u-u^0\big\rangle \geqslant 0 \quad \forall u \in U_\partial. \tag{8.3}$$

We now apply the Lagrange principle to the optimal control problem (1.10)–(1.13).

Theorem 8.2. *A triplet*

$$(a^0,p^0,y^0) \in \big[H^2(\Omega)\cap H_0^1(\Omega)\big] \times L^2(\Omega) \times \big[H^2(\Omega)\cap H_0^1(\Omega)\big], \quad y^0 - a^0 \in L^2(\Omega,d\mu^*)$$

is an optimal solution to the problem (1.10)–(1.13) *if and only if there exists a function $\psi \in H^2(\Omega)\cap H_0^1(\Omega)\cap L^2(\Omega,d\mu^*)$ such that the quaternary (a^0,p^0,y^0,ψ) satisfies the following optimality system*:

$$\left.\begin{aligned} &\textstyle\int_\Omega (\nabla y^0\cdot\nabla\varphi)\,dx + \rho^*\int_\Omega (y^0-a^0)\varphi\,d\mu^* + (1+k_0|\partial Q|_H)\int_\Omega y^0\varphi\,dx \\ &\quad = \textstyle\int_\Omega f\varphi\,dx + |\partial Q|_H\int_\Omega p^0\varphi\,dx, \quad \forall\varphi\in H_0^1(\Omega)\cap L^2(\Omega,d\mu^*). \end{aligned}\right\} \tag{8.4}$$

$$\left.\begin{aligned} &\textstyle\int_\Omega (\nabla\psi + 2\nabla y^0)\cdot\nabla\phi\,dx + \int_\Omega \big[2(y^0-z^\partial) + (1+k_0|\partial Q|)\psi\big]\phi\,dx \\ &\quad +\rho^*\textstyle\int_\Omega (\psi + 2(y^0-a^0))\phi\,d\mu^* = 0 \quad \forall\phi\in H_0^1(\Omega)\cap L^2(\Omega,d\mu^*), \end{aligned}\right\} \tag{8.5}$$

$$p^0 = \psi/2 \quad \text{a.e. in } \Omega, \tag{8.6}$$

$$\left.\begin{aligned} &2|K\cap\partial\Lambda|_H\textstyle\int_\Omega a^0(a-a^0)\,dx - \rho^*\int_\Omega \psi(a-a^0)\,d\mu^* - 2\rho^*\int_\Omega (y^0-a^0)(a-a^0)\,d\mu^* \geqslant 0, \\ &\quad \forall a \in \big\{a\in H^2(\Omega)\cap H_0^1(\Omega)\colon \|a\|_{H^2(\Omega)} \leqslant C_0\big\}. \end{aligned}\right\} \tag{8.7}$$

Proof. Let (a^0,p^0,y^0) be an optimal solution to problem (1.10)–(1.13). To apply the Lagrange principle, we set $\mathbb{Y} = H^2(\Omega)\cap H_0^1(\Omega)$, $\mathbb{U} = [H^2(\Omega)\cap H_0^1(\Omega)]\times L^2(\Omega)$, $\mathbb{V} = L^2(\Omega)$, and $F(a,p,y) = -\Delta y + (1+k_0|\partial Q|_H)y + \rho^*(y-a)\mu^* - f - |\partial Q|_H p$. Since $f\in L^2(\Omega)$, it follows that the boundary value problem (6.31) has a unique solution $y\in H^2(\Omega)\cap H_0^1(\Omega)$ for any $a\in H^2(\Omega)\cap H_0^1(\Omega)$ and $p\in L^2(\Omega)$, and moreover, in this case $y-a\in L^2(\Omega,d\mu^*)$ (see [21,14]). Hence $\mathrm{Im}F'_y = \mathbb{V}$. Thus all the assumptions of Theorem 8.1 are fulfilled. We now define the Lagrange function as follows

$$\begin{aligned} \mathcal{L}(a,p,y,\psi) &= \int_\Omega |\nabla y|^2\,dx + \int_\Omega |y-z^\partial|^2\,dx + \rho^*\int_\Omega (y-a)^2 d\mu^* \\ &\quad + |\partial Q|_H\int_\Omega p^2\,dx + |K\cap\partial\Lambda|_H\int_\Omega a^2\,dx + \int_\Omega (\nabla y\cdot\nabla\psi)\,dx - \int_\Omega f\psi\,dx \\ &\quad + \rho^*\int_\Omega (y-a)\psi\,d\mu^* + \big(1+k_0|\partial Q|_H\big)\int_\Omega y\psi\,dx - |\partial Q|_H\int_\Omega p\psi\,dx \\ &\forall\psi\in H_0^1(\Omega)\cap L^2(\Omega,d\mu^*). \end{aligned}$$

In accordance with Theorem 8.1, there exists a function $\psi\in H_0^1(\Omega)\cap L^2(\Omega,d\mu^*)$ such that relations (8.2)–(8.3) are valid. In this case relation (8.3) takes the form (8.6)–(8.7), whereas (8.2) can be written as (8.5). Since $y^0\in H^2(\Omega)\cap H_0^1(\Omega)$ and $z^\partial\in L^2(\Omega)$, it follows that the bilinear form $\langle\nabla\psi,\nabla\phi\rangle_{L^2(\Omega)} + (1+k_0|\partial Q|_H)\langle\psi,\phi\rangle_{L^2(\Omega)} + \rho^*\langle\psi,\phi\rangle_{L^2(\Omega,d\mu^*)}$ is coercive on the space $H_0^1(\Omega)\cap L^2(\Omega,d\mu^*)$. So, by the Riesz representation theorem we immediately conclude that there exists a unique function $\psi\in H_0^1(\Omega)\cap L^2(\Omega)$ satisfying equality (8.5) and such that $\psi\in H^2(\Omega)$. Thus, the first part of Theorem 8.1 is proved, i.e. (8.4)–(8.7) are the necessary optimality conditions. Since the mapping $y\to I_0(a,p,y)$ is convex and the mapping $(a,p)\to F(a,p,y)$ is continuous and affine, relations (8.4)–(8.7) are also sufficient optimality conditions for the problem (1.10)–(1.13). As this problem is uniquely solvable, the proof is complete. □

As an evident consequence of this theorem we have the following result.

Corollary 8.3. *If* (a^0, p^0, y^0) *is an optimal solution to* (1.10)–(1.13) *then*

$$p^0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^2(\Omega, d\mu^*). \tag{8.8}$$

Now using (8.8) and applying Lemmas 4.2 and 4.4, we immediately establish the following approximation property for the optimal controls:

Proposition 8.4. *If* $p^0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^2(\Omega, d\mu^*)$ *and* $a^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *are the optimal controls to the homogenized problem* (1.10)–(1.13)*, then*

$$(a^0, p^0) \in L^2(\Omega, d\mu_\varepsilon^{\lambda,h}) \times L^2(\Omega, d\nu_\varepsilon^{\lambda,h}) \quad \forall \varepsilon > 0, \tag{8.9}$$

$$a^0 \rightharpoonup a^0 \quad \text{in } L^2(\Omega, d\mu_\varepsilon^{\lambda,h}), \qquad \lim_{\varepsilon\to 0} \int_\Omega (a^0)^2 d\mu_\varepsilon^{\lambda,h} = \int_\Omega (a^0)^2\, dx, \tag{8.10}$$

$$p^0 \rightharpoonup p^0 \quad \text{in } L^2(\Omega, d\nu_\varepsilon^{\lambda,h}), \quad \lim_{\varepsilon\to 0} \int_\Omega (p^0)^2\, d\nu_\varepsilon^{\lambda,h} = \int_\Omega (p^0)^2\, dx. \tag{8.11}$$

The next question we are going to consider in this section concerns the approximation of the optimal solutions of the original problem $\hat{\mathbb{P}}_\varepsilon$ for ε small enough. We focus our attention on the possibility to define the so-called suboptimal solutions which have to guarantee the closeness of the corresponding value of the cost functional $I_\varepsilon(a_\varepsilon^{\mathrm{sub}}, p_\varepsilon^{\mathrm{sub}}, y_\varepsilon^{\mathrm{sub}})$ to its minimum if ε is small enough. To do so, we introduce the following concept:

Definition 8.5. We say that a sequence of pairs $\{(\tilde{a}_\varepsilon^0, \tilde{p}_\varepsilon^0)\}_{\varepsilon>0}$ is asymptotically suboptimal for the problem $\hat{\mathbb{P}}_\varepsilon$ if for every $\delta > 0$, there is $\varepsilon_0 > 0$ such that

$$\Bigl| \inf_{(a_\varepsilon, p_\varepsilon, y_\varepsilon) \in \hat{\Xi}_\varepsilon} \hat{I}_\varepsilon(a_\varepsilon, p_\varepsilon, y_\varepsilon) - \hat{I}_\varepsilon(\tilde{a}_\varepsilon^0, \tilde{p}_\varepsilon^0, \tilde{y}_\varepsilon) \Bigr| < \delta \quad \forall \varepsilon < \varepsilon_0, \tag{8.12}$$

where $\tilde{y}_\varepsilon = \tilde{y}_\varepsilon(\tilde{a}_\varepsilon^0, \tilde{p}_\varepsilon^0)$ denote the corresponding solutions of the boundary value problem (1.5).

Proposition 8.4 leads to the following final result:

Theorem 8.6. *Let* $p^0 \in H^2(\Omega) \cap H_0^1(\Omega) \cap L^2(\Omega, d\mu^*)$ *and* $a^0 \in H^2(\Omega) \cap H_0^1(\Omega)$ *be the optimal controls for the homogenized problem* (1.10)–(1.13). *Then the sequence of the pairs* $\{(a^0, p^0)\}_{\varepsilon>0}$ *is asymptotically suboptimal for the original optimal control problem* $\hat{\mathbb{P}}_\varepsilon$.

Proof. Let us consider the sequence of triplets $\{(a^0, p^0, \breve{\tilde{y}}_\varepsilon) \in \mathbb{X}_\varepsilon\}_{\varepsilon>0}$, where $\tilde{y}_\varepsilon = \tilde{y}_\varepsilon(a^0, p^0)$ are the corresponding solutions of the boundary value problem (1.5). Each of these triplets is admissible for the problem $\hat{\mathbb{P}}_\varepsilon$. Moreover, due to estimate (3.15), this sequence is equibounded in $\mathbb{X}_\varepsilon$. By Theorem 4.11 it is relatively compact with respect to the w-convergence in $\mathbb{X}_\varepsilon$. Hence, taking into account Proposition 8.4 and Theorem 6.8, we deduce: this sequence is w-compact and $(a^0, p^0, \tilde{y}_\varepsilon) \xrightarrow{\varepsilon\to 0} (a^0, p^0, y^0)$, where (a^0, p^0, y^0) is an optimal solution to the homogenized problem (1.10)–(1.13).

Let $\{(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) \in \hat{\Xi}_\varepsilon\}_{\varepsilon>0}$ be the optimal solutions to the $\hat{\mathbb{P}}_\varepsilon$. We observe that

$$\begin{aligned}
&\Bigl| \inf_{(a_\varepsilon, p_\varepsilon, y_\varepsilon) \in \hat{\Xi}_\varepsilon} \hat{I}_\varepsilon(a_\varepsilon, p_\varepsilon, y_\varepsilon) - \hat{I}_\varepsilon(a^0, p^0, \tilde{y}_\varepsilon) \Bigr| = \bigl| \hat{I}_\varepsilon(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) - \hat{I}_\varepsilon(a^0, p^0, \tilde{y}_\varepsilon) \bigr| \\
&\quad \leqslant \bigl| \hat{I}_\varepsilon(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) - I_0(a^0, p^0, y^0) \bigr| + \bigl| I_0(a^0, p^0, y^0) - \hat{I}_\varepsilon(a^0, p^0, \tilde{y}_\varepsilon) \bigr| \\
&\quad \leqslant \bigl| \hat{I}_\varepsilon(a_\varepsilon^0, p_\varepsilon^0, y_\varepsilon^0) - I_0(a^0, p^0, y^0) \bigr| + \Bigl| \int_\Omega |\nabla y^0|^2\, dx + \rho^* \int_\Omega (y^0 - a^0)^2\, d\mu^* - \int_\Omega \chi_\varepsilon |\nabla \breve{\tilde{y}}_\varepsilon|^2\, dx \Bigr| \\
&\qquad + \Bigl| \int_\Omega |y^0 - z^\partial|^2\, dx - \int_\Omega \chi_\varepsilon |\breve{\tilde{y}}_\varepsilon - z^\partial|^2\, dx \Bigr| + \Bigl| |\partial Q|_H \int_\Omega (p^0)^2\, dx - \varepsilon^{-n} \sigma(\varepsilon) \int_\Omega (p^0)^2\, d\nu_\varepsilon^{\lambda,h} \Bigr|
\end{aligned}$$

$$+\left| |K\cap\partial\Lambda|_H \int_\Omega (a^0)^2\,dx - |K\cap\partial Q^{\varsigma(\varepsilon)}|_H \int_\Omega (a^0)^2\,d\mu_\varepsilon^{\lambda,h}\right| = J_1+J_2+J_3+J_4+J_5.$$

To conclude the proof, we note that for a given $\delta>0$ one can always find: (1) $\varepsilon_1>0$ such that $J_1<\delta/5$ for all $\varepsilon<\varepsilon_1$ by Theorem 7.4; (2) $\varepsilon_2>0$ such that $J_2<\delta/5$ for all $\varepsilon<\varepsilon_2$ by Lemma 7.1; (3) $\varepsilon_3>0$ such that $J_3<\delta/5$ for all $\varepsilon<\varepsilon_3$ by the w-convergence $(a^0,p^0,\tilde y_\varepsilon)$ to (a^0,p^0,y^0); (4) $\varepsilon_4>0$ such that $J_4<\delta/5$ for all $\varepsilon<\varepsilon_4$ by (3.9) and (8.11); (5) $\varepsilon_5>0$ such that $J_5<\delta/5$ for all $\varepsilon<\varepsilon_5$ by (8.10) and (3.17). Thus, as expected, estimate (8.12) is valid for all $\varepsilon<\min\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5\}$.

It should be stressed that a sequence of asymptotically suboptimal controls for the problem $\hat{\mathbb{P}}_\varepsilon$ has a particularly simple and attractive form if the optimal control pair $(a_\varepsilon^0,p_\varepsilon^0)$ for the problem $\hat{\mathbb{P}}_\varepsilon$ is such that $a_\varepsilon^0(x)=a^*(\frac{x}{\varepsilon\lambda(\varepsilon)})$, $p_\varepsilon^0(x)=p^*(\frac{x}{\varepsilon h(\varepsilon)})$, where $a^*\in H^1(K)$ and $p^*\in L^2(\partial Q,d\mathcal{H}^{n-1})$ are some Y-periodic functions. Indeed, in this case by Proposition 4.13 and Theorem 7.4, we have

$$a_\varepsilon^0 \rightharpoonup \left(\frac{1}{|K\cap\partial\Lambda|_H}\int_{K\cap\partial\Lambda} a^*\,d\mathcal{H}^{n-1}\right)=a^0,\qquad p_\varepsilon^0 \rightharpoonup \left(\frac{1}{|\partial Q|_H}\int_{\partial Q} p^*\,d\mathcal{H}^{n-1}\right)=p^0$$

in $L^2(\Omega,d\mu_\varepsilon^{\lambda,h})$ and $L^2(\Omega,d\nu_\varepsilon^{\lambda,h})$, respectively, where (a^0,p^0) are optimal controls for the homogenized problem (1.10)–(1.13). Hence, the conclusion of Theorem 8.6 can be reformulated as follows: the constant sequence of pairs $\{(a^0,p^0)\in\mathbb{R}^2\}_{\varepsilon>0}$, where

$$p^0=\left(\frac{1}{|\partial Q|_H}\int_{\partial Q} p^*d\mathcal{H}^{n-1}\right)\quad\text{and}\quad a^0=\left(\frac{1}{|K\cap\partial\Lambda|_H}\int_{K\cap\partial\Lambda} a^*d\mathcal{H}^{n-1}\right),$$

is a sequence of asymptotically suboptimal controls for the original problem $\hat{\mathbb{P}}_\varepsilon$.

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