



Partial differential equations. — *Entire solutions of autonomous equations on \mathbb{R}^n with nontrivial asymptotics*, by ANDREA MALCHIODI.

ABSTRACT. — We prove existence of a new type of solutions for the semilinear equation $-\Delta u + u = u^p$ on \mathbb{R}^n , with $1 < p < (n+2)/(n-2)$. These solutions are positive, bounded, decay exponentially to zero away from three half-lines with a common origin, and at infinity are asymptotically periodic.

KEY WORDS: Semilinear elliptic equations; entire solutions; Lyapunov–Schmidt reduction; weighted spaces.

MATHEMATICS SUBJECT CLASSIFICATION (2000): 35B40, 35J10, 35J60, 58C15.

1. INTRODUCTION

This note summarizes the results of [15], where new positive entire solutions of the equation

$$(E_p) \quad -\Delta u + u = u^p \quad \text{in } \mathbb{R}^n$$

are constructed, assuming that $p \in (1, \frac{n+2}{n-2})$. These new solutions decay exponentially away from three half-lines and are asymptotically periodic in these three directions.

The study of (E_p) has several motivations: as basic examples we have nonlinear scalar field equations like the *Nonlinear Klein–Gordon* or the *Nonlinear Schrödinger*. More precisely a special class of solutions of the latter, $i\hbar \frac{\partial \tilde{\psi}}{\partial t} = -\hbar^2 \Delta \tilde{\psi} + V(x)\tilde{\psi} - |\tilde{\psi}|^{p-1}\tilde{\psi}$, called *standing waves*, are complex-valued functions $\psi(x, t)$, $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, of the form $\psi(x, t) = e^{-i\omega t}u(x)$, where ω is a real constant and $u : \mathbb{R}^n \rightarrow \mathbb{R}$ a real-valued function which satisfies the equation (adding ω to V)

$$(NLS) \quad -\Delta u + V(x)u = u^p \quad \text{in } \mathbb{R}^n$$

($V : \mathbb{R}^n \rightarrow \mathbb{R}$ is the potential and $p > 1$). Under the above restriction on p , problem (NLS) is variational but noncompact, since the domain, \mathbb{R}^n , is unbounded. An important step to understand how compactness is lost along Palais–Smale sequences is to consider problem (E_p) . Another motivation for the study of (E_p) is the semiclassical limit of (NLS) , namely

$$(NLS_\varepsilon) \quad -\varepsilon^2 \Delta u + V(x)u = u^p \quad \text{in } \mathbb{R}^n,$$

where ε is a small positive parameter which stands for the Planck constant \hbar . By a scaling of the form $x \mapsto \varepsilon x$, the equation becomes just (NLS) , but with $V(x)$ replaced by

$V(\varepsilon x)$, a potential which now has a slow dependence on its argument. Solutions of (NLS_ε) localized near some point $x_0 \in \mathbb{R}^n$ solve in the limit (after rescaling) $-\Delta u + V(x_0)u = u^p$, and can be obtained from solutions of (E_p) by easy algebraic manipulations. The localization phenomenon, also related to the quantum-mechanical requirement of getting wave functions with finite probability, corresponds to looking for solutions to (E_p) which decay to zero at infinity, for example solutions of

$$(1) \quad \begin{cases} -\Delta U + U = U^p & \text{in } \mathbb{R}^n, \\ U > 0, \quad U \in H^1(\mathbb{R}^2). \end{cases}$$

Problem (1) possesses *ground states* which have exponential decay, are radial (up to translation), radially decreasing and unique.

Still other reasons for considering (E_p) arise in the study of models from biology: for example, the *Gierer–Meinhardt* system (see [23]), can be approached by studying first the equation $-\varepsilon^2 \Delta u + u = u^p$ in a domain $\Omega \subseteq \mathbb{R}^n$, with Neumann boundary conditions. There is a broad literature on this problem, concerning existence and multiplicity results on *spike layers*, namely solutions u_ε which concentrate at a finite number of points of $\bar{\Omega}$, with the profile $u_\varepsilon(x) \simeq U\left(\frac{x-x_0}{\varepsilon}\right)$, $x_0 \in \bar{\Omega}$.

For the above issues we refer the reader to [1], where a rather complete list of references is given.

Recently, a different kind of solutions (whose existence has been conjectured for some time, see [23]) has been shown to exist, both for (NLS_ε) and for the above Neumann problem. These have a different profile and scale only in one direction (or, more generally, in k directions, with $k \in \{1, \dots, n-1\}$), corresponding to solutions of the equation in (1) which are independent of some of the variables (see [2]–[4], [7], [9], [14], [16], [17], [22]).

Except when some symmetry is present, this kind of result asserts that concentration occurs provided we restrict ourselves to a suitable sequence $\varepsilon_j \rightarrow 0$: the reason is that these solutions have a larger and larger Morse index, and therefore resonance occurs. As a consequence, if one wishes to employ local inversion arguments, it is necessary to avoid some values of the parameter ε , so that the linearized equation is invertible. Under symmetry assumptions one can work in spaces of invariant functions and obtain existence for all ε ; however, the resonance phenomenon still occurs, and this generates bifurcation phenomena (see [2]).

This bifurcation is indeed also present for a class of solutions of (E_p) . For example, one can start from entire (decaying) solutions of the equation in lower dimension, say in \mathbb{R}^{n-1} , and extend them (with obvious notation) to the whole \mathbb{R}^n by setting $\tilde{U}(x_1, x') = U_{n-1}(x')$. In [6] N. Dancer proved bifurcation of noncylindrical solutions from \tilde{U} which are periodic in x_1 , considering the Morse index of \tilde{U} restricted to the strip $D_L := \{-L/2 \leq x_1 \leq L/2\}$, and showing that this diverges when $L \rightarrow +\infty$.

A similar strategy was previously used by R. Schoen to prove multiplicity of solutions for the Yamabe problem (see [24]), and in fact other geometric problems exhibit this kind of phenomenon, like that of finding surfaces in \mathbb{R}^3 which have constant mean curvature. Considering for example axially-symmetric objects, it turns out that from the cylinder bifurcates a family of surfaces, the *Delaunay unduloids*, which have constant mean curvature and are periodic along the axis of the cylinder. A similar behavior is present when considering conformal Yamabe metrics defined in $\mathbb{R}^n \setminus \{0\}$, $n \geq 3$, which are singular

at the origin. Besides $|x|^{-(n-2)/2}dx^2$, there are other metrics whose conformal factor is radial and periodic in $|x|$ after a logarithmic change of variables.

Delaunay unduloids are used as *building blocks* to produce complete surfaces in \mathbb{R}^3 with constant mean curvature which are unions of a compact set and a finite number of *ends*, subsets with the topology of the cylinder which are asymptotically close to Delaunay surfaces. We refer for example to the papers [10]–[12], [18]–[20] for details. Analogous constructions can be done with Yamabe metrics which are defined on domains of \mathbb{R}^n with a finite number of points removed, and which are singular at these points (see e.g. [13], [21] and references therein). The aim of [15] is to show that a similar structure is present for solutions to (E_p) : to our knowledge there are no previous examples which arise in a pure PDE context. Some related results are given in [9], [8] (also for the Allen–Cahn equation), but there the profile of solutions is homogeneous, or nearly homogeneous, along the transitions, in strong contrast with our case.

Denoting points of \mathbb{R}^n by couples $(x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$, we consider first a family of solutions u_L to (E_p) which are periodic in the x_1 variable and which decay to zero at an exponential rate away from $x' = 0$, counterparts of the Delaunay surfaces. We focus on the case of large period L , which allows us to construct the solutions of [6] using perturbative methods. In fact, set $\mathbf{z}_i = (iL, 0, \dots, 0)$. Then the function $u_{0,L} = \sum_{i \in \mathbb{N}} U(\cdot - \mathbf{z}_i)$ satisfies the Neumann boundary conditions on ∂D_L and is an approximate solution of (E_p) for L large. Using the implicit function theorem, one can add a correction \bar{w}_L to $u_{0,L}$ so that $u_L = u_{0,L} + \bar{w}_L$ solves (E_p) exactly.

To state our result, we introduce some extra notation: set $\Pi = \{(z_1, z_2, 0, \dots, 0) : (z_1, z_2) \in \mathbb{R}^2\} \subseteq \mathbb{R}^n$ and also, given $\theta \in S^{n-1} (\subseteq \mathbb{R}^n) \cap \Pi$, we define the ray $l_\theta = \{t\theta : t \geq 0\}$. We also let R_θ denote the rotation in the plane Π (extended naturally to all of \mathbb{R}^n) of angle θ . The distance function between two points (or between two sets) of \mathbb{R}^n is denoted by $\text{dist}(\cdot, \cdot)$. In the statement of Theorem 1.1 below, u_L stands for the solution of (E_p) periodic in x_1 just described.

THEOREM 1.1. *Problem (E_p) admits a three-dimensional (up to rotations and translations) family of solutions which decay exponentially away from three rays originating from a common point, and which have an asymptotic periodic profile along the rays. More precisely, there exist a positive constant C , a neighborhood \mathcal{U} of 0 in \mathbb{R}^3 , smooth functions $\theta_1, \theta_2, \theta_3 : \mathcal{U} \rightarrow S^{n-1} \cap \Pi$, $L_1, L_2, L_3 : \mathcal{U} \rightarrow \mathbb{R}$, $y_1, y_2, y_3 : \mathcal{U} \rightarrow \Pi$ and a map from \mathcal{U} into $L^\infty(\mathbb{R}^n)$, $\zeta \in \mathcal{U} \mapsto u_\zeta$, such that the following properties hold:*

- (i) u_ζ is a positive solution of (E_p) ;
- (ii) if $l_{\theta_1}, l_{\theta_2}, l_{\theta_3}$ are the rays corresponding to the directions θ_1, θ_2 and θ_3 respectively, then

$$u_\zeta(x) \leq C e^{-\frac{1}{C} \text{dist}(x, l_{\theta_1} \cup l_{\theta_2} \cup l_{\theta_3})} \quad \text{for every } x \in \mathbb{R}^n;$$

- (iii) for any $t_i \rightarrow +\infty$, given any compact set K of \mathbb{R}^n ,

$$\|u(\cdot - t_i \theta_a) - u_{L_a}(R_{\theta_a}(\cdot - y_a))\|_{C^2(K)} \leq C_K e^{\frac{1}{C} |t_i|} \quad \text{for } a = 1, 2, 3.$$

We can indeed characterize more precisely these solutions in terms of their asymptotic behavior at infinity. In our construction the values of the numbers L_a , $a = 1, 2, 3$, can be chosen arbitrarily large, but the differences $|L_a - L_b|$, with $a \neq b$, stay uniformly bounded.

Also, we have $\theta_a \angle \theta_b > \pi/3$ for every $a \neq b$, where $\theta_a \angle \theta_b$ stands for the angle between the two versors θ_a and θ_b . It is also possible to prove that the following function is positive and monotone in L ($L \gg 1$):

$$G(L) := \frac{1}{4} \int_{\partial D_L} (|\nabla u_L|^2 + u_L^2) d\sigma - \frac{1}{2(p+1)} \int_{\partial D_L} |u_L|^{p+1} d\sigma,$$

and that it determines uniquely the asymptotic period and profile of the functions u_L . In analogy with a *balance condition* for CMC surfaces or singular Yamabe metrics we have the following result.

THEOREM 1.2. *Let u be a function with properties (i)–(iii) of Theorem 1.1, and let θ_a , L_a , $a = 1, 2, 3$, be the corresponding quantities. Assume that the angle $\theta_a \angle \theta_b$ between any two different θ 's is greater than $\pi/3$. Then $\sum_{a=1,2,3} \theta_a G(L_a) = 0$.*

Theorem 1.2 follows from properties (i)–(iii) above and some integration by parts, while the proof of Theorem 1.1 is rather involved, and will be sketched in the next section.

REMARK 1.3. (a) Existence of solutions of semilinear elliptic equations with infinitely many bumps has been considered in other works, but from other points of view. For example, in [5], similar equations in the presence of a slowly oscillating potential have been considered. While in that paper it is the potential that mainly determines the locations of the *bumps*, here precisely their mutual interactions allow us to perform the construction of Theorem 1.1.

(b) Concerning the Neumann problem mentioned above, we believe that the functions constructed in Theorem 1.1, scaled in ε , might lead to the existence of solutions concentrating at a singular set in Ω , with a triple point. This would be a new type of phenomenon, since so far concentration at sets of dimension greater than zero has been proved for smooth curves or manifolds only.

2. SOME DETAILS ABOUT THE PROOF OF THEOREM 1.1

First we recall some basic properties of the solution U to (1): its asymptotic behavior is

$$(2) \quad \lim_{r \rightarrow \infty} e^r r^{(n-1)/2} U(r) = \alpha_{n,p}; \quad \lim_{r \rightarrow \infty} \frac{U'(r)}{U(r)} = -1 \quad (r = |x|).$$

Moreover, the kernel of the operator $L_0 v := -\Delta v + v - pU^{p-1}v$ (the linearization of (1) at U) is spanned by $\partial U / \partial x_1, \dots, \partial U / \partial x_n$. We will work within the space of functions which are rotationally invariant in the last $n - 2$ variables, so under this condition the elements of $\ker(L_0)$ will be linear combinations of $\partial U / \partial x_1, \partial U / \partial x_2$.

Our strategy consists in starting with approximate solutions which have the desired behavior at infinity, and then using a Lyapunov–Schmidt reduction to fully solve the equation.

We introduce three half-spaces $V_a = \{x \in \mathbb{R}^n : \langle x, \theta_a \rangle \geq L/2 - 1\}$, and also $\psi_a(x) = \psi(d(x, V_a))$, where ψ is a fixed smooth cutoff function defined on \mathbb{R} with values in $[0, 1]$

such that $\psi(t) = 1$ for $t \leq 0$, and $\psi(t) = 0$ for $t \geq 1$. Let $\theta_1, \theta_2, \theta_3$ be three unit vectors in $\Pi := \{(x_1, x_2, 0, \dots, 0)\} \subseteq \mathbb{R}^n$ which satisfy

$$(3) \quad \theta_1 \angle \theta_2 \geq \frac{\pi}{3} + \theta_0; \quad \theta_2 \angle \theta_3 \geq \frac{\pi}{3} + \theta_0; \quad \theta_1 \angle \theta_3 \geq \frac{\pi}{3} + \theta_0$$

for some $\theta_0 > 0$. Recall that for any $a = 1, 2, 3$, R_{θ_a} stands for the rotation in Π by angle θ_a . If \bar{w}_L is as above, we define $\bar{w}_{L_a, \theta_a} = R_{\theta_a} \bar{w}_L$. Next we choose three large numbers L_1, L_2, L_3 (with $|L_a - L_b| + |L_a - L|$ uniformly bounded by a fixed constant C), points $y_a, a = 1, 2, 3$, and $(P_{a,i})_{a,i}$ such that

$$(4) \quad |y_a| \leq c_{\theta_0}, \quad |P_{a,i} - i\theta_i L_a - y_a| \leq C_{\theta_0} e^{-\tau|P_i|}, \quad a = 1, 2, 3, i = 1, 2, \dots,$$

for some constants $c_{\theta_0}, C_{\theta_0}$ and τ (uniformly bounded in L). We set for simplicity $\{P_I\} = \{0\} \cup \bigcup_{a,i} \{P_{a,i}\}$, $X = \bigcup_I \{P_I\}_I$, $Y = (y_1, y_2, y_3)$ and $U_I(\cdot) = (\cdot - P_I)$ for any index I . We finally define

$$(5) \quad u_{X,Y}(x) = \sum_I U_I(x) + \sum_{a=1}^3 \psi_a(x) \bar{w}_{L_a, \theta_a}(x - y_a).$$

By our choice, this function is exponentially close to a rotation of u_{L_a} along each direction θ_a : indeed, it is possible to prove the following quantitative estimate on $u_{X,Y}$.

LEMMA 2.1. *Let $S_0(u_{X,Y}) = -\Delta u_{X,Y} + u_{X,Y} - u_{X,Y}^p$. If $(y_a)_a, (P_I)_I$ satisfy (4) and $(\theta_a)_a$ satisfy (3), then for any $\gamma \in (0, 1)$*

$$(6) \quad \|S_0(u_{X,Y})\|_{C^\gamma(B_1(x))} \leq C e^{-(1+\xi)L/2} e^{-\sigma d(x, \cup_i P_i)} [e^{-\eta|x|} + C_{\theta_0} e^{-\tau|x|}], \quad x \in \mathbb{R}^n,$$

where ξ, σ and η are positive constants depending only on n, p and θ_0 , but not on L , and where C is a fixed constant (depending only on n, p, γ and θ_0) also independent of L .

The Lyapunov–Schmidt reduction consists in transferring problem (E_p) into determining the appropriate location of the points $\{P_I\}_I$. For doing this we can exploit the linear properties of L_0 , and as a first step solve the equation up to, basically, a sequence of Lagrange multipliers in the kernel of $L_I := -\Delta + 1 - pU_I^{p-1}$. Precisely, one can prove the following result.

PROPOSITION 2.2. *Suppose $(y_a)_a, (P_I)_I$ satisfy (4), and let $u_{X,Y}$ be as in (5). Then, for L sufficiently large, there exists a function $w_{X,Y}$ and a sequence $(\alpha^I)_I$ of elements of \mathbb{R}^2 , $\alpha^I = (\alpha^{I,j})_j, j = 1, 2$, such that*

$$(a) \quad -\Delta(u_{X,Y} + w_{X,Y}) + (u_{X,Y} + w_{X,Y}) - (u_{X,Y} + w_{X,Y})^p = \sum_{I,j} \alpha^{I,j} U_I^{p-1} \partial_j U_I;$$

$$(b) \quad \int_{\mathbb{R}^n} w_{X,Y} U_I^{p-1} \frac{\partial U_I}{\partial z_j} = 0 \quad \text{for every } I \text{ and for every } j = 1, 2.$$

While this method is rather standard when dealing with a *finite* number of solitons, some technical difficulties arise when dealing with infinitely many ones: our proof uses

crucially weighted spaces and Toeplitz type operators. The final step of the proof consists in adjusting the positions of the points $(P_I)_I$ in order to make all the coefficients $(\alpha^I)_I$ vanish. First of all, using Lemma 2.1 with $C_{\theta_0} = 0$, one can estimate the α^I 's corresponding to the function $u_{X(Y),Y}$ where $X(Y)$ denotes the special configuration of points satisfying

$$(7) \quad P_{a,i} = y_a + i\theta_a L_a \quad \text{for every } a = 1, 2, 3 \text{ and every } i \in \mathbb{N},$$

and where the symbol Y stands for the triple (y_1, y_2, y_3) .

LEMMA 2.3. *For X and Y satisfying (4) and (7) we have the following estimates:*

$$\begin{aligned} \alpha_{X(Y),Y}^0 &= - \sum_{a=1,2,3} F_1(|P_{a,1}|) \frac{P_{a,1}}{|P_{a,1}|} + O(e^{-(1+\xi)L}); \\ \alpha_{X(Y),Y}^I &= \left[F_1(|P_{a,1}|) \frac{P_{a,1}}{|P_{a,1}|} + F_1(|P_{a,1} - P_{a,2}|) \frac{P_{a,1} - P_{a,2}}{|P_{a,1} - P_{a,2}|} \right] + O(e^{-(1+\xi)L}) \\ &\quad \text{if } P_I = P_{a,1}, a = 1, 2, 3; \\ |\alpha_{X(Y),Y}^I| &\leq C e^{-(1+\xi)L} e^{-\eta|P_I|} + C C_{\theta_0} e^{-\tau(|P_{a,h-1}|)} F_0(L) \\ &\quad \text{if } P_I = P_{a,h} \text{ for } a = 1, 2, 3, \text{ and } h > 1, \end{aligned}$$

where F_1 satisfies $F_1(t) = (1 + o_t(1))F_0(t)$, and where C, η, ξ are constants depending on n and p .

Next, we study the variation of the α^I 's depending on the points $(P_I)_I$ and $(y_a)_a$. To understand this, looking at the expansions in Lemma 2.3, one can imagine α^I to behave like

$$\alpha^I \simeq - \sum_{S \neq I} \frac{P_S - P_I}{|P_S - P_I|} e^{-|P_S - P_I|}.$$

By the presence of the exponential term, the main contribution to the above expression will be given by the points closest to P_I : three when $P_I = 0$ and two for $P_I \neq 0$ (here condition (3) is also used). In particular, along each l_{θ_a} when the configuration of points $P_{a,i}$ is nearly periodic the linearization looks like a *Toda operator* which, in matrix form with respect to the index i , qualitatively looks like

$$\begin{pmatrix} \ddots & \dots & 0 & -1 & 0 & \dots & \dots & \dots & \vdots \\ \vdots & 0 & -1 & 2 & -1 & 0 & \dots & \dots & \vdots \\ \vdots & \dots & 0 & -1 & 2 & -1 & 0 & \dots & \vdots \\ \vdots & \dots & \dots & 0 & -1 & 2 & -1 & 0 & \vdots \\ \dots & \dots \end{pmatrix}.$$

The latter operator can be viewed as a discretization of the Laplacian in one dimension, and it is indeed possible to invert it via convolution with a kernel which is piecewise affine in the index i . If ξ, η are given by Lemmas 2.1 and 2.3, using the above invertibility, one finds the following result.

PROPOSITION 2.4. *Suppose $\theta_1, \theta_2, \theta_3$ satisfy (3), and L_1, L_2, L_3 satisfy $|L_a - L_b| + |L_a - L| \leq C$ for C fixed and L sufficiently large. If we choose $\tau < \min\{\xi, \eta/2\}$ then there exist $(y_a)_a$ and $(P_I)_I = (P_{a,i})_{a,i}$ such that (4) holds true for some uniformly bounded (in L) $c_{\theta_0}, C_{\theta_0}$, and with $\alpha^I = 0$ for all $I \neq 0$.*

Notice that we have a six-dimensional family of configurations satisfying the properties of Proposition 2.4. The final step consists in choosing the L_a 's and the θ_a 's so that also α^0 vanishes, which leaves four parameters free: taking the quotient with respect to rotations in Π , we obtain a genuine three-dimensional family of solutions.

ACKNOWLEDGMENTS. The author has been supported by M.U.R.S.T. within the PRIN 2006 *Variational methods and nonlinear differential equations*.

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Received 4 September 2007,
and in revised form 17 October 2007.

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