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The ground state energy of the two dimensional Ginzburg–Landau functional with variable magnetic field

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Abstract

We consider the Ginzburg–Landau functional with a variable applied magnetic field in a bounded and smooth two dimensional domain. We determine an accurate asymptotic formula for the minimizing energy when the Ginzburg–Landau parameter and the magnetic field are large and of the same order. As a consequence, it is shown how bulk superconductivity decreases in average as the applied magnetic field increases.

MSC: 82D55

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1. Introduction

1.1. The functional and main results

We consider a bounded open simply connected set $\Omega \subset \mathbb{R}^2$ with smooth boundary. We suppose that Ω models a superconducting sample submitted to an applied external magnetic field. The energy of the sample is given by the Ginzburg–Landau functional,

$$\mathcal{E}_{\kappa,H}(\psi,\mathbf{A}) = \int_\Omega \left[\left|(\nabla - i\kappa H\mathbf{A})\psi\right|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4 \right] dx + \kappa^2 H^2 \int_\Omega |\operatorname{curl}\mathbf{A} - B_0|^2\, dx. \tag{1.1}$$

Here κ and H are two positive parameters; κ (the Ginzburg–Landau constant) is a material parameter and H measures the intensity of the applied magnetic field. The wave function (order parameter) $\psi \in H^1(\Omega;\mathbb{C})$ describes the superconducting properties of the material. The induced magnetic field is $\operatorname{curl}\mathbf{A}$, where the potential $\mathbf{A} \in H^1_{\mathrm{div}}(\Omega)$, with $H^1_{\mathrm{div}}(\Omega)$ is the space defined in (1.4) below. Finally, $B_0 \in C^\infty(\overline{\Omega})$ is the intensity of the external variable magnetic field and satisfies

$$|B_0| + |\nabla B_0| > 0 \quad \text{in } \overline{\Omega}. \tag{1.2}$$

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The assumption in (1.2) implies that for any open set ω relatively compact in Ω the set $\{x \in \omega,\ B_0(x) = 0\}$ will be either empty, or consists of a union of smooth curves. Let $\mathbf{F} : \Omega \to \mathbb{R}^2$ be the unique vector field such that

$$\operatorname{div} \mathbf{F} = 0 \quad \text{and} \quad \operatorname{curl} \mathbf{F} = B_0 \quad \text{in } \Omega, \qquad \nu \cdot \mathbf{F} = 0 \quad \text{on } \partial\Omega. \tag{1.3}$$

The vector ν is the unit interior normal vector of $\partial\Omega$. The construction of $\mathbf{F}$ is recalled in Appendix A. We define the space

$$H^1_{\mathrm{div}}(\Omega) = \{\mathbf{A} = (\mathbf{A}_1, \mathbf{A}_2) \in H^1(\Omega)^2\colon \operatorname{div} \mathbf{A} = 0 \text{ in } \Omega,\ \mathbf{A} \cdot \nu = 0 \text{ on } \partial\Omega\}. \tag{1.4}$$

Critical points $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ of $\mathcal{E}_{\kappa,H}$ are weak solutions of the Ginzburg–Landau equations

$$\begin{cases} -(\nabla - i\kappa H\mathbf{A})^2\psi = \kappa^2(1 - |\psi|^2)\psi & \text{in } \Omega, \\ -\nabla^\perp \operatorname{curl}(\mathbf{A} - \mathbf{F}) = \dfrac{1}{\kappa H} \operatorname{Im}(\overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi) & \text{in } \Omega, \\ \nu \cdot (\nabla - i\kappa H\mathbf{A})\psi = 0 & \text{on } \partial\Omega, \\ \operatorname{curl} \mathbf{A} = \operatorname{curl} \mathbf{F} & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Here, $\operatorname{curl} \mathbf{A} = \partial_{x_1} \mathbf{A}_2 - \partial_{x_2} \mathbf{A}_1$ and $\nabla^\perp \operatorname{curl} \mathbf{A} = (\partial_{x_2}(\operatorname{curl} \mathbf{A}), -\partial_{x_1}(\operatorname{curl} \mathbf{A}))$. If $\operatorname{div} \mathbf{A} = 0$, then $\nabla^\perp \operatorname{curl} \mathbf{A} = \Delta \mathbf{A}$. In this paper, we study the ground state energy defined as follows:

$$\mathrm{E_g}(\kappa, H) = \inf\{\mathcal{E}_{\kappa,H}(\psi, \mathbf{A})\colon (\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)\}. \tag{1.6}$$

More precisely, we give an asymptotic estimate which is valid in the simultaneous limit $\kappa \to \infty$ and $H \to \infty$ in such a way that $\frac{H}{\kappa}$ remains asymptotically constant. The behavior of $\mathrm{E_g}(\kappa, H)$ involves an auxiliary function $g : [0, \infty) \to [-\frac{1}{2}, 0]$ introduced in [9] whose definition will be recalled in (2.5) below. The function g is increasing, continuous, $g(b) = 0$ for all $b \geqslant 1$ and $g(0) = -\frac{1}{2}$.

Theorem 1.1. *Let $0 < \Lambda_{\min} < \Lambda_{\max}$. Under assumption (1.2), there exist positive constants C, κ_0 and $\tau_0 \in (1, 2)$ such that if*

$$\kappa_0 \leqslant \kappa, \qquad \Lambda_{\min} \leqslant \frac{H}{\kappa} \leqslant \Lambda_{\max},$$

then the ground state energy in (1.6) satisfies

$$\left| \mathrm{E_g}(\kappa, H) - \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}|B_0(x)|\right) dx \right| \leqslant C\kappa^{\tau_0}. \tag{1.7}$$

Theorem 1.1 was proved in [9] when the magnetic field is constant ($B_0(x) = 1$). However, the estimate of the remainder is not explicitly given in [9].

The approach used in the proof of Theorem 1.1 is slightly different from the one in [9], and is closer to that in [4] which studies the same problem when $\Omega \subset \mathbb{R}^3$ and B_0 constant.

Corollary 1.2. *Suppose that the assumptions of Theorem 1.1 are satisfied. Then the magnetic energy of the minimizer satisfies, for some positive constant C*

$$(\kappa H)^2 \int_\Omega |\operatorname{curl} \mathbf{A} - B_0|^2\, dx \leqslant C\kappa^{\tau_0}. \tag{1.8}$$

Remark 1.3. The value of τ_0 depends on the properties of B_0: we find $\tau_0 = \frac{7}{4}$ when B_0 does not vanish in $\overline{\Omega}$ and $\tau_0 = \frac{15}{8}$ in the general case.

Theorem 1.4. *Suppose the assumptions of Theorem 1.1 are satisfied. There exist positive constants C, κ_0 and a negative constant $\tau_1 \in (-1, 0)$ such that, if $\kappa \geqslant \kappa_0$, and D is regular set such that $\overline{D} \subset \Omega$, then the following are true.*

(1) *If* $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ *is a solution of* (1.5)*, then*

$$\frac{1}{2}\int_D |\psi|^4\,dx \leqslant -\int_D g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right)dx + C\kappa^{\tau_1}. \tag{1.9}$$

(2) *If* $(\psi, \mathbf{A}) \in H^1(\Omega;\mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ *is a minimizer of* (1.1)*, then*

$$\left|\int_D |\psi|^4\,dx + 2\int_D g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right)dx\right| \leqslant C\kappa^{\tau_1}. \tag{1.10}$$

Remark 1.5. The value of τ_1 depends on the properties of B_0: we find $\tau_1 = -\frac{1}{4}$ when B_0 does not vanish in $\overline{\Omega}$ and $\tau_1 = -\frac{1}{8}$ in the general case.

1.2. Discussion of main result

If $\{x \in \overline{\Omega}\colon\ B_0(x) = 0\} \neq \varnothing$ and $H = b\kappa$, $b > 0$, then $g(\frac{H}{\kappa}|B_0(x)|) \neq 0$ in $D = \{x \in \Omega\colon\ \frac{H}{\kappa}|B_0(x)| < 1\}$, and $|D| \neq 0$. Consequently, for κ sufficiently large, the restriction of ψ on D is not zero in $L^2(\Omega)$. This is a significant difference between our result and the one for constant magnetic field. When the magnetic field is a non-zero constant, then (see [3]), there is a universal constant $\Theta_0 \in (\frac{1}{2}, 1)$ such that, if $H = b\kappa$ and $b > \Theta_0^{-1}$, then $\psi = 0$ in $\overline{\Omega}$. Moreover, in the same situation, when $H = b\kappa$ and $1 < b < \Theta_0^{-1}$, then ψ is small everywhere except in a thin tubular neighborhood of $\partial\Omega$ (see [6]). Our result goes in the same spirit as in [8], where the authors established under the assumption (1.2) that when $H = b\kappa^2$ and $b > b_0$, then $\psi = 0$ in $\overline{\Omega}$ (b_0 is a constant).

1.3. Notation

Throughout the paper, we use the following notation:

- We write $\mathcal{E}$ for the functional $\mathcal{E}_{\kappa,H}$ in (1.1).
- The letter C denotes a positive constant that is independent of the parameters κ and H, and whose value may change from a formula to another.
- If $a(\kappa)$ and $b(\kappa)$ are two positive functions, we write $a(\kappa) \ll b(\kappa)$ if $a(\kappa)/b(\kappa) \to 0$ as $\kappa \to \infty$.
- If $a(\kappa)$ and $b(\kappa)$ are two functions with $b(\kappa) \neq 0$, we write $a(\kappa) \sim b(\kappa)$ if $a(\kappa)/b(\kappa) \to 1$ as $\kappa \to \infty$.
- If $a(\kappa)$ and $b(\kappa)$ are two positive functions, we write $a(\kappa) \approx b(\kappa)$ if there exist positive constants c_1, c_2 and κ_0 such that $c_1 b(\kappa) \leqslant a(\kappa) \leqslant c_2 b(\kappa)$ for all $\kappa \geqslant \kappa_0$.
- If $x \in \mathbb{R}$, we let $[x]_+ = \max(x, 0)$.
- Given $R > 0$ and $x = (x_1, x_2) \in \mathbb{R}^2$, we denote by $Q_R(x) = (-R/2 + x_1, R/2 + x_1) \times (-R/2 + x_2, R/2 + x_2)$ the square of side length R centered at x.
- We will use the standard Sobolev spaces $W^{s,p}$. For integer values of s these are given by

$$W^{n,p}(\Omega) := \left\{u \in L^p(\Omega)\colon\ D^\alpha u \in L^p(\Omega) \text{ for all } |\alpha| \leqslant n\right\}.$$

- Finally we use the standard symbol $H^n(\Omega) = W^{n,2}(\Omega)$.

2. The limiting energy

2.1. Two-dimensional limiting energy

Given a constant $b \geqslant 0$ and an open set $\mathcal{D} \subset \mathbb{R}^2$, we define the following Ginzburg–Landau energy

$$G^\sigma_{b,\mathcal{D}}(u) = \int_{\mathcal{D}} \left(b\left|(\nabla - i\sigma\mathbf{A}_0)u\right|^2 - |u|^2 + \frac{1}{2}|u|^4\right)dx, \quad \forall u \in H^1_0(\mathcal{D}). \tag{2.1}$$

Here $\sigma \in \{-1, +1\}$ and $\mathbf{A}_0$ is the canonical magnetic potential

$$\mathbf{A}_0(x) = \frac{1}{2}(-x_2, x_1), \quad \forall x = (x_1, x_2) \in \mathbb{R}^2, \tag{2.2}$$

that satisfies

$$\operatorname{curl} \mathbf{A}_0 = 1 \quad \text{in } \mathbb{R}^2.$$

We write $Q_R = Q_R(0)$ and let

$$m_0(b, R) = \inf_{u \in H_0^1(Q_R;\mathbb{C})} G_{b,Q_R}^{+1}(u). \tag{2.3}$$

Remark 2.1. As $G_{b,\mathcal{D}}^{+1}(u) = G_{b,\mathcal{D}}^{-1}(\bar{u})$, it is immediate that

$$\inf_{u \in H_0^1(Q_R;\mathbb{C})} G_{b,Q_R}^{-1}(u) = \inf_{u \in H_0^1(Q_R;\mathbb{C})} G_{b,Q_R}^{+1}(u). \tag{2.4}$$

The main part of the next theorem was obtained by Sandier and Serfaty [9] and Aftalion and Serfaty [1, Lemma 2.4]. However, the estimate in (2.7) is obtained by Fournais and Kachmar [4].

Theorem 2.2. *Let $m_0(b, R)$ be as defined in (2.3).*

(1) *For all $b \geqslant 1$ and $R > 0$, we have $m_0(b, R) = 0$.*
(2) *For any $b \in [0, \infty)$, there exists a constant $g(b) \leqslant 0$ such that*

$$g(b) = \lim_{R \to \infty} \frac{m_0(b, R)}{|Q_R|} \quad \text{and} \quad g(0) = -\frac{1}{2}. \tag{2.5}$$

(3) *The function $[0, +\infty) \ni b \mapsto g(b)$ is continuous, non-decreasing, concave and its range is the interval $[-\frac{1}{2}, 0]$.*
(4) *There exists a constant $\alpha \in (0, \frac{1}{2})$ such that*

$$\forall b \in [0, 1], \quad \alpha(b-1)^2 \leqslant |g(b)| \leqslant \frac{1}{2}(b-1)^2. \tag{2.6}$$

(5) *There exist constants C and R_0 such that*

$$\forall R \geqslant R_0,\ \forall b \in [0, 1], \quad g(b) \leqslant \frac{m_0(b, R)}{R^2} \leqslant g(b) + \frac{C}{R}. \tag{2.7}$$

3. A priori estimates

The aim of this section is to give *a priori* estimates for solutions of the Ginzburg–Landau equations (1.5). These estimates play an essential role in controlling the errors resulting from various approximations. The starting point is the following L^∞-bound resulting from the maximum principle. Actually, if $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ is a solution of (1.5), then

$$\|\psi\|_{L^\infty(\Omega)} \leqslant 1. \tag{3.1}$$

The set of estimates below is proved in [2, Theorem 3.3 and Eq. (3.35)] (see also [7] for an earlier version).

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be bounded and smooth and $B_0 \in C^\infty(\overline{\Omega})$.*

(1) *For all $p \in (1, \infty)$ there exists $C_p > 0$ such that, if $(\psi, \mathbf{A}) \in H^1(\Omega, C) \times H^1_{\text{div}}(\Omega)$ is a solution of (1.5), then*

$$\|\operatorname{curl} \mathbf{A} - B_0\|_{W^{1,p}(\Omega)} \leqslant C_p \frac{1 + \kappa H + \kappa^2}{\kappa H} \|\psi\|_{L^\infty(\Omega)} \|\psi\|_{L^2(\Omega)}. \tag{3.2}$$

(2) *For all* $\alpha \in (0,1)$ *there exists* $C_\alpha > 0$ *such that, if* $(\psi, \mathbf{A}) \in H^1(\Omega, C) \times H^1_{\text{div}}(\Omega)$ *is a solution of* (1.5), *then*

$$\|\operatorname{curl}\mathbf{A} - B_0\|_{C^{0,\alpha}(\overline{\Omega})} \leqslant C_\alpha \frac{1+\kappa H+\kappa^2}{\kappa H}\|\psi\|_{L^\infty(\Omega)}\|\psi\|_{L^2(\Omega)}. \tag{3.3}$$

(3) *For all* $p \in [2,\infty)$ *there exists* $C > 0$ *such that, if* $\kappa > 0$, $H > 0$ *and* $(\psi, \mathbf{A}) \in H^1(\Omega, C) \times H^1_{\text{div}}(\Omega)$ *is a solution of* (1.5), *then*

$$\|(\nabla - i\kappa H\mathbf{A})^2\psi\|_p \leqslant \kappa^2\|\psi\|_p, \tag{3.4}$$

$$\|(\nabla - i\kappa H\mathbf{A})\psi\|_2 \leqslant \kappa\|\psi\|_2, \tag{3.5}$$

$$\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{W^{1,p}(\Omega)} \leqslant \frac{C}{\kappa H}\|\psi\|_\infty\|(\nabla - i\kappa H\mathbf{A})\psi\|_p. \tag{3.6}$$

Remark 3.2.

(1) Using the $W^{k,p}$-regularity of the curl–div system [3, Appendix A, Proposition A.5.1], we obtain from (3.2),

$$\|\mathbf{A}-\mathbf{F}\|_{W^{2,p}(\Omega)} \leqslant C_p\frac{1+\kappa H+\kappa^2}{\kappa H}\|\psi\|_{L^\infty(\Omega)}\|\psi\|_{L^2(\Omega)}. \tag{3.7}$$

The estimate is true for any $p \in [2,\infty)$.

(2) Using the Sobolev embedding theorem we get, for all $\alpha \in (0,1)$

$$\|\mathbf{A}-\mathbf{F}\|_{C^{1,\alpha}(\overline{\Omega})} \leqslant C_\alpha\frac{1+\kappa H+\kappa^2}{\kappa H}\|\psi\|_{L^\infty(\Omega)}\|\psi\|_{L^2(\Omega)}. \tag{3.8}$$

(3) Combining (3.5) and (3.6) (with $p = 2$) yields

$$\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)} \leqslant \frac{C}{H}\|\psi\|_{L^\infty(\Omega)}\|\psi\|_{L^2(\Omega)}. \tag{3.9}$$

Theorem 3.1 is needed in order to obtain the improved *a priori* estimates of the next theorem. Similar estimates are given in [7].

Theorem 3.3. *Suppose that* $0 < \Lambda_{\min} \leqslant \Lambda_{\max}$. *There exist constants* $\kappa_0 > 1$, $C_1 > 0$ *and for any* $\alpha \in (0,1)$, $C_\alpha > 0$ *such that, if*

$$\kappa \geqslant \kappa_0, \quad \Lambda_{\min} \leqslant \frac{H}{\kappa} \leqslant \Lambda_{\max}, \tag{3.10}$$

and $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\text{div}}(\Omega)$ *is a solution of* (1.5), *then*

$$\|(\nabla - i\kappa H\mathbf{A})\psi\|_{C(\overline{\Omega})} \leqslant C_1\sqrt{\kappa H}\|\psi\|_{L^\infty(\Omega)}, \tag{3.11}$$

$$\|\mathbf{A}-\mathbf{F}\|_{H^2(\Omega)} \leqslant C_1\left(\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)} + \frac{1}{\sqrt{\kappa H}}\|\psi\|_{L^2(\Omega)}\|\psi\|_{L^\infty(\Omega)}\right), \tag{3.12}$$

$$\|\mathbf{A}-\mathbf{F}\|_{C^{0,\alpha}(\overline{\Omega})} \leqslant C_\alpha\left(\|\operatorname{curl}(\mathbf{A}-\mathbf{F})\|_{L^2(\Omega)} + \frac{1}{\sqrt{\kappa H}}\|\psi\|_{L^2(\Omega)}\|\psi\|_{L^\infty(\Omega)}\right). \tag{3.13}$$

Proof. *Proof of* (3.11): See [3, Proposition 12.4.4].

Proof of (3.12): Let $a = \mathbf{A} - \mathbf{F}$. Since $\operatorname{div} a = 0$ and $a \cdot \nu = 0$ on $\partial\Omega$, we get by regularity of the curl–div system (see Appendix A, Proposition A.1),

$$\|a\|_{H^2(\Omega)} \leqslant C\|\operatorname{curl} a\|_{H^1(\Omega)}. \tag{3.14}$$

The second equation in (1.5) reads as follows

$$-\nabla^\perp \operatorname{curl} a = \frac{1}{\kappa H}\operatorname{Im}\big(\overline{\psi}(\nabla - i\kappa H\mathbf{A})\psi\big).$$

The estimates in (3.11) and (3.14) now give

$$\|a\|_{H^2(\Omega)} \leqslant C\left(\|\operatorname{curl} a\|_{L^2(\Omega)} + \frac{1}{\sqrt{\kappa H}}\|\psi\|_{L^2(\Omega)}\|\psi\|_{L^\infty(\Omega)}\right).$$

Proof of (3.13): This is a consequence of the Sobolev embedding of $H^2(\Omega)$ into $C^{0,\alpha}(\overline{\Omega})$ for any $\alpha \in (0,1)$ and (3.12). □

4. Energy estimates in small squares

If $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$, we introduce the energy density

$$e(\psi, \mathbf{A}) = \left|(\nabla - i\kappa H\mathbf{A})\psi\right|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4.$$

We also introduce the local energy of $(\psi, \mathbf{A})$ in a domain $D \subset \Omega$:

$$\mathcal{E}_0(u, \mathbf{A}; D) = \int_D e(\psi, \mathbf{A})\,dx. \tag{4.1}$$

Furthermore, we define the Ginzburg–Landau energy of $(\psi, \mathbf{A})$ in a domain $D \subset \Omega$ as follows,

$$\mathcal{E}(\psi, \mathbf{A}; D) = \mathcal{E}_0(\psi, \mathbf{A}; D) + (\kappa H)^2 \int_\Omega \left|\operatorname{curl}(\mathbf{A} - \mathbf{F})\right|^2 dx. \tag{4.2}$$

If $D = \Omega$, we sometimes omit the dependence on the domain and write $\mathcal{E}_0(\psi, \mathbf{A})$ for $\mathcal{E}_0(\psi, \mathbf{A}; \Omega)$. We start with a lemma that will be useful in the proof of Proposition 4.2 below. Before we start to state the lemma, we define for all (ℓ, x_0) such that $\overline{Q_\ell(x_0)} \subset \Omega$,

$$\overline{B}_{Q_\ell(x_0)} = \sup_{x \in Q_\ell(x_0)} \left|B_0(x)\right|, \tag{4.3}$$

where B_0 is introduced in (1.2). Later x_0 will be chosen in a lattice of $\mathbb{R}^2$.

Lemma 4.1. *For any $\alpha \in (0,1)$, there exist positive constants C and κ_0 such that if (3.10) holds, $0 < \delta < 1$, $0 < \ell < 1$, and $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a critical point of (1.1) (i.e. a solution of (1.5)), then, for any square $Q_\ell(x_0)$ relatively compact in $\Omega \cap \{|B_0| > 0\}$, there exists $\varphi \in H^1(\Omega)$, such that*

$$\begin{aligned}\mathcal{E}_0\big(\psi, \mathbf{A}; Q_\ell(x_0)\big) &\geqslant (1-\delta)\mathcal{E}_0\big(e^{-i\kappa H\varphi}\psi, \sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0), Q_\ell(x_0)\big) \\ &\quad - C\kappa^2\big(\delta^{-1}\ell^{2\alpha} + \delta^{-1}\ell^4\kappa^2 + \delta\big) \int_{Q_\ell(x_0)} |\psi|^2\,dx,\end{aligned} \tag{4.4}$$

where σ_ℓ denotes the sign of B_0 in $Q_\ell(x_0)$.

Proof. *Construction of φ*: Let $\phi_{x_0}(x) = (\mathbf{A}(x_0) - \mathbf{F}(x_0)) \cdot x$, where $\mathbf{F}$ is the magnetic potential introduced in (1.3). Using the estimate in (3.13), we get for all $x \in Q_\ell(x_0)$ and $\alpha \in (0,1)$,

$$\begin{aligned}\left|\mathbf{A}(x) - \nabla\phi_{x_0} - \mathbf{F}(x)\right| &= \left|(\mathbf{A} - \mathbf{F})(x) - (\mathbf{A} - \mathbf{F})(x_0)\right| \\ &\leqslant \|\mathbf{A} - \mathbf{F}\|_{C^{0,\alpha}} \cdot |x - x_0|^\alpha \\ &\leqslant C\frac{\sqrt{\lambda}}{\kappa H}\ell^\alpha,\end{aligned} \tag{4.5}$$

where

$$\lambda = (\kappa H)^2\left(\left\|\operatorname{curl}(\mathbf{A} - \mathbf{F})\right\|^2_{L^2(\Omega)} + \frac{1}{\kappa H}\|\psi\|^2_{L^2(\Omega)}\right).$$

Using the bound $\|\psi\|_\infty \leqslant 1$ and the estimate in (3.9), we get

$$\lambda \leqslant C\kappa^2, \tag{4.6}$$

which implies that

$$\left|\mathbf{A}(x) - \nabla\phi_{x_0}(x) - \mathbf{F}(x)\right| \leqslant C\frac{\ell^\alpha}{H}. \tag{4.7}$$

We estimate the energy $\mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0))$ from below. We will need the function φ_0 introduced in Lemma A.3 and satisfying

$$\left|\mathbf{F}(x) - \sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) - \nabla\varphi_0(x)\right| \leqslant C\ell^2 \quad \text{in } Q_\ell(x_0).$$

Let

$$u = e^{-i\kappa H\varphi}\psi, \tag{4.8}$$

where $\varphi = \varphi_0 + \phi_{x_0}$.

Lower bound: We start with estimating the kinetic energy from below as follows. For any $\delta \in (0, 1)$, we write

$$\begin{aligned}
\left|(\nabla - i\kappa H\mathbf{A})\psi\right|^2 &= \left|\left(\nabla - i\kappa H\left(\sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) + \nabla\varphi\right)\right)\psi - i\kappa H\left(\mathbf{A} - \sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) - \nabla\varphi\right)\psi\right|^2 \\
&\geqslant (1 - \delta)\left|\left(\nabla - i\kappa H\left(\sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) + \nabla\varphi\right)\right)\psi\right|^2 \\
&\quad + \left(1 - \delta^{-1}\right)(\kappa H)^2\left|(\mathbf{A} - \nabla\phi_{x_0} - \mathbf{F})\psi + \left(\mathbf{F} - \sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) - \nabla\varphi_0\right)\psi\right|^2.
\end{aligned}$$

Using the estimates in (4.7), (A.3) and the assumption in (3.10), we get, for any $\alpha \in (0, 1)$

$$\begin{aligned}
\left|(\nabla - i\kappa H\mathbf{A})\psi\right|^2 &\geqslant (1 - \delta)\left|\left(\nabla - i\kappa H\left(\sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) + \nabla\varphi\right)\right)\psi\right|^2 \\
&\quad - C\kappa^2\left(\delta^{-\frac{1}{2}}\ell^\alpha + \delta^{-\frac{1}{2}}\ell^2 H\right)^2|\psi|^2.
\end{aligned} \tag{4.9}$$

Remembering the definition of u in (4.8), then, we deduce the lower bound of $\mathcal{E}_0$

$$\begin{aligned}
\mathcal{E}_0\left(\psi, \mathbf{A}; Q_\ell(x_0)\right) &\geqslant \int_{Q_\ell(x_0)} \left[(1 - \delta)\left|\left(\nabla - i\kappa H\left(\sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0)\right)\right)u\right|^2 - \kappa^2|u|^2 + \frac{\kappa^2}{2}|u|^4\right]dx \\
&\quad - C\kappa^2\left(\delta^{-\frac{1}{2}}\ell^2\kappa + \delta^{-\frac{1}{2}}\ell^\alpha\right)^2 \int_{Q_\ell(x_0)} |\psi|^2\,dx \\
&\geqslant (1 - \delta)\mathcal{E}_0\left(u, \sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0); Q_\ell(x_0)\right) \\
&\quad - \widehat{C}\kappa^2\left(\delta^{-1}\ell^4\kappa^2 + \delta^{-1}\ell^{2\alpha} + \delta\right) \int_{Q_\ell(x_0)} |\psi|^2\,dx.
\end{aligned} \tag{4.10}$$

This finishes the proof of Lemma 4.1. □

Proposition 4.2. *For all $\alpha \in (0, 1)$, there exist positive constants C, ϵ_0 and κ_0 such that, if (3.10) holds, $\kappa \geqslant \kappa_0$, $\ell \in (0, \frac{1}{2})$, $\epsilon \in (0, \epsilon_0)$, $\ell^2\kappa^2\epsilon > 1$, $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a critical point of (1.1), and $\overline{Q_\ell(x_0)} \subset \Omega \cap \{|B_0| > \epsilon\}$, then*

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0\left(\psi, \mathbf{A}; Q_\ell(x_0)\right) \geqslant g\left(\frac{H}{\kappa}\overline{B}_{Q_\ell(x_0)}\right)\kappa^2 - C\left(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa\epsilon)^{-1} + \ell\epsilon^{-1}\right)\kappa^2.$$

Here $g(\cdot)$ is the function introduced in (2.5), and $\overline{B}_{Q_\ell(x_0)}$ is introduced in (4.3).

Proof. Using the inequality $\|\psi\|_\infty \leqslant 1$ and (4.4) to obtain

$$\begin{aligned}
\mathcal{E}_0\left(\psi, \mathbf{A}; Q_\ell(x_0)\right) &\geqslant (1 - \delta)\mathcal{E}_0\left(u, \sigma_\ell \overline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0); Q_\ell(x_0)\right) \\
&\quad - C\kappa^2\left(\delta^{-1}\ell^4\kappa^2 + \delta^{-1}\ell^{2\alpha} + \delta\right)\left|Q_\ell(x_0)\right|,
\end{aligned} \tag{4.11}$$

where u is defined in (4.8).

Let

$$b = \frac{H}{\kappa} \overline{B}_{Q_\ell(x_0)}, \qquad R = \ell \sqrt{\kappa H \overline{B}_{Q_\ell(x_0)}}. \tag{4.12}$$

Define the rescaled function

$$v(x) = u\left(\frac{\ell}{R} x + x_0\right), \quad \forall x \in Q_R. \tag{4.13}$$

Remember that σ_ℓ denotes the sign of B_0 in $Q_\ell(x_0)$. The change of variable $y = \frac{R}{\ell}(x - x_0)$ gives

$$\begin{aligned}
&\mathcal{E}_0\big(u, \sigma_\ell \overline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x - x_0); Q_\ell(x_0)\big) \\
&\quad = \int_{Q_R} \left(\left| \left(\frac{R}{\ell} \nabla_y - i\sigma_\ell \frac{R}{\ell} \mathbf{A}_0(y) \right) v \right|^2 - \kappa^2 |v|^2 + \frac{\kappa^2}{2} |v|^4 \right) \frac{\ell}{R} \, dy \\
&\quad = \int_{Q_R} \left(\big|(\nabla_y - i\sigma_\ell \mathbf{A}_0) v\big|^2 - \frac{\kappa}{H \overline{B}_{Q_\ell(x_0)}} |v|^2 + \frac{\kappa}{2H \overline{B}_{Q_\ell(x_0)}} |v|^4 \right) dy \\
&\quad = \frac{\kappa}{H \overline{B}_{Q_\ell(x_0)}} \int_{Q_R} b \left(\big|(\nabla_y - i\sigma_\ell \mathbf{A}_0) v\big|^2 - |v|^2 + \frac{1}{2} |v|^4 \right) dy \\
&\quad = \frac{1}{b} G^{\sigma_\ell}_{b, Q_R}(v).
\end{aligned} \tag{4.14}$$

We still need to estimate from below the reduced energy $G^{\sigma_\ell}_{b,Q_R}(v)$. Since v is not in $H^1_0(Q_R)$, we introduce a cut-off function $\chi_R \in C_c^\infty(\mathbb{R}^2)$ such that

$$0 \leqslant \chi_R \leqslant 1 \quad \text{in } \mathbb{R}^2, \qquad \operatorname{supp} \chi_R \subset Q_R, \qquad \chi_R = 1 \quad \text{in } Q_{R-1}, \qquad \text{and} \quad |\nabla \chi_R| \leqslant M \quad \text{in } \mathbb{R}^2. \tag{4.15}$$

The constant M is universal.

Let

$$u_R = \chi_R v. \tag{4.16}$$

We have

$$\begin{aligned}
G^{\sigma_\ell}_{b,Q_R}(v) &= \int_{Q_R} \left(b\big|(\nabla - i\sigma_\ell \mathbf{A}_0) v\big|^2 - |v|^2 + \frac{1}{2}|v|^4 \right) dx \\
&\geqslant \int_{Q_R} \left(b\big|\chi_R(\nabla - i\sigma_\ell \mathbf{A}_0) v\big|^2 - |\chi_R v|^2 + \frac{1}{2}|v|^4 + \big(\chi_R^2 - 1\big)|v|^2 \right) dx \\
&\geqslant G^{\sigma_\ell}_{b,Q_R}(\chi_R v) - \int_{Q_R} \big(1 - \chi_R^2\big)|v|^2 \, dx - 2 \int_{Q_R} \big|\big\langle (\nabla - i\sigma_\ell \mathbf{A}_0)\chi_R v, \nabla \chi_R v \big\rangle\big| \, dy.
\end{aligned} \tag{4.17}$$

Having in mind (4.13) and (4.8), we get

$$\big|\big(\nabla_y - i\sigma_\ell \mathbf{A}_0(y)\big) v(y)\big| = \frac{\ell}{R} \big|\big(\nabla_x - i\kappa H \sigma_\ell \overline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x - x_0)\big) u(x)\big|.$$

Using the estimate in (3.11), (4.7) and (A.3) we get

$$\begin{aligned}
\big|\big(\nabla_y - i\sigma_\ell \mathbf{A}_0(y)\big) v(y)\big| &\leqslant \frac{\ell}{R} \big|\big(\nabla_x - i\kappa H \sigma_\ell \overline{B}_{Q_\ell(x_0)} (\mathbf{A} + \nabla\varphi)\big) u(x)\big| \\
&\quad + \frac{\kappa H \ell}{R} \big|\big(\mathbf{A} - \sigma_\ell \overline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x - x_0) - \nabla\varphi\big) u(x)\big| \\
&\leqslant \frac{C_1 \ell}{R} \big(\kappa + \kappa \ell^\alpha + \kappa^2 \ell^2\big).
\end{aligned} \tag{4.18}$$

From the definition of u_R in (4.16) and χ_R in (4.15) we get

$$|v| \leqslant 1. \tag{4.19}$$

Using (4.19), (4.18) and the definition of χ_R in (4.15), we get

$$\begin{aligned}\int_{Q_R} \big|\langle(\nabla - i\sigma_\ell \mathbf{A}_0)\chi_R v, \nabla\chi_R v\rangle\big|\,dy &\leqslant \frac{C_1\ell}{R}\big(\kappa + \kappa\ell^\alpha + \kappa^2\ell^2\big)\int_{Q_R\setminus Q_{R-1}} |\nabla\chi_R|\,dx\\ &\leqslant C_1\big(\kappa\ell + \kappa\ell^{\alpha+1} + \kappa^2\ell^3\big),\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}\int_{Q_R} \big(1-\chi_R^2\big)|v|^2\,dx &\leqslant |Q_R \setminus Q_{R-1}|\\ &\leqslant R.\end{aligned} \tag{4.21}$$

Inserting (4.20) and (4.21) into (4.17), we get

$$\begin{aligned}G^{\sigma_\ell}_{b,Q_R}(v) &\geqslant G^{\sigma_\ell}_{b,Q_R}(u_R) - C_2\big(\kappa\ell + \kappa\ell^{\alpha+1} + \kappa^2\ell^3 + \kappa\ell\sqrt{\epsilon}\big)\\ &\geqslant G^{\sigma_\ell}_{b,Q_R}(u_R) - C_2\big(\kappa\ell(\sqrt{\epsilon}+1) + \kappa^2\ell^3\big).\end{aligned}$$

There are two cases:

Case 1: $\sigma_\ell = +1$, when $B_0 > 0$, in $Q_\ell(x_0)$.

Case 2: $\sigma_\ell = -1$, when $B_0 < 0$, in $Q_\ell(x_0)$.

In Case 1, after recalling the definition of $m_0(b,R)$ introduced in (2.3), where b is introduced in (4.12) we get

$$G^{+1}_{b,Q_R}(v) \geqslant m_0(b,R) - C_2\big(\kappa\ell(\sqrt{\epsilon}+1) + \kappa^2\ell^3\big). \tag{4.22}$$

We get by collecting the estimates in (4.11)–(4.22):

$$\begin{aligned}\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0\big(\psi,\mathbf{A};Q_\ell(x_0)\big) &\geqslant \frac{(1-\delta)}{b\ell^2}\big(m_0(b,R) - C_2\big(\kappa\ell + \kappa^2\ell^3(\epsilon+1)\big)\big) - C\big(\delta^{-1}\ell^4\kappa^2 + \delta^{-1}\ell^{2\alpha} + \delta\big)\kappa^2\\ &\geqslant \frac{(1-\delta)}{b\ell^2}m_0(b,R) - r(\kappa),\end{aligned} \tag{4.23}$$

where

$$r(\kappa) = C_3\bigg(\delta^{-1}\ell^4\kappa^4 + \delta^{-1}\ell^{2\alpha}\kappa^2 + \delta\kappa^2 + \frac{1}{b\ell^2}\big(\kappa\ell(\sqrt{\epsilon}+1) + \kappa^2\ell^3\big)\bigg). \tag{4.24}$$

Theorem 2.2 tells us that $m_0(b,R) \geqslant R^2 g(b)$ for all $b \in [0,1]$ and R sufficiently large. Here $g(b)$ is introduced in (2.5). Therefore, we get from (4.23) the estimate

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0\big(\psi,\mathbf{A};Q_\ell(x_0)\big) \geqslant \bigg(\frac{(1-\delta)R^2}{b\ell^2}\bigg)g(b) - r(\kappa), \tag{4.25}$$

with b defined in (4.12). By choosing $\delta = \ell$ and using that $\overline{Q_\ell(x_0)} \subset \{|B_0| > \epsilon\}$, we get

$$r(\kappa) = \mathcal{O}\bigg(\ell^3\kappa^2 + \ell^{2\alpha-1} + \frac{1}{\epsilon}\big((\ell\kappa)^{-1} + \ell\big)\bigg)\kappa^2. \tag{4.26}$$

This implies that

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0\big(\psi,\mathbf{A};Q_\ell(x_0)\big) \geqslant g\bigg(\frac{H}{\kappa}\overline{B}_{Q_\ell(x_0)}\bigg)\kappa^2 - C\big(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa\epsilon)^{-1} + \ell\epsilon^{-1}\big)\kappa^2.$$

Similarly, in Case 2, according to Remark 2.1, we get that

$$G^{-1}_{b,Q_R}(v) \geqslant m_0(b,R) - C_2\big(\kappa\ell + \kappa^2\ell^3(\epsilon+1)\big),$$

and the rest of the proof is as for Case 1. □

5. Proof of Theorem 1.1

5.1. Upper bound

Proposition 5.1. *There exist positive constants C and κ_0 such that, if (3.10) holds, then the ground state energy* $\mathrm{E_g}(\kappa, H)$ *in (1.6) satisfies*

$$\mathrm{E_g}(\kappa, H) \leqslant \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx + C\kappa^{\frac{15}{8}}.$$

Proof. Let $\ell = \ell(\kappa)$ and $\epsilon = \epsilon(\kappa)$ be positive parameters such that $\kappa^{-1} \ll \ell \ll 1$ and $\kappa^{-1} \ll \epsilon \ll 1$ as $\kappa \to \infty$. For some $\beta \in (0, 1)$, $\mu \in (0, 1)$ to be determined later, we will choose

$$\ell = \kappa^{-\beta}, \qquad \epsilon = \kappa^{-\mu}. \tag{5.1}$$

Consider the lattice $\Gamma_\ell := \ell\mathbb{Z} \times \ell\mathbb{Z}$ and write for $\gamma \in \Gamma_\ell$, $Q_{\gamma,\ell} = Q_\ell(\gamma)$. For any $\gamma \in \Gamma_\ell$ such that $\overline{Q_{\gamma,\ell}} \subset \Omega \cap \{|B_0| > \epsilon\}$ let

$$\underline{B}_{\gamma,\ell} = \inf_{x \in Q_{\gamma,\ell}} \left|B_0(x)\right|. \tag{5.2}$$

Let

$$\mathcal{I}_{\ell,\epsilon} = \left\{\gamma\colon\ \overline{Q_{\gamma,\ell}} \subset \Omega \cap \left\{|B_0| > \epsilon\right\}\right\},$$
$$N = \operatorname{Card} \mathcal{I}_{\ell,\epsilon},$$

and

$$\Omega_{\ell,\epsilon} = \operatorname{int}\left(\bigcup_{\gamma \in \mathcal{I}_{\ell,\epsilon}} \overline{Q_{\gamma,\ell}}\right).$$

It follows from (1.2) that

$$N = |\Omega|\ell^{-2} + \mathcal{O}\left(\epsilon\ell^{-2}\right) + \mathcal{O}\left(\ell^{-1}\right) \quad \text{as } \ell \to 0 \text{ and } \epsilon \to 0.$$

Let

$$b = \frac{H}{\kappa}\underline{B}_{\gamma,\ell}, \qquad R = \ell\sqrt{\kappa H \underline{B}_{\gamma,\ell}}, \tag{5.3}$$

and u_R be a minimizer of the functional in (2.1), i.e.

$$m_0(b, R) = \int_{Q_R} \left(b\left|(\nabla - i\mathbf{A}_0)u_R\right|^2 - |u_R|^2 + \frac{1}{2}|u_R|^4\right) dx.$$

We will need the function φ_γ introduced in Lemma A.3 which satisfies

$$\left|\mathbf{F}(x) - \sigma_{\gamma,\ell}\underline{B}_{\gamma,\ell}\mathbf{A}_0(x - \gamma) - \nabla\varphi_\gamma(x)\right| \leqslant C\ell^2 \quad \text{in } Q_{\gamma,\ell},$$

where $\sigma_{\gamma,\ell}$ is the sign of B_0 in $Q_{\gamma,\ell}$.

We define the function

$$v(x) = \begin{cases} e^{-i\kappa H\varphi_\gamma} u_R(\frac{R}{\ell}(x - \gamma)) & \text{if } x \in Q_{\gamma,\ell} \subset \{B_0 > \epsilon\}, \\ e^{-i\kappa H\varphi_\gamma} \overline{u_R}(\frac{R}{\ell}(x - \gamma)) & \text{if } x \in Q_{\gamma,\ell} \subset \{B_0 < -\epsilon\}, \\ 0 & \text{if } x \in \Omega \setminus \Omega_{\ell,\epsilon}. \end{cases}$$

Since $u_R \in H_0^1(Q_R)$, then $v \in H^1(\Omega)$. We compute the energy of the configuration $(v, \mathbf{F})$. We get

$$\mathcal{E}(v,\mathbf{F}) = \int_\Omega \left(\left|(\nabla - i\kappa H\mathbf{F})v\right|^2 - \kappa^2|v|^2 + \frac{\kappa^2}{2}|v|^4 \right) dx$$
$$= \sum_{\gamma\in\mathcal{I}_{\ell,\epsilon}} \mathcal{E}_0(v,\mathbf{F}; Q_{\gamma,\ell}). \tag{5.4}$$

We estimate the term $\mathcal{E}_0(v,\mathbf{F}; Q_{\gamma,\ell})$ from above and we write

$$\begin{aligned}
\mathcal{E}_0(v,\mathbf{F}; Q_{\gamma,\ell}) &= \int_{Q_{\gamma,\ell}} \left|(\nabla - i\kappa H\mathbf{F})v\right|^2 - \kappa^2|v|^2 + \frac{\kappa^2}{2}|v|^4\,dx \\
&= \int_{Q_{\gamma,\ell}} \big|\big(\nabla - i\kappa H\big(\sigma_{\gamma,\ell}\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma) + \nabla\varphi_\gamma(x)\big)\big)v \\
&\qquad - i\kappa H\big(\mathbf{F} - \sigma_{\gamma,\ell}\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma) - \nabla\varphi_\gamma(x)\big)v\big|^2 - \kappa^2|v|^2 + \frac{\kappa^2}{2}|v|^4\,dx \\
&\leqslant \int_{Q_{\gamma,\ell}} (1+\delta)\big|\big(\nabla - i\kappa H\big(\sigma_{\gamma,\ell}\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma) + \nabla\varphi_\gamma(x)\big)\big)v\big|^2 - \kappa^2|v|^2 + \frac{\kappa^2}{2}|v|^4\,dx \\
&\qquad + C\big(1+\delta^{-1}\big)(\kappa H)^2 \int_{Q_{\gamma,\ell}} \big|\big(\mathbf{F} - \sigma_{\gamma,\ell}\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma) - \nabla\varphi_\gamma(x)\big)v\big|^2\,dx \\
&\leqslant (1+\delta)\mathcal{E}_0\big(e^{-i\kappa H\varphi_\gamma}v, \sigma_{\gamma,\ell}\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma); Q_{\gamma,\ell}\big) + C\big(\delta\kappa^2 + \delta^{-1}\kappa^4\ell^4\big)\int_{Q_{\gamma,\ell}} |v|^2\,dx.
\end{aligned} \tag{5.5}$$

Having in mind that u_R is a minimizer of the functional in (2.1), and using the estimate in (3.1) we get

$$\int_{Q_{\gamma,\ell}} |v|^2\,dx \leqslant |Q_{\gamma,\ell}|.$$

Remark 2.1 and a change of variables give us

$$\int_{Q_{\gamma,\ell}} \left(\left|\big(\nabla - i\kappa H\sigma_{\gamma,\ell}\big(\underline{B}_{\gamma,\ell}\mathbf{A}_0(x-\gamma)\big)e^{-i\kappa H\varphi_\gamma}\big)v\right|^2 - \kappa^2|v|^2 + \frac{\kappa^2}{2}|v|^4 \right) dx = \frac{m_0(b,R)}{b}.$$

We insert this into (5.5) to obtain

$$\mathcal{E}_0(v,\mathbf{F}; Q_{\gamma,\ell}) \leqslant (1+\delta)\frac{m_0(b,R)}{b} + C\big(\delta\kappa^2 + \delta^{-1}\kappa^4\ell^4\big)\ell^2. \tag{5.6}$$

We know from Theorem 2.2 that $m_0(b,R) \leqslant g(b)R^2 + CR$ for all $b \in [0,1]$ and R sufficiently large, where b introduced in (5.3). We choose $\delta = \ell$ in (5.6). That way we get

$$\mathcal{E}_0(v,\mathbf{F}; Q_{\gamma,\ell}) \leqslant g\left(\frac{H}{\kappa}\underline{B}_{\gamma,\ell}\right)\ell^2\kappa^2 + C\left(\frac{1}{\kappa\ell\sqrt{\epsilon}} + \ell + \kappa^2\ell^3\right)\ell^2\kappa^2. \tag{5.7}$$

Summing (5.7) over γ in $I_{\ell,\epsilon}$, we recognize the lower Riemann sum of $x \to g(\frac{H}{\kappa}|B_0(x)|)$. By monotonicity of g, g is Riemann-integrable and its integral is larger than any lower Riemann sum. Thus

$$\mathcal{E}(v,\mathbf{F}) \leqslant \left(\int_{\Omega_{\ell,\epsilon}} g\left(\frac{H}{\kappa}\big|B_0(x)\big|\right) dx\right)\kappa^2 + C\left(\frac{1}{\kappa\ell\sqrt{\epsilon}} + \ell + \kappa^2\ell^3\right)\kappa^2. \tag{5.8}$$

Notice that using the regularity of $\partial\Omega$ and (1.2), there exists $C > 0$ such that

$$|\Omega \setminus \Omega_{\ell,\epsilon}| = \mathcal{O}\big(\ell|\partial\Omega| + C\epsilon\big), \tag{5.9}$$

as ϵ and ℓ tend to 0.

Thus, we get by using the properties of g in Theorem 2.2,

$$\int_{\Omega_{\ell,\epsilon}} g\left(\frac{H}{\kappa}\big|B_0(x)\big|\right) dx \leqslant \int_{\Omega} g\left(\frac{H}{\kappa}\big|B_0(x)\big|\right) dx + \frac{1}{2}|\Omega \setminus \Omega_{\ell,\epsilon}|.$$

This implies that

$$\mathcal{E}(v, \mathbf{F}) \leqslant \int_{\Omega} g\left(\frac{H}{\kappa}\big|B_0(x)\big|\right) dx + C\left(\frac{1}{\kappa\ell\sqrt{\epsilon}} + \ell + \epsilon + \kappa^2\ell^3\right)\kappa^2. \tag{5.10}$$

We choose in (5.1)

$$\beta = \frac{3}{4} \quad \text{and} \quad \mu = \frac{1}{8}. \tag{5.11}$$

With this choice, we infer from (5.10),

$$\mathcal{E}(v, \mathbf{F}) \leqslant \int_{\Omega} g\left(\frac{H}{\kappa}\big|B_0(x)\big|\right) dx + C_1\kappa^{\frac{15}{8}}, \tag{5.12}$$

and

$$\ell^2\kappa^2\epsilon = \kappa^{\frac{3}{8}} > 1. \tag{5.13}$$

This finishes the proof of Proposition 5.1. □

Remark 5.2. In the case when B_0 does not vanish in Ω, ϵ disappears and $\{x \in \Omega;\ |B_0(x)| > 0\} = \Omega$. Consequently, the Ginzburg–Landau energy of $(v, \mathbf{F})$ in (4.2) satisfies

$$\mathcal{E}(v, \mathbf{F}) \leqslant \int_{\Omega} g\left(\frac{H}{\kappa}\big|B_0(x)\big|\right) dx + C\left(\frac{1}{\kappa\ell} + \ell + \kappa^2\ell^3\right)\kappa^2.$$

We take the same choice of β as in (5.11), then the ground state energy $\mathrm{E_g}(\kappa, H)$ in (1.6) satisfies

$$\mathrm{E_g}(\kappa, H) \leqslant \kappa^2 \int_{\Omega} g\left(\frac{H}{\kappa}\big|B_0(x)\big|\right) dx + C\kappa^{\frac{7}{4}}.$$

5.2. *Lower bound*

We now establish a lower bound for the ground state energy $\mathrm{E_g}(\kappa, H)$ in (1.6). The parameters ϵ and ℓ have the same form as in (5.1).

Let

$$\overline{B}_{\gamma,\ell} = \sup_{x \in Q_{\gamma,\ell}} \big|B_0(x)\big|, \tag{5.14}$$

and

$$b_{\gamma,\ell} = \frac{H}{\kappa}\overline{B}_{\gamma,\ell}, \qquad R = \ell\sqrt{\kappa H \overline{B}_{\gamma,\ell}}. \tag{5.15}$$

If $(\psi, \mathbf{A})$ is a minimizer of (1.1), we have

$$\mathrm{E_g}(\kappa, H) = \mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon}) + \mathcal{E}_0(\psi, \mathbf{A}; \Omega \setminus \Omega_{\ell,\epsilon}) + (\kappa H)^2 \int_{\Omega} \big|\mathrm{curl}(\mathbf{A} - \mathbf{F})\big|^2 dx, \tag{5.16}$$

where, for any $D \subset \Omega$, the energy $\mathcal{E}_0(\psi, \mathbf{A}; D)$ is introduced in (4.1). Since the magnetic energy term is positive, we may write

$$\mathrm{E_g}(\kappa, H) \geqslant \mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon}) + \mathcal{E}_0(\psi, \mathbf{A}; \Omega \setminus \Omega_{\ell,\epsilon}). \tag{5.17}$$

Thus, we get by using (3.1), (3.11), and (5.9):

$$\begin{aligned}\left|\mathcal{E}_0(\psi, \mathbf{A}; \Omega \setminus \Omega_{\ell,\epsilon})\right| &\leqslant \int_{\Omega\setminus\Omega_{\ell,\epsilon}} \left|(\nabla - i\kappa H\mathbf{A})\psi\right|^2 + \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4\, dx \\ &\leqslant |\Omega \setminus \Omega_{\ell,\epsilon}|\left(C_1\kappa^2\|\psi\|^2_{L^\infty(\Omega)} + \kappa^2\|\psi\|^2_{L^\infty(\Omega)} + \frac{\kappa^2}{2}\|\psi\|^4_{L^\infty(\Omega)}\right) \\ &\leqslant C_2(\ell + \epsilon)\kappa^2. \end{aligned} \tag{5.18}$$

To estimate $\mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon})$, we notice that

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon}) = \sum_{\gamma\in\mathcal{I}_{\ell,\epsilon}} \mathcal{E}_0(\psi, \mathbf{A}; Q_{\gamma,\ell}).$$

Using Proposition 4.2 with $\alpha = \frac{2}{3}$ and (5.18) with $\beta = \frac{3}{4}$ and $\mu = \frac{1}{8}$ in (5.1), we get

$$\begin{aligned}\mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon}) &\geqslant \sum_{\gamma\in\mathcal{I}_{\ell,\epsilon}} g\left(\frac{H}{\kappa}\overline{B}_{Q_\ell(x_0)}\right)\ell^2\kappa^2 - C\left(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa\epsilon)^{-1} + \ell\epsilon^{-1}\right)\kappa^2 \\ &\geqslant \kappa^2 \sum_{\gamma\in\mathcal{I}_{\ell,\epsilon}} g\left(\frac{H}{\kappa}\overline{B}_{Q_\ell(x_0)}\right)\ell^2 - C_1\kappa^{\frac{15}{8}},\end{aligned}$$

and

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega \setminus \Omega_{\ell,\epsilon}) \geqslant -C_2\kappa^{\frac{15}{8}}. \tag{5.19}$$

As for the upper bound, we can use the monotonicity of g and recognize that the sum above is an upper Riemann sum of g. In that way, we get

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon}) \geqslant \kappa^2 \int_{\Omega_{\ell,\epsilon}} g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx - C_1\kappa^{\frac{15}{8}}.$$

Notice that $\Omega_{\ell,\epsilon} \subset \Omega$ and that $g \leqslant 0$, we deduce that

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon}) \geqslant \kappa^2 \int_{\Omega} g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx - C_1\kappa^{\frac{15}{8}}. \tag{5.20}$$

Finally, putting (5.19) and (5.20) into (5.17), we obtain

$$\mathrm{E_g}(\kappa, H) \geqslant \kappa^2 \int_{\Omega} g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx - C\kappa^{\frac{15}{8}}. \tag{5.21}$$

Remark 5.3. When B_0 does not vanish, the local energy in $Q_\ell(x_0)$ in Proposition 4.2 becomes

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0\big(\psi, \mathbf{A}; Q_\ell(x_0)\big) \geqslant g\left(\frac{H}{\kappa}\overline{B}_{Q_\ell(x_0)}\right)\kappa^2 - C\left(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa)^{-1}\right)\kappa^2.$$

Similarly, we choose $\alpha = \frac{2}{3}$ and $\ell = \kappa^{-\frac{3}{4}}$, we get

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega \setminus \Omega_{\ell,\epsilon}) \geqslant -C_2\kappa^{\frac{7}{4}}, \tag{5.22}$$

and

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega_{\ell,\epsilon}) \geqslant \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx - C_1\kappa^{\frac{7}{4}}. \tag{5.23}$$

As a consequence of (5.22) and (5.23), (5.21) becomes

$$\mathrm{E_g}(\kappa, H) \geqslant \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx - C\kappa^{\frac{7}{4}}. \tag{5.24}$$

5.3. *Proof of Corollary 1.2*

If $(\psi, \mathbf{A})$ is a minimizer of (1.1), we have

$$\mathcal{E}(\psi, \mathbf{A}; \Omega) = \mathcal{E}_0(\psi, \mathbf{A}; \Omega) + (\kappa H)^2 \int_\Omega \left|\mathrm{curl}(\mathbf{A} - \mathbf{F})\right|^2 dx. \tag{5.25}$$

Theorem 1.1 tells us that

$$\mathcal{E}(\psi, \mathbf{A}; \Omega) \leqslant \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx + C\kappa^{\tau_0}.$$

This implies that

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega) + (\kappa H)^2 \int_\Omega \left|\mathrm{curl}(\mathbf{A} - \mathbf{F})\right|^2 dx \leqslant \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx + C_1\kappa^{\tau_0}. \tag{5.26}$$

Using (5.19), (5.20), (5.22) and (5.23), we get

$$\mathcal{E}_0(\psi, \mathbf{A}; \Omega) \geqslant \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx - C_2\kappa^{\tau_0}. \tag{5.27}$$

Putting (5.27) into (5.26), we get

$$\begin{aligned} &-C_2\kappa^{\tau_0} + \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx + (\kappa H)^2 \int_\Omega \left|\mathrm{curl}(\mathbf{A} - \mathbf{F})\right|^2 dx \\ &\quad \leqslant \kappa^2 \int_\Omega g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx + C_1\kappa^{\tau_0}. \end{aligned} \tag{5.28}$$

By simplification, we obtain

$$(\kappa H)^2 \int_\Omega \left|\mathrm{curl}(\mathbf{A} - \mathbf{F})\right|^2 dx \leqslant C'\kappa^{\tau_0}. \tag{5.29}$$

6. Local energy estimates

The object of this section is to give an estimate to the Ginzburg–Landau energy (4.2) in the open set $D \subset \Omega$.

6.1. *Main statements*

Theorem 6.1. *There exist positive constants κ_0 such that if (3.10) is true and $D \subset \Omega$ is an open set, then the local energy of the minimizer satisfies*

$$\left|\mathcal{E}(\psi, \mathbf{A}; D) - \kappa^2 \int_D g\left(\frac{H}{\kappa}\left|B_0(x)\right|\right) dx\right| = o\left(\kappa^2\right). \tag{6.1}$$

For all (ℓ, x_0) such that $\overline{Q_\ell(x_0)} \subset \Omega \cap \{|B_0| > \epsilon\}$, we define

$$\underline{B}_{Q_\ell(x_0)} = \inf_{x \in Q_\ell(x_0)} \left|B_0(x)\right|, \tag{6.2}$$

where B_0 is introduced in (1.2).

Proposition 6.2. *For all $\alpha \in (0,1)$, there exist positive constants C, ϵ_0 and κ_0 such that if (3.10) is true, $\kappa \geqslant \kappa_0$, $\ell \in (0, \frac{1}{2})$, $\epsilon \in (0, \epsilon_0)$, $\ell^2\kappa^2\epsilon > 1$, $(\psi, \mathbf{A}) \in H^1(\Omega; \mathbb{C}) \times H^1_{\mathrm{div}}(\Omega)$ is a minimizer of (1.1), and $\overline{Q_\ell(x_0)} \subset \Omega \cap \{|B_0| > \epsilon\}$, then*

$$\frac{1}{|Q_\ell(x_0)|}\mathcal{E}_0\big(\psi, \mathbf{A}; Q_\ell(x_0)\big) \leqslant g\left(\frac{H}{\kappa}\underline{B}_{Q_\ell(x_0)}\right)\kappa^2 + C\big(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa\sqrt{\epsilon})^{-1}\big)\kappa^2.$$

Here $g(\cdot)$ is the function introduced in (2.5) and $\mathcal{E}_0$ is the functional in (4.1).

Proof. As explained earlier in the proof of Lemma 4.1 in (4.5), we may suppose after performing a gauge transformation that the magnetic potential $\mathbf{A}$ satisfies

$$\left|\mathbf{A}(x) - \mathbf{F}(x)\right| \leqslant C\frac{\ell^\alpha}{H}, \quad \forall x \in Q_\ell(x_0). \tag{6.3}$$

Let

$$b = \frac{H}{\kappa}\underline{B}_{Q_\ell(x_0)}, \qquad R = \ell\sqrt{\kappa H \underline{B}_{Q_\ell(x_0)}}, \tag{6.4}$$

and $u_R \in H^1_0(Q_R)$ be the minimizer of the functional G^{+1}_{b,Q_R} introduced in (2.1). Let $\chi_R \in C_c^\infty(\mathbb{R}^2)$ be a cut-off function such that

$$0 \leqslant \chi_R \leqslant 1 \quad \text{in } \mathbb{R}^2, \qquad \operatorname{supp}\chi_R \subset Q_{R+1}, \qquad \chi_R = 1 \quad \text{in } Q_R,$$

and $|\nabla\chi_R| \leqslant C$ for some universal constant C.

Let $\eta_R(x) = 1 - \chi_R(\frac{R}{\ell}(x - x_0))$ for all $x \in \mathbb{R}^2$ and $\widetilde{\ell} = \ell(1 + \frac{1}{R})$.

This implies that

$$\eta_R(x) = 0 \quad \text{in } Q_\ell(x_0), \tag{6.5}$$

$$0 \leqslant \eta_R(x) \leqslant 1 \quad \text{in } Q_{\widetilde{\ell}}(x_0) \setminus Q_\ell(x_0), \tag{6.6}$$

$$\eta_R(x) = 1 \quad \text{in } \Omega \setminus Q_{\widetilde{\ell}}(x_0). \tag{6.7}$$

Consider the function $w(x)$ defined as follows,

$$w(x) = \eta_R(x)\psi(x) \quad \text{in } \Omega \setminus Q_\ell(x_0),$$

and, if $x \in Q_\ell(x_0)$,

$$w(x) = \begin{cases} e^{i\kappa H\varphi}u_R(\frac{R}{\ell}(x - x_0)) & \text{if } Q_\ell(x_0) \subset \{B_0 > \epsilon\} \cap \Omega, \\ e^{i\kappa H\varphi}\bar{u}_R(\frac{R}{\ell}(x - x_0)) & \text{if } Q_\ell(x_0) \subset \{B_0 < -\epsilon\} \cap \Omega. \end{cases}$$

Notice that by construction, $w = \psi$ in $\Omega \setminus Q_{\widetilde{\ell}}(x_0)$. We will prove that, for any $\delta \in (0,1)$ and $\alpha \in (0,1)$,

$$\mathcal{E}(w, \mathbf{A}; \Omega) \leqslant \mathcal{E}\big(\psi, \mathbf{A}; \Omega \setminus Q_\ell(x_0)\big) + (1+\delta)\frac{\ell}{bR}m_0(b, R) + r_0(\kappa)\ell^2, \tag{6.8}$$

and for some constant C, $r_0(\kappa)$ is given as follows,

$$r_0(\kappa) = C\left(\delta + \delta^{-1}\ell^4\kappa^2 + \delta^{-1}\ell^{2\alpha} + \frac{1}{\ell\kappa\sqrt{\epsilon}}\right)\kappa^2. \tag{6.9}$$

Proof of (6.8): With $\mathcal{E}_0$ defined in (4.1), we write

$$\mathcal{E}_0(w, \mathbf{A}; \Omega) = \mathcal{E}_1 + \mathcal{E}_2, \tag{6.10}$$

where

$$\mathcal{E}_1 = \mathcal{E}_0\big(w, \mathbf{A}; \Omega \setminus Q_\ell(x_0)\big), \qquad \mathcal{E}_2 = \mathcal{E}_0\big(w, \mathbf{A}; Q_\ell(x_0)\big). \tag{6.11}$$

We estimate $\mathcal{E}_1$ and $\mathcal{E}_2$ from above. Starting with $\mathcal{E}_1$ and using (6.7), we get

$$\begin{aligned}
\mathcal{E}_1 &= \int\limits_{\Omega\setminus Q_\ell(x_0)} \big|(\nabla - i\kappa H\mathbf{A})\eta_R\psi\big|^2 - \kappa^2|\eta_R\psi|^2 + \frac{\kappa^2}{2}|\eta_R\psi|^4\,dx \\
&= \int\limits_{\Omega\setminus Q_\ell(x_0)} \eta_R^2\big|(\nabla - i\kappa H\mathbf{A})\psi\big|^2 + |\nabla\eta_R\psi|^2 + 2R\big\langle\eta_R(\nabla - i\kappa H\mathbf{A})\psi, \nabla\eta_R\psi\big\rangle - \kappa^2\eta_R^2|\psi|^2 + \frac{\kappa^2}{2}\eta_R^4|\psi|^4\,dx \\
&= \mathcal{E}_0\big(\psi, \mathbf{A}; \Omega \setminus Q_\ell(x_0)\big) + \mathcal{R}(\psi, \mathbf{A}),
\end{aligned} \tag{6.12}$$

where

$$\begin{aligned}
\mathcal{R}(\psi, \mathbf{A}) = \int\limits_{Q_{\tilde\ell}(x_0)\setminus Q_\ell(x_0)} &\Big(\big(\eta_R^2 - 1\big)\big(\big|(\nabla - i\kappa H\mathbf{A})\psi\big|^2 - \kappa^2|\psi|^2\big) + |\psi\nabla\eta_R|^2 + \frac{\kappa^2}{2}\big(\eta_R^4 - 1\big)|\psi|^4 \\
&+ 2\Re\big\langle\eta_R(\nabla - i\kappa H\mathbf{A})\psi, \psi\nabla\eta_R\big\rangle\Big)\,dx.
\end{aligned}$$

Noticing that $|Q_{\tilde\ell}(x_0) \setminus Q_\ell(x_0)| \leqslant \frac{\ell}{\sqrt{\kappa H \underline{B}_{Q_\ell(x_0)}}}$ and using (6.6) together with the estimates in (3.1), (3.10), (3.11) and $|\nabla\eta_R| \leqslant C\frac{R}{\ell}$, we get

$$\big|\mathcal{R}(\psi, \mathbf{A})\big| \leqslant C\frac{\ell\kappa}{\sqrt{\epsilon}}. \tag{6.13}$$

Inserting (6.13) in (6.12), we get the following estimate

$$\mathcal{E}_1 \leqslant \mathcal{E}_0\big(\psi, \mathbf{A}; \Omega \setminus Q_\ell(x_0)\big) + C\frac{\ell\kappa}{\sqrt{\epsilon}}. \tag{6.14}$$

We estimate the term $\mathcal{E}_2$ in (6.11). We will need the function φ_0 introduced in Lemma A.3 and satisfying $|\mathbf{F}(x) - \sigma_\ell\underline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) - \nabla\varphi_0(x)| \leqslant C\ell^2$ in $Q_\ell(x_0)$, where σ_ℓ denotes the sign of B_0. We start with the kinetic energy term and write for any $\delta \in (0, 1)$:

$$\begin{aligned}
\mathcal{E}_2 = \int\limits_{Q_\ell(x_0)} &\big|\big(\nabla - i\kappa H\big(\sigma_\ell\underline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) + \nabla\varphi(x)\big)\big)w \\
&- i\kappa H\big(\mathbf{A} - \big(\sigma_\ell\underline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) + \nabla\varphi(x)\big)\big)\big|^2 + \Big(-\kappa^2|w|^2 + \frac{\kappa^2}{2}|w|^4\Big)\,dx \\
\leqslant \int\limits_{Q_\ell(x_0)} &(1 + \delta)\big|\big(\nabla - i\kappa H\big(\sigma_\ell\underline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) + \nabla\varphi(x)\big)\big)w\big|^2 - \kappa^2|w|^2 + \frac{\kappa^2}{2}|w|^4\,dx \\
&+ \big(1 + \delta^{-1}\big)(\kappa H)^2 \int\limits_{Q_\ell(x_0)} \big|(\mathbf{A} - \nabla\phi_{x_0} - \mathbf{F})w + \big(\mathbf{F} - \sigma_\ell\underline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0) - \nabla\varphi_0(x)\big)w\big|^2\,dx.
\end{aligned} \tag{6.15}$$

Using the estimate in (6.3) together with (3.10) and (3.1), we deduce the upper bound

$$\mathcal{E}_2 \leqslant (1 + \delta)\mathcal{E}_0\big(e^{-i\kappa H\varphi}w, \sigma_\ell\underline{B}_{Q_\ell(x_0)}\mathbf{A}_0(x - x_0); Q_\ell(x_0)\big) + C\big(\delta^{-1}\ell^{2\alpha} + \delta^{-1}\ell^4\kappa^2 + \delta\big)\kappa^2\ell^2, \tag{6.16}$$

where $\alpha \in (0, 1)$.

There are two cases:

Case 1: If $B_0 > \epsilon$ in $Q_\ell(x_0)$, then $\sigma_\ell = +1$ and

$$w(x) = \begin{cases} e^{i\kappa H\varphi} u_R(\frac{R}{\ell}(x - x_0)) & \text{in } Q_\ell(x_0), \\ \eta_R(x)\psi(x) & \text{in } \Omega \setminus Q_\ell(x_0). \end{cases}$$

The change of variable $y = \frac{R}{\ell}(x - x_0)$ and (4.12) gives us

$$\begin{aligned} &\mathcal{E}_0\big(e^{-i\kappa H\varphi} w, \sigma_\ell \underline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x - x_0); Q_\ell(x_0)\big) \\ &= \int_{Q_R} \left(\left| \left(\frac{R}{\ell} \nabla_y - i \frac{R}{\ell} \mathbf{A}_0(y) \right) u_R \right|^2 - \kappa^2 |u_R|^2 + \frac{\kappa^2}{2} |u_R|^4 \right) \frac{\ell}{R}\, dy \\ &= \int_{Q_R} \left(\big| (\nabla_y - i \mathbf{A}_0(y)) u_R \big|^2 - \frac{\kappa}{H \underline{B}_{Q_\ell(x_0)}} |u_R|^2 + \frac{\kappa}{2H \underline{B}_{Q_\ell(x_0)}} |u_R|^4 \right) dy \\ &= \frac{\kappa}{H \underline{B}_{Q_\ell(x_0)}} \int_{Q_R} b \left(\big| (\nabla_y - i \mathbf{A}_0(y)) u_R \big|^2 - |u_R|^2 + \frac{1}{2} |u_R|^4 \right) dy \\ &= \frac{1}{b} G^{+1}_{b, Q_R}(u_R), \end{aligned} \tag{6.17}$$

where G^{+1}_{b,Q_R} is the functional from (2.1).

Case 2: If $B_0 < -\epsilon$ in $Q_\ell(x_0)$, then $\sigma_\ell = -1$ and

$$w(x) = \begin{cases} e^{i\kappa H\varphi} \bar{u}_R(\frac{R}{\ell}(x - x_0)) & \text{in } Q_\ell(x_0), \\ \eta_R(x)\psi(x) & \text{in } \Omega \setminus Q_\ell(x_0). \end{cases}$$

Similarly, like in Case 1, we have

$$\mathcal{E}_0\big(e^{-i\kappa H\varphi} w, \sigma_\ell \underline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x - x_0); Q_\ell(x_0)\big) = \frac{1}{b} G^{-1}_{b,Q_R}(\bar{u}_R) = \frac{1}{b} G^{+1}_{b,Q_R}(u_R).$$

In both cases we see that

$$\mathcal{E}_0\big(e^{-i\kappa H\varphi} w, \sigma_\ell \underline{B}_{Q_\ell(x_0)} \mathbf{A}_0(x - x_0); Q_\ell(x_0)\big) = \frac{1}{b} G^{+1}_{b,Q_R}(u_R) = \frac{m_0(b, R)}{b}. \tag{6.18}$$

Inserting (6.18) into (6.16), we get

$$\mathcal{E}_2 \leqslant (1 + \delta) \frac{1}{b} m_0(b, R) + C\big(\delta + \delta^{-1} \ell^4 \kappa^2 + \delta^{-1} \ell^{2\alpha}\big) \kappa^2 \ell^2. \tag{6.19}$$

Inserting (6.14) and (6.19) into (6.10), we deduce that

$$\begin{aligned} \mathcal{E}_0(\varphi, \mathbf{A}) &\leqslant \mathcal{E}_0\big(\psi, \mathbf{A}; \Omega \setminus Q_\ell(x_0)\big) + (1 + \delta) \frac{1}{b} m_0(b, R) \\ &\quad + C\big(\delta + \delta^{-1} \ell^4 \kappa^2 + \delta^{-1} \ell^{2\alpha} \kappa^2 + (\ell \kappa \sqrt{\epsilon})^{-1}\big) \ell^2 \kappa^2. \end{aligned} \tag{6.20}$$

This proves (6.8). Now, we show how (6.8) proves Proposition 6.2. By definition of the minimizer $(\psi, \mathbf{A})$, we have

$$\mathcal{E}(\psi, \mathbf{A}) \leqslant \mathcal{E}(\varphi, \mathbf{A}; \Omega).$$

Since $\mathcal{E}(\psi, \mathbf{A}; \Omega) = \mathcal{E}(\psi, \mathbf{A}; \Omega \setminus Q_\ell(x_0)) + \mathcal{E}_0(\psi, \mathbf{A}; Q_\ell(x_0))$, the estimate (6.8) gives us

$$\mathcal{E}_0\big(\psi, \mathbf{A}; Q_\ell(x_0)\big) \leqslant \frac{(1 + \delta)}{b} m_0(b, R) + r_0(\kappa),$$

where $r_0(\kappa)$ is defined in (6.9).

Dividing both sides by $|Q_\ell(x_0)| = \ell^2$, we get

$$\frac{1}{|Q_\ell(x_0)|} \mathcal{E}_0\big(\psi, \mathbf{A}, Q_\ell(x_0)\big) \leqslant \frac{(1 + \delta)}{b \ell^2} m_0(b, R) + C\left(\delta + \delta^{-1} \ell^4 \kappa^2 + \frac{1}{\ell \kappa \sqrt{\epsilon}} + \delta^{-1} \ell^{2\alpha}\right) \kappa^2. \tag{6.21}$$

The inequality in (2.7) tell us that $m_0(b,R) \leqslant R^2 g(b) + CR$ for all $b \in [0,1]$ and R sufficiently large. We substitute this into (6.21) and we select $\delta = \ell$, so that

$$r_0(\kappa) = \kappa^2 \mathcal{O}\big((\ell\kappa\sqrt{\epsilon})^{-1} + \ell^3\kappa^2 + \ell^{2\alpha-1}\big).$$

Using (4.12) we get

$$\begin{aligned}\frac{1}{|Q_\ell(x_0)|}\mathcal{E}\big(\psi,\mathbf{A},Q_\ell(x_0)\big) &\leqslant \frac{(1+\delta)R^2}{b\ell^2}g(b) + \frac{CR}{b\ell^2} + \kappa^2\mathcal{O}\big((\ell\kappa\sqrt{\epsilon})^{-1} + \ell^3\kappa^2 + \ell^{2\alpha-1}\big)\\ &\leqslant g\left(\frac{H}{\kappa}\underline{B}_{Q_\ell(x_0)}\right)\kappa^2 + C\big((\ell\kappa\sqrt{\epsilon})^{-1} + \ell^3\kappa^2 + \ell^{2\alpha-1}\big)\kappa^2.\end{aligned}$$

This establishes the result of Proposition 6.2. □

6.2. *Proof of Theorem 6.1, upper bound*

The parameters ℓ and ϵ have the same form as in (5.1) and we take the same choice of β and μ as in (5.11). Consider the lattice $\Gamma_\ell := \ell\mathbb{Z} \times \ell\mathbb{Z}$ and write, for $\gamma \in \Gamma_\ell$, $Q_{\gamma,\ell} = Q_\ell(\gamma)$. For any $\gamma \in \Gamma_\ell$ such that $\overline{Q_\ell(\gamma)} \subset \Omega \cap \{|B_0| > \epsilon\}$, let

$$\mathcal{I}_{\ell,\epsilon}(D) = \big\{\gamma\colon \overline{Q_{\gamma,\ell}} \subset D \cap \big\{|B_0| > \epsilon\big\}\big\}, \qquad N = \operatorname{Card}\mathcal{I}_{\ell,\epsilon}(D),$$

and

$$D_{\ell,\epsilon} = \operatorname{int}\bigg(\bigcup_{\gamma\in\mathcal{I}_{\ell,\epsilon}(D)} \overline{Q_{\gamma,\ell}}\bigg).$$

Notice that, by (1.2),

$$N = |D|\ell^{-2} + \mathcal{O}\big(\epsilon\ell^{-2}\big) + \mathcal{O}\big(\ell^{-1}\big) \quad \text{as } \ell \to 0 \text{ and } \epsilon \to 0.$$

If $(\psi,\mathbf{A})$ is a minimizer of (1.1), we have

$$\mathcal{E}(\psi,\mathbf{A};D) = \mathcal{E}_0(\psi,\mathbf{A};D_{\ell,\epsilon}) + \mathcal{E}_0(\psi,\mathbf{A};D\setminus D_{\ell,\epsilon}) + (\kappa H)^2\int_\Omega \big|\operatorname{curl}(\mathbf{A}-\mathbf{F})\big|^2\,dx. \tag{6.22}$$

Using Corollary 1.2, we may write

$$\mathcal{E}(\psi,\mathbf{A};D) \leqslant \mathcal{E}_0(\psi,\mathbf{A};D_{\ell,\epsilon}) + \mathcal{E}_0(\psi,\mathbf{A};D\setminus D_{\ell,\epsilon}) + C\kappa^{\tau_0}. \tag{6.23}$$

Here $\tau_0 \in (1,2)$. Notice that

$$|D\setminus D_{\ell,\epsilon}| = \mathcal{O}\big(\ell|\partial D_{\ell,\epsilon}| + \epsilon\big). \tag{6.24}$$

We get by using (3.1) and (3.11):

$$\begin{aligned}\big|\mathcal{E}_0(\psi,\mathbf{A};D\setminus D_{\ell,\epsilon})\big| &\leqslant |D\setminus D_{\ell,\epsilon}|\left(C_1\kappa^2\|\psi\|^2_{L^\infty(D)} + \kappa^2\|\psi\|^2_{L^\infty(D)} + \frac{\kappa^2}{2}\|\psi\|^4_{L^\infty(D)}\right)\\ &\leqslant C_2(\ell+\epsilon)\kappa^2.\end{aligned} \tag{6.25}$$

To estimate $\mathcal{E}_0(\psi,\mathbf{A};D_{\ell,\epsilon})$, we notice that

$$\mathcal{E}_0(\psi,\mathbf{A};D_{\ell,\epsilon}) = \sum_{\gamma\in\mathcal{I}_{\ell,\epsilon}(D)} \mathcal{E}_0(\psi,\mathbf{A};Q_{\gamma,\ell}).$$

Using Proposition 6.2 and the estimates in (6.25) with $\beta = \frac{3}{4}$, $\alpha = \frac{2}{3}$ and $\mu = \frac{1}{8}$, we get

$$\begin{aligned}\mathcal{E}_0(\psi,\mathbf{A};D) &\leqslant \sum_{\gamma\in\mathcal{I}_{\ell,\epsilon}(D)} g\left(\frac{H}{\kappa}\underline{B}_{Q_\ell(x_0)}\right)\kappa^2\ell^2 + C\big(\ell^3\kappa^2 + \ell^{2\alpha-1} + (\ell\kappa\sqrt{\epsilon})^{-1} + \epsilon\big)\kappa^2 + C_1\kappa^{\tau_0}\\ &\leqslant \kappa^2\sum_{\gamma\in\mathcal{I}_{\ell,\epsilon}(D)} g\left(\frac{H}{\kappa}\underline{B}_{Q_\ell(x_0)}\right)\ell^2 + C_2\kappa^{\tau_0},\end{aligned}$$

where

$$\underline{B}_{Q_\ell(x_0)} = \sup_{x \in Q_\ell(x_0)} B_0(x).$$

Recognizing the lower Riemann sum of $x \mapsto g(\frac{H}{\kappa} B_0(x))$, and using the monotonicity of g we get

$$\mathcal{E}_0(\psi, \mathbf{A}; D) \leqslant \kappa^2 \int_{D_{\ell,\epsilon}} g\left(\frac{H}{\kappa} B_0(x)\right) dx + C_2 \kappa^{\tau_0}. \tag{6.26}$$

Thus, we get by using (6.24) and the property of g in Theorem 2.2,

$$\kappa^2 \int_{D_{\ell,\epsilon}} g\left(\frac{H}{\kappa} B_0(x)\right) dx \leqslant \kappa^2 \int_D g\left(\frac{H}{\kappa} B_0(x)\right) dx + C_3 \kappa^{\tau_0}.$$

This finishes the proof of the upper bound.

6.3. *Lower bound*

We keep the same notation as in the derivation of the upper bound. We start with (6.22) and write

$$\mathcal{E}(\psi, \mathbf{A}; D) \geqslant \mathcal{E}_0(\psi, \mathbf{A}; D_{\ell,\epsilon}) + \mathcal{E}_0(\psi, \mathbf{A}; D \setminus D_{\ell,\epsilon}). \tag{6.27}$$

Similarly, as we did for the Lower bound Section 5.2, we get

$$\mathcal{E}(\psi, \mathbf{A}; D) \geqslant \kappa^2 \int_D g\left(\frac{H}{\kappa} B_0(x)\right) dx - C \kappa^{\tau_0}. \tag{6.28}$$

This finishes the proof of Theorem 6.1.

7. Proof of Theorem 1.4

7.1. *Proof of (1.9)*

Let $(\psi, \mathbf{A})$ be a solution of (1.5) and $\tau_1 = \tau_0 - 2$. Then ψ satisfies

$$-(\nabla - i\kappa H\mathbf{A})^2 \psi = \kappa^2 \big(1 - |\psi|^2\big) \psi \quad \text{in } \Omega. \tag{7.1}$$

We multiply both sides of the equation in (7.1) by $\overline{\psi}$ then we integrate over D. An integration by parts gives us

$$\int_D \big(\big|(\nabla - i\kappa H\mathbf{A})\psi\big|^2 - \kappa^2 |\psi|^2 + \kappa^2 |\psi|^4\big) dx - \int_{\partial D} \nu \cdot (\nabla - i\kappa H\mathbf{A})\psi \overline{\psi}\, d\sigma(x) = 0. \tag{7.2}$$

Using the estimates (3.1), (3.10) and (3.11), we get that the boundary term which is not necessary 0 if $D \neq \Omega$ above is $\mathcal{O}(\kappa)$. So, we rewrite (7.2) as follows,

$$-\frac{1}{2}\kappa^2 \int_D |\psi|^4 dx = \mathcal{E}_0(\psi, \mathbf{A}; D) + \mathcal{O}(\kappa). \tag{7.3}$$

Using (6.28), we conclude that

$$\frac{1}{2} \int_D |\psi|^4 dx \leqslant -\int_D g\left(\frac{H}{\kappa} B_0(x)\right) dx + C \kappa^{\tau_1}. \tag{7.4}$$

7.2. Proof of (1.10)

If $(\psi, \mathbf{A})$ is a minimizer of (1.1), then (7.3) is still true. We apply in this case Theorem 6.1 to write an upper bound of $\mathcal{E}_0(\psi, \mathbf{A}; D)$. Consequently, we deduce that

$$\frac{1}{2}\int_D |\psi|^4\,dx \geqslant -\int_D g\left(\frac{H}{\kappa}B_0(x)\right)dx - C\kappa^{\tau_1}. \tag{7.5}$$

Combining the upper bound in (7.5) with the lower bound in (7.4) finishes the proof of Theorem 1.4.

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Appendix A

A.1. L^p-regularity for the curl–div system

We consider the two dimensional case. We denote, for $k \in \mathbb{N}$, by $W^{k,p}_{\mathrm{div}}(\Omega)$ the space

$$W^{k,p}_{\mathrm{div}}(\Omega) = \left\{\mathbf{A} \in W^{k,p}(\Omega),\ \operatorname{div}\mathbf{A} = 0 \text{ and } \mathbf{A}\cdot\nu = 0 \text{ on } \partial\Omega\right\}.$$

Then we have the following L^p regularity for the curl–div system.

Proposition A.1. *Let $1 \leqslant p < \infty$. If $\mathbf{A} \in W^{1,p}_{\mathrm{div}}(\Omega)$ satisfies $\operatorname{curl}\mathbf{A} \in W^{k,p}(\Omega)$, for some $k \geqslant 0$, then $\mathbf{A} \in W^{k+1,p}_{\mathrm{div}}(\Omega)$.*

Proof. If $\mathbf{A}$ belongs to $W^{1,p}_{\mathrm{div}}(\Omega)$ and $\operatorname{curl}\mathbf{A} \in L^p(\Omega)$, then there exists $\psi \in W^{2,p}(\Omega)$ such that $\mathbf{A} = (-\partial_{x_2}\psi, \partial_{x_1}\psi)$, $-\Delta\psi = \operatorname{curl}\mathbf{A}$, with $\psi = 0$ on $\partial\Omega$. This is simply the Dirichlet L^p problem for the Laplacian (see [2, Section A.1]). The result we need for proving the proposition is then that if $-\Delta\psi$ is in addition in $W^{k,p}(\Omega)$ then $\psi \in W^{k+2,p}(\Omega)$. This is simply an L^p regularity result for the Dirichlet problem for the Laplacian which is described in [2, Section F.4]. □

A.2. Construction of φ_{x_0}

Lemma A.2. *If $B_0 \in L^2(\Omega)$, then there exists a unique $\mathbf{F} \in H^1_{\mathrm{div}}(\Omega)$ such that*

$$\operatorname{curl}\mathbf{F} = B_0. \tag{A.1}$$

Proof. The proof is standard, see [5]. Let $\mathbf{F} = \begin{bmatrix} \partial_{x_2} f \\ -\partial_{x_1} f \end{bmatrix}$, where $f \in H^2(\Omega) \cap H^1_0(\Omega)$ is the unique solution of

$$-\Delta f = B_0 \quad \text{in } \Omega. \tag{A.2}$$

Then we deduce from the Dirichlet condition satisfied by f that $\tau\cdot\nabla f = 0$ on $\partial\Omega$ which is equivalent to $\nu\cdot\mathbf{F} = 0$ on $\partial\Omega$. This finishes the proof of Lemma A.2. □

We continue with a lemma that will be useful in estimating the Ginzburg–Landau functional.

Lemma A.3. *There exists a positive constant C such that, if $\ell \in (0,1)$ and $x_0 \in \Omega$ are such that $\overline{Q_\ell(x_0)} \subset \Omega$, then for any $\widetilde{x_0} \in \overline{Q_\ell(x_0)}$, there exists a function $\varphi_0 \in H^1(\Omega)$ such that the magnetic potential $\mathbf{F}$ satisfies*

$$\left|\mathbf{F}(x) - \nabla\varphi_0(x) - B_0(\widetilde{x_0})\mathbf{A}_0(x - x_0)\right| \leqslant C\ell^2 \quad \bigl(x \in Q_\ell(x_0)\bigr), \tag{A.3}$$

where B_0 is the function introduced in (1.2) and $\mathbf{A}_0$ is the magnetic potential introduced in (2.2).

Proof. We use Taylor formula near $\widetilde{x_0}$ to order 2 and get

$$\mathbf{F}(x) = \mathbf{F}(\widetilde{x_0}) + M(x - \widetilde{x_0}) + \mathcal{O}\big(|x - \widetilde{x_0}|^2\big), \quad \forall x \in Q_\ell(x_0), \tag{A.4}$$

where

$$M = D\mathbf{F}(\widetilde{x_0}) = \begin{bmatrix} \frac{\partial \mathbf{F}^1}{\partial x_1}\big|_{\widetilde{x_0}} & \frac{\partial \mathbf{F}^1}{\partial x_2}\big|_{\widetilde{x_0}} \\ \frac{\partial \mathbf{F}^2}{\partial x_1}\big|_{\widetilde{x_0}} & \frac{\partial \mathbf{F}^2}{\partial x_2}\big|_{\widetilde{x_0}} \end{bmatrix}.$$

We can write M as the sum of two matrices, $M = M^s + M^{as}$, where $M^s = \frac{M+M^t}{2}$ is symmetric and $M^{as} = \frac{M-M^t}{2}$ is antisymmetric.

Notice that $\operatorname{curl} \mathbf{F}(\widetilde{x_0}) = \frac{\partial \mathbf{F}^2}{\partial x_1}|_{\widetilde{x_0}} - \frac{\partial \mathbf{F}^1}{\partial x_2}|_{\widetilde{x_0}} = B_0(\widetilde{x_0})$. Consequently,

$$M^{as} = \begin{bmatrix} 0 & -B_0/2 \\ B_0/2 & 0 \end{bmatrix}.$$

Substitution into M gives as that

$$M(x - x_0) = \nabla \phi_0(x) + B_0(\widetilde{x_0})\mathbf{A}_0(x - x_0),$$

where $\mathbf{A}_0(x) = \frac{1}{2}(-x_2, x_1)$ and the function ϕ_0 is defined by

$$\phi_0(x) = \frac{1}{2}\bigg\langle \bigg(\frac{M + M^t}{2}\bigg)(x - x_0), (x - x_0)\bigg\rangle.$$

Let $\varphi_0(x) = \phi_0(x) + (\mathbf{F}(\widetilde{x_0}) + M(x_0 - \widetilde{x_0})) \cdot x$. Substitution into (A.4) gives as that

$$\mathbf{F} = B_0(\widetilde{x_0})\mathbf{A}_0(x - x_0) + \nabla \varphi_0(x) + \mathcal{O}\big(|x - \widetilde{x_0}|^2\big).$$

Notice that, if $x \in Q_\ell(x_0)$, then $|x - \widetilde{x_0}| \leqslant \ell\sqrt{2}$. This finishes the proof of Lemma A.3. □

Remark A.4. We will apply this lemma by considering $\widetilde{x_0}$ such that $B_0(\widetilde{x_0}) = \sup_{Q_\ell(x_0)} B_0(x)$ or $B_0(\widetilde{x_0}) = \inf_{Q_\ell(x_0)} B_0(x)$.

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