Polynomial propagation of moments and global existence for a Vlasov–Poisson system with a point charge

Laurent Desvillettes [a], Evelyne Miot [b,*], Chiara Saffirio [c]

[a] *CMLA, ENS Cachan & CNRS, 61 Av. du Pdt. Wilson, 94235 Cachan Cedex, France*
[b] *Université Paris-Sud, Département de mathématiques, Bâtiment 425, 91405 Orsay, France*
[c] *Institute for Applied Mathematics, University of Bonn, 53115 Bonn, Germany*

Abstract

In this paper, we extend to the case of initial data constituted of a Dirac mass plus a bounded density (with finite moments) the theory of Lions and Perthame [8] for the Vlasov–Poisson equation. Our techniques also provide polynomially growing in time estimates for moments of the bounded density.

© 2014 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

1. Introduction

The aim of this article is to propose new estimates for moments of the three-dimensional Vlasov–Poisson system, which enable us to improve the existing theory in the following directions: first, the possibility of considering initial data which are the sum of a Dirac mass and a continuous density which does not vanish in the vicinity of the Dirac mass; second, the opportunity of bounding polynomially (in an explicit computable way) with respect to time the high order moments of the distribution (when the initial datum is not assumed to be compactly supported). The three-dimensional Vlasov–Poisson system for initial data containing one Dirac mass writes

$$
\begin{cases}
\partial_t f + v \cdot \nabla_x f + (E + F) \cdot \nabla_v f = 0, \\
E(t, x) = \int\limits_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \rho(t, y)\, dy, \\
\rho(t, x) = \int\limits_{\mathbb{R}^3} f(t, x, v)\, dv, \\
F(t, x) = \frac{x - \xi(t)}{|x - \xi(t)|^3}.
\end{cases}
\tag{1}
$$

* Corresponding author.
  E-mail addresses: desville@cmla.ens-cachan.fr (L. Desvillettes), Evelyne.Miot@math.u-psud.fr (E. Miot), chiara.saffirio@hausdorff-center.uni-bonn.de (C. Saffirio).

Here $f := f(t, x, v) \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ corresponds to a nonnegative density of charged particles in a plasma, subjected to a self-induced electric force field $E := E(t, x)$. The plasma interacts with a point charge, located at $\xi(t)$ with velocity $\eta(t)$, which induces the singular electric field $F := F(t, x)$. The evolution of the charge is itself given by

$$\begin{cases} \dot{\xi}(t) = \eta(t), \\ \dot{\eta}(t) = E(t, \xi(t)). \end{cases} \tag{2}$$

The initial conditions associated to (1)–(2) are

$$(\xi(0), \eta(0)) = (\xi_0, \eta_0), \qquad f(0, x, v) = f_0(x, v). \tag{3}$$

When there is no point charge, then (1)–(2) reduces to (1) with $F = 0$, which is the standard Vlasov–Poisson system. As announced, (1)–(2) can also be thought of as the standard Vlasov–Poisson system for the total density $f(t) + \delta_{\xi(t)} \otimes \delta_{\eta(t)}$.

In this paper we will focus on weak solutions to (1), which we define as couples (f, ξ) such that the first equation in (1) is satisfied in the sense of distributions and such that (2) holds in the classical sense. Our main global existence and propagation of moments result of such solutions with finite moments is stated in the following

Theorem 1.1. *There exists* $0 < \lambda \leqslant 1$ *with the following property*: *If* $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *is nonnegative,* $(\xi_0, \eta_0) \in \mathbb{R}^3 \times \mathbb{R}^3$, *and*

(i) $\mathcal{M}_0 = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x, v) \, dx \, dv < \lambda$;
(ii) *There exists* $m_0 > 6$ *such that for all* $m < m_0$

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(|v|^2 + \frac{1}{|x - \xi_0|} \right)^{m/2} f_0(x, v) \, dx \, dv < +\infty;$$

Then there exists a global weak solution (f, ξ) *to the system* (1)–(3), *with* $f \in C(\mathbb{R}_+, L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+, L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ *for any* $1 \leqslant p < +\infty$, $E \in L^\infty([0, T], L^\infty(\mathbb{R}^3))$ *for all* $T > 0$ *and* $\xi \in C^2(\mathbb{R}_+)$.

Moreover, for all $t \in \mathbb{R}_+$ *and for all* $m < \min(m_0, 7)$,

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(|v|^2 + \frac{1}{|x - \xi(t)|} \right)^{m/2} f(t, x, v) \, dx \, dv \leqslant C(1 + t)^c,$$

where C *and* c *only depend on* $(f_0, (\xi_0, \eta_0))$ *and can be estimated explicitly* (*cf. Remark* 5).

Remark 1. Since the density f constructed in Theorem 1.1 has bounded moments of order higher than 6 and belongs to $C(\mathbb{R}_+, L^p(\mathbb{R}^3 \times \mathbb{R}^3))$, the field E has some additional regularity: it belongs to $C(\mathbb{R}_+, C^{0,\alpha}(\mathbb{R}^3))$ for some $0 < \alpha < 1$ (see Corollary 2 in [8]). In particular the ODE (2) holds in the classical sense.

We stress that if we modified the assumption $m_0 > 6$ in Theorem 1.1 (for example if $3 < m_0 < 6$), then the electric field E might be not continuous and we would not be able to prove the existence of the trajectory $t \mapsto (\xi(t), \eta(t))$ for all initial datum (ξ_0, η_0).

Note however that if we considered the standard Vlasov–Poisson system (i.e. system (1) with $F = 0$), we would not need $m_0 > 6$ (the assumption $m_0 > 3$ would be enough). We would also not need assumption (i), so that we would be left with the assumptions of [8]. In that case, we would still get the explicit, polynomial with respect to time estimate of the moments of order $m < \min(m_0, 7)$, which provides an improvement of the result in [8], where the growth with respect to time of the moments is of the form $\exp(\exp(t))$.

Note finally that if we fix the point charge $\xi = \xi_0$, which corresponds to the evolution of a plasma density under the influence of a stationary exterior singular field:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E + F_*) \cdot \nabla_v f = 0, \\ E(t, x) = \displaystyle\int_{\mathbb{R}^3} \frac{x - y}{|x - y|^3} \rho(t, y) \, dy, \\ \rho(t, x) = \displaystyle\int_{\mathbb{R}^3} f(t, x, v) \, dv, \\ F_*(t, x) = C \dfrac{x - \xi_0}{|x - \xi_0|^3}, \end{cases} \tag{4}$$

then in this situation also, conditions (i) and (ii) of Theorem 1.1 are not needed and we can obtain the following

Proposition 1.2. *Let $f_0 \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ be nonnegative, let $\xi_0 \in \mathbb{R}^3$ such that there exists $m_0 > 3$ for which for all $m < m_0$*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(|v|^2 + \frac{C}{|x - \xi_0|} \right)^{m/2} f_0(x, v) \, dx \, dv < +\infty.$$

Then there exists a global weak solution (f, ξ) to the system (4) (here C can be $C = 0$, that is the standard Vlasov–Poisson system, or $C > 0$), with $f \in C(\mathbb{R}_+, L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+, L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ and $E \in C(\mathbb{R}_+, L^q(\mathbb{R}^3))$, for all $2 \leqslant q < 3(3 + m_0)/(6 - m_0)$ if $m_0 < 6$, and $2 \leqslant q \leqslant +\infty$ if $m_0 > 6$.

Moreover, for all $t \in \mathbb{R}_+$ and for all $m < \min(m_0, 7)$,

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(|v|^2 + \frac{1}{|x - \xi(t)|} \right)^{m/2} f(t, x, v) \, dx \, dv \leqslant C(1 + t)^c,$$

where C and c only depend on (f_0, ξ_0) and can be estimated explicitly.

The Cauchy problem for the Vlasov–Poisson system (1), with or without point charge, has been the object of several works in the last decades. For the pure Vlasov–Poisson system without charge, namely $F = 0$, global existence and uniqueness of classical solutions where obtained by Ukai and Okabe [10] in two dimensions. The three-dimensional case is more delicate; global weak solutions with finite energy were first built by Arsenev [1].

Global existence and, in some cases, uniqueness, of more regular solutions were then separately established by Lions and Perthame [8] and by Pfaffelmoser [12] by different techniques. In both works the main issue consists in controlling the large plasma velocities for all time in order to propagate regularity properties of the solution.

In [8], this is achieved by constructing weak solutions with finite velocity moments of order higher than three

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^m f(t, x, v) \, dx \, dv < \infty, \quad m > 3,$$

which, by Sobolev embeddings, implies further bounds on the spatial density and on the electric field. In particular, if the solution admits finite moments of order $m > 6$ then the electric field is uniformly bounded and uniqueness holds under some additional regularity assumptions on the initial density. On the other hand, the theory of DiPerna and Lions [6] ensures that such solutions are constant along the trajectories of a "generalized flow" defined in a weak sense.

In contrast with the Eulerian approach of [8], the strategy of [12] relies on a careful analysis of the characteristics to control the growth of the velocity support and thereby obtain global existence and uniqueness of classical compactly supported solutions, which moreover propagate the regularity of the initial density.

We refer to the further improvements and developments by Schaeffer [15], Wollman [16], Castella [4], Loeper [9], Chen and Zhang [5]. Moreover, Gasser, Jabin and Perthame [7] established propagation of the velocity moments for $m > 2$ with an additional assumption on the space moments, and in [14], Salort proved existence and uniqueness of weak solutions even if $m < 6$. Finally, Pallard [11] recently combined Eulerian and Lagrangian points of view to establish existence of solutions propagating velocity moments for $m > 2$ and obtained an explicit polynomial in time bound on the moments.

The study of the modified Vlasov–Poisson system with macroscopic point charges was initiated more recently by Caprino and Marchioro [2]. In two dimensions, they proved global existence and uniqueness of solutions à la

Pfaffelmoser. This was then extended to the three-dimensional case by Marchioro, Miot and Pulvirenti [13]. The results of [2] and [13] hold for initial plasma densities that do not overlap the charge. Thanks to the repulsive nature of the plasma-charge interaction, this property remains true at later times so that the field induced by the charge is bounded on the support of the density and the velocities of the plasma particles do not blow up. The analysis of [2] and [13] exploits the notion of microscopic energy, defined in this context by

$$h(t, x, v) = \frac{|v - \eta(t)|^2}{2} + \frac{1}{|x - \xi(t)|}.$$

It turns out that the variation of the energy along the plasma characteristics is controlled by the electric field (see Lemma 2.7), exactly as for the velocity in the absence of charge. On the other hand, the energy controls both the velocity and the distance to the charge. This makes it possible to adapt Pfaffelmoser's arguments by replacing the largest velocity of the plasma particles by the largest energy $\sup_{\mathrm{supp}(f(t))} h$, which by assumption is initially finite. Note that when the plasma density overlaps the charge this last quantity is not finite, so that the method fails.

In order to treat densities with unbounded energy, which is the purpose of the present paper, we adapt the PDE point of view from [8] and we show existence of a solution propagating the energy moments (see Definition 2.6 hereafter). In particular, since the energy moments control the velocity moments, we recover all additional regularity properties on the electric field which have been established in [8]. The new estimates that we provide are somehow based on an iteration of some of the arguments used in [8], which leads to the improvement of both the assumptions made on the initial datum (that is the possibility of introducing a Dirac mass) and the dependence with respect to large time of the moments estimates (which become polynomial and explicit). Moreover, we also tried to provide fully detailed proofs for the most technical part of [8], for the sake of readability of our paper.

We emphasize the fact that Theorem 1.1 allows for initial densities that do not necessarily vanish in a neighborhood of the charge. They nevertheless have to decay close to it somehow; unfortunately it does not include the "generic" densities that are constant close to the charge.

On the other hand, our techniques do not enable to obtain uniqueness because of the singularity of F in the neighborhood of the charge. Finally, we believe that the limitation $m < 7$ (appearing in the proof of Proposition 2.15) is purely technical. We also hope to extend Theorem 1.1 to the case of several point charges being all positively charged, as is the case in [2] and [13].

Among the most interesting open questions connected to this paper, we would like to mention: i) the issues related to the gravitational case (which are the most interesting from the modeling point of view), where $|v|^2 + \frac{1}{|x-\xi|}$ is replaced by $|v|^2 - \frac{1}{|x-\xi|}$, which makes much more difficult the use of this quantity (see [3] for the two-dimensional case); ii) the study of the Cauchy problem associated to system (1) with general initial densities around the charge, for instance constant (note that this situation appears as a limit critical case of the present analysis); iii) the improvement of large time estimates for moments, indeed in our work the polynomials appearing in the estimates have a very high degree (typically from 100 to 1000).

The remainder of this paper is organized as follows. Next section is devoted to the proof of Theorem 1.1. The general idea, which follows the lines of [8], consists in deriving a priori estimates for the moments for a sequence of smooth solutions to (1)–(3) obtained by regularizing the initial density in order to obtain a global solution by compactness arguments. Concerning Proposition 1.2 we omit the proof, since it can be obtained by mimicking the steps of the proof of Theorem 1.1.

In Section 2.2 we gather some basic facts and a priori estimates for the modified Vlasov–Poisson system (1). We also derive some first estimates for the energy moments. In Section 2.3 we introduce the flow associated to a relevant cutoff of the field $E + F$, which enables to express the solution of (1) by means of Duhamel's formula with a suitable source term. Then, in Section 2.4 we establish intermediate a priori estimates for the moments, which as a byproduct ensure that the moments are uniformly bounded for small times. These estimates are exploited to show that the moments are uniformly bounded for all times in Section 2.5. They eventually provide a global solution satisfying the assumptions of Theorem 1.1, as explained in Section 2.6. Appendix A is also devoted to the proof of technical estimates on the flow.

2. Proof of Theorem 1.1

Notations. Throughout the paper, $|z|$ denotes the usual (Euclidean) norm of $z \in \mathbb{R}^3$ and $|A|$ is the operator norm of the matrix $A \in \mathcal{M}_3(\mathbb{R})$. When $F : [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ (or $\mathcal{M}_3(\mathbb{R})$) and $1 \leqslant p, q \leqslant +\infty$, we set $\|F\|_{L^p([0,T], L^q(\mathbb{R}^3 \times \mathbb{R}^3))} = \|\|F\|\|_{L^p([0,T], L^q(\mathbb{R}^3 \times \mathbb{R}^3))}$.

The notation C will refer to a constant depending only on the quantities $\mathcal{H}(0)$, $\mathcal{M}(0)$, $\|f_0\|_\infty$, $|\xi_0|$, $|\eta_0|$, m_0, and $H_m(0)$, for $m < m_0$ (those moments are defined in Section 2.2, cf. formula (15)), but not on T (when a constant depends on another parameter, we state it explicitly). Under the assumptions of Theorem 1.1, all these quantities are finite (in the approximation process, they will be bounded with respect to the regularization parameter).

2.1. Interpolation estimates

We first recall a collection of well-known interpolation and Sobolev (or Young's) inequalities that we shall apply later to the solutions of (1)–(3). All of them may be found in [8].

Proposition 2.1. *Let $f := f(x, v) \geqslant 0$. Let $b > a \geqslant 0$. Then for all $x \in \mathbb{R}^3$,*

$$\int_{\mathbb{R}^3} |v|^a f(x, v) \, dv \leqslant C \|f\|_{L^\infty}^{\frac{b-a}{3+b}} \left(\int_{\mathbb{R}^3} |v|^b f(x, v) \, dv \right)^{\frac{3+a}{3+b}}, \tag{5}$$

with C depending only on a, b. In particular, setting $\rho(x) = \int_{\mathbb{R}^3} f(x, v) \, dv$, we have for any $b > 0$,

$$\|\rho\|_{L^{\frac{b+3}{3}}} \leqslant C \|f\|_{L^\infty}^{\frac{b}{3+b}} \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^b f(x, v) \, dx \, dv \right)^{\frac{3}{3+b}}. \tag{6}$$

Proof. For all $R \geqslant 0$

$$\int_{\mathbb{R}^3} |v|^a f(x, v) \, dv \leqslant R^{a-b} \int_{\mathbb{R}^3} |v|^b f(x, v) \, dv + C R^{3+a} \|f\|_\infty, \tag{7}$$

and estimate (5) is obtained by optimizing $R > 0$ (cf. also the proof of estimate (14) in [8]). Setting $a = 0$ and taking the $L^{\frac{b+3}{3}}$ norm in the x-variable, we obtain (6). $\quad\square$

Proposition 2.2. *Let $f \geqslant 0$ be in $L^1(\mathbb{R}^3 \times \mathbb{R}^3)$, assume that $\rho(x) = \int_{\mathbb{R}^3} f(x, v) \, dv \in L^s(\mathbb{R}^3)$ for some $1 \leqslant s \leqslant +\infty$, and consider $E = \rho * (x \mapsto x/|x|^3)$. Then for $1 < s < 3$,*

$$\|E\|_{L^{\frac{3s}{3-s}}} \leqslant C \|\rho\|_{L^s}, \tag{8}$$

and for $s > 3$,

$$\|E\|_{L^\infty} \leqslant C \|\rho\|_{L^s}, \tag{9}$$

where C only depends on s.

Proof. The inequalities are direct consequences of Sobolev inequalities and the fact that $E = 4\pi \nabla_x \Delta_x^{-1} \rho$. $\quad\square$

2.2. First dynamical estimates

We now turn to the study of system (1)–(3). In the remainder of this article, we fix $T > 0$. We will call *classical solution* on $[0, T]$ any solution (f, ξ) of (1)–(3) on $[0, T]$, with initial condition $(f_0, (\xi_0, \eta_0))$ satisfying the assumptions in Theorem 1.1, such that *moreover* f_0 is C^1, compactly supported, and vanishes in a neighborhood of ξ_0, which satisfies $f \in C_c^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ and $\xi \in C^2([0, T])$. Moreover,

$$f\big(t, \boldsymbol{x}(t, x, v), \boldsymbol{v}(t, x, v)\big) = f_0(x, v), \quad t \in \mathbb{R}_+,$$

where for all $t \in \mathbb{R}_+$, the map $(x, v) \in \mathbb{R}^3 \setminus \{\xi_0\} \times \mathbb{R}^3 \mapsto (\boldsymbol{x}(t, x, v), \boldsymbol{v}(t, x, v)) \in \mathbb{R}^3 \setminus \{\xi(t)\} \times \mathbb{R}^3$ is invertible and:

(i) For all $(x, v) \in \mathbb{R}^3 \setminus \{\xi_0\} \times \mathbb{R}^3$, $t \mapsto (\boldsymbol{x}(t, x, v), \boldsymbol{v}(t, x, v)) \in C^1(\mathbb{R}_+)$ is the solution of

$$\begin{cases} \dfrac{d}{dt} \boldsymbol{x}(t, x, v) = \boldsymbol{v}(t, x, v), \\ \dfrac{d}{dt} \boldsymbol{v}(t, x, v) = E\big(t, \boldsymbol{x}(t, x, v)\big) + \dfrac{\boldsymbol{x}(t, x, v) - \xi(t)}{|\boldsymbol{x}(t, x, v) - \xi(t)|^3}, \quad (\boldsymbol{x}, \boldsymbol{v})(0, x, v) = (x, v). \end{cases} \tag{10}$$

In particular, we have for all $x \neq \xi_0$ and $v \in \mathbb{R}^3$,

$$\big|\boldsymbol{x}(t, x, v) - \xi(t)\big| > 0, \quad \forall t \in \mathbb{R}_+.$$

(ii) For all $t \in \mathbb{R}_+$, the map $(x, v) \mapsto (\boldsymbol{x}(t, x, v), \boldsymbol{v}(t, x, v))$ preserves the Lebesgue's measure on $\mathbb{R}^3 \times \mathbb{R}^3$.

In the following we will sometimes write $(\boldsymbol{x}(t), \boldsymbol{v}(t)) = (\boldsymbol{x}(t, x, v), \boldsymbol{v}(t, x, v))$.

The existence (and uniqueness) of classical solutions corresponding to such initial data is ensured by [13]. Our purpose is to establish relevant a priori estimates for (f, ξ) on $[0, T]$, which will eventually lead to the existence of a solution to (1)–(3) by compactness. Such a priori estimates will concern the moments of order $m < m_0$, which are defined in Definition 2.6 below.

We start with a few basic properties of the Vlasov–Poisson system.

Proposition 2.3. *Let (f, ξ) be a classical solution of (1)–(3) on $[0, T]$.*

Then, the norms

$$\big\| f(t) \big\|_{L^p(\mathbb{R}^3 \times \mathbb{R}^3)}, \quad 1 \leqslant p \leqslant \infty,$$

and the energy

$$\mathcal{H}(t) = \frac{1}{2} \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f(t, x, v) \, dv \, dx + \frac{1}{2} \big|\eta(t)\big|^2 + \frac{1}{2} \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho(t, x) \rho(t, y)}{|x - y|} \, dx \, dy + \int\limits_{\mathbb{R}^3} \frac{\rho(t, x)}{|x - \xi(t)|} \, dx$$

are conserved in time. In particular, the mass

$$\mathcal{M}(t) = \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v) \, dx \, dv$$

is conserved in time.

Proof. The conservation of the L^p norms is an immediate consequence of the fact that f is constant along the trajectories of a Lebesgue's measure preserving flow.

We only detail the computation for the energy:

$$\begin{aligned} \frac{d}{dt} &\bigg\{ \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f \frac{|v|^2}{2} \, dx \, dv + \frac{1}{2} |\eta|^2 + \frac{1}{2} \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\rho(x) \rho(y)}{|x - y|} \, dx \, dy + \int\limits_{\mathbb{R}^3} \frac{\rho(x)}{|x - \xi|} \, dx \bigg\} \\ &= \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} v \cdot (E + F) f \, dx \, dv + \eta \cdot E(\xi) - \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\nabla_x \cdot (\int_{\mathbb{R}^3} v f \, dv)}{|x - y|} \rho(y) \, dx \, dy \\ &\quad - \int\limits_{\mathbb{R}^3} \frac{\nabla_x \cdot \int v f \, dv}{|x - \xi|} \, dx - \int\limits_{\mathbb{R}^3} \rho(x) \eta \cdot \frac{\xi - x}{|\xi - x|^3} \, dx = 0. \quad \square \end{aligned}$$

For the initial data $(f_0, (\xi_0, \eta_0))$ considered in the setting of Theorem 1.1, the energy is initially finite; indeed Proposition 2.1 yields $\rho_0 \in L^1 \cap L^{5/3}$, so that $\iint \rho(x) \rho(y) / |x - y| \, dx \, dy$ is finite by Hölder estimates; on the other hand the other terms are clearly finite by assumption (ii). So we immediately get the

Proposition 2.4. *Let* (f, ξ) *be a classical solution of* (1)–(3) *on* $[0, T]$. *We have*

$$\sup_{t \in [0,T]} |\eta(t)| \leqslant \sqrt{2\mathcal{H}(0)} \tag{11}$$

and

$$\sup_{t \in [0,T]} |\xi(t)| \leqslant |\xi_0| + \sqrt{2\mathcal{H}(0)}\,T. \tag{12}$$

Proof. The first inequality is a consequence of the conservation of the energy. The second one comes out of the integration w.r.t. time of the first one. $\quad\square$

Another well-known consequence of the conservation of the energy is the following

Proposition 2.5. *Let* (f, ξ) *be a classical solution of* (1)–(3) *on* $[0, T]$. *We have*

$$\sup_{t \in [0,T]} \|\rho(t)\|_{L^{5/3}} \leqslant C,$$

and for all $\frac{3}{2} < r \leqslant \frac{15}{4}$,

$$\sup_{t \in [0,T]} \|E(t)\|_{L^r} \leqslant C,$$

with C *a constant depending only on* $\mathcal{H}(0)$, $\|f_0\|_\infty$ *and* r.

Proof. The first estimate is a consequence of (6) with $b = 2$ and the fact that the velocity moment of order 2 is controlled by the energy. The second estimate is deduced from the first one and (8), using the fact that $\rho \in L^\infty([0, T], L^1 \cap L^{5/3}(\mathbb{R}^3))$. $\quad\square$

We now give our definition of energy moments.

Definition 2.6. We define the energy function

$$h(t, x, v) = \frac{|v - \eta(t)|^2}{2} + \frac{1}{|x - \xi(t)|} + \mathcal{H}(0) + (\mathcal{M}_0)^{-1} + 1.$$

In view of Proposition 2.4, we have

$$|v| \leqslant 2\sqrt{h(t, x, v)}, \quad \forall (t, x, v) \in [0, T] \times \mathbb{R}^3 \times \mathbb{R}^3.$$

Then we set for $k \in \mathbb{R}_+$

$$\tilde{H}_k(t) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} h(t, x, v)^{k/2} f(t, x, v) \, dx \, dv, \tag{13}$$

and

$$H_k(t) = \sup_{s \in [0,t]} \tilde{H}_k(s) = \sup_{s \in [0,t]} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} h(s, x, v)^{k/2} f(s, x, v) \, dx \, dv. \tag{14}$$

In particular

$$H_k(t) \geqslant 1 \quad \text{if } k/2 \geqslant 1, \quad \text{and} \quad H_k(t) \geqslant \mathcal{M}_0, \quad \forall k \geqslant 0.$$

A first basic observation is that the energy moments H_k control the velocity moments M_k defined in [8], namely

$$M_k(t) := \sup_{s \in [0,t]} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(s, x, v) \, dx \, dv \leqslant 2^k H_k(t). \tag{15}$$

Lemma 2.7. *Let* (f, ξ) *be a classical solution of* (1)–(3) *on* $[0, T]$. *For all plasma trajectory* $(\mathbf{x}(t), \mathbf{v}(t)) = (\mathbf{x}(t, x, v), \mathbf{v}(t, x, v))$ *solution of* (10)*, we have*

$$\frac{d}{dt} \sqrt{h}\big(t, \mathbf{x}(t), \mathbf{v}(t)\big) \leqslant \big|E\big(t, \mathbf{x}(t)\big)\big| + \big|E\big(t, \xi(t)\big)\big|.$$

Proof. A simple computation using (10) and (2) yields

$$\frac{d}{dt} h\big(t, \mathbf{x}(t), \mathbf{v}(t)\big) = \big(\mathbf{v}(t) - \eta(t)\big) \cdot \big(E\big(t, \mathbf{x}(t)\big) - E\big(t, \xi(t)\big)\big).$$

Note that the singular field F does not appear in the equality. □

Lemma 2.8. *Let* (f, ξ) *be a classical solution of* (1)–(3) *on* $[0, T]$. *We have for all* $t \in [0, T]$ *and for all* $k \in \mathbb{R}_+$

$$\frac{d}{dt} \tilde{H}_k(t) \leqslant C(k)\big(\|E(t)\|_{L^{k+3}} + \big|E\big(t, \xi(t)\big)\big|\big) H_k(t)^{\frac{k+2}{k+3}} \tag{16}$$

and therefore,

$$H_k(t) \leqslant C(k) \left\{ H_k(0) + \left(\int_0^t \big\{ \|E(s)\|_{L^{k+3}} + \big|E\big(s, \xi(s)\big)\big| \big\} \, ds \right)^{k+3} \right\}. \tag{17}$$

Proof. Since $f \in C_c^1([0, T] \times \mathbb{R}^3 \times \mathbb{R}^3)$ is a classical solution of (1), we may compute

$$\frac{d}{dt} \tilde{H}_k(t) = \frac{k}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} h^{k/2-1} f \big\{ \partial_t h + v \cdot \nabla_x h + (E + F) \cdot \nabla_v h \big\}(t, x, v) \, dx \, dv$$

$$= \frac{k}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} h^{k/2-1} f \big\{ (v - \eta(t)) \cdot \big(E(t, x) - E\big(t, \xi(t)\big)\big) \big\} \, dx \, dv.$$

We remark again that the choice of the energy function h enabled to get rid of the singular field in the second equality. Therefore

$$\frac{d}{dt} \tilde{H}_k(t) \leqslant C(k) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \big|E(t, x)\big| h^{(k-1)/2} f(t, x, v) \, dx \, dv$$

$$+ C(k)\big|E(t, \xi)\big| \iint_{\mathbb{R}^3 \times \mathbb{R}^3} h^{(k-1)/2} f(t, x, v) \, dx \, dv. \tag{18}$$

We assume first that $k \geqslant 1$. In order to bound the first term of the right-hand side in (18), we use interpolation arguments from [8] that we recall here for sake of clarity. First, we have thanks to Hölder's inequality

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \big|E(t, x)\big| h^{(k-1)/2} f(t, x, v) \, dx \, dv \leqslant \|E(t)\|_{L^{k+3}} \left\| \int_{\mathbb{R}^3} h^{(k-1)/2} f(t, \cdot, v) \, dv \right\|_{L^{\frac{k+3}{k+2}}}.$$

Next, we have for $x \in \mathbb{R}^3$ and for $R > 0$,

$$\int_{\mathbb{R}^3} h^{(k-1)/2} f(t, x, v) \, dv = \int_{h^{1/2} \leqslant R} h^{(k-1)/2} f(t, x, v) \, dv + \int_{h^{1/2} \geqslant R} h^{(k-1)/2} f(t, x, v) \, dv$$

$$\leqslant R^{k-1} \int_{|v| \leqslant CR} f(t, x, v) \, dv + R^{-1} \int_{h^{1/2} \geqslant R} h^{k/2} f(t, x, v) \, dv$$

$$\leqslant C \|f(t)\|_{L^\infty} R^{k+2} + R^{-1} \int_{\mathbb{R}^3} h^{k/2} f(t, x, v) \, dv.$$

We used the fact that $|v| \leqslant Ch^{1/2}$ in the second inequality. Now, optimizing with respect to R, and using the identity $\|f(t)\|_{L^\infty} = \|f_0\|_{L^\infty}$, we find

$$\int_{\mathbb{R}^3} h^{(k-1)/2} f(t,x,v)\,dv \leqslant C(k)\bigg(\int_{\mathbb{R}^3} h^{k/2} f(t,x,v)\,dv\bigg)^{(k+2)/(k+3)}.$$

So finally, integrating in x, we obtain

$$\bigg\|\int_{\mathbb{R}^3} h^{(k-1)/2} f(t,\cdot,v)\,dv\bigg\|_{L^{\frac{k+3}{k+2}}} \leqslant C(k) H_k(t)^{(k+2)/(k+3)},$$

and we are led to

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |E(t,x)| h^{(k-1)/2} f(t,x,v)\,dx\,dv \leqslant C(k) \|E(t)\|_{L^{k+3}} H_k(t)^{(k+2)/(k+3)}. \tag{19}$$

We next estimate the second term in (18). Applying again Hölder's inequality yields

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} h^{(k-1)/2} f(t,x,v)\,dx\,dv \leqslant \bigg(\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)\,dx\,dv\bigg)^{1/k} \bigg(\iint_{\mathbb{R}^3\times\mathbb{R}^3} h^{k/2} f(t,x,v)\,dx\,dv\bigg)^{(k-1)/k}$$

so that, since $\mathcal{M}(t) = \mathcal{M}_0$,

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} h^{(k-1)/2} f(t,x,v)\,dx\,dv \leqslant C(k) H_k(t)^{(k-1)/k}.$$

Since $(k-1)/k \leqslant (k+2)/(k+3)$ and $H_k(t) \geqslant \mathcal{M}_0$, it follows that

$$|E(t,\xi)| \iint_{\mathbb{R}^3\times\mathbb{R}^3} h^{(k-1)/2} f(t,x,v)\,dx\,dv \leqslant C(k) |E(t,\xi)| H_k(t)^{(k+2)/(k+3)}. \tag{20}$$

Gathering estimates (19) and (20) we are led to the conclusion of Lemma 17.

When $k < 1$, since $h(t,x,v) \geqslant 1$ we directly obtain

$$\frac{d}{dt}\tilde{H}_k(t) \leqslant C(k)\big(\|E(t)\|_{L^{k+3}}\|\rho(t)\|_{L^{\frac{k+3}{k+2}}} + \mathcal{M}_0|E(t,\xi)|\big)$$

$$\leqslant C(k)\big(\|E(t)\|_{L^{k+3}} + |E(t,\xi)|\big) \leqslant C(k)\big(\|E(t)\|_{L^{k+3}} + |E(t,\xi)|\big) H_k(t)^{\frac{k+2}{k+3}},$$

because (since $1 \leqslant (k+3)/(k+2) \leqslant 5/3$) we know that $\|\rho(t)\|_{L^{\frac{k+3}{k+2}}} \leqslant C$ by Proposition 2.5, and $H_k(t) \geqslant \mathcal{M}_0$. $\quad\square$

In order to exploit Lemma 17 to estimate the moments $H_k(t)$ by a Gronwall inequality, we now need to control the electric fields $\|E(t)\|_{L^{k+3}}$ and $|E(t,\xi)|$. The next subsection will be devoted to the control of $\|E(t)\|_{L^{k+3}}$. On the other hand, when $\mathcal{M}_0$ is not too large, one can control the quantity $|E(t,\xi)|$ by the following virial-type argument.

Proposition 2.9. *Let (f,ξ) be a classical solution of (1)–(3) on $[0,T]$. We have*

$$\int_0^t |E(s,\xi(s))|\,ds \leqslant C(1+t),$$

and

$$\int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{f(s,x,v)}{|x-\xi(s)|^2}\,dx\,dv\,ds \leqslant C(1+t).$$

Remark 2. This is the only point in the proof of Theorem 1.1 where we have to use assumption (i).

Proof. Let $(\boldsymbol{x}(s), \boldsymbol{v}(s)) = (\boldsymbol{x}(s, x, v), \boldsymbol{v}(s, x, v))$ be a plasma trajectory on $[0, T]$. Using the system of ODE (2) and (10), we compute

$$\frac{d^2}{ds^2}|\boldsymbol{x}(s) - \xi(s)| = \frac{|\boldsymbol{v}(s) - \eta(s)|^2}{|\boldsymbol{x}(s) - \xi(s)|} + \frac{1}{|\boldsymbol{x}(s) - \xi(s)|^2}$$
$$+ \frac{(\boldsymbol{x}(s) - \xi(s)) \cdot (E(s, \boldsymbol{x}(s)) - E(s, \xi(s)))}{|\boldsymbol{x}(s) - \xi(s)|} - \frac{[(\boldsymbol{x}(s) - \xi(s)) \cdot (\boldsymbol{v}(s) - \eta(s))]^2}{|\boldsymbol{x}(s) - \xi(s)|^3}.$$

Therefore

$$\frac{1}{|\boldsymbol{x}(s) - \xi(s)|^2} \leqslant \frac{d^2}{ds^2}|\boldsymbol{x}(s) - \xi(s)| + |E(s, \boldsymbol{x}(s))| + |E(s, \xi(s))|. \tag{21}$$

On the other hand, since f is constant along the trajectories of the measure-preserving flow $(\boldsymbol{x}, \boldsymbol{v})$, we have by changing variable

$$|E(s, \xi(s))| \leqslant \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{f(s, x, v)}{|x - \xi(s)|^2}\, dx\, dv = \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{f_0(x, v)}{|\boldsymbol{x}(s, x, v) - \xi(s)|^2}\, dx\, dv.$$

Therefore inserting (21) we get

$$\int\limits_0^t |E(s, \xi(s))|\, ds \leqslant \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x, v) \left(\int\limits_0^t \frac{d^2}{ds^2}|\boldsymbol{x}(s) - \xi(s)|\, ds \right) dx\, dv$$
$$+ \int\limits_0^t \left(\iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x, v) |E(s, \boldsymbol{x}(s, x, v))|\, dx\, dv \right) ds + \mathcal{M}_0 \int\limits_0^t |E(s, \xi(s))|\, ds. \tag{22}$$

For the first term in the right-hand side of (22), we have

$$\iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x, v) \left(\int\limits_0^t \frac{d^2}{ds^2}|\boldsymbol{x}(s) - \xi(s)|\, ds \right) dx\, dv = \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x, v) \left[\frac{d}{ds}|\boldsymbol{x} - \xi| \right]_{s=0}^{s=t} dx\, dv$$
$$\leqslant \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x, v) \left(\left| \frac{d}{ds}|\boldsymbol{x} - \xi| \right|(t) + \left| \frac{d}{ds}|\boldsymbol{x} - \xi| \right|(0) \right) dx\, dv$$
$$\leqslant 2 \sup_{t \in [0, T]} \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x, v) |\boldsymbol{v}(t, x, v) - \eta(t)|\, dx\, dv$$
$$= 2 \sup_{t \in [0, T]} \iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f(t, x, v) |v - \eta(t)|\, dx\, dv.$$

Hence, by Hölder's inequality, we obtain

$$\iint\limits_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x, v) \left(\int\limits_0^t \frac{d^2}{ds^2}|\boldsymbol{x}(s) - \xi(s)|\, ds \right) dx\, dv \leqslant C \sup_{t \in [0, T]} \mathcal{M}(t)^{1/2} \left(\mathcal{H}(t) + \mathcal{H}(t)\mathcal{M}(t) \right)^{1/2} \leqslant C. \tag{23}$$

We turn to the second term in (22). We have by changing variable backwards

$$
\int_0^t \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} f_0(x,v) \big| E(s, \boldsymbol{x}(s,x,v)) \big| \, dx\, dv \right) ds = \int_0^t \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} f(s,x,v) \big| E(s,x) \big| \, dx\, dv \right) ds
$$

$$
= \int_0^t \left(\int_{\mathbb{R}^3} \rho(s,x) \big| E(s,x) \big| \, dx \right) ds
$$

$$
\leqslant C \int_0^t \big\| \rho(s) \big\|_{L^{5/3}} \big\| E(s) \big\|_{L^{5/2}} \, ds \leqslant Ct. \tag{24}
$$

We used Proposition 2.5 in the last inequality.

Therefore coming back to (22), we find

$$
\int_0^t \big| E(s, \xi(s)) \big| \, ds \leqslant C(1+t) + \mathcal{M}_0 \int_0^t \big| E(s, \xi(s)) \big| \, ds.
$$

The first inequality of Proposition 2.9 then follows from the assumption (i) on $\mathcal{M}_0$ since $\lambda \leqslant 1$. For the second inequality, we come back to (21), integrate with respect to the measure $f_0(x,v)\, dx\, dv\, ds$, and use (23) and (24). The conclusion follows. $\square$

2.3. A Duhamel formula

The purpose of this subsection is to establish estimates for the norms $\|E(t)\|_{L^{k+3}}$.

Let χ_0 be a smooth cutoff function such that $\chi_0(x) = 1$ on $B(0,1)$, $\chi_0(x) = 0$ on $B(0,2)^c$ and $0 < \chi_0(x) < 1$ on $\mathbb{R}^3$, $\|\nabla \chi\|_{L^\infty} \leqslant 2$, $\|D^2 \chi\|_{L^\infty} \leqslant 20$. Let $R := R(T) > 1$ be sufficiently large (depending only on $\|f_0\|_{L^1}$ and T). Such an $R := R(T)$ will be determined by the condition (A.3) in Appendix A, by

$$
R(T) = 24^{1/3} K_0^{1/3} \big(1 + \|f_0\|_{L^1}\big)^{1/3} (1+T),
$$

with K_0 a given number such that $K_0 \geqslant 100$.

We set $\chi_R(z) = \chi_0(z/R)$.

We decompose the electric field and the force field into two parts:

$$
E = E_{\text{int}} + E_{\text{ext}}, \qquad F = F_{\text{int}} + F_{\text{ext}},
$$

where

$$
E_{\text{int}} = \rho * \big(x \mapsto \chi_R(x)\, x/|x|^3 \big), \qquad F_{\text{int}}(t,x) = F(t,x) \chi_R\big(x - \xi(t)\big).
$$

We have (denoting by superscripts the components of the fields)

$$
\max_{i=1,2,3} \big\| E_{\text{ext}}^{(i)} + F_{\text{ext}}^{(i)} \big\|_{L^\infty} \leqslant \frac{1 + \|f_0\|_{L^1}}{R^2}, \tag{25}
$$

$$
\max_{i,j=1,2,3} \big\| \partial_{x_j} \big(E_{\text{ext}}^{(i)} + F_{\text{ext}}^{(i)} \big) \big\|_{L^\infty} \leqslant 6 \frac{1 + \|f_0\|_{L^1}}{R^3}, \tag{26}
$$

and

$$
\max_{i,j,k=1,2,3} \big\| \partial_{x_j x_k}^2 \big(E_{\text{ext}}^{(i)} + F_{\text{ext}}^{(i)} \big) \big\|_{L^\infty} \leqslant 60 \frac{1 + \|f_0\|_{L^1}}{R^4}. \tag{27}
$$

As in [8], we write the Vlasov–Poisson equation using the internal part of E and F as a source term:

$$
\partial_t f + v \cdot \nabla_x f + (E_{\text{ext}} + F_{\text{ext}}) \cdot \nabla_v f = -(E_{\text{int}} + F_{\text{int}}) \cdot \nabla_v f. \tag{28}
$$

The reason why we do not consider the free transport (namely we do not consider the full field as a source term) will appear in Section 2.5.

In all this subsection, we fix $t \in [0, T]$. We define the flow map $(x, v) \mapsto (X^t, V^t)(x, v)$ such that for all $s \in [0, t]$,

$$
\begin{cases}
\dfrac{d}{ds} X^t(s, x, v) = -V^t(s, x, v), & X^t(0, x, v) = x, \\[2mm]
\dfrac{d}{ds} V^t(s, x, v) = -(E_{\text{ext}} + F_{\text{ext}})(t - s, X^t(s, x, v)), & V^t(0, x, v) = v.
\end{cases}
\tag{29}
$$

This is the backward flow associated to the field $E_{\text{ext}} + F_{\text{ext}}$. More precisely, if $(X_f, V_f) : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is the forward flow of $E_{\text{ext}} + F_{\text{ext}}$, defined by $\frac{d}{dt} X_f(t, x, v) = V_f(t, x, v)$, $\frac{d}{dt} V_f(t, x, v) = (E_{\text{ext}} + F_{\text{ext}})(t, X_f(t, x, v))$ and $(X_f, V_f)(0, x, v) = (x, v)$, then for all $t \in \mathbb{R}_+$, we have $(X_f, V_f)(t, \cdot, \cdot)^{-1} = (X^t, V^t)(t, \cdot, \cdot)$.

In the sequel we shall omit the dependence upon t in the notations. Moreover, we shall sometimes write

$$
\big(X(s), V(s)\big) = \big(X^t(s, x, v), V^t(s, x, v)\big) = \big(X(s, x, v), V(s, x, v)\big).
$$

Then, $(X(s), V(s))$ preserves the Lebesgue's measure on $\mathbb{R}^3 \times \mathbb{R}^3$. Note that if the external field vanished we would obtain the free flow $X(s, x, v) = x - vs$, $V(s, x, v) = v$, and if we considered the total field $E + F$ in (29), we would obtain $(X, V)(t) = (\boldsymbol{x}, \boldsymbol{v})(t)^{-1}$.

Using the invertibility properties of the flow listed in Appendix A, one can establish the analogue of Proposition 2.1:

Proposition 2.10. *Let (f, ξ) be a classical solution of* (1)–(3) *on $[0, T]$. Let $t \in [0, T]$ and let $(\tau, s) \in [0, t]^2$. Let $b > a \geqslant 0$. Then*

$$
\int_{\mathbb{R}^3} |v|^a f\big(\tau, X(s, x, v), V(s, x, v)\big) \, dv \leqslant C(a, b) \|f_0\|_{L^\infty}^{\frac{b-a}{3+b}} \left(\int_{\mathbb{R}^3} |v|^b f\big(\tau, X(s, x, v), V(s, x, v)\big) \, dv \right)^{\frac{3+a}{3+b}}.
\tag{30}
$$

Moreover, setting $\tilde{\rho}(\tau, s, x) = \int_{\mathbb{R}^3} f(\tau, X(s, x, v), V(s, x, v)) \, dv$, we have

$$
\big\| \tilde{\rho}(\tau, s, \cdot) \big\|_{L^{\frac{b+3}{3}}} \leqslant C(b)(1 + s)^{\frac{3b}{3+b}} \|f_0\|_{L^\infty}^{\frac{b}{3+b}} \left(1 + \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^b f(\tau, x, v) \, dx \, dv \right)^{\frac{3}{3+b}}.
\tag{31}
$$

Proof. For the first inequality this is exactly the same proof as for Proposition 2.1, estimate (5). The second inequality is obtained thanks to the bound $|V(s, x, v) - v| \leqslant s$ (see (A.2)), and using the fact that $(X(s), V(s))$ preserves the Lebesgue's measure. Recall also that $\|f(\tau)\|_{L^\infty} = \|f_0\|_{L^\infty}$. $\quad\square$

A consequence of Propositions 2.2 and 2.10 is

Proposition 2.11. *Let (f, ξ) be a classical solution of* (1)–(3) *on $[0, T]$. Let $t \in [0, T]$ and let $(\tau, s) \in [0, t]^2$. Let $\tilde{\rho}(\tau, s, x) = \int_{\mathbb{R}^3} f(\tau, X(s, x, v), V(s, x, v)) \, dv$ and $\tilde{E}(\tau, s, \cdot) = \tilde{\rho}(\tau, s, \cdot) * (x \mapsto x/|x|^3)$. If $0 < m < 6$, we have*

$$
\big\| \tilde{E}(\tau, s) \big\|_{L^{\frac{3(m+3)}{6-m}}} \leqslant C(1 + s)^{\frac{3m}{3+m}} \left(1 + \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^m f(\tau, x, v) \, dx \, dv \right)^{\frac{3}{m+3}} \leqslant C(1 + s)^{\frac{3m}{3+m}} H_m(\tau)^{\frac{3}{m+3}},
\tag{32}
$$

and if $m > 6$,

$$
\big\| \tilde{E}(\tau, s) \big\|_{L^\infty} \leqslant C(1 + s)^{\frac{3m}{3+m}} \left(1 + \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^m f(\tau, x, v) \, dx \, dv \right)^{\frac{3}{m+3}} \leqslant C(1 + s)^{\frac{3m}{3+m}} H_m(\tau)^{\frac{3}{m+3}},
\tag{33}
$$

where $C > 0$ depends only on $\|f_0\|_{L^\infty}$ and m.

Proof. We apply (8) and (9) with $s = (m + 3)/3$, and conclude thanks to (31) with $b = m$. $\quad\square$

Using Duhamel formula we can express the solution of (28) as follows:

$$f(t, x, v) = f_0\big(X(t, x, v), V(t, x, v)\big) - \int_0^t \big(\mathrm{div}_v\big[(E_{\mathrm{int}} + F_{\mathrm{int}})f\big]\big)\big(t - s, X(s, x, v), V(s, x, v)\big) \, ds. \tag{34}$$

Proposition 2.12. *Let* (f, ξ) *be a classical solution of* (1)–(3) *on* $[0, T]$. *We have for* $3 \leqslant m < m_0$,

$$\big\| E(t) \big\|_{L^{m+3}} \leqslant C(1 + T)^{\sup(2, \frac{3m}{3+m})} + C \left\| \int_0^t s \int_{\mathbb{R}^3} \big(|E_{\mathrm{int}} + F_{\mathrm{int}}| f\big)\big(t - s, X(s), V(s)\big) \, dv \, ds \right\|_{L^{m+3}}, \tag{35}$$

where C depends on m and the initial data.

Proof. By (34), we have

$$\rho(t, x) = \int_{\mathbb{R}^3} f_0\big(X(t, x, v), V(t, x, v)\big) \, dv$$

$$- \int_0^t \int_{\mathbb{R}^3} \big(\mathrm{div}_v\big((E_{\mathrm{int}} + F_{\mathrm{int}})f\big)\big)\big(t - s, X(s, x, v), V(s, x, v)\big) \, dv \, ds$$

$$:= \rho_1(t, x) + \rho_2(t, x). \tag{36}$$

We set $E_1(t) = \rho_1(t) * (x \mapsto x/|x|^3)$. We apply Propositions 2.10 and 2.11 with $\tau = 0$ and $s = t$: first, since $H_m(0)$ is finite, we have (with $\frac{3(m+3)}{6-m}$ replaced by ∞ if $m > 6$)

$$\sup_{t \in [0, T]} \big\| E_1(t) \big\|_{L^{\frac{3(m+3)}{6-m}}} \leqslant C(1 + T)^{\frac{3m}{3+m}}.$$

On the other hand, by (31), we have $\|\rho_1(t)\|_{L^{5/3}} \leqslant C(1 + T)^{6/5}(1 + H_2(0)^{3/5}) \leqslant C(1 + T)^{6/5}$. Therefore by interpolation, observing that $5/3 \leqslant m + 3 \leqslant 3(m + 3)/(6 - m)$ because $m \geqslant 3$, we get

$$\sup_{t \in [0, T]} \big\| E_1(t) \big\|_{L^{m+3}} \leqslant C(1 + T)^{\frac{3m}{3+m}}. \tag{37}$$

For the term ρ_2 and the corresponding field E_2, we have to work more. More precisely, we will establish in Appendix A, see (A.23), that

$$\rho_2(t, x) = \mathrm{div}_x \int_0^t \int_{\mathbb{R}^3} N(s, x, v)\big[(E_{\mathrm{int}} + F_{\mathrm{int}})f\big]\big(t - s, X(s, x, v), V(s, x, v)\big) \, dv \, ds$$

$$+ \int_0^t \int_{\mathbb{R}^3} \big[\mathrm{div}_v({}^t M)(x, v) - \mathrm{div}_x({}^t N)(x, v)\big] \cdot \big[(E_{\mathrm{int}} + F_{\mathrm{int}})f\big]\big(t - s, X(s, x, v), V(s, x, v)\big) \, dv \, ds$$

$$:= \rho_{2,1}(t, x) + \rho_{2,2}(t, x)$$

$$:= \mathrm{div}_x \, a_{2,1}(t, x) + \rho_{2,2}(t, x), \tag{38}$$

where

$$M = M(s, x, v) = \big[D_v V - D_x V (D_x X)^{-1} D_v X\big]^{-1},$$

$$N = N(s, x, v) = (D_x X)^{-1}(D_v X) M(s, x, v).$$

We first estimate $E_{2,1}(t) = \rho_{2,1}(t) * (x \mapsto x/|x|^3)$. By elliptic regularity we have

$$\big\| E_{2,1}(t) \big\|_{L^{m+3}} \leqslant C \big\| a_{2,1}(t) \big\|_{L^{m+3}}.$$

Next, by (A.18) we have $\|N(s)\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \leqslant Cs$, so that

$$\big\|E_{2,1}(t)\big\|_{L^{m+3}} \leqslant C\left\|\int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}} + F_{\text{int}}|f\big)\big(t-s, X(s,x,v), V(s,x,v)\big) \, dv \, ds\right\|_{L^{m+3}}. \tag{39}$$

We next estimate $E_{2,2}(t) = \rho_{2,2}(t) * (x \mapsto x/|x|^3)$. By interpolation inequality (8) with $s = 3(m+3)/(m+6) \in \,]1, 3[$, we have $\|E_{2,2}(t)\|_{L^{m+3}} \leqslant C\|\rho_{2,2}(t)\|_{L^{\frac{3(m+3)}{m+6}}}$ and, since $1 < 3(m+3)/(m+6) < m+3$, we obtain

$$\big\|E_{2,2}(t)\big\|_{L^{m+3}} \leqslant C\big(\big\|\rho_{2,2}(t)\big\|_{L^{m+3}} + \big\|\rho_{2,2}(t)\big\|_{L^1}\big).$$

On the other hand, by (A.15) and (A.21) (Section A.1 in Appendix A), we have $M^{-1} = I_3 + sP_5$ with $\max_i \|\partial_{v_i} P_5\|_{L^\infty([0,t] \times \mathbb{R}^3 \times \mathbb{R}^3)} \leqslant C$ and with $\|M\|_{L^\infty([0,t] \times \mathbb{R}^3 \times \mathbb{R}^3)} \leqslant 2$ by (A.16), so that

$$\big\|\operatorname{div}_v({}^t M)(s)\big\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \leqslant Cs, \quad \forall s \in [0, t]. \tag{40}$$

Similarly, by (A.17) and (A.22) we have $N = sP_6$ with $\max_i \|\partial_{x_i} P_6\|_{L^\infty([0,t] \times \mathbb{R}^3 \times \mathbb{R}^3)} \leqslant C$, hence

$$\big\|\operatorname{div}_x({}^t N)(s)\big\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \leqslant Cs, \quad \forall s \in [0, t]. \tag{41}$$

Therefore

$$\big\|\rho_{2,2}(t)\big\|_{L^{m+3}} \leqslant C\left\|\int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}} + F_{\text{int}}|f\big)\big(t-s, X(s,x,v), V(s,x,v)\big) \, dv \, ds\right\|_{L^{m+3}}. \tag{42}$$

Next, we have by changing variable

$$\big\|\rho_{2,2}(t)\big\|_{L^1} \leqslant C\int_0^t s \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \big[\big(|E_{\text{int}}| + |F_{\text{int}}|\big)f\big](t-s, x, v) \, dx \, dv \, ds. \tag{43}$$

By an argument similar to that of (24), we have for $s \in [0, t]$

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \big(|E_{\text{int}}|f\big)(s, x, v) \, dx \, dv \leqslant \big\|E_{\text{int}}(s)\big\|_{L^{5/2}} \big\|\rho(s)\big\|_{L^{5/3}} \leqslant C.$$

On the other hand, Proposition 2.9 entails that

$$\int_0^t t - s \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \big(|F_{\text{int}}|f\big)(s, x, v) \, dx \, dv \, ds \leqslant Ct(1+t).$$

We deduce from (42)–(43) and the two inequalities above the estimate for $E_{2,2}$:

$$\big\|E_{2,2}(t)\big\|_{L^{m+3}} \leqslant C t(1+t) + C\left\|\int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}} + F_{\text{int}}|f\big)\big(t-s, X(s,x,v), V(s,x,v)\big) \, dv \, ds\right\|_{L^{m+3}}. \tag{44}$$

Gathering (39) and (44), we can conclude. $\quad\square$

Next sections are devoted to the control of the right-hand side of (35).

2.4. Intermediate small time estimates for the moments

The purpose of this paragraph is to obtain estimates for the moments on small but uniform intervals of time in $[0, T]$.

Proposition 2.13. *Let* (f, ξ) *be a classical solution of* (1)–(3) *on* $[0, T]$. *Let* $t \in [0, \min(1, T)]$ *and let* $m \geqslant 3$.
For any $\gamma \in \,]0, 1[$, *we have*

$$\left\| \int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}} + F_{\text{int}}| f \big) \big(t - s, X(s), V(s) \big) \, dv \, ds \right\|_{L^{m+3}} \leqslant C(\gamma, m) t^\delta H_k(t)^{\frac{1}{m+3}},$$

where k *is defined by* $k + 3 = (m + 3)(1 + \gamma)$ *(note that* $k > m$*), and where* $0 < \delta < 1$ *is defined by* $\delta = \frac{\gamma}{1 + (\gamma+1)(m+3)}$.

Remark 3. Taking γ small in Proposition 2.13, we realize that $k > m$ may be chosen as close as we want to m, therefore the estimate $\|E(t)\|_{L^{m+3}} \leqslant C H_m(t)^{1/(m+3)}$, which in view of Lemma 17 would be enough to obtain an estimate on $H_m(t)$, is close to be achieved.

Proof. *Step* 1: *Estimate for* E_{int}.
By Proposition 2.5, the field E_{int} belongs to $L^\infty([0, T], L^{r_1}(\mathbb{R}^3))$ for all $3/2 < r_1 \leqslant 15/4$. We can proceed as in the proof of the estimates (31)–(32) and (28')–(40) of [8] to get for all $3/2 < r_1 \leqslant 15/4$

$$\left\| \int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}}| f \big) \big(t - s, X(s), V(s) \big) \, dv \, ds \right\|_{L^{m+3}} \leqslant C(r_1, m) t^{2 - \frac{3}{r_1}} M_{k_1}(t)^{\frac{1}{m+3}}, \tag{45}$$

where $k_1 > m$ is defined by $k_1 + 3 = (m + 3)(3 - 3/r_1)$, which by (15) yields

$$\left\| \int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}}| f \big) \big(t - s, X(s), V(s) \big) \, dv \, ds \right\|_{L^{m+3}} \leqslant C(r_1, m) t^{2 - \frac{3}{r_1}} H_{k_1}(t)^{\frac{1}{m+3}}. \tag{46}$$

Since (31)–(32) and (28')–(40) of [8] are not exactly stated as above, we provide the full proof of estimate (45) here for the sake of completeness. Let us consider the quantity

$$\left\| \int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}}| f \big) \big(t - s, X(s), V(s) \big) \, dv \, ds \right\|_{L^{m+3}}. \tag{47}$$

By adapting the argument in [8], we bound (47) applying Hölder's inequality with exponents r_1 and r_1' such that $\frac{1}{r_1'} = 1 - \frac{1}{r_1}$ (and such that $3/2 < r_1 \leqslant 15/4$):

$$\left\| \int_0^t s \left(\int_{\mathbb{R}^3} |E_{\text{int}}|^{r_1} (t - s, y) \big| \det(D_v X) \big|^{-1} dy \right)^{\frac{1}{r_1}} \left(\int_{\mathbb{R}^3} f \big(t - s, X(s), V(s) \big) \, dv \right)^{\frac{1}{r_1'}} ds \right\|_{L^{m+3}} \|f\|_{L^\infty}^{\frac{r_1' - 1}{r_1'}},$$

where we perform the change of variable $X(s, x, v) \mapsto y$, with Jacobian $|\det(D_v X)|^{-1}$. Thanks to estimate (A.14) in Appendix A, we can bound the above quantity by:

$$\left\| \int_0^t s \sup_{\tau \in [0, t]} \left(\int_{\mathbb{R}^3} |E_{\text{int}}|^{r_1}(\tau, y) \frac{8}{s^3} \, dy \right)^{\frac{1}{r_1}} \left(\int_{\mathbb{R}^3} f \big(t - s, X(s), V(s) \big) \, dv \right)^{\frac{1}{r_1'}} ds \right\|_{L^{m+3}} \|f\|_{L^\infty}^{\frac{r_1' - 1}{r_1'}}$$

$$\leqslant C \sup_{\tau \in [0, t]} \big\| E_{\text{int}}(\tau) \big\|_{L^{r_1}} \left\| \int_0^t s^{1 - 3/r_1} \left(\int_{\mathbb{R}^3} f \big(t - s, X(s), V(s) \big) \, dv \right)^{\frac{1}{r_1'}} ds \right\|_{L^{m+3}}$$

$$\leqslant C \sup_{\tau \in [0,t]} \| E_{\text{int}}(\tau) \|_{L^{r_1}} \int_0^t s^{1-3/r_1} \left\| \left(\int_{\mathbb{R}^3} f\big(t-s, X(s), V(s)\big) \, dv \right)^{\frac{1}{r_1'}} \right\|_{L^{m+3}} ds$$

$$\leqslant C t^{2-\frac{3}{r_1}} \sup_{\tau \in [0,t]} \| E_{\text{int}}(\tau) \|_{L^{r_1}} \sup_{(\tau,s) \in [0,t]^2} \left\| \int f\big(\tau, X(s), V(s)\big) \, dv \right\|_{L^{\frac{m+3}{r_1'}}}^{\frac{1}{r_1'}}.$$

Applying Proposition 2.5, we obtain (since $3/2 < r_1 \leqslant 15/4$)

$$\sup_{\tau \in [0,t]} \| E_{\text{int}}(\tau) \|_{L^{r_1}} \leqslant C(r_1).$$

We are left with the estimate of the second factor in the above inequality. We apply the formula (31) in Proposition 2.10, observing that $t \leqslant 1$, where b is chosen in such a way that $b + 3 = \frac{3(m+3)}{r_1'}$. Note that $b \geqslant 0$ because $m \geqslant r_1'$ since $m \geqslant 3$ and $r_1' \leqslant 3$. So we get

$$\sup_{(\tau,s) \in [0,t]^2} \left\| \int_{\mathbb{R}^3} f\big(\tau, X(s), V(s)\big) \, dv \right\|_{L^{\frac{m+3}{r_1'}}}^{\frac{1}{r_1'}} \leqslant C \| f_0 \|_{L^\infty}^{\frac{b}{r_1'(b+3)}} \big(1 + M_{k_1}(t) \big)^{\frac{1}{m+3}},$$

where $k_1 = b$. By using Eq. (15), we obtain the bound (46). We observe that (since $k_1 + 3 = \frac{3(m+3)}{r_1'}$ and $\frac{1}{r_1'} = 1 - \frac{1}{r_1}$), we have the identity $k_1 = (m+3)(3 - 3/r_1) - 3$.

Step 2: *Estimate for* F_{int}.

Let

$$I(x) = \int_0^t s \int_{\mathbb{R}^3} \big(|F_{\text{int}}| f \big) \big(t-s, X(s), V(s) \big) \, dv \, ds.$$

In the following we will write ξ instead of $\xi(t-s)$ when not misleading.

• *Local estimate for* I.

We recall that by Proposition 2.4, there exists $R_b \geqslant 4$ such that $\sup_{t \in [0, \min(1,T)]} |\xi(t)| \leqslant R_b$. We set $B = B(0, 3R_0)$, where $R_0 := \sup(R_b, 2R(1))$ (where $R(T)$ is defined in Appendix A, formula (A.3)).

Let $0 < \varepsilon < 2/(m+3)$ be a small parameter and let us pick $3/(2+\varepsilon) < r_2 < 3/2$. By Hölder's inequality, we get for all $x \in B$

$$I(x) = \int_0^t s^{1+\varepsilon} \int_{\mathbb{R}^3} \frac{|F_{\text{int}}|(t-s, X(s))}{|X(s) - x|^\varepsilon} \left(\frac{|X(s) - x|}{s} \right)^\varepsilon f\big(t-s, X(s), V(s)\big) \, dv \, ds$$

$$\leqslant \| f_0 \|_{L^\infty}^{1-\frac{1}{r_2}} \int_0^t s^{1+\varepsilon} \left(\int_{\mathbb{R}^3} \frac{|F_{\text{int}}|^{r_2}(t-s, X(s))}{|X(s) - x|^{\varepsilon r_2}} \, dv \right)^{\frac{1}{r_2}} \left(\int_{\mathbb{R}^3} \left(\frac{|X(s) - x|}{s} \right)^{\varepsilon r_2'} f\big(t-s, X(s), V(s)\big) \, dv \right)^{\frac{1}{r_2'}} ds.$$

For all fixed $x \in B$ and $0 \leqslant s \leqslant \min(1, T)$ we perform the change of variable $v \mapsto y$ with

$$y = x - X(s, x, v).$$

Then, by formula (A.1) in Appendix A, $|y| \leqslant s|v| + C \frac{s^2}{R(1)^2} \leqslant s(|v| + C)$. Moreover we have

$$\int_{\mathbb{R}^3} \frac{|F_{\text{int}}|^{r_2}(t-s, X(s))}{|X(s) - x|^{\varepsilon r_2}} \, dv = \int_{\mathbb{R}^3} \frac{|F_{\text{int}}|^{r_2}(t-s, x-y)}{|y|^{\varepsilon r_2}} \big| \det(D_v X(s)) \big|^{-1} \, dy$$

and, according to estimate (A.14) in Appendix A, we have $|\det(D_v X(s))|^{-1} \leqslant 8/s^3$.

So we obtain

$$
\|I\|_{L^{m+3}(B)} \leqslant C \left\| \int_0^t s^{1-\frac{3}{r_2}+\varepsilon} \left(\int_{\mathbb{R}^3} \frac{|F_{\text{int}}|^{r_2}(t-s, x-y)}{|y|^{\varepsilon r_2}} \, dy \right)^{\frac{1}{r_2}} \right.
$$

$$
\left. \times \left(\int_{\mathbb{R}^3} \left(1 + |v|^{\varepsilon r_2'}\right) f\left(t-s, X(s), V(s)\right) dv \right)^{\frac{1}{r_2'}} ds \right\|_{L^{m+3}(B)}
$$

$$
\leqslant C \int_0^t s^{1-\frac{3}{r_2}+\varepsilon} J(s)^{\frac{1}{m+3}} \, ds,
$$

where

$$
J(s) = \int_{|x| \leqslant 3R_0} \left(\int_{|x-\xi-y| \leqslant 2R(1)} \frac{dy}{|y|^{\varepsilon r_2} |x-\xi-y|^{2r_2}} \right)^{\frac{m+3}{r_2}} \left(\int_{\mathbb{R}^3} \left(1+|v|^{\varepsilon r_2'}\right) f\left(t-s, X(s), V(s)\right) dv \right)^{\frac{(m+3)}{r_2'}} dx
$$

$$
\leqslant C(r_2, \varepsilon) \int_{|x| \leqslant 3R_0} |x-\xi|^{(\frac{3}{r_2}-2-\varepsilon)(m+3)} \left(\int_{\mathbb{R}^3} \left(1+|v|^{\varepsilon r_2'}\right) f\left(t-s, X(s), V(s)\right) dv \right)^{\frac{(m+3)}{r_2'}} dx.
$$

Note that in the estimate above, we used the fact that

$$
\int_{\mathbb{R}^3} \frac{dy}{|y|^{\varepsilon r_2} |x-\xi-y|^{2r_2}} = C(r_2, \varepsilon) |x-\xi|^{3-\varepsilon r_2 - 2r_2}.
$$

We now set

$$
r_2 = \frac{3}{2 + \varepsilon/2}
$$

and we define

$$
p = \frac{2}{\varepsilon(m+3)}.
$$

Note that $p > 1$ and

$$
\frac{3}{2 + \frac{1}{m+3}} < r_2 < \frac{3}{2}. \tag{48}
$$

Moreover

$$
-\left(\frac{3}{r_2} - 2 - \varepsilon \right)(m+3)p = \frac{\varepsilon}{2}(m+3)p = 1 < 3.
$$

Applying Hölder's inequality, we obtain

$$
J(s) \leqslant C \left(\int_{|x| \leqslant 3R_0} |x-\xi|^{-\frac{\varepsilon}{2}(m+3)p} \, dx \right)^{1/p} \left(\int_{|x| \leqslant 3R_0} \left(\int_{\mathbb{R}^3} \left(1+|v|^{\varepsilon r_2'}\right) f\left(t-s, X(s), V(s)\right) dv \right)^{\frac{(m+3)}{r_2'}p'} dx \right)^{1/p'},
$$

therefore

$$
\|I\|_{L^{m+3}(B)} \leqslant C R_0^\varepsilon t^{\frac{\varepsilon}{2}} \sup_{\tau, s \in [0,t]} \left\{ \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \left(1+|v|^{\varepsilon r_2'}\right) f\left(\tau, X(s), V(s)\right) dv \right)^{\frac{(m+3)}{r_2'}p'} dx \right\}^{\frac{1}{(m+3)p'}}.
$$

We now focus on the right-hand side

$$\sup_{\tau,s\in[0,t]}\left\{\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}\left(1+|v|^{\varepsilon r_2'}\right)f\big(\tau,X(s),V(s)\big)\,dv\right)^{\frac{(m+3)}{r_2'}p'}dx\right\}^{\frac{1}{(m+3)p'}}.$$

Let us introduce k_2 such that

$$\left(\frac{3+\varepsilon r_2'}{3+k_2}\right)\left(\frac{m+3}{r_2'}p'\right)=1$$

and apply (30) with this choice and $a=\varepsilon r_2'$, $b=k_2$. Note that $b>a$ since

$$\frac{(m+3)}{r_2'}p'>1\quad\Longleftrightarrow\quad(m+3)\left(\frac{1}{3}-\frac{\varepsilon}{6}\right)>1-\varepsilon\frac{m+3}{2}\quad\Longleftrightarrow\quad\frac{1}{3}>\frac{1}{m+3}-\frac{\varepsilon}{3},$$

which holds as soon as $m>0$ (remember that $m\geqslant 3$ in this proposition).

We obtain

$$\sup_{\tau,s\in[0,t]}\left\{\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}|v|^{\varepsilon r_2'}f\big(\tau,X(s),V(s)\big)\,dv\right)^{\frac{(m+3)}{r_2'}p'}dx\right\}^{\frac{1}{(m+3)p'}}$$

$$\leqslant C\sup_{\tau,s\in[0,t]}\left\{\iint_{\mathbb{R}^3\times\mathbb{R}^3}|v|^{k_2}f\big(\tau,X(s),V(s)\big)\,dx\,dv\right\}^{\frac{1}{(m+3)p'}}.$$

Similarly, we introduce k_2' such that

$$\left(\frac{3}{3+k_2'}\right)\left(\frac{m+3}{r_2'}p'\right)=1,$$

and apply (30) with this choice and $a=0$, $b=k_2'>a$. Since $k_2'<k_2$, we obtain

$$\sup_{\tau,s\in[0,t]}\left\{\int_{\mathbb{R}^3}\left(\int_{\mathbb{R}^3}f\big(\tau,X(s),V(s)\big)\,dv\right)^{\frac{(m+3)}{r_2'}p'}dx\right\}^{\frac{1}{(m+3)p'}}$$

$$\leqslant C\sup_{\tau,s\in[0,t]}\left\{\iint_{\mathbb{R}^3\times\mathbb{R}^3}|v|^{k_2'}f\big(\tau,X(s),V(s)\big)\,dx\,dv\right\}^{\frac{1}{(m+3)p'}}$$

$$\leqslant C\sup_{\tau,s\in[0,t]}\left\{\iint_{\mathbb{R}^3\times\mathbb{R}^3}\left(1+|v|^{k_2}\right)f\big(\tau,X(s),V(s)\big)\,dx\,dv\right\}^{\frac{1}{(m+3)p'}}$$

$$\leqslant C\sup_{\tau,s\in[0,t]}\left\{\iint_{\mathbb{R}^3\times\mathbb{R}^3}\left(1+\big|V(s)\big|^{k_2}\right)f\big(\tau,X(s),V(s)\big)\,dx\,dv\right\}^{\frac{1}{(m+3)p'}}$$

$$\leqslant C+C\sup_{\tau,s\in[0,t]}\left\{\iint_{\mathbb{R}^3\times\mathbb{R}^3}\big|V(s)\big|^{k_2}f\big(\tau,X(s),V(s)\big)\,dx\,dv\right\}^{\frac{1}{(m+3)p'}}.$$

We have used (A.2) together with the condition (A.3) in the last inequalities.

Finally, using that $(X(s),V(s))$ preserves the Lebesgue's measure we obtain

$$\|I\|_{L^{m+3}(B)}\leqslant CR_0^\varepsilon t^{\frac{\varepsilon}{2}}H_{k_2}(t)^{\frac{1}{(m+3)p'}}.$$

Since $p'\geqslant 1$, making explicit the dependence of the constants, we get for any r_2 satisfying the condition (48),

$$\|I\|_{L^{m+3}(B)} \leqslant C(r_2, m) R_0^{2(\frac{3}{r_2}-2)} t^{\frac{3}{r_2}-2} H_{k_2}(t)^{\frac{1}{m+3}}, \tag{49}$$

where $k_2 > m$ satisfies

$$3 + k_2 = (3+m) \frac{\frac{3}{r_2} - 1}{1 - (m+3)(\frac{3}{r_2} - 2)}. \tag{50}$$

- *Estimate for I at infinity.*

In this step we estimate the norm of I on the exterior of $B = B(0, 3R_0)$. Observe that when $|x| \geqslant 3R_0$ and when $|x - \xi(t-s) - y| \leqslant 2R(1)$, we have

$$|y| \geqslant |x| - |\xi(t-s)| - 2R(1) \geqslant 2R_0 - 2R(1) \geqslant 2R(1) \geqslant 1.$$

We use again the parameters $0 < \varepsilon < 2/(m+3)$ and $r_2 = 3/(2 + \varepsilon/2)$. By similar computations we find

$$
\begin{aligned}
\|I\|_{L^{m+3}(B^c)} &\leqslant C \left\| \int_0^t s^{1-\frac{3}{r_2}+\varepsilon} \left(\int_{|x-\xi-y| \leqslant 2R(1)} \frac{dy}{|y|^{\varepsilon r_2} |x-\xi-y|^{2r_2}} \right)^{\frac{1}{r_2}} \right. \\
&\qquad \left. \times \left(\int_{\mathbb{R}^3} \left(1 + |v|^{\varepsilon r_2'}\right) f\left(t-s, X(s), V(s)\right) dv \right)^{\frac{1}{r_2'}} \right\|_{L^{m+3}(B^c)} ds \\
&\leqslant C \int_0^t s^{1-\frac{3}{r_2}+\varepsilon} \left\{ \int_{\mathbb{R}^3} \left(\int_{|x-\xi-y| \leqslant 2R(1)} \frac{dy}{|x-\xi-y|^{2r_2}} \right)^{\frac{m+3}{r_2}} \right. \\
&\qquad \left. \times \left(\int_{\mathbb{R}^3} \left(1 + |v|^{\varepsilon r_2'}\right) f\left(t-s, X(s), V(s)\right) dv \right)^{\frac{(m+3)}{r_2'}} dx \right\}^{\frac{1}{m+3}} ds \\
&\leqslant C R(1)^{3/r_2 - 2} \int_0^t s^{1-\frac{3}{r_2}+\varepsilon} \left\{ \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \left(1 + |v|^{\varepsilon r_2'}\right) f\left(t-s, X(s), V(s)\right) dv \right)^{\frac{(m+3)}{r_2'}} dx \right\}^{\frac{1}{m+3}} ds.
\end{aligned}
$$

We now introduce k_3 such that

$$\left(\frac{3 + \varepsilon r_2'}{3 + k_3} \right) \left(\frac{m+3}{r_2'} \right) = 1.$$

Note that $k_3 > \varepsilon r_2'$ when $m + 3 > r_2'$, which is always true if (like in our case) $m \geqslant 3$ (remember that $\varepsilon < \frac{2}{m+3}$ so that $r' \leqslant \frac{3}{1 - \frac{1}{m+3}}$). It follows from (30) with $a = \varepsilon r_2'$ and $b = k_3$ that

$$\|I\|_{L^{m+3}(B^c)} \leqslant C R(1)^{\frac{3}{r_2}-2} t^{\frac{3}{r_2}-2} H_{k_3}(t)^{\frac{1}{m+3}}.$$

Since $k_3 < k_2$, we have $H_{k_3} \leqslant C H_{k_2}$. Finally, making explicit the dependence of the constant:

$$\|I\|_{L^{m+3}(B^c)} \leqslant C(r_2, m) R(1)^{\frac{3}{r_2}-2} t^{\frac{3}{r_2}-2} H_{k_2}(t)^{\frac{1}{m+3}}, \tag{51}$$

with $k_2 = \frac{(3+m)(3/r_2-1)}{1-(m+3)(3/r_2-2)} - 3$, for all r_2 such that $\frac{3}{2+1/(m+3)} < r_2 < 3/2$.

Step 3: *End of the proof of* Proposition 2.13.

Gathering the estimates (49) and (51), we find

$$\left\| \int_0^t s \int_{\mathbb{R}^3} \left(|F_{\text{int}}| f\right)\left(t-s, X(s), V(s)\right) dv \, ds \right\|_{L^{m+3}} \leqslant C(r_2, m) R_0^{2(\frac{3}{r_2}-2)} t^{\frac{3}{r_2}-2} H_{k_2}(t)^{\frac{1}{m+3}}, \tag{52}$$

hence (dropping the dependence w.r.t. R_0 and $R(1)$)

$$\left\| \int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}} + F_{\text{int}}| f \big)\big(t - s, X(s), V(s)\big) \, dv \, ds \right\|_{L^{m+3}}$$
$$\leqslant C(r_1, m) t^{2 - \frac{3}{r_1}} H_{k_1}(t)^{\frac{1}{m+3}} + C(r_2, m) t^{\frac{3}{r_2} - 2} H_{k_2}(t)^{\frac{1}{m+3}}. \tag{53}$$

We recall that $3/2 < r_1 \leqslant 15/4$ and $\frac{3}{2 + 1/(m+3)} < r_2 < 3/2$ can be taken as close as necessary to $3/2$, and that $k_1, k_2 > m$ are defined by $k_1 + 3 = (m + 3)(3 - 3/r_1)$ and by $k_2 + 3 = (m + 3)(\frac{3}{r_2} - 1)/(1 - (m + 3)(3/r_2 - 2))$ (see (50)).

We next choose r_1 and r_2 so that $k = k_1 = k_2$ in the following way. We consider a small parameter $0 < \gamma < 1$. We define r_1 so that

$$2 - \frac{3}{r_1} = \gamma.$$

Note that $3/2 < r_1 < 3 \leqslant 15/4$ by choice of γ.

We next define r_2 so that

$$\frac{3}{r_2} - 2 = \frac{\gamma}{1 + (m + 3)(\gamma + 1)},$$

which implies that $k = k_2 = k_1$. Then the condition (48) on r_2 is satisfied. Noticing that $k + 3 = (m + 3)(1 + \gamma)$, and using that $t \leqslant 1$, (53) rewrites

$$\left\| \int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}} + F_{\text{int}}| f \big)\big(t - s, X(s), V(s)\big) \, dv \, ds \right\|_{L^{m+3}} \leqslant C(\gamma, m)\big(t^\gamma + t^{\frac{\gamma}{1+(m+3)(\gamma+1)}} \big) H_k(t)^{\frac{1}{m+3}}$$
$$\leqslant C(\gamma, m) t^{\frac{\gamma}{1+(m+3)(\gamma+1)}} H_k(t)^{\frac{1}{m+3}}$$
$$\leqslant C(\gamma, m) t^\delta H_k(t)^{\frac{1}{m+3}}.$$

The conclusion follows. $\square$

Proposition 2.14 (*Intermediate small time estimates*). *Let* (f, ξ) *be a classical solution of* (1)–(3) *on* $[0, T]$. *For* $t \in [0, \min(1, T)]$ *and* $3 < m < m_0$, *the following estimate holds*:

$$\| E(t) \|_{m+3} \leqslant C + C \left\| \int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}} + F_{\text{int}}| f \big)\big(t - s, X(s), V(s)\big) \, dv \, ds \right\|_{L^{m+3}}$$
$$\leqslant C + C t^\delta + C t^{1+\gamma+\delta} H_m(t)^{\frac{3(k+3)}{(m+3)^2}}. \tag{54}$$

Here γ *is any number in* $]0, 1[$ *such that* $\gamma < \frac{m_0 - m}{m + 3}$ *if* $m \geqslant 6$, *and any number in* $]0, 1[$ *such that*

$$\gamma \leqslant \frac{m - 3}{6 - m}, \qquad \gamma < \frac{m_0 - m}{m + 3} \tag{55}$$

if $m < 6$. *The parameter* $k > m$ *is defined by* $k + 3 = (m + 3)(1 + \gamma)$, *and* $0 < \delta < 1$ *by*

$$\delta = \frac{\gamma}{1 + (m + 3)(\gamma + 1)}.$$

Remark 4. We stress that the constants depend on k, or equivalently, on γ (in fact some of them blow up when $k \to m$).

Proof. Thanks to Propositions 2.12 and 2.13, we obtain (remember that $t \leqslant 1$)

$$\| E(t) \|_{L^{m+3}} \leqslant C + C \left\| \int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}} + F_{\text{int}}| f \big)\big(t - s, X(s), V(s)\big) \, dv \, ds \right\|_{L^{m+3}}$$
$$\leqslant C + C t^\delta H_k(t)^{\frac{1}{m+3}}, \tag{56}$$

with $k > m$ such that $k + 3 = (m + 3)(1 + \gamma)$ and $\delta = \frac{\gamma}{1+(m+3)(\gamma+1)}$, and for all $0 < \gamma < 1$.

On the other hand, we infer from Lemma 17 and from Proposition 2.9 that

$$H_k(t)^{\frac{1}{m+3}} \leqslant C\Big(H_k(0)^{\frac{1}{m+3}} + t^{\frac{k+3}{m+3}} \sup_{s\in[0,t]} \|E(s)\|_{L^{k+3}}^{\frac{k+3}{m+3}} + 1\Big).$$

Therefore, since $k < m_0$ because of the second condition on γ, we get

$$H_k(t)^{\frac{1}{m+3}} \leqslant C\Big(1 + t^{\frac{k+3}{m+3}} \sup_{s\in[0,t]} \|E(s)\|_{L^{k+3}}^{\frac{k+3}{m+3}}\Big). \tag{57}$$

Next, combining Propositions 2.1 and 2.2 we have $\|E(s)\|_{L^{3(m+3)/(6-m)}} \leqslant C H_m(s)^{\frac{3}{m+3}}$ if $m < 6$, and $\|E(s)\|_{L^\infty} \leqslant C H_m(s)^{\frac{3}{m+3}}$ if $m > 6$. Observing that $k + 3 \leqslant 3(m+3)/(6-m)$ by the first condition of (55), this yields

$$\big\|E(s)\big\|_{L^{k+3}} \leqslant C H_m(s)^{\frac{3}{3+m}}. \tag{58}$$

Therefore, we infer from (56), (57) and (58), that

$$\big\|E(t)\big\|_{L^{m+3}} \leqslant C + C\left\|\int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}} + F_{\text{int}}|f\big)\big(t-s, X(s), V(s)\big)\, dv\, ds\right\|_{L^{m+3}}$$

$$\leqslant C + C t^\delta + C t^{1+\gamma+\delta} H_m(t)^{\frac{3(k+3)}{(m+3)^2}}. \tag{59}$$

This completes the proof of Proposition 2.14. $\quad\square$

2.5. Bound on the moments

This paragraph is devoted to the proof of the propagation of the moments, formulated in the following

Proposition 2.15. *Let* (f, ξ) *be a classical solution of* (1)–(3) *on* $[0, T]$. *Then we have for all* $0 \leqslant m < \min(m_0, 7)$

$$H_m(T) \leqslant C(1+T)^c, \tag{60}$$

where C *only depends on the quantities* $\mathcal{H}(0)$, $\mathcal{M}(0)$, $\|f_0\|_\infty$, ξ_0, m_0, $H_m(0)$ *for* $m < m_0$; *and* c *only depends on* m.

Remark 5. The constant c can be estimated in terms of m. Indeed, it is possible to take in (60) any real number strictly bigger than

$$c_0(m) := \frac{31}{10} \min_{\gamma\in]0,1[} \max\left\{\frac{m+3}{\gamma}, \frac{5}{2}\left[\frac{1}{m-2} - \frac{2+4\gamma}{(m+3)(1+\gamma+\gamma/(1+(\gamma+1)(m+3)))}\right]^{-1}\right\}.$$

One can check that $c_0(6)$ is of order 500, while $c_0(m) \to \infty$ when $m \to 7$.

Proof. We begin with the case $\frac{16}{3} < m < \min(m_0, 7)$.

Let $t \in [0, T]$. In view of Lemma 17 and Proposition 2.12 it is enough to control the quantity

$$\left\|\int_0^t s \int_{\mathbb{R}^3} \big(|E_{\text{int}} + F_{\text{int}}|f\big)\big(t-s, X(s), V(s)\big)\, dv\, ds\right\|_{L^{m+3}}$$

in terms of $H_m^{1/(m+3)}$. Unfortunately, the bound obtained in Proposition 2.14 does not allow to conclude, since it provides an exponent $3(k+3)/(m+3)^2$, which is much too large. However this enables to show that there exist $C > 0$ and $T_1 > 0$, depending only on the usual quantities, such that $H_m(T_1) \leqslant C$ for all $0 \leqslant m < m_0$. We can assume that $T_1 < 1$ and that $T > T_1$. In order to obtain a bound for $H_m(T)$, we shall use, as in [8], two kinds of estimates: for small times we will use the estimate of Proposition 2.14; note indeed that the right-hand side is small when t is small. On the other hand for large times we will perform other estimates.

More precisely, let $0 < t_0 < \min(1, T)$ be sufficiently small, to be determined later on. Let $\alpha \in \,]0, 1/4[$.
First case: $t \in [t_0, T]$.
We have thanks to (A.14),

$$\left\| \int_{t_0}^{t} s \int_{\mathbb{R}^3} \big(|F_{\text{int}}| f \big) \big(t - s, X(s), V(s)\big) \, dv \, ds \right\|_{L^{m+3}}$$

$$\leqslant \left\| \int_{t_0}^{t} s \left\{ \int_{\mathbb{R}^3} |F_{\text{int}}|^{3/2-\alpha} \big(t - s, X(s)\big) \, dv \right\}^{\frac{2}{3-2\alpha}} \left\{ \int_{\mathbb{R}^3} f^{\frac{3-2\alpha}{1-2\alpha}} \big(t - s, X(s), V(s)\big) \, dv \right\}^{\frac{1-2\alpha}{3-2\alpha}} ds \right\|_{L^{m+3}}$$

$$\leqslant C \int_{t_0}^{t} s \left\{ \int_{\mathbb{R}^3} \chi_{R(T)}(y) \frac{1}{|y|^{3-2\alpha}} \frac{dy}{s^3} \right\}^{\frac{2}{3-2\alpha}} ds \sup_{\tau, \tau' \in [0,t]} \left\| \int_{\mathbb{R}^3} f^{\frac{3-2\alpha}{1-2\alpha}} \big(\tau, X(\tau'), V(\tau')\big) \, dv \right\|_{L^{(m+3)\frac{1-2\alpha}{3-2\alpha}}}^{\frac{1-2\alpha}{3-2\alpha}}$$

$$\leqslant C R(T)^{\frac{4\alpha}{3-2\alpha}} \|f_0\|_{L^\infty}^{\frac{2}{3-2\alpha}} \int_{t_0}^{t} s^{1-\frac{6}{3-2\alpha}} ds \sup_{\tau, \tau' \in [0,t]} \left\| \int_{\mathbb{R}^3} f\big(\tau, X(\tau'), V(\tau')\big) \, dv \right\|_{L^{(m+3)\frac{1-2\alpha}{3-2\alpha}}}^{\frac{1-2\alpha}{3-2\alpha}} .$$

We now use Proposition 2.10, more precisely estimate (31), with $a = 0$ and b such that $\frac{b+3}{3} = (m + 3)\frac{1-2\alpha}{3-2\alpha}$. Note that $b > 0$ since $\alpha \in \,]0, 1/4[$ and $m > 2$. We obtain

$$\left\| \int_{t_0}^{t} s \int_{\mathbb{R}^3} \big(|F_{\text{int}}| f \big) \big(t - s, X(s), V(s)\big) \, dv \, ds \right\|_{L^{m+3}} \leqslant C R(T)^{\frac{4\alpha}{3-2\alpha}} t_0^{-\frac{4\alpha}{3-2\alpha}} (1 + T)^{\frac{b}{m+3}} H_{\frac{m-\alpha(2m+4)}{1-\frac{2}{3}\alpha}}(t)^{\frac{1}{m+3}} .$$

We now apply the interpolation inequality

$$H_b(t) \leqslant H_2(t)^{\frac{m-b}{m-2}} H_m(t)^{\frac{b-2}{m-2}}, \quad b \in [2, m], \tag{61}$$

with $b = \frac{m-\alpha(2m+4)}{1-\frac{2}{3}\alpha}$. Note that $b \in [2, m]$ since $m > 16/3$ and $\alpha < 1/4$. Since $\sup_{t \in [0,T]} H_2(t) \leqslant C$ thanks to the conservation of energy, this yields

$$\left\| \int_{t_0}^{t} s \int_{\mathbb{R}^3} \big(|F_{\text{int}}| f \big) \big(t - s, X(s), V(s)\big) \, dv \, ds \right\|_{L^{m+3}} \leqslant C R(T)^{\frac{4\alpha}{3-2\alpha}} t_0^{-\frac{4\alpha}{3-2\alpha}} (1 + T)^{\frac{b}{m+3}} H_m(t)^{\frac{1}{m+3} - \frac{4\alpha}{(3-2\alpha)(m-2)}} . \tag{62}$$

We obtain an analogous estimate for the internal part of the electric field. Since ρ belongs to $L^\infty([0, T], L^{5/3}(\mathbb{R}^3))$, the internal part E_{int} is bounded in $L^\infty([0, T], L^{3/2-\alpha}(\mathbb{R}^3))$ for all $0 < \alpha < 1/4$ (thanks to a direct convolution inequality). So by exactly the same computations as before we find

$$\left\| \int_{t_0}^{t} s \int_{\mathbb{R}^3} \big(|E_{\text{int}}| f \big) \big(t - s, X(s), V(s)\big) \, dv \, ds \right\|_{L^{m+3}}$$

$$\leqslant \int_{t_0}^{t} s^{1-\frac{6}{3-2\alpha}} ds \sup_{\tau, \tau' \in [0,t]} \left(\|E_{\text{int}}(\tau)\|_{3/2-\alpha} \left\| \int_{\mathbb{R}^3} f\big(\tau, X(\tau'), V(\tau')\big) \, dv \right\|_{L^{(m+3)(\frac{1-2\alpha}{3-2\alpha})}}^{\frac{1-2\alpha}{3-2\alpha}} \|f_0\|_{L^\infty}^{\frac{2}{3-2\alpha}} \right)$$

$$\leqslant C R(T)^{\frac{4\alpha}{3-2\alpha}} t_0^{-\frac{4\alpha}{3-2\alpha}} (1 + T)^{\frac{b}{m+3}} H_m(t)^{\frac{1}{m+3} - \frac{4\alpha}{(3-2\alpha)(m-2)}} . \tag{63}$$

Combining (62) and (63), we are led to (for all $\alpha \in \,]0, 1/4[$ and $m \in \,]16/3, \min(m_0, 7)[$),

$$\left\| \int_{t_0}^{t} s \int_{\mathbb{R}^3} \big(|E_{\text{int}} + F_{\text{int}}| f \big) \big(t - s, X(s), V(s)\big) \, dv \, ds \right\|_{L^{m+3}}$$

$$\leqslant C R(T)^{\frac{4\alpha}{3-2\alpha}} t_0^{-\frac{4\alpha}{3-2\alpha}} (1 + T)^{\frac{b}{m+3}} H_m(t)^{\frac{1}{m+3} - \frac{4\alpha}{(3-2\alpha)(m-2)}} . \tag{64}$$

Second case: $t \in [0, t_0]$.

By Proposition 2.14, we have

$$\left\| \int_0^t s \int_{\mathbb{R}^3} \left(|E_{\text{int}} + F_{\text{int}}| f \right) \left(t - s, X(s), V(s) \right) dv \, ds \right\|_{L^{m+3}}$$

$$\leqslant C \left(1 + t^\delta + t^{1+\gamma+\delta} H_m(t)^{\frac{3(k+3)}{(m+3)^2}} \right), \tag{65}$$

where $k + 3 = (m + 3)(1 + \gamma)$ and where $\delta = \frac{\gamma}{1+(m+3)(\gamma+1)}$, with any $\gamma \in]0, 1[$ such that (55) is satisfied if $m < 6$ and such that $\gamma < \frac{m_0 - m}{m+3}$ if $m \geqslant 6$. In particular,

$$\left\| \int_0^{t_0} s \int_{\mathbb{R}^3} \left(|E_{\text{int}} + F_{\text{int}}| f \right) \left(t - s, X(s), V(s) \right) dv \, ds \right\|_{L^{m+3}}$$

$$\leqslant C \left(1 + t_0^\delta + t_0^{1+\gamma+\delta} H_m(t_0)^{\frac{3(k+3)}{(m+3)^2}} \right). \tag{66}$$

We deduce from (64), (66) and Proposition 2.12 that if $t \in [t_0, T]$,

$$\left\| E(t) \right\|_{L^{m+3}} \leqslant C(1 + T)^{21/10} + C \left\| \int_0^{t_0} s \int_{\mathbb{R}^3} \left(|E_{\text{int}} + F_{\text{int}}| f \right) \left(t - s, X(s), V(s) \right) dv \, ds \right\|_{L^{m+3}}$$

$$+ C \left\| \int_{t_0}^t s \int_{\mathbb{R}^3} \left(|E_{\text{int}} + F_{\text{int}}| f \right) \left(t - s, X(s), V(s) \right) dv \, ds \right\|_{L^{m+3}}$$

$$\leqslant C(1 + T)^{21/10} + Ct_0^\delta + Ct_0^{1+\gamma+\delta} H_m(t_0)^{\frac{3(k+3)}{(m+3)^2}}$$

$$+ CR(T)^{\frac{4\alpha}{3-2\alpha}} t_0^{-\frac{4\alpha}{3-2\alpha}} (1 + T)^{\frac{b}{m+3}} H_m(t)^{\frac{1}{m+3} - \frac{4\alpha}{(3-2\alpha)(m-2)}}.$$

On the other hand, if $t \in [0, t_0]$, (65) yields

$$\left\| E(t) \right\|_{L^{m+3}} \leqslant C(1 + T)^{21/10} + C \left\| \int_0^t s \int_{\mathbb{R}^3} \left(|E_{\text{int}} + F_{\text{int}}| f \right) \left(t - s, X(s), V(s) \right) dv \, ds \right\|_{L^{m+3}}$$

$$\leqslant C(1 + T)^{21/10} + Ct_0^\delta + Ct_0^{1+\gamma+\delta} H_m(t)^{\frac{3(k+3)}{(m+3)^2}}.$$

In all cases, for $t \in [0, T]$, we have

$$\left\| E(t) \right\|_{L^{m+3}} \leqslant C(1 + T)^{21/10} + Ct_0^\delta + Ct_0^{1+\gamma+\delta} H_m(t)^{\frac{3(k+3)}{(m+3)^2}}$$

$$+ CR(T)^{\frac{4\alpha}{3-2\alpha}} t_0^{-\frac{4\alpha}{3-2\alpha}} (1 + T)^{\frac{b}{m+3}} H_m(t)^{\frac{1}{m+3} - \frac{4\alpha}{(3-2\alpha)(m-2)}}. \tag{67}$$

We then specify our choice for t_0: we set (for example)

$$t_0^{1+\gamma+\delta} = H_m(t)^{-\frac{3(k+3)-(m+3)(1-\gamma)}{(m+3)^2}}$$

hence

$$t_0 = H_m(t)^{-\frac{3(k+3)-(m+3)(1-\gamma)}{(1+\gamma+\delta)(m+3)^2}} < 1, \tag{68}$$

because $m \geqslant 2$. On the other hand since $T \geqslant T_1$ we can assume that $t_0 < T$ otherwise the proposition is already proved. It follows that

$$t_0^{-\frac{4\alpha}{3-2\alpha}} H_m(t)^{-\frac{4\alpha}{(3-2\alpha)(m-2)}} = H_m(t)^{\frac{4\alpha}{3-2\alpha} e(m)},$$

where we denote by $e(m)$ the term appearing in the exponent above,

$$e(m) = \frac{3(k+3) - (m+3)(1-\gamma)}{(1+\delta+\gamma)(m+3)^2} - \frac{1}{m-2} = \frac{2+4\gamma}{(1+\delta+\gamma)(m+3)} - \frac{1}{m-2}.$$

We recall that $0 < \gamma < 1$ is a parameter that can be chosen as small as wanted such that (55) is satisfied if $m < 6$ and such that $\gamma < \frac{m_0 - m}{m+3}$ if $m \geqslant 6$, and that $\delta = \gamma / (1 + (\gamma + 1)(m + 3))$.

We now use the assumption $m < 7$. Then

$$\frac{2}{m+3} - \frac{1}{m-2} < 0,$$

and we can choose $\gamma > 0$ sufficiently small so that $e(m) < 0$ and condition (55) is satisfied.

Next, invoking Lemma 17 we have

$$\frac{d}{dt} \tilde{H}_m(t) \leqslant C \big(\|E(t)\|_{L^{m+3}} H_m^{-\frac{1}{m+3}} + |E(t, \xi(t))| H_m^{-\frac{1}{m+3}} \big) H_m(t).$$

Therefore using (67) and our choice (68) of t_0, we obtain, since $\frac{4\alpha}{3-2\alpha} + \frac{b}{m+3} = 1 - \frac{3}{m+3} \leqslant 1$,

$$\frac{d}{dt} \tilde{H}_m(t) \leqslant C \big((1+T)^{\frac{21}{10}} H_m^{-\frac{1}{m+3}} + H_m^{-\frac{\gamma}{m+3}} + R(T)^{\frac{4\alpha}{3-2\alpha}} (1+T)^{\frac{b}{m+3}} H_m^{\frac{4\alpha}{3-2\alpha} e(m)} + |E(t, \xi(t))| H_m^{-\frac{1}{m+3}} \big) H_m(t)$$

$$\leqslant C \big((1+T)^{\frac{21}{10}} H_m^{-\min\{\frac{\gamma}{m+3}, \frac{4\alpha}{3-2\alpha} |e(m)|\}} + |E(t, \xi(t))| H_m^{-\frac{1}{m+3}} \big) H_m(t).$$

Setting

$$a = \min \left\{ \frac{\gamma}{m+3}, \frac{4\alpha}{3-2\alpha} |e(m)| \right\} \in (0, 1),$$

we then have for all $t \in [0, T]$

$$\frac{d}{dt} \tilde{H}_m \leqslant C \big((1+T)^{\frac{21}{10}} + |E(t, \xi(t))| \big) H_m(t)^{1-a}.$$

Since $H_m(0)$ is bounded, by a Gronwall argument using Proposition 2.9, we conclude the proof of Proposition 2.15 in the case $\frac{16}{3} < m < \min(m_0, 7)$ by choosing $c = a^{-1}(\frac{21}{10} + 1)$.

Finally, by interpolation, we have for all $m \leqslant \frac{16}{3}$

$$H_m(T) \leqslant C \big(1 + H_{m'}(T) \big)$$

for some $m' \in (\frac{16}{3}, \min(m_0, 7))$, and we conclude thanks to the previous step. □

2.6. Proof of Theorem 1.1

We complete here the proof of Theorem 1.1. Let $T > 0$. We first consider a sequence of mollifiers $f_{0,\varepsilon}$ of f_0 that moreover vanish in a small ε-neighborhood of ξ_0. We then consider the unique classical solution $(f_\varepsilon(t), \xi_\varepsilon(t))$ of (1)–(2) on $[0, T]$ provided by [13]. We infer from the uniform estimates derived in Proposition 2.15 for moments of order larger than 6 and from Proposition 2.11 ((33) with $s = 0$ and $\tau = t$) that $(E_\varepsilon)_{0<\varepsilon<1}$ is uniformly bounded in $L^\infty([0, T] \times \mathbb{R}^3)$. By Ascoli's theorem there exists a subsequence such that $((\xi_{\varepsilon_n}, \eta_{\varepsilon_n}))_{n \in \mathbb{N}}$ converges to some $(\xi, \eta) \in C^1([0, T])$ as $n \to \infty$, where $\dot{\xi}(t) = \eta(t)$.

On the other hand, by usual compactness arguments using the bounds for suitable norms of f_ε and $\partial_t \rho_\varepsilon$ and on the moments, we can pass to the limit to obtain a weak solution f to (1), with $f \in C(\mathbb{R}_+, L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$, and with the same polynomial bounds on the moments. In particular (extracting again if necessary), $(E_{\varepsilon_n})_{n \in \mathbb{N}}$ converges to $E = \rho * (x \mapsto x/|x|^3)$ uniformly on $[0, T] \times \mathbb{R}^3$. Therefore we finally obtain $\dot{\eta}(t) = E(t, \xi(t)), t \in [0, T]$, and we conclude that (f, ξ, η) satisfies all the conclusions of Theorem 1.1.

Acknowledgements

Appendix A. Estimates for the flow

In the following, for a given map $F:(x,v)\in\mathbb{R}^3\times\mathbb{R}^3\mapsto F(x,v)\in\mathbb{R}^3$, we denote by $(D_xF)(x,v)$, $(D_vF)(x,v)$, the Jacobian matrices (belonging to $\mathcal{M}_3(\mathbb{R})$) with respect to the variables x and v, or, for short, D_xF or D_vF when not misleading, and for $F:(x,v)\in\mathbb{R}^3\times\mathbb{R}^3\mapsto F(x,v)\in\mathbb{R}$ we denote by D_x^2F and D_v^2F the Hessian matrices of F.

A.1. Some estimates for the flow

The purpose of this paragraph is to collect a few estimates for the flow defined in (29). We consider $t\in[0,T]$.
First we can compare $(X(s),V(s))$ with the free flow by using (25):

$$\big|X(s,x,v)-(x-vs)\big|\leqslant\frac{(1+\|f_0\|_{L^1})}{R^2}\left(\frac{s^2}{2}\right),\quad\forall(s,x,v)\in[0,t]\times\mathbb{R}^3\times\mathbb{R}^3,\tag{A.1}$$

and

$$\big|V(s,x,v)-v\big|\leqslant\frac{(1+\|f_0\|_{L^1})}{R^2}s,\quad\forall(s,x,v)\in[0,t]\times\mathbb{R}^3\times\mathbb{R}^3.\tag{A.2}$$

From now on we choose $R:=R(T)$ large with respect to T in the following way:

$$24K_0\big(1+\|f_0\|_{L^1}\big)(1+T)^3=R(T)^3,\tag{A.3}$$

where $K_0\geqslant100$. Next, differentiating (29) with respect to v we have for $s\in[0,t]$ and $(x,v)\in\mathbb{R}^3\times\mathbb{R}^3$,

$$\begin{cases}\dfrac{d}{ds}(D_vX)(s,x,v)=-(D_vV)(s,x,v),&(D_vX)(0,x,v)=0,\\[2mm]\dfrac{d}{ds}(D_vV)(s,x,v)=-D_x(E_{\text{ext}}+F_{\text{ext}})\big(t-s,X(s,x,v)\big)D_vX(s,x,v),&(D_vV)(0,x,v)=I_3.\end{cases}$$

Integrating the system above and using (26) and the definition (A.3) of $R=R(T)$ we obtain

$$\|D_vX\|_{L^\infty([0,t]\times\mathbb{R}^3\times\mathbb{R}^3)}+\|D_vV\|_{L^\infty([0,t]\times\mathbb{R}^3\times\mathbb{R}^3)}$$
$$\leqslant T\exp\left(\frac{12(1+\|f_0\|_{L^1})T}{R^3}\right)+1+\frac{12(1+\|f_0\|_{L^1})T}{R^3}T^2\exp\left(\frac{12(1+\|f_0\|_{L^1})T}{R^3}\right)$$
$$\leqslant4(1+T).\tag{A.4}$$

The same argument used for spatial derivatives ensures that

$$\|D_xX\|_{L^\infty([0,t]\times\mathbb{R}^3\times\mathbb{R}^3)}+\|D_xV\|_{L^\infty([0,t]\times\mathbb{R}^3\times\mathbb{R}^3)}\leqslant4(1+T).\tag{A.5}$$

Then, using (A.4)–(A.5) and (26), for any $s\in[0,t]$ (remembering (A.3)), we get

$$D_xX(s,x,v)=I_3+\int_0^s\int_0^\sigma D_x(E_{\text{ext}}+F_{\text{ext}})\big(t-\tau,X(\tau)\big)D_xX(\tau)\,d\tau\,d\sigma:=I_3+P_1(s,x,v),\tag{A.6}$$

where

$$\|P_1\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\leqslant24(1+T)\frac{1+\|f_0\|_{L^1}}{R^3}T^2\leqslant\frac{1}{K_0}.\tag{A.7}$$

Similarly,

$$D_vX(s,x,v)=sI_3+\int_0^s\int_0^\sigma D_x(E_{\text{ext}}+F_{\text{ext}})\big(t-\tau,X(\tau)\big)D_vX(\tau)\,d\tau\,d\sigma:=s\big(I_3+P_2(s,x,v)\big),\tag{A.8}$$

where

$$\|P_2\|_{L^\infty(\mathbb{R}^3\times\mathbb{R}^3)}\leqslant24(1+T)\frac{1+\|f_0\|_{L^1}}{R^3}T\leqslant\frac{1}{K_0}.\tag{A.9}$$

In the same way, using the definition of R we obtain

$$D_x V(s, x, v) = s P_3(s, x, v),$$

$$\|P_3\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \leqslant 24(1 + T) \frac{1 + \|f_0\|_{L^1}}{R^3} \leqslant \frac{1}{K_0} \min\left(1, \frac{1}{T^2}\right) \tag{A.10}$$

and

$$D_v V(s, x, v) = I_3 + s P_4(s, x, v), \qquad \|P_4\|_{L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)} \leqslant \frac{1}{K_0} \min\left(1, \frac{1}{T^2}\right). \tag{A.11}$$

We now fix $s \in (0, t]$ and $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and we omit the dependence w.r.t. (s, x, v) in the matrices. We deduce from (A.6) and (A.7) that (since $1/K_0 < 1$) $D_x X(s, x, v)$ is invertible and (since $K_0 > 2$),

$$\left|(D_x X)^{-1}\right| \leqslant \frac{1}{1 - |P_1|} \leqslant 2. \tag{A.12}$$

We now recall that $K_0 \geqslant 100$, so that

$$\left|\det(I_3 + P)\right| \geqslant \frac{1}{2} \quad \text{for any } |P| \leqslant \frac{10}{K_0} \left(\leqslant \frac{1}{10}\right). \tag{A.13}$$

Hence by (A.9) $D_v X(s, x, v)$ is invertible and

$$\left|\det(D_v X)\right|^{-1} = s^{-3} \left|\det(I_3 + P_2)\right|^{-1} \leqslant 8s^{-3}. \tag{A.14}$$

For later purposes (see the subsection hereafter) we next show that $D_v V - (D_x V)(D_x X)^{-1}(D_v X)$ is invertible. Combining (A.6)–(A.11), we obtain

$$D_v V - (D_x V)(D_x X)^{-1}(D_v X) = I_3 + s P_5, \quad |P_5| \leqslant \frac{1}{K_0} \min\left(1, \frac{1}{T^2}\right) + \frac{4T}{K_0} \min\left(1, \frac{1}{T^2}\right). \tag{A.15}$$

Hence $T|P_5| < 5/K_0$, so that $(D_v V - (D_x V)(D_x X)^{-1}(D_v X))(s, x, v)$ is indeed invertible and

$$\left\|\left[D_v V - (D_x V)(D_x X)^{-1}(D_v X)\right]^{-1}\right\|_{L^\infty([0,t] \times \mathbb{R}^3 \times \mathbb{R}^3)} \leqslant \frac{1}{1 - \|T P_5\|_{L^\infty([0,t] \times \mathbb{R}^3 \times \mathbb{R}^3)}} \leqslant 2. \tag{A.16}$$

On the other hand, we will also need the following estimate, which is obtained by using (A.6)–(A.8) and (A.16):

$$(D_x X)^{-1}(D_v X)\left[D_v V - (D_x V)(D_x X)^{-1}(D_v X)\right]^{-1} = s P_6, \quad |P_6| \leqslant 8, \tag{A.17}$$

so that

$$\left\|(D_x X)^{-1}(D_v X)\left[D_v V - (D_x V)(D_x X)^{-1}(D_v X)\right]^{-1}\right\|_{L^\infty([0,s] \times \mathbb{R}^3 \times \mathbb{R}^3)} \leqslant 8s. \tag{A.18}$$

Finally, writing the differential system satisfied by the second-order derivatives with respect to x, v of (X, V), and using (27), we obtain

$$\max_{j,k=1,2,3} \left(\left\|\partial^2_{(x,v)_j(x,v)_k} X\right\|_{L^\infty([0,t] \times \mathbb{R}^3 \times \mathbb{R}^3)} + \left\|\partial^2_{(x,v)_j(x,v)_k} V\right\|_{L^\infty([0,t] \times \mathbb{R}^3 \times \mathbb{R}^3)}\right) \leqslant 2. \tag{A.19}$$

Indeed, differentiating again (29) we have

$$\partial_{v_j v_k} V^{(i)}(s) = -\int_0^s \left[D_x^2 \left(E_{\text{ext}}^{(i)} + F_{\text{ext}}^{(i)}\right)\left(t - \tau, X(\tau)\right) \partial_{v_j} X(\tau)\right] \cdot \partial_{v_k} X(\tau) \, d\tau$$

$$+ \int_0^s \int_0^\tau \nabla\left(E_{\text{ext}}^{(i)} + F_{\text{ext}}^{(i)}\right)\left(t - \tau, X(\tau)\right) \cdot \partial_{v_j v_k} V(\sigma) \, d\sigma \, d\tau$$

so that

$$\big|\partial_{v_j v_k} V(s)\big| \leqslant 16(1+T)^2 T \frac{120(1+\|f_0\|_{L^1})T}{R^4} \exp\left(\frac{12(1+\|f_0\|_{L^1})T}{R^3}\right) \leqslant 1,$$

and the other terms can be treated in the same way.

Then, we observe that

$$\partial_{v_i} P_4(s) = -\frac{1}{s} \int_0^s \partial_{v_i}\big(D_x(E_{\text{ext}} + F_{\text{ext}})(t-\tau, X(\tau)) D_v X(\tau)\big) d\tau$$

so that in view of the previous estimates, we obtain

$$\max_{i=1,2,3} \|\partial_{v_i} P_4\|_{L^\infty([0,t]\times\mathbb{R}^3\times\mathbb{R}^3)} \leqslant 2.$$

On the other hand, since

$$P_2(s) = \frac{1}{s} \int_0^s \int_0^\sigma D_x(E_{\text{ext}} + F_{\text{ext}})(t-\tau, X(\tau)) D_v X(\tau) \, d\tau \, d\sigma,$$

by the same arguments, we get

$$\max_{i=1,2,3} \big(\|\partial_{x_i} P_2\|_{L^\infty([0,t]\times\mathbb{R}^3\times\mathbb{R}^3)} + \|\partial_{v_i} P_2\|_{L^\infty([0,t]\times\mathbb{R}^3\times\mathbb{R}^3)}\big) \leqslant 2. \tag{A.20}$$

Next, recall that

$$P_5 = P_4 - (D_x V)(D_x X)^{-1}(I_3 + P_2),$$

therefore

$$\begin{aligned}
\partial_{v_i} P_5 = {}& \partial_{v_i} P_4 - \partial_{v_i}(D_x V)(D_x X)^{-1}(I_3 + P_2) \\
& + (D_x V)(D_x X)^{-1}\partial_{v_i}(D_x X)(D_x X)^{-1}(I + P_2) - (D_x V)(D_x X)^{-1}\partial_{v_i} P_2,
\end{aligned}$$

and arguing in the same way for the spatial derivatives, we obtain

$$\max_{i=1,2,3} \big(\|\partial_{v_i} P_5\|_{L^\infty([0,t]\times\mathbb{R}^3\times\mathbb{R}^3)} + \|\partial_{x_i} P_5\|_{L^\infty([0,t]\times\mathbb{R}^3\times\mathbb{R}^3)}\big) \leqslant 8. \tag{A.21}$$

Also,

$$P_6 = (D_x X)^{-1}(I_3 + P_2)(I_3 + sP_5)^{-1},$$

therefore

$$\begin{aligned}
\partial_{x_i} P_6 = {}& -(D_x X)^{-1}\partial_{x_i}(D_x X)(D_x X)^{-1}(I_3 + P_2)(I_3 + sP_5)^{-1} + (D_x X)^{-1}(\partial_{x_i} P_2)(I_3 + sP_5)^{-1} \\
& - (D_x X)^{-1}(D_v X)(I_3 + sP_5)^{-1}(\partial_{x_i} P_5)(I_3 + sP_5)^{-1},
\end{aligned}$$

where we have used that $s(I_3 + P_2) = D_v X$. Therefore combining the previous estimates (in particular (A.9), (A.12), (A.16), (A.19) (A.20) and (A.21)) we obtain

$$\max_{i=1,2,3} \|\partial_{x_i} P_6\|_{L^\infty([0,t]\times\mathbb{R}^3\times\mathbb{R}^3)} \leqslant 400. \tag{A.22}$$

A.2. The flow and the Duhamel formula

In this paragraph we establish the formula (38). Let $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ be C^1. We set

$$h(s, x, v) = g\big(X(s, x, v), V(s, x, v)\big).$$

Then

$$(D_x h)(s, x, v) = (D_x g)\big(X(s, x, v), V(s, x, v)\big)(D_x X)(s, x, v) + (D_v g)\big(X(s, x, v), V(s, x, v)\big)(D_x V)(s, x, v),$$

$$(D_v h)(s, x, v) = (D_x g)\big(X(s, x, v), V(s, x, v)\big)(D_v X)(s, x, v) + (D_v g)\big(X(s, x, v), V(s, x, v)\big)(D_v V)(s, x, v).$$

We next recall that $D_x X$ and $D_v V - (D_x V)(D_x X)^{-1}(D_v X)$ are invertible and we set

$$M = M(x, v) = \big(D_v V - D_x V (D_x X)^{-1} D_v X\big)^{-1}, \qquad N = N(x, v) = (D_x X)^{-1}(D_v X) M(x, v).$$

Then, we obtain

$$(D_v g)\big(X(s, x, v), V(s, x, v)\big) = (D_v h)(x, v) M(x, v) - (D_x h)(x, v) N(x, v)$$

so that, denoting for a matrix A by $\mathrm{div}(A)$ the column vector such that $[\mathrm{div}(A)]_i = \mathrm{div}(A_i)$ where A_i is the i-th line of A,

$$(\mathrm{div}_v g)\big(X(s, x, v), V(s, x, v)\big) = \mathrm{tr}(D_v g)\big(X(s, x, v), V(s, x, v)\big) = \mathrm{tr}\big((D_v h) M\big) - \mathrm{tr}\big((D_x h) N\big)$$
$$= \mathrm{div}_v(Mh) - \mathrm{div}_v(^t M) \cdot h - \mathrm{div}_x(Nh) + \mathrm{div}_x(^t N) \cdot h.$$

Applying the formula above with $g = (E_{\mathrm{int}} + F_{\mathrm{int}}) f(t - s, \cdot, \cdot)$, we get after integrating with respect to v:

$$\rho_2(t, x) = \mathrm{div}_x \int\limits_0^t \int\limits_{\mathbb{R}^3} N(s, x, v)\big[(E_{\mathrm{int}} + F_{\mathrm{int}}) f\big](t - s, X(s, x, v), V(s, x, v)) \, dv \, ds$$

$$+ \int\limits_0^t \int\limits_{\mathbb{R}^3} \big[\mathrm{div}_v(^t M)(x, v) - \mathrm{div}_x(^t N)(x, v)\big] \cdot \big[(E_{\mathrm{int}} + F_{\mathrm{int}}) f\big](t - s, X(s, x, v), V(s, x, v)) \, dv \, ds.$$

$$\text{(A.23)}$$

References

[1] A.A. Arsenev, Global existence of a weak solution of Vlasov's system of equations, USSR Comput. Math. Math. Phys. 15 (1975) 131–143.
[2] S. Caprino, C. Marchioro, On the plasma-charge model, Kinet. Relat. Models 3 (2) (2010) 241–254.
[3] S. Caprino, C. Marchioro, E. Miot, M. Pulvirenti, On the 2D attractive plasma-charge model, Commun. Partial Differ. Equ. 37 (7) (2012) 1237–1272.
[4] F. Castella, Propagation of space moments in the Vlasov–Poisson equation and further results, Ann. Inst. Henri Poincaré 16 (4) (1999) 503–533.
[5] Z. Chen, X. Zhang, Sub-linear estimate of large velocity in a collisionless plasma, Commun. Math. Sci. 12 (2) (2014) 279–291.
[6] R.J. Di Perna, P.L. Lions, Ordinary differential equations, transport equations and Sobolev spaces, Invent. Math. 98 (1989) 511–547.
[7] I. Gasser, P.E. Jabin, B. Perthame, Regularity and propagation of moments in some nonlinear Vlasov systems, Proc. R. Soc. Edinb. A 130 (2000) 1259–1273.
[8] P.L. Lions, B. Perthame, Propagation of moments and regularity for the 3-dimensional Vlasov–Poisson system, Invent. Math. 105 (1991) 415–430.
[9] G. Loeper, Uniqueness of the solution to the Vlasov–Poisson system with bounded density, J. Math. Pures Appl. (9) 86 (1) (2006) 68–79.
[10] S. Okabe, T. Ukai, On classical solutions in the large in time of the two-dimensional Vlasov equation, Osaka J. Math. 15 (1978) 245–261.
[11] C. Pallard, Moment propagation for weak solutions to the Vlasov–Poisson system, Commun. Partial Differ. Equ. 37 (7) (2012) 1273–1285.
[12] K. Pfaffelmoser, Global existence of the Vlasov–Poisson system in three dimensions for general initial data, J. Differ. Equ. 95 (1992) 281–303.
[13] C. Marchioro, E. Miot, M. Pulvirenti, The Cauchy problem for the 3-D Vlasov–Poisson system with point charges, Arch. Ration. Mech. Anal. 201 (2011) 1–26.
[14] D. Salort, Transport equations with unbounded force fields and application to the Vlasov–Poisson equation, Math. Models Methods Appl. Sci. 19 (2) (2009) 199–228.
[15] J. Schaeffer, Global existence of smooth solutions to the Vlasov–Poisson system in three dimensions, Commun. Partial Differ. Equ. 16 (8–9) (1991) 1313–1335.
[16] S. Wollman, Global in time solution to the three-dimensional Vlasov–Poisson system, J. Math. Anal. Appl. 176 (1) (1996) 76–91.