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Superlensing using complementary media

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Abstract

This paper studies magnifying superlens using complementary media. Superlensing using complementary media was suggested by Veselago in [16] and innovated by Nicorovici et al. in [9] and Pendry in [10]. The study of this problem is difficult due to two facts. Firstly, this problem is unstable since the equations describing the phenomena have sign changing coefficients; hence the ellipticity is lost. Secondly, the phenomena associated might be localized resonant, i.e., the field explodes in some regions and remains bounded in some others. This makes the problem difficult to analyze. In this paper, we develop the technique of removing of localized singularity introduced in [6] and make use of the reflecting technique in [5] to overcome these two difficulties. More precisely, we suggest a class of lenses which has root from [9] and [14] and inspired from [6] and give a proof of superlensing for this class. To our knowledge, this is the first rigorous proof on the magnification of an *arbitrary inhomogeneous* object using complementary media.

1. Introduction

Negative index materials (NIMs) were first investigated theoretically by Veselago in [16] and innovated by Nicorovici et al. in [9] and Pendry in [10]. The existence of such materials was confirmed by Shelby et al. in [15]. NIMs have been intensively studied recently thanks to their many applications and surprising properties. One of the appealing ones is superlensing. The construction of a slab superlens using NIMs was suggested by Veselago in [16] via the ray theory. Later, this was developed by Nicorovici et al. in [9] and Pendry in [10]. In [9] the authors studied a cylindrical lens in the two dimensional quasistatic regime, and in [10] the author studied the Veselago slab in the finite frequency one. These works have been developed further, see, e.g., in [4,11–14] where cylindrical and spherical superlenses were investigated. The reader can find an interesting review and many recent results on superlensing using complementary media in [4].

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The study of superlensing has been concentrating a lot on the image of dipoles in homogeneous media, see [4, 10–14]. There are a few works devoted to the image of an object. It seems for us that [9], in which the authors gave a proof on the magnification of a constant material disk, is the only work in this direction. Even though, the methods in the papers mentioned above can be used to obtain the magnification of radial objects having constant materials in two or three dimensions, the magnification of an arbitrary inhomogeneous object is out of the scope of these methods, which are strongly based on the separation of variables. Let us mention two difficulties related to the study of this problem. Firstly, this problem is unstable since the equations describing the phenomena have sign changing coefficients; hence the ellipticity is lost. Secondly, the phenomena associated might be localized resonant, i.e., the field explodes in some regions and remains bounded in some others. This makes the problem difficult to analyze.

In this paper, we study magnifying superlens using complementary media. More precisely, given $m > 1$ the magnification, we suggest a class of lenses, which has root from [9] and [14] and inspired from [6], and show that one can magnify m times an *arbitrary inhomogeneous* object in the quasistatic and finite frequency regimes using a lens in this class. To overcome the difficulties mentioned above, we develop the technique of removing localized singularity introduced in [6], and make use of the reflecting technique in [5]. To our knowledge, the results of this paper are new even in the two dimensional quasistatic regime.

Let us describe how to magnify the region B_{r_0} for some $r_0 > 0$ in which the medium is characterized by a matrix-valued function a and a real function σ using complementary media. Here and in what follows given $r > 0$, B_r denotes the ball centered at the origin of radius r in $\mathbb{R}^d$ ($d = 2$ or 3). The assumption on the geometry of the object by all means imposes no restriction since any region can be placed in such a ball provided that the radius and the origin are appropriately chosen. We first concentrate on the quasistatic regime. The idea suggested in [9,11,14] is to put a lens in $B_{r_2} \setminus B_{r_0}$ whose medium is characterized by matrix $-b$ with $r_2^2/r_0^2 = m$. Here $b = I$, the identity matrix, in two dimensions and $b = (r_2^2/|x|^2)I$ in three dimensions.

In this paper, we slightly change the strategy discussed above and take into account the suggestion in [6]. Our lens contains two parts. The first one is given by

$$-\left(r_2^2/|x|^2\right)^{d-2}I \quad \text{in } B_{r_2} \setminus B_{r_1} \tag{1.1}$$

and the second one is the matrix

$$m^{d-2}I \quad \text{in } B_{r_1} \setminus B_{r_0}. \tag{1.2}$$

Here

$$r_1 = m^{1/4}r_0 \tag{1.3}$$

and

$$r_2 = \sqrt{m}r_1 = m^{3/4}r_0. \tag{1.4}$$

Set

$$r_3 := r_2^2/r_1 = m^{5/4}r_0. \tag{1.5}$$

It is clear that

$$m = r_2^2/r_1^2. \tag{1.6}$$

We will give some comments on this construction later.

Since materials have some loss, the correct approach is to allow some loss in the medium and investigate the limit as the loss goes to 0. With the loss, the medium is characterized by $s_\delta A$, where

$$A = \begin{cases} (r_2^2/|x|^2)^{d-2}I & \text{in } B_{r_2} \setminus B_{r_1}, \\ m^{d-2}I & \text{in } B_{r_1} \setminus B_{r_0}, \\ a & \text{in } B_{r_0}, \\ I & \text{otherwise,} \end{cases} \tag{1.7}$$

and

$$s_\delta = \begin{cases} -1 + i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ 1 & \text{otherwise.} \end{cases} \tag{1.8}$$

Physically, the imaginary part of $s_\delta A$ is the loss of the medium (more precisely the loss of the medium in $B_{r_2} \setminus B_{r_1}$). In what follows, we assume that

$$\frac{1}{\Lambda}|\xi|^2 \leqslant \langle a(x)\xi, \xi\rangle \leqslant \Lambda|\xi|^2, \quad \forall \xi \in \mathbb{R}^d, \text{ for a.e. } x \in B_{R_3}, \tag{1.9}$$

for some constant $\Lambda \geqslant 1$.

We next make some comments on the construction. We first note that $-(r_2^2/|x|^2)^{d-2}I$ in $B_{r_2} \setminus B_{r_1}$ and I in $B_{r_3} \setminus B_{r_2}$ are complementary or more precisely reflecting complementary via the Kelvin transform $F : B_{r_2} \to \mathbb{R}^d \setminus \bar{B}_{r_2}$ w.r.t. ∂B_{r_2}, i.e.,

$$F(x) = r_2^2 x/|x|^2 \quad \text{and} \quad F_*A = I \quad \text{in } B_{r_3} \setminus B_{r_2} \tag{1.10}$$

(see [5] for the definition of reflecting complementary media and their properties). Here

$$T_*M(y) = \frac{DT(x)M(x)DT^T(x)}{J(x)} \quad \text{where } x = T^{-1}(y) \text{ and } J(x) = |\det DT(x)|, \tag{1.11}$$

for a diffeomorphism T and a matrix M. Given r_1, the choice of r_2 follows from (1.6) since a superlens of m times magnification is considered as in [9,11,14] (see also (1.22) and Theorem 1). The choice of r_1 and A in $B_{r_1} \setminus B_{r_0}$ are inspired from [5,6] as follows. Let $G : \mathbb{R}^d \setminus \bar{B}_{r_3} \to B_{r_3} \setminus \{0\}$ be the Kelvin transform w.r.t. ∂B_{r_3}, i.e.,

$$G(x) = r_3^2 x/|x|^2. \tag{1.12}$$

Then $G \circ F : B_{r_1} \to B_{r_3}$ satisfies

$$G \circ F(x) = mx \quad \text{in } B_{r_1}. \tag{1.13}$$

This implies, since $A = m^{d-2}I$ in $B_{r_1} \setminus B_{r_0}$,

$$G_*F_*A = I \quad \text{in } B_{r_3} \setminus B_{r_*}. \tag{1.14}$$

Here

$$r_* := mr_0 = \sqrt{r_2 r_3} = \sqrt{r_2^3/r_1}. \tag{1.15}$$

In the last identity, we use the fact that $r_3 = r_2^2/r_1$. Using (1.6) and (1.15), we derive the formula for r_1 and r_2 as in (1.3) and (1.4). The choice of A in $B_{r_1} \setminus B_{r_0}$ follows from (1.14).

In the finite frequency regime, the medium is also characterized by $s_\delta \Sigma$ where

$$\Sigma = \begin{cases} (r_2^2/|x|^2)^d & \text{if } x \in B_{r_2} \setminus B_{r_1}, \\ m^d & \text{if } x \in B_{r_1} \setminus B_{r_0}, \\ \sigma & \text{in } B_{r_0}, \\ 1 & \text{otherwise.} \end{cases} \tag{1.16}$$

The construction of Σ for the lens is given in $B_{r_2} \setminus B_{r_0}$. This construction is based on the requirements

$$F_*\Sigma = 1 \quad \text{in } B_{r_3} \setminus B_{r_2} \quad \text{and} \quad G_*F_*\Sigma = 1 \quad \text{in } B_{r_3} \setminus B_{r_*}. \tag{1.17}$$

Here

$$T_*h(y) = \frac{h(x)}{J(x)} \quad \text{where } x = T^{-1}(y) \text{ and } J(x) = |\det DT(x)|, \tag{1.18}$$

for a diffeomorphism T and a function h. These requirements are not easy to predict but follow naturally from the study of reflecting complementary media in [5]. We will assume that

$$1/\Lambda \leqslant \sigma(x) \leqslant \Lambda, \quad \text{for a.e. } x \in B_{r_0}, \tag{1.19}$$

for some $\Lambda \geqslant 1$.

This paper deals with the bounded setting equipped the zero Dirichlet boundary condition. Let $k \geqslant 0$ and Ω be a smooth open subset of $\mathbb{R}^d$ $(d = 2, 3)$ such that $B_{r_3} \subset \Omega$. Given $f \in L^2(\Omega)$, let $u_\delta, u \in H^1_0(\Omega)$ be resp. the unique solution (the well-posedness follows from (1.23) and (1.24) below) to

$$\operatorname{div}(s_\delta A \nabla u_\delta) + s_\delta k^2 \Sigma = f \quad \text{in } \Omega, \tag{1.20}$$

and

$$\operatorname{div}(\hat{A} \nabla u) + k^2 \hat{\Sigma} u = f \quad \text{in } \Omega. \tag{1.21}$$

Here

$$\hat{A},\ \hat{\Sigma} = \begin{cases} m^{2-d} a(x/m),\ m^{-d} \sigma(x/m) & \text{in } B_{mr_0}, \\ I,\ 1 & \text{otherwise.} \end{cases} \tag{1.22}$$

When $k > 0$, we will assume in addition that, as in [5],

$$\text{(1.21) is well-posed in } H^1_0(\Omega) \tag{1.23}$$

and

$$\text{the equation } \Delta v + k^2 v = 0 \text{ in } \Omega \setminus B_{r_2} \text{ has only zero solution in } H^1_0(\Omega \setminus B_{r_2}). \tag{1.24}$$

Here is one of the two main results of this paper (the second one is Theorem 2 in Section 3).

Theorem 1. *Let $d = 2, 3$, $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset \Omega \setminus B_{r_3}$ and let $u, u_\delta \in H^1_0(\Omega)$ be the unique solutions to (1.20) and (1.21) resp. We have*

$$u_\delta \to u \quad \textit{weakly in } H^1(\Omega \setminus B_{r_3}) \textit{ as } \delta \to 0. \tag{1.25}$$

For an observer outside B_{r_3}, the object (a, σ) in B_{r_0} would act like

$$\big(m^{2-d} a(x/m), m^{-d} \sigma(x/m)\big) \quad \text{in } B_{mr_0}$$

by (1.25): one has a superlens whose magnification is m.

The key ingredient of the proof of Theorem 1 is the removing of localized singularity technique which is introduced in [6] to study cloaking using complementary media. The reflecting technique, which is presented in [5] also plays an important role in our analysis. In [7], these techniques will be developed for the context of cloaking due to anomalous localized resonance. To make use of these techniques, we require that $A = m^{d-2} I$ and $\Sigma = m^{d-2}$ in $B_{r_1} \setminus B_{r_0}$ (which is the second part of our lens construction). Indeed, in the proof we use interpolation inequalities in which the conditions $r_* \leqslant \sqrt{r_2 r_3}$, $G_* F_A = I$, $G_* F_* \Sigma = 1$ are required, see, e.g., (2.9) and (2.27). It was argued in [4] that in the two dimensional quasistatic regime, to be successfully imaged, a conducting object has to be placed in the circle B_r with $r \leqslant \sqrt{r_1^3 / r_2}$. In our notations, it is required that $r_1 \geqslant m^{1/4} r_0$; hence the layer $B_{r_1} \setminus B_{r_0}$ might be necessary. Nevertheless, we do not know how to prove or disprove the necessity of this layer.

It was shown in [5, Theorem 1] that (1.25) holds if $\|u_\delta\|_{H^1}$ remains bounded (this is equivalent to the compatibility condition on f in [5, Definition 2]). The goal of this paper is to show that (1.25) holds without the compatibility assumption. It is clear that the localized resonance appears if the compatibility does not hold. The localized resonance appearing in this situation would be anomalous one whose concept is introduced in [3] since it seems that the boundary of the resonant regions would vary with the position of the source, and their boundary do not coincide with any discontinuity in moduli. We do not verify this property in this work. We note that there are plasmonic structures for which either localized resonance or else complete resonance takes places whenever resonance appears, see [8]. The localized resonance is related to the geometry of the problem.

The lens in the region $B_{r_2} \setminus B_{r_1}$ discussed above is given by I in two dimensions and $(r_2^2 / |x|^2) I$ in three dimensions. The construction in three dimensions from [13,14] is more involved than the one in two dimensions and based on the search of isotropic radial forms. In Section 3, we will extend this construction to a class of lenses containing anisotropic ones (Theorem 2). In particular, we will point out a construction for which r_3 can be arbitrary close to mr_0 (see Remark 1). This extension is based on the study of reflecting complementary media in [5]. The concept of complementary media was originally suggested in [12,13] (see also [2,9,10,14]), where various examples were mentioned, and played an important role in the study of NIMs. In [5], the author provides a precise definition of a class of complementary media, reflecting complementary media, generated by reflections and investigates the properties of this class.

The paper is organized as follows. The proof of Theorem 1 will be given in Section 2. Theorem 2, a generalization of Theorem 1 which allows anisotropic lenses, will be given in Section 3.

2. Proof of Theorem 1

This section is devoted to the proof of Theorem 1. We first present the proof in the two dimensional quasistatic case (Section 2.1). We will profit the notational ease in this case to present clearly the ideas of the proof. The proof in the three dimensional quasistatic case is briefly sketched in Section 2.2. In Section 2.3, we consider the finite frequency case. The proof in this case is similar to the one in the quasistatic one though more involved, in particular, for low modes.

2.1. *Proof of Theorem 1 in the two dimensional quasistatic regime*

In this section, $k=0$ and $d=2$. Multiplying (1.20) by $\bar{u}_\delta$ (the conjugate of u_δ), integrating over Ω, and using the fact that $u_\delta = 0$ on $\partial\Omega$, we have

$$\int_\Omega s_\delta \langle A\nabla u_\delta, \nabla u_\delta\rangle = -\int_\Omega f\bar{u}_\delta.$$

Considering first the imaginary part and then the real part, we obtain, by (1.9),

$$\|u_\delta\|^2_{H^1(\Omega)} \leqslant \frac{C}{\delta}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}\|f\|_{L^2}. \tag{2.1}$$

Here and in what follows in the proof, C denotes a positive constant independent of δ and f.

As in [5,6], let $u_{1,\delta} \in H^1_{loc}(\mathbb{R}^d \setminus B_{r_2})$ be the reflection of u_δ through ∂B_{r_2} by F, i.e.,

$$u_{1,\delta} = u_\delta \circ F^{-1},$$

and let $u_{2,\delta} \in H^1(B_{r_3})$ be the reflection of $u_{1,\delta}$ through ∂B_{r_3} by G, i.e.,

$$u_{2,\delta} = u_{1,\delta}\circ G^{-1} = u_\delta \circ F^{-1}\circ G^{-1}.$$

We recall that F and G are given in (1.10) and (1.12). Since $G\circ F(x) = (r_3^2/r_2^2)x$, it follows from (1.11) that

$$\hat{A} = G_*F_*A \quad \text{in } B_{r_3}.$$

Applying [5, Lemma 2], we have

$$\Delta u_{1,\delta} = 0 \quad \text{in } B_{r_3}\setminus B_{r_2} \tag{2.2}$$

and

$$\operatorname{div}(\hat{A}\nabla u_{2,\delta}) = 0 \quad \text{in } B_{r_3}. \tag{2.3}$$

We derive from (2.3) that

$$\Delta u_{2,\delta} = 0 \quad \text{in } B_{r_3}\setminus B_{r_*}. \tag{2.4}$$

From the transmission conditions on ∂B_{r_2}, we have

$$u_{1,\delta} = u_\delta \quad \text{and} \quad (1-i\delta)\partial_r u_{1,\delta} = \partial_r u_\delta|_{\text{ext}} \quad \text{on } \partial B_{r_2} \tag{2.5}$$

and, from the transmission conditions on ∂B_{r_3}, we obtain

$$u_{2,\delta} = u_{1,\delta} \quad \text{and} \quad \partial_r u_{2,\delta} = (1-i\delta)\partial_r u_{1,\delta}|_{\text{int}} \quad \text{on } \partial B_{r_3}. \tag{2.6}$$

Since $\Delta u_\delta = \Delta u_{1,\delta} = 0$ in $B_{r_3} \setminus B_{r_2}$, by (2.2), and $\Delta u_{2,\delta} = 0$ in $B_{r_3} \setminus B_{r_*}$, by (2.4),[2] one can represent u_δ, $u_{1,\delta}$, and $u_{2,\delta}$ in the forms

$$u_\delta = a_0 + b_0 \ln r + \sum_{n\geqslant 1} (a_n r^n + b_n r^{-n}) e^{in\theta} \quad \text{in } B_{r_3} \setminus B_{r_2}, \tag{2.7}$$

$$u_{1,\delta} = c_0 + d_0 \ln r + \sum_{n\geqslant 1} (c_n r^n + d_n r^{-n}) e^{in\theta} \quad \text{in } B_{r_3} \setminus B_{r_2}, \tag{2.8}$$

and

$$u_{2,\delta} = e_0 + f_0 \ln r + \sum_{n\geqslant 1} (e_n r^n + f_n r^{-n}) e^{in\theta} \quad \text{in } B_{r_3} \setminus B_{r_*}, \tag{2.9}$$

for some $(a_n), (b_n), (c_n), (d_n), (e_n)$, and $(f_n) \subset \mathbb{C}$. We derive from (2.5), (2.7), and (2.8) that

$$\begin{cases} a_n r_2^n + b_n r_2^{-n} = c_n r_2^n + d_n r_2^{-n}, \\ a_n r_2^n - b_n r_2^{-n} = (1 - i\delta)(c_n r_2^n - d_n r_2^{-n}), \end{cases} \quad \text{for } n \geqslant 1,$$

and

$$\begin{cases} a_0 + b_0 \ln r_2 = c_0 + d_0 \ln r_2, \\ b_0 = (1 - i\delta) d_0. \end{cases}$$

This implies

$$\begin{cases} a_n = \dfrac{2 - i\delta}{2} c_n + \dfrac{i\delta}{2} d_n r_2^{-2n}, \\ b_n = \dfrac{i\delta}{2} c_n r_2^{2n} + \dfrac{2 - i\delta}{2} d_n, \end{cases} \quad \text{for } n \geqslant 1, \tag{2.10}$$

and

$$\begin{cases} a_0 = c_0 + i\delta d_0 \ln r_2, \\ b_0 = (1 - i\delta) d_0. \end{cases} \tag{2.11}$$

Since

$$u_\delta - u_{1,\delta} = a_0 + b_0 \ln r + \sum_{n\geqslant 1} (a_n r^n + b_n r^{-n}) e^{in\theta} - c_0 - d_0 \ln r - \sum_{n\geqslant 1} (c_n r^n + d_n r^{-n}) e^{in\theta}$$

in $B_{r_3} \setminus B_{r_2}$, it follows from (2.10) and (2.11) that, in $B_{r_3} \setminus B_{r_2}$,

$$u_\delta - u_{1,\delta} = i\delta d_0 (\ln r_2 - \ln r) - \frac{i\delta}{2} \sum_{n\geqslant 1} (c_n - d_n r_2^{-2n}) r^n e^{in\theta} + \frac{i\delta}{2} \sum_{n\geqslant 1} (c_n r_2^{2n} - d_n) r^{-n} e^{in\theta}. \tag{2.12}$$

Similarly, we derive from (2.6), (2.8), and (2.9) that

$$\begin{cases} e_n r_3^n + f_n r_3^{-n} = c_n r_3^n + d_n r_3^{-n}, \\ e_n r_3^n - f_n r_3^{-n} = (1 - i\delta)(c_n r_3^n - d_n r_3^{-n}), \end{cases}$$

and

$$\begin{cases} e_0 + f_0 \ln r_3 = c_0 + d_0 \ln r_3, \\ f_0 = (1 - i\delta) d_0. \end{cases}$$

This implies

$$\begin{cases} e_n = \dfrac{2 - i\delta}{2} c_n + \dfrac{i\delta}{2} d_n r_3^{-2n}, \\ f_n = \dfrac{i\delta}{2} c_n r_3^{2n} + \dfrac{2 - i\delta}{2} d_n, \end{cases} \quad \text{for } n \geqslant 1, \tag{2.13}$$

[2] We recall that $r_* = \sqrt{r_2 r_3}$ by (1.15).

and

$$\begin{cases} e_0 = c_0 + i\delta d_0 \ln r_3, \\ f_0 = (1 - i\delta) d_0. \end{cases} \tag{2.14}$$

Since

$$u_{1,\delta} - u_{2,\delta} = c_0 + d_0 \ln r + \sum_{n \geqslant 1} (c_n r^n + d_n r^{-n}) e^{in\theta} - e_0 - f_0 \ln r - \sum_{n \geqslant 1} (e_n r^n + f_n r^{-n}) e^{in\theta}$$

in $B_{r_3} \setminus B_{r_*}$, it follows from (2.13) and (2.14) that, in $B_{r_3} \setminus B_{r_*}$,

$$u_{1,\delta} - u_{2,\delta} = -i\delta d_0 (\ln r_3 - \ln r) + \frac{i\delta}{2} \sum_{n \geqslant 1} (c_n - d_n r_3^{-2n}) r^n e^{in\theta} - \frac{i\delta}{2} \sum_{n \geqslant 1} (c_n r_3^{2n} - d_n) r^{-n} e^{in\theta}. \tag{2.15}$$

A combination of (2.12) and (2.15) yields, in $B_{r_3} \setminus B_{r_*}$,

$$u_\delta - u_{2,\delta} = i\delta d_0 (\ln r_2 - \ln r_3) + \frac{i\delta}{2} \sum_{n \geqslant 1} c_n (r_2^{2n} - r_3^{2n}) r^{-n} e^{in\theta} + \frac{i\delta}{2} \sum_{n \geqslant 1} d_n (r_2^{-2n} - r_3^{-2n}) r^n e^{in\theta}. \tag{2.16}$$

We now use the removing of localized singularity technique introduced in [6]. Set

$$U_\delta = \begin{cases} u_\delta - \hat{u}_\delta & \text{if } x \in \Omega \setminus B_{r_*}, \\ u_{2,\delta} & \text{if } x \in B_{r_*}, \end{cases}$$

where

$$\hat{u}_\delta = i\delta d_0 (\ln r_2 - \ln r_3) + \frac{i\delta}{2} \sum_{n \geqslant 1} (c_n r_2^{2n} - c_n r_3^{2n}) r^{-n} e^{in\theta}, \quad \text{for } |x| \geqslant r_*. \tag{2.17}$$

As in [6], we remove $\hat{u}_\delta$ from u_δ in $\Omega \setminus B_{r_*}$. The function $\hat{u}_\delta$ contains very high modes and creates a trouble for estimating $u_\delta - u_{2,\delta}$ on ∂B_{r_*} (to obtain an estimate for u_δ). However this term can be negligible for large $|x|$ since r^{-n} is small for large r and large n; hence $u_\delta - \hat{u}_\delta$ well approximates u_δ for $|x|$ large enough. This is the spirit of the removing of regularized singularity technique.

We next estimate

$$[U_\delta] \quad \text{and} \quad [\hat{A} \nabla U_\delta \cdot x/|x|] \quad \text{on } \partial B_{r_*}.$$

Here and in what follows $[U]$ and $[\hat{A} \nabla U_\delta \cdot x/|x|]$ denote the jumps of U_δ and $\hat{A} \nabla U_\delta \cdot x/|x|$ on ∂B_{r_*}.

From (2.16) and (2.17), we have

$$[U_\delta] = \frac{i\delta}{2} \sum_{n \geqslant 1} d_n (r_2^{-2n} - r_3^{-2n}) r_*^n e^{in\theta} \quad \text{on } \partial B_{r_*}. \tag{2.18}$$

This implies

$$\| [U_\delta] \|_{H^{1/2}(\partial B_{r_*})}^2 \leqslant C\delta^2 \sum_{n \geqslant 1} n |d_n|^2 r_2^{-4n} r_*^{2n}. \tag{2.19}$$

Since, by (2.1),

$$\| u_{1,\delta} \|_{H^1(B_{r_3} \setminus B_{r_2})}^2 \leqslant \frac{C}{\delta} \| f \|_{L^2} \| u_\delta \|_{L^2(\Omega \setminus B_{r_3})}, \tag{2.20}$$

and $\Delta u_{1,\delta} = 0$ in $B_{r_3} \setminus B_{r_2}$, it follows that

$$\| u_{1,\delta} \|_{H^{1/2}(\partial B_{r_2})}^2 + \| \partial_r u_{1,\delta} \|_{H^{-1/2}(\partial B_{r_2})}^2 \leqslant \frac{C}{\delta} \| f \|_{L^2} \| u_\delta \|_{L^2(\Omega \setminus B_{r_3})} \tag{2.21}$$

and

$$\| u_{1,\delta} \|_{H^{1/2}(\partial B_{r_3})}^2 + \| \partial_r u_{1,\delta} \|_{H^{-1/2}(\partial B_{r_3})}^2 \leqslant \frac{C}{\delta} \| f \|_{L^2} \| u_\delta \|_{L^2(\Omega \setminus B_{r_3})}. \tag{2.22}$$

A combination of (2.8) and (2.21) yields

$$\sum_{n\geqslant 0}(n+1)\big(|c_n|^2 r_2^{2n} + |d_n|^2 r_2^{-2n}\big) \leqslant \frac{C}{\delta}\|f\|_{L^2}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}, \tag{2.23}$$

and a combination of (2.8) and (2.22) implies

$$\sum_{n\geqslant 0}(n+1)\big(|c_n|^2 r_3^{2n} + |d_n|^2 r_3^{-2n}\big) \leqslant \frac{C}{\delta}\|f\|_{L^2}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}. \tag{2.24}$$

Similarly,

$$\sum_{n\geqslant 0}(n+1)\big(|a_n|^2 r_3^{2n} + |b_n|^2 r_3^{-2n}\big) \leqslant \frac{C}{\delta}\|f\|_{L^2}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}. \tag{2.25}$$

We derive from (2.10), (2.11), (2.24), and (2.25) that

$$\sum_{n\geqslant 0}(n+1)\big(|c_n|^2 r_3^{2n} + \delta^2|d_n|^2 r_3^{2n} r_2^{-4n}\big) \leqslant \frac{C}{\delta}\|f\|_{L^2}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}. \tag{2.26}$$

Since $r_* = \sqrt{r_2 r_3}$, by the Hölder inequality, we have

$$\delta^2\sum_{n\geqslant 1} n|d_n|^2 r_2^{-4n} r_*^{2n} \leqslant \delta\bigg(\delta^2\sum_{n\geqslant 1} n|d_n|^2 r_2^{-4n} r_3^{2n}\bigg)^{1/2}\bigg(\sum_{n\geqslant 1} n|d_n|^2 r_2^{-2n}\bigg)^{1/2}. \tag{2.27}$$

A combination of (2.19), (2.23), (2.26), and (2.27) yields

$$\big\|[U_\delta]\big\|^2_{H^{1/2}(\partial B_{r_*})} \leqslant C\|f\|_{L^2}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}. \tag{2.28}$$

Similarly, we have

$$\big\|\big[\hat{A}\nabla U_\delta\cdot x/|x|\big]\big\|^2_{H^{-1/2}(\partial B_{r_*})} \leqslant C\|f\|_{L^2}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}. \tag{2.29}$$

On the other hand, from (2.17), we have

$$\|\hat{u}_\delta\|^2_{H^1(\Omega\setminus B_{r_3})} \leqslant C\bigg(\delta^2 d_0^2 + \delta^2\sum_{n\geqslant 1} n|c_n|^2 r_3^{2n}\bigg). \tag{2.30}$$

We derive from (2.23), (2.24), and (2.30) that

$$\|\hat{u}_\delta\|^2_{H^1(\Omega\setminus B_{r_3})} \leqslant C\delta\|f\|_{L^2}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}, \tag{2.31}$$

which implies, since $U_\delta = u_\delta - \hat{u}_\delta$ in $\Omega\setminus B_{r_3}$,

$$\|\hat{u}_\delta\|_{H^1(\Omega\setminus B_{r_3})} \leqslant C\delta^{1/2}\big(\|f\|_{L^2} + \|U_\delta\|_{L^2(\Omega\setminus B_{r_3})}\big). \tag{2.32}$$

It follows from (2.28), (2.29), and (2.31) that

$$\big\|[U_\delta]\big\|^2_{H^{1/2}(\partial B_{r_*})} + \big\|\big[\hat{A}\nabla U_\delta\cdot x/|x|\big]\big\|^2_{H^{-1/2}(\partial B_{r_*})} \leqslant C\big(\|f\|_{L^2}\|U_\delta\|_{L^2(\Omega\setminus B_{r_3})} + \delta^{1/2}\|f\|^2_{L^2}\big). \tag{2.33}$$

Since $\operatorname{div}(\hat{A}U_\delta) = f$ in $\Omega\setminus\partial B_{r_*}$, $U_\delta\in H^1(\Omega\setminus\partial B_{r_*})$, and $U_\delta = -\hat{u}_\delta$ on $\partial\Omega$, we derive from (2.32) and (2.33) that

$$\|U_\delta\|_{H^1(\Omega\setminus B_{r_*})} + \|U_\delta\|_{H^1(B_{r_*})} \leqslant C\|f\|_{L^2}. \tag{2.34}$$

A combination of (2.32) and (2.34) yields

$$\|\hat{u}_\delta\|_{H^1(\Omega\setminus B_{r_3})} \to 0 \quad \text{as } \delta\to 0. \tag{2.35}$$

We claim that

$$[U_\delta] \to 0 \quad \text{weakly in } H^{1/2}(\partial B_{r_*}) \quad \text{and} \quad \big[\hat{A}\nabla U_\delta \cdot x/|x|\big] \to 0 \quad \text{weakly in } H^{-1/2}(\partial B_{r_*}). \tag{2.36}$$

Assuming (2.36) holds, we have

$$U_\delta \to U_0 \quad \text{weakly in } H^1(\Omega \setminus B_{r_3})$$

where $U_0 \in H_0^1(\Omega)$ (by (2.35)) is the unique solution to the equation

$$\operatorname{div}(\hat{A}\nabla U_0) = f \quad \text{in } \Omega.$$

The conclusion now follows from (2.32).

It remains to prove (2.36). We only prove that

$$[U_\delta] \to 0 \quad \text{weakly in } H^{1/2}(\partial B_{r_*}),$$

the proof of the statement

$$\big[\hat{A}\nabla U_\delta \cdot x/|x|\big] \to 0 \quad \text{weakly in } H^{-1/2}(\partial B_{r_*})$$

follows similarly. Indeed, since $\|U_\delta\|_{H^1(\Omega\setminus B_{r_3})} \leqslant C\|f\|_{L^2}$, it follows from (2.32) that

$$\sum_{n\geqslant 0}(n+1)\big(|a_n|^2 r_3^{2n} + |b_n|^2 r_3^{-2n}\big) \leqslant C\|f\|_{L^2}.$$

We derive from (2.10) and (2.11) that

$$|d_n| \leqslant C(n)\|f\|_{L^2},$$

for some $C(n)$ depending only on n, r_2, and r_3. Since

$$\big\|[U_\delta]\big\|_{H^{1/2}(\partial B_{r_*})} \leqslant C\|f\|_{L^2},$$

by (2.19), (2.31), and (2.34), it follows from (2.18) that

$$[U_\delta] \to 0 \quad \text{weakly in } H^{1/2}(\partial B_{r_*}).$$

The proof is complete. □

2.2. *Proof of Theorem 1 in the three dimensional quasistatic regime*

The proof in the three dimensional quasistatic case follows similarly as the one in two dimensions. We also have $\Delta u_\delta = \Delta u_{1,\delta} = 0$ in $B_{r_3} \setminus B_{r_2}$, and $\Delta u_{2,\delta} = 0$ in $B_{r_3} \setminus B_{r_*}$. Hence u_δ, $u_{1,\delta}$, and $u_{2,\delta}$ can be written in the forms

$$u_\delta(x) = \sum_{n=0}^{\infty}\sum_{l=-n}^{n}\big(a_{n,l}|x|^n + b_{n,l}|x|^{-n}\big)Y_n^l\big(x/|x|\big) \quad \text{in } B_{r_3} \setminus B_{r_2}, \tag{2.37}$$

$$u_{1,\delta}(x) = \sum_{n=0}^{\infty}\sum_{l=-n}^{n}\big(c_{n,l}|x|^n + d_{n,l}|x|^{-n}\big)Y_n^l\big(x/|x|\big) \quad \text{in } B_{r_3} \setminus B_{r_2}, \tag{2.38}$$

and

$$u_{2,\delta}(x) = \sum_{n=0}^{\infty}\sum_{l=-n}^{n}\big(e_{n,l}|x|^n + f_{n,l}|x|^{-n}\big)Y_n^l\big(x/|x|\big) \quad \text{in } B_{r_3} \setminus B_{r_*}, \tag{2.39}$$

for some $(a_{n,l})$, $(b_{n,l})$, $(c_{n,l})$, $(d_{n,l})$, $(e_{n,l})$, and $(f_{n,l}) \subset \mathbb{C}$. Here Y_n^l is the spherical harmonic function of degree n and of order l. The details are left to the reader. □

2.3. *Proof of Theorem 1 in the finite frequency regime*

The proof in this case is similar to the one in the quasistatic case though it is more complicated. We will present necessary modifications in the two dimensional case. The three dimensional case follows similarly. For notational ease, we will assume $k = 1$.

Let $d = 2$ and $k = 1$. Using (1.23) and (1.24) and applying the same method used in the proof of [5, Lemma 1], we obtain, for small δ,

$$\|u_\delta\|^2_{H^1(\Omega)} \leqslant C\left(\frac{1}{\delta}\left|\int_\Omega f\bar{u}_\delta\right| + \|f\|^2_{L^2}\right).$$

This implies

$$\|u_\delta\|^2_{H^1(\Omega)} \leqslant C\left(\frac{1}{\delta}\|f\|_{L^2}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})} + \|f\|^2_{L^2}\right). \tag{2.40}$$

We have

$$\Delta u_\delta + k^2 u_\delta = \Delta u_{1,\delta} + k^2 u_{1,\delta} = 0 \quad \text{in } B_{r_3}\setminus B_{r_2} \quad \text{and} \quad \Delta u_{2,\delta} + k^2 u_{2,\delta} = 0 \quad \text{in } B_{r_3}\setminus B_{r_*} \tag{2.41}$$

by (1.10), (1.14), and (1.17). From (2.41), one can represent u_δ, $u_{1,\delta}$, and $u_{2,\delta}$ in the forms

$$u_\delta = \sum_{n=0}^{\infty}\left[a_n \hat{J}_n\big(|x|\big) + b_n \hat{Y}_n\big(|x|\big)\right]e^{in\theta} \quad \text{in } B_{r_3}\setminus B_{r_2}, \tag{2.42}$$

$$u_{1,\delta} = \sum_{n=0}^{\infty}\left[c_n \hat{J}_n\big(|x|\big) + d_n \hat{Y}_n\big(|x|\big)\right]e^{in\theta} \quad \text{in } B_{r_3}\setminus B_{r_2}, \tag{2.43}$$

and

$$u_{2,\delta} = \sum_{n=0}^{\infty}\left[e_n \hat{J}_n\big(|x|\big) + f_n \hat{Y}_n\big(|x|\big)\right]e^{in\theta} \quad \text{in } B_{r_3}\setminus B_{r_*}, \tag{2.44}$$

for some $(a_n), (b_n), (c_n), (d_n), (e_n)$, and $(f_n) \subset \mathbb{C}$. Here

$$\hat{J}_n(r) = 2^n n! J_n(r) \quad \text{and} \quad \hat{Y}_n(r) = \frac{\pi i}{2^n(n-1)!}Y_n(r),$$

where J_n and Y_n are the Bessel and Neumann functions of order n. It follows from [1, (3.57) and (3.58)] that

$$\hat{J}_n(t) = t^n\left[1 + o(1)\right] \tag{2.45}$$

and

$$\hat{Y}_n(t) = t^{-n}\left[1 + o(1)\right], \tag{2.46}$$

as $n \to +\infty$. Similar to (2.10), we have

$$\begin{cases} a_n = c_n + i\delta c_n AC_n + i\delta d_n AD_n, \\ b_n = i\delta c_n BC_n + d_n + i\delta d_n BD_n, \end{cases} \quad \text{for } n \geqslant 0, \tag{2.47}$$

and similar to (2.13), we obtain

$$\begin{cases} e_n = c_n + i\delta c_n EC_n + i\delta d_n ED_n, \\ f_n = i\delta c_n FC_n + d_n + i\delta d_n FD_n, \end{cases} \quad \text{for } n \geqslant 0. \tag{2.48}$$

Here

$$AC_n = \frac{\hat{J}_n' \hat{Y}_n}{\hat{J}_n \hat{Y}_n' - \hat{J}_n' \hat{Y}_n}(r_2), \qquad AD_n = \frac{\hat{Y}_n \hat{Y}_n'}{\hat{J}_n \hat{Y}_n' - \hat{J}_n' \hat{Y}_n}(r_2), \tag{2.49}$$

$$BC_n = \frac{\hat{J}_n \hat{J}_n'}{\hat{Y}_n \hat{J}_n' - \hat{Y}_n' \hat{J}_n}(r_2), \qquad BD_n = \frac{\hat{Y}_n' \hat{J}_n}{\hat{Y}_n \hat{J}_n' - \hat{Y}_n' \hat{J}_n}(r_2), \tag{2.50}$$

$$EC_n = \frac{\hat{J}_n' \hat{Y}_n}{\hat{J}_n \hat{Y}_n' - \hat{J}_n' \hat{Y}_n}(r_3), \qquad ED_n = \frac{\hat{Y}_n \hat{Y}_n'}{\hat{J}_n \hat{Y}_n' - \hat{J}_n' \hat{Y}_n}(r_3),$$

and

$$FC_n = \frac{\hat{J}_n \hat{J}_n'}{\hat{Y}_n \hat{J}_n' - \hat{Y}_n' \hat{J}_n}(r_3), \qquad FD_n = \frac{\hat{Y}_n' \hat{J}_n}{\hat{Y}_n \hat{J}_n' - \hat{Y}_n' \hat{J}_n}(r_3).$$

Then, in $B_{r_3} \setminus B_{r_2}$,

$$u_\delta - u_{1,\delta} = \sum_{n \geqslant 0} i\delta(AC_n c_n + AD_n d_n)\hat{J}_n\big(|x|\big)e^{in\theta} + \sum_{n \geqslant 0} i\delta(BC_n c_n + BD_n d_n)\hat{Y}_n\big(|x|\big)e^{in\theta} \tag{2.51}$$

and, in $B_{r_3} \setminus B_{r_*}$,

$$u_{1,\delta} - u_{2,\delta} = -\sum_{n \geqslant 0} i\delta(EC_n c_n + ED_n d_n)\hat{J}_n\big(|x|\big)e^{in\theta} - \sum_{n \geqslant 0} i\delta(FC_n c_n + FD_n d_n)\hat{Y}_n\big(|x|\big)e^{in\theta}. \tag{2.52}$$

A combination of (2.51) and (2.52) yields, in $B_{r_3} \setminus B_{r_*}$,

$$\begin{aligned} u_\delta - u_{2,\delta} = {} & \sum_{n \geqslant 0} i\delta\big[c_n(AC_n - EC_n) + d_n(AD_n - ED_n)\big]\hat{J}_n\big(|x|\big)e^{in\theta} \\ & + \sum_{n \geqslant 0} i\delta\big[(BC_n - FC_n)c_n + (BD_n - FD_n)d_n\big]\hat{Y}_n\big(|x|\big)e^{in\theta}. \end{aligned} \tag{2.53}$$

We now use the removing of localized singularity technique as in the quasistatic case. Set

$$\hat{u}_\delta(x) = \sum_{n \geqslant 0} i\delta\big[(BC_n - FC_n)c_n + (BD_n - FD_n)d_n\big]\hat{Y}_n\big(|x|\big)e^{in\theta},$$

and define

$$U_\delta = \begin{cases} u_\delta - \hat{u}_\delta & \text{if } x \in \Omega \setminus B_{r_*}, \\ u_{2,\delta} & \text{if } x \in B_{r_*}. \end{cases}$$

Using (2.45) and (2.46), we have

$$AC_n = -\frac{1}{2}\big[1 + o(1)\big], \qquad AD_n = \frac{1}{2}r_2^{-2n}\big[1 + o(1)\big], \tag{2.54}$$

and

$$BC_n = \frac{1}{2}r_2^{2n}\big[1 + o(1)\big], \qquad BD_n = -\frac{1}{2}\big[1 + o(1)\big]. \tag{2.55}$$

Similarly, we obtain

$$EC_n = -\frac{1}{2}\big[1 + o(1)\big], \qquad ED_n = \frac{1}{2}r_3^{-2n}\big[1 + o(1)\big], \tag{2.56}$$

and

$$FC_n = \frac{1}{2}r_3^{2n}\big[1 + o(1)\big], \qquad FD_n = -\frac{1}{2}\big[1 + o(1)\big]. \tag{2.57}$$

Since (see, e.g., [1, (3.56)])

$$\hat{J}_n'(r)\hat{Y}_n(r) - \hat{J}_n(r)\hat{Y}_n'(r) = C_n r^{-1},$$

it follows that

$$|c_n|^2 + |d_n|^2 \leqslant C_{n,r}\Big(\big|c_n \hat{J}_n(r) + d_n \hat{Y}_n(r)\big|^2 + \big|c_n \hat{J}_n'(r) + d_n \hat{Y}_n'(r)\big|^2\Big). \tag{2.58}$$

Combining (2.45), (2.46), and (2.58), as in (2.24), we obtain

$$\sum_{n\geqslant 0}(n+1)\big(|c_n|^2 r_3^{2n} + |d_n|^2 r_3^{-2n}\big) \leqslant C\left(\frac{1}{\delta}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}\|f\|_{L^2} + \|f\|_{L^2}^2\right) \tag{2.59}$$

and

$$\sum_{n\geqslant 0}(n+1)\big(|c_n|^2 r_2^{2n} + |d_n|^2 r_2^{-2n}\big) \leqslant C\left(\frac{1}{\delta}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}\|f\|_{L^2} + \|f\|_{L^2}^2\right) \tag{2.60}$$

Similarly,

$$\sum_{n\geqslant 0}(n+1)\big(|a_n|^2 r_3^{2n} + |b_n|^2 r_3^{-2n}\big) \leqslant C\left(\frac{1}{\delta}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}\|f\|_{L^2} + \|f\|_{L^2}^2\right). \tag{2.61}$$

We derive from (2.59) and (2.61) that

$$\sum_{n\geqslant n_0}(n+1)\big(|c_n|^2 r_3^{2n} + \delta^2|d_n|^2 r_3^{2n} r_2^{-4n}\big) \leqslant C\left(\frac{1}{\delta}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})}\|f\|_{L^2} + \|f\|_{L^2}^2\right), \tag{2.62}$$

for some n_0 large enough.

As in the proof of Theorem 1 in the quasistatic case, using (2.60) and (2.62), we have

$$\big\|[U_\delta]\big\|_{H^{1/2}(\partial B_{r_*})}^2 + \big\|[\hat{A}\nabla U_\delta\cdot\eta]\big\|_{H^{-1/2}(\partial B_{r_*})}^2 \leqslant C\big(\|f\|_{L^2}\|u_\delta\|_{L^2(\Omega\setminus B_{r_3})} + \delta\|f\|_{L^2}^2\big).$$

The proof is now similar to the one in the quasistatic case. The uniqueness of the limit of U_δ follows from (1.23). The details are left to the reader. □

3. Other constructions of superlenses

The construction of the superlens given by (1.7) and (1.8) is not restricted to the Kelvin transform F w.r.t. ∂B_{r_2}. In fact, using the study of reflecting complementary media in [5], we can extend this construction further. We confine ourselves to a class of radial reflections for which the formulae for A and Σ are explicit even though general reflections as in [5] can be used.

Fix $\alpha, \beta > 1$ such that[3]

$$\alpha\beta - \alpha - \beta = 0. \tag{3.1}$$

Let $F_1 : B_{r_2}\setminus\{0\} \to \mathbb{R}^d \setminus \bar{B}_{r_2}$ and $G_1 : \mathbb{R}^d\setminus \bar{B}_{r_3} \to B_{r_3}\setminus\{0\}$ be defined as follows:

$$F_1(x) = r_2^\alpha x/|x|^\alpha \quad\text{and}\quad G_1(x) = r_3^\beta x/|x|^\beta.$$

Here, r_1, r_2, and r_3 are chosen such that

$$r_3/r_1 = r_2^\alpha/r_1^\alpha = m \quad\text{and}\quad \sqrt{r_2 r_3} = m r_0,$$

which yields

$$r_1 = r_0 m^{\frac{\alpha-1}{2\alpha}}, \qquad r_2 = r_0 m^{\frac{\alpha+1}{2\alpha}}, \quad\text{and}\quad r_3 = r_0 m^{\frac{3\alpha-1}{2\alpha}}. \tag{3.2}$$

It follows from (3.1) and (3.2) that $G_1\circ F_1 : B_{r_1} \to B_{r_3}$ satisfies

$$G_1\circ F_1(x) = mx.$$

[3] One can choose α and β such that $\alpha\beta - \alpha - \beta \geqslant 0$. However, the expression of A_1 and $\hat{A}_1$ below are more involved in this case.

Define

$$A_1,\ \Sigma_1 = \begin{cases} (F_1^{-1})_* I,\ (F_1^{-1})_* 1 & \text{in } B_{r_2} \setminus B_{r_1}, \\ (F_1^{-1})_*(G_1^{-1})_* I,\ (F_1^{-1})_*(G_1^{-1})_* I & \text{in } B_{r_1} \setminus B_{r_0}, \\ a,\ \sigma & \text{in } B_{r_0}, \\ I,\ 1 & \text{otherwise}, \end{cases} \tag{3.3}$$

and

$$\hat{A}_1,\ \hat{\Sigma}_1 = \begin{cases} I,\ 1 & \text{in } \Omega \setminus B_{mr_0}, \\ (G_1) * (F_1)_* a,\ (G_1)_*(F_1)_* 1 = m^{2-d} a(x/m),\ m^{-d}\sigma(x/m) & \text{in } B_{mr_0}. \end{cases}$$

One can verify that, in $B_{r_2} \setminus B_{r_1}$,

$$A_1,\ \Sigma_1 = \frac{r_2^\alpha}{r^\alpha}\left[\frac{1}{\alpha-1} e_r \otimes e_r + (\alpha - 1)(e_\theta \otimes e_\theta + e_\theta \otimes e_\varphi)\right],\ (\alpha-1)\frac{r_2^{3\alpha}}{r^{3\alpha}} \quad \text{if } d = 3, \tag{3.4}$$

and

$$A_1,\ \Sigma_1 = \frac{1}{\alpha-1} e_r \otimes e_r + (\alpha-1) e_\theta \otimes e_\theta,\ (\alpha-1)\frac{r_2^{2\alpha}}{r^{2\alpha}} \quad \text{if } d = 2, \tag{3.5}$$

and, in $B_{r_1} \setminus B_{r_0}$,

$$A_1,\ \Sigma_1 = m^{d-2} I,\ m^d. \tag{3.6}$$

We will assume that (1.23) holds for $(\hat{A}_1, \hat{\Sigma}_1)$ instead of $(\hat{A}, \hat{\Sigma})$ and

$$\text{equation } \Delta v + k^2 v = 0 \text{ in } \Omega \setminus B_{r_2} \text{ has only zero solution in } H_0^1(\Omega \setminus B_{r_2}). \tag{3.7}$$

The following result is a generalization of Theorem 1.

Theorem 2. *Let $d = 2, 3$, $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset \Omega \setminus B_{r_3}$ and let $u, u_\delta \in H_0^1(\Omega)$ be respectively the unique solutions to*

$$\operatorname{div}(s_\delta A_1 \nabla u_\delta) + s_0 k^2 \Sigma_1 u_\delta = f \quad \text{in } \Omega$$

and

$$\operatorname{div}(\hat{A}_1 \nabla u) + k^2 \hat{\Sigma}_1 u = f \quad \text{in } \Omega.$$

We have

$$u_\delta \to u \quad \text{weakly in } H^1(\Omega \setminus B_{r_3}) \text{ as } \delta \to 0. \tag{3.8}$$

By taking $\alpha = \beta = 2$, we obtain Theorem 1 from Theorem 2.

Remark 1. We have $\beta = \alpha/(\alpha - 1)$ by (3.1). Letting $\alpha \to 1_+$, we derive from (3.2) that

$$r_1 \to r_0 \quad \text{and} \quad r_3 \to mr_0. \tag{3.9}$$

Thus for any $\varepsilon > 0$, there exists a construction such that the magnification of m times for an object in B_{r_0} takes place for any $\operatorname{supp} f \subset \Omega \setminus B_{mr_0+\varepsilon}$.

Proof. We have

$$(F_1)_* A_1 = I \quad \text{in } B_{r_3} \setminus B_{r_2} \quad \text{and} \quad (G_1)_*(F_1)_* A_1 = I \quad \text{in } B_{r_3} \setminus B_{r_*},$$

and

$$(F_1)_* \Sigma_1 = 1 \quad \text{in } B_{r_3} \setminus B_{r_2} \quad \text{and} \quad (G_1)_*(F_1)_* \Sigma_1 = 1 \quad \text{in } B_{r_3} \setminus B_{r_*},$$

by the definition of (A_1, Σ_1). We recall that $r_* = \sqrt{r_2 r_3} = m r_0$ by (3.2). This implies, by [5, Lemma 4],

$$\Delta u_{1,\delta} + k^2 u_{1,\delta} = 0 \quad \text{in } B_{r_3} \setminus B_{r_2}$$

and

$$\Delta u_{2,\delta} + k^2 u_{2,\delta} = 0 \quad \text{in } B_{r_3} \setminus B_{r_*}.$$

Here, as in the proof of Theorem 1, we define

$$u_{1,\delta} = u \circ F_1^{-1} \quad \text{in } \mathbb{R}^d \setminus B_{r_3} \quad \text{and} \quad u_{2,\delta} = u_{1,\delta} \circ G_1^{-1} \quad \text{in } B_{r_3}.$$

Similar to (2.5) and (2.6), by [5, Lemma 4], we obtain

$$u_{1,\delta} = u_\delta|_+ \quad \text{on } \partial B_{r_2} \quad \text{and} \quad (1 - i\delta) A_1 \nabla u_{1,\delta} \cdot \eta = A_1 \nabla u_\delta \cdot \eta|_+ \quad \text{on } \partial B_{r_2},$$

and

$$u_{1,\delta} = u_{2,\delta} \quad \text{on } \partial B_{r_3} \quad \text{and} \quad (1 - i\delta) \partial_\eta u_{1,\delta}|_- = \partial_\eta u_{2,\delta} \quad \text{on } \partial B_{r_3}.$$

This proof is now similar to the one of Theorem 1. The details are left to the reader. □

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