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The Cauchy problem for the modified two-component Camassa–Holm system in critical Besov space

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Abstract

In this paper, we are concerned with the Cauchy problem for the modified two-component Camassa–Holm system in the Besov space with data having critical regularity. The key elements in our paper are the real interpolations and logarithmic interpolation among inhomogeneous Besov space and Lemma 5.2.1 of [7] which is also called Osgood Lemma and the Fatou Lemma. The new ingredient that we introduce in this paper can be seen on pages 453–457.

MSC: 35G25; 35L05; 35R25

Keywords: Cauchy problem; Modified two-component Camassa–Holm system; Critical Besov space; Osgood Lemma

1. Introduction

In this paper, we consider the Cauchy problem for the modified two-component Camassa–Holm equation (MCH2)

$$m_t + um_x + 2u_x m = -\rho\overline{\rho}_x, \quad t > 0,\ x \in \mathbf{R}, \tag{1.1}$$

$$\rho_t + (u\rho)_x = 0, \quad t > 0,\ x \in \mathbf{R}, \tag{1.2}$$

$$m(x,0) = m_0(x), \quad x \in \mathbf{R}, \tag{1.3}$$

$$\rho(x,0) = \rho_0(x), \quad x \in \mathbf{R}, \tag{1.4}$$

where $m(x,t) = u(x,t) - u_{xx}(x,t)$, $\rho(x) = (1-\partial_x^2)(\overline{\rho} - \overline{\rho}_0)$. (1.1)–(1.2) are proposed by Holm et al. in [33] to find a model that describes the motion of shallow water waves other than Camassa–Holm (CH) or two-component Camassa–Holm (CH2) and has both similar and different dynamics of singular solutions compared with CH or CH2. These equations allow a dependence on not only the pointwise density $\overline{\rho}$ but also the average density $\overline{\rho}_0$.

When the evolution of the density is ignored, i.e. $\rho = 0$, the MCH2 reduces to the CH equation

$$m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}, \tag{1.5}$$

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which models the unidirectional propagation of shallow water waves over a flat bottom, where $u(x,t)$ represents the fluid velocity at time t in the spatial x direction [6,18]. (1.5) possesses the bi-Hamiltonian structures and is completely integrable [6,8,9,25] and has attracted the attention of many researchers, e.g. [2,6,10–17,19–21,31,32,36–40]. (1.5) possesses two remarkable properties. One is that (1.5) possesses the peaked solitary waves $u(x,t) = ce^{-|x-ct|}$, $c \neq 0$ which is orbitally stable [15,16]. The other is breaking waves. More precisely, the solution remains bounded while its slope becomes unbounded in finite time [10–12]. The existence and uniqueness of the weak solutions to the Cauchy problem for (1.5) has been studied in [13,14,28,43,44]. The results of [2] and [32] implied that $s = \frac{3}{2}$ is the critical Sobolev index for the well-posedness in H^s in the sense of Hadamard. The local well-posedness of (1.5) in Besov space $B^{3/2}_{2,1}$ has been proved by Danchin [20]. Bressan and Constantin [3,4] showed that after wave breaking, the solutions can be continued uniquely as either global conservative or global dissipative solutions. Recently, the initial-boundary value problem for (1.5) has been studied by Escher and Yin [23,24]. Very recently, the new and direct proof for Mckean's theorem [41] on wave-breaking of the Camassa–Holm equation has been given by Jiang et al. [34].

With $m = u - u_{xx}$, $\rho = \gamma - \gamma_{xx}$ and $\gamma = \bar{\rho} - \bar{\rho}_0$, we can rewrite (1.1)–(1.4)

$$m_t + um_x + 2mu_x = -\rho\gamma_x, \tag{1.6}$$

$$\rho_t + (u\rho)_x = 0, \tag{1.7}$$

$$m(x,0) = m_0(x) = u_{0x} - u_{0xx}, \tag{1.8}$$

$$\rho(x,0) = \gamma_0(x) - \gamma_{0xx}. \tag{1.9}$$

Let $G(x) = \frac{1}{2}e^{-|x|}$, $x \in \mathbf{R}$, then $(1-\partial_x^2)^{-1} f = G * f$ for all $f \in L^2(\mathbf{R})$ and $g * m = u$, where $*$ denotes the spatial convolution. Defining $P_1(D) = -\partial_x(1-\partial_x^2)^{-1}$ and $P_2(D) = -(1-\partial_x^2)^{-1}$, we can rewrite (1.6)–(1.9) equivalently as follows:

$$u_t + uu_x = P_1(D)\left[u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right], \quad t > 0,\ x \in \mathbf{R}, \tag{1.10}$$

$$\gamma_t + u\gamma_x = P_2(D)\big[(u_x\gamma_x)_x + u_x\gamma\big], \quad t > 0,\ x \in \mathbf{R}, \tag{1.11}$$

$$u(x,0) = u_0(x), \quad x \in \mathbf{R} \tag{1.12}$$

$$\gamma(x,0) = \gamma_0(x), \quad x \in \mathbf{R}. \tag{1.13}$$

The mathematical properties of (1.10)–(1.13) have been investigated in many works, e.g. [27,29,30,45]. Guan and Yin [27] proved that the system (1.10)–(1.13) is locally well-posed in $H^s(\mathbf{R}) \times H^{s-1}(\mathbf{R})$ with $s > \frac{5}{2}$ and presented some blow-up results. By using Helly theorem and some a priori one-sided upper bound and higher integrability space–time estimates on the first-order derivatives of approximation solutions, Guan and Yin [30] obtained the existence of global-in-time weak solutions. Guo and Zhu [29] established sufficient conditions on the initial data to guarantee blow-up solutions. Recently, by using the transport equation theory and the inhomogeneous Besov spaces, Yan et al. [45] established the local well-posedness in $B^s_{p,r} \times B^s_{p,r}$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$, $1 \leqslant p, r \leqslant \infty$.

In this paper we will show that the system (1.10)–(1.13) is locally well-posed in $B^{3/2}_{2,1} \times B^{3/2}_{2,1}$ via the iterative method and give a blow-up criterion. Now we give the outline of the well-posedness and blow-up of the system (1.10)–(1.13) in our paper. Firstly, we construct a sequence of the approximate solution $(u^{(n)}, \gamma^{(n)})$ via the iterative method. Secondly, by using the real interpolations and logarithmic interpolation among inhomogeneous Besov space, we prove that $(u^{(n)}, \gamma^n)$ is a Cauchy sequence in $C([0,T]; B^{1/2}_{2,\infty}) \times C([0,T]; B^{1/2}_{2,\infty})$. This is the key step to prove that $(u^{(n)}, \gamma^{(n)})$ is a Cauchy sequence in $C([0,T]; B^{1/2}_{2,1}) \times C([0,T]; B^{1/2}_{2,1})$. The limit is the solution to (1.10)–(1.13). Thirdly, we use the Osgood Lemma and the logarithmic interpolation and among the Besov spaces to establish the uniqueness of the solution to the system (1.10)–(1.13). Finally, we give a criterion for the blow-up of the solution. However, we notice that we cannot obtain the solution simply extracting a convergent subsequence since $(u^{(n)}, \gamma^{(n)})$ is an iterative sequence.

For $s \in \mathbf{R}$, we introduce

$$X_s = B^s_{2,1} \times B^s_{2,1}, \qquad Y_s = B^s_{2,\infty} \times B^s_{2,\infty},$$

$$E^s_{2,1}(T) = C\big([0,T]; B^s_{2,1}\big) \cap C^1\big([0,T]; B^{s-1}_{2,1}\big),$$

$$\big\|(u,v)\big\|_{X_s} = \|u\|_{B^s_{2,1}} + \|v\|_{B^s_{2,1}}.$$

In this paper, C denotes the generic positive constant which may vary from line to line and $C(\theta) = \frac{C}{\theta(1-\theta)}$, where $\theta \in (0,1)$. In this paper, we denote by *Lip* the space of bounded functions with bounded first derivatives.

The main results of this paper are as follows:

Theorem 1.1. *Let $(u_0, \gamma_0) \in X_{3/2}$. Then the system (1.10)–(1.13) is locally well-posed. More precisely, for any $(u_0, \gamma_0) \in X_{3/2}$ and any $T > 0$, there exists a unique solution in $E^{3/2}_{2,1}(T) \times E^{3/2}_{2,1}(T)$. The solution map which maps (u_0, γ_0) to (u, γ) is Hölder continuous from $X_{3/2}$ to $E^{3/2}_{2,1}(T) \times E^{3/2}_{2,1}(T)$.*

Moreover,

$$\|u(t)\|_{B^{3/2}_{2,1}} + \|\gamma(t)\|_{B^{3/2}_{2,1}} \leqslant \frac{\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}}}{1 - C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t}$$

for $t \in [0, T]$.

Remark. We say that $X_{3/2}$ is the critical Besov space for the well-posedness of modified Camassa–Holm system based on the following two facts.

(1) The Cauchy problem for the CH equation is locally well-posed $H^s(\mathbf{R})$ with $s > 3/2$ [2,40] and is not locally well-posed in $H^s(\mathbf{R})$ with $s < 3/2$ [32].
(2) Danchin [20] proved that the Cauchy problem for the CH equation is not locally well-posed in $B^{3/2}_{2,\infty}$ in the following sense.

There exist two solutions $u, v \in L^\infty(0, T; B^{3/2}_{2,\infty})$ such that for any $\epsilon > 0$

$$\|u(0) - v(0)\|_{B^{3/2}_{2,\infty}} \leqslant \epsilon \quad \text{and} \quad \|u - v\|_{L^\infty(0,T;B^{3/2}_{2,\infty})} \geqslant 1.$$

These imply that the exponent $s = 3/2$ is a critical regularity exponent of Besov spaces $B^s_{2,r}$, $r \in [1, +\infty]$.

Theorem 1.2. *Let $(u_0, \gamma_0) \in X_{3/2}$ as in Theorem 1.1 and (u, γ) be the corresponding solution to (1.10)–(1.13). Assume that T^* is the maximal existence time. If $T^* < \infty$, then*

$$\int_0^{T^*} \left[\|u_x\|_{L^\infty} + \|\gamma_x\|_{L^\infty}\right] d\tau = +\infty.$$

Moreover, $T^\star \geqslant \frac{1}{C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})}$.

The remainder of this paper is organized as follows. In Section 2, we give some preliminaries. In Section 3, we prove the existence of the solution to the problem (1.10)–(1.13) with the initial data $(u_0, \gamma_0) \in X_{3/2}$. In Section 4, we establish the uniqueness of the solution in $X_{3/2}$. In Section 5, we obtain continuity of the solution in $C([0, T]; X_{3/2})$ with the initial data $(u_0, \gamma_0) \in X_{3/2}$. In Section 6, we prove the blow-up criterion.

2. Preliminaries

In this section, we will state some preliminaries. The proof of Lemmas 2.1, 2.3–2.5 can be seen in [5,19–22,42].

Let (χ, ϕ) be two smooth radial functions, $0 \leqslant (\chi, \phi) \leqslant 1$, such that χ is supported in the ball $B = \{\xi \in \mathbf{R}^n, |\xi| \leqslant \frac{4}{3}\}$ and ϕ is supported in the ring $C = \{\xi \in \mathbf{R}^n, \frac{3}{4} \leqslant |\xi| \leqslant \frac{8}{3}\}$. Moreover,

$$\chi(\xi) + \sum_{j=0}^{\infty} \phi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbf{R}^n$$

and

$$\operatorname{supp}\phi(2^{-j}\cdot)\cap\operatorname{supp}\phi(2^{-j'}\cdot)=\varnothing,\quad \text{if } |j-j'|\geqslant 2,$$
$$\operatorname{supp}\chi(\cdot)\cap\operatorname{supp}\phi(2^{-j}\cdot)=\varnothing,\quad \text{if } j\geqslant 1.$$

For $u\in\mathscr{S}'(\mathbf{R})$, define the nonhomogeneous dyadic block operators

$$\Delta_j u=0,\quad \text{if } j\leqslant -2,$$
$$\Delta_{-1}u=\chi(D)u=\mathscr{F}_x^{-1}\chi\mathscr{F}_x u,$$
$$\Delta_j u=\phi(2^{-j}D)=\mathscr{F}_x^{-1}\phi(2^{-j}\xi)\mathscr{F}_x u,\quad \forall j\in N,\ \text{if } j\geqslant 0.$$

Lemma 2.1 *(Littlewood–Paley decomposition).*

(i) *For* $u\in\mathscr{S}'(\mathbf{R})$,
$$u=\sum_{j=-1}^{\infty}\Delta_j u\quad \textit{converges in } \mathscr{S}'(\mathbf{R}).$$
(ii) *For* $u\in H^s(\mathbf{R})$,
$$u=\sum_{j=-1}^{\infty}\Delta_j u\quad \textit{converges in } H^s(\mathbf{R}).$$

Remark. The low frequency cut-off S_j is defined by

$$S_j u=\sum_{p=-1}^{j-1}\Delta_p u=\chi(2^{-j}D)u=\mathscr{F}_x^{-1}\chi(2^{-j}\xi)\mathscr{F}_x u,\quad \forall j\in N.$$

Obviously, $\forall u,v\in\mathscr{S}'(\mathbf{R})$

$$\Delta_i\Delta_j u\equiv 0,\quad \text{if } |i-j|\geqslant 2,$$
$$\Delta_j(S_{i-1}u\Delta_i v)\equiv 0,\quad \text{if } |i-j|\geqslant 5,$$
$$\|\Delta_j u\|_{L^p}\leqslant\|u\|_{L^p},\quad \forall u\in L^p(\mathbf{R}),$$
$$\|S_j u\|_{L^p}\leqslant C\|u\|_{L^p},\quad \forall u\in L^p(\mathbf{R})$$

where C is a positive constant independent of j.

Definition 2.2 *(Besov spaces).* Let $s\in\mathbf{R}$, $1\leqslant p\leqslant+\infty$. The nonhomogeneous Besov space $B^s_{p,r}(\mathbf{R}^)$ is defined by

$$B^s_{p,r}=B^s_{p,r}(\mathbf{R})=\{f\in\mathscr{S}'(\mathbf{R}):\ \|f\|_{B^s_{p,r}}<\infty\}$$

where $\|f\|_{B^s_{p,r}}=\|(2^{qs}\|\Delta_q f\|_{L^p})_{q\geqslant -1}\|_{l^r}$.

In particular, $B^\infty_{p,r}=\bigcap_{s\in\mathbf{R}}B^s_{p,r}$.

Lemma 2.3. *Let* $s\in\mathbf{R}$, $1\leqslant p,r,p_j,r_j\leqslant\infty$, $j=1,2$, *then*

(1) $B^s_{p,r}$ *is a Banach space and is continuously embedded in* $\mathscr{S}'(\mathbf{R})$.

(2) $B^{s_1}_{p_1,r_1}\hookrightarrow B^{s_2}_{p_2,r_2}$, *if* $p_1\leqslant p_2$ *and* $r_1\leqslant r_2$ *and* $s_2=s_1-n(\frac{1}{p_1}-\frac{1}{p_2})$,
$$B^{s_1}_{p,r_2}\hookrightarrow B^{s_2}_{p,r_1}\quad \textit{locally compact if } s_2<s_1.$$

(3) $\forall s>0$, $B^s_{p,r}\cap L^\infty$ *is a Banach algebra.* $B^s_{p,r}$ *is a Banach algebra iff* $B^s_{p,r}\hookrightarrow L^\infty$ *and iff* $s>\frac{1}{p}$ *or* ($s\geqslant\frac{1}{p}$ *and* $r=1$).

(4) (i) *For* $s>0$,

$$\|fg\|_{B^s_{p,r}} \leqslant C\big(\|f\|_{B^s_{p,r}}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{B^s_{p,r}}\big), \quad \forall f,g\in B^s_{p,r}\cap L^\infty.$$

(ii) $\forall s_1 \leqslant \frac{1}{p} < s_2$ ($s_2 \geqslant \frac{1}{p}$ *if* $r=1$) *and* $s_1+s_2>0$,

$$\|fg\|_{B^{s_1}_{p,r}} \leqslant C\|f\|_{B^{s_1}_{p,r}}\|g\|_{B^{s_2}_{p,r}}, \quad \forall f\in B^{s_1}_{p,r},\ g\in B^{s_2}_{p,r}.$$

(5) $\forall\theta\in[0,1]$ *and* $s=\theta s_1+(1-\theta)s_2$,

$$\|f\|_{B^s_{p,r}} \leqslant C\|f\|^{\theta}_{B^{s_1}_{p,r}}\|f\|^{1-\theta}_{B^{s_2}_{p,r}}, \quad \forall f\in B^{s_1}_{p,r}\cap B^{s_2}_{p,r}.$$

(6) $\forall\theta\in(0,1)$, $s_1>s_2$, $s=\theta s_1+(1-\theta)s_2$, *there exists a constant* C *such that*

$$\|u\|_{B^s_{p,1}} \leqslant \frac{C(\theta)}{s_1-s_2}\|u\|^{\theta}_{B^{s_1}_{p,\infty}}\|u\|^{1-\theta}_{B^{s_2}_{p,\infty}}, \quad \forall u\in B^{s_1}_{p,\infty}.$$

(7) *If* $(u_n)_{n\in\mathbf{N}}$ *is bounded in* $B^s_{p,r}$ *and* $u_n\to u$ *in* $\mathscr{S}'(\mathbf{R})$, *then* $u\in B^s_{p,r}$ *and*

$$\|u\|_{B^s_{p,r}} \leqslant \liminf_{n\to\infty}\|u_n\|_{B^s_{p,r}}.$$

(8) *Let* $m\in\mathbf{R}$ *and* Ψ *be an* S^m*-multiplier. Then the operator* $\Psi(D)$ *is continuous from* $B^s_{p,r}$ *into* $B^{s-m}_{p,r}$.

(9) *The multiplication is continuous from* $B^{-1/2}_{2,1}\times(B^{1/2}_{2,\infty}\cap L^\infty)$ *to* $B^{-1/2}_{2,\infty}$.

(10) *There exists a constant* $C>0$ *such that for all* $s\in\mathbf{R}$, $\epsilon>0$ *and* $1\leqslant p\leqslant\infty$,

$$\|f\|_{B^s_{p,1}} \leqslant C\frac{1+\epsilon}{\epsilon}\|f\|_{B^s_{p,\infty}}\ln\left(e+\frac{\|f\|_{B^{s+\epsilon}_{p,\infty}}}{\|f\|_{B^s_{p,\infty}}}\right), \quad \forall f\in B^{s+\epsilon}_{p,\infty}.$$

Remark.

(i) From (3) and (4) we see that $B^{1/2}_{2,1}$ is continuously embedded in $B^{1/2}_{2,\infty}\cap L^\infty$.

(ii) A special case of (6) that we shall frequently use is that for any $0<\theta<1$,

$$\|u\|_{B^{1/2}_{2,1}} \leqslant \|u\|_{B^{\frac{3}{2}-\theta}_{2,1}} \leqslant C(\theta)\|u\|^{\theta}_{B^{1/2}_{2,\infty}}\|u\|^{1-\theta}_{B^{3/2}_{2,\infty}}. \tag{2.1}$$

(iii) (8) is the lower semi-continuity of the norm of $B^s_{p,r}$. It is also often called the Fatou Lemma.

(iv) We recall that a smooth function Ψ is said to be an S^m-multiplier if $\forall\alpha\in N^n$, there exists a constant $C_\alpha>0$ s.t. $|\partial_\alpha\Psi(\xi)|\leqslant C_\alpha(1+|\xi|)^{m-|\alpha|}$ for all $\xi\in\mathbf{R}^n$.

(v) For any $m\geqslant 0$, S_j is an S^m-multiplier. That is to say, for any $m\geqslant 0$, S_j is continuous from $B^s_{p,r}$ into $B^{s+m}_{p,r}$, $P_2(D)$ is an S^{-2}-multiplier. That is, $P_2(D)$ is continuous from $B^s_{p,r}$ into $B^{s+2}_{p,r}$.

Below are some *a priori* estimates in Besov spaces of transport equation.

Lemma 2.4. *Let* $1\leqslant p,r\leqslant\infty$ *and* $s>-\min\{\frac{1}{p},1-\frac{1}{p}\}$. *Assume that* $f_0\in B^s_{p,r}$, $F\in L^1(0,T;B^s_{p,r})$ *and* $\partial_x v$ *belongs to* $L^1(0,T;B^{s-1}_{p,r})$ *if* $s>1+\frac{1}{p}$ *or to* $L^1(0,T;B^{1/p}_{p,r}\cap L^\infty)$ *otherwise. If* $f\in L^\infty(0,T;B^s_{p,r})\cap C([0,T];\mathscr{S}'(\mathbf{R}))$ *solves the following* 1*-D linear transport equation*:

$$f_t+vf_x=F, \tag{2.2}$$

$$f(x,0)=f_0, \tag{2.3}$$

then there exists a constant C *depending only on* s,p,r *such that the following statements hold*:

(1) *If* $r=1$ *or* $s\neq 1+\frac{1}{p}$,

$$\|f\|_{B^s_{p,r}} \leqslant e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}}+\int_0^t e^{-CV(\tau)}\big\|F(\tau)\big\|_{B^s_{p,r}}\,d\tau\right) \tag{2.4}$$

where

$$V(t)=\int_0^t \left\|v_x(\tau)\right\|_B \,d\tau \tag{2.5}$$

with $B=B^{1/p}_{p,r}\cap L^\infty$ *if* $s<1+\frac{1}{p}$ *and* $B=B^{s-1}_{p,r}$ *else.*

(2) *If* $s\leqslant 1+\frac{1}{p}$, $f_0'\in L^\infty$ *and* $f_x\in L^\infty((0,T)\times\mathbf{R})$ *and* $F_x\in L^1(0,T;L^\infty)$, *then*

$$\begin{aligned}&\left\|f(t)\right\|_{B^s_{p,r}}+\left\|f_x(t)\right\|_{L^\infty}\\&\quad\leqslant e^{CV(t)}\left(\|f_0\|_{B^s_{p,r}}+\left\|f_0'\right\|_{L^\infty}+\int_0^t e^{-CV(\tau)}\left[\left\|F(\tau)\right\|_{B^s_{p,r}}+\left\|F_x(\tau)\right\|_{L^\infty}\right]d\tau\right)\end{aligned}$$

where $V(t)$ *is defined by* (2.5) *with* $B=B^{1/p}_{p,r}\cap L^\infty$.

(3) *If* $v=f$, *then* $\forall s>0$, (2.4) *holds with* $V(t)$ *being as in* (2.5) *and* $B=L^\infty$.

(4) *If* $r<\infty$, *then* $f\in C([0,T];B^s_{p,r})$. *If* $r=\infty$, *then* $f\in C([0,T];B^{s'}_{p,1})$ *for all* $s'<s$.

Lemma 2.5 *(Existence and uniqueness). Let* p,r,s,f_0 *and* F *be as in the statement of Lemma 2.4. Assume that* $v\in L^\rho(0,T;B^{-M}_{\infty,\infty})$ *for some* $\rho>1$ *and* $M>0$ *and* $v_x\in L^1(0,T;B^{s-1}_{p,r})$ *if* $s>1+\frac{1}{p}$ *or* $s=1+\frac{1}{p}$ *and* $r=1$ *and* $v_x\in L^1(0,T;B^{1/p}_{p,\infty}\cap L^\infty)$ *if* $s<1+\frac{1}{p}$. *Then the problem* (2.1)–(2.2) *has a unique solution* $f\in L^\infty(0,T;B^s_{p,r})\cap(\bigcap_{s'<s}C([0,T];B^{s'}_{p,1}))$ *and the inequalities of Lemma 2.4 are true. Moreover, if* $r<\infty$, *then* $f\in C([0,T];B^s_{p,r})$.

Below we state some basic properties with respect to logarithm function that we shall use in the sequel. The proofs are easy and thus are omitted.

Lemma 2.6.

(1) *Let* $f(x)=x(1-\ln x)$, $x\in(0,1]$, *then* $f(x)$ *is a monotonic increasing function for* $x\in(0,1]$.

(2) *For* $x\in(0,1]$ *and* $\alpha>0$, *we have*

$$\ln\left(e+\frac{\alpha}{x}\right)\leqslant\ln(e+\alpha)(1-\ln x). \tag{2.6}$$

(3) *For* $x>0$ *and* $\alpha>0$, *we have that*

$$f(x)=x\ln\left(e+\frac{\alpha}{x}\right) \tag{2.7}$$

is a monotonic increasing function in $x>0$.

Lemma 2.7. *Let* ρ *be a measurable, nonnegative function,* γ *a positive, locally integrable function and* μ *a continuous, increasing function.* $a\geqslant 0$ *is a real number. Assume that* ρ *satisfies*

$$\rho(t)\leqslant a+\int_{t_0}^t \gamma(s)\mu\big(\rho(s)\big)\,ds. \tag{2.8}$$

If $a>0$, *then we have*

$$-\Omega\big(\rho(t)\big)+\Omega(a)\leqslant\int_{t_0}^t\gamma(s)\,ds, \tag{2.9}$$

where

$$\Omega(x) = \int_x^1 \frac{dr}{\mu(r)}. \tag{2.10}$$

If $a = 0$ and if μ satisfies

$$\int_0^1 \frac{dr}{\mu(r)} = +\infty, \tag{2.11}$$

then the function $\rho \equiv 0$.

Lemma 2.7 can be seen in Lemma 5.2.1 of [7] and [26]. Lemma 2.7 is also called Osgood Lemma.

Remark. In Lemma 2.7, when $0 \leqslant a, \rho \leqslant R$, where $R > 0$ is real constant and $\mu(r) = r\ln(e + \frac{C}{r})$, $r > 0$, $C > 0$ which is a real constant, we claim that

$$\frac{\rho(t)}{eR} \leqslant \left[\frac{a}{eR}\right]^{\exp(-\ln(e+\frac{C}{R})\int_{t_0}^t \gamma(\tau)\,d\tau)}. \tag{2.12}$$

Now we prove the claim. When $a = 0$, since

$$\int_0^1 \frac{dr}{r\ln(e+\frac{C}{r})} = +\infty,$$

by Lemma 2.7, we know that $\rho \equiv 0$.

Now we consider the case $a > 0$.

From (2.9), we derive that

$$\int_a^{\rho(t)} \frac{dr}{r\ln(e+\frac{C}{r})} \leqslant \int_{t_0}^t \gamma(\tau)\,d\tau. \tag{2.13}$$

Combining (2.6) with (2.13), we have that

$$\int_a^{\rho(t)} \frac{dr}{\ln(e+\frac{C}{R})r(1-\ln\frac{r}{R})} \leqslant \int_a^{\rho(t)} \frac{dr}{r\ln(e+\frac{\frac{C}{R}}{\frac{r}{R}})} = \int_a^{\rho(t)} \frac{dr}{r\ln(e+\frac{C}{r})} \leqslant \int_{t_0}^t \gamma(\tau)\,d\tau. \tag{2.14}$$

By (2.14), we derive that

$$\int_a^{\rho(t)} \frac{dr}{r(1-\ln\frac{r}{R})} \leqslant \ln\left(e+\frac{C}{R}\right)\int_{t_0}^t \gamma(\tau)\,d\tau. \tag{2.15}$$

Solving (2.15) yields

$$\frac{\rho(t)}{eR} \leqslant \left[\frac{a}{eR}\right]^{\exp(-\ln(e+\frac{C}{R})\int_{t_0}^t \gamma(\tau)\,d\tau)}. \tag{2.16}$$

We have completed the proof of the claim.

3. Existence of solution with data in $X_{3/2}$

In this section, we prove the existence of solution to (1.10)–(1.13) with data in $X_{3/2}$ with the following four steps.

3.1. Approximate solution

Let $(u_0, \gamma_0) \in X_{3/2}$, via the iterative method, we will construct a solution. Starting from $(u^{(0)}, \gamma^{(0)}) = (0, 0)$, we define inductively a sequence of smooth functions $\{(u^{(n)}, \gamma^{(n)})\}_{n\in\mathbf{N}}$ by solving the following linear transport equations:

$$[\partial_t + u^{(n)}\partial_x]u^{(n+1)} = P_1(D)\Big[(u^{(n)})^2 + \frac{1}{2}(u_x^{(n)})^2 + \frac{1}{2}(\gamma^{(n)})^2 - \frac{1}{2}(\gamma_x^{(n)})^2\Big], \tag{3.1}$$

$$[\partial_t + u^{(n)}\partial_x]\gamma^{(n+1)} = P_2(D)\big[(u_x^{(n)}\gamma_x^{(n)})_x + u_x^{(n)}\gamma^{(n)}\big], \tag{3.2}$$

$$u^{(n+1)}(x,0) = u_0^{(n+1)}(x) = S_{n+1}u_0(x), \tag{3.3}$$

$$\gamma^{(n+1)}(x,0) = \gamma_0^{(n+1)}(x) = S_{n+1}\gamma_0(x). \tag{3.4}$$

By the remark after Lemma 2.3, we have that $(S_{n+1}u_0, S_{n+1}\gamma_0) \in \bigcap_{s\in\mathbf{R}} X_s$. From Lemma 2.4, for all $n \in \mathbf{N}$, we can show by induction that the above system has a global solution $(u^{(n+1)}, \gamma^{(n+1)}) \in \bigcap_{s\in\mathbf{R}} C(\mathbf{R}^+, X_s)$.

3.2. Uniform bounds

For all $n \in \mathbf{N}$, let $H^{(n)}(t) = \|u^{(n)}(t)\|_{B^{3/2}_{2,1}} + \|\gamma^{(n)}(t)\|_{B^{3/2}_{2,1}}$ and $H(0) = \|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}}$. We claim that

$$H^{(n+1)}(t) \leqslant \exp\{CU^n(t)\}\left(H(0) + C\int_0^t \exp\{-CU^n(\tau)\}\big[H^{(n)}(\tau)\big]^2\,d\tau\right), \tag{3.5}$$

with $U^n = \int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau$.

Applying (2.4) of Lemma 2.4 to (3.1), we derive

$$\begin{aligned}\|u^{(n+1)}(t)\|_{B^{3/2}_{2,1}} &\leqslant \exp\left\{C\int_0^t \|u^{(n)}(t')\|_{B^{3/2}_{2,1}}\,dt'\right\}\|u_0\|_{B^{3/2}_{2,1}} \\ &\quad + \int_0^t \exp\left\{C\int_\tau^t \|u^n(t')\|_{B^{3/2}_{2,1}}\,dt'\right\}\|F_1(u^{(n)}, u_x^{(n)}, \gamma^{(n)}, \gamma_x^{(n)})\|_{B^{3/2}_{2,1}}\,d\tau,\end{aligned} \tag{3.6}$$

where

$$F_1(u^{(n)}, u_x^{(n)}, \gamma^{(n)}, \gamma_x^{(n)}) = P_1(D)\Big[(u^{(n)})^2 + \frac{1}{2}(u_x^{(n)})^2 + \frac{1}{2}(\gamma^{(n)})^2 - \frac{1}{2}(\gamma_x^{(n)})^2\Big].$$

Since $P_1(D)$ is continuous from $B^s_{p,r}$ into $B^{s+1}_{p,r}$ and $B^{\frac{1}{2}}_{2,1}$ is a Banach algebra, $B^{3/2}_{2,1} \hookrightarrow B^{1/2}_{2,1}$, we have

$$\begin{aligned}&\|F_1(u^{(n)}, u_x^{(n)}, \gamma^{(n)}, \gamma_x^{(n)})\|_{B^{3/2}_{2,1}} \\ &\quad\leqslant C\left\|(u^{(n)})^2 + \frac{1}{2}(u_x^{(n)})^2 + \frac{1}{2}(\gamma^{(n)})^2 - \frac{1}{2}(\gamma_x^{(n)})^2\right\|_{B^{\frac{1}{2}}_{2,1}} \\ &\quad\leqslant C\|u^{(n)}\|^2_{B^{1/2}_{2,1}} + C\|u_x^{(n)}\|^2_{B^{1/2}_{2,1}} + C\|\gamma^{(n)}\|^2_{B^{1/2}_{2,1}} + C\|\gamma_x^{(n)}\|^2_{B^{1/2}_{2,1}} \\ &\quad\leqslant C\|u^{(n)}\|^2_{B^{3/2}_{2,1}} + C\|\gamma^{(n)}\|^2_{B^{3/2}_{2,1}}.\end{aligned} \tag{3.7}$$

Similarly, applying (2.4) of Lemma 2.4 to (3.2), we have

$$\begin{aligned}\|\gamma^{(n+1)}(t)\|_{B^{3/2}_{2,1}} &\leqslant \exp\left\{C\int_0^t \|u^{(n)}(t')\|_{B^{3/2}_{2,1}}\,dt'\right\}\|\gamma_0\|_{B^{3/2}_{2,1}} \\ &\quad + \int_0^t \exp\left\{C\int_\tau^t \|u^{(n)}(t')\|_{B^{3/2}_{2,1}}\,dt'\right\}\big\|P_2(D)\big[\big(u_x^{(n)}\gamma_x^{(n)}\big)_x + u_x^{(n)}\gamma^{(n)}\big]\big\|_{B^{\frac{3}{2}}_{2,1}}\,d\tau \\ &\leqslant \exp\left\{C\int_0^t \|u^{(n)}(t')\|_{B^{3/2}_{2,1}}\,dt'\right\}\|\gamma_0\|_{B^{3/2}_{2,1}} \\ &\quad + \int_0^t \exp\left\{C\int_\tau^t \|u^{(n)}(t')\|_{B^{3/2}_{2,1}}\,dt'\right\}\|u^{(n)}\|_{B^{3/2}_{2,1}}\|\gamma^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau. \end{aligned} \tag{3.8}$$

Combining (3.6), (3.7) with (3.8), we derive (3.5). Thus we prove the claim.

Fix a $T>0$ such that

$$T \leqslant \frac{1}{4C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})} \tag{3.9}$$

and assume that $\forall t \in [0,T]$

$$\|u^{(n)}\|_{B^{3/2}_{2,1}} + \|\gamma^{(n)}\|_{B^{3/2}_{2,1}} \leqslant \frac{(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})}{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t}. \tag{3.10}$$

Recalling that $U^n = \int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau$, by using (3.10), we have

$$\begin{aligned}&\exp\{CU^n(t) - CU^n(\tau)\} \\ &= \exp\left\{C\int_\tau^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\} \leqslant \exp\left[\int_\tau^t \frac{C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})}{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t'}\,dt'\right] \\ &= \left(\frac{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})\tau}{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t}\right)^{1/2}\end{aligned} \tag{3.11}$$

and

$$\exp\{CU^n(t)\} = \exp\left\{C\int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\} \leqslant \left[\frac{1}{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t}\right]^{1/2}. \tag{3.12}$$

Inserting (3.10)–(3.12) into (3.5), we have

$$\begin{aligned}H^{n+1}(t) &\leqslant \frac{\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}}}{[1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t]^{1/2}} \\ &\quad + \frac{1}{[1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t]^{1/2}}\int_0^t \frac{C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})^2}{[1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})\tau]^{3/2}}\,d\tau \\ &\leqslant \frac{\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}}}{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t}.\end{aligned} \tag{3.13}$$

Consequently, we derive that $\{(u^{(n)},\gamma^{(n)})\}_{n\in N}$ is uniformly bounded in $C([0,T];X_{3/2})$. Noting that $B^{1/2}_{2,1}$ is a Banach algebra and $B^{3/2}_{2,1} \hookrightarrow B^{1/2}_{2,1}$, we have

$$\|u^{(n)}u_x^{(n+1)}\|_{B^{1/2}_{2,1}} \leqslant C\|u^{(n)}\|_{B^{1/2}_{2,1}}\|u_x^{(n+1)}\|_{B^{1/2}_{2,1}} \leqslant C\|u^{(n)}\|_{B^{3/2}_{2,1}}\|u^{(n+1)}\|_{B^{3/2}_{2,1}}$$
$$\leqslant C\left[\frac{(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})}{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t}\right]^2. \tag{3.14}$$

Thus, from (3.7), (3.8) and (3.14), we have

$$\|u_t^{(n+1)}\|_{B^{1/2}_{2,1}} \leqslant \|u^{(n)}u_x^{(n+1)}\|_{B^{1/2}_{2,1}} + \|F_1(u^{(n)}, u_x^{(n)}, \gamma^{(n)}, \gamma_x^{(n)})\|_{B^{1/2}_{2,1}}$$
$$\leqslant C\left[\frac{(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})}{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t}\right]^2. \tag{3.15}$$

Similarly, from (3.2), we have

$$\|\gamma_t^{(n+1)}\|_{B^{1/2}_{2,1}} = \|P_2(D)[(u_x^{(n)}\gamma_x^{(n)})_x + u_x^{(n)}\gamma^{(n)}]\|_{B^{1/2}_{2,1}} + \|u^{(n)}\partial_x\gamma^{(n+1)}\|_{B^{1/2}_{2,1}}$$
$$\leqslant \|P_1(D)(u_x^{(n)}\gamma_x^{(n)})\|_{B^{1/2}_{2,1}} + \|P_2(D)(u_x^{(n)}\gamma^{(n)})\|_{B^{1/2}_{2,1}} + C\|u^{(n)}\|_{B^{1/2}_{2,1}}\|\partial_x\gamma^{(n+1)}\|_{B^{1/2}_{2,1}}$$
$$\leqslant C\|u^{(n)}\|_{B^{3/2}_{2,1}}\|\gamma^{(n)}\|_{B^{3/2}_{2,1}}$$
$$\leqslant C\left[\frac{(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})}{1-2C(\|u_0\|_{B^{\frac{3}{2}}_{2,1}} + \|\gamma_0\|_{B^{\frac{3}{2}}_{2,1}})t}\right]^2$$

Consequently, for $\forall n \in \mathbf{N}$, we have

$$\big(u^{(n)}, \gamma^{(n)}\big) \in E^{3/2}_{2,1}(T) \times E^{3/2}_{2,1}(T). \tag{3.16}$$

Remark. For $t \in [0, T]$ and $\forall (n, k) \in \mathbf{N^2}$, from (3.13) and (3.9), we have that

$$\exp\left\{C\int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\}$$
$$\leqslant \left[\frac{1}{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t}\right]^{1/2} \leqslant \left[\frac{1}{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})T}\right]^{1/2} \leqslant 2 \tag{3.17}$$

and

$$\|(u^{(n+k)} - u^{(n)})(t)\|_{B^{3/2}_{2,\infty}} + \|(\gamma^{(n+k)} - \gamma^{(n)})(t)\|_{B^{3/2}_{2,\infty}}$$
$$\leqslant \|(u^{(n+k)} - u^{(n)})(t)\|_{B^{3/2}_{2,1}} + \|(\gamma^{(n+k)} - \gamma^{(n)})(t)\|_{B^{3/2}_{2,1}}$$
$$\leqslant \|u^{(n+k)}(t)\|_{B^{3/2}_{2,1}} + \|u^{(n)}(t)\|_{B^{3/2}_{2,1}} + \|\gamma^{(n+k)}(t)\|_{B^{3/2}_{2,1}} + \|\gamma^{(n)}(t)\|_{B^{3/2}_{2,1}}$$
$$\leqslant \frac{2(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})}{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t}$$
$$\leqslant \frac{2(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})}{1-2C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})T}$$
$$\leqslant 4\big[\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}}\big] \tag{3.18}$$

since $B^{3/2}_{2,1} \hookrightarrow B^{3/2}_{2,\infty}$.

We define

$$M = 4\big[\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}}\big]. \tag{3.19}$$

3.3. Convergence of $(u^{(n)}, \gamma^{(n)})$

Now we prove that $(u^{(n)}, \gamma^{(n)})_{n\in\mathbf{N}}$ is a Cauchy sequence in $C([0,T]; X_{1/2})$. We prove first that $(u^{(n)}, \gamma^{(n)})_{n\in\mathbf{N}}$ is a Cauchy sequence in $C([0,T]; Y_{1/2})$. For $(n,k)\in\mathbf{N}^2$, we derive that

$$\begin{aligned}
&\big(\partial_t + u^{(n+k)}\partial_x\big)\big(u^{(n+1+k)} - u^{(n+1)}\big)\\
&\quad= \big(u^{(n)} - u^{(n+k)}\big)\partial_x u^{(n+1)} + P_1(D)\big[\big(u^{(n+k)} - u^{(n)}\big)\big(u^{(n+k)} + u^{(n)}\big)\big]\\
&\qquad+ \frac{1}{2}P_1(D)\big[\big(u^{(n+k)} - u^{(n)}\big)_x\big(u^{(n+k)} + u^{(n)}\big)_x\big] + \frac{1}{2}P_1(D)\big[\big(\gamma^{(n+k)} - \gamma^{(n)}\big)\big(\gamma^{(n+k)} + \gamma^{(n)}\big)\big]\\
&\qquad+ \frac{1}{2}P_1(D)\big[\big(\gamma^{(n+k)} - \gamma^{(n)}\big)_x\big(\gamma^{(n+k)} + \gamma^{(n)}\big)_x\big],
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&\big(\partial_t + u^{(n+k)}\partial_x\big)\big(\gamma^{(n+1+k)} - \gamma^{(n+1)}\big)\\
&\quad= \big(u^{(n)} - u^{(n+k)}\big)\partial_x \gamma^{(n+1)} + P_1(D)\big[\big(u^{(n+k)} - u^{(n)}\big)_x\gamma_x^{(n+k)} + u_x^{(n)}\big(\gamma^{(n+k)} - \gamma^{(n)}\big)_x\big]\\
&\qquad+ P_2(D)\big[\big(u^{(n+k)} - u^{(n)}\big)_x\gamma^{(n+k)} + u_x^{(n)}\big(\gamma^{(n+k)} - \gamma^{(n)}\big)\big].
\end{aligned} \tag{3.21}$$

We define

$$W_{n,k}(t) = \big\|\big(u^{(n+k)} - u^{(n)}\big)(t)\big\|_{B^{1/2}_{2,\infty}} + \big\|\big(\gamma^{(n+k)} - \gamma^{(n)}\big)(t)\big\|_{B^{1/2}_{2,\infty}} \tag{3.22}$$

and

$$W_n(t) = \sup_{k\in\mathbf{N}} W_{n,k}(t), \tag{3.23}$$

as well as

$$\widetilde{W}(t) = \limsup_{n\to\infty} W_n(t). \tag{3.24}$$

We will prove

$$\widetilde{W}(t) = 0, \quad t\in[0,T] \tag{3.25}$$

and that $\lim_{n\to+\infty} W_n(t) = 0$.

By (3.24), for $\forall\epsilon>0$, we derive that $\forall n > N_\epsilon$, there exists $N_\epsilon\in\mathbf{N}$ such that

$$W_n(t) < \widetilde{W}(t) + \epsilon. \tag{3.26}$$

By using (2) and (3), (9) of Lemma 2.3 and (3.18), (3.19), we have

$$\begin{aligned}
&\big\|\big(u^{(n)} - u^{(n+k)}\big)\partial_x u^{(n+1)}(t)\big\|_{B^{1/2}_{2,\infty}}\\
&\quad\leqslant C\big\|\big(u^{(n)} - u^{(n+k)}\big)\partial_x u^{(n+1)}(t)\big\|_{B^{1/2}_{2,\infty}\cap L^\infty}\\
&\quad\leqslant C\big\|\big(u^{(n)} - u^{(n+k)}\big)(t)\big\|_{B^{1/2}_{2,\infty}\cap L^\infty}\big\|\partial_x u^{(n+1)}(t)\big\|_{B^{1/2}_{2,\infty}\cap L^\infty}\\
&\quad\leqslant C\big\|\big(u^{(n)} - u^{(n+k)}\big)(t)\big\|_{B^{1/2}_{2,1}}\big\|u^{(n+1)}(t)\big\|_{B^{3/2}_{2,\infty}\cap Lip}\\
&\quad\leqslant C\big\|\big(u^{(n)} - u^{(n+k)}\big)(t)\big\|_{B^{1/2}_{2,1}}\big\|u^{(n+1)}(t)\big\|_{B^{3/2}_{2,1}}\\
&\quad\leqslant CM\big\|\big(u^{(n)} - u^{(n+k)}\big)(t)\big\|_{B^{1/2}_{2,1}}
\end{aligned} \tag{3.27}$$

and

$$\begin{aligned}
&\big\|P_1(D)\big[\big(u^{(n+k)}-u^{(n)}\big)\big(u^{(n+k)}+u^{(n)}\big)\big](t)\big\|_{B^{1/2}_{2,\infty}}+\big\|P_1(D)\big[\big(u^{(n+k)}-u^{(n)}\big)_x\big(u^{(n+k)}+u^{(n)}\big)_x\big](t)\big\|_{B^{1/2}_{2,\infty}}\\
&\quad+\big\|P_1(D)\big[\big(\gamma^{(n+k)}-\gamma^{(n)}\big)\big(\gamma^{(n+k)}+\gamma^{(n)}\big)\big](t)\big\|_{B^{1/2}_{2,\infty}}+\big\|P_1(D)\big[\big(\gamma^{(n+k)}-\gamma^{(n)}\big)_x\big(\gamma^{(n+k)}+\gamma^{(n)}\big)_x\big]\big\|_{B^{1/2}_{2,\infty}}\\
&\leqslant C\big\|\big(u^{(n+k)}-u^{(n)}\big)(t)\big\|_{B^{-1/2}_{2,1}}\big[\big\|u^{(n+k)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty}+\big\|u^{(n)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty}\big]\\
&\quad+C\big\|\big(u^{(n+k)}-u^{(n)}\big)(t)\big\|_{B^{1/2}_{2,1}}\big[\big\|u^{(n+k)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}+\big\|u^{(n)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}\big]\\
&\quad+C\big\|\big(\gamma^{(n+k)}-\gamma^{(n)}\big)(t)\big\|_{B^{-1/2}_{2,1}}\big[\big\|\gamma^{(n+k)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty}+\big\|\gamma^{(n)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty}\big]\\
&\quad+C\big\|\big(\gamma^{(n+k)}-\gamma^{(n)}\big)(t)\big\|_{B^{1/2}_{2,1}}\big[\big\|\gamma^{(n+k)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}+\big\|\gamma^{(n)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}\big]\\
&\leqslant CM\big[\big\|\big(u^{(n+k)}-u^{(n)}\big)(t)\big\|_{B^{1/2}_{2,1}}+\big\|\big(\gamma^{(n+k)}-\gamma^{(n)}\big)(t)\big\|_{B^{1/2}_{2,1}}\big]
\end{aligned}\tag{3.28}$$

and

$$\begin{aligned}
&\big\|\big(u^{(n)}-u^{(n+k)}\big)\partial_x\gamma^{(n+1)}\big\|_{B^{1/2}_{2,\infty}}\\
&\leqslant C\big\|\big(u^{(n)}-u^{(n+k)}\big)\partial_x\gamma^{(n+1)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty}\leqslant C\big\|u^{(n)}-u^{(n+k)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty}\big\|\partial_x\gamma^{(n+1)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty}\\
&\leqslant C\big\|u^{(n)}-u^{(n+k)}\big\|_{B^{1/2}_{2,1}}\big\|\gamma^{(n+1)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}\leqslant C\big\|u^{(n)}-u^{(n+k)}\big\|_{B^{1/2}_{2,1}}\big\|\gamma^{(n+1)}\big\|_{B^{3/2}_{2,1}}\\
&\leqslant CM\big\|u^{(n)}-u^{(n+k)}\big\|_{B^{1/2}_{2,1}}
\end{aligned}\tag{3.29}$$

as well as

$$\begin{aligned}
&\big\|P_1(D)\big[\big(u^{(n+k)}-u^{(n)}\big)_x\gamma_x^{(n+k)}+u_x^{(n)}\big(\gamma^{(n+k)}-\gamma^{(n)}\big)_x\big]\big\|_{B^{1/2}_{2,\infty}}\\
&\quad+\big\|P_2(D)\big[\big(u^{(n+k)}-u^{(n)}\big)_x\gamma^{(n+k)}+u_x^{(n)}\big(\gamma^{(n+k)}-\gamma^{(n)}\big)\big]\big\|_{B^{1/2}_{2,\infty}}\\
&\leqslant C\big[\big\|u^{(n+k)}-u^{(n)}\big\|_{B^{1/2}_{2,1}}\big\|\gamma^{(n+k)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}\big]+C\big[\big\|\gamma^{(n+k)}-\gamma^{(n)}\big\|_{B^{1/2}_{2,1}}\big\|u^{(n)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}\big]\\
&\leqslant CM\big[\big\|u^{(n)}-u^{(n+k)}\big\|_{B^{1/2}_{2,1}}+\big\|\gamma^{(n+k)}-\gamma^{(n)}\big\|_{B^{1/2}_{2,1}}\big].
\end{aligned}\tag{3.30}$$

By using (i) of Lemma 2.4 and (3.27)–(3.30) as well as $B^{3/2}_{2,1}\hookrightarrow B^{3/2}_{2,\infty}\cap Lip$, for $t\in[0,T]$, we derive that

$$\begin{aligned}
&\big\|\big(u^{(n+1+k)}-u^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,\infty}}\\
&\leqslant\exp\left\{C\int_0^t\big\|u^{(n+k)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}d\tau\right\}\big\|u_0^{(n+1+k)}-u_0^{(n+1)}\big\|_{B^{1/2}_{2,\infty}}\\
&\quad+CM\int_0^t\exp\left\{C\int_\tau^t\big\|u^{(n+k)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}d\tau'\right\}\big\|\big(u^{(n+k)}-u^{(n)}\big)(\tau)\big\|_{B^{1/2}_{2,1}}d\tau\\
&\quad+CM\int_0^t\exp\left\{C\int_\tau^t\big\|u^{(n+k)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}d\tau'\right\}\big\|\big(\gamma^{(n+k)}-\gamma^{(n)}\big)(\tau)\big\|_{B^{1/2}_{2,1}}d\tau\\
&\leqslant C\big[\|u_0^{(n+k)}-u_0^{(n)}\|_{B^{1/2}_{2,\infty}}+\|\gamma_0^{(n+k)}-\gamma_0^{(n)}\|_{B^{1/2}_{2,\infty}}\big]+CM\int_0^t\big\|\big(u^{(n+k)}-u^{(n)}\big)(\tau)\big\|_{B^{1/2}_{2,1}}d\tau\\
&\quad+CM\int_0^t\big\|\big(\gamma^{(n+k)}-\gamma^{(n)}\big)(\tau)\big\|_{B^{1/2}_{2,1}}d\tau
\end{aligned}\tag{3.31}$$

and

$$
\begin{aligned}
&\big\|\big(\gamma^{(n+1+k)}-\gamma^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,\infty}}\\
&\quad\leqslant \exp\Bigg\{C\int_0^t \|u^{(n+k)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau\Bigg\}\big\|\gamma_0^{(n+1+k)}-\gamma_0^{(n+1)}\big\|_{B^{1/2}_{2,\infty}}\\
&\qquad+CM\int_0^t \exp\Bigg\{C\int_\tau^t \|u^{(n+k)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\Bigg\}\big[\|u^{(n+k)}-u^{(n)}\|_{B^{1/2}_{2,1}}+\|\gamma^{(n+k)}-\gamma^{(n)}\|_{B^{1/2}_{2,1}}\big]\,d\tau\\
&\quad\leqslant C\big[\|u_0^{(n+1+k)}-u_0^{(n+1)}\|_{B^{1/2}_{2,\infty}}+\|\gamma_0^{(n+1+k)}-\gamma_0^{(n+1)}\|_{B^{1/2}_{2,\infty}}\big]+CM\int_0^t\big\|\big(u^{(n+k)}-u^{(n)}\big)(\tau)\big\|_{B^{1/2}_{2,1}}\,d\tau\\
&\qquad+CM\int_0^t\big\|\big(\gamma^{(n+k)}-\gamma^{(n)}\big)(\tau)\big\|_{B^{1/2}_{2,1}}\,d\tau.
\end{aligned}
\tag{3.32}
$$

Combining (3.31) with (3.32), we have

$$
\begin{aligned}
&\big\|\big(u^{(n+1+k)}-u^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,\infty}}+\big\|\big(\gamma^{(n+1+k)}-\gamma^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,\infty}}\\
&\quad\leqslant C\big[\|u_0^{(n+1+k)}-u_0^{(n+1)}\|_{B^{1/2}_{2,\infty}}+\|\gamma_0^{(n+1+k)}-\gamma_0^{(n+1)}\|_{B^{1/2}_{2,\infty}}\big]\\
&\qquad+CM\int_0^t\big\|\big(u^{(n+k)}-u^{(n)}\big)(\tau)\big\|_{B^{1/2}_{2,1}}\,d\tau+CM\int_0^t\big\|\big(\gamma^{(n+k)}-\gamma^{(n)}\big)(\tau)\big\|_{B^{1/2}_{2,1}}\,d\tau.
\end{aligned}
\tag{3.33}
$$

By a direct computation, we easily obtain that

$$
\begin{aligned}
&\|u_0^{(n+1+k)}-u_0^{(n+1)}\|_{B^{1/2}_{2,\infty}}\\
&\quad\leqslant \|u_0^{(n+1+k)}-u_0^{(n+1)}\|_{B^{1/2}_{2,1}}=\Bigg\|\sum_{i=n+1}^{n+k}\Delta_i u_0\Bigg\|_{B^{1/2}_{2,1}}\leqslant\sum_{j\geqslant -1}2^{j/2}\Bigg\|\Delta_j\Bigg(\sum_{i=n+1}^{n+k}\Delta_i u_0\Bigg)\Bigg\|_{L^2}\\
&\quad\leqslant C\sum_{j=n}^{n+1+k}2^{-j}2^{3j/2}\|\Delta_j u_0\|_{L^2}\leqslant C2^{-n}\|u_0\|_{B^{3/2}_{2,1}}
\end{aligned}
\tag{3.34}
$$

and similarly,

$$
\|\gamma_0^{(n+1+k)}-\gamma_0^{(n+1)}\|_{B^{1/2}_{2,\infty}}\leqslant C2^{-n}\|\gamma_0\|_{B^{3/2}_{2,1}}
\tag{3.35}
$$

since $B^{1/2}_{2,1}\hookrightarrow B^{1/2}_{2,\infty}$. From (3.34)–(3.35) and (3.24), we know that

$$
\widetilde{W}(0)=0.
\tag{3.36}
$$

For any $(n,k)\in\mathbf{N}^2$, from (3.33)–(3.35) and (3) of Lemma 2.6, we derive that

$$
\begin{aligned}
W_{n+1,k}&=\big\|\big(u^{(n+1+k)}-u^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,\infty}}+\big\|\big(\gamma^{(n+1+k)}-\gamma^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,\infty}}\\
&\leqslant C2^{-n}\big[\|u_0\|_{B^{3/2}_{2,1}}+\|\gamma_0\|_{B^{3/2}_{2,1}}\big]+CM\int_0^t\big[\big\|\big(u^{(n+k)}-u^{(n)}\big)(\tau)\big\|_{B^{1/2}_{2,1}}+\big\|\big(\gamma^{(n+k)}-\gamma^{(n)}\big)(\tau)\big\|_{B^{1/2}_{2,1}}\big]\,d\tau\\
&\leqslant C2^{-n}\big[\|u_0\|_{B^{3/2}_{2,1}}+\|\gamma_0\|_{B^{3/2}_{2,1}}\big]\\
&\quad+CM\int_0^t\big\|\big(u^{(n+k)}-u^{(n)}\big)(\tau)\big\|_{B^{1/2}_{2,\infty}}\ln\Bigg(e+\frac{\|(u^{(n+k)}-u^{(n)})(\tau)\|_{B^{3/2}_{2,\infty}}}{\|(u^{(n+k)}-u^{(n)})(\tau)\|_{B^{1/2}_{2,\infty}}}\Bigg)\,d\tau
\end{aligned}
$$

$$+CM\int_0^t \big\|\big(\gamma^{(n)}-\gamma^{(n+k)}\big)(\tau)\big\|_{B^{1/2}_{2,\infty}} \ln\bigg(e+\frac{\|(\gamma^{(n)}-\gamma^{(n+k)})(\tau)\|_{B^{3/2}_{2,\infty}}}{\|(\gamma^{(n)}-\gamma^{(n+k)})(\tau)\|_{B^{1/2}_{2,\infty}}}\bigg)\,d\tau$$

$$\leqslant C2^{-n}\big[\|u_0\|_{B^{3/2}_{2,1}}+\|\gamma_0\|_{B^{3/2}_{2,1}}\big]+CM\int_0^t W_{n,k}(\tau)\ln\bigg(e+\frac{M}{W_{n,k}(\tau)}\bigg)\,d\tau$$

$$\leqslant C2^{-n}\big[\|u_0\|_{B^{3/2}_{2,1}}+\|\gamma_0\|_{B^{3/2}_{2,1}}\big]+CM\int_0^t W_{n}(\tau)\ln\bigg(e+\frac{M}{W_{n}(\tau)}\bigg)\,d\tau. \tag{3.37}$$

From (3.37), we have

$$W_{n+1}(t)\leqslant C2^{-n}\big[\|u_0\|_{B^{3/2}_{2,1}}+\|\gamma_0\|_{B^{3/2}_{2,1}}\big]+CM\int_0^t W_{n}(\tau)\ln\bigg(e+\frac{M}{W_{n}(\tau)}\bigg)\,d\tau. \tag{3.38}$$

By using (3.24), for $\forall\epsilon>0$, for large enough $N_\epsilon\in\mathbf{N}$, from (3.38), we have that

$$W_{n+1}(t)\leqslant C2^{-n}\big[\|u_0\|_{B^{3/2}_{2,1}}+\|\gamma_0\|_{B^{3/2}_{2,1}}\big]+CM\int_0^t \big[\widetilde{W}(\tau)+\epsilon\big]\ln\bigg(e+\frac{M}{[\widetilde{W}(\tau)+\epsilon]}\bigg)\,d\tau. \tag{3.39}$$

From (3.39), we have that

$$\widetilde{W}(t)=\limsup_{n\to+\infty}W_{n+1}(t)\leqslant CM\int_0^t \big[\widetilde{W}(\tau)+\epsilon\big]\ln\bigg(e+\frac{M}{[\widetilde{W}(\tau)+\epsilon]}\bigg)\,d\tau. \tag{3.40}$$

Letting $\epsilon\to 0$ in (3.40) yields

$$\widetilde{W}(t)\leqslant CM\int_0^t \widetilde{W}(\tau)\ln\bigg(e+\frac{M}{\widetilde{W}(\tau)}\bigg)\,d\tau. \tag{3.41}$$

We define $\mu(r)=r\ln(e+\frac{M}{r})$. It is easily checked that $\mu(r)$ satisfies (2.11). From Lemma 2.7, we have that

$$\widetilde{W}(t)=0 \tag{3.42}$$

for $\forall t\in[0,T]$. Combining the definition of $\widetilde{W}(t)$ with (3.42), we have that

$$\lim_{n\to+\infty}W_n=0. \tag{3.43}$$

Thus, $\{(u^{(n)},\gamma^{(n)})\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $C([0,T];Y_{1/2})$.

Now we prove that $\{(u^{(n)},\gamma^{(n)})_{n\in\mathbf{N}}\}$ is a Cauchy sequence in $C([0,T];X_{1/2})$. By using (2.1), we have that

$$\big\|\big(u^{(n+1+k)}-u^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,1}}\leqslant C(\theta)\big\|\big(u^{(n+1+k)}-u^{(n+1)}\big)(t)\big\|^{\theta}_{B^{1/2}_{2,\infty}}\big\|\big(u^{(n+1+k)}-u^{(n+1)}\big)(t)\big\|^{1-\theta}_{B^{3/2}_{2,\infty}} \tag{3.44}$$

and

$$\big\|\big(\gamma^{(n+1+m)}-\gamma^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,1}}\leqslant C(\theta)\big\|\big(\gamma^{(n+1+m)}-\gamma^{(n+1)}\big)(t)\big\|^{\theta}_{B^{1/2}_{2,\infty}}\big\|\big(\gamma^{(n+1+m)}-\gamma^{(n+1)}\big)(t)\big\|^{1-\theta}_{B^{3/2}_{2,\infty}}. \tag{3.45}$$

Combining (3.44) with (3.45), for $\forall t\in[0,T]$, we have that

$$\big\|\big(u^{(n+1+k)}-u^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,1}}+\big\|\big(\gamma^{(n+1+k)}-\gamma^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,1}}\leqslant (CM)^{1-\theta}C(\theta)W^{\theta}_{n,k}(t). \tag{3.46}$$

For $\forall t\in[0,T]$, from (3.46) and (3.27), we have that

$$\widetilde{I}(t) := \lim_{n\to\infty} \sup\big[\big\|\big(u^{(n+1+k)} - u^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,1}} + \big\|\big(\gamma^{(n+1+k)} - \gamma^{(n+1)}\big)(t)\big\|_{B^{1/2}_{2,1}}\big]$$
$$\leqslant (CM)^{1-\theta} C(\theta)\widetilde{W}(t) = 0. \tag{3.47}$$

From (3.47), for $\forall t \in [0, T]$, we have that

$$\widetilde{I}(t) = 0. \tag{3.48}$$

From (3.48) and the definition of $\widetilde{I}(t)$, we know that $\{(u^{(n)}, \gamma^{(n)})\}_{n\in\mathbf{N}}$ is a Cauchy sequence in $C([0, T]; X_{1/2})$, consequently, $(u^{(n)}, \gamma^{(n)})_{n\in\mathbf{N}}$ converges to some limit $(u, \gamma) \in C([0, T]; X_{1/2})$.

3.4. Existence of solution in $E^{3/2}_{2,1}(T) \times E^{3/2}_{2,1}(T)$

Now we need only to check that (u, γ) belongs to $E^{3/2}_{2,1}(T) \times E^{3/2}_{2,1}(T)$ and satisfies (1.10)–(1.13). By using the fact that $(u^{(n)}, \gamma^{(n)})_{n\in N}$ is uniformly bounded in $L^\infty(0, T; X_{3/2})$, from (7) in Lemma 2.3, we derive that $(u, \gamma) \in L^\infty(0, T; X_{3/2})$. By taking the limit in (3.1)–(3.4), we obtain that (u, γ) is indeed a solution to (1.10)–(1.13). From Lemma 2.4, we infer that $(u, \gamma) \in C([0, T]; X_{3/2})$. From (1.10) and (1.11), we have that $(u_t, \gamma_t) \in C([0, T]; X_{1/2})$.

4. Uniqueness of solution in $E^{3/2}_{2,1}(T) \times E^{3/2}_{2,1}(T)$

Now we establish the uniqueness of the solution to the problem (1.10)–(1.13).

The uniqueness of the solution to the problems (1.10)–(1.13) is a corollary of the following.

Lemma 4.1. *Let $(u^{(j)}, \gamma^{(j)}) \in E^{3/2}_{2,1}(T) \times E^{3/2}_{2,1}(T)$ be a solution to (1.10)–(1.13) with initial data $(u^{(j)}(x, 0), \gamma^{(j)}(x, 0)) \in B^{3/2}_{2,1} \times B^{3/2}_{2,1}$ $(j = 1, 2)$. Let $v(x, t) = u^{(1)}(x, t) - u^{(2)}(x, t)$, $\eta(x, t) = \gamma^{(1)}(x, t) - \gamma^{(2)}(x, t)$, $v_0 = v(x, 0) = u^{(1)}(x, 0) - u^{(2)}(x, 0)$, $\eta_0 = \eta(x, 0) = \gamma^{(1)}(x, 0) - \gamma^{(2)}(x, 0)$. We define*

$$Z(t) = \big\|u^{(1)}(\cdot, t)\big\|_{B^{3/2}_{2,\infty}\cap Lip} + \big\|u^{(2)}(\cdot, t)\big\|_{B^{3/2}_{2,\infty}\cap Lip}\big\|\gamma^{(1)}(\cdot, t)\big\|_{B^{3/2}_{2,\infty}\cap Lip} + \big\|\gamma^{(2)}(\cdot, t)\big\|_{B^{3/2}_{2,\infty}\cap Lip}$$

and

$$\widetilde{Z} = 4\big[\big\|u^{(1)}(\cdot, 0)\big\|_{B^{3/2}_{2,1}} + \big\|u^{(2)}(\cdot, 0)\big\|_{B^{3/2}_{2,1}} + \big\|\gamma^{(1)}(\cdot, 0)\big\|_{B^{3/2}_{2,1}} + \big\|\gamma^{(2)}(\cdot, 0)\big\|_{B^{3/2}_{2,1}}\big].$$

For some $T^\star \leqslant T$, then

$$\frac{\|v(t)\|_{B^{1/2}_{2,\infty}} + \|\eta(t)\|_{B^{1/2}_{2,\infty}}}{e\widetilde{Z}} \leqslant e^{C\int_0^t \|u^{(1)}\|_{B^{1/2}_{2,\infty}\cap L^\infty} d\tau} \left(\frac{\|v_0\|_{B^{1/2}_{2,\infty}} + \|\eta_0\|_{B^{1/2}_{2,\infty}}}{e\widetilde{Z}}\right)^{F(t)} \tag{4.1}$$

for $t \in [0, T^\star]$, where

$$F(t) = \exp[-C\widetilde{Z}t].$$

If

$$\|v_0\|_{B^{1/2}_{2,\infty}} + \|\eta_0\|_{B^{1/2}_{2,\infty}} \leqslant (e\widetilde{Z})^{1-\exp[C\widetilde{Z}T]}, \tag{4.2}$$

then (4.1) is valid on $[0, T]$. In particular, if $v_0(x) = 0$, $\eta_0(x) = 0$, then we have that $u^{(1)}(x, t) = u^{(2)}(x, t)$, $\gamma^{(1)}(x, t) = \gamma^{(2)}(x, t)$.

Proof. Obviously, (v, η) solves the following Cauchy problems of the transport equations:

$$\partial_t v + u^{(1)}\partial_x v = -v\partial_x u^{(2)} + P_1(D)\Big[v\big(u^{(1)} + u^{(2)}\big) + \frac{1}{2}\partial_x v\big(\partial_x u^{(1)} + \partial_x u^{(2)}\big)\Big]$$
$$+ \frac{1}{2}P_1(D)\big[\eta\big(\gamma^{(1)} + \gamma^{(2)}\big) - \eta_x\big(\gamma^{(1)} + \gamma^{(2)}\big)_x\big], \tag{4.3}$$

$$\partial_t\eta + u^{(1)}\partial_x\eta = -v\partial_x\gamma^{(2)} + P_1(D)\big[v_x\gamma_x^{(2)} + u_x^{(2)}\eta_x\big] + P_2(D)\big[\gamma^{(2)}v_x + \eta u_x^{(1)}\big], \tag{4.4}$$
$$v_0(x) = u^{(1)}(x,0) - u^{(2)}(x,0),$$
$$\eta_0(x) = \gamma^{(1)}(x,0) - \gamma^{(2)}(x,0).$$

By using (1) of Lemma 2.4 and (4.3), we have

$$\|v(t)\|_{B^{1/2}_{2,\infty}} \leqslant \|v_0\|_{B^{1/2}_{2,\infty}} \exp\left\{C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau\right\} + \int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\left[\sum_{j=1}^4 T_j\right]d\tau \tag{4.5}$$

where

$$T_1 = \big\|-v\partial_x u^{(2)} + P_1(D)\big[v\big(u^{(1)} + u^{(2)}\big)\big]\big\|_{B^{1/2}_{2,\infty}},$$
$$T_2 = \big\|P_1(D)\big[\partial_x v\partial_x\big(u^{(1)} + u^{(2)}\big)\big]\big\|_{B^{1/2}_{2,\infty}},$$
$$T_3 = \big\|P_1(D)\big[\eta\big(\gamma^{(1)} + \gamma^{(2)}\big)\big]\big\|_{B^{1/2}_{2,\infty}},$$
$$T_4 = \big\|\big[P_1(D)\big[\eta_x\big(\gamma^{(1)} + \gamma^{(2)}\big)_x\big]\big]\big\|_{B^{1/2}_{2,\infty}}.$$

By using (2), (3) and (9), (8) of Lemma 2.3, we have that

$$\begin{aligned}
T_1 &\leqslant \big\|v\partial_x u^{(2)}\big\|_{B^{1/2}_{2,\infty}} + \big\|P_1(D)\big[v\big(u^{(1)} + u^{(2)}\big)\big]\big\|_{B^{1/2}_{2,\infty}} \\
&\leqslant \big\|v\partial_x u^{(2)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty} + \big\|P_1(D)\big[v\big(u^{(1)} + u^{(2)}\big)\big]\big\|_{B^{1/2}_{2,\infty}} \\
&\leqslant C\|v\|_{B^{1/2}_{2,1}}\big\|\partial_x u^{(2)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty} + C\big\|v\big(u^{(1)} + u^{(2)}\big)\big\|_{B^{-1/2}_{2,\infty}} \\
&\leqslant C\|v\|_{B^{1/2}_{2,1}}\big\|u^{(2)}\big\|_{B^{3/2}_{2,\infty}\cap L^\infty} + C\|v\|_{B^{-1/2}_{2,1}}\big\|u^{(1)} + u^{(2)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty} \\
&\leqslant C\|v\|_{B^{1/2}_{2,1}}\big\|u^{(2)}\big\|_{B^{3/2}_{2,\infty}\cap Lip} + C\|v\|_{B^{1/2}_{2,1}}\big[\big\|u^{(1)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty} + \big\|u^{(2)}\big\|_{B^{1/2}_{2,\infty}\cap L^\infty}\big] \\
&\leqslant C\|v\|_{B^{1/2}_{2,1}}\big[\big\|u^{(1)}\big\|_{B^{3/2}_{2,\infty}\cap Lip} + \big\|u^{(2)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}\big]
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
T_2 &\leqslant C\big\|\partial_x v\partial_x\big(u^{(1)} + u^{(2)}\big)\big\|_{B^{-1/2}_{2,\infty}} \leqslant C\|\partial_x v\|_{B^{-1/2}_{2,1}}\big\|\partial_x\big(u^{(1)} + u^{(2)}\big)\big\|_{B^{1/2}_{2,\infty}\cap L^\infty} \\
&\leqslant C\|v\|_{B^{1/2}_{2,1}}\big[\big\|u^{(1)}\big\|_{B^{3/2}_{2,\infty}\cap Lip} + \big\|u^{(2)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}\big]
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
T_3 &\leqslant C\big\|\eta\big(\gamma^{(1)} + \gamma^{(2)}\big)\big\|_{B^{-1/2}_{2,\infty}} \leqslant C\|\eta\|_{B^{-1/2}_{2,1}}\big\|\big(\gamma^{(1)} + \gamma^{(2)}\big)\big\|_{B^{1/2}_{2,\infty}\cap L^\infty} \\
&\leqslant C\|\eta\|_{B^{1/2}_{2,1}}\big[\big\|\gamma^{(1)}\big\|_{B^{3/2}_{2,\infty}\cap Lip} + \big\|\gamma^{(2)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}\big]
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}
T_4 &\leqslant C\big\|\eta_x\big(\gamma^{(1)} + \gamma^{(2)}\big)_x\big\|_{B^{-1/2}_{2,\infty}} \leqslant C\|\eta_x\|_{B^{-1/2}_{2,1}}\big\|\big(\gamma^{(1)} + \gamma^{(2)}\big)_x\big\|_{B^{1/2}_{2,\infty}\cap L^\infty} \\
&\leqslant C\|\eta\|_{B^{1/2}_{2,1}}\big[\big\|\gamma^{(1)}\big\|_{B^{3/2}_{2,\infty}\cap Lip} + \big\|\gamma^{(2)}\big\|_{B^{3/2}_{2,\infty}\cap Lip}\big].
\end{aligned} \tag{4.9}$$

Combining (4.6)–(4.9) with (4.5), we have that

$$\begin{aligned}
\|v(t)\|_{B^{1/2}_{2,\infty}} &\leqslant \|v_0\|_{B^{1/2}_{2,\infty}} \exp\left\{C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau\right\} \\
&\quad + C\int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\big[\|v\|_{B^{1/2}_{2,1}} + \|\eta\|_{B^{1/2}_{2,1}}\big]Z(\tau)\,d\tau.
\end{aligned} \tag{4.10}$$

By using (1) of Lemma 2.4 and (4.4), we have

$$\begin{aligned}
\|\eta(t)\|_{B^{1/2}_{2,\infty}} &\leqslant \|\eta_0\|_{B^{1/2}_{2,\infty}} \exp\left\{C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau\right\} \\
&\quad + \int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\|v\partial_x\gamma^{(2)}\|_{B^{1/2}_{2,\infty}}\,d\tau \\
&\quad + \int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\big\|P_1(D)\big[v_x\gamma_x^{(2)} + u_x^{(2)}\eta_x\big]\big\|_{B^{1/2}_{2,\infty}}\,d\tau \\
&\quad + \int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\big\|P_2(D)\big[\gamma^{(2)}v_x + \eta u_x^{(1)}\big]\big\|_{B^{1/2}_{2,\infty}}\,d\tau \\
&\leqslant \|\eta_0\|_{B^{1/2}_{2,\infty}} \exp\left\{C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau\right\} \\
&\quad + \int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\|v\|_{B^{1/2}_{2,\infty}\cap L^\infty}\|\gamma^{(2)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau \\
&\quad + \int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\big\|\big[v_x\gamma_x^{(2)} + u_x^{(2)}\eta_x\big]\big\|_{B^{-1/2}_{2,\infty}}\,d\tau \\
&\quad + \int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\big\|\big[\gamma^{(2)}v_x + \eta u_x^{(1)}\big]\big\|_{B^{-1/2}_{2,\infty}}\,d\tau \\
&\leqslant \|\eta_0\|_{B^{1/2}_{2,\infty}} \exp\left\{C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau\right\} \\
&\quad + C\int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\big(\|v\|_{B^{1/2}_{2,1}} + \|\eta\|_{B^{1/2}_{2,1}}\big)Z(\tau)\,d\tau. \qquad (4.11)
\end{aligned}$$

Combining (4.10) with (4.11), we have

$$\begin{aligned}
\|v(t)\|_{B^{1/2}_{2,\infty}} + \|\eta(t)\|_{B^{1/2}_{2,\infty}} &\leqslant \exp\left\{C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau\right\}\big[\|v_0\|_{B^{1/2}_{2,\infty}} + \|\eta_0\|_{B^{1/2}_{2,\infty}}\big] \\
&\quad + C\int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\big(\|v\|_{B^{1/2}_{2,1}} + \|\eta\|_{B^{1/2}_{2,1}}\big)Z(\tau)\,d\tau. \qquad (4.12)
\end{aligned}$$

For $\forall t \in [0, T^\star]$, from (4.12) and (10) of Lemma 2.3 and (3) of Lemma 2.6, we have that

$$\begin{aligned}
&\|v(t)\|_{B^{1/2}_{2,\infty}} + \|\eta(t)\|_{B^{1/2}_{2,\infty}} \\
&\quad \leqslant \exp\left\{C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau\right\}\big[\|v_0\|_{B^{1/2}_{2,\infty}} + \|\eta_0\|_{B^{1/2}_{2,\infty}}\big]
\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}Z(\tau)\|v\|_{B^{1/2}_{2,\infty}}\ln\left[e+\frac{\|v\|_{B^{3/2}_{2,\infty}}}{\|v\|_{B^{1/2}_{2,\infty}}}\right]d\tau\\
&+C\int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}Z(\tau)\|\eta\|_{B^{1/2}_{2,\infty}}\ln\left[e+\frac{\|\eta\|_{B^{3/2}_{2,\infty}}}{\|\eta\|_{B^{1/2}_{2,\infty}}}\right]d\tau\\
\leqslant{}& \exp\left\{C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau\right\}\left[\|v_0\|_{B^{1/2}_{2,\infty}}+\|\eta_0\|_{B^{1/2}_{2,\infty}}\right]\\
&+C\int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}Z(\tau)\|v\|_{B^{1/2}_{2,\infty}}\ln\left[e+\frac{\|v\|_{B^{3/2}_{2,\infty}}}{\exp\{-C\int_0^\tau \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\}\|v\|_{B^{1/2}_{2,\infty}}}\right]d\tau\\
&+C\int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}Z(\tau)\|\eta\|_{B^{1/2}_{2,\infty}}\ln\left[e+\frac{\|\eta\|_{B^{3/2}_{2,\infty}}}{\exp\{-C\int_0^\tau \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\}\|\eta\|_{B^{1/2}_{2,\infty}}}\right]d\tau\\
\leqslant{}& C\exp\left\{C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau\right\}\left[\|v_0\|_{B^{1/2}_{2,\infty}}+\|\eta_0\|_{B^{1/2}_{2,\infty}}\right]\\
&+C\int_0^t \exp\left\{C\int_\tau^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}Z(\tau)\left[\|v\|_{B^{1/2}_{2,\infty}}+\|\eta\|_{B^{1/2}_{2,\infty}}\right]A(\tau)\,d\tau.
\end{aligned}
\tag{4.13}
$$

where

$$
A(\tau)=\ln\left[e+\frac{Z(\tau)}{\exp\{-C\int_0^\tau \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\}[\|v\|_{B^{1/2}_{2,\infty}}+\|\eta\|_{B^{1/2}_{2,\infty}}]}\right].
$$

From (4.13), we have that

$$
\begin{aligned}
&\exp\left\{-C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\left[\|v(t)\|_{B^{1/2}_{2,\infty}}+\|\eta(t)\|_{B^{1/2}_{2,\infty}}\right]\\
&\quad\leqslant\left[\|v_0\|_{B^{1/2}_{2,\infty}}+\|\eta_0\|_{B^{1/2}_{2,\infty}}\right]+C\int_0^t Z(\tau)\exp\left\{-C\int_0^\tau \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\left[\|v\|_{B^{1/2}_{2,\infty}}+\|\eta\|_{B^{1/2}_{2,\infty}}\right]A(\tau)\,d\tau.
\end{aligned}
\tag{4.14}
$$

Let

$$
W(t)=\exp\left\{-C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau'\right\}\left[\|v(t)\|_{B^{1/2}_{2,\infty}}+\|\eta(t)\|_{B^{1/2}_{2,\infty}}\right].
\tag{4.15}
$$

By (4.14) and (4.15), we derive that

$$
W(t)\leqslant W(0)+C\int_0^t Z(\tau)W(\tau)\ln\left(e+\frac{Z(\tau)}{W(\tau)}\right)d\tau.
\tag{4.16}
$$

From the definition of $Z(t)$ and (3.18) and Fatou's Lemma, we have that

$$
\begin{aligned}
W(t) &\leqslant \sup_{t\in[0,T^\star]}\left[\exp\left\{-C\int_0^t \|u^{(1)}\|_{B^{3/2}_{2,\infty}\cap Lip}\,d\tau\right\}\left(\|v\|_{B^{\frac12}_{2,\infty}}+\|\eta\|_{B^{\frac12}_{2,\infty}}\right)\right]\\
&\leqslant \|v\|_{B^{\frac12}_{2,\infty}}+\|\eta\|_{B^{\frac12}_{2,\infty}} \leqslant \|u^{(1)}\|_{B^{\frac12}_{2,\infty}}+\|u^{(2)}\|_{B^{\frac12}_{2,\infty}}+\|\gamma^1\|_{B^{\frac12}_{2,\infty}}+\|\gamma^{(2)}\|_{B^{\frac12}_{2,\infty}}\\
&\leqslant Z(\tau)\leqslant 4\left[\|u^{(1)}(\cdot,0)\|_{B^{3/2}_{2,1}}+\|u^{(2)}(\cdot,0)\|_{B^{3/2}_{2,1}}+\|\gamma^{(1)}(\cdot,0)\|_{B^{3/2}_{2,1}}+\|\gamma^{(2)}(\cdot,0)\|_{B^{3/2}_{2,1}}\right]\\
&:= \widetilde{Z}.
\end{aligned}
\tag{4.17}
$$

Inserting (4.17) into (4.16), we know that

$$
W(t)\leqslant W(0)+C\int_0^t \widetilde{Z}W(\tau)\ln\left(e+\frac{\widetilde{Z}}{W(\tau)}\right)d\tau. \tag{4.18}
$$

Applying Lemma 2.7 and (2.12) →(4.18) yields

$$
\frac{\|v(t)\|_{B^{1/2}_{2,\infty}}+\|\eta(t)\|_{B^{1/2}_{2,\infty}}}{e\widetilde{Z}}\leqslant e^{C\int_0^t\|u^1\|_{B^{1/2}_{2,\infty}\cap L^\infty}d\tau}\left(\frac{\|v_0\|_{B^{1/2}_{2,\infty}}+\|\eta_0\|_{B^{1/2}_{2,\infty}}}{e\widetilde{Z}}\right)^{F(t)} \tag{4.19}
$$

for $\forall t\in[0,T^\star]$, where

$$
F(t)=\exp[-C\widetilde{Z}t].
$$

(4.2) implies that (4.1) is valid with $T^\star=T$.

We have completed the proof of Lemma 4.1. □

5. Continuity with respect to the initial data in $X_{3/2}$

To prove the continuous dependence of solutions with the initial data in $X_{3/2}$, we shall use the following proposition.

Proposition 5.1. *Assume that $w^{(n)}\in C([0,T];B^{1/2}_{2,1})$ with $n\geqslant 1$ is the solution to*

$$
w_t^{(n)}+a^{(n)}w_x^{(n)}=h, \tag{5.1}
$$

$$
w^{(n)}\big|_{t=0}=w_0, \tag{5.2}
$$

with $w_0\in B^{1/2}_{2,1}$, $h\in L^1(0,T;B^{1/2}_{2,1})$. If there exists some function $\alpha\in L^1(0,T)$ such that

$$
\sup_{n\in\mathbf{N}}\|a^{(n)}\|_{B^{3/2}_{2,1}}\leqslant\alpha(t)
$$

and $a^{(n)}$ tends to $a^{(\infty)}$ in $L^1(0,T;B^{1/2}_{2,1})$ then $w^{(n)}$ tends to $w^{(\infty)}$ in $C([0,T];B^{1/2}_{2,1})$.

For the proof of Proposition 5.1, we refer the readers to [21].

Theorem 5.2. *For any $\xi_0:=(u_0,\gamma_0)\in X_{3/2}$, there exists a $T>0$ and a neighborhood V of ξ_0 in $X_{3/2}$ such that the solution map $\Phi:(u_0,\gamma_0)\to(u,\gamma)$ where (u,γ) is the solution to (1.10)–(1.13) with initial data (u_0,γ_0) in $X_{3/2}$, is continuous from V into $C([0,T];X_{3/2})$.*

Proof. We will prove Theorem 5.2 with the following two steps.

First step: We firstly establish continuity in $C([0,T];X_{1/2})$. If $(u_0,\gamma_0)\in X_{3/2}$ and $r>0$, we show that for $(u_0',\gamma_0')\in X_{3/2}$ with

$$
\|u_0'-u_0\|_{B^{3/2}_{2,1}}+\|\gamma_0'-\gamma_0\|_{B^{3/2}_{2,1}}\leqslant r, \tag{5.3}
$$

there exist $T>0$ and $M>0$ such that $(u',\gamma')=\Phi(u_0',\gamma_0')$ of (1.10)–(1.11) with (u_0',γ_0') satisfies the following inequality

$$\|u'\|_{L^\infty(0,T;B^{3/2}_{2,1})}+\|\gamma'\|_{L^\infty(0,T;B^{3/2}_{2,1})}\leqslant M.$$

Indeed, following the proof of (3.9) and (3.10), we can choose

$$T=\frac{1}{4C(\|u_0\|_{B^{3/2}_{2,1}}+\|\gamma_0\|_{B^{3/2}_{2,1}}+2r)}\tag{5.4}$$

and

$$M=2\big(\|u_0\|_{B^{3/2}_{2,1}}+\|\gamma_0\|_{B^{3/2}_{2,1}}\big)+2r.\tag{5.5}$$

Combining the above conclusions with Lemma 4.1, we have

$$\begin{aligned}&\big\|\Phi(u_0,\gamma_0)-\Phi\big(u_0',\gamma_0'\big)\big\|_{B^{1/2}_{2,\infty}\times B^{1/2}_{2,\infty}}\\&\quad=\big\|u'-u\big\|_{B^{1/2}_{2,\infty}}+\big\|\gamma'-\gamma\big\|_{B^{1/2}_{2,\infty}}\leqslant CMe^{CMT}\left(\frac{\|u_0'-u_0\|_{B^{1/2}_{2,\infty}}+\|\gamma_0'-\gamma_0\|_{B^{1/2}_{2,\infty}}}{eM}\right)^{\exp[-CMT]}.\end{aligned}\tag{5.6}$$

Combining (5.6) with (2.1), we obtain

$$\begin{aligned}&\big\|\Phi(u_0,\gamma_0)-\Phi\big(u_0',\gamma_0'\big)\big\|_{B^{1/2}_{2,1}\times B^{1/2}_{2,1}}\\&\quad=\big\|u'-u\big\|_{B^{1/2}_{2,1}}+\big\|\gamma'-\gamma\big\|_{B^{1/2}_{2,1}}\\&\quad\leqslant CMe^{CMT}\big[\big\|u_0'\big\|_{B^{3/2}_{2,\infty}}+\|u_0\|_{B^{3/2}_{2,\infty}}+\big\|\gamma_0'\big\|_{B^{3/2}_{2,\infty}}+\|\gamma_0\|_{B^{3/2}_{2,\infty}}\big]^{1-\theta}\\&\qquad\times\left(\frac{\|u_0'-u_0\|_{B^{1/2}_{2,\infty}}+\|\gamma_0'-\gamma_0\|_{B^{1/2}_{2,\infty}}}{eM}\right)^{\theta\exp[-CMT]}\\&\quad\leqslant CM^{2-\theta}e^{CMT}\left(\frac{\|u_0'-u_0\|_{B^{1/2}_{2,\infty}}+\|\gamma_0'-\gamma_0\|_{B^{1/2}_{2,\infty}}}{eM}\right)^{\theta\exp[-CMT]}\end{aligned}$$

if

$$\big\|u_0'-u_0\big\|_{B^{1/2}_{2,\infty}}+\big\|u_0'-u_0\big\|_{B^{1/2}_{2,\infty}}\leqslant(eM)^{1-e^{[-CMT]}}.$$

Thus Φ is Hölder continuous from $X_{3/2}$ into $C([0,T];X_{1/2})$.

Second step: We establish the continuity in $C([0,T];X_{3/2})$. We assume that $(u_0^{(\infty)},\gamma_0^{(\infty)})\in X_{3/2}$ and $(u_0^{(n)},\gamma_0^{(n)})_{n\in\mathbf{N}}$ tend to $(u_0^{(\infty)},\gamma_0^{(\infty)})$ in $X_{3/2}$ and $(u^{(n)},\gamma^{(n)})$ is the solution with the initial data $(u_0^{(n)},\gamma_0^{(n)})$. By the first step, it is easy to find $T,M>0$ such that for all $n\in\mathbf{N}$,

$$\sup_{n\in\mathbf{N}}\big[\big\|u^{(n)}\big\|_{L^\infty_T(B^{3/2}_{2,1})}+\big\|\gamma^{(n)}\big\|_{L^\infty_T(B^{3/2}_{2,1})}\big]\leqslant M,\tag{5.7}$$

where $(u^{(n)},\gamma^{(n)})$ is defined on $[0,T]\times[0,T]$. To prove that $(u^{(n)},\gamma^{(n)})$ tends to $(u^{(\infty)},\gamma^{(\infty)})$ in $C([0,T];X_{3/2})$, it suffices to prove that $(v^{(n)},y^{(n)})=(\partial_x u^{(n)},\partial_x\gamma^{(n)})$ tends to

$$\big(v^{(\infty)},y^{(\infty)}\big)=\big(\partial_x u^{(\infty)},\partial_x\gamma^{(\infty)}\big)$$

in $C([0,T];X_{1/2})$. It is easily checked that $(v^{(n)},y^{(n)})$ solves

$$\partial_t v^{(n)}+u^{(n)}\partial_x v^{(n)}=F_1^{(n)},\tag{5.8}$$

$$\partial_t y^{(n)}+u^{(n)}\partial_x y^{(n)}=F_2^{(n)},\tag{5.9}$$

$$v^{(n)}\big|_{t=0}=\partial_x u_0^{(n)},\tag{5.10}$$

$$y^{(n)}\big|_{t=0}=\partial_x\gamma_0^{(n)}\tag{5.11}$$

where

$$F_1^{(n)} = -\big(u_x^{(n)}\big)^2 + P_1(D)\bigg[\big(u^{(n)}\big)^2 + \frac{1}{2}\big(\gamma^{(n)}\big)^2 - \frac{1}{2}\gamma_x^{(n)}\bigg]_x + P_1(D)\bigg[\frac{1}{2}\big(u_x^{(n)}\big)^2\bigg]_x, \tag{5.12}$$

$$F_2^{(n)} = -u_x^{(n)}\gamma_x^{(n)} + P_1(D)\big[\big(u_x^{(n)}\gamma_x^{(n)}\big)_x + u_x^{(n)}\gamma^{(n)}\big]. \tag{5.13}$$

By the method of [35], we can decompose

$$\big(v^{(n)}, y^{(n)}\big) = \big(v_1^{(n)}, y_1^{(n)}\big) + \big(v_2^{(n)}, y_2^{(n)}\big)$$

with

$$\partial_t v_1^{(n)} + u^{(n)}\partial_x v_1^{(n)} = F_1^{(n)} - F_1^{(\infty)}, \tag{5.14}$$

$$\partial_t y_1^{(n)} + u^{(n)}\partial_x y_1^{(n)} = F_2^{(n)} - F_2^{(\infty)}, \tag{5.15}$$

$$v_1^{(n)}\big|_{t=0} = \partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}, \tag{5.16}$$

$$y_1^{(n)}\big|_{t=0} = \partial_x \gamma_0^{(n)} - \partial_x \gamma_0^{(\infty)}, \tag{5.17}$$

and

$$\partial_t v_2^{(n)} + u^{(n)}\partial_x v_2^{(n)} = F_1^{(\infty)}, \tag{5.18}$$

$$\partial_t y_2^{(n)} + u^{(n)}\partial_x y_2^{(n)} = F_2^{(\infty)}, \tag{5.19}$$

$$v_2^{(n)}\big|_{t=0} = \partial_x u_0^{(\infty)}, \tag{5.20}$$

$$y_2^{(n)}\big|_{t=0} = \partial_x \gamma_0^{(\infty)}. \tag{5.21}$$

By using the fact that $B_{2,1}^{1/2}$ is a Banach algebra, for $\forall n \in \mathbf{N}$, we derive that

$$\begin{aligned}
\big\|F_1^{(n)}\big\|_{B_{2,1}^{1/2}} &= \bigg\| -\big(u_x^{(n)}\big)^2 + P_1(D)\bigg[\big(u^{(n)}\big)^2 + \frac{1}{2}\big(\gamma^{(n)}\big)^2 - \frac{1}{2}\gamma_x^{(n)}\bigg]_x + P_1(D)\bigg[\frac{1}{2}\big(u_x^{(n)}\big)^2\bigg]_x\bigg\|_{B_{2,1}^{1/2}} \\
&\leqslant \big\|\big(u_x^{(n)}\big)^2\big\|_{B_{2,1}^{1/2}} + \bigg\|\partial_x P_1(D)\bigg[\big(u^{(n)}\big)^2 + \frac{1}{2}\big(\gamma^{(n)}\big)^2 - \frac{1}{2}\big(\gamma_x^{(n)}\big)^2\bigg]\bigg\|_{B_{2,1}^{1/2}} + \bigg\|\partial_x P_1(D)\bigg[\frac{1}{2}\big(u_x^{(n)}\big)^2\bigg]\bigg\|_{B_{2,1}^{1/2}} \\
&\leqslant C\big\|u^{(n)}\big\|_{B_{2,1}^{3/2}}^2 + C\big\|\gamma^{(n)}\big\|_{B_{2,1}^{3/2}}^2
\end{aligned} \tag{5.22}$$

and

$$\big\|F_2^{(n)}\big\|_{B_{2,1}^{1/2}} = \big\|-u_x^{(n)}\gamma_x^{(n)} + P_1(D)\big[\big(u_x^{(n)}\gamma_x^{(n)}\big)_x + u_x^{(n)}\gamma^{(n)}\big]\big\|_{B_{2,1}^{1/2}} \leqslant C\big\|u^{(n)}\big\|_{B_{2,1}^{3/2}}\big\|\gamma^{(n)}\big\|_{B_{2,1}^{3/2}}. \tag{5.23}$$

In addition,

$$\begin{aligned}
F_1^{(n)} - F_1^{(\infty)} &= \bigg(\frac{\partial_x P(D)}{2} - 1\bigg)\big(\partial_x u^{(n)} - \partial_x u^{(\infty)}\big)\big(\partial_x u^{(n)} + \partial_x u^{(\infty)}\big) \\
&\quad + \partial_x P_1(D)\bigg[\big(u^{(n)} - u^{(\infty)}\big)\big(u^{(n)} + u^{(\infty)}\big) + \frac{1}{2}\big(\gamma^{(n)} - \gamma^{(\infty)}\big)\big(\gamma^{(n)} + \gamma^{(\infty)}\big)\bigg] \\
&\quad - \partial_x P_1(D)\bigg[\frac{1}{2}\big(\gamma^{(n)} - \gamma^{(\infty)}\big)_x\big(\gamma^{(n)} + \gamma^{(\infty)}\big)_x\bigg]
\end{aligned} \tag{5.24}$$

and

$$\begin{aligned}
F_2^{(n)} - F_2^{(\infty)} &= \big(u_x^{(\infty)} - u_x^{(n)}\big)\gamma_x^{(\infty)} + u_x^{(n)}\big(\gamma_x^{(\infty)} - \gamma_x^{(n)}\big) + P_1(D)\big[\big(u^{(n)} - u^{(\infty)}\big)_x\big(\gamma_x^{(n)} + \gamma^{(n)}\big)\big] \\
&\quad + P_1(D)\big[u_x^{(\infty)}\big[\big(\gamma^{(n)} - \gamma^{(\infty)}\big)_x + \big(\gamma^{(n)} - \gamma^{(\infty)}\big)\big]\big].
\end{aligned} \tag{5.25}$$

By (1) of Lemma 2.4, (5.14) and (5.15), we derive that

$$\begin{aligned}\|v_1^{(n)}(t)\|_{B^{1/2}_{2,1}} \leqslant{}& \exp\left\{C\int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\}\|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\|_{B^{1/2}_{2,1}}\\ &+ C\exp\left\{C\int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\}\int_0^t \|u^{(n)} - u^{(\infty)}\|_{B^{1/2}_{2,1}}\left[\|u^{(n)}\|_{B^{1/2}_{2,1}} + \|u^{(\infty)}\|_{B^{1/2}_{2,1}}\right]d\tau\\ &+ C\exp\left\{C\int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\}\int_0^t \|\partial_x u^{(n)} - \partial_x u^{(\infty)}\|_{B^{1/2}_{2,1}}\left[\|\partial_x u^{(n)}\|_{B^{1/2}_{2,1}} + \|\partial_x u^{(\infty)}\|_{B^{1/2}_{2,1}}\right]d\tau\\ &+ C\exp\left\{C\int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\}\int_0^t \|\gamma^{(n)} - \gamma^{(\infty)}\|_{B^{1/2}_{2,1}}\left[\|\gamma^{(n)}\|_{B^{1/2}_{2,1}} + \|\gamma^{(\infty)}\|_{B^{1/2}_{2,1}}\right]d\tau\\ &+ C\exp\left\{C\int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\}\int_0^t \|(\gamma^{(n)} - \gamma^{(\infty)})_x\|_{B^{1/2}_{2,1}}\left[\|\gamma_x^{(n)}\|_{B^{1/2}_{2,1}} + \|\gamma_x^{(\infty)}\|_{B^{1/2}_{2,1}}\right]d\tau \end{aligned} \tag{5.26}$$

and

$$\begin{aligned}\|y_1^{(n)}(t)\|_{B^{1/2}_{2,1}} \leqslant{}& \exp\left\{C\int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\}\|\partial_x \gamma_0^{(n)} - \partial_x \gamma_0^{(\infty)}\|_{B^{1/2}_{2,1}}\\ &+ C\exp\left\{C\int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\}\int_0^t \|(u^{(n)} - u^{(\infty)})_x\|_{B^{1/2}_{2,1}}\left[\|\gamma^{(n)}\|_{B^{3/2}_{2,1}} + \|\gamma^{(\infty)}\|_{B^{3/2}_{2,1}}\right]d\tau\\ &+ C\exp\left\{C\int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\}\int_0^t \|(\gamma^{(\infty)} - \gamma^{(n)})\|_{B^{1/2}_{2,1}}\|u_x^{(\infty)}\|_{B^{1/2}_{2,1}}\,d\tau\\ &+ C\exp\left\{C\int_0^t \|u^{(n)}\|_{B^{3/2}_{2,1}}\,d\tau\right\}\int_0^t \|(\gamma^{(\infty)} - \gamma^{(n)})_x\|_{B^{1/2}_{2,1}}\left[\|u_x^{(\infty)}\|_{B^{1/2}_{2,1}} + \|u_x^{(n)}\|_{B^{1/2}_{2,1}}\right]d\tau. \end{aligned} \tag{5.27}$$

Combining (5.26) with (5.27), we have

$$\begin{aligned}&\|v_1^{(n)}(t)\|_{B^{1/2}_{2,1}} + \|y_1^{(n)}(t)\|_{B^{1/2}_{2,1}}\\ &\quad\leqslant e^{CMT}\left[\|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\|_{B^{1/2}_{2,1}} + \|\partial_x \gamma_0^{(n)} - \partial_x \gamma_0^{(\infty)}\|_{B^{1/2}_{2,1}}\right]\\ &\qquad+ Ce^{CMT}\int_0^t \left[\|u^{(n)} - u^{(\infty)}\|_{B^{1/2}_{2,1}} + \|\partial_x u^{(n)} - \partial_x u^{(\infty)}\|_{B^{1/2}_{2,1}}\right]d\tau\\ &\qquad+ Ce^{CMT}\int_0^t \left[\|\gamma^{(n)} - \gamma^{(\infty)}\|_{B^{1/2}_{2,1}} + \|\partial_x \gamma^{(n)} - \partial_x \gamma^{(\infty)}\|_{B^{1/2}_{2,1}}\right]d\tau. \end{aligned} \tag{5.28}$$

Since $(u^{(n)})_{n\in\mathbf{N}}$ is uniformly bounded in $C([0,T]; B^{3/2}_{2,1})$ and tends to $u^{(\infty)}$ in $C([0,T]; B^{1/2}_{2,1})$, Proposition 5.1 tells us that

$$\left(v_2^{(n)}, y_2^{(n)}\right) \to \left(v^{(\infty)}, y^{(\infty)}\right) \tag{5.29}$$

in $C([0,T]; X_{1/2})$. That is to say, for $\forall\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $\forall n > \mathbf{N}$,

$$\|v_2^{(n)} - v^{(\infty)}\|_{B^{1/2}_{2,1}} + \|y_2^{(n)} - y^{(\infty)}\|_{B^{1/2}_{2,1}} < \epsilon. \tag{5.30}$$

Thus, for $\forall\epsilon > 0$, for large enough $n \in \mathbf{N}$, since $(u^{(n)}, \gamma^{(n)}) \to (u^{(\infty)}, \gamma^{(\infty)})$ in $C([0,T]; X_{1/2})$, we have

$$\begin{aligned}
&\left\|\left(\partial_x u^{(n)} - \partial_x u^{(\infty)}\right)(t)\right\|_{B^{1/2}_{2,1}} + \left\|\left(\partial_x \gamma^{(n)} - \partial_x \gamma^{(\infty)}\right)(t)\right\|_{B^{1/2}_{2,1}} \\
&\quad\leqslant \epsilon + Ce^{CMT}\left[\left\|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\right\|_{B^{1/2}_{2,1}} + \left\|\partial_x \gamma_0^{(n)} - \partial_x \gamma_0^{(\infty)}\right\|_{B^{1/2}_{2,1}}\right] \\
&\qquad + Ce^{CMT}\left[\int_0^t \left\|\partial_x u^{(n)} - \partial_x u^{(\infty)}\right\|_{B^{1/2}_{2,1}} + \left\|\partial_x \gamma^{(n)} - \partial_x \gamma^{(\infty)}\right\|_{B^{1/2}_{2,1}} d\tau\right].
\end{aligned} \tag{5.31}$$

Applying Gronwall's inequality to (5.31) yields

$$\begin{aligned}
&\left\|\left(\partial_x u^{(n)} - \partial_x u^{(\infty)}\right)(t)\right\|_{B^{1/2}_{2,1}} + \left\|\left(\partial_x \gamma^{(n)} - \partial_x \gamma^{(\infty)}\right)(t)\right\|_{B^{1/2}_{2,1}} \\
&\quad\leqslant Ce^{Ce^{C(M+1)T}}\left[\epsilon + \left\|\partial_x u_0^{(n)} - \partial_x u_0^{(\infty)}\right\|_{B^{1/2}_{2,1}} + \left\|\partial_x \gamma_0^{(n)} - \partial_x \gamma_0^{(\infty)}\right\|_{B^{1/2}_{2,1}}\right].
\end{aligned} \tag{5.32}$$

Combining the first step with the second step, we have completed the proof of Theorem 5.2. □

6. Proof of Theorem 1.2

We define

$$E(u,\gamma) = \int_{\mathbf{R}} \left[u^2 + u_x^2 + \gamma^2 + \gamma_x^2\right] dx. \tag{6.1}$$

Proof of Theorem 1.2. Applying Δ_q to (1.10) yields

$$\partial_t \Delta_q u + u\Delta_q u_x = [u, \Delta_q]u_x + \Delta_q P_1(D)\left[u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right]. \tag{6.2}$$

Multiplying (6.2) by $2\Delta_q u$ and integrating by parts and by using the Cauchy–Schwartz inequality, we have that

$$\begin{aligned}
\frac{d}{dt}\|\Delta_q u\|_{L^2}^2 &= \int_{\mathbf{R}} u_x[\Delta_q u]^2\,dx + 2\int_{\mathbf{R}} \Delta_q u[u,\Delta_q]u_x\,dx \\
&\quad + 2\int_{\mathbf{R}} \Delta_q u \Delta_q P_1(D)\left[u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right] dx \\
&\leqslant \|u_x\|_{L^\infty}\|\Delta_q u\|_{L^2}^2 + 2\|\Delta_q u\|_{L^2}\left\|[u,\Delta_q]u_x\right\|_{L^2} \\
&\quad + 2\|\Delta_q u\|_{L^2}\left\|\Delta_q P_1(D)\left[u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right]\right\|_{L^2}.
\end{aligned} \tag{6.3}$$

From (6.3), we know that

$$\frac{d}{dt}\|\Delta_q u\|_{L^2} \leqslant \|u_x\|_{L^\infty}\|\Delta_q u\|_{L^2} + \left\|[u,\Delta_q]u_x\right\|_{L^2} + \left\|\Delta_q P_1(D)\left[u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right]\right\|_{L^2}. \tag{6.4}$$

Applying (A.9) of A.2 in [19] to (6.4) yields

$$\begin{aligned}
\frac{d}{dt}\|\Delta_q u\|_{L^2} &\leqslant \|u_x\|_{L^\infty}\|\Delta_q u\|_{L^2} + Cc_q 2^{-\frac{3}{2}q}\|u_x\|_{L^\infty}\|u\|_{B^{3/2}_{2,1}} \\
&\quad + \left\|\Delta_q P_1(D)\left[u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right]\right\|_{L^2},
\end{aligned} \tag{6.5}$$

where $\sum_{q\geqslant -1} c^q = 1$. Integrating with respect to the variable t yields

$$\|\Delta_q u\|_{L^2} \leqslant \|\Delta_q u_0\|_{L^2} + \int_0^t \|u_x\|_{L^\infty} \|\Delta_q u\|_{L^2}\, d\tau + Cc_q 2^{-\frac{3}{2}q} \int_0^t \|u_x\|_{L^\infty} \|u\|_{B^{3/2}_{2,1}}\, d\tau$$
$$+ \int_0^t \left\| \Delta_q P_1(D)\left[u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right]\right\|_{L^2} d\tau. \tag{6.6}$$

Multiplying both sides of (6.6) by $2^{\frac{3}{2}q}$, taking ℓ^1 norm yield

$$\|u\|_{B^{3/2}_{2,1}} \leqslant \|u_0\|_{B^{3/2}_{2,1}} + \int_0^t \|u_x\|_{L^\infty} \|u(\tau)\|_{B^{3/2}_{2,1}}\, dx + C\int_0^t \left\| P_1(D)\left[u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right]\right\|_{B^{3/2}_{2,1}} d\tau. \tag{6.7}$$

By using (4) of Lemma 2.3, we have that

$$\left\| P_1(D)\left[u^2 + \frac{1}{2}u_x^2 + \frac{1}{2}\gamma^2 - \frac{1}{2}\gamma_x^2\right]\right\|_{B^{3/2}_{2,1}}$$
$$\leqslant C\left[\|u\|_{B^{3/2}_{2,1}} + \|\gamma\|_{B^{3/2}_{2,1}}\right]\left[\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|\gamma\|_{L^\infty} + \|\gamma_x\|_{L^\infty}\right]. \tag{6.8}$$

Inserting (6.8) into (6.7) yields

$$\|u\|_{B^{3/2}_{2,1}} \leqslant \|u_0\|_{B^{3/2}_{2,1}} + C\int_0^t \left[\|u\|_{B^{3/2}_{2,1}} + \|\gamma\|_{B^{3/2}_{2,1}}\right]\left[\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|\gamma\|_{L^\infty} + \|\gamma_x\|_{L^\infty}\right] d\tau. \tag{6.9}$$

Applying Δ_q to (1.11) yields

$$\partial_t \Delta_q \gamma + u\Delta_q \gamma_x = [u, \Delta_q]\gamma_x + \Delta_q P_2(D)\left[(u_x\gamma_x)_x + u_x\gamma\right]. \tag{6.10}$$

Multiplying (6.10) by $2\Delta_q\gamma$ and integrating by parts with respect to the variable x yield

$$\frac{d}{dt}\|\Delta_q\gamma\|_{L^2}^2 = 2\int_{\mathbf{R}} u_x[\Delta_q\gamma]^2\, dx + 2\int_{\mathbf{R}} \Delta_q\gamma[u, \Delta_q]\gamma_x\, dx + 2\int_{\mathbf{R}} \Delta_q\gamma\Delta_q P_2(D)\left[(u_x\gamma_x)_x + u_x\gamma\right] dx$$
$$\leqslant 2\|u_x\|_{L^\infty}\|\Delta_q\gamma\|_{L^2}^2 + 2\|\Delta_q\gamma\|_{L^2}\left\|[u, \Delta_q]\gamma_x\right\|_{L^2}$$
$$+ 2\|\Delta_q\gamma\|_{L^2}\left\|\Delta_q P_2(D)\left[(u_x\gamma_x)_x + u_x\gamma\right]\right\|_{L^2}. \tag{6.11}$$

From (6.11), we know that

$$\frac{d}{dt}\|\Delta_q\gamma\|_{L^2} \leqslant \|u_x\|_{L^\infty}\|\Delta_q\gamma\|_{L^2} + \left\|[u, \Delta_q]\gamma_x\right\|_{L^2} + \left\|\Delta_q P_2(D)\left[(u_x\gamma_x)_x + u_x\gamma\right]\right\|_{L^2}. \tag{6.12}$$

Integrating with respect to the variable t yields

$$\|\Delta_q\gamma\|_{L^2} \leqslant \|\Delta_q\gamma_0\|_{L^2} + \int_0^t \|u_x\|_{L^\infty}\|\Delta_q\gamma\|_{L^2}\, d\tau + \int_0^t \left\|[u, \Delta_q]\gamma_x\right\|_{L^2} d\tau$$
$$+ \int_0^t \left\|\Delta_q P_2(D)\left[(u_x\gamma_x)_x + u_x\gamma\right]\right\|_{L^2} d\tau. \tag{6.13}$$

Multiplying both sides of (6.13) by $2^{\frac{3}{2}q}$, taking ℓ^1 norm yield

$$\|\gamma\|_{B^{3/2}_{2,1}} \leqslant \|\gamma_0\|_{B^{3/2}_{2,1}} + \int_0^t \|u_x\|_{L^\infty}\|\gamma\|_{B^{3/2}_{2,1}} + C\int_0^t \left\|[u, \Delta_q]\gamma_x\right\|_{B^{3/2}_{2,1}} d\tau + \int_0^t \left\|P_2(D)\left[(u_x\gamma_x)_x + u_x\gamma\right]\right\|_{B^{3/2}_{2,1}} d\tau. \tag{6.14}$$

By using (2.54) of Lemma 2.100 in [1] and (4) of Lemma 2.3, respectively, we have that

$$\left\|[u, \Delta_q]\gamma_x\right\|_{B^{3/2}_{2,1}} \leqslant C\left[\|u_x\|_{L^\infty}\|\gamma\|_{B^{3/2}_{2,1}} + \|\gamma_x\|_{L^\infty}\|u\|_{B^{3/2}_{2,1}}\right] \tag{6.15}$$

and

$$\left\|P_2(D)\left[(u_x\gamma_x)_x + u_x\gamma\right]\right\|_{B^{3/2}_{2,1}} \leqslant C\left[\|u_x\|_{L^\infty}\|\gamma\|_{B^{3/2}_{2,1}} + \|\gamma_x\|_{L^\infty}\|u\|_{B^{3/2}_{2,1}} + \|\gamma\|_{L^\infty}\|u\|_{B^{3/2}_{2,1}}\right]. \tag{6.16}$$

Inserting (6.15), (6.16) into (6.14) yields

$$\|\gamma\|_{B^{3/2}_{2,1}} \leqslant \|\gamma_0\|_{B^{3/2}_{2,1}} + C\int_0^t \left[\|u_x\|_{L^\infty} + \|u\|_{L^\infty} + \|\gamma_x\|_{L^\infty} + \|\gamma\|_{L^\infty}\right]\left[\|u\|_{B^{1/2}_{2,1}} + \|\gamma\|_{B^{1/2}_{2,1}}\right]d\tau. \tag{6.17}$$

Combining (6.9) with (6.17), we know that

$$\begin{aligned}\|u\|_{B^{3/2}_{2,1}} + \|\gamma\|_{B^{3/2}_{2,1}} \leqslant \left[\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}}\right] + C\int_0^t \left[\|u\|_{B^{3/2}_{2,1}} + \|\gamma\|_{B^{3/2}_{2,1}}\right] \\ \times\left[\|u\|_{L^\infty} + \|u_x\|_{L^\infty} + \|\gamma\|_{L^\infty} + \|\gamma_x\|_{L^\infty}\right]d\tau.\end{aligned} \tag{6.18}$$

By Gronwall's inequality to (6.18), we have that

$$\|u\|_{B^{3/2}_{2,1}} + \|\gamma\|_{B^{3/2}_{2,1}} \leqslant \left[\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}}\right]e^{C\int_0^t[\|u\|_{L^\infty}+\|u_x\|_{L^\infty}+\|\gamma\|_{L^\infty}+\|\gamma_x\|_{L^\infty}]\,d\tau}. \tag{6.19}$$

From [29], we know that

$$\|u\|^2_{H^1} + \|\gamma\|^2_{H^1} = E(u,\gamma) = E(u_0,\gamma_0) = \|u_0\|^2_{H^1} + \|\gamma_0\|^2_{H^1}. \tag{6.20}$$

Since $H^1(\mathbf{R}) \hookrightarrow L^\infty(\mathbf{R})$, from (6.20), we have that

$$\|u\|_{L^\infty} + \|\gamma\|_{L^\infty} \leqslant \|u\|_{H^1} + \|\gamma\|_{H^1} \leqslant 2E^{1/2}(u_0,\gamma_0). \tag{6.21}$$

Inserting (6.21) into (6.19) yields

$$\|u\|_{B^{3/2}_{2,1}} + \|\gamma\|_{B^{3/2}_{2,1}} \leqslant \left[\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}}\right]e^{C\int_0^t[2E^{1/2}(u_0,\gamma_0)+\|u_x\|_{L^\infty}+\|\gamma_x\|_{L^\infty}]\,d\tau}. \tag{6.22}$$

From (6.22), we have that if $T^* < \infty$, then $\int_0^{T^*}[\|u_x\|_{L^\infty} + \|\gamma_x\|_{L^\infty}]\,d\tau < \infty$ which yields that

$$\left\|u\left(T^*\right)\right\|_{B^{3/2}_{2,1}} + \left\|\gamma\left(T^*\right)\right\|_{B^{3/2}_{2,1}} < \infty. \tag{6.23}$$

(6.23) contradicts with the fact that T^* is the maximal time of existence. Thus, if $T^* < \infty$, then $\int_0^{T^*}[\|u_x\|_{L^\infty} + \|\gamma_x\|_{L^\infty}]\,d\tau = \infty$. Since $B^{1/2}_{2,1} \hookrightarrow L^\infty$, from (6.18), we have that

$$\|u\|_{B^{3/2}_{2,1}} + \|\gamma\|_{B^{3/2}_{2,1}} \leqslant \left[\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}}\right] + C\int_0^t \left[\|u\|_{B^{3/2}_{2,1}} + \|\gamma\|_{B^{3/2}_{2,1}}\right]^2 d\tau. \tag{6.24}$$

Solving (6.24) leads to

$$\left\|u(t)\right\|_{B^{3/2}_{2,1}} + \left\|\gamma(t)\right\|_{B^{3/2}_{2,1}} \leqslant \frac{\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}}}{1 - C(\|u_0\|_{B^{3/2}_{2,1}} + \|\gamma_0\|_{B^{3/2}_{2,1}})t}. \tag{6.25}$$

From (6.25) and (6.22), we know that $T^\star \geqslant \frac{1}{C(\|u_0\|_{B^{3/2}_{2,1}}+\|\gamma_0\|_{B^{3/2}_{2,1}})}$.

We have completed the proof of Theorem 1.2. □

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