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Fractional elliptic equations, Caccioppoli estimates and regularity ☆

Luis A. Caffarelli *, Pablo Raúl Stinga

Department of Mathematics, The University of Texas at Austin, 1 University Station, C1200, Austin, TX 78712-1202, United States

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Abstract

Let $L=-\operatorname{div}_x(A(x)\nabla_x)$ be a uniformly elliptic operator in divergence form in a bounded domain Ω. We consider the fractional nonlocal equations

$$\begin{cases} L^s u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} L^s u = f, & \text{in } \Omega, \\ \partial_A u = 0, & \text{on } \partial\Omega. \end{cases}$$

Here L^s, $0<s<1$, is the fractional power of L and $\partial_A u$ is the conormal derivative of u with respect to the coefficients $A(x)$. We reproduce Caccioppoli type estimates that allow us to develop the regularity theory. Indeed, we prove interior and boundary Schauder regularity estimates depending on the smoothness of the coefficients $A(x)$, the right hand side f and the boundary of the domain. Moreover, we establish estimates for fundamental solutions in the spirit of the classical result by Littman–Stampacchia–Weinberger and we obtain nonlocal integro-differential formulas for $L^s u(x)$. Essential tools in the analysis are the semigroup language approach and the extension problem.
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1. Introduction

In a bounded Lipschitz domain $\Omega\subset\mathbb{R}^n$, $n\geq 1$, we consider an elliptic operator in divergence form

$$Lu=-\operatorname{div}_x(A(x)\nabla_x u),$$

with boundary condition

$$\text{Dirichlet: } u=0, \quad \text{or} \quad \text{Neumann: } \partial_A u := A(x)\nabla_x u\cdot\nu=0, \quad \text{on } \partial\Omega,$$

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* Corresponding author.
E-mail addresses: caffarel@math.utexas.edu (L.A. Caffarelli), stinga@math.utexas.edu (P.R. Stinga).

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where ν is the exterior unit normal to $\partial\Omega$. The coefficients are symmetric $A(x)=A^{ij}(x)=A^{ji}(x)$, $i,j=1,\dots,n$, bounded and measurable in Ω and satisfy the uniform ellipticity condition $\Lambda_1|\xi|^2\le A(x)\xi\cdot\xi\le\Lambda_2|\xi|^2$, for all $\xi\in\mathbb{R}^n$ and almost every $x\in\Omega$, for some ellipticity constants $0<\Lambda_1\le\Lambda_2$. In this paper we study interior and boundary regularity estimates for the fractional nonlocal problem $L^su=f$ in Ω, subject to the boundary conditions above, in the cases when f is Hölder continuous, see Theorems 1.1, 1.3 and 1.4, and when f is just L^p integrable, see Theorems 1.2 and 1.5. These estimates of course depend on the regularity of the coefficients A and of the boundary of Ω. Our main tools are the semigroup language approach as developed in [32] and the extension problem as introduced in [7]. We obtain Caccioppoli type estimates that are combined with a compactness and approximation argument based on the ideas of [5] to prove the regularity results.

Let us begin by considering the case of Dirichlet boundary condition. By using the L^2-Dirichlet eigenvalues and eigenfunctions $(\lambda_k,\phi_k)_{k=0}^\infty$, $\phi_k\in H_0^1(\Omega)$, of L we can define the fractional powers L^su, $0<s<1$, for u in the domain $\operatorname{Dom}(L^s)\equiv\mathcal{H}^s$ (see Remark 2.1) in the natural way. If $u(x)=\sum_{k=0}^\infty u_k\phi_k(x)$, $x\in\Omega$, then

$$L^su(x)=\sum_{k=0}^\infty\lambda_k^su_k\phi_k(x).$$

Observe that $u=0$ on $\partial\Omega$. Equivalently, we have the semigroup formula

$$L^su(x)=\frac{1}{\Gamma(-s)}\int_0^\infty\left(e^{-tL}u(x)-u(x)\right)\frac{dt}{t^{1+s}},\tag{1.1}$$

where $\{e^{-tL}u\}_{t>0}$ is the heat diffusion semigroup generated by L with the Dirichlet boundary condition and Γ is the Gamma function. See Section 2. It is clear that for f in the dual space $\mathcal{H}^{-s}\equiv(\mathcal{H}^s)'$ there exists a unique solution $u\in\mathcal{H}^s$ to the fractional nonlocal equation

$$\begin{cases}L^su=f, & \text{in }\Omega,\\ u=0, & \text{on }\partial\Omega.\end{cases}\tag{1.2}$$

Starting from (1.1) and by using the heat kernel for e^{-tL} we are able to obtain integro-differential formulas for $L^su(x)$ of the form

$$\langle L^su,\psi\rangle=\int_\Omega\int_\Omega\big(u(x)-u(z)\big)\big(\psi(x)-\psi(z)\big)K_s(x,z)\,dx\,dz+\int_\Omega u(x)\psi(x)B_s(x)\,dx,\tag{1.3}$$

where $\psi\in\mathcal{H}^s$, see Theorem 2.3. Observe that this formula is parallel to the weak form interpretation of Lu in $H_0^1(\Omega)$. Estimates for the kernel $K_s(x,z)$ and the fundamental solution $G_s(x,z)$ of L^s are contained in Theorems 2.4, 2.6 and 2.7. In particular, we show that the fundamental solution satisfies the interior estimate

$$G_s(x,z)\sim\frac{1}{|x-z|^{n-2s}},\qquad\text{for }x,z\in\Omega,$$

when $n\neq 2s$, with a logarithmic estimate when $n=2s$.

Similarly, we can define the fractional powers of L_N, the operator L subject to Neumann boundary conditions. In this case we use the Neumann eigenvalues and eigenfunctions $(\mu_k,\varphi_k)_{k=0}^\infty$, $\varphi_k\in H^1(\Omega)$, to define L_N^su as

$$L_N^su(x)=\sum_{k=1}^\infty\mu_k^su_k\varphi_k(x).\tag{1.4}$$

The formula in (1.1) is also valid for L_N in place of L. Then we obtain the integro-differential formula

$$\langle L_N^su,\psi\rangle=\int_\Omega\int_\Omega\big(u(x)-u(z)\big)\big(\psi(x)-\psi(z)\big)K_s^N(x,z)\,dx\,dz,\tag{1.5}$$

where the kernel $K_s^N(x,z)$ is given in terms of the heat kernel for e^{-tL_N}. Notice the difference between this formula and the one in (1.3) for the Dirichlet case. This is so because for the Neumann boundary condition we have

$e^{-tL_N}1 \equiv 1$, while for the Dirichlet condition $e^{-tL}1 \neq 1$. Now if $\int_\Omega f\,dx = 0$ then there exists a unique solution $u \in \mathrm{Dom}(L_N^s) = H^s(\Omega)$ to

$$\begin{cases} L^s u = f, & \text{in } \Omega, \\ \partial_A u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$

with $\int_\Omega u\,dx = 0$. For the details see Section 7.

It is already known, see [33], that the fractional operators (1.1) can be described as Dirichlet-to-Neumann maps for an extension problem in the spirit of the extension problem for the fractional Laplacian on $\mathbb{R}^n$ of [7]. In fact, let $U = U(x,y) : \Omega \times (0,\infty) \to \mathbb{R}$ be the solution to the degenerate elliptic equation with A_2 weight

$$\begin{cases} \mathrm{div}(y^a B(x)\nabla U) = 0, & \text{in } \Omega \times (0,\infty), \\ U = 0, & \text{on } \partial\Omega \times [0,\infty), \\ U(x,0) = u(x), & \text{on } \Omega, \end{cases} \tag{1.7}$$

where

$$B(x) := \begin{bmatrix} A(x) & 0 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}, \quad \text{and} \quad a := 1 - 2s \in (-1,1). \tag{1.8}$$

Then, for $c_s = |\Gamma(-s)|/(4^s\Gamma(s)) > 0$,

$$-\frac{1}{2s}\lim_{y\to 0^+} y^a U_y(x,y) = -\lim_{y\to 0^+}\frac{U(x,y)-U(x,0)}{y^{2s}} = c_s L^s u(x), \quad x \in \Omega.$$

Moreover, there are explicit formulas for U in terms of the semigroup e^{-tL}. See Theorem 2.5 and the comments before it. When $A(x) = I$ and $\Omega = \mathbb{R}^n$ in (1.7) we recover the extension problem for the fractional Laplacian of [7]. By replacing the second equation for U in (1.7) by $\partial_A U(x,y) = 0$ for all $y \geq 0$ we get the extension problem for the fractional operator L_N^s.

Fractional powers of elliptic operators as those above arise naturally in applications, for instance, in nonlinear elasticity, probability and mathematical biology. Consider for example the following thin obstacle problem for an elastic membrane $U(x,y) : \Omega \times (0,\infty) \to \mathbb{R}$ and an obstacle $\varphi : \Omega \to \mathbb{R}$ such that $\varphi \leq 0$ on $\partial\Omega$:

$$\begin{cases} U_{yy} - LU = 0 = \mathrm{div}(B(x)\nabla U), & \text{in } \Omega \times (0,\infty), \\ U(x,0) \geq \varphi(x), & \text{on } \Omega, \\ U_y(x,0) \leq 0, & \text{on } \{U(x,0) = \varphi(x)\}, \\ U_y(x,0) = 0, & \text{on } \{U(x,0) > \varphi(x)\}. \end{cases}$$

We also require for the membrane U to be at level zero (Dirichlet) or to have zero flux (Neumann) on $\partial\Omega \times [0,\infty)$. The classical case of the Signorini problem is when $A(x) = I$, so the membrane is a harmonic function in $\Omega \times (0,\infty)$. It is cleat that the solution $U(x,y)$ to the first equation above with the boundary datum $u(x) := U(x,0)$ and the Dirichlet (or Neumann) boundary condition on $\partial\Omega \times [0,\infty)$ is given by the Poisson semigroup generated by L (or L_N):

$$U(x,y) = e^{-yL^{1/2}}u(x), \quad x \in \Omega,\ y \geq 0.$$

Observe that $U_y(x,y) = -L^{1/2}e^{-yL^{1/2}}u(x)$ (see [32]). Therefore we readily see that the membrane solves the thin obstacle problem if and only if its trace u solves the fractional obstacle problem

$$\begin{cases} u \geq \varphi, & \text{in } \Omega, \\ L^{1/2}u \geq 0, & \text{in } \{u = \varphi\}, \\ L^{1/2}u = 0, & \text{in } \{u > \varphi\}, \end{cases}$$

with $u = 0$ (or $\partial_A u = 0$) on $\partial\Omega$, see [32]. This obstacle problem for $L = -\Delta$ and $\Omega = \mathbb{R}^n$ was studied in [6,29]. Another application comes from the theory of stochastic processes. It is known that there is a Markov process Y_t having as generator the fractional power $(-\Delta_D)^s$ of the Dirichlet Laplacian $-\Delta_D$ on Ω. Indeed, we first kill the Wiener process X_t at τ_Ω, the first exit time of X_t from Ω, and then we subordinate the killed Wiener process with an s-stable subordinator T_t. Hence $Y_t = X_{T_t}$ is the desired process, see for example [30] and the references therein. For a semilinear problem involving the fractional Dirichlet Laplacian see [9] and the references therein. By considering nonlocal chemical diffusion in the Keller–Segel model one is led to a semilinear problem for the fractional Neumann

Laplacian, see [34]. Finally, we mention that finite element approximations for the fractional problem (1.2) were studied in [22] by using the extension problem.

We now present the interior regularity estimates.

Theorem 1.1 *(Interior regularity for f in C^α). Assume that Ω is a bounded Lipschitz domain and that $f \in C^{0,\alpha}(\Omega)$, for some $0<\alpha<1$. Let u be a solution to (1.2) or (1.6).*

(1) *Suppose that $0<\alpha+2s<1$ and that $A(x)$ is continuous in Ω. Then $u \in C^{0,\alpha+2s}(\Omega)$ and*

$$[u]_{C^{0,\alpha+2s}(\Omega)} \le C\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + \|f\|_{C^{0,\alpha}(\Omega)}\big).$$

(2) *Suppose that $1<\alpha+2s<2$ and that $A(x)$ is in $C^{0,\alpha+2s-1}(\Omega)$. Then $u \in C^{1,\alpha+2s-1}(\Omega)$ and*

$$[u]_{C^{1,\alpha+2s-1}(\Omega)} \le C\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + \|f\|_{C^{0,\alpha}(\Omega)}\big).$$

The constants C above depend only on ellipticity, n, Ω, α, s and the modulus of continuity of $A(x)$.

Theorem 1.2 *(Interior regularity for f in L^p). Assume that Ω is a bounded Lipschitz domain and that $f \in L^p(\Omega)$, for some $1<p<\infty$. Let u be a solution to (1.2) or (1.6).*

(1) *Suppose that $n/(2s)<p<n/(2s-1)^+$ and that $A(x)$ is continuous in Ω. Then $u \in C^{0,\alpha}(\Omega)$, for $\alpha = 2s - n/p \in (0,1)$, and*

$$[u]_{C^{0,\alpha}(\Omega)} \le C\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + \|f\|_{L^p(\Omega)}\big).$$

(2) *Suppose that $s>1/2$, $p>n/(2s-1)$ and that $A(x)$ is in $C^{0,\alpha}(\Omega)$, for $\alpha = 2s-n/p-1 \in (0,1)$. Then $u \in C^{1,\alpha}(\Omega)$ and*

$$[u]_{C^{1,\alpha}(\Omega)} \le C\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + \|f\|_{L^p(\Omega)}\big).$$

The constants C above depend only on ellipticity, n, Ω, α, s and the modulus of continuity of $A(x)$.

The results above should be compared with the classical regularity estimates for the fractional Laplacian and with the classical Schauder estimates for divergence form elliptic operators. If $(-\Delta)^s u = f$ in $\mathbb{R}^n$ and $f \in C^\alpha$ then $u \in C^{\alpha+2s}$. On the other hand, if $Lu = f$ and the coefficients $A(x)$ and the right hand side f are in C^α, then $u \in C^{1,\alpha}$ in the interior. If the coefficients are just continuous and f is in L^p, for some $n/2<p<n$, then $u \in C^{2-n/p}$, while if $p>n$ and the coefficients are Hölder continuous with exponent $\alpha = 2-n/p-1$ then $u \in C^{1,\alpha}$ in the interior. See Proposition 5.1 and [15,18,29,31].

Notice that in Theorems 1.1 and 1.2 we require the coefficients to be continuous in part (1) and Hölder continuous in part (2). The idea behind these results is to compare the solution u with the solution of the equation with frozen coefficients. In (1) we notice that $u-c$ is still a solution in the interior for any constant c, so the regularity basically comes from the right hand side as in the case of the fractional Laplacian. For part (2), if ℓ is a linear function then $u-\ell$ is not a solution of the same equation. Then Hölder regularity in the coefficients is needed in order to gain a decay in the oscillation of the remainder error in the right hand side.

Next we establish the boundary regularity in the case of Dirichlet boundary condition.

Theorem 1.3 *(Boundary regularity for f in C^α – Dirichlet). Assume that Ω is a bounded domain and that $f \in C^{0,\alpha}(\overline{\Omega})$, for some $0<\alpha<1$. Let u be a solution to (1.2).*

(1) *Suppose that $0<\alpha+2s<1$, Ω is a C^1 domain and that $A(x)$ is continuous in $\overline{\Omega}$. Then*

$$u(x) \sim \operatorname{dist}(x,\partial\Omega)^{2s} + v(x), \quad \text{for } x \text{ close to } \partial\Omega,$$

where $v \in C^{0,\alpha+2s}(\overline{\Omega})$. Moreover,

$$[v]_{C^{0,\alpha+2s}(\overline{\Omega})} \le C\big(1 + \|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + \|f\|_{C^{0,\alpha}(\overline{\Omega})}\big).$$

(2) *Suppose that $s \geq 1/2$, $1 < \alpha + 2s < 2$, Ω is a $C^{1,\alpha+2s-1}$ domain and that $A(x)$ is in $C^{0,\alpha+2s-1}(\overline{\Omega})$. Then*

$$u(x) \sim \operatorname{dist}(x, \partial\Omega) + v(x), \quad \text{for } x \text{ close to } \partial\Omega,$$

where $v \in C^{1,\alpha+2s-1}(\overline{\Omega})$. Moreover,

$$[v]_{C^{1,\alpha+2s-1}(\overline{\Omega})} \leq C\big(1 + \|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + \|f\|_{C^{0,\alpha}(\overline{\Omega})}\big).$$

In both cases, if $f(x_0) = 0$ for some $x_0 \in \partial\Omega$, then $u(x_0) = v(x_0)$ and u has the same regularity as v at $x_0 \in \partial\Omega$. The constants C above depend only on ellipticity, n, Ω, α, s and the modulus of continuity of $A(x)$.

The result above should be compared with the boundary regularity estimates for the fractional Dirichlet Laplacian $(-\Delta_D^+)^s$ in the half space $\mathbb{R}^n_+$ contained in Theorem 5.3. Observe that here an odd reflection can be performed to compare with the global problem.

For the case of Neumann boundary condition the global regularity is the same as the interior regularity.

Theorem 1.4 *(Global regularity for f in C^α – Neumann). Assume that Ω is a bounded domain and that $f \in C^{0,\alpha}(\overline{\Omega})$, for some $0 < \alpha < 1$. Let u be a solution to (1.6).*

(1) *Suppose that $0 < \alpha + 2s < 1$, Ω is a C^1 domain and that $A(x)$ is continuous in $\overline{\Omega}$. Then $u \in C^{0,\alpha+2s}(\overline{\Omega})$ and*

$$[u]_{C^{0,\alpha+2s}(\overline{\Omega})} \leq C\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + \|f\|_{C^{0,\alpha}(\overline{\Omega})}\big).$$

(2) *Suppose that $s \geq 1/2$, $1 < \alpha + 2s < 2$, Ω is a $C^{1,\alpha+2s-1}$ domain and that $A(x)$ is in $C^{0,\alpha+2s-1}(\overline{\Omega})$. Then $u \in C^{1,\alpha+2s-1}(\overline{\Omega})$ and*

$$[u]_{C^{1,\alpha+2s-1}(\overline{\Omega})} \leq C\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + \|f\|_{C^{0,\alpha}(\overline{\Omega})}\big).$$

The constants C above depend only on ellipticity, n, Ω, α, s and the modulus of continuity of $A(x)$.

Again this Theorem should be compared with the boundary regularity estimates for the fractional Neumann Laplacian $(-\Delta_N^+)^s$ in the half space $\mathbb{R}^n_+$, see Theorem 7.2. In this case even reflections can be used to relate the problem with the global one.

Finally, for L^p right hand side, both Dirichlet and Neumann cases have the same regularity up to the boundary.

Theorem 1.5 *(Boundary regularity for f in L^p). Assume that Ω is a bounded domain and that $f \in L^p(\Omega)$, for some $1 < p < \infty$. Let u be a solution to (1.2) or (1.6).*

(1) *Suppose that $n/(2s) < p < n/(2s-1)^+$, Ω is a C^1 domain and that $A(x)$ is continuous in $\overline{\Omega}$. Then $u \in C^{0,\alpha}(\overline{\Omega})$, for $\alpha = 2s - n/p \in (0,1)$, and*

$$[u]_{C^{0,\alpha}(\overline{\Omega})} \leq C\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + \|f\|_{L^p(\Omega)}\big).$$

(2) *Suppose that $s > 1/2$, $p > n/(2s-1)$, Ω is a $C^{1,\alpha}$ domain and that $A(x)$ is in $C^{0,\alpha}(\Omega)$, for $\alpha = 2s - n/p - 1 \in (0,1)$. Then $u \in C^{1,\alpha}(\overline{\Omega})$ and*

$$[u]_{C^{1,\alpha}(\overline{\Omega})} \leq C\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + \|f\|_{L^p(\Omega)}\big).$$

The constants C above depend only on ellipticity, n, Ω, α, s and the modulus of continuity of $A(x)$.

We recall that if $Lu = f$ and the coefficients $A(x)$ and the right hand side f are in C^α up to the boundary of Ω then $u \in C^{1,\alpha}$ up to the boundary. If the coefficients are just continuous up to the boundary and f is in L^p for some $n/2 < p < n$ then u is globally in $C^{2-n/p}$, while if $p > n$ and the coefficients are Hölder continuous up to the boundary with exponent $\alpha = 2 - n/p - 1$ then $u \in C^{1,\alpha}$ up to the boundary. See [15,18].

Some bounds at the boundary for the fractional Dirichlet Laplacian when f is just bounded and Ω is smooth where obtained in [9]. For the particular case $s = 1/2$ in a smooth domain with a right hand side vanishing at the boundary see also [4]. The regularity estimates for the Neumann case generalize the results for the fractional Neumann Laplacian $(-\Delta_N)^{1/2}$ obtained in [34, Theorem 3.5]. When the coefficients $A(x)$ and the domain Ω are smooth, the L^p-domain and regularity of fractional powers of strongly elliptic operators was considered in [27,28], see also the recent preprint [17].[1] For the fractional Laplacian on $\mathbb{R}^n$ we know that the unique solution $u \in H^s(\mathbb{R}^n)$ of the Dirichlet problem

$$\begin{cases} (-\Delta)^s u = 1, & \text{in } B_1, \\ u = 0, & \text{in } \mathbb{R}^n \setminus B_1, \end{cases}$$

is given by $u(x) = c_{n,s}(1 - |x|^2)_+^s$, see [14]. Thus in general u is globally in C^s but not in any C^α for $\alpha > s$, see also [25]. For this case the boundary regularity in fractional Sobolev spaces and in Hölder spaces on smooth domains was studied in [16].

Throughout the paper we will mainly focus on the case of the Dirichlet boundary condition. We explain only in Section 7 how the case of the Neumann condition works by pointing out the main differences with the Dirichlet case. In Section 2 we define in a precise way the fractional operator L^s. By using the heat semigroup e^{-tL} and (1.1) we obtain the integro-differential formula (1.3), with estimates on the kernel. The extension problem is explained. We also include in this section the estimates for the fundamental solutions and comment about the Harnack inequality of [35] and the De Giorgi–Nash–Moser theory for the case of bounded measurable coefficients. Section 3 contains a Caccioppoli inequality that we use to prove an approximation lemma via a compactness argument. Here we also prove a trace inequality on balls with explicit dependence on the radius that will be useful to prove regularity. Then Section 4 is devoted to the proof of the interior regularity results (Dirichlet case). The case of the fractional Dirichlet Laplacian in a half space is studied in detail in Section 5. We collect in Section 6 the proof of the boundary estimates for the Dirichlet case.

Notation. The notation we will use in this paper is the following. The upper half space is given by $\mathbb{R}^n_+ := \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1}, x_n > 0\}$. For the extension problem we use the notation $\mathbb{R}^{n+1}_+ = \{(x, y) \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n, y > 0\}$. We usually write $X = (x, y) \in \mathbb{R}^{n+1}_+$. For $x_0 \in \mathbb{R}^n$ and $r > 0$ we denote

$$\begin{aligned} B_r(x_0) &= \{x \in \mathbb{R}^n : |x - x_0| < r\}, \\ B_r^+(x_0) &= B_r(x_0) \cap \mathbb{R}^n_+, \\ B_r(x_0)^* &= B_r(x_0) \times (0, r) \subset \mathbb{R}^{n+1}_+, \\ B_r^+(x_0)^* &= B_r^+(x_0) \times (0, r) \subset \mathbb{R}^n_+ \times (0, \infty). \end{aligned}$$

We will just put B_r, B_r^+, etc. whenever $x_0 = 0$. The letters C, c and d will denote positive constants that may change at each occurrence. We will add subscripts to them whenever we want to stress their dependence on other constants, domains, etc. The matrix $B(x)$ and the parameter a are given by (1.8). The notation div and ∇ stand for the divergence and the gradient with respect to the variable $X = (x, y) \in \Omega \times (0, \infty)$.

2. Fractional divergence form elliptic operators

Throughout this section, unless explicitly stated, Ω will be a bounded Lipschitz domain of $\mathbb{R}^n$ and the matrix of coefficients $A(x)$ will be uniformly elliptic, bounded and measurable.

2.1. Definition of L^s

The operator L is nonnegative and selfadjoint in the Sobolev space $H_0^1(\Omega)$. Therefore there exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions $\phi_k \in H_0^1(\Omega)$, $k = 0, 1, 2, \ldots$, that correspond to eigenvalues $0 < \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \nearrow \infty$. Let us define the domain $\mathcal{H}^s \equiv \mathrm{Dom}(L^s)$ of the fractional operator L^s, $0 < s < 1$, as the Hilbert space of functions

[1] We are grateful to Gerd Grubb for several interesting comments about the smooth case.

$$u=\sum_{k=0}^{\infty}u_k\phi_k=\sum_{k=0}^{\infty}\langle u,\phi_k\rangle_{L^2(\Omega)}\phi_k\in L^2(\Omega),$$

with inner product

$$\langle u,\psi\rangle_{\mathcal{H}^s}^2:=\sum_{k=0}^{\infty}\lambda_k^s u_k d_k,$$

where $\psi=\sum_{k=0}^{\infty}d_k\phi_k\in\mathcal{H}^s$. Observe that for some positive constant C we have $\|u\|_{L^2(\Omega)}\leq C\langle u,u\rangle_{\mathcal{H}^s}=C\|u\|_{\mathcal{H}^s}$, for $u\in\mathcal{H}^s$, so that $\langle\cdot,\cdot\rangle_{\mathcal{H}^s}$ defines indeed an inner product in $\mathcal{H}^s$. For $u\in\mathcal{H}^s$, let $L^s u$ be the element in the dual space $\mathcal{H}^{-s}:=(\mathcal{H}^s)'$ given by

$$L^s u=\sum_{k=0}^{\infty}\lambda_k^s u_k\phi_k,\quad \text{in } \mathcal{H}^{-s}.$$

Namely,

$$\langle L^s u,\psi\rangle=\sum_{k=0}^{\infty}\lambda_k^s u_k d_k=\langle u,\psi\rangle_{\mathcal{H}^s},$$

where $\langle\cdot,\cdot\rangle$ denotes the paring between $\mathcal{H}^s$ and $\mathcal{H}^{-s}$. By the Riesz representation theorem any functional $f\in\mathcal{H}^{-s}$ can be written as $f=\sum_{k=0}^{\infty}f_k\varphi_k$ in $\mathcal{H}^{-s}$, where the coefficients f_k satisfy $\sum_{k=0}^{\infty}\lambda_k^{-s}f_k^2<\infty$. With these definitions and observations, if $f=\sum_{k=0}^{\infty}f_k\phi_k\in\mathcal{H}^{-s}$ then the unique solution $u\in\mathcal{H}^s$ to the Dirichlet problem (1.2) is given by $u=\sum_{k=0}^{\infty}\lambda_k^{-s}f_k\phi_k\in\mathcal{H}^s$. More generally, if $f\in\mathcal{H}^r$ for $r\geq 0$ (here $\mathcal{H}^0:=L^2(\Omega)$), then there exists a unique solution $u\in\mathcal{H}^{r+2s}$.

Remark 2.1. We use the following notation:

$$H^s:=\begin{cases}H^s(\Omega), & \text{when } 0<s<1/2,\\ H_{00}^{1/2}(\Omega), & \text{when } s=1/2,\\ H_0^s(\Omega), & \text{when } 1/2<s<1.\end{cases}\tag{2.1}$$

The spaces $H^s(\Omega)$ and $H_0^s(\Omega)$, $s\neq 1/2$, are the classical fractional Sobolev spaces given by the completion of $C_c^\infty(\Omega)$ under the norm

$$\|u\|_{H^s(\Omega)}^2=\|u\|_{L^2(\Omega)}^2+[u]_{H^s(\Omega)}^2,$$

where

$$[u]_{H^s(\Omega)}^2=\int_\Omega\int_\Omega\frac{(u(x)-u(z))^2}{|x-z|^{n+2s}}\,dx\,dz,\quad 0<s<1.$$

The space $H_{00}^{1/2}(\Omega)$ is the Lions–Magenes space which consists of functions u in $L^2(\Omega)$ such that $[u]_{H^{1/2}(\Omega)}<\infty$ and

$$\int_\Omega\frac{u(x)^2}{\operatorname{dist}(x,\partial\Omega)}\,dx<\infty.$$

See [20, Chapter 1], also [22, Section 2] for a discussion. The norm in any of these spaces is denoted by $\|\cdot\|_{H^s}$. We will later see, by using the extension problem, that in fact we have $\mathcal{H}^s=H^s$ as Hilbert spaces.

2.2. Heat semigroup and pointwise formula

Given a function $u=\sum_{k=0}^{\infty}u_k\phi_k$ in $L^2(\Omega)$, the weak solution $v(x,t)$ to the heat equation for L with Dirichlet boundary condition

$$\begin{cases} v_t = -Lv, & \text{in } \Omega \times (0,\infty), \\ v(x,t) = 0, & \text{on } \partial\Omega \times [0,\infty), \\ v(x,0) = u(x), & \text{on } \Omega, \end{cases}$$

is given by

$$v(x,t) \equiv e^{-tL}u(x) = \sum_{k=0}^{\infty} e^{-t\lambda_k} u_k \phi_k(x),$$

in the sense that, for every test function $\psi = \sum_{k=0}^{\infty} d_k \phi_k \in H_0^1(\Omega)$,

$$\langle e^{-tL}u, \psi \rangle_{L^2(\Omega)} = \sum_{k=0}^{\infty} e^{-t\lambda_k} u_k d_k.$$

In particular, $e^{-tL}u \in L^2((0,\infty); H_0^1(\Omega)) \cap C([0,\infty); L^2(\Omega))$, and $\partial_t e^{-tL}u \in L^2((0,\infty); H^{-1}(\Omega))$.

Lemma 2.2. *Let $u \in \mathcal{H}^s$. Then*

$$L^s u = \frac{1}{\Gamma(-s)} \int_0^{\infty} \left(e^{-tL}u - u\right) \frac{dt}{t^{1+s}}, \quad \text{in } \mathcal{H}^{-s}.$$

More precisely, if $\psi \in \mathcal{H}^s$ then

$$\langle L^s u, \psi \rangle = \frac{1}{\Gamma(-s)} \int_0^{\infty} \left(\langle e^{-tL}u, \psi \rangle_{L^2(\Omega)} - \langle u, \psi \rangle_{L^2(\Omega)}\right) \frac{dt}{t^{1+s}}. \tag{2.2}$$

Proof. We have the following numerical formula with the Gamma function:

$$\lambda^s = \frac{1}{\Gamma(-s)} \int_0^{\infty} \left(e^{-t\lambda} - 1\right) \frac{dt}{t^{1+s}}, \quad \text{for } \lambda > 0,\ 0 < s < 1.$$

Then, if $\psi = \sum_{k=0}^{\infty} d_k \phi_k$,

$$\begin{aligned} \langle L^s u, \psi \rangle = \sum_{k=0}^{\infty} \lambda_k^s u_k d_k &= \frac{1}{\Gamma(-s)} \sum_{k=0}^{\infty} \int_0^{\infty} \left(e^{-t\lambda_k} u_k d_k - u_k d_k\right) \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)} \int_0^{\infty} \left(\sum_{k=0}^{\infty} e^{-t\lambda_k} u_k d_k - \sum_{k=0}^{\infty} u_k d_k\right) \frac{dt}{t^{1+s}}, \end{aligned}$$

which is the desired formula. The last identity follows from Fubini's theorem, since $u, \psi \in \mathcal{H}^s$. □

Let $W_t(x,z)$ be the distributional heat kernel for L with the Dirichlet boundary condition, that is,

$$W_t(x,z) = \sum_{k=0}^{\infty} e^{-t\lambda_k} \phi_k(x)\phi_k(z) = W_t(z,x), \quad t > 0,\ x, z \in \Omega. \tag{2.3}$$

It is clear that $W_t(x,z) \geq 0$ (see [10]) and that if $u, \psi \in L^2(\Omega)$ then

$$\langle e^{-tL}u, \psi \rangle_{L^2(\Omega)} = \int_\Omega \int_\Omega W_t(x,z) u(z) \psi(x)\, dz\, dx = \langle u, e^{-tL}\psi \rangle_{L^2(\Omega)}, \quad t \geq 0.$$

Theorem 2.3 *(Pointwise/energy formula). Let $u, \psi \in \mathcal{H}^s$. Then (1.3) holds, where*

$$0 \le K_s(x,z) := \frac{1}{2|\Gamma(-s)|} \int_0^\infty W_t(x,z)\,\frac{dt}{t^{1+s}} \le \frac{c_{n,s}}{|x-z|^{n+2s}}, \quad x \neq z, \tag{2.4}$$

and

$$B_s(x) := \frac{1}{2|\Gamma(-s)|} \int_0^\infty \left(1 - e^{-tL}1(x)\right) \frac{dt}{t^{1+s}} \ge 0. \tag{2.5}$$

Proof. By plugging the heat kernel into (2.2),

$$\begin{aligned}
\Gamma(-s)\langle L^s u, \psi\rangle &= \int_0^\infty \int_\Omega \left[\int_\Omega W_t(x,z)u(x)\psi(z)\,dx - u(z)\psi(z) \right] dz\,\frac{dt}{t^{1+s}} \\
&= \int_0^\infty \int_\Omega \left[\int_\Omega W_t(x,z)\big(u(x)-u(z)\big)\psi(z)\,dx + u(z)\psi(z)\left(\int_\Omega W_t(x,z)\,dx - 1 \right) \right] dz\,\frac{dt}{t^{1+s}} \\
&= \int_0^\infty \int_\Omega \int_\Omega W_t(x,z)\big(u(x)-u(z)\big)\psi(z)\,dx\,dz\,\frac{dt}{t^{1+s}} \\
&\quad + \int_0^\infty \int_\Omega u(z)\psi(z)\left(e^{-tL}1(z) - 1\right) dz\,\frac{dt}{t^{1+s}} =: I.
\end{aligned}$$

By exchanging the roles of x and z and using the symmetry of the heat kernel, we also have that

$$I = -\int_0^\infty \int_\Omega \int_\Omega W_t(x,z)\big(u(x)-u(z)\big)\psi(x)\,dx\,dz\,\frac{dt}{t^{1+s}} + \int_0^\infty \int_\Omega u(z)\psi(z)\left(e^{-tL}1(z)-1\right) dz\,\frac{dt}{t^{1+s}}.$$

Therefore, by adding both identities for I,

$$\begin{aligned}
2|\Gamma(-s)|\langle L^s u, \psi\rangle &= \int_0^\infty \int_\Omega \int_\Omega W_t(x,z)\big(u(x)-u(z)\big)\big(\psi(x)-\psi(z)\big)\,dx\,dz\,\frac{dt}{t^{1+s}} \\
&\quad + \int_0^\infty \int_\Omega u(z)\psi(z)\left(1 - e^{-tL}1(z)\right) dz\,\frac{dt}{t^{1+s}}.
\end{aligned} \tag{2.6}$$

To reach the final expression with the kernel $K_s(x,z)$ and the function $B_s(x)$ we need to interchange the order of integration in (2.6). The estimate for the kernel $K_s(x,z)$ is contained in Theorem 2.4. Since $u, \psi \in H^s$ (see Remark 2.1) it follows that Fubini's theorem can be applied to the first term in the right hand side of (2.6). For the second term in (2.6), take $\psi = u$. Observe that $0 \le e^{-tL}1(z) \le 1$, which follows from the maximum principle. This and the fact that $K_s(x,z) \ge 0$ imply in (2.6) that

$$\begin{aligned}
0 &\le \int_0^\infty \int_\Omega |u(z)|^2\big(1 - e^{-tL}1(z)\big)\,dz\,\frac{dt}{t^{1+s}} \\
&= 2|\Gamma(-s)|\langle L^s u, u\rangle - \int_\Omega \int_\Omega \big(u(x)-u(z)\big)^2 K_s(x,z)\,dx\,dz \le 2|\Gamma(-s)|\|u\|_{\mathcal{H}^s} < \infty.
\end{aligned}$$

Then, by Fubini's theorem,

$$0 \le \int_0^\infty \int_\Omega |u(z)|^2 \big(1 - e^{-tL}1(z)\big)\, dz \, \frac{dt}{t^{1+s}} = \int_\Omega |u(z)|^2 B_s(z)\, dz < \infty,$$

with $B_s(z)$ as in the statement. The same is true for when we replace u by ψ and by $u-\psi$. Thus, by writing $u\psi = \frac{1}{2}(u^2 + \psi^2 - (u-\psi)^2)$, it follows that we can apply Fubini's theorem to the second term of (2.6). □

Theorem 2.4 *(Estimates for $K_s(x,z)$). Let $K_s(x,z) \ge 0$ be the kernel in (2.4).*

(1) *If the coefficients $A(x)$ are bounded and measurable then*

$$K_s(x,z) \le \frac{c_{n,s}}{|x-z|^{n+2s}}, \quad x, z \in \Omega,\ x \ne z.$$

(2) *If the coefficients $A(x)$ are bounded and measurable in $\Omega = \mathbb{R}^n$ then*

$$K_s(x,z) \sim \frac{c_{n,s}}{|x-z|^{n+2s}}, \quad x, z \in \mathbb{R}^n,\ x \ne z.$$

In this case the function $B_s(x)$ of (2.5) is identically zero.

(3) *If the coefficients $A(x)$ are Hölder continuous in Ω with exponent $\alpha \in (0,1)$ then there exist positive constants c and $\eta \le 1 \le \rho$ depending only on n, α, Ω and ellipticity, with c depending also on s, such that*

$$c^{-1} \min\left(1, \frac{\phi_0(x)\phi_0(z)}{|x-z|^{2\eta}}\right) \frac{1}{|x-z|^{n+2s}} \le K_s(x,z) \le c \min\left(1, \frac{\phi_0(x)\phi_0(z)}{|x-z|^{2\rho}}\right) \frac{1}{|x-z|^{n+2s}},$$

where λ_0 and ϕ_0 are the first eigenvalue and the first eigenfunction of L. Here, for some constant $C > 0$ depending on α, n, Ω and ellipticity,

$$C^{-1} \operatorname{dist}(x, \partial\Omega)^\rho \le \phi_0(x) \le C \operatorname{dist}(x, \partial\Omega)^\eta.$$

(4) *Under the hypothesis of* (3)*, if in addition Ω is a $C^{1,\gamma}$ domain for some $0 < \gamma < 1$, then the estimate in* (3) *is true for $\eta = \rho = 1$ and c depending also on γ. In particular, the estimate holds when $L^s = (-\Delta_D)^s$, the fractional Dirichlet Laplacian in a $C^{1,\gamma}$ domain.*

Proof. We use the following known estimates for the heat kernel (2.3) and then integrate in t in (2.4) via the change of variables $r = |x-z|^2/t \in (0,\infty)$.

(1) In this case there exist constants $C, c > 0$ depending on ellipticity, n and Ω such that

$$W_t(x,z) \le C \frac{e^{-|x-z|^2/(ct)}}{t^{n/2}},$$

for all $x, z \in \Omega$, $t > 0$, see [10, p. 89], also [3].

(2) For the case of bounded measurable coefficients in the whole space, the result of Aronson [2] establishes that for some positive constants $c_1, \dots, c_4$ depending on ellipticity and n,

$$c_1 \frac{e^{-|x-z|^2/(c_2 t)}}{t^{n/2}} \le W_t(x,z) \le c_3 \frac{e^{-|x-z|^2/(c_4 t)}}{t^{n/2}},$$

for all $x, z \in \mathbb{R}^n$, $t > 0$. See also [10, p. 97]. Moreover, in this case we have $e^{-tL}1(x) \equiv 1$, so $B_s(x) \equiv 0$.

(3) Under these hypotheses it is proved in [23, Theorem 2.2] that there exist positive constants $\eta \le 1 \le \rho$ and c, c_1, c_2 depending only on n, α, Ω and ellipticity such that

$$c^{-1} \min\left(1, \frac{\phi_0(x)\phi_0(z)}{1 \wedge t^\eta}\right) e^{-\lambda_0 t} \frac{e^{-c_1|x-z|^2/t}}{1 \wedge t^{n/2}} \le K_t(x,z)$$
$$\le c^{-1} \min\left(1, \frac{\phi_0(x)\phi_0(z)}{1 \wedge t^\rho}\right) e^{-\lambda_0 t} \frac{e^{-c_2|x-z|^2/t}}{1 \wedge t^{n/2}},$$

for all $x, z \in \Omega$, $t > 0$. The behavior of ϕ_0 is also known, see [23, (1.2)].

(4) This follows from the fact that in the heat kernel estimate written in (3) above we can take $\eta = \rho = 1$, see [23, Remark 1, p. 123]. □

2.3. Extension problem

We particularize to our situation the extension problem of Stinga–Torrea [33], which is in turn a generalization of the Caffarelli–Silvestre extension problem of [7]. Let us explain the details, which can be found in [32,33].

Let $u \in \mathcal{H}^s$. Consider the solution $U = U(x, y) : \Omega \times [0, \infty) \to \mathbb{R}$ to the extension problem

$$\begin{cases} -LU + \frac{a}{y}U_y + U_{yy} = 0, & \text{in } \Omega \times (0, \infty), \\ U(x, y) = 0, & \text{on } \partial\Omega \times [0, \infty), \\ U(x, 0) = u(x), & \text{on } \Omega. \end{cases} \tag{2.7}$$

The equation above is in principle understood in the sense that U belongs to $C^\infty((0, \infty); H_0^1(\Omega)) \cap C([0, \infty); L^2(\Omega))$ and

$$\int_\Omega A(x)\nabla_x U(x, y)\nabla_x \eta(x)\,dx = \int_\Omega \big(\tfrac{a}{y}U_y + U_{yy}\big)\eta(x)\,dx, \quad \text{for each } y > 0,$$

for any test function $\eta \in H_0^1(\Omega)$. The boundary conditions in y read

$$\lim_{y\to 0^+} U(x, y) = u(x) \text{ in } L^2(\Omega), \quad \text{and} \quad \lim_{y\to\infty} U(x, y) = 0 \text{ weakly in } L^2(\Omega).$$

Notice that problem (2.7) can also be written in divergence form as (1.7). It was shown in [32,33] that if $u = \sum_{k=0}^\infty u_k\phi_k$ then the solution to this problem is

$$U(x, y) = y^s \frac{2^{1-s}}{\Gamma(s)} \sum_{k=0}^\infty \lambda_k^{s/2} u_k \mathcal{K}_s(\lambda_k^{1/2} y)\phi_k(x), \tag{2.8}$$

where $\mathcal{K}_s$ is the modified Bessel function of the second kind and parameter s. Equivalently,

$$\begin{aligned} U(x, y) &= \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-tL} u(x) \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} e^{-tL} (L^s u)(x) \frac{dt}{t^{1-s}}. \end{aligned} \tag{2.9}$$

By using the heat kernel one can show that

$$U(x, y) = \int_\Omega P_y^s(x, z)u(z)\,dz, \tag{2.10}$$

where the Poisson kernel $P_y^s(x, z)$ is given by

$$P_y^s(x, z) = \frac{y^{2s}}{4^s\Gamma(s)} \int_0^\infty e^{-y^2/(4t)} W_t(x, z) \frac{dt}{t^{1+s}}. \tag{2.11}$$

In addition, by letting $c_s = \frac{\Gamma(1-s)}{4^{s-1/2}\Gamma(s)} > 0$, we have

$$-\lim_{y\to 0^+} y^a U_y(x, y) = c_s L^s u, \quad \text{in } \mathcal{H}^{-s}. \tag{2.12}$$

It is easy to show, by using the representation with eigenfunctions and Bessel functions of (2.8), that U belongs to the space $H_0^1(\Omega \times (0, \infty), y^a dX)$, which is the completion of $C_c^\infty(\Omega \times [0, \infty))$ under the norm

$$\|U\|^2_{H_0^1(\Omega\times(0,\infty), y^a dX)} = \int_\Omega \int_0^\infty y^a\big(U^2 + |\nabla U|^2\big)\,dX.$$

See [11,13], also [36], for the theory of weighted Sobolev spaces. It is known (see [9, Proposition 2.1]) that these weighted Sobolev spaces have the fractional Sobolev spaces H^s defined in (2.1) as trace spaces, that is,

$$\|U(\cdot,0)\|_{H^s} \le C_{\Omega,a}\|U\|_{H_0^1(\Omega\times(0,\infty),y^a dX)}.$$

Therefore, $u(x)=U(x,0)\in H^s$. This and the fact that the norm in $H_0^1(\Omega\times(0,\infty),y^a dX)$ is comparable to the natural energy for the extension equation given by (2.13) show that $\mathcal{H}^s=H^s$, for all $0<s<1$, as we already mentioned at the end of Remark 2.1. We summarize all these considerations in the following result. See [32,33].

Theorem 2.5 *(Extension problem). Let $u\in\mathcal{H}^s$. There exists a unique weak solution $U\in H_0^1(\Omega\times(0,\infty),y^a dX)$ to the extension problem (1.7), where $B(x)$ and a are as in (1.8). Moreover, U is given by (2.9), which can also be written as (2.10), and it satisfies (2.12). More precisely, for each $\varphi\in H_0^1(\Omega\times(0,\infty),y^a dX)$,*

$$\int_\Omega\int_0^\infty y^a B(x)\nabla U\nabla\varphi\,dX = c_s\int_\Omega L^s u(x)\varphi(x,0)\,dx.$$

In particular, U is the unique minimizer of the energy functional

$$\mathcal{J}(U)=\int_\Omega\int_0^\infty y^a B(x)\nabla U\nabla U\,dX, \tag{2.13}$$

over the set $\mathcal{U}=\{U\in H_0^1(\Omega\times(0,\infty),y^a dX): U(x,0)=u(x)\}$, and for the minimizer U we have the identity

$$\int_\Omega\int_0^\infty y^a B(x)\nabla U\nabla U\,dX=\|L^{s/2}u\|_{L^2(\Omega)}^2=\|u\|_{\mathcal{H}^s}^2,$$

and the inequality

$$\int_\Omega\int_0^\infty y^a|U|^2\,dX\le C_{\Omega,s}\|u\|_{L^2(\Omega)}^2.$$

2.4. Scaling

For $u\in\mathcal{H}^s$ and $\lambda>0$, let

$$A_\lambda(x):=A(\lambda x),\qquad u_\lambda(x):=u(\lambda x),$$

and call L_λ the operator with coefficients A_λ in $\Omega_\lambda:=\{x:x/\lambda\in\Omega\}$. Then

$$(L_\lambda^s u_\lambda)(x)=\lambda^{2s}(L^s u)(\lambda x),\qquad\text{in }\Omega_\lambda.$$

In particular, if L has constant coefficients (as in the case of the Dirichlet Laplacian) then $L^s u_\lambda(x)=\lambda^{2s}L^s u(\lambda x)$, in Ω_λ. We present two different proofs.

Proof using the semigroup. Let $v(x,t)=e^{-tL}u(x)$. Since L is a linear second order divergence form elliptic operator, it follows that v satisfies the usual parabolic scaling. This immediately implies that the heat semigroup for L_λ is given by

$$e^{-tL_\lambda}u_\lambda(x)=v(\lambda x,\lambda^2 t),\qquad x\in\Omega_\lambda,\ t>0.$$

Now, by Lemma 2.2 and the change of variables $r=\lambda^2 t$, we see that the following identities hold in the weak sense:

$$(L_\lambda^s u_\lambda)(x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{-tL_\lambda}u_\lambda(x) - u_\lambda(x)\right)\frac{dt}{t^{1+s}}$$
$$= \frac{1}{\Gamma(-s)}\int_0^\infty \left(v(\lambda x,\lambda^2 t) - u(\lambda x)\right)\frac{dt}{t^{1+s}}$$
$$= \frac{\lambda^{2s}}{\Gamma(-s)}\int_0^\infty \left(v(\lambda x,r) - u(\lambda x)\right)\frac{dr}{r^{1+s}} = \lambda^{2s}(L^s u)(\lambda x). \qquad \square$$

Proof using the extension problem. Let U be the solution to the extension problem (1.7). Consider $U_\lambda(x,y) = U(\lambda x,\lambda y)$. This function is defined for x in Ω_λ and $y>0$. Then, by using the weak formulation of the extension problem it is easy to check that

$$\begin{cases} \operatorname{div}(y^a B_\lambda(x)\nabla U_\lambda) = 0, & \text{for } x\in\Omega_\lambda,\ y>0,\\ U_\lambda(x,y) = 0, & \text{for } x\in\partial\Omega_\lambda,\ y\geq 0,\\ U_\lambda(x,0) = u_\lambda(x), & \text{for } x\in\Omega_\lambda, \end{cases} \tag{2.14}$$

where $B_\lambda(x) = B(\lambda x)$, $x\in\Omega_\lambda$. Since (2.14) is the extension problem for u_λ and the operator L_λ^s,

$$-y^a\partial_y U_\lambda(x,y)\Big|_{y=0} = c_s(L_\lambda^s u_\lambda)(x),$$

in $L^2(\Omega)$. To conclude notice that

$$-y^a\partial_y U_\lambda(x,y)\Big|_{y=0} = -\lambda^{2s}(\lambda y)^a U_y(\lambda x,\lambda y)\Big|_{(\lambda y)=0} = c_s\lambda^{2s}(L^s u)(\lambda x). \qquad \square$$

2.5. Fundamental solution

The fundamental solution $G_s(x,z) = G_s^z(x)$ (Green function) of L^s with pole at $z\in\Omega$ is defined as the weak solution to

$$\begin{cases} L^s G^z = \delta_z,\ G\geq 0, & \text{in } \Omega,\\ G^z = 0, & \text{on } \partial\Omega. \end{cases}$$

Then $G_s(x,z)$ is the distributional kernel of L^{-s}, namely,

$$G_s(x,z) = \sum_{k=0}^\infty \frac{1}{\lambda_k^s}\phi_k(z)\phi_k(x) = G_s(z,x), \quad \text{in } \mathcal{H}^{-s}. \tag{2.15}$$

Indeed, for any $\psi = \sum_{k=0}^\infty d_k\phi_k \in \mathcal{H}^s$, by the symmetry of L^s,

$$\langle L_x^s G_s^z,\psi\rangle = \langle G_s^z, L_x^s\psi\rangle = \sum_{k=0}^\infty \frac{1}{\lambda_k^s}\phi_k(z)\lambda_k^s d_k = \psi(z).$$

The following result is in the spirit of Littman–Stampacchia–Weinberger [21]. The proof is done by using the extension problem.

Theorem 2.6 *(Littman–Stampacchia–Weinberger-type estimate). Fix the ellipticity constants $0<\Lambda_1\leq\Lambda_2$. Then the fundamental solutions G_s of any of the operators L^s that have ellipticity constants between Λ_1 and Λ_2 satisfy the following property. For any compact subset $\mathcal{K}\subset\Omega$ there exist positive constants C_1, C_2, depending only on $\mathcal{K}$, Ω, Λ_1, Λ_2 and s such that, when $n>2s$,*

$$\frac{C_1}{|x-z|^{n-2s}} \leq G_s(x,z) \leq \frac{C_2}{|x-z|^{n-2s}}, \quad x,z\in\mathcal{K},\ x\neq z.$$

In the case $n=2s$ we must replace $|x-z|^{-(n-2s)}$ by $-\ln|x-z|$.

Proof. We do the proof for $G_s(x,0)$, the fundamental solution of L^s with pole at the origin, and for $\Omega = Q_1$, the cube with center at the origin an side length 1. We take $\mathcal{K}$ to be $Q_{1/4} \subset Q_1$. Let U be the solution to the following extension problem

$$\begin{cases} \operatorname{div}(y^a B(x)\nabla U) = 0, & \text{in } Q_1 \times (0,\infty), \\ U(x,y) = 0, & \text{on } \partial Q_1 \times [0,\infty), \\ -\lim_{y\to 0} y^a U_y(x,y) = c_s \delta_{(0,0)}, & \text{on } Q_1. \end{cases}$$

Then, $U(x,0) = G_s(x,0)$, see [32,33]. Because of the Neumann boundary condition at $y=0$, it is easy to check (see the technique in [7] or [35]) that the even reflection $\widetilde{U}(x,y) = U(x,|y|)$, $x \in Q_1$, $y \in \mathbb{R}$, is a weak solution to the equation

$$\begin{cases} \operatorname{div}(|y|^a B(x)\nabla \widetilde{U}) = c_s\delta_{(0,0)}, & \text{in } Q_1 \times \mathbb{R}, \\ \widetilde{U} = 0, & \text{on } \partial Q_1 \times \mathbb{R}. \end{cases}$$

This is a degenerate elliptic equation with A_2 weight $\omega(x,y) = |y|^a$ in $\mathbb{R}^{n+1}$. By the result of Fabes, Jerison and Kenig [12] (see also Fabes [11]), the Green function $\widetilde{U}(x,y)$ is comparable in $Q_{1/4}$ to the quantity

$$\int_{|x-z|}^{1} \frac{s}{s^{n+a+1}}\,ds \sim c_{n,s} \begin{cases} |x-z|^{-(n-2s)}, & \text{if } n > 2s, \\ \ln \frac{1}{|x-z|}, & \text{if } n = 2s. \end{cases} \qquad \square$$

One can also apply the language of semigroups to study the fundamental solution of L^s as explained in [32,33]. If we use the numerical formula

$$\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} \frac{dt}{t^{1-s}}, \quad \lambda, s > 0,$$

in (2.15), we see that the fundamental solution can be written as

$$G_s(x,z) = \frac{1}{\Gamma(s)} \int_0^\infty W_t(x,z) \frac{dt}{t^{1-s}}, \quad \text{in } \mathcal{H}^{-s}, \tag{2.16}$$

where $W_t(x,z)$ is the heat kernel for L, see (2.3). The next estimates should be compared with those of Theorem 2.4.

Theorem 2.7 *(Estimates for $G_s(x,z)$). Let $G_s(x,z) \geq 0$ be the fundamental solution of L^s.*

(1) *If the coefficients $A(x)$ are bounded and measurable then*
$$G_s(x,z) \leq \frac{c_{n,s}}{|x-z|^{n-2s}}, \quad x,z \in \Omega,\ x \neq z.$$
(2) *If the coefficients $A(x)$ are bounded and measurable in $\Omega = \mathbb{R}^n$ then*
$$G_s(x,z) \sim \frac{c_{n,s}}{|x-z|^{n-2s}}, \quad x,z \in \mathbb{R}^n,\ x \neq z.$$
(3) *If the coefficients $A(x)$ are Hölder continuous in Ω with exponent $\alpha \in (0,1)$ then there exist positive constants c and $\eta \leq 1 \leq \rho$ depending only on n, α, Ω and ellipticity, with c depending also on s, such that*
$$c^{-1}\min\left(1, \frac{\phi_0(x)\phi_0(z)}{|x-z|^{2\eta}}\right)\frac{1}{|x-z|^{n-2s}} \leq G_s(x,z) \leq c\min\left(1, \frac{\phi_0(x)\phi_0(z)}{|x-z|^{2\rho}}\right)\frac{1}{|x-z|^{n-2s}},$$
for $x,z \in \Omega$, $x \neq z$, where λ_0 and ϕ_0 are the first eigenvalue and the first eigenfunction of L.
(4) *Under the hypothesis of (3), if in addition Ω is a $C^{1,\gamma}$ domain for some $0 < \gamma < 1$, then the estimate in (3) is true for $\eta = \rho = 1$ and c depending also on γ. In particular, the estimate holds when G^s is the fundamental solution of the fractional Dirichlet Laplacian $(-\Delta_D)^s$ in a $C^{1,\gamma}$ domain.*

Proof. The proof is parallel to that of Theorem 2.4 by using the heat kernel estimates given there and then integrating in t in the identity (2.16) via the change of variables $r = |x-z|^2/t$. $\square$

2.6. Harnack inequality and De Giorgi–Nash–Moser theory

Let $u \in \mathcal{H}^s$, $u \geq 0$ in Ω such that $L^s u = 0$ in some ball $B \subset\subset \Omega$. Then there exists a constant C depending on B, Ω, n and s such that

$$\sup_{\frac{1}{2}B} u \leq C \inf_{\frac{1}{2}B} u.$$

Moreover, u is α-Hölder continuous in $\frac{1}{2}B$, for some exponent $0 < \alpha < 1$. This result can be proved by using the extension problem of [33] as stated in Theorem 2.5. For details see [35].

3. Caccioppoli estimate, approximation, regularity of harmonic functions and a trace inequality

In this section we consider solutions $U \in H^1(B_1^*, y^a dX)$ to

$$\begin{cases} \operatorname{div}(y^a B(x)\nabla U) = \operatorname{div}(y^a F), & \text{in } B_1^*, \\ -y^a U_y\big|_{y=0} = f, & \text{on } B_1, \end{cases} \tag{3.1}$$

where $B(x)$ is given by (1.8) and $F = (F_1, \dots, F_{n+1})$ is a vector field on B_1^* such that

$$F_i(x) \in L^2(B_1^*, y^a dX),\ i = 1, \dots, n, \quad \text{and} \quad F_{n+1} = 0. \tag{3.2}$$

Definition 3.1. A function $U \in H^1(B_1^*, y^a dX)$ is a weak solution to (3.1) if

$$\int_{B_1^*} y^a B(x)\nabla U \nabla\psi\, dX = \int_{B_1^*} y^a F\nabla\psi\, dX + \int_{B_1} \psi(x,0) f(x)\, dx,$$

for every $\psi \in H^1(B_1^*, y^a dX)$ such that $\psi = 0$ on $\partial B_1^* \setminus (\overline{B_1} \times \{0\})$.

By a change of coordinates we can always assume that

$$B(0) = I.$$

Lemma 3.2 *(Caccioppoli inequality). Let U be a weak solution to (3.1) in the sense of Definition 3.1 with F as in (3.2). Then, for every $\eta \in C^\infty(\overline{B_1^*})$ that vanishes on $\partial B_1^* \setminus (\overline{B_1} \times \{0\})$,*

$$\int_{B_1^*} y^a \eta^2 |\nabla U|^2\, dX \leq C\left(\int_{B_1^*} y^a \left(|\nabla\eta|^2 U^2 + |F|^2\eta^2\right) dX + \int_{B_1} (\eta(x,0))^2 |U(x,0)||f(x)|\, dx\right),$$

where $C = C(\lambda, \Lambda)$.

Proof. Take $\psi = \eta^2 U \in H^1(B_1^*, y^a dX)$ as a test function. Then

$$\int y^a B(x)\eta^2 \nabla U\nabla U\, dX = -2\int y^a \eta U B(x)\nabla U\nabla\eta\, dX + \int (\eta(x,0))^2 U(x,0) f(x)\, dx + \int y^a \eta^2 F\nabla U\, dX + 2\int y^a U\eta F\nabla\eta\, dX.$$

Using the ellipticity and the Cauchy inequality with $\varepsilon > 0$,

$$\begin{aligned} \lambda\int y^a\eta^2|\nabla U|^2\, dX \leq{}& \frac{\Lambda}{2\varepsilon}\int y^a U^2|\nabla\eta|^2\, dX + 2\Lambda\varepsilon\int y^a\eta^2|\nabla U|^2\, dX + \int \eta^2|U||f|\, dx \\ &+ \frac{1}{4\varepsilon}\int y^a\eta^2|F|^2\, dX + \varepsilon\int y^a\eta^2|\nabla U|^2\, dX \\ &+ \frac{1}{2\varepsilon}\int y^a\eta^2|F|^2\, dX + 2\varepsilon\int y^a U^2|\nabla\eta|^2\, dX. \end{aligned}$$

The inequality follows by choosing ε such that $(2\Lambda + 1)\varepsilon/\lambda < 1/2$. $\square$

By a compactness argument we get the following consequence of the Caccioppoli inequality.

Corollary 3.3 *(Approximation lemma). Let U be a weak solution to (3.1) in the sense of Definition 3.1 with F as in (3.2). Suppose that U is normalized so that*

$$\int_{B_1} U(x,0)^2\,dx + \int_{B_1^*} y^a U^2\,dX \le 1.$$

Then for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if

$$\int_{B_1} f^2\,dx + \int_{B_1^*} y^a |F|^2\,dX + \int_{B_1} |A(x) - I|^2\,dx < \delta^2,$$

then there exists a solution W to

$$\begin{cases} \operatorname{div}(y^a \nabla W) = 0, & \text{in } B_{3/4}^*, \\ -y^a W_y\big|_{y=0} = 0, & \text{on } B_{3/4}, \end{cases} \tag{3.3}$$

such that

$$\int_{B_{3/4}^*} |U - W|^2 y^a\,dX < \varepsilon^2.$$

Proof. We prove it by contradiction. Suppose that there exist $\varepsilon_0 > 0$, coefficients A_k, weak solutions U_k in B_1^*, Neumann type data f_k and right hand sides F^k, such that

$$\int_{B_1} U_k^2\,dx + \int_{B_1^*} y^a U_k^2\,dX \le 1,$$

and

$$\int_{B_1} f_k^2\,dx + \int_{B_1^*} y^a |F^k|^2\,dX + \int_{B_1} |A_k(x) - I|^2\,dx < \frac{1}{k^2},$$

so that for any solution W to (3.3),

$$\int_{B_{3/4}^*} |U_k - W|^2 y^a\,dX \ge \varepsilon_0^2, \tag{3.4}$$

for all $k \ge 1$. Let η be a test function which is equal to 1 in $\overline{B_{3/4}^*}$ and vanishes outside B_1^*. Then the Caccioppoli estimate and the hypotheses imply that

$$\int_{B_{3/4}^*} y^a |\nabla U_k|^2\,dX \le C, \quad \text{for all } k.$$

Therefore, $\{U_k\}_{k\ge1}$ is a bounded sequence in $H^1(B_{3/4}^*, y^a dX)$. Hence, by compactness of the Sobolev embedding, there exists a subsequence, that we still denote by U_k, and a function U_∞ such that

$$\begin{cases} U_k \to U_\infty, & \text{weakly in } H^1(B_{3/4}^*, y^a dX), \text{ and} \\ U_k \to U_\infty, & \text{strongly in } L^2(B_{3/4}^*, y^a dX). \end{cases}$$

We show now that U_∞ is a solution to (3.3), which will give us a contradiction. Indeed, for any suitable test function ψ,

$$\int_{B^*_{3/4}} y^a B_k(x)\nabla U_k \nabla\psi\, dX = \int_{B^*_{3/4}} y^a F^k \nabla\psi\, dX + \int_{B_{3/4}} \psi(x,0) f_k(x)\, dx.$$

By taking the limit as $k \to \infty$ along the subsequence found above we get

$$\int_{B^*_{3/4}} y^a \nabla U_\infty \nabla\psi\, dX = 0.$$

This contradicts (3.4) for $W = U_\infty$ and k sufficiently large. $\square$

Remark 3.4 *(Approximation up to the boundary).* The following observation will be useful when studying the boundary regularity for the fractional problem with Dirichlet boundary condition. We say that $U \in H^1((B_1^+)^*, y^a dX)$ is a weak solution to the half ball problem

$$\begin{cases} \operatorname{div}(y^a B(x)\nabla U) = \operatorname{div}(y^a F), & \text{in } (B_1^+)^*, \\ -y^a U_y\big|_{y=0} = f, & \text{on } B_1^+, \\ U = 0, & \text{on } B_1 \cap \{x_n = 0\}, \end{cases} \tag{3.5}$$

if U satisfies the identity in Definition 3.1 with B_1 replaced by B_1^+. The test functions vanish on $\partial(B_1^+)^* \setminus (\overline{B_1^+} \times \{0\})$. With this definition then it is clear that the Caccioppoli inequality of Lemma 3.2 holds for solutions U of (3.5) with B_1^+ in place of B_1 in the statement. This allows to prove an approximation lemma parallel to Corollary 3.3. Namely, given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that if U is a solution of (3.5) that satisfies the hypotheses of Corollary 3.3 with B_1^+ in place of B_1, then there exists a solution $\mathcal{W}$ to

$$\begin{cases} \operatorname{div}(y^a \nabla \mathcal{W}) = 0, & \text{in } (B_{3/4}^+)^*, \\ -y^a \mathcal{W}_y\big|_{y=0} = 0, & \text{on } B_{3/4}^+, \\ \mathcal{W} = 0, & \text{on } B_{3/4} \cap \{x_n = 0\}, \end{cases} \tag{3.6}$$

such that

$$\int_{(B^+_{3/4})^*} y^a |V - \mathcal{W}|^2\, dX < \varepsilon^2.$$

Since we will apply Corollary 3.3, we need to understand the regularity of solutions to (3.3).

Proposition 3.5. *Let $W \in H^1(B_1^*, y^a dX)$ be a weak solution to*

$$\begin{cases} \operatorname{div}(y^a \nabla W) = 0, & \text{in } B_1^*, \\ -y^a W_y\big|_{y=0} = 0, & \text{on } B_1. \end{cases}$$

(1) *For each integer $k \geq 0$ and each $B_r(x_0) \subset B_1$,*

$$\sup_{B_{r/2}(x_0)\times[0,r/2)} |D_x^k W| \leq \frac{C}{r^k} \operatorname*{osc}_{B_r(x_0)\times[0,r)} W,$$

where C depends only on n, k and s.

(2) *For each $B_r(x_0) \subset B_1$,*

$$\max_{B_{r/2}(x_0)\times[0,r/2)} |W| \leq M \left(\frac{1}{r^{n+1+a}} \int_{B_r(x_0)^*} y^a |W|^2\, dX \right)^{1/2},$$

where M depends only on n and s.

(3) *We have*

$$\sup_{x\in B_{1/2}} |W_y(x,y)| \le Cy,$$

where C depends only on n and s.

Proof. It can be seen that $\widetilde{W}(x,y) := W(x,|y|)$, $y \in (-1,1)$, is a weak solution to

$$\operatorname{div}(|y|^a \nabla W) = 0, \quad \text{in } B_1 \times (-1,1),$$

see [7, Lemma 4.1]. Now (1) is contained in [6, Corollary 2.5], and (2) follows from [13, Corollary 2.3.4]. Finally (3) is due to the fact that the Neumann type condition that W satisfies has a zero right hand side. Indeed, this follows from the proof of Lemma 4.2 in [7, p. 1254]. $\square$

Remark 3.6 *(Regularity up to the boundary – Dirichlet).* For a solution $\mathcal{W}$ to (3.6) we can perform an odd reflection in the x_n variable, that we call $\mathcal{W}_o$, which satisfies

$$\begin{cases} \operatorname{div}(y^a \nabla \mathcal{W}_o) = 0, & \text{in } B_{3/4}^*, \\ -y^a (\mathcal{W}_o)_y\big|_{y=0} = 0, & \text{on } B_{3/4}. \end{cases}$$

Therefore $\mathcal{W}$ is smooth in $\overline{B_{1/2}^+}$. Also, $(\mathcal{W}_o)_y$ grows like y near $y=0$, $x_n=0$. Hence $\mathcal{W}$ satisfies the same estimates as those for harmonic functions contained in Proposition 3.5 up to the boundary $B_{1/2} \cap \{x_n = 0\}$.

In the proof of the regularity estimates we will need to use the following trace inequality on balls with explicit dependence of the constant in terms of the radius.

Lemma 3.7 *(Trace inequality in balls). There exists a constant $C > 0$ depending only n and s such that*

$$r^{1-s} \|U(\cdot,0)\|_{L^2(B_r)} \le C \|U\|_{H^1(B_r^*, y^a dX)}, \tag{3.7}$$

for all $U \in H^1(B_1^*, y^a dX)$ and for any $0 < r \le 1$. The inequality is also true if we replace B_r by B_r^+.

Proof. It is enough to consider $r = 1$. For if (3.7) is true in this case then for the general case we need to take the rescaled function $V(x,y) = U(rx, ry)$. Recall that there exists a linear extension operator $E : H^1(B_1^*, |y|^a dX) \to H^1(\mathbb{R}^{n+1}, |y|^a dX)$, such that $EU = U$ in B_1^* and

$$\|EU\|_{H^1(\mathbb{R}^{n+1},|y|^a dX)} \le C_0 \|U\|_{H^1(B_1^*,|y|^a dX)}, \tag{3.8}$$

where C_0 depends only on n and s. Also, EU has compact support. See for example [36, Chapter 2, Theorem 2.1.13]. Moreover, the following trace inequality of Lions [19, Paragraph 5]

$$\|F(\cdot,0)\|_{H^s(\mathbb{R}^n)}^2 = \|F(\cdot,0)\|_{L^2(\mathbb{R}^n)}^2 + [F(\cdot,0)]_{H^s(\mathbb{R}^n)}^2 \le c^2 \|F\|_{H^1(\mathbb{R}^{n+1}_+, y^a dX)}^2,$$

holds for any $F \in H^1(\mathbb{R}^{n+1}_+, y^a dX)$, with a constant c depending only on n and s. Using this trace inequality with $F = EU$ and (3.8) we get

$$\begin{aligned} \|U(\cdot,0)\|_{L^2(B_1)} &= \|(EU)(\cdot,0)\|_{L^2(B_1)} \le \|(EU)(\cdot,0)\|_{H^s(\mathbb{R}^n)} \\ &\le c\|EU\|_{H^1(\mathbb{R}^{n+1}_+, y^a dX)} \le cC_0 \|U\|_{H^1(B_1^*,|y|^a dX)}. \qquad \square \end{aligned}$$

4. Interior regularity

Theorems 1.1 and 1.2 are in fact corollaries of the more general results that we state and prove in this section.

We say that a function $f : B_1 \to \mathbb{R}$ is in $L^{2,\alpha}(0)$, for $0 \le \alpha < 1$, whenever

$$[f]^2_{L^{2,\alpha}(0)} := \sup_{0<r\le 1} \frac{1}{r^{n+2\alpha}} \int_{B_r} |f(x) - f(0)|^2\, dx < \infty,$$

where $f(0)$ is defined as $f(0) := \lim_{r\to 0} \frac{1}{|B_r|} \int_{B_r} f(x)\, dx$. It is clear that if f is Hölder continuous of order $0 < \alpha < 1$ at 0 then $f \in L^{2,\alpha}(0)$. If the condition above holds uniformly in balls centered at points close to the origin, then f is α-Hölder continuous around the origin, see [8].

Theorem 4.1. *Let u be a solution to (1.2). Assume that Ω is a bounded Lipschitz domain containing the ball B_1 and let $f \in L^{2,\alpha}(0)$, for some $0 < \alpha < 1$.*

(1) *Suppose that $0 < \alpha + 2s < 1$. There exist $0 < \delta < 1$, depending only on n, ellipticity, α and s, and a constant $C_0 > 0$ such that if*

$$\sup_{0<r\le 1} \frac{1}{r^n} \int_{B_r} |A(x) - A(0)|^2\, dx < \delta^2,$$

then there exists a constant c such that

$$\frac{1}{r^n} \int_{B_r} |u(x) - c|^2\, dx \le C_1 r^{2(\alpha+2s)}, \quad \text{for all } r > 0 \text{ sufficiently small},$$

where $C_1 + |c| \le C_0\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + [f]_{L^{2,\alpha}(0)} + |f(0)|\big)$.

(2) *Suppose that $1 < \alpha + 2s < 2$. There exist $0 < \delta < 1$, depending only on n, ellipticity, α and s, and a constant $C_0 > 0$ such that if*

$$\sup_{0<r\le 1} \frac{1}{r^{n+2(\alpha+2s-1)}} \int_{B_r} |A(x) - A(0)|^2\, dx < \delta^2,$$

then there exists a linear function $\ell(x) = \mathcal{A} + \mathcal{B} \cdot x$ such that

$$\frac{1}{r^n} \int_{B_r} |u(x) - \ell(x)|^2\, dx \le C_1 r^{2(\alpha+2s)}, \quad \text{for all } r > 0 \text{ sufficiently small},$$

where $C_1 + |\mathcal{A}| + |\mathcal{B}| \le C_0\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + [f]_{L^{2,\alpha}(0)} + |f(0)|\big)$.

The constants C_0 above depend only on $[A]_{L^{2,0}(0)}$ (resp. $[A]_{L^{2,\alpha+2s-1}(0)}$), ellipticity, n, α and s.

It is clear then that Theorem 1.1 for the Dirichlet case is a direct consequence of Theorem 4.1 after a dilation of the variables if necessary. Indeed, the conditions on f and on the coefficients hold everywhere in Ω and therefore the estimate for u can be obtained around every interior point.

We say that a function $f : B_1 \to \mathbb{R}$ is in $L^{2,-2s+\alpha}(0)$, $0 < \alpha < 1$, whenever

$$[f]^2_{L^{2,-2s+\alpha}(0)} := \sup_{0<r\le 1} \frac{1}{r^{n+2(-2s+\alpha)}} \int_{B_r} |f(x)|^2\, dx < \infty.$$

Also, f is in $L^{2,-2s+\alpha+1}(0)$, $0 < \alpha < 1$, whenever

$$[f]^2_{L^{2,-2s+\alpha+1}(0)} := \sup_{0<r\le 1} \frac{1}{r^{n+2(-2s+\alpha+1)}} \int_{B_r} |f(x)|^2\, dx < \infty.$$

We have the following consequences of Hölder's inequality.

- If $f \in L^p(B_1)$, for $n/(2s) < p < n/(2s-1)^+$, then $f \in L^{2,-2s+\alpha}(0)$ and
$$[f]_{L^{2,-2s+\alpha}(0)} \leq \|f\|_{L^p(B_1)},$$
for $\alpha = 2s - n/p$.
- If $s > 1/2$ and $f \in L^p(B_1)$, for $p > n/(2s-1)$, then $f \in L^{2,-2s+\alpha+1}(0)$ and
$$[f]_{L^{2,-2s+\alpha+1}(0)} \leq \|f\|_{L^p(B_1)},$$
for $\alpha = 2s - n/p - 1$.

Theorem 4.2. *Let u be a solution to (1.2). Assume that Ω is a bounded Lipschitz domain containing the ball B_1 and let $0 < \alpha < 1$.*

(1) *Suppose that $f \in L^{2,-2s+\alpha}(0)$. There exist $0 < \delta < 1$, depending only on n, ellipticity, α and s, and a constant $C_0 > 0$ such that if*
$$\sup_{0<r\leq 1} \frac{1}{r^n} \int_{B_r} |A(x) - A(0)|^2\,dx < \delta^2,$$
then there exists a constant c such that
$$\frac{1}{r^n} \int_{B_r} |u(x) - c|^2\,dx \leq C_1 r^{2\alpha}, \quad \text{for all } r > 0 \text{ sufficiently small},$$
where $C_1 + |c| \leq C_0\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + [f]_{L^{2,-2s+\alpha}(0)}\big)$.

(2) *Suppose that $f \in L^{2,-2s+\alpha+1}(0)$. There exist $0 < \delta < 1$, depending only on n, ellipticity, α and s, and a constant $C_0 > 0$ such that if*
$$\sup_{0<r\leq 1} \frac{1}{r^{n+2\alpha}} \int_{B_r} |A(x) - A(0)|^2\,dx < \delta^2,$$
then there exists a linear function $\ell(x) = \mathcal{A} + \mathcal{B} \cdot x$ such that
$$\frac{1}{r^n} \int_{B_r} |u(x) - \ell(x)|^2\,dx \leq C_1 r^{2(1+\alpha)}, \quad \text{for all } r > 0 \text{ sufficiently small},$$
where $C_1 + |\mathcal{A}| + |\mathcal{B}| \leq C_0\big(\|u\|_{L^2(\Omega)} + [u]_{H^s(\Omega)} + [f]_{L^{2,-2s+\alpha+1}(0)}\big)$.

The constants C_0 above depend only on $[A]_{L^{2,0}(0)}$ (resp. $[A]_{L^{2,\alpha}(0)}$), ellipticity, n, α and s.

In view of the comments above, Theorem 1.2 is a direct corollary of Theorem 4.2 after a dilation of the variables if necessary.

The rest of this section is devoted to the proof of Theorems 4.1 and 4.2.

4.1. *Proof of Theorem 4.1(1)*

It is enough to prove the regularity for $u(x) = U(x, 0)$, where $U \in H^1(B_1^*, y^a\,dX)$ is a solution to
$$\begin{cases} \operatorname{div}(y^a B(x) \nabla U) = \operatorname{div}(y^a F), & \text{in } B_1^*, \\ -y^a U_y\big|_{y=0} = f, & \text{on } B_1. \end{cases} \tag{4.1}$$
Here we take F to be a B_1^*-valued vector field in an appropriate Morrey space (see (3) below) such that $F_{n+1} = 0$. Theorem 4.1(1) then follows by taking into account Theorem 2.5, where $F \equiv 0$.

It is clear that after an orthogonal change of variables we can assume that $A(0) = I$. We can also assume that $f(0) = 0$. For if $f(0) \neq 0$ we take

$$\widetilde{U}(x, y) = U(x, y) + \tfrac{1}{1-a} y^{1-a} f(0),$$

that solves (4.1) with Neumann data $\widetilde{f}(x) = f(x) - f(0)$ (recall that $B_{n+1,n+1}(x) = 1$) and $\widetilde{f}(0) = 0$.

Let $\delta > 0$. By scaling and by considering

$$\widetilde{U}(x, y) = U(x, y)\left(\int_{B_1} U(x,0)^2 dx + \int_{B_1^*} y^a U^2\,dX + \frac{1}{\delta}\left([f]_{L^{2,\alpha}(0)} + [F]_{\alpha,s}\right)\right)^{-1},$$

we can suppose the following.

(1) (A has small $L^{2,0}(0)$ seminorm) $\displaystyle\sup_{0<r\le 1} \frac{1}{r^n}\int_{B_r} |A(x) - I|^2\,dx < \delta^2$;

(2) (f has small $L^{2,\alpha}(0)$ seminorm) $\displaystyle[f]^2_{L^{2,\alpha}(0)} = \sup_{0<r\le 1} \frac{1}{r^{n+2\alpha}}\int_{B_r} |f|^2\,dx < \delta^2$;

(3) (F has small Morrey seminorm at 0) $\displaystyle[F]_{\alpha,s} := \sup_{0<r\le 1} \frac{1}{r^{n+1+a+2(\alpha+2s-1)}}\int_{B_r^*} y^a|F|^2\,dX < \delta^2$;

(4) (U has bounded L^2 norms) $\displaystyle\int_{B_1} U(x,0)^2\,dx + \int_{B_1^*} y^a U^2\,dX \le 1.$

Given $\delta > 0$, a solution U to (4.1) is called *normalized* if $f(0) = 0$, $A(0) = I$ and (1)–(4) above holds.

Now we prove that given $0 < \alpha + 2s < 1$ there exists $0 < \delta < 1$, depending only on n, ellipticity, α and s, such that for any normalized solution U to (4.1) there exists a constant c_∞ such that

$$\frac{1}{r^n}\int_{B_r} |U(x,0) - c_\infty|^2\,dx \le C_0 r^{2(\alpha+2s)}, \quad \text{for all } r > 0 \text{ sufficiently small,} \tag{4.2}$$

and $|c_\infty| \le C_0$, for some constant C_0 depending only on n, ellipticity, α and s.

Lemma 4.3. *Given* $0 < \alpha + 2s < 1$ *there exist* $0 < \delta, \lambda < 1$, *a constant* c *and a universal constant* $D > 0$ *such that for any normalized solution* U *to* (4.1) *we have*

$$\frac{1}{\lambda^n}\int_{B_\lambda} |U(x,0) - c|^2\,dx + \frac{1}{\lambda^{n+1+a}}\int_{B_\lambda^*} |U - c|^2 y^a\,dX < \lambda^{2(\alpha+2s)},$$

and $|c| \le D$.

Proof. Let $0 < \varepsilon < 1$ to be fixed. Then there exist $0 < \delta < 1$ and a harmonic function W that satisfy Corollary 3.3. We have

$$\int_{B_{1/2}^*} |W|^2 y^a\,dX \le 2\int_{B_{1/2}^*} |U - W|^2 y^a\,dX + 2\int_{B_{1/2}^*} U^2 y^a\,dX \le 2\varepsilon^2 + 2 \le 4.$$

Define $c = W(0,0)$. By the estimates on harmonic functions given in Proposition 3.5(2), there exists a universal constant D such that $|c| \le D$. Moreover, for any $X \in B_{1/4}^*$, by Proposition 3.5(1)–(3),

$$\begin{aligned}|W(X) - c| &\le |W(x,y) - W(x,0)| + |W(x,0) - W(0,0)|\\ &\le |W_y(x,\xi)|y + \|\nabla_x W\|_{L^\infty(B_{1/4})}|x| \le N(y^2 + |x|) \le N|X|,\end{aligned}$$

for some universal constant N. For any $0 < \lambda < 1/4$,

$$\begin{aligned}
&\frac{1}{\lambda^{n+1+a}}\int_{B_\lambda^*}|U-c|^2y^a\,dX \\
&\leq \frac{2}{\lambda^{n+1+a}}\int_{B_\lambda^*}|U-W|^2y^a\,dX+\frac{2}{\lambda^{n+1+a}}\int_{B_\lambda^*}|W-c|^2y^a\,dX \\
&\leq \frac{2\varepsilon^2}{\lambda^{n+1+a}}+\frac{2N^2}{\lambda^{n+1+a}}\int_{B_\lambda^*}|X|^2y^a\,dX\leq \frac{2\varepsilon^2}{\lambda^{n+1+a}}+c_{n,a}\lambda^2. \qquad (4.3)
\end{aligned}$$

On the other hand, we apply the trace inequality (3.7) to $U-c\in H^1(B_1^*,y^a dX)$ to get, for any $0<\lambda<1/8$,

$$\lambda^{1+a}\int_{B_\lambda}|U(x,0)-c|^2\,dx\leq C\left(\int_{B_\lambda^*}|U-c|^2y^a\,dX+\int_{B_\lambda^*}|\nabla U|^2y^a\,dX\right). \qquad (4.4)$$

Next we need to control the gradient in the right hand side of (4.4). To that end we use the Caccioppoli inequality in Lemma 3.2 that ensures that, since $U-c$ is also a solution to the extension equation with the same Neumann type datum f,

$$\begin{aligned}
\int_{B_\lambda^*}|\nabla U|^2y^a\,dX&\leq C\left(\int_{B_{2\lambda}^*}\big(|U-c|^2+|F|^2\big)y^a\,dX+\int_{B_{2\lambda}}|U(x,0)-c||f(x)|\,dx\right)\\
&\leq C\int_{B_{2\lambda}^*}|U-c|^2y^a\,dX+\|F\|^2_{L^2(B_{2\lambda}^*,y^a dX)}+C\big(\|U(\cdot,0)\|_{L^2(B_{2\lambda})}+|c||B_{2\lambda}|^{1/2}\big)\|f\|_{L^2(B_{2\lambda})}.
\end{aligned}$$

Then, since we are in the normalization situation, in (4.4) we get

$$\lambda^{1+a}\int_{B_\lambda}|U(x,0)-c|^2\,dx\leq C\int_{B_{2\lambda}^*}|U-c|^2y^a\,dX+\delta^2+C(1+|c|)\delta, \qquad (4.5)$$

where C depends only on ellipticity, n and a. Therefore, for any $0<\lambda<1/8$, from (4.5) and (4.3) we get

$$\begin{aligned}
\frac{1}{\lambda^n}\int_{B_\lambda}|U(x,0)-c|^2\,dx&\leq \frac{C}{\lambda^{n+1+a}}\int_{B_{2\lambda}^*}|U-c|^2y^a\,dX+\frac{C(1+|c|)}{\lambda^{n+1+a}}(\delta+\delta^2)\\
&\leq \frac{C\varepsilon^2}{\lambda^{n+1+a}}+c_{n,a}\lambda^2+\frac{C(1+D)}{\lambda^{n+1+a}}\delta.
\end{aligned}$$

Hence, for any $0<\lambda<1/8$, from this and (4.3),

$$\frac{1}{\lambda^n}\int_{B_\lambda}|U(x,0)-c|^2\,dx+\frac{1}{\lambda^{n+1+a}}\int_{B_\lambda^*}|U-c|^2y^a\,dX<\frac{C\varepsilon^2}{\lambda^{n+1+a}}+c_{n,a}\lambda^2+\frac{C\delta}{\lambda^{n+1+a}},$$

where C depends only on ellipticity, n and a and it is universal for any W. We first take $0<\lambda<1/8$ sufficiently small in such a way that the second term in the right hand side above is less than $\frac{1}{3}\lambda^{2(\alpha+2s)}$. Then we let $\varepsilon>0$ small enough so that the first term is less than $\frac{1}{3}\lambda^{2(\alpha+2s)}$. For this choice of ε we take $0<\delta<1$ in the approximation lemma (Corollary 3.3) to be so small in such a way that the third term above is smaller than $\frac{1}{3}\lambda^{2(\alpha+2s)}$. Hence there exists a constant c bounded by a universal constant $D>0$ and $0<\delta<1$ such that for any normalized solution U and for some fixed $0<\lambda<1/8$,

$$\frac{1}{\lambda^n}\int_{B_\lambda}|U(x,0)-c|^2\,dx+\frac{1}{\lambda^{n+1+a}}\int_{B_\lambda^*}|U-c|^2y^a\,dX<\lambda^{2(\alpha+2s)}. \qquad \square$$

Lemma 4.4. *In the situation of Lemma 4.3 there is a sequence of constants c_k, $k \geq 0$, such that*

$$|c_k - c_{k+1}| \leq D\lambda^{k(\alpha+2s)},$$

and

$$\frac{1}{\lambda^{kn}}\int_{B_{\lambda^k}} |U(x,0) - c_k|^2\,dx + \frac{1}{\lambda^{k(n+1+a)}}\int_{B^*_{\lambda^k}} |U - c_k|^2 y^a\,dX < \lambda^{2k(\alpha+2s)}.$$

Lemma 4.4 is enough to get (4.2). Indeed, let $c_\infty = \lim_{k\to\infty} c_k$, which is well defined because of the estimate for c_k. For any $r < 1/8$, take $k \geq 0$ such that $\lambda^{k+1} < r \leq \lambda^k$. Then

$$\begin{aligned}\frac{1}{r^n}\int_{B_r} |U(x,0) - c_\infty|^2\,dx &\leq \frac{2}{r^n}\int_{B_r} |U(x,0) - c_k|^2\,dx + \frac{2}{r^n}\int_{B_r} |c_k - c_\infty|^2\,dx \\ &\leq 2\left(\frac{\lambda^{kn}}{r^n}\right)\frac{1}{\lambda^{kn}}\int_{B_{\lambda^k}} |U(x,0) - c_k|^2\,dx + C_n D^2 \lambda^{2k(\alpha+2s)} \\ &\leq C_{n,\lambda,D}\lambda^{2k(\alpha+2s)} \leq C_{n,\lambda,D} r^{2(\alpha+2s)}. \end{aligned} \tag{4.6}$$

Proof of Lemma 4.4. The proof is done by induction. When $k = 0$, we take $c_0 = c_1 = 0$. Then the conclusion is true because U is a normalized solution. Assume that the claim is true for some $k \geq 0$. Consider

$$\widetilde{U}(X) = \frac{U(\lambda^k X) - c_k}{\lambda^{(\alpha+2s)k}}, \quad X \in B_1^*,$$

where λ is as in Lemma 4.3. By applying the change of variables $X = \lambda^k Z$ in the weak formulation

$$\int_{B^*_{\lambda^k}} y^a B(x)\nabla U \nabla\psi\,dX = \int_{B^*_{\lambda^k}} y^a F\nabla\psi\,dX + \int_{B_{\lambda^k}} f(x)\psi(x,0)\,dx,$$

we get, for $\widetilde{B}(x) := B(\lambda^k x)$, $\widetilde{\psi}(X) = \psi(\lambda^k X)$, $\widetilde{f}(x) = \lambda^{-k\alpha} f(\lambda^k x)$, $\widetilde{F}(x) = \lambda^{-k(\alpha+2s-1)} F(\lambda^k x)$ and $\widetilde{U}$ as above,

$$\int_{B_1^*} y^a \widetilde{B}(x)\nabla\widetilde{U}\nabla\widetilde{\psi}\,dX = \int_{B_1^*} y^a \widetilde{F}\nabla\widetilde{\psi}\,dX + \int_{B_1} \widetilde{f}(x)\widetilde{\psi}(x,0)\,dx.$$

Thus $\widetilde{U}$ is a weak solution to

$$\begin{cases} \operatorname{div}(y^a \widetilde{B}(x)\nabla\widetilde{U}) = \operatorname{div}(y^a\widetilde{F}), & \text{in } B_1^*, \\ -y^a\widetilde{U}_y\big|_{y=0} = \widetilde{f}, & \text{on } B_1. \end{cases} \tag{4.7}$$

Notice that $\widetilde{A}(0) = I$, $\widetilde{F}_{n+1} = 0$ and $\widetilde{f}(0) = 0$. Moreover, by changing variables back and using the induction hypothesis,

$$\frac{1}{r^n}\int_{B_r} (\widetilde{A}(x) - I)^2\,dx = \frac{1}{(\lambda^k r)^n}\int_{B_{\lambda^k r}} (A(x) - I)^2\,dx < \delta^2;$$

$$\frac{1}{r^{n+2\alpha}}\int_{B_r} |\widetilde{f}|^2\,dx = \frac{1}{(\lambda^k r)^{n+2\alpha}}\int_{B_{\lambda^k r}} |f|^2\,dx \leq [f]^2_{L^{2,\alpha}(0)} < \delta^2;$$

$$\frac{1}{r^{n+1+a+2(\alpha+2s-1)}}\int_{B_r} y^a|\widetilde{F}|^2\,dX = \frac{1}{(\lambda^k r)^{n+1+a+2(\alpha+2s-1)}}\int_{B_{\lambda^k r}} y^a|F|^2\,dX < \delta^2;$$

$$\int_{B_1} \widetilde{U}(x,0)^2\,dx = \frac{1}{\lambda^{kn}}\int_{B_{\lambda^k}} \frac{|U(x,0) - c_k|^2}{\lambda^{2k(\alpha+2s)}}\,dx \leq 1;$$

$$\int_{B_1^*} y^a \widetilde{U}^2\, dX = \frac{1}{\lambda^{k(n+1+a)}} \int_{B_{\lambda^k}^*} y^a \frac{|U-c_k|^2}{\lambda^{2k(\alpha+2s)}}\, dX \leq 1.$$

Therefore $\widetilde{U}$ is a normalized solution to (4.7). Hence we can apply Lemma 4.3 to $\widetilde{U}$ in order to get

$$\frac{1}{\lambda^n}\int_{B_\lambda} |\widetilde{U}(x,0)-c|^2\,dx + \frac{1}{\lambda^{n+1+a}}\int_{B_\lambda^*} |\widetilde{U}-c|^2 y^a\,dX < \lambda^{2(\alpha+2s)}.$$

Now we change variables back in the definition of $\widetilde{U}$ to obtain

$$\frac{1}{\lambda^{(k+1)n}}\int_{B_{\lambda^{k+1}}} |U(x,0)-c_{k+1}|^2\,dx < \lambda^{2(k+1)(\alpha+2s)},$$

and

$$\frac{1}{\lambda^{(k+1)(n+1+a)}}\int_{B_{\lambda^{k+1}}^*} |\widetilde{U}-c_{k+1}|^2 y^a\,dX < \lambda^{2(k+1)(\alpha+2s)},$$

where

$$c_{k+1} = c_k + \lambda^{k(\alpha+2s)}c.$$

Obviously, we have $|c_{k+1}-c_k| = |c\lambda^{k(\alpha+2s)}| \leq D\lambda^{\alpha+2s}$. This proves the induction step. □

4.2. *Proof of Theorem 4.1(2)*

As in the previous subsection, it is enough to prove the regularity for $u(x)=U(x,0)$, where $U\in H^1(B_1^*, y^a dX)$ is a solution to

$$\begin{cases} \operatorname{div}(y^aB(x)\nabla U) = \operatorname{div}(y^aF), & \text{in } B_1^*, \\ -y^aU_y\big|_{y=0} = f, & \text{on } B_1. \end{cases} \tag{4.8}$$

Here F is now a B_1^*-valued vector field that belongs to an appropriate Campanato space (see (3) below) and such that $F(0)=0$.

As before, we can assume that $A(0)=I$ and $f(0)=0$. We can also suppose the following for $\delta>0$.

(1) (A has small $L^{2,\alpha+2s-1}(0)$ seminorm) $\displaystyle\sup_{0<r\leq1}\frac{1}{r^{n+2(\alpha+2s-1)}}\int_{B_r}|A(x)-I|^2\,dx < \delta^2$;

(2) (f has small $L^{2,\alpha}(0)$ seminorm) $\displaystyle[f]^2_{L^{2,\alpha}(0)} = \sup_{0<r\leq1}\frac{1}{r^{n+2\alpha}}\int_{B_r}|f|^2\,dx < \delta^2$;

(3) (F has small Campanato seminorm at 0) $\displaystyle\sup_{0<r\leq1}\frac{1}{r^{n+1+a+2(\alpha+2s-1)}}\int_{B_r^*}y^a|F|^2\,dX < \delta^2$;

(4) (U has bounded L^2 norms) $\displaystyle\int_{B_1}U(x,0)^2\,dx + \int_{B_1^*}y^aU^2\,dX \leq 1$.

Observe that assumption (1) on the coefficients $A(x)$ is equivalent to ask for the matrix $B(x)$ that

$$\sup_{0<r\leq1}\frac{1}{r^{n+1+a+2(\alpha+2s-1)}}\int_{B_r^*}y^a|B(x)-I|^2\,dX < \delta^2.$$

Now we prove that given $1<\alpha+2s<2$ there exists $0<\delta<1$, depending only on n, ellipticity, α and s, such that for any normalized solution U to (4.8) there exists a linear function $\ell_\infty(x) = A_\infty + B_\infty\cdot x$ such that

$$\frac{1}{r^n}\int_{B_r} |U(x,0)-\ell_\infty|^2\,dx \le C_0 r^{2(\alpha+2s)}, \quad \text{for all } r>0 \text{ sufficiently small,} \tag{4.9}$$

and $|A_\infty|+|B_\infty|\le C_0$, for some constant C_0 depending only on n, ellipticity, α and s.

Lemma 4.5. *Given* $1<\alpha+2s<2$ *there exist* $0<\delta,\ \lambda<1$, *a linear function*

$$\ell(x)=A+B\cdot x,$$

and a universal constant $D>0$ *such that for any normalized solution* U *to* (4.8) *we have*

$$\frac{1}{\lambda^n}\int_{B_\lambda}|U(x,0)-\ell(x)|^2\,dx+\frac{1}{\lambda^{n+1+a}}\int_{B_\lambda^*}|U-\ell|^2y^a\,dX<\lambda^{2(\alpha+2s)},$$

and $|A|+|B|\le D$.

Proof. Let $0<\varepsilon<1$ to be fixed. Then there exist $0<\delta<1$ and a harmonic function W that satisfy Corollary 3.3. As before we have

$$\int_{B_{1/2}^*}|W|^2y^a\,dX\le 4.$$

Define $\ell(x)=W(0,0)+\nabla_xW(0,0)\cdot x=:A+B\cdot x$. By the estimates on harmonic functions given in Proposition 3.5, there exists a universal constant D such that $|A|+|B|\le D$. Moreover, for any $X\in B_{1/4}^*$, by Proposition 3.5,

$$\begin{aligned}|W(x,y)-\ell(x)|&=\big|\big(W(x,y)-W(x,0)\big)+\big(W(x,0)-W(0,0)-\nabla_xW(0,0)\cdot x\big)\big|\\&\le|W_y(x,\xi)|y+\tfrac12|D_x^2W(\xi,0)||x|^2\\&\le C\xi y+\tfrac12\|D_x^2W\|_{L^\infty(B_{1/4}^*)}|x|^2\le N|X|^2,\end{aligned}$$

for some universal constant N. For any $0<\lambda<1/4$,

$$\begin{aligned}&\frac{1}{\lambda^{n+1+a}}\int_{B_\lambda^*}|U-\ell|^2y^a\,dX\\&\le\frac{2}{\lambda^{n+1+a}}\int_{B_\lambda^*}|U-W|^2y^a\,dX+\frac{2}{\lambda^{n+1+a}}\int_{B_\lambda^*}|W-\ell|^2y^a\,dX\\&\le\frac{2\varepsilon^2}{\lambda^{n+1+a}}+\frac{2N^2}{\lambda^{n+1+a}}\int_{B_\lambda^*}|X|^4y^a\,dX\le\frac{2\varepsilon^2}{\lambda^{n+1+a}}+c_{n,a}\lambda^4.\end{aligned} \tag{4.10}$$

On the other hand, apply the trace inequality (3.7) to $U-\ell\in H^1(B_1^*,y^a dX)$ to get, for any $0<\lambda<1/8$,

$$\lambda^{1+a}\int_{B_\lambda}|U(x,0)-\ell(x)|^2\,dx\le C\left(\int_{B_\lambda^*}|U-\ell|^2y^a\,dX+\int_{B_\lambda^*}|\nabla(U-\ell)|^2y^a\,dX\right). \tag{4.11}$$

Next we control the gradient in the right hand side of (4.11) by using the Caccioppoli inequality. Notice that $U-\ell$ is a solution (in the sense of Definition 3.1) to

$$\begin{cases}\operatorname{div}(y^aB(x)\nabla(U-\ell))=\operatorname{div}(y^a(F+G)), & \text{in } B_1^*,\\ -y^a(U-\ell)_y\big|_{y=0}=f, & \text{on } B_1,\end{cases}$$

where the vector field G is given by

$$G=\big((I-A(x))\nabla_x\ell,0\big)\in\mathbb{R}^{n+1}, \quad \text{and } G(0)=0.$$

Then, by the Caccioppoli inequality in Lemma 3.2,

$$\int_{B_\lambda^*} |\nabla(U-\ell)|^2 y^a\,dX \leq C\left(\int_{B_{2\lambda}^*} \big(|U-\ell|^2+|F+G|^2\big)y^a\,dX + \int_{B_{2\lambda}} |U(x,0)-\ell(x)||f(x)|\,dx\right)$$

$$\leq C\int_{B_{2\lambda}^*} |U-\ell|^2 y^a\,dX + C\|F+G\|^2_{L^2(B_{2\lambda}^*,y^a dX)} + C\big(\|U(\cdot,0)\|_{L^2(B_{2\lambda})}+\|\ell\|_{L^2(B_{2\lambda})}\big)\|f\|_{L^2(B_{2\lambda})}$$

$$\leq C\int_{B_{2\lambda}^*} |U-\ell|^2 y^a\,dX + C\delta^2 + C(1+D)\delta.$$

Plugging this into (4.11) and taking into account (4.10) we see that, for any $0<\lambda<1/8$,

$$\frac{1}{\lambda^n}\int_{B_\lambda} |U(x,0)-\ell(x)|^2\,dx \leq \frac{C}{\lambda^{n+1+a}}\int_{B_{2\lambda}^*} |U-\ell|^2 y^a\,dX + \frac{C}{\lambda^{n+1+a}}(\delta^2+\delta)$$

$$\leq \frac{C\varepsilon^2}{\lambda^{n+1+a}} + c_{n,a}\lambda^4 + \frac{C\delta}{\lambda^{n+1+a}}.$$

Hence, for any $0<\lambda<1/8$, from this and (4.10),

$$\frac{1}{\lambda^n}\int_{B_\lambda} |U(x,0)-\ell(x)|^2\,dx + \frac{1}{\lambda^{n+1+a}}\int_{B_\lambda^*} |U-\ell|^2 y^a\,dX < \frac{C\varepsilon^2}{\lambda^{n+1+a}} + c_{n,a}\lambda^4 + \frac{C\delta}{\lambda^{n+1+a}},$$

where C depends only on ellipticity, n and a and it is universal for any W. We first take $0<\lambda<1/8$ sufficiently small in such a way that the second term in the right hand side above is less than $\frac{1}{3}\lambda^{2(\alpha+2s)}$ (recall that we are in the situation where $1<\alpha+2s<2$). Then we let $\varepsilon>0$ small enough so that the first term is less than $\frac{1}{3}\lambda^{2(\alpha+2s)}$. For this choice of ε we take $\delta>0$ in the approximation lemma to be so small in such a way that the third term above is smaller than $\frac{1}{3}\lambda^{2(\alpha+2s)}$. Hence, there exists a linear function $\ell(x)$, whose coefficients are bounded by a universal constant D, and $0<\delta<1$ such that for any normalized solution and for some fixed $0<\lambda<1/8$,

$$\frac{1}{\lambda^n}\int_{B_\lambda} |U(x,0)-\ell(x)|^2\,dx + \frac{1}{\lambda^{n+1+a}}\int_{B_\lambda^*} |U-\ell|^2 y^a\,dX < \lambda^{2(\alpha+2s)}. \qquad \square$$

Lemma 4.6. *In the situation of Lemma 4.5, there exists a sequence of linear functions*

$$\ell_k(x) = A_k + B_k\cdot x,$$

for $k\geq 0$, such that

$$\frac{1}{\lambda^{kn}}\int_{B_{\lambda^k}} |U(x,0)-\ell_k(x)|^2\,dx + \frac{1}{\lambda^{k(n+1+a)}}\int_{B_{\lambda^k}^*} |U-\ell_k|^2 y^a\,dX < \lambda^{2k(\alpha+2s)},$$

and

$$|A_k - A_{k+1}|,\ \lambda^k|B_k-B_{k+1}| \leq D\lambda^{k(\alpha+2s)}.$$

Before proceeding with the proof, let us show how this claim already implies (4.9). Let

$$\ell_\infty(x) = A_\infty + B_\infty\cdot x := \Big(\lim_{k\to\infty} A_k\Big) + \Big(\lim_{k\to\infty} B_k\Big)\cdot x.$$

Notice that A_∞ and B_∞ are well defined because of the Cauchy property they verify. Observe also that for any $k\geq 0$, since $1<\alpha+2s<2$, we have

$$|\ell_\infty(x) - \ell_k(x)| \le C_{\alpha,s} D\lambda^{k(\alpha+2s)}, \quad |x| \le \lambda^k.$$

For any $0 < r < 1/8$, take $k \ge 0$ such that $\lambda^{k+1} < r \le \lambda^k$. Then, in a parallel way to (4.6), (4.9) follows for small r.

Proof of Lemma 4.6. The proof is done by induction in $k \ge 0$. When $k = 0$, we take $\ell_0(x) = \ell_1(x) = 0$ and the conclusion is true because U is a normalized solution. Assume that the claim is true for some $k \ge 0$. Consider

$$\widetilde{U}(x, y) = \frac{U(\lambda^k x, \lambda^k y) - \ell_k(\lambda^k x)}{\lambda^{(\alpha+2s)k}}, \quad (x, y) \in B_1^*,$$

where λ is as in Lemma 4.5. Then, for $\widetilde{B}(x) := B(\lambda^k x)$, $\widetilde{\psi}(x) = \psi(\lambda^k x)$, $\widetilde{f}(x) = \lambda^{-k\alpha} f(\lambda^k x)$ and $\widetilde{F}(X) = \lambda^{-k(\alpha+2s-1)} F(\lambda^k X)$ we have

$$\int_{B_1^*} y^a \widetilde{B}(x) \nabla\widetilde{U}\nabla\widetilde{\psi}\, dX = \int_{B_1} \widetilde{f}(x)\widetilde{\psi}(x, 0)\, dx + \int_{B_1^*} y^a \left(\widetilde{F} + \frac{I - \widetilde{B}(x)}{\lambda^{k(\alpha+2s-1)}}\nabla\ell_k\right)\nabla\widetilde{\psi}\, dX.$$

Thus $\widetilde{U}$ is a weak solution to

$$\begin{cases} \operatorname{div}(y^a \widetilde{B}(x)\nabla\widetilde{U}) = \operatorname{div}(y^a(\widetilde{F} + \widetilde{G})), & \text{in } B_1^* \\ -y^a \widetilde{U}_y\big|_{y=0} = \widetilde{f}, & \text{on } B_1, \end{cases}$$

where

$$\widetilde{G} = \left(\frac{I - B(\lambda^k x)}{\lambda^{k(\alpha+2s-1)}}\nabla\ell_k, 0\right) \in \mathbb{R}^{n+1}, \quad \text{and } \widetilde{G}(0) = 0.$$

Certainly $\widetilde{A}(0) = I$ and $\widetilde{f}(0) = 0$. By changing variables and using the hypotheses,

$$\begin{aligned} &\frac{1}{r^{n+1+a+2(\alpha+2s-1)}} \int_{B_r^*} y^a |\widetilde{F} + \widetilde{G}|^2\, dX \\ &\quad \le \frac{1}{(\lambda^k r)^{n+1+a+2(\alpha+2s-1)}} \int_{B_{\lambda^k r}^*} y^a \left(|F|^2 + |I - B(x)|^2 |B_k|^2\right) dX < (1 + D^2c^2)\delta^2, \end{aligned}$$

where we used that

$$|B_k| \le \sum_{j=1}^{k} |B_j - B_{j-1}| \le D \sum_{j=0}^{\infty} \lambda^{j(\alpha+2s-1)} = Dc.$$

Also,

$$\frac{1}{r^{n+2\alpha}} \int_{B_r} |\widetilde{f}|^2\, dx = \frac{1}{(\lambda^k r)^{2\alpha+n}} \int_{B_{\lambda^k r}} f^2\, dx \le [f]^2_{L^{2,\alpha}(0)} < \delta^2;$$

$$\int_{B_1} \widetilde{U}(x, 0)^2\, dx = \frac{1}{\lambda^{kn}} \int_{B_{\lambda^k}} \frac{|U(x, 0) - \ell_k(x)|^2}{\lambda^{2k(\alpha+2s)}}\, dx \le 1;$$

$$\int_{B_1^*} y^a \widetilde{U}^2\, dX = \frac{1}{\lambda^{k(n+1+a)}} \int_{B_{\lambda^k}^*} y^a \frac{|U - \ell_k|^2}{\lambda^{2k(\alpha+2s)}}\, dX \le 1.$$

By Lemma 4.5 (that can be applied to $\widetilde{U}/(1 + D^2c^2)$), there is a linear function ℓ such that

$$\frac{1}{\lambda^n} \int_{B_\lambda} |\widetilde{U}(x, 0) - \ell(x)|^2\, dx + \frac{1}{\lambda^{n+1+a}} \int_{B_\lambda^*} |\widetilde{U} - \ell|^2 y^a\, dX < \lambda^{2(\alpha+2s)}.$$

So changing variables back in the definition of $\widetilde{U}$ we get

$$\frac{1}{\lambda^{(k+1)n}}\int_{B_{\lambda^{k+1}}}|U(x,0)-\ell_{k+1}(x)|^2\,dx+\frac{1}{\lambda^{(k+1)(n+1+a)}}\int_{B^*_{\lambda^{k+1}}}|U-\ell_{k+1}|^2y^a\,dX<\lambda^{2(k+1)(\alpha+2s)},$$

where $\ell_{k+1}(x)=\ell_k(x)+\lambda^{k(\alpha+2s)}\ell(\lambda^{-k}x)$. By construction,

$$|\ell_{k+1}(x)-\ell_k(x)|=\lambda^{k(\alpha+2s)}|\ell(\lambda^{-k}x)|\le D\lambda^{k(\alpha+2s)}(1+\lambda^{-k}|x|).$$

When $x=0$ we get $|A_{k+1}-A_k|\le D\lambda^{k(\alpha+2s)}$. On the other hand, again by construction, $|B_{k+1}-B_k|\le D\lambda^{-k}\lambda^{k(\alpha+2s)}$. This finishes the proof. □

4.3. *Proof of Theorem 4.2*

This proof is done by following exactly the same lines of the proof of Theorem 4.1, but with easy changes in the exponents. Indeed, for part (1) we have to replace in the proof of Theorem 4.1(1) the exponent α that appears everywhere there by $-2s+\alpha$. For part (2), along the proof of Theorem 4.1(2) we need to replace the exponent α by the new exponent $-2s+\alpha+1$. Observe that we do not need the reduction to the case $f(0)=0$.

5. Case study: the fractional Dirichlet Laplacian in the half space

In this section we study the global regularity and the growth near the boundary for solutions to the fractional Dirichlet Laplacian of the half space.

5.1. *Global regularity*

Let us recall the Schauder estimates for the fractional Laplacian on $\mathbb{R}^n$.

Proposition 5.1. *Let $0<s<1$ and $0<\alpha\le 1$. Assume that $f\in C^{0,\alpha}(\mathbb{R}^n)$ and that $u\in L^\infty(\mathbb{R}^n)$ is a solution to*

$$(-\Delta)^s u=f,\quad \text{in } \mathbb{R}^n.$$

(1) *If $\alpha+2s\le 1$, then $u\in C^{0,\alpha+2s}(\mathbb{R}^n)$ and*

$$\|u\|_{C^{0,\alpha+2s}(\mathbb{R}^n)}\le C\big(\|u\|_{L^\infty(\mathbb{R}^n)}+\|f\|_{C^{0,\alpha}(\mathbb{R}^n)}\big).$$

(2) *If $1<\alpha+2s\le 2$, then $u\in C^{1,\alpha+2s-1}(\mathbb{R}^n)$ and*

$$\|u\|_{C^{1,\alpha+2s-1}(\mathbb{R}^n)}\le C\big(\|u\|_{L^\infty(\mathbb{R}^n)}+\|f\|_{C^{0,\alpha}(\mathbb{R}^n)}\big).$$

(3) *If $2<\alpha+2s<3$, then $u\in C^{2,\alpha+2s-2}(\mathbb{R}^n)$ and*

$$\|u\|_{C^{2,\alpha+2s-2}(\mathbb{R}^n)}\le C\big(\|u\|_{L^\infty(\mathbb{R}^n)}+\|f\|_{C^{0,\alpha}(\mathbb{R}^n)}\big).$$

The constants C above depend only on n, α and s. In particular, if $f=0$ in a ball B_r, then u is smooth in $B_{r/2}$.

Proof. For (1) and (2) see [29, Proposition 2.8]. The statement in (3) is proved analogously by taking into account the range of the exponents. The details are omitted. □

Recall that the Zygmund space $\Lambda_*(\mathbb{R}^n)$ consists of all bounded functions u on $\mathbb{R}^n$ such that

$$[u]_{\Lambda_*(\mathbb{R}^n)}:=\sup_{x,h\in\mathbb{R}^n}\frac{|u(x+h)-2u(x)+u(x-h)|}{|h|}<\infty,$$

under the norm $\|u\|_{\Lambda_*(\mathbb{R}^n)}:=\|u\|_{L^\infty(\mathbb{R}^n)}+[u]_{\Lambda_*(\mathbb{R}^n)}$, see [37] (also [31, Chapter V] or [24]).

Proposition 5.2. *Let $0 < s < 1$. Assume that $f \in L^\infty(\mathbb{R}^n)$ and that $u \in L^\infty(\mathbb{R}^n)$ is a solution to*

$$(-\Delta)^s u = f, \quad \text{in } \mathbb{R}^n.$$

Namely, assume that $u \in L^\infty(\mathbb{R}^n)$ is given by

$$u(x) = (-\Delta)^{-s} f(x) = \frac{1}{\Gamma(s)} \int_0^\infty e^{t\Delta} f(x) \frac{dt}{t^{1-s}},$$

where the integral is well defined for almost all $x \in \mathbb{R}^n$ and for some $f \in L^\infty(\mathbb{R}^n)$.

(1) *If $0 < 2s < 1$ then $u \in C^{0,2s}(\mathbb{R}^n)$ and*

$$\|u\|_{C^{0,2s}(\mathbb{R}^n)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\big).$$

(2) *If $2s = 1$ then u is in the Zygmund space $\Lambda_*(\mathbb{R}^n)$ and*

$$\|u\|_{\Lambda_*(\mathbb{R}^n)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\big).$$

(3) *If $1 < 2s < 2$ then $u \in C^{1,2s-1}(\mathbb{R}^n)$ and*

$$\|u\|_{C^{1,2s-1}(\mathbb{R}^n)} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n)} + \|f\|_{L^\infty(\mathbb{R}^n)}\big).$$

The constants C above depend only on n and s.

Proof. Parts (1) and (3) of this result, that is, when $2s \neq 1$, are already contained in [1, Theorem 6.4].[2] Here we present a proof that works for every $0 < s < 1$ and includes the Zygmund space.

For $\alpha > 0$, let Λ_α be the space of bounded functions u on $\mathbb{R}^n$ for which

$$[u]_{\Lambda_\alpha} := \sup_{x\in\mathbb{R}^n, t>0} |t^{1-\alpha/2} \partial_t e^{t\Delta} u(x)| < \infty.$$

It is known that

$$\Lambda_\alpha = \begin{cases} C^{0,\alpha}(\mathbb{R}^n), & \text{if } 0 < \alpha < 1, \\ \Lambda_*(\mathbb{R}^n), & \text{if } \alpha = 1, \\ C^{1,\alpha-1}(\mathbb{R}^n), & \text{if } 1 < \alpha < 2. \end{cases}$$

Moreover, the norm on all these spaces is equivalent to $\|u\|_{L^\infty(\mathbb{R}^n)} + [u]_{\Lambda_\alpha}$. See [24] where this result is proved for the torus. In [31] a similar characterization is proved by using the Poisson semigroup instead of the heat semigroup. The proof in [24] can be easily adapted to the case of $\mathbb{R}^n$. It is enough to show that

$$[u]_{\Lambda_{2s}} = \sup_{x\in\mathbb{R}^n, t>0} |t^{1-s} \partial_t e^{t\Delta} (-\Delta)^{-s} f(x)| \le C\|f\|_{L^\infty(\mathbb{R}^n)},$$

for some constant C depending only on n and s. Consider the heat kernel $W_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/(4t)}$, $x \in \mathbb{R}^n$, $t > 0$. Notice that the following simple estimate

$$\int_{\mathbb{R}^n} |\partial_t W_t(x)|\, dx \le \frac{c}{t}, \quad t > 0,$$

implies that

$$|\partial_t e^{t\Delta} f(x)| \le \frac{c}{t} \|f\|_{L^\infty(\mathbb{R}^n)}, \quad \text{for all } x \in \mathbb{R}^n,\ t > 0.$$

Thus, with this and the semigroup property $e^{t\Delta} e^{r\Delta} f = e^{(t+r)\Delta} f$ we obtain

[2] We are grateful to Mark Allen for pointing out this to us.

$$
\begin{aligned}
|t\partial_t e^{t\Delta}(-\Delta)^{-s}f(x)| &\le Ct\int_0^\infty \left|\partial_w e^{w\Delta}f(x)\big|_{w=t+r}\right|\frac{dr}{r^{1-s}} \\
&\le Ct\|f\|_{L^\infty(\mathbb{R}^n)}\int_0^\infty \frac{1}{t+r}\frac{dr}{r^{1-s}} \\
&= Ct^s\|f\|_{L^\infty(\mathbb{R}^n)}\int_0^\infty \frac{1}{1+\rho}\frac{d\rho}{\rho^{1-s}} = C\|f\|_{L^\infty(\mathbb{R}^n)}t^s,
\end{aligned}
$$

where C is a constant that depends only on n and s. □

In the half space $\mathbb{R}^n_+$ we consider the Laplacian with homogeneous Dirichlet boundary condition on $\partial\mathbb{R}^n_+ = \{x_n = 0\}$. We denote this operator by $-\Delta_D^+$. Then $-\Delta_D^+$ is a nonnegative and selfadjoint operator on $H_0^1(\mathbb{R}^n_+)$ for which the spectral theorem applies. For a function u defined on $\overline{\mathbb{R}^n_+}$ with $u(x',0) = 0$ and for $0 < s < 1$ we have

$$
(-\Delta_D^+)^s u(x) = \frac{1}{\Gamma(-s)}\int_0^\infty \left(e^{t\Delta_D^+}u(x) - u(x)\right)\frac{dt}{t^{1+s}}, \quad x \in \mathbb{R}^n_+. \tag{5.1}
$$

Here $v(x,t) \equiv e^{t\Delta_D^+}u(x)$ is the heat semigroup generated by $-\Delta_D^+$ on the half space, namely, v is the solution to

$$
\begin{cases}
v_t = \Delta v, & \text{for } x \in \mathbb{R}^n_+, t > 0, \\
v(x,0) = u(x), & \text{on } \mathbb{R}^n_+, \\
v(x',0,t) = 0, & \text{for } t \ge 0.
\end{cases}
$$

Let $x^* = (x', -x_n)$, for $x \in \mathbb{R}^n$. Denote by u_o the odd extension of u to $\mathbb{R}^n$ with respect to x_n:

$$
u_o(x) = \begin{cases}
u(x), & \text{if } x_n \ge 0, \\
-u(x^*) = -u(x', -x_n), & \text{if } x_n < 0.
\end{cases}
$$

By using the reflection method we see that

$$
e^{t\Delta_D^+}u(x) = e^{t\Delta}u_o(x), \quad t > 0, \ x \in \mathbb{R}^n_+,
$$

so that, from (5.1) we observe that

$$
(-\Delta_D^+)^s u(x) = (-\Delta)^s u_o(x), \quad x \in \mathbb{R}^n_+, \tag{5.2}
$$

where $(-\Delta)^s$ is the fractional Laplacian on $\mathbb{R}^n$. Moreover, since

$$
e^{t\Delta_D^+}u(x) = \frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n_+}\left(e^{-|x-z|^2/(4t)} - e^{-|x-z^*|^2/(4t)}\right)u(z)\,dz, \quad x \in \mathbb{R}^n_+,
$$

from (5.1) we obtain the following pointwise formula:

$$
(-\Delta_D^+)^s u(x) = c_{n,s}\int_{\mathbb{R}^n_+}\left(u(x) - u(z)\right)\left(\frac{1}{|x-z|^{n+2s}} - \frac{1}{|x-z^*|^{n+2s}}\right)dz, \quad x \in \mathbb{R}^n_+.
$$

Also, from the fact that

$$
(-\Delta_D^+)^{-s} f(x) = \frac{1}{\Gamma(s)}\int_0^\infty e^{t\Delta_D}f(x)\,\frac{dt}{t^{1-s}},
$$

we get, when $n \neq 2s$,

$$(-\Delta_D^+)^{-s} f(x) = d_{n,s} \int_{\mathbb{R}^n_+} f(z) \left(\frac{1}{|x-z|^{n-2s}} - \frac{1}{|x-z^*|^{n-2s}} \right) dz, \quad x \in \mathbb{R}^n_+. \tag{5.3}$$

The constants $c_{n,s}$ and $d_{n,s}$ above can be computed explicitly.

Theorem 5.3 *(Global regularity in half space – Dirichlet case). Let u be a bounded solution to*

$$\begin{cases} (-\Delta_D^+)^s u = f, & \text{in } \mathbb{R}^n_+, \\ u = 0, & \text{on } \partial\mathbb{R}^n_+ = \{x_n = 0\}, \end{cases}$$

where $f \in C^{0,\alpha}(\overline{\mathbb{R}^n_+})$, $0 < \alpha \le 1$.

- *Suppose that* $f(x',0) = 0$*, for all* $x' \in \mathbb{R}^{n-1}$*. Then*
  (1) *If* $\alpha + 2s \le 1$ *then* $u \in C^{0,\alpha+2s}(\overline{\mathbb{R}^n_+})$ *and*
  $$\|u\|_{C^{0,\alpha+2s}(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{C^{0,\alpha}(\overline{\mathbb{R}^n_+})}\big).$$
  (2) *If* $1 < \alpha + 2s \le 2$ *then* $u \in C^{1,\alpha+2s-1}(\overline{\mathbb{R}^n_+})$ *and*
  $$\|u\|_{C^{1,\alpha+2s-1}(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{C^{0,\alpha}(\overline{\mathbb{R}^n_+})}\big).$$
  (3) *If* $2 < \alpha + 2s \le 3$ *then* $u \in C^{2,\alpha+2s-2}(\overline{\mathbb{R}^n_+})$ *and*
  $$\|u\|_{C^{2,\alpha+2s-2}(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{C^{0,\alpha}(\overline{\mathbb{R}^n_+})}\big).$$
- *If* $f(x',0) \neq 0$ *at some* $x' \in \mathbb{R}^{n-1}$ *then*
  (i) *If* $0 < 2s < 1$ *then* $u \in C^{0,2s}(\overline{\mathbb{R}^n_+})$ *and*
  $$\|u\|_{C^{0,2s}(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{L^\infty(\mathbb{R}^n_+)}\big).$$
  (ii) *If* $2s = 1$ *then u is in the Zygmund space* $\Lambda_*(\overline{\mathbb{R}^n_+})$ *and*
  $$\|u\|_{\Lambda_*(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{L^\infty(\mathbb{R}^n_+)}\big).$$
  (iii) *If* $1 < 2s < 2$ *then* $u \in C^{1,2s-1}(\overline{\mathbb{R}^n_+})$ *and*
  $$\|u\|_{C^{1,2s-1}(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{L^\infty(\mathbb{R}^n_+)}\big).$$

All the constants C above depend only on n, α and s.

Proof. From (5.2) we see that $(-\Delta)^s u_o(x) = (-\Delta_D^+)^s u(x) = f(x)$ when $x \in \mathbb{R}^n_+$. On the other hand, for $x = (x', x_n)$ with $x_n < 0$ we have

$$\begin{aligned}(-\Delta)^s u_o(x) &= \frac{1}{\Gamma(-s)} \int_0^\infty \big(e^{t\Delta} u_o(x) - u_o(x)\big) \frac{dt}{t^{1+s}} \\ &= \frac{1}{\Gamma(-s)} \int_0^\infty \big(u(x^*) - e^{t\Delta_D^+} u(x^*)\big) \frac{dt}{t^{1+s}} \\ &= -(-\Delta_D^+)^s u(x^*) = -f(x^*).\end{aligned}$$

Hence,

$$(-\Delta)^s u_o(x) = f_o(x), \quad \text{for all } x \in \mathbb{R}^n. \tag{5.4}$$

Now we apply the results by Silvestre to u_o (which coincides with u when $x_n \ge 0$). For (1)–(3), we notice that the condition $f(x',0) = 0$ ensures that the odd extension f_o is globally in $C^{0,\alpha}(\mathbb{R}^n)$. Then we can recall Proposition 5.1. As for (i)–(iii), we can only assure that f_o is just bounded (it has a jump discontinuity at $x_n = 0$) and the conclusion follows from Proposition 5.2. □

5.2. Particular one dimensional solutions

In this subsection we study the growth near the boundary of solutions to the one dimensional fractional problem

$$\begin{cases} (-D_{xx}^2)^s u(x) = f(x), & \text{for } x > 0, \\ u(0) = 0, \end{cases}$$

where $-D_{xx}^2$ denotes the Dirichlet Laplacian on the half line $[0, \infty)$ and

$$f = \begin{cases} 1, & \text{when } 0 < s < 1/2, \\ \chi_{[0,1]}(x), & \text{when } 1/2 \le s < 1. \end{cases}$$

5.2.1. Case $0 < s < 1/2$ and $f \equiv 1$

From (5.3),

$$\begin{aligned} u(x) &= c_s \int_0^\infty \left(\frac{1}{|x-z|^{1-2s}} - \frac{1}{|x+z|^{1-2s}} \right) dz \\ &= \frac{c_s}{x^{1-2s}} \int_0^\infty \left(\frac{1}{|1-z/x|^{1-2s}} - \frac{1}{|1+z/x|^{1-2s}} \right) dz \\ &= c_s x^{2s} \int_0^\infty \left(\frac{1}{|1-\omega|^{1-2s}} - \frac{1}{|1+\omega|^{1-2s}} \right) d\omega. \end{aligned}$$

The last integral above is finite. Indeed, since $s > 0$, the integral converges at the origin. On the other hand, let $\omega > 2$. Consider the function $\varphi(t) = (\omega - t)^{2s-1}$, for $-1 \le t \le 1$. Then

$$\varphi(1) - \varphi(-1) = |1-\omega|^{2s-1} - |1+\omega|^{2s-1} = 2\varphi'(\xi) \le C_s \omega^{2s-2},$$

which implies that the integral converges at infinity for $s < 1/2$. We conclude that

$$u(x) = c_s x^{2s}, \quad x \in \mathbb{R}^+, \ 0 < s < 1/2,$$

for some positive constant c_s that can be computed explicitly.

5.2.2. Case $s = 1/2$ and $f = \chi_{[0,1]}$

Let $0 < x < 1/2$. We can write

$$\begin{aligned} u(x) &= c \int_0^1 \big(\ln|x+z| - \ln|x-z| \big) dz = c \int_0^1 \big(\ln|x(1+z/x)| - \ln|x(1-z/x)| \big) dz \\ &= c \int_0^1 \big(\ln|1+z/x| - \ln|1-z/x| \big) dz = cx \int_0^{1/x} \big(\ln|1+\omega| - \ln|1-\omega| \big) d\omega \\ &= cx \left(C + \int_2^{1/x} \big(\ln(\omega+1) - \ln(\omega-1) \big) d\omega \right) \\ &= cx \left[C + \left(\tfrac{1}{x}+1\right) \ln\left(\tfrac{1}{x}+1\right) - \left(\tfrac{1}{x}-1\right) \ln\left(\tfrac{1}{x}-1\right) \right] \\ &= c \left[Cx + (1+x)\ln(1+x) - (1-x)\ln(1-x) - 2x\ln x \right]. \end{aligned}$$

It is clear that

$$u(x) = -2cx \ln x + w(x), \quad 0 < x < 1/2,$$

where w is smooth up to $x = 0$. Recall that, $\ln(1+x) \sim x$ and $-\ln(1-x) \sim x$, when $0 < x < 1/2$. Also $\ln x \sim x - 1$, when $0 < x < 1/2$. Therefore,

$$u(x) \sim x, \quad \text{as } x \to 0^+.$$

5.2.3. Case $1/2 < s < 1$ and $f = \chi_{[0,1]}$

From (5.3),

$$\begin{aligned} u(x) &= c_s \int_0^1 \left(|x-y|^{2s-1} - (x+y)^{2s-1}\right) dy \\ &= c_s \left(\int_0^x (x-y)^{2s-1}\, dy + \int_x^1 (y-x)^{2s-1}\, dy - \tfrac{1}{2s}\left((x+1)^{2s} - x^{2s}\right) \right) \\ &= \frac{c_s}{2s} \left(2x^{2s} + (1-x)^{2s} - (1+x)^{2s} \right). \end{aligned}$$

It is clear that, for some constant $c > 0$,

$$u(x) = cx^{2s} + w(x), \quad 0 < x < 1/2,$$

where w is smooth up to $x = 0$. By taking into account the series expansions of $(1 \pm x)^{2s}$ it is easy to check that

$$u(x) \sim x, \quad \text{as } x \to 0^+.$$

5.3. Behavior near the boundary for half space solutions

Our next step is to consider the problem for the fractional Dirichlet Laplacian in the half space $\mathbb{R}^n_+$, $n \geq 2$,

$$\begin{cases} (-\Delta_D^+)^s w = g, & \text{in } \mathbb{R}^n_+, \\ w(x', 0) = 0, & \text{on } \partial\mathbb{R}^n_+, \end{cases} \tag{5.5}$$

in the cases where

$$g(x) = g(x', x_n) = \begin{cases} 1, & \text{when } 0 < s < 1/2, \\ \chi_{[0,1]}(x_n), & \text{when } 1/2 \leq s < 1. \end{cases} \tag{5.6}$$

To that end we apply the following result.

Lemma 5.4. *Let $g : \mathbb{R}^n \to \mathbb{R}$ be a function depending only on the x_n-variable, that is, $g(x) = \phi(x_n)$ for some function $\phi : \mathbb{R} \to \mathbb{R}$, for all $x \in \mathbb{R}^n$. Then the solution to*

$$(-\Delta)^s w = g, \quad \textit{in } \mathbb{R}^n,$$

is a function that depends only on x_n. More precisely, $w(x) = \psi(x_n)$ for all $x \in \mathbb{R}^n$, where $\psi : \mathbb{R} \to \mathbb{R}$ is the solution to the one dimensional problem

$$(-\partial_{xx}^2)^s \psi = \phi, \quad \textit{in } \mathbb{R}^n.$$

Here $-\partial_{xx}^2$ is the Laplacian on the real line $\mathbb{R}$.

Proof. Notice first that

$$e^{t\Delta}g(x) = \int_{-\infty}^{\infty} \frac{e^{-|x_n-z_n|^2/(4t)}}{(4\pi t)^{n/2}}\phi(z_n)\left(\int_{\mathbb{R}^{n-1}} e^{-|x'-z'|^2/(4t)}\,dz'\right)dz_n$$
$$= \int_{-\infty}^{\infty} \frac{e^{-|x_n-z_n|^2/(4t)}}{(4\pi t)^{1/2}}\phi(z_n)\,dz_n = e^{-t\partial_{xx}^2}\phi(x_n),$$

where $\{e^{-t\partial_{xx}^2}\}_{t>0}$ denotes the heat semigroup on the real line. Hence,

$$w(x) = (-\Delta)^{-s}g(x) = \frac{1}{\Gamma(s)}\int_0^\infty e^{t\Delta}g(x)\,\frac{dt}{t^{1-s}}$$
$$= \frac{1}{\Gamma(s)}\int_0^\infty e^{-t\partial_{xx}^2}\phi(x_n)\,\frac{dt}{t^{1-s}} = (-\partial_{xx}^2)^{-s}\phi(x_n) = \psi(x_n). \qquad \Box$$

In view of the previous Lemma, the one dimensional results and the relations (5.2) and (5.4), we get that the solution w to (5.5) with g as in (5.6) satisfies the following properties:

$$w(x) = \begin{cases} cx_n^{2s}, & \text{for all } x \in \mathbb{R}^n_+, \text{ when } 0 < s < 1/2, \\ -cx_n \ln x_n + \eta_{1/2}(x_n), & \text{for all } x \in \mathbb{R}^n_+,\ x_n < 1/2, \text{ when } s = 1/2, \\ cx_n^{2s} + \eta(x_n), & \text{for all } x \in \mathbb{R}^n_+,\ x_n < 1/2, \text{ when } 1/2 < s < 1. \end{cases} \tag{5.7}$$

In the last two cases, $\eta_{1/2}$ and η are smooth up to $x_n = 0$. Also,

$$w(x) \sim x_n^{\min\{2s,1\}}, \quad \text{as } x_n \to 0^+, \text{ uniformly in } x' \in \mathbb{R}^{n-1}. \tag{5.8}$$

Finally, it is clear that the solution W to

$$\begin{cases} \operatorname{div}(y^a\nabla W) = 0, & \text{in } \mathbb{R}^n_+ \times (0,\infty), \\ -y^a W_y\big|_{y=0} = \theta g, & \text{on } \mathbb{R}^n_+, \\ W = 0, & \text{on } \partial\mathbb{R}^n_+ \times [0,\infty), \end{cases} \tag{5.9}$$

with g as in (5.6) and $\theta \in \mathbb{R}$ satisfies

$$W(x,0) = \theta w(x), \quad x \in \mathbb{R}^n_+.$$

6. Boundary regularity – Dirichlet case

Theorems 1.3 and 1.5 for solutions to (1.2) are consequences of the more general results that we state and prove here.

Throughout this section we assume that $\Omega \subset \mathbb{R}^n_+$ is a bounded domain whose boundary $\partial\Omega$ contains a flat portion on $\{x_n = 0\}$ in such a way that $B_1^+ \subset \Omega$.

We say that a function $f : B_1 \cap \{x_n \geq 0\} \to \mathbb{R}$ is in $L^{2,\alpha}_{\partial\Omega}(0)$, for $0 \leq \alpha < 1$, whenever

$$[f]^2_{L^{2,\alpha}_{\partial\Omega}(0)} := \sup_{0<r\leq 1} \frac{1}{r^{n+2\alpha}} \int_{B_r^+} |f(x) - f(0)|^2\,dx < \infty,$$

where $f(0)$ is defined as $f(0) := \lim_{r\to 0} \frac{1}{|B_r^+|}\int_{B_r^+} f(x)\,dx$. It is clear that if f is Hölder continuous of order α at 0 then $f \in L^{2,\alpha}_{\partial\Omega}(0)$. If the condition above holds uniformly in balls centered at points of $\partial\Omega$ close to the origin, then f is α-Hölder continuous at the boundary near the origin, see [8].

Theorem 6.1. *Consider the half space solutions w given in (5.7). Let u be a solution to (1.2). Assume that $f \in L^{2,\alpha}_{\partial\Omega}(0)$, for some $0<\alpha<1$.*

(1) *Suppose that $0<\alpha+2s<1$. There exist $0<\delta<1$, depending only on n, ellipticity, α and s, and a constant $C_0>0$ such that if*

$$\sup_{0<r\le 1}\frac{1}{r^n}\int_{B_r^+}|A(x)-A(0)|^2\,dx<\delta^2,$$

then

$$\frac{1}{r^n}\int_{B_r^+}|u(x)-f(0)w(x)|^2\,dx\le C_1r^{2(\alpha+2s)},\quad \text{for all } r>0 \text{ sufficiently small},$$

where $C_1\le C_0\big(1+\|u\|_{L^2(\Omega)}+[u]_{H^s(\Omega)}+[f]_{L^{2,\alpha}_{\partial\Omega}(0)}+|f(0)|\big)$.

(2) *Suppose that $s\ge 1/2$ and $1<\alpha+2s<2$. There exist $0<\delta<1$, depending only on n, ellipticity, α and s, and a constant $C_0>0$ such that if*

$$\sup_{0<r\le 1}\frac{1}{r^{n+2(\alpha+2s-1)}}\int_{B_r^+}|A(x)-A(0)|^2\,dx<\delta^2,$$

then there exists a linear function $\ell(x)=\mathcal{B}\cdot x$ such that

$$\frac{1}{r^n}\int_{B_r^+}|u(x)-f(0)w(x)-\ell(x)|^2\,dx\le C_1r^{2(\alpha+2s)},\quad \text{for all } r>0 \text{ sufficiently small},$$

where $C_1+|\mathcal{B}|\le C_0\big(1+\|u\|_{L^2(\Omega)}+[u]_{H^s(\Omega)}+[f]_{L^{2,\alpha}_{\partial\Omega}(0)}+|f(0)|\big)$.

The constants C_0 above depend only on $[A]_{L^{2,0}_{\partial\Omega}(0)}$ (resp. $[A]_{L^{2,\alpha+2s-1}_{\partial\Omega}(0)}$), ellipticity, n, α and s.

Observe the extra term 1 in the estimates for C_1 and $C_1+|\mathcal{B}|$ that comes from the H^s norm of w.

Theorem 1.3 is a consequence of Theorem 6.1. Indeed, first notice that the conclusion of Theorem 6.1 can be translated to any point x_0 at $\partial\Omega$. We can first flatten the boundary of Ω at x_0 and then rescale (and rotate if necessary) the resulting domain so that $B_1^+(x_0)\subset\Omega$ with $B_1(x_0)\cap\{x_n=0\}\subset\partial\Omega$. Then $v:=u-f(x_0)w$ has the desired regularity around x_0 (remember that $u(x_0)-f(x_0)w(x_0)=0$). By taking into account the growth of w near the boundary (5.8) and going back to the initial variables the conclusion follows.

In a similar way as we did for $L^{2,\alpha}_{\partial\Omega}(0)$, we can define $L^{2,-2s+\alpha}_{\partial\Omega}(0)$ and $L^{2,-2s+\alpha+1}_{\partial\Omega}(0)$. It is clear that if $f\in L^p(B_1^+)$ then parallel remarks as those preceding Theorem 4.2 hold for $L^{2,-2s+\alpha}_{\partial\Omega}(0)$ and $L^{2,-2s+\alpha+1}_{\partial\Omega}(0)$, with the same exponents p and α.

Theorem 6.2. *Let u be a solution to (1.2).*

(1) *Suppose that $f\in L^{2,-2s+\alpha}_{\partial\Omega}(0)$. There exist $0<\delta<1$, depending only on n, ellipticity, α and s, and a constant $C_0>0$ such that if*

$$\sup_{0<r\le 1}\frac{1}{r^n}\int_{B_r^+}|A(x)-A(0)|^2\,dx<\delta^2,$$

then

$$\frac{1}{r^n}\int_{B_r^+}|u(x)|^2\,dx\le C_1r^{2\alpha},\quad \text{for all } r>0 \text{ sufficiently small},$$

where $C_1\le C_0\big(\|u\|_{L^2(\Omega)}+[u]_{H^s(\Omega)}+[f]_{L^{2,-2s+\alpha}_{\partial\Omega}(0)}\big)$.

(2) *Suppose that* $f \in L^{2,-2s+\alpha+1}_{\partial\Omega}(0)$. *There exist* $0<\delta<1$, *depending only on* n, *ellipticity,* α *and* s, *and a constant* $C_0>0$ *such that if*

$$\sup_{0<r\leq 1}\frac{1}{r^{n+2\alpha}}\int_{B_r^+}|A(x)-A(0)|^2\,dx<\delta^2,$$

then there exists a linear function $\ell(x)=\mathcal{B}\cdot x$ *such that*

$$\frac{1}{r^n}\int_{B_r^+}|u(x)-\ell(x)|^2\,dx\leq C_1r^{2(1+\alpha)},\quad \text{for all } r>0 \text{ sufficiently small},$$

where $C_1+|\mathcal{B}|\leq C_0\big(\|u\|_{L^2(\Omega)}+[u]_{H^s(\Omega)}+[f]_{L^{2,-2s+\alpha+1}_{\partial\Omega}(0)}\big)$.

The constants C_0 *above depend only on* $[A]_{L^{2,0}_{\partial\Omega}(0)}$ *(resp.* $[A]_{L^{2,\alpha}_{\partial\Omega}(0)}$*), ellipticity,* n*,* α *and* s.

Notice that in Theorem 6.2 we do not need to subtract w from u to obtain the regularity up to the origin. Observe that Theorem 1.5 in the Dirichlet case follows from this last result after flattening the boundary.

The rest of this section is devoted to the proof of Theorems 6.1 and 6.2.

6.1. *Proof of Theorem 6.1(1)*

It is enough to prove the result for $u(x)=U(x,0)$, where $U\in H^1((B_1^+)^*,y^a\,dX)$ is a solution to

$$\begin{cases}\operatorname{div}(y^aB(x)\nabla U)=\operatorname{div}(y^aF), & \text{in } (B_1^+)^*,\\ -y^aU_y\big|_{y=0}=f, & \text{on } B_1^+,\\ U=0, & \text{on } B_1\cap\{x_n=0\}.\end{cases}\tag{6.1}$$

Here F is a $(B_1^+)^*$-valued vector field such that $F_{n+1}=0$ and satisfies the Morrey condition

$$\sup_{0<r\leq 1}\frac{1}{r^{n+1+a+2(\alpha+2s-1)}}\int_{(B_r^+)^*}y^a|F|^2\,dX<\infty.$$

After a change of variables we can assume that $B(0)=I$.

We compare $U(x,0)$ with $W(x,0)$, where W is the solution to (5.9) with $\theta=f(0)$. In particular, $W\in H^1((B_1^+)^*,y^a\,dX)$ is a solution to

$$\begin{cases}\operatorname{div}(y^a\nabla W)=0, & \text{in } (B_1^+)^*,\\ -y^aW_y\big|_{y=0}=f(0), & \text{on } B_1^+,\\ W=0, & \text{on } B_1\cap\{x_n=0\},\end{cases}$$

and $W(x,0)=f(0)w(x)$, for $x\in B_1^+$, with w as in (5.7).

Let $V=U-W$. Then

$$\begin{cases}\operatorname{div}(y^aB(x)\nabla V)=\operatorname{div}(y^aH), & \text{in } (B_1^+)^*,\\ -y^aV_y\big|_{y=0}=h, & \text{on } B_1^+,\\ V=0, & \text{on } B_1\cap\{x_n=0\},\end{cases}\tag{6.2}$$

where $h=f-f(0)$, so that $h(0)=0$, and $H=F+(I-B(x))\nabla W$, with $H_{n+1}=0$. Given $\delta>0$ we can always assume that the conditions (1)–(4) in Subsection 4.1 hold with the appropriate changes: B_r^+, h, H, V, B_1^+ and $(B_1^+)^*$ in place of B_r, f, F, U, B_1 and B_1^*, respectively. Under those hypotheses and the proper normalization for the L^2 norms of V and its trace, we call V a normalized solution.

Now we prove that given $0<\alpha+2s<1$ there exists $0<\delta<1$, depending only on n, ellipticity, α and s, such that for any normalized solution V of (6.2) (recall that $V(0,0)=0$)

$$\frac{1}{r^n}\int_{B_r^+}|V(x,0)|^2\,dx \le Cr^{2(\alpha+2s)}, \quad \text{for all } r>0 \text{ sufficiently small,}$$

and $C \le C_0$, for some constant C_0 depending only on n, ellipticity, α and s. The strategy of the proof is the same as the proof of Theorem 4.1(1) presented in Subsection 4.1. Here we explain the changes that need to be made. We follow parallel steps as those in the proof of Lemma 4.3. Indeed, by using Remarks 3.4 and 3.6 we can prove that there exist $0<\delta, \lambda<1$ such that

$$\frac{1}{\lambda^n}\int_{B_\lambda^+}|V(x,0)|^2\,dx + \frac{1}{\lambda^{n+1+a}}\int_{(B_\lambda^+)^*} y^a|V|^2\,dX < \lambda^{2(\alpha+2s)}.$$

Notice that in the present case we have in Lemma 4.3 that $c=0$. Now the rescaling process of Lemma 4.4 comes into play, but now we must take $c_k=0$ for all $k\ge 0$. Every step goes through by just changing balls by half balls, because when we rescale we always get a normalized solution to (6.2).

6.2. *Proof of Theorem 6.1(2)*

As in the proof of part (1), we just have to check that there exists $0<\delta<1$ such that in the proper normalized situation for (6.2) there is a linear function $\ell_\infty(x) = B_\infty\cdot x$ such that

$$\frac{1}{r^n}\int_{B_r^+}|V(x,0)-\ell_\infty(x)|^2\,dx \le Cr^{2(\alpha+2s)},$$

for $r>0$ sufficiently small. Now we suppose that $F(0)=0$ and that we have the Campanato condition

$$\sup_{0<r\le 1}\frac{1}{r^{n+1+a+2(\alpha+2s-1)}}\int_{(B_r^+)^*} y^a|F|^2\,dX < \delta^2.$$

But again we can follow parallel steps as those of the proof of Theorem 4.1(2). Observe that in our case the independent term A in the linear function that appears in Lemma 4.5 is 0 since for the approximating harmonic function $\mathcal{W}$ we have $\mathcal{W}(0,0)=0$. This is essential in the iteration process in order to always have a rescaled solution that is 0 on $B_1\cap\{x_n=0\}$. Further details are omitted.

6.3. *Proof of Theorem 6.2*

As before, it is enough to prove the result for $u(x)=U(x,0)$, where $U\in H^1((B_1^+)^*, y^a dX)$ is a solution to (6.1), where $F_{n+1}=0$. For part (1) we assume the Morrey condition

$$\sup_{0<r\le 1}\frac{1}{r^{n+1+a+2(\alpha-1)}}\int_{(B_r^+)^*} y^a|F|^2\,dX < \infty,$$

while for part (2) we suppose that $F(0)=0$ and that we have the Campanato condition

$$\sup_{0<r\le 1}\frac{1}{r^{n+1+a+2\alpha}}\int_{(B_r^+)^*} y^a|F|^2\,dX < \infty.$$

Now the proof follows exactly the same steps of the proof of Theorem 4.2, by just replacing B_r by B_r^+. We also observe that in this case the constant c and the independent term A that come from the approximating harmonic function $\mathcal{W}$ are both 0. We omit further details.

7. The case of Neumann boundary condition

In this section we sketch the proof of the results in the case of the Neumann problem (1.6). Recall that the domain of L_N is the Sobolev space $H^1(\Omega)$. There exists an orthonormal basis of $L^2(\Omega)$ consisting of eigenfunctions $\varphi_k \in H^1(\Omega)$, $k = 0, 1, 2, \ldots$, that correspond to eigenvalues $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \cdots \nearrow \infty$ of L_N. The domain of the fractional operator L_N^s is the Hilbert space $\mathcal{H}_N^s$ of functions $u \in L^2(\Omega)$ such that $\int_\Omega u\,dx = 0$ and $\sum_{k=1}^\infty \mu_k^s u_k^2 \equiv \sum_{k=1}^\infty \mu_k^s |\langle u, \varphi_k\rangle_{L^2(\Omega)}|^2 < \infty$. We define $L_N^s u$ by (1.4) in the dual space $\mathcal{H}_N^{-s} := (\mathcal{H}_N^s)'$. Notice that $\langle L_N^s u, 1\rangle = 0$. The heat semigroup generated by L_N is given by

$$e^{-tL_N}u(x) = \sum_{k=0}^\infty e^{-t\mu_k} u_k \varphi_k(x) = \int_\Omega W_t^N(x,z)u(z)\,dz,$$

where $W_t^N(x,z)$ is the corresponding heat kernel. Observe that $e^{-tL_N}1(x) = 1$ for all $x \in \Omega$, $t > 0$. As in Theorem 2.3 we can prove that (1.5) holds for $u, \psi \in \mathcal{H}_N^s$ and

$$K_s^N(x,z) := \frac{1}{2|\Gamma(-s)|}\int_0^\infty W_t^N(x,z)\,\frac{dt}{t^{1+s}}.$$

It is well known that if Ω is an exterior domain or the region lying above the graph of a Lipschitz function, then the heat kernel $W_t^N(x,z)$ as global upper Gaussian estimates. If the domain is bounded then the Gaussian estimate holds only for short times and the heat kernel is bounded in x, z and t as $t \to \infty$. For this see [3] and the references therein. Hence, as we did in Section 2 we can prove that the kernel $K_s^N(x,z)$ satisfies the estimate

$$0 \leq K_s^N(x,z) \leq \frac{c_{\Omega,n,s}}{|x-z|^{n+2s}}, \quad x, z \in \Omega.$$

For the heat kernel related to the Neumann Laplacian we have two-sided Gaussian estimates (see [26], also [34], and the references therein), which imply that in this case the estimate for $K_s^N(x,z)$ from above also holds from below with a different constant. The extension problem for L_N^s is given as follows. The solution U to

$$\begin{cases} \operatorname{div}_x(A(x)\nabla_x U) + \frac{a}{y}U_y + U_{yy} = 0, & \text{in } \Omega \times (0,\infty), \\ \partial_A U(x,y) = 0, & \text{on } \partial\Omega \times [0,\infty), \\ U(x,0) = u(x), & \text{on } \Omega, \end{cases} \tag{7.1}$$

satisfies, for $c_s = |\Gamma(-s)|/(4^s\Gamma(s)) > 0$,

$$-\frac{1}{2s}\lim_{y\to 0^+} y^a U_y(x,y) = -\lim_{y\to 0^+}\frac{U(x,y)-U(x,0)}{y^{2s}} = c_s L_N^s u(x), \quad \text{in } \mathcal{H}_N^{-s},$$

see [32,33]. Similar scaling properties as those of Section 2 hold for L_N^s. Parallel to Theorem 2.6, the fundamental solution $G_s^N(x,z)$ of L_N^s verifies the interior estimate

$$G_s^N(x,z) \sim \frac{c_{\Omega,n,s}}{|x-z|^{n-2s}}, \quad n \neq 2s,$$

and it is logarithmic when $n = 2s$. The interior Harnack inequality for L_N^s is also true.

Next we show that the domain of L_N^s is the fractional Sobolev space $H^s(\Omega)$, which is the closure of $C^\infty(\Omega)$ with respect to the norm $\|u\|_{H^s(\Omega)}^2 = \|u\|_{L^2(\Omega)}^2 + [u]_{H^s(\Omega)}^2$. Notice that the solution U to the extension problem (7.1) belongs to $H^1(\Omega \times (0,\infty), y^a\,dX)$ and that for each fixed $y \geq 0$ we have $\int_\Omega U(x,y)\,dx = 0$. Since the trace of U is an element of the fractional Sobolev space $H^s(\Omega)$ (see for example [20]), it turns out that $\mathcal{H}_N^s \subset H^s(\Omega)$. For the other inclusion, we notice that the energy for the extension problem for L_N^s is comparable to the energy for the extension problem for the Neumann Laplacian $-\Delta_N$. This and the following Lemma give that if $\int_\Omega u\,dx = 0$ then $u \in \mathcal{H}_N^s$ if and only if $u \in H^s(\Omega)$.

Lemma 7.1. *Let $u:\Omega\to\mathbb{R}$ such that $\int_\Omega u\,dx=0$. For any $0<s<1$, $u\in\mathrm{Dom}((-\Delta_N)^s)$ if and only if $u\in H^s(\Omega)$. In this case we have*

$$\|(-\Delta_N)^{s/2}u\|_{L^2(\Omega)}\sim[u]_{H^s(\Omega)}.$$

Proof. The idea is similar to the proof of Theorem 2.4 in [34] and here we sketch the steps. Let U be the solution to the extension problem for the fractional Neumann Laplacian in Ω. Then

$$-\frac{\langle U(\cdot,y),u\rangle_{L^2(\Omega)}-\langle u,u\rangle_{L^2(\Omega)}}{y^{2s}}\to c_s\langle(-\Delta_N)^s u,u\rangle_{L^2(\Omega)}=c_s\|(-\Delta_N)^{s/2}u\|^2_{L^2(\Omega)},$$

as $y\to0^+$. Now, with the Poisson kernel $P_y^{s,N}$ for the Neumann Laplacian extension problem (as in (2.10)–(2.11) but with the Neumann heat kernel in place of $W_t(x,z)$) we get

$$\langle U(\cdot,y),u\rangle_{L^2(\Omega)}-\langle u,u\rangle_{L^2(\Omega)}=\int_\Omega\int_\Omega P_y^{s,N}(x,z)\big(u(z)u(x)-u(x)^2\big)\,dx\,dz.$$

Here we used the fact that

$$\int_\Omega P_y^{s,N}(x,z)\,dz=1,\qquad\text{for all } x\in\Omega,\ t>0.$$

By exchanging the roles of x and z and using that $P_y^{s,N}(x,z)=P_y^{s,N}(z,x)$, we get

$$\langle U(\cdot,y),u\rangle_{L^2(\Omega)}-\langle u,u\rangle_{L^2(\Omega)}=\frac12\int_\Omega\int_\Omega\big(u(z)-u(x)\big)P_y^{s,N}(x,z)\,dz\,dx.$$

But now, since the heat kernel for the Neumann Laplacian has two sided Gaussian estimates [26,34], we readily get

$$P_y^{s,N}(x,z)\sim\frac{y^{2s}}{(|x-z|^2+y^2)^{(n+2s)/2}}.$$

Therefore, by dividing by y^{2s} and taking $y\to0^+$ above we arrive to $\|(-\Delta_N)^{s/2}u\|_{L^2(\Omega)}\sim[u]_{H^s(\Omega)}$. □

Given $f\in\mathcal{H}_N^{-s}(\Omega)$ (observe that $\langle f,1\rangle=0$), there exists a unique solution $u\in H^s(\Omega)$ to (1.6) with $\int_\Omega u\,dx=0$.

It is clear that the interior regularity results of Theorems 1.1 and 1.2 hold for the case of the fractional Neumann operator L_N^s. Indeed, the proof presented in Section 4 is based on a Caccioppoli estimate for the (localized) extension problem and the Dirichlet boundary condition plays no significant role there.

The boundary results deserve some attention. The reason why the interior regularity should hold up to the boundary becomes apparent once we look at the half space case. Consider the fractional Neumann Laplacian $-\Delta_N^+$ in a half space $\mathbb{R}^n_+$. For u having mean zero we have

$$(-\Delta_N^+)^s u(x)=\frac{1}{\Gamma(-s)}\int_0^\infty\big(e^{t\Delta_N^+}u(x)-u(x)\big)\frac{dt}{t^{1+s}},\qquad x\in\mathbb{R}^n_+.$$

We denote by u_e the even reflection of u with respect to the variable x_n:

$$u_e(x)=\begin{cases}u(x), & \text{if } x_n\ge0,\\ u(x^*), & \text{if } x_n<0.\end{cases}$$

Then the method of reflections gives $e^{-t\Delta_N^+}u(x)=e^{t\Delta}u_e(x)$, for $x\in\mathbb{R}^n_+$. Therefore,

$$(-\Delta_N^+)^s u(x)=c_{n,s}\int_{\mathbb{R}^n_+}\big(u(x)-u(z)\big)\left(\frac{1}{|x-z|^{n+2s}}+\frac{1}{|x-z^*|^{n+2s}}\right)dz,\qquad x\in\mathbb{R}^n_+.$$

Now, we easily see that if $(-\Delta_N^+)^s u(x)=f(x)$, with f having zero mean in $\mathbb{R}^n_+$, then $(-\Delta)^s u_e(x)=f_e(x)$, for $x\in\mathbb{R}^n$. As a conclusion, by applying Propositions 5.1 and 5.2 we get the following.

Theorem 7.2 *(Global regularity in half space – Neumann case). Let u be a bounded solution to*

$$\begin{cases} (-\Delta_N^+)^s u = f, & \text{in } \mathbb{R}^n_+, \\ \partial_{x_n} u = 0, & \text{on } \partial\mathbb{R}^n_+ = \{x_n = 0\}, \end{cases}$$

where f has zero mean.

- *Suppose that $f \in C^{0,\alpha}(\overline{\mathbb{R}^n_+})$, $0 < \alpha \le 1$. Then*
  (1) *If $\alpha + 2s \le 1$ then $u \in C^{0,\alpha+2s}(\overline{\mathbb{R}^n_+})$ and*
  $$\|u\|_{C^{0,\alpha+2s}(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{C^{0,\alpha}(\overline{\mathbb{R}^n_+})}\big).$$
  (2) *If $1 < \alpha + 2s \le 2$ then $u \in C^{1,\alpha+2s-1}(\overline{\mathbb{R}^n_+})$ and*
  $$\|u\|_{C^{1,\alpha+2s-1}(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{C^{0,\alpha}(\overline{\mathbb{R}^n_+})}\big).$$
  (3) *If $2 < \alpha + 2s \le 3$ then $u \in C^{2,\alpha+2s-2}(\overline{\mathbb{R}^n_+})$ and*
  $$\|u\|_{C^{2,\alpha+2s-2}(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{C^{0,\alpha}(\overline{\mathbb{R}^n_+})}\big).$$
- *Suppose that $f \in L^\infty(\mathbb{R}^n_+)$. Then*
  (i) *If $0 < 2s < 1$ then $u \in C^{0,2s}(\overline{\mathbb{R}^n_+})$ and*
  $$\|u\|_{C^{0,2s}(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{L^\infty(\mathbb{R}^n_+)}\big).$$
  (ii) *If $2s = 1$ then u is in the Zygmund space $\Lambda_*(\overline{\mathbb{R}^n_+})$ and*
  $$\|u\|_{\Lambda_*(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{L^\infty(\mathbb{R}^n_+)}\big).$$
  (iii) *If $1 < 2s < 2$ then $u \in C^{1,2s-1}(\overline{\mathbb{R}^n_+})$ and*
  $$\|u\|_{C^{1,2s-1}(\overline{\mathbb{R}^n_+})} \le C\big(\|u\|_{L^\infty(\mathbb{R}^n_+)} + \|f\|_{L^\infty(\mathbb{R}^n_+)}\big).$$

All the constants C above depend only on n, α and s.

For the general case, as we did before for the Dirichlet case, it is enough to prove the regularity at the origin for the solution U to the extension problem

$$\begin{cases} \operatorname{div}(y^a B(x)\nabla U) = \operatorname{div}(y^a F), & \text{in } (B_1^+)^*, \\ -y^a U_y\big|_{y=0} = f, & \text{on } B_1^+, \\ \partial_A U(x,y) = 0, & \text{on } B_1^* \cap \{x_n = 0\}, \end{cases}$$

where A and F have the corresponding regularity. We can assume that $B(0) = I$, so that $\partial_A U(0,0) = -\partial_{x_n} U(0,0) = 0$. The same Caccioppoli estimate holds in this case. The approximation lemma can then be proved to get an approximating harmonic function $\mathcal{W} \in H^1((B_{3/4}^*)^+, y^a dX)$, see also Remark 3.4. By performing an even reflexion instead of an odd one we can reproduce the argument in Remark 3.6 and conclude that $\mathcal{W}$ has the desired regularity. From here on we can repeat the arguments given in Section 6. Details are left to the interested reader.

Conflict of interest statement

We confirm that we do not have any conflict.

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