



Energy estimates and symmetry breaking in attractive Bose–Einstein condensates with ring-shaped potentials

Yujin Guo, Xiaoyu Zeng, Huan-Song Zhou*

Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, PR China

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Abstract

This paper is concerned with the properties of L^2 -normalized minimizers of the Gross–Pitaevskii (GP) functional for a two-dimensional Bose–Einstein condensate with attractive interaction and ring-shaped potential. By establishing some delicate estimates on the least energy of the GP functional, we prove that symmetry breaking occurs for the minimizers of the GP functional as the interaction strength $a > 0$ approaches a critical value a^* , each minimizer of the GP functional concentrates to a point on the circular bottom of the potential well and then is non-radially symmetric as $a \nearrow a^*$. However, when $a > 0$ is suitably small we prove that the minimizers of the GP functional are unique, and this unique minimizer is radially symmetric.

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1. Introduction

Since the remarkable experiments on Bose–Einstein condensates (BEC) in dilute gases of alkali atoms in 1995 [1,6], much attention has been attracted to the experimental studies on BEC over the last two decades, and many new phenomena of BEC have been observed in experiments [6]. These new experimental progresses also inspired the theoretical research in BEC, especially, the theory of Gross–Pitaevskii (GP) equations proposed by Gross and Pitaevskii [11,12,29]. There has been a growing interest in the mathematical theories and numerical methods of GP equations [2]. Several rigorous mathematical verifications of GP theory were established, see e.g. [8,22–25]. It is known that the classical trapping potential used in the study of BEC is the harmonic potential. With the advance of experimental techniques for BEC, some different trapping potentials have been used in the experiments [4,14,16,31,32]. Theoretically, it is also interesting to discuss mathematically how the shapes of trapping potentials affect the behavior

* Corresponding author. Tel.: +86 27 87199196; fax: +86 27 87199291.

E-mail addresses: yjguo@wipm.ac.cn (Y. Guo), xyzeng@wipm.ac.cn (X. Zeng), hszhou@wipm.ac.cn (H.-S. Zhou).

of BEC. Very recently, Guo and Seiringer [13] studied the BEC with attractive interactions in \mathbb{R}^2 described by the following GP functional

$$E_a(u) := \int_{\mathbb{R}^2} (|\nabla u(x)|^2 + V(x)|u(x)|^2) dx - \frac{a}{2} \int_{\mathbb{R}^2} |u(x)|^4 dx, \quad u \in \mathcal{H}, \quad (1.1)$$

where $a > 0$ describes the strength of the attractive interactions, and \mathcal{H} is a real-valued function space defined by

$$\mathcal{H} := \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)|u(x)|^2 dx < \infty \right\}, \quad (1.2)$$

with a trapping potential of the form

$$V(x) = h(x) \prod_{i=1}^n |x - x_i|^{p_i}, \quad p_i > 0 \text{ and } C < h(x) < 1/C \quad (1.3)$$

for some $C > 0$ and all $x \in \mathbb{R}^2$. The authors in [2,13] proved that there exists $a^* > 0$ such that the constrained minimization problem

$$e(a) := \inf \left\{ E_a(u) : u \in \mathcal{H} \text{ and } \int_{\mathbb{R}^2} u^2 dx = 1 \right\} \quad (1.4)$$

has at least one minimizer if and only if $a \in [0, a^*)$. Moreover,

$$a^* = \int_{\mathbb{R}^2} |Q(x)|^2 dx, \quad (1.5)$$

and $Q(x)$ is the unique positive solution (up to translations) of the scalar field equation

$$-\Delta u + u - u^3 = 0 \quad \text{in } \mathbb{R}^2, \quad u \in H^1(\mathbb{R}^2). \quad (1.6)$$

The existence of $Q(x)$ is well known and $Q(x)$ is actually radially symmetric, see e.g. [9,19,20,27].

In what follows, we call $e(a)$ the GP energy, which is also the least energy of a BEC system. As mentioned in [13], the parameter a in (1.1) has to be interpreted as the particle number times the interaction strength, the existence of the threshold value a^* described above shows that there exists a critical particle number for collapse of the BEC [6]. Theorem 1 of [13] implies that the shape of trapping potential does not affect the critical particle number. However, the behavior of the minimizers for (1.4) as $a \nearrow a^*$ does depend on the shape of potentials. In fact, for the trapping potential (1.3), a detailed description of the behavior of the minimizers for (1.4) is given in Theorem 2 of [13], which shows that minimizers of (1.4) must concentrate at one of the flattest minima x_{i_0} ($1 \leq i \leq n$) of $V(x)$ as $a \nearrow a^*$. This also implies the presence of symmetry breaking of the minimizer. Note that the method of [13] depends heavily on the potential $V(x)$ of (1.3) having a finite number of minima $\{x_i \in \mathbb{R}^2, i = 1, \dots, n\}$.

It is natural to ask what would happen if $V(x)$ has infinitely many minima. Hence, in this paper we are mainly interested in studying the GP functional with a trapping potential $V(x)$ with infinitely many minima and analyzing the detailed behavior of its minimizers as $a \nearrow a^*$. For this purpose, we focus on the following ring-shaped trapping potential:

$$V(x) = (|x| - A)^2, \quad \text{where } A > 0, \quad x \in \mathbb{R}^2, \quad (1.7)$$

which is essentially an important potential used in BEC experiments, see e.g. [15,16,31]. Clearly, all points in the set $\{x \in \mathbb{R}^2 : |x| = A\}$ are minima of the potential given by (1.7). Concerning the existence of minimizers of problem (1.4), much more general potentials $V(x)$ than (1.7) are allowed, see [13, Theorem 1]. But to demonstrate clearly that symmetry breaking does occur in the minimizers of problem (1.4), the uniqueness of the minimizers of problem (1.4) is used in our Corollary 1.4. So, we first give the theorem as follows.

Theorem 1.1. *Let a^* be given by (1.5), and let $V(x)$ be such that*

$$0 \leq V(x) \in L_{loc}^\infty(\mathbb{R}^2), \quad \lim_{|x| \rightarrow \infty} V(x) = \infty \quad \text{and} \quad \inf_{x \in \mathbb{R}^2} V(x) = 0.$$

Then

- (i) *For all $a \in [0, a^*)$ Eq. (1.4) has at least one minimizer, and there is no minimizer for (1.4) if $a \geq a^*$. Moreover, $e(a) > 0$ if $a < a^*$, $\lim_{a \nearrow a^*} e(a) = e(a^*) = 0$ and $e(a) = -\infty$ if $a > a^*$.*
- (ii) *When $a \in [0, a^*)$ is suitably small, Eq. (1.4) has a unique non-negative minimizer in \mathcal{H} .*

Part (i) of the above theorem is just Theorem 1 of [13]. For part (ii), a proof based on an implicit function theorem is given in Appendix A.

To analyze the detailed behavior of the minimizers for problem (1.4), a delicate estimate on the GP functional is required. As far as we know, it is usually not easy to derive directly the optimal energy estimates for the GP functional (1.1) under general trapping potentials. Although the authors in [13] developed an approach to establish this kind of energy estimates for the potential (1.3), it does not work well for our potential (1.7). In fact, by following the method of [13] we are only able to get the following type of estimates

$$C_1(a^* - a)^{\frac{2}{3}} \leq e(a) \leq C_2(a^* - a)^{\frac{1}{2}} \quad \text{as } a \nearrow a^*, \tag{1.8}$$

see Lemma 2.1 in Section 2. Therefore, one of the aims of the paper is to provide some new ways to estimate precisely the GP energy under the potential (1.7), which may be used effectively to handle some general type potentials. Based on the estimates, we may improve the power $\frac{2}{3}$ at the left of (1.8) to be the same as that at the right, namely $\frac{1}{2}$, see our Theorem 2.1 for the details. Then, we may continue to analyze in detail the behavior of the minimizers of (1.4), and we finally have the following theorem.

Theorem 1.2. *Let $V(x)$ be given by (1.7) and let u_a be a non-negative minimizer of (1.4) for $a < a^*$. For any given sequence $\{a_k\}$ with $a_k \nearrow a^*$ as $k \rightarrow \infty$, there exists a subsequence, still denoted by $\{a_k\}$, such that each u_{a_k} has a unique maximum point x_k and $x_k \rightarrow y_0$ as $k \rightarrow \infty$ for some $y_0 \in \mathbb{R}^2$ satisfying $|y_0| = A > 0$. Moreover,*

$$\lim_{k \rightarrow \infty} \frac{|x_k| - A}{(a^* - a_k)^{\frac{1}{4}}} = 0, \tag{1.9}$$

and

$$(a^* - a_k)^{\frac{1}{4}} u_{a_k}(x_k + (a^* - a_k)^{\frac{1}{4}} x) \xrightarrow{k} \frac{\lambda_0 Q(\lambda_0 x)}{\|Q\|_2} \quad \text{strongly in } H^1(\mathbb{R}^2), \tag{1.10}$$

where $\lambda_0 > 0$ satisfies

$$\lambda_0 = \left(\frac{1}{2} \int_{\mathbb{R}^2} |x|^2 Q^2(x) dx \right)^{\frac{1}{4}}. \tag{1.11}$$

As we mentioned above, our Theorem 2.1 gives the optimal power of the estimates of the GP energy $e(a)$ as $a \nearrow a^*$. Can we determine precisely the coefficients of the estimates in Theorem 2.1? Our following theorem answers the question.

Theorem 1.3. *Let $V(x)$ be given by (1.7), then the GP energy $e(a)$ satisfies*

$$\lim_{a \nearrow a^*} \frac{e(a)}{(a^* - a)^{\frac{1}{2}}} = \frac{2\lambda_0^2}{\|Q\|_2^2} \tag{1.12}$$

where λ_0 is given by (1.11).

Since the potential $V(x)$ in (1.7) has infinitely many global minima, the method used in [13] cannot be applied directly in our case. For this reason, we have to introduce some new tricks to prove Theorem 1.2. Moreover, there are also some new difficulties to be overcome in finding the exact value of λ_0 in Theorem 1.2. Noting that the trapping potential $V(x)$ of (1.7) is radially symmetric, it then follows from Theorem 1.1(ii) that $e(a)$ has a unique non-negative minimizer which is also radially symmetric for small $a > 0$. On the other hand, Theorem 1.2 shows that any non-negative minimizer of $e(a)$ concentrates at a point on the ring $\{x \in \mathbb{R}^2 : |x| = A\}$ as $a \nearrow a^*$, and thus it cannot be radially symmetric. This implies that, as the strength of the interaction a increases from 0 to a^* , symmetry breaking occurs in the minimizers of $e(a)$. Therefore, the above arguments yield immediately the following corollary.

Corollary 1.4. *Let $V(x)$ be given by (1.7). Then there exist $a_* > 0$ and $a_{**} > 0$ satisfying $a_{**} \leq a_* < a^*$ such that*

- (i) $e(a)$ has a unique non-negative minimizer which is radially symmetric about the origin if $a \in [0, a_{**})$.
- (ii) $e(a)$ has infinitely many different non-negative minimizers, which are not radially symmetric if $a \in [a_*, a^*)$.

We end this section by recalling some useful information related to the unique positive solution $Q = Q(|x|)$ of (1.6). Taking $N = 2$ in (I.2) of [35], we then have the following Gagliardo–Nirenberg inequality

$$\int_{\mathbb{R}^2} |u(x)|^4 dx \leq \frac{2}{\|Q\|_2^2} \left(\int_{\mathbb{R}^2} |\nabla u(x)|^2 dx \right) \int_{\mathbb{R}^2} |u(x)|^2 dx, \quad u \in H^1(\mathbb{R}^2), \quad (1.13)$$

which can be an equality when $u(x) = Q(|x|)$. Since Q is a solution of (1.6), it is easy to see that

$$\int_{\mathbb{R}^2} |\nabla Q|^2 dx = \int_{\mathbb{R}^2} |Q|^2 dx = \frac{1}{2} \int_{\mathbb{R}^2} |Q|^4 dx, \quad (1.14)$$

see also [3, Lemma 8.1.2] for the details. Furthermore, by the results of [9, Proposition 4.1], we know that

$$Q(x), \quad |\nabla Q(x)| = O(|x|^{-\frac{1}{2}} e^{-|x|}) \quad \text{as } |x| \rightarrow \infty. \quad (1.15)$$

Throughout the paper, we denote the norm of $L^p(\mathbb{R}^2)$ by $\|\cdot\|_p$ for $p \in (1, +\infty)$, and define the norms of the real-valued function spaces \mathcal{H} and $H^1(\mathbb{R}^2)$ by

$$\|u\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)|u|^2) dx \quad \text{for } u \in \mathcal{H},$$

and

$$\|u\|^2 = \int_{\mathbb{R}^2} (|\nabla u|^2 + |u|^2) dx \quad \text{for } u \in H^1(\mathbb{R}^2), \text{ respectively.}$$

Also, the scalar product of \mathcal{H} is given by

$$\langle u, v \rangle_{\mathcal{H}, \mathcal{H}} = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv) dx \quad \text{for any real-valued functions } u, v \in \mathcal{H}.$$

This paper is organized as follows: in Section 2 we first establish some preparatory energy estimates and then prove our Theorem 2.1 which gives the refined estimates of the energy $e(a)$. Theorems 1.2 and 1.3 are showed in Section 3, where the phenomena of concentration and symmetry breaking of the minimizers of (1.4) are also discussed. Finally, by using an implicit function theorem, Theorem 1.1(ii) is proved in Appendix A.

2. Estimates in the energy $e(a)$ as $a \nearrow a^*$

In this section, we mainly establish the following estimates on the energy $e(a)$.

Theorem 2.1. *Let $V(x)$ be given by (1.7). Then, there exist two positive constants C_1 and C_2 , independent of a , such that*

$$C_1(a^* - a)^{\frac{1}{2}} \leq e(a) \leq C_2(a^* - a)^{\frac{1}{2}} \quad \text{as } a \nearrow a^*. \tag{2.1}$$

In order to prove Theorem 2.1, we first give a rough estimates as (1.8) for the energy $e(a)$ of (1.4) by using some ideas of [13]. Based on these estimates, some detailed properties of the minimizers of $e(a)$ can be obtained. Finally, we can complete the proof of Theorem 2.1.

Lemma 2.1. *Let $V(x)$ be given by (1.7). Then, there exist two positive constants C_1 and C_2 , independent of a , such that*

$$C_1(a^* - a)^{\frac{2}{3}} \leq e(a) \leq C_2(a^* - a)^{\frac{1}{2}} \quad \text{as } a \nearrow a^*. \tag{2.2}$$

Proof. For any $\lambda > 0$ and $u \in \mathcal{H}$ with $\|u\|_2^2 = 1$, using (1.13),

$$\begin{aligned} E_a(u) &\geq \int_{\mathbb{R}^2} (|x| - A)^2 |u(x)|^2 dx + \frac{a^* - a}{2} \int_{\mathbb{R}^2} |u(x)|^4 dx \\ &= \lambda + \int_{\mathbb{R}^2} [(|x| - A)^2 - \lambda] |u(x)|^2 dx + \frac{a^* - a}{2} \int_{\mathbb{R}^2} |u(x)|^4 dx \\ &\geq \lambda - \frac{1}{2(a^* - a)} \int_{\mathbb{R}^2} [\lambda - (|x| - A)^2]_+^2 dx, \end{aligned} \tag{2.3}$$

where $A > 0$ and $[\cdot]_+ = \max\{0, \cdot\}$ denotes the positive part. For $\lambda > 0$ small enough, we have

$$\begin{aligned} \int_{\mathbb{R}^2} [\lambda - (|x| - A)^2]_+^2 dx &= 2\pi \int_{A-\sqrt{\lambda}}^{A+\sqrt{\lambda}} [\lambda - (r - A)^2]_+^2 r dr \\ &= 2\pi \lambda^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 \beta (A + \sqrt{\lambda} \sin \beta) \sqrt{\lambda} \cos \beta d\beta \leq C\lambda^{\frac{5}{2}}, \end{aligned}$$

where we change the variable $r = A + \sqrt{\lambda} \sin \beta$ with $-\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}$ in the second identity. The lower estimate of (2.2) therefore follows from the above estimate and (2.3) by taking $\lambda = [4(a^* - a)/(5C)]^{2/3}$ and $a \nearrow a^*$.

We next prove the upper estimate of (2.2) as follows. For this purpose, we let

$$u(x) = \frac{\tau}{\|Q\|_2} Q(\tau(x - x_0)), \quad \text{for any } \tau > 0. \tag{2.4}$$

Then $\int_{\mathbb{R}^2} u^2(x) dx = 1$, and it follows from (1.14) that

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx - \frac{a}{2} \int_{\mathbb{R}^2} u^4(x) dx &= \frac{\tau^2}{\|Q\|_2^2} \left[\int_{\mathbb{R}^2} |\nabla Q(x)|^2 dx - \frac{a}{2\|Q\|_2^2} \int_{\mathbb{R}^2} Q^4(x) dx \right] \\ &= \frac{\tau^2}{2\|Q\|_2^2} \left[\left(1 - \frac{a}{\|Q\|_2^2}\right) \int_{\mathbb{R}^2} Q^4(x) dx \right]. \end{aligned} \tag{2.5}$$

Moreover, by the exponential decay of (1.15), we have

$$\begin{aligned} \int_{\mathbb{R}^2} (|x| - A)^2 |u|^2 dx &= \frac{1}{\|Q\|_2^2} \int_{\mathbb{R}^2} \left(\left| \frac{x}{\tau} + x_0 \right| - |x_0| \right)^2 Q^2(x) dx \\ &\leq \frac{1}{\|Q\|_2^2} \int_{\mathbb{R}^2} \left| \frac{x}{\tau} \right|^2 Q^2(x) dx = \frac{C}{\tau^2}. \end{aligned} \quad (2.6)$$

It then follows from (2.5) and (2.6) that

$$e(a) \leq C(a^* - a)\tau^2 + \frac{C}{\tau^2}.$$

By taking $\tau = (a^* - a)^{-\frac{1}{4}}$, the above inequality implies the desired upper estimate of (2.2). \square

Motivated by [13, Lemma 4], we have the following lemma.

Lemma 2.2. *Let $V(x)$ be given by (1.7) and suppose u_a is a non-negative minimizer of (1.4), then there exists a positive constant K , independent of a , such that*

$$0 < K(a^* - a)^{-\frac{1}{4}} \leq \int_{\mathbb{R}^2} |u_a|^4 dx \leq \frac{1}{K}(a^* - a)^{-\frac{1}{2}} \quad \text{as } a \nearrow a^*. \quad (2.7)$$

Proof. Noting from (2.3) that

$$e(a) = E_a(u_a) \geq \frac{a^* - a}{2} \int_{\mathbb{R}^2} |u_a(x)|^4 dx,$$

and the upper bound of (2.7) then follows from Lemma 2.1.

To prove the lower bound of (2.7), we choose $0 < b < a < a^*$ so that

$$e(b) \leq E_b(u_a) = e(a) + \frac{a - b}{2} \int_{\mathbb{R}^2} |u_a(x)|^4 dx.$$

It then follows from Lemma 2.1 that

$$\frac{1}{2} \int_{\mathbb{R}^2} |u_a(x)|^4 dx \geq \frac{e(b) - e(a)}{a - b} \geq \frac{C_1(a^* - b)^{\frac{2}{3}} - C_2(a^* - a)^{\frac{1}{2}}}{a - b}.$$

Taking $b = a - C_0(a^* - a)^{\frac{3}{4}}$, where $C_0 > 0$ is large enough such that $C_1 C_0^{\frac{2}{3}} > 2C_2$, we then obtain from the above inequality that

$$\int_{\mathbb{R}^2} |u_a|^4 dx \geq C(a^* - a)^{-\frac{1}{4}},$$

which therefore implies the lower bound of (2.7). \square

Remark 2.1. Once our Theorem 2.1 is proved, then we can improve the power $-1/4$ to $-1/2$ in estimate of left hand side of (2.7). In fact, by applying Theorem 2.1, instead of using Lemma 2.1 in the proof of Lemma 2.2, and taking $b = a - C_0(a^* - a)$, we then have

$$\int_{\mathbb{R}^2} |u_a|^4 dx \geq C(a^* - a)^{-\frac{1}{2}}.$$

This will be used in Section 3.

Lemma 2.3. For $V(x)$ satisfying (1.7), let u_a be a non-negative minimizer of (1.4), and set

$$\epsilon_a^{-2} := \int_{\mathbb{R}^2} |\nabla u_a(x)|^2 dx. \tag{2.8}$$

Then

(i) $\epsilon_a \rightarrow 0$ as $a \nearrow a^*$.

(ii) There exist a sequence $\{y_{\epsilon_a}\} \subset \mathbb{R}^2$ and positive constants R_0, η such that the sequence

$$w_a(x) := \epsilon_a u_a(\epsilon_a x + y_{\epsilon_a}) \tag{2.9}$$

satisfies

$$\liminf_{a \nearrow a^*} \int_{B_{R_0}(0)} |w_a|^2 dx \geq \eta > 0. \tag{2.10}$$

(iii) The sequence $\{\epsilon_a y_{\epsilon_a}\}$ is bounded uniformly for $\epsilon_a \rightarrow 0$. Moreover, for any sequence $\{a_k\}$ with $a_k \nearrow a^*$, there exists a convergent subsequence, still denoted by $\{a_k\}$, such that

$$\bar{x}_k := \epsilon_{a_k} y_{\epsilon_{a_k}} \rightarrow x_0 \quad \text{as } a_k \nearrow a^* \tag{2.11}$$

for some $x_0 \in \mathbb{R}^2$ being a global minimum point of $V(x)$, i.e., $|x_0| = A > 0$. Furthermore, we also have

$$w_{a_k} \xrightarrow{k} \frac{\beta_1}{\|Q\|_2} Q(\beta_1|x - \bar{y}_0|) \quad \text{in } H^1(\mathbb{R}^2) \text{ for some } \bar{y}_0 \in \mathbb{R}^2 \text{ and } \beta_1 > 0. \tag{2.12}$$

Proof. (i): Applying (1.13), it follows from Lemma 2.1 that

$$\int_{\mathbb{R}^2} V(x) |u_a(x)|^2 dx \leq e(a) \leq C_1 (a^* - a)^{\frac{1}{2}} \quad \text{as } a \nearrow a^*, \tag{2.13}$$

and

$$0 \leq \int_{\mathbb{R}^2} |\nabla u_a(x)|^2 dx - \frac{a}{2} \int_{\mathbb{R}^2} |u_a(x)|^4 dx = \epsilon_a^{-2} - \frac{a}{2} \int_{\mathbb{R}^2} |u_a(x)|^4 dx \leq e(a) \xrightarrow{a \nearrow a^*} 0.$$

By Lemma 2.2,

$$\int_{\mathbb{R}^2} |u_a(x)|^4 dx \rightarrow +\infty \quad \text{as } a \nearrow a^*,$$

then we see that

$$0 \leq \frac{\epsilon_a^{-2}}{\int_{\mathbb{R}^2} |u_a(x)|^4 dx} - \frac{a}{2} \leq \frac{e(a)}{\int_{\mathbb{R}^2} |u_a(x)|^4 dx} \xrightarrow{a \nearrow a^*} 0,$$

i.e.,

$$\frac{\epsilon_a^{-2}}{\int_{\mathbb{R}^2} |u_a(x)|^4 dx} \rightarrow \frac{a^*}{2} \quad \text{as } a \nearrow a^*.$$

So, by taking $m = \max\{\frac{4}{a^*}, \frac{3a^*}{4}\}$ we have

$$0 < \frac{1}{m} \epsilon_a^{-2} \leq \int_{\mathbb{R}^2} |u_a(x)|^4 dx \leq m \epsilon_a^{-2} \quad \text{as } a \nearrow a^*, \tag{2.14}$$

this and (2.7) imply that there exist $C_2 > 0$ and $C_3 > 0$ such that

$$C_2(a^* - a)^{-\frac{1}{4}} \leq \int_{\mathbb{R}^2} |\nabla u_a(x)|^2 dx \leq C_3(a^* - a)^{-\frac{1}{2}} \quad \text{as } a \nearrow a^*. \quad (2.15)$$

Hence, $\epsilon_a \rightarrow 0$ as $a \nearrow a^*$, and part (i) is proved.

(ii): Let

$$\tilde{w}_a(x) := \epsilon_a u_a(\epsilon_a x). \quad (2.16)$$

From (2.8) and (2.14), we see that

$$\int_{\mathbb{R}^2} |\nabla \tilde{w}_a|^2 dx = \int_{\mathbb{R}^2} |\tilde{w}_a|^2 dx = 1, \quad \frac{1}{m} \leq \int_{\mathbb{R}^2} |\tilde{w}_a|^4 dx \leq m. \quad (2.17)$$

We claim that there exist a sequence $\{y_{\epsilon_a}\} \subset \mathbb{R}^2$ and $R_0 > 0$, $\eta > 0$ such that

$$\liminf_{\epsilon_a \rightarrow 0} \int_{B_{R_0}(y_{\epsilon_a})} |\tilde{w}_a|^2 dx \geq \eta > 0. \quad (2.18)$$

We argue this by contradiction. If (2.18) is not true, then, for any $R > 0$, there exists a sequence $\{\tilde{w}_{a_k}\}$ with $a_k \nearrow a^*$ such that

$$\limsup_{k \rightarrow \infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y)} |\tilde{w}_{a_k}|^2 dx = 0.$$

By Lemma I.1 in [26] (or, Theorem 8.10 in [21]), we know that $\tilde{w}_{a_k} \xrightarrow{k} 0$ in $L^p(\mathbb{R}^2)$ for all $p \in (2, +\infty)$, which however contradicts (2.17) if we take $p = 4$. Thus, Eq. (2.18) holds. By applying (2.16) and (2.18), we therefore conclude (2.10), which gives part (ii).

(iii): By (2.13), we see that

$$\int_{\mathbb{R}^2} V(x) |u_a(x)|^2 dx = \int_{\mathbb{R}^2} V(\epsilon_a x + \epsilon_a y_{\epsilon_a}) |w_a(x)|^2 dx \rightarrow 0 \quad \text{as } a \nearrow a^*. \quad (2.19)$$

We first claim that

$$\lim_{\epsilon_a \rightarrow 0} |\epsilon_a y_{\epsilon_a}| = A.$$

Indeed, if this is false, then there exist a constant $\alpha > 0$ and a subsequence $\{a_n\}$ with $a_n \nearrow a^*$ as $n \rightarrow \infty$, such that

$$\epsilon_n := \epsilon_{a_n} \rightarrow 0 \quad \text{and} \quad | |\epsilon_n y_{\epsilon_n}| - A | \geq \alpha > 0 \quad \text{as } n \rightarrow \infty.$$

Hence,

$$V(\epsilon_n y_{\epsilon_n}) = (|\epsilon_n y_{\epsilon_n}| - A)^2 \geq \alpha^2 > 0 \quad \text{as } n \rightarrow \infty.$$

Therefore, it follows from (2.10) and Fatou's Lemma that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^2} V(\epsilon_n x + \epsilon_n y_{\epsilon_n}) |w_{a_n}(x)|^2 dx \geq \int_{\mathbb{R}^2} \lim_{n \rightarrow \infty} V(\epsilon_n x + \epsilon_n y_{\epsilon_n}) |w_{a_n}(x)|^2 dx \geq \frac{\alpha^2}{2} \eta > 0,$$

which contradicts (2.19). So, the above claim is proved. This claim implies that $\{\epsilon_a y_{\epsilon_a}\}$ is bounded uniformly as $\epsilon_a \rightarrow 0$, and (2.11) follows from these conclusions.

We now turn to proving (2.12). Since u_a is a non-negative minimizer of (1.4), it satisfies the Euler–Lagrange equation

$$-\Delta u_a(x) + V(x)u_a(x) = \mu_a u_a(x) + a u_a^3(x) \quad \text{in } \mathbb{R}^2, \quad (2.20)$$

where $\mu_a \in \mathbb{R}$ is a Lagrange multiplier, and

$$\mu_a = e(a) - \frac{a}{2} \int_{\mathbb{R}^2} |u_a|^4 dx.$$

It then follows from Lemma 2.1 and (2.14) that there exist two positive constants C_1 and C_2 , independent of a , such that

$$-C_2 < \epsilon_a^2 \mu_a < -C_1 < 0 \quad \text{as } a \nearrow a^*.$$

Using (2.20), we know that $w_a(x)$ defined in (2.9) satisfies the elliptic equation

$$-\Delta w_a(x) + \epsilon_a^2 V(\epsilon_a x + \epsilon_a y_{\epsilon_a}) w_a(x) = \epsilon_a^2 \mu_a w_a(x) + a w_a^3(x) \quad \text{in } \mathbb{R}^2. \tag{2.21}$$

Therefore, for the convergent subsequence $\{a_k\}$ obtained in (2.11), we may assume that $\epsilon_k^2 \mu_{a_k} \xrightarrow{k} -\beta_1^2 < 0$ for some $\beta_1 > 0$, and $w_{a_k} \xrightarrow{k} w_0 \geq 0$ weakly in $H^1(\mathbb{R}^2)$ for some $w_0 \in H^1(\mathbb{R}^2)$. Since $\{\epsilon_a y_{\epsilon_a}\}$ is bounded uniformly in ϵ_a , by passing to the weak limit of (2.21), we see that $w_0 \geq 0$ satisfies

$$-\Delta w_0(x) = -\beta_1^2 w_0(x) + a^* w_0^3(x) \quad \text{in } \mathbb{R}^2. \tag{2.22}$$

Furthermore, it follows from (2.10) that $w_0 \not\equiv 0$, and therefore we have $w_0 > 0$ by the strong maximum principle. By a simple rescaling, the uniqueness (up to translations) of positive solutions for the nonlinear scalar field equation (1.6) implies that

$$w_0(x) = \frac{\beta_1}{\|Q\|_2} Q(\beta_1 |x - \bar{y}_0|) \quad \text{for some } \bar{y}_0 \in \mathbb{R}^2, \tag{2.23}$$

where $\|w_0\|_2^2 = 1$. By the norm preservation we further conclude that w_{a_k} converges to w_0 strongly in $L^2(\mathbb{R}^2)$ and in fact, strongly in $L^p(\mathbb{R}^2)$ for any $2 \leq p < \infty$ because of $H^1(\mathbb{R}^2)$ boundedness. Also, since w_{a_k} and w_0 satisfy (2.21) and (2.22), respectively, a simple analysis shows that w_{a_k} converges to w_0 strongly in $H^1(\mathbb{R}^2)$, and thus (2.12) holds. \square

Lemma 2.4. *Under the assumptions of Lemma 2.3, let $\{a_k\}$ be the convergent subsequence given by Lemma 2.3(iii). Then, for any $R > 0$, there exists $C_0(R) > 0$, independent of a_k , such that*

$$\lim_{\epsilon_{a_k} \rightarrow 0} \frac{1}{\epsilon_{a_k}^2} \int_{B_R(0)} V(\epsilon_{a_k} x + \epsilon_{a_k} y_{\epsilon_{a_k}}) |w_{a_k}(x)|^2 dx \geq C_0(R). \tag{2.24}$$

Proof. Since $V(x) = (|x| - A)^2$ with $A > 0$, we have

$$V(\epsilon_a x + \epsilon_a y_{\epsilon_a}) = \epsilon_a^2 \left(|x + y_{\epsilon_a}| - \frac{A}{\epsilon_a} \right)^2, \tag{2.25}$$

where the term $|x + y_{\epsilon_a}|$ can be rewritten as

$$|x + y_{\epsilon_a}| = \sqrt{|x|^2 + |y_{\epsilon_a}|^2} \sqrt{1 + \frac{2x \cdot y_{\epsilon_a}}{|x|^2 + |y_{\epsilon_a}|^2}}. \tag{2.26}$$

For the convergent sequence $\{a_k\}$ given by Lemma 2.3(iii), since $\epsilon_k y_{\epsilon_k} \xrightarrow{k} x_0$ with $|x_0| = A > 0$, we have $|y_{\epsilon_k}| \xrightarrow{k} \infty$, and hence $\frac{2x \cdot y_{\epsilon_k}}{|x|^2 + |y_{\epsilon_k}|^2} \xrightarrow{k} 0$ uniformly for $x \in B_R(0)$. Using the Taylor expansion we obtain that

$$\sqrt{1 + \frac{2x \cdot y_{\epsilon_k}}{|x|^2 + |y_{\epsilon_k}|^2}} = 1 + \frac{x \cdot y_{\epsilon_k}}{|x|^2 + |y_{\epsilon_k}|^2} + O\left(\frac{1}{|y_{\epsilon_k}|^2}\right) \quad \text{for all } x \in B_R(0),$$

which, together with (2.25) and (2.26), then implies that

$$\frac{1}{\epsilon_k^2} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) = \left| \sqrt{|x|^2 + |y_{\epsilon_k}|^2} + \frac{x \cdot y_{\epsilon_k}}{\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} - \frac{A}{\epsilon_k} + O\left(\frac{1}{|y_{\epsilon_k}|}\right) \right|^2. \tag{2.27}$$

For any $x \in \mathbb{R}^2$, let $\arg x$ be the angle between x and the positive x -axis, and $\langle x, y \rangle$ be the angle between the vectors x and y . Without loss of generality, we may assume that $x_0 = (A, 0)$, and it then follows that $\arg y_{\epsilon_k} \xrightarrow{k} 0$. Thus, we can choose $0 < \delta < \frac{\pi}{16}$ small enough such that

$$-\delta < \arg y_{\epsilon_k} < \delta \quad \text{as } \epsilon_k \rightarrow 0. \tag{2.28}$$

Denote

$$\begin{aligned} \Omega_{\epsilon_k}^1 &= \left\{ x \in B_R(0) : \sqrt{|x|^2 + |y_{\epsilon_k}|^2} \leq \frac{A}{\epsilon_k} \right\} \\ &= \left\{ x \in B_R(0) : |x|^2 \leq \left(\frac{A}{\epsilon_k}\right)^2 - |y_{\epsilon_k}|^2 \right\}, \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} \Omega_{\epsilon_k}^2 &= \left\{ x \in B_R(0) : \sqrt{|x|^2 + |y_{\epsilon_k}|^2} > \frac{A}{\epsilon_k} \right\} \\ &= \left\{ x \in B_R(0) : \left(\frac{A}{\epsilon_k}\right)^2 - |y_{\epsilon_k}|^2 < |x|^2 < R^2 \right\}, \end{aligned} \tag{2.30}$$

so that $B_R(0) = \Omega_{\epsilon_k}^1 \cup \Omega_{\epsilon_k}^2$ and $\Omega_{\epsilon_k}^1 \cap \Omega_{\epsilon_k}^2 = \emptyset$. Since

$$|\Omega_{\epsilon_k}^1| + |\Omega_{\epsilon_k}^2| = |B_R(0)| = \pi R^2,$$

there exists a subsequence, still denoted by $\{\epsilon_k\}$, of $\{\epsilon_k\}$ such that

$$\text{either } |\Omega_{\epsilon_k}^1| \geq \frac{\pi R^2}{2} \quad \text{or} \quad |\Omega_{\epsilon_k}^2| \geq \frac{\pi R^2}{2}.$$

We finish the proof by considering the following two cases:

Case 1: $|\Omega_{\epsilon_k}^1| \geq \frac{\pi R^2}{2}$. In this case, we have $B_{\frac{R}{\sqrt{2}}}(0) \subset \Omega_{\epsilon_k}^1$, and set

$$\Omega_1 := \left(B_{\frac{R}{\sqrt{2}}}(0) \setminus B_{\frac{R}{2}}(0) \right) \cap \left\{ x : \frac{\pi}{2} + 2\delta < \arg x < \frac{3\pi}{2} - 2\delta \right\} \subset \Omega_{\epsilon_k}^1.$$

Then

$$|\Omega_1| = \frac{(\pi - 4\delta)}{8} R^2. \tag{2.31}$$

By (2.28), one can easily check that for any $x \in \Omega_1$,

$$x \cdot y_{\epsilon_k} = |x||y_{\epsilon_k}| \cos \langle x, y_{\epsilon_k} \rangle < 0 \quad \text{and} \quad |\cos \langle x, y_{\epsilon_k} \rangle| > -\cos\left(\frac{\pi}{2} + \delta\right) > 0. \tag{2.32}$$

We thus derive from (2.27) that

$$\begin{aligned} &\sqrt{|x|^2 + |y_{\epsilon_k}|^2} + \frac{x \cdot y_{\epsilon_k}}{\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} - \frac{A}{\epsilon_k} + O\left(\frac{1}{|y_{\epsilon_k}|}\right) \\ &\leq \frac{x \cdot y_{\epsilon_k}}{\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} + O\left(\frac{1}{|y_{\epsilon_k}|}\right) \leq \frac{x \cdot y_{\epsilon_k}}{2\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} \leq \frac{|x||y_{\epsilon_k}| \cos\left(\frac{\pi}{2} + \delta\right)}{2\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} < 0 \quad \text{for } x \in \Omega_1. \end{aligned}$$

Noting that $\lim_{\epsilon_k \rightarrow 0} |y_{\epsilon_k}| = \infty$, we thus have

$$\frac{1}{\epsilon_k^2} V(\epsilon_k x + \epsilon y_{\epsilon_k}) \geq \frac{\cos^2\left(\frac{\pi}{2} + \delta\right)|x|^2}{8} \quad \text{for } x \in \Omega_1. \tag{2.33}$$

Taking $\delta = \frac{\pi}{20}$, the above estimate implies that

$$\begin{aligned} \lim_{\epsilon_k \rightarrow 0} \frac{1}{\epsilon_k^2} \int_{B_R} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_a(x)|^2 dx &\geq \lim_{\epsilon_k \rightarrow 0} \frac{1}{\epsilon_k^2} \int_{\Omega_1} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_a(x)|^2 dx \\ &\geq \frac{\cos^2 \frac{11\pi}{20}}{8} \int_{\Omega_1} |x|^2 |w_0(x)|^2 dx := C(R) > 0, \end{aligned} \tag{2.34}$$

and then (2.24) is proved.

Case 2: $|\Omega_{\epsilon_k}^2| \geq \frac{\pi R^2}{2}$. In this case, we deduce that the annular region $D_R := B_R \setminus B_{\frac{R}{\sqrt{2}}} \subset \Omega_{\epsilon_k}^2$. Set $\Omega_2 := D_R \cap \{x; -\frac{\pi}{2} + 2\delta < \arg x < \frac{\pi}{2} - 2\delta\} \subset \Omega_{\epsilon_k}^2$. Then,

$$|\Omega_2| = \frac{(\pi - 4\delta)}{4} R^2. \tag{2.35}$$

One can check that for any $x \in \Omega_2$,

$$x \cdot y_{\epsilon_k} = |x| |y_{\epsilon_k}| \cos \langle x, y_{\epsilon_k} \rangle > 0 \quad \text{and} \quad \cos \langle x, y_{\epsilon_k} \rangle > \cos \left(\frac{\pi}{2} - \delta \right) > 0. \tag{2.36}$$

It then follows from (2.27) and (2.36) that

$$\begin{aligned} &\sqrt{|x|^2 + |y_{\epsilon_k}|^2} + \frac{x \cdot y_{\epsilon_k}}{\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} - \frac{A}{\epsilon_k} + O\left(\frac{1}{|y_{\epsilon_k}|}\right) \\ &\geq \frac{x \cdot y_{\epsilon_k}}{\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} + O\left(\frac{1}{|y_{\epsilon_k}|}\right) \geq \frac{x \cdot y_{\epsilon_k}}{2\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} \geq \frac{|x| |y_{\epsilon_k}| \cos(\frac{\pi}{2} - \delta)}{2\sqrt{|x|^2 + |y_{\epsilon_k}|^2}} > 0 \quad \text{for } x \in \Omega_2. \end{aligned}$$

Hence

$$\frac{1}{\epsilon_k^2} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) \geq \frac{\cos^2(\frac{\pi}{2} - \delta) |x|^2}{8} \quad \text{for } x \in \Omega_2.$$

Thus, by taking $\delta = \frac{\pi}{20}$, the above estimate gives that

$$\begin{aligned} \lim_{\epsilon_k \rightarrow 0} \frac{1}{\epsilon_k^2} \int_{B_R} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_a(x)|^2 dx &\geq \lim_{\epsilon_k \rightarrow 0} \frac{1}{\epsilon_k^2} \int_{\Omega_2} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_a(x)|^2 dx \\ &\geq \frac{\cos^2 \frac{9\pi}{20}}{8} \int_{\Omega_2} |x|^2 |w_0(x)|^2 dx := C_0(R) > 0. \end{aligned} \tag{2.37}$$

Therefore, Eq. (2.24) also follows from (2.34) and (2.37) in this case. \square

We end this section by proving Theorem 2.1, which gives the refined estimates for $e(a)$.

Proof of Theorem 2.1. By Lemma 2.1, it suffices to prove that there exists a positive $C > 0$, independent of a , such that

$$e(a) \geq C(a^* - a)^{\frac{1}{2}} \quad \text{as } a \nearrow a^*. \tag{2.38}$$

In fact, by the proof of Lemma 2.3(iii), we see that for any sequence $\{a_k\}$ with $a_k \nearrow a^*$, there exists a convergent subsequence, still denoted by $\{a_k\}$, such that $w_{a_k} \rightarrow w_0 > 0$ strongly in $L^4(\mathbb{R}^2)$, where w_0 satisfies (2.23). This implies that there exists a constant $M_1 > 0$, independent of a_k , such that

$$\int_{\mathbb{R}^2} |w_{a_k}(x)|^4 dx \geq M_1 \quad \text{as } a_k \nearrow a^*.$$

Moreover, applying (2.24) with $R = 1$ yields that there exists a constant $M_2 > 0$, independent of a_k , such that

$$\int_{B_1(0)} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_{a_k}(x)|^2 dx \geq M_2 \epsilon_k^2 \quad \text{as } a_k \nearrow a^*.$$

Thus,

$$\begin{aligned} e(a_k) = E_{a_k}(u_{a_k}) &= \frac{1}{\epsilon_k^2} \left[\int_{\mathbb{R}^2} |\nabla w_{a_k}(x)|^2 dx - \frac{a^*}{2} \int_{\mathbb{R}^2} |w_{a_k}(x)|^4 dx \right] \\ &\quad + \frac{a^* - a_k}{2\epsilon_k^2} \int_{\mathbb{R}^2} |w_{a_k}(x)|^4 dx + \int_{\mathbb{R}^2} V(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_{a_k}(x)|^2 dx \\ &\geq \frac{a^* - a_k}{2\epsilon_k^2} M_1 + M_2 \epsilon_k^2 \geq \sqrt{2M_1 M_2} (a^* - a_k)^{\frac{1}{2}} \quad \text{as } a_k \nearrow a^*, \end{aligned} \tag{2.39}$$

and (2.38) therefore holds for the subsequence $\{a_k\}$.

Actually, the above argument can be carried out for any subsequence $\{a_k\}$ satisfying $a_k \nearrow a^*$, which then implies that (2.38) holds for all $a \nearrow a^*$. This completes the proof of Theorem 2.1. \square

3. Mass concentration and symmetry breaking

In this section we complete the proof of Theorems 1.2 and 1.3 under the ring-shaped potential $V(x) = (|x| - A)^2$ with $A > 0$, which addresses the mass concentration and symmetry breaking of minimizers as $a \nearrow a^*$. Let u_a be a non-negative minimizer of (1.4). By Remark 2.1, it is easy to see that there exists a constant $M > 0$, independent of a , such that

$$0 < M(a^* - a)^{-\frac{1}{2}} \leq \int_{\mathbb{R}^2} |u_a|^4 dx \leq \frac{1}{M}(a^* - a)^{-\frac{1}{2}} \quad \text{as } a \nearrow a^*. \tag{3.1}$$

Stimulated by above estimates, we define

$$\varepsilon_a := (a^* - a)^{\frac{1}{4}} > 0. \tag{3.2}$$

From (1.13) we conclude that

$$e(a) \geq \left(1 - \frac{a}{a^*}\right) \int_{\mathbb{R}^2} |\nabla u_a(x)|^2 dx + \int_{\mathbb{R}^2} (|x| - A)^2 u_a^2(x) dx,$$

and it hence follows from Theorem 2.1 that

$$\int_{\mathbb{R}^2} |\nabla u_a(x)|^2 dx \leq C \varepsilon_a^{-2} \quad \text{and} \quad \int_{\mathbb{R}^2} (|x| - A)^2 u_a^2(x) dx \leq C \varepsilon_a^2. \tag{3.3}$$

Similar to Lemma 2.3(ii), for ε_a given by (3.2), we know that there exist a sequence $\{y_{\varepsilon_a}\} \subset \mathbb{R}^2$ and positive constants R_0 and η such that

$$\liminf_{a \nearrow a^*} \int_{B_{R_0}(0)} |w_a|^2 dx \geq \eta > 0, \tag{3.4}$$

where we define the $L^2(\mathbb{R}^2)$ -normalized function

$$w_a(x) = \varepsilon_a u_a(\varepsilon_a x + \varepsilon_a y_{\varepsilon_a}). \tag{3.5}$$

Note from (3.1) and (3.3) that

$$\int_{\mathbb{R}^2} |\nabla w_a|^2 dx \leq C, \quad M \leq \int_{\mathbb{R}^2} |w_a|^4 dx \leq \frac{1}{M}, \tag{3.6}$$

where the positive constants C and M are independent of a .

Lemma 3.1. For any given sequence $\{a_k\}$ with $a_k \nearrow a^*$, let $\varepsilon_k := \varepsilon_{a_k} = (a^* - a_k)^{\frac{1}{4}} > 0$, $u_k(x) := u_{a_k}(x)$ be a non-negative minimizer of (1.4), and $w_k := w_{a_k} \geq 0$ be defined by (3.5). Then, there is a subsequence, still denoted by $\{a_k\}$, such that

$$z_k := \varepsilon_k y_{\varepsilon_k} \xrightarrow{k} y_0 \quad \text{for some } y_0 \in \mathbb{R}^2 \text{ and } |y_0| = A. \tag{3.7}$$

Moreover, for any $\delta > 0$ small enough, we have

$$u_k(x) = \frac{1}{\varepsilon_k} w_k \left(\frac{x - z_k}{\varepsilon_k} \right) \xrightarrow{k} 0, \quad \forall x \in B_\delta^c(y_0). \tag{3.8}$$

Proof. By (2.20) and (3.5), we see that w_k satisfies

$$-\Delta w_k(x) + \varepsilon_k^2 (|\varepsilon_k x + \varepsilon_k y_{\varepsilon_k}| - A)^2 w_k(x) = \mu_k \varepsilon_k^2 w_k(x) + a_k w_k^3(x) \quad \text{in } \mathbb{R}^2, \tag{3.9}$$

where $\mu_k \in \mathbb{R}^2$ is a Lagrange multiplier. Similar to the proof of Lemma 2.3(iii), we can prove that there exists a subsequence of $\{w_k\}$, still denoted by $\{w_k\}$, such that (3.7) holds and $w_k \xrightarrow{k} w_0$ strongly in $H^1(\mathbb{R}^2)$ for some positive function w_0 satisfying

$$-\Delta w_0(x) = -\beta^2 w_0(x) + a^* w_0^3(x) \quad \text{in } \mathbb{R}^2, \tag{3.10}$$

where $\beta > 0$ is a positive constant. Hence, for any $\alpha > 2$,

$$\int_{|x| \geq R} |w_k|^\alpha dx \rightarrow 0 \quad \text{as } R \rightarrow \infty \text{ uniformly for large } k. \tag{3.11}$$

Note from (3.9) that $-\Delta w_k - c(x)w_k \leq 0$, where $c(x) = a_k w_k^2(x)$. By applying De Giorgi–Nash–Moser theory, see e.g. [17, Theorem 4.1], we have

$$\max_{B_1(\xi)} w_k \leq C \left(\int_{B_2(\xi)} |w_k|^\alpha dx \right)^{\frac{1}{\alpha}},$$

where ξ is an arbitrary point in \mathbb{R}^2 , and C is a constant depending only on the bound of $\|w_k\|_{L^\alpha(B_2(\xi))}$. We hence deduce from (3.11) that

$$w_k(x) \rightarrow 0 \quad \text{as } |x| \rightarrow \infty \text{ uniformly in } k. \tag{3.12}$$

Since w_k satisfies (3.9), one can use the comparison principle as in [18] to compare w_k with $Ce^{-\frac{\beta}{2}|x|}$, which then shows that there exists a large constant $R > 0$, independent of k , such that

$$w_k(x) \leq Ce^{-\frac{\beta}{2}|x|} \quad \text{for } |x| > R \text{ as } k \rightarrow \infty. \tag{3.13}$$

For any $x \in B_\delta^c(y_0)$, it then follows from (3.7) that

$$\frac{|x - z_k|}{\varepsilon_k} \geq \frac{1}{2} \frac{|x - y_0|}{\varepsilon_k} \geq \frac{\delta}{2\varepsilon_k} \xrightarrow{k} +\infty,$$

which, together with (3.13), yields that

$$u_k(x) = \frac{1}{\varepsilon_k} w_k \left(\frac{x - z_k}{\varepsilon_k} \right) \leq \frac{1}{\varepsilon_k} e^{-\frac{\beta\delta}{2\varepsilon_k} k} \xrightarrow{k} 0, \quad \forall x \in B_\delta^c(y_0),$$

i.e., Eq. (3.8) holds. \square

Remark 3.1. The estimate (3.8) shows that, for any $x \in \mathbb{R}^2$ and out of any small ball centered at y_0 , $u_k(x)$ vanishes as $k \rightarrow \infty$. That is, Eq. (3.8) implies that, for any given sequence $\{x_k\} \subset \mathbb{R}^2$, if there exists $\alpha > 0$ such that $u_k(x_k) \geq \alpha > 0$, then, $x_k \rightarrow y_0$ as $k \rightarrow \infty$. This fact is used to prove (3.15) below.

We are now ready to prove [Theorem 1.2](#), which is partially motivated by [\[13,34\]](#).

Proof of Theorem 1.2. We still set $\varepsilon_k = (a^* - a_k)^{\frac{1}{4}} > 0$, where $a_k \nearrow a^*$, and $u_k(x) := u_{a_k}(x)$ is a non-negative minimizer of [\(1.4\)](#). We start the proof by establishing first the detailed concentration behavior of u_k .

Let \bar{z}_k be any local maximum point of u_k . It then follows from [\(2.20\)](#) that

$$u_k(\bar{z}_k) \geq \left(\frac{-\mu_k}{a_k}\right)^{\frac{1}{2}} \geq C\varepsilon_k^{-1}. \tag{3.14}$$

This estimate and [\(3.8\)](#) (see [Remark 3.1](#)) imply that, by passing to a subsequence,

$$\bar{z}_k \xrightarrow{k} y_0 \in \mathbb{R}^2 \quad \text{with } |y_0| = A. \tag{3.15}$$

Set

$$\bar{w}_k = \varepsilon_k u_k(\varepsilon_k x + \bar{z}_k). \tag{3.16}$$

It then follows from [\(3.9\)](#) that

$$-\Delta \bar{w}_k(x) + \varepsilon_k^2(|\varepsilon_k x + \bar{z}_k| - A)^2 \bar{w}_k(x) = \mu_k \varepsilon_k^2 \bar{w}_k(x) + a_k \bar{w}_k^3(x) \quad \text{in } \mathbb{R}^2. \tag{3.17}$$

We claim that \bar{w}_k satisfies [\(3.4\)](#) for some positive constants R_0 and η . For this purpose, we first show that $\{\frac{\bar{z}_k - z_k}{\varepsilon_k}\} \subset \mathbb{R}^2$ is bounded uniformly in k . Otherwise, if $|\frac{\bar{z}_k - z_k}{\varepsilon_k}| \rightarrow \infty$ as $k \rightarrow \infty$, it then follows from the exponential decay [\(3.13\)](#) that

$$u_k(\bar{z}_k) = \frac{1}{\varepsilon_k} w_k\left(\frac{\bar{z}_k - z_k}{\varepsilon_k}\right) \leq \frac{C}{\varepsilon_k} e^{-\frac{\beta}{2}|\frac{\bar{z}_k - z_k}{\varepsilon_k}|} = o(\varepsilon_k^{-1}) \quad \text{as } k \rightarrow \infty,$$

which however contradicts [\(3.14\)](#). Therefore, there exists a constant $R_1 > 0$, independent of k , such that $|\frac{\bar{z}_k - z_k}{\varepsilon_k}| < \frac{R_1}{2}$. Note from [\(3.16\)](#) and [\(3.5\)](#) that

$$\bar{w}_k(x) = w_k\left(x + \frac{\bar{z}_k - z_k}{\varepsilon_k}\right).$$

Since w_k satisfies [\(3.4\)](#), we know that

$$\lim_{k \rightarrow \infty} \int_{B_{R_0+R_1}(0)} |\bar{w}_k|^2 dx = \lim_{k \rightarrow \infty} \int_{B_{R_0+R_1}(\frac{\bar{z}_k - z_k}{\varepsilon_k})} |w_k|^2 dx \geq \int_{B_{R_0}(0)} |w_k|^2 dx \geq \eta > 0, \tag{3.18}$$

and the claim is proved, that is, Eq. [\(3.4\)](#) holds also for \bar{w}_k .

As in the proof of [Lemma 3.1](#) (see also [Lemma 2.3\(iii\)](#)), one can further derive that there exists a subsequence, still denoted by $\{\bar{w}_k\}$, of $\{\bar{w}_k\}$ such that

$$\bar{w}_k \xrightarrow{k} \bar{w}_0 \quad \text{strongly in } H^1(\mathbb{R}^2) \quad \text{and} \quad \mu_k \varepsilon_k^2 \xrightarrow{k} -\beta^2, \tag{3.19}$$

for some $0 \leq \bar{w}_0 \in H^1(\mathbb{R}^2)$ and some $\beta > 0$, where \bar{w}_0 satisfies [\(3.10\)](#). Note from [\(3.18\)](#) that $\bar{w}_0 \not\equiv 0$. Thus, the strong maximum principle yields that $\bar{w}_0(x) > 0$ in \mathbb{R}^2 . Since the origin is a critical point of \bar{w}_k for all $k > 0$, it is also a critical point of \bar{w}_0 . We therefore conclude from the uniqueness (up to translations) of positive radial solutions for [\(1.6\)](#) that \bar{w}_0 is spherically symmetric about the origin, and for the above $\beta > 0$,

$$\bar{w}_0 = \frac{\beta}{\|Q\|_2} Q(\beta|x|). \tag{3.20}$$

Using [\(3.17\)](#) and [\(3.19\)](#), we know that $\bar{w}_k \geq (\frac{\beta^2}{2a^*})^{\frac{1}{2}}$ at each local maximum point. Since \bar{w}_k decays to zero uniformly in k as $|x| \rightarrow \infty$, all local maximum points of \bar{w}_k stay in a finite ball in \mathbb{R}^2 . We claim that $\bar{w}_k \rightarrow \bar{w}_0$ in $C_{loc}^2(\mathbb{R}^2)$ as $k \rightarrow \infty$. In fact, by [\(3.12\)](#) and the definition of $\bar{w}_k(x)$ we see that $\{\bar{w}_k\}$ is bounded in $L^\infty(\mathbb{R}^2)$, uniformly in k . Applying L^p -theory (see e.g., Theorem 9.11 of [\[10\]](#)), it follows from [\(3.17\)](#) that $\{\bar{w}_k\}$ is bounded uniformly in $W_{loc}^{2,q}(\mathbb{R}^2)$ for any $q > 2$. Thus, the standard Sobolev embedding theorem implies that $\{\bar{w}_k\}$ is bounded uniformly in

$C_{\text{loc}}^{1,\alpha}(\mathbb{R}^2)$ for some $\alpha \in (0, 1)$. Furthermore, since $\bar{V}_k(x) := \varepsilon_k^2(|\varepsilon_k x + \bar{z}_k| - A)^2$ is locally Lipschitz continuous in \mathbb{R}^2 , it follows from (3.17) and Theorem 6.2 in [10] that $\{\bar{w}_k\}$ is bounded uniformly in $C_{\text{loc}}^{2,\alpha}(\mathbb{R}^2)$. Therefore, there exists $\tilde{w}_0 \in C_{\text{loc}}^{2,\alpha}(\mathbb{R}^2)$ such that

$$\bar{w}_k \rightarrow \tilde{w}_0 \quad \text{in } C_{\text{loc}}^2(\mathbb{R}^2) \text{ as } k \rightarrow \infty.$$

Further, by using (3.19) we conclude that $\tilde{w}_0 = \bar{w}_0$, and the claim is therefore established.

Note that the origin is the only critical point of \bar{w}_0 , then the above claim shows that all local maximum points of $\{\bar{w}_k\}$ must approach the origin and hence stay in a small ball $B_\epsilon(0)$ as $k \rightarrow \infty$. One can take ϵ small enough such that $\bar{w}_0''(r) < 0$ for $0 \leq r \leq \epsilon$. It then follows from Lemma 4.2 in [28] that for large k , each \bar{w}_k has no critical points other than the origin. This gives the uniqueness of local maximum points for each $\bar{w}_k(x)$, which therefore implies that there exists a subsequence of $\{u_k\}$ concentrating at a *unique* global minimum point of potential $V(x) = (|x| - A)^2$.

To complete the proof of Theorem 1.2, we need to determine the exact value of β in (3.20). From (3.16), we have

$$\begin{aligned} e(a_k) = E_{a_k}(u_k) &= \frac{1}{\varepsilon_k^2} \left[\int_{\mathbb{R}^2} |\nabla \bar{w}_k(x)|^2 dx - \frac{a^*}{2} \int_{\mathbb{R}^2} \bar{w}_k^4(x) dx \right] \\ &\quad + \frac{\varepsilon_k^2}{2} \int_{\mathbb{R}^2} \bar{w}_k^4(x) dx + \int_{\mathbb{R}^2} (|\varepsilon_k x + \bar{z}_k| - |y_0|)^2 \bar{w}_k^2(x) dx, \end{aligned} \tag{3.21}$$

where \bar{z}_k is the unique global maximum point of u_k , and $\bar{z}_k \rightarrow y_0 \in \mathbb{R}^2$ as $k \rightarrow \infty$ for some $|y_0| = A > 0$. The term in square brackets is non-negative which can be ignored for the lower bound of $e(a_k)$. The $L^4(\mathbb{R}^2)$ norm of \bar{w}_k converges to that of \bar{w}_0 as $k \rightarrow \infty$.

To estimate the last term of (3.21), we claim that $\left\{ \frac{|\bar{z}_k| - |y_0|}{\varepsilon_k} \right\} \subset \mathbb{R}$ is bounded uniformly for $k \rightarrow \infty$. Otherwise, there must exist a subsequence of $\{a_k\}$, still denoted by $\{a_k\}$, such that $\left| \frac{|\bar{z}_k| - |y_0|}{\varepsilon_k} \right| \rightarrow \infty$ as $k \rightarrow \infty$, then, for any constant $C > 0$, using (3.18) we see that

$$\lim_{k \rightarrow \infty} \varepsilon_k^{-2} \int_{\mathbb{R}^2} (|\varepsilon_k x + \bar{z}_k| - |y_0|)^2 \bar{w}_k^2(x) dx = \lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} \left(\left| x + \frac{\bar{z}_k}{\varepsilon_k} \right| - \frac{|y_0|}{\varepsilon_k} \right)^2 \bar{w}_k^2(x) dx \geq C.$$

This estimate and (3.21) then imply that

$$e(a_k) \geq C \varepsilon_k^2 = C (a^* - a_k)^{\frac{1}{2}}$$

holds for any constant $C > 0$, which however contradicts Theorem 2.1, and the claim is proved.

So, by our above claim we know that there exists a subsequence still denoted by $\{a_k\}$ such that

$$\frac{|\bar{z}_k| - |y_0|}{\varepsilon_k} \rightarrow C_0 \quad \text{as } k \rightarrow \infty \tag{3.22}$$

for some constant C_0 . Since Q is a radially symmetric function and decays exponentially as $|x| \rightarrow \infty$, we then deduce from (3.20) that

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{1}{\varepsilon_k^2} \int_{\mathbb{R}^2} (|\varepsilon_k x + \bar{z}_k| - |y_0|)^2 \bar{w}_k^2(x) dx &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} \left(\frac{|\varepsilon_k x + \bar{z}_k|}{\varepsilon_k} - \frac{|y_0|}{\varepsilon_k} \right)^2 \bar{w}_k^2(x) dx \\ &= \lim_{k \rightarrow \infty} \int_{\mathbb{R}^2} \left(\frac{|\varepsilon_k x + \bar{z}_k| - |\bar{z}_k|}{\varepsilon_k} + \frac{|\bar{z}_k| - |y_0|}{\varepsilon_k} \right)^2 \bar{w}_k^2(x) dx \\ &= \int_{\mathbb{R}^2} \left(\frac{y_0 \cdot x}{|y_0|} + C_0 \right)^2 \bar{w}_0^2(x) dx \geq \int_{\mathbb{R}^2} \frac{|y_0 \cdot x|^2}{A^2} \bar{w}_0^2(x) dx, \end{aligned} \tag{3.23}$$

where the equality holds if and only if $C_0 = 0$. We hence infer from (3.21) and (3.23) that

$$\begin{aligned} \lim_{k \rightarrow \infty} \frac{e(a_k)}{(a^* - a_k)^{1/2}} &\geq \frac{1}{2} \|\bar{w}_0\|_4^4 + \frac{1}{A^2} \int_{\mathbb{R}^2} |y_0 \cdot x|^2 \bar{w}_0^2(x) dx \\ &= \frac{1}{a^*} \left(\beta^2 + \frac{1}{A^2 \beta^2} \int_{\mathbb{R}^2} |y_0 \cdot x|^2 Q^2(x) dx \right), \end{aligned} \quad (3.24)$$

where (1.14) is used in the equality. So, for any $\beta > 0$, we have

$$\lim_{k \rightarrow \infty} \frac{e(a_k)}{(a^* - a_k)^{1/2}} \geq \frac{2}{a^*} \left(\frac{\int_{\mathbb{R}^2} |y_0 \cdot x|^2 Q^2(x) dx}{A^2} \right)^{\frac{1}{2}}, \quad (3.25)$$

where the equality is achieved at

$$\beta = \lambda_0 := \left(\frac{\int_{\mathbb{R}^2} |y_0 \cdot x|^2 Q^2(x) dx}{A^2} \right)^{\frac{1}{4}} = \left(\frac{1}{2} \int_{\mathbb{R}^2} |x|^2 Q^2(x) dx \right)^{\frac{1}{4}},$$

here a suitable rotation and $Q(x) = Q(|x|)$ in \mathbb{R}^2 are used in getting the last equality.

We finally note that the limit in (3.25) actually exists, and it is equal to the right hand side of (3.25). To see this, one simply takes

$$u(x) = \frac{\beta}{\varepsilon \|Q\|_2} Q\left(\frac{\beta|x - y_0|}{\varepsilon}\right)$$

as a trial function for $E_a(\cdot)$ and minimizes over $\beta > 0$. By applying (3.25), this leads to

$$\lim_{a_k \nearrow a^*} \frac{e(a_k)}{(a^* - a_k)^{1/2}} = \frac{2}{a^*} \left(\frac{\int_{\mathbb{R}^2} |y_0 \cdot x|^2 Q^2(x) dx}{A^2} \right)^{\frac{1}{2}}. \quad (3.26)$$

The equality (3.26) leads to two conclusions. Firstly, β is unique, which is independent of the choice of the subsequence, and takes the value of λ_0 as above. Secondly, Eq. (3.23) is indeed an equality, and thus $C_0 = 0$, i.e., Eq. (1.9) holds. Moreover, combining (3.15), (3.19) and (3.20), we see that

$$\bar{w}_k(x) = \varepsilon_k u(\varepsilon_k x + \bar{z}_k) \xrightarrow{k} \frac{\lambda_0}{\|Q\|_2} Q(\lambda_0 |x|) \quad \text{strongly in } H^1(\mathbb{R}^2),$$

where \bar{z}_k is the unique maximum point of u_k and $\bar{z}_k \xrightarrow{k} y_0$ for some $y_0 \in \mathbb{R}^2$ satisfying $|y_0| = A > 0$. This completes the proof of Theorem 1.2. \square

Proof of Theorem 1.3. By (3.26), it is clear that (1.12) holds for the subsequence $\{a_k\}$. In fact, by the proof of Theorem 1.2, we know that (3.26) is essentially true for any subsequence $\{a_k\}$ with $a_k \nearrow a^*$, this implies that (1.12) holds also for all $a \nearrow a^*$. \square

Remark 3.2. Theorem 1.2 gives a detailed description on the concentration behavior of the minimizers of $e(a)$ when a is close to a^* , upon which the phenomena of symmetry breaking of the minimizers of $e(a)$ can be demonstrated by Corollary 1.4. That is, when a increases from 0 to a^* , the minimizers of GP energy $e(a)$ have essentially different properties: the GP energy $e(a)$ has a unique non-negative minimizer which is radially symmetric if $a > 0$ is small, but $e(a)$ has infinity many minimizers which are non-radially symmetric if a approaches a^* .

Conflict of interest statement

We declare that we have no conflict of interest.

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Appendix A. Proof of Theorem 1.1(ii)

This appendix is devoted to the proof of Theorem 1.1(ii), for which we always assume that

$$0 \leq V(x) \in L^\infty_{loc}(\mathbb{R}^2), \quad \lim_{|x| \rightarrow \infty} V(x) = \infty \quad \text{and} \quad \inf_{x \in \mathbb{R}^2} V(x) = 0. \tag{A.1}$$

The properties of the Schrödinger operator $-\Delta + V(x)$ with $V(x) \in L^\infty(\mathbb{R}^2)$ are well known, see e.g. [33], but we could not find a reference for that of $V(x)$ satisfying (A.1) although we guess it should exist somewhere. For the sake of completeness, we begin this appendix by giving some properties of the Schrödinger operator $-\Delta + V(x)$ under conditions (A.1), which are required in proving the uniqueness of non-negative minimizers. Before going to the properties of $-\Delta + V(x)$, we recall the following embedding lemma, which can be found in [30, Theorem XIII.67], [7, Lemma 2.1] or [2, Lemma 2.1], etc.

Lemma A.1. *Suppose $V(x)$ satisfies (A.1). Then, the embedding $\mathcal{H} \hookrightarrow L^q(\mathbb{R}^2)$ is compact for all $q \in [2, \infty)$. \square*

Define

$$\mu_1 = \inf \left\{ \int_{\mathbb{R}^2} |\nabla u|^2 + V(x)u^2 dx : u \in \mathcal{H} \text{ and } \int_{\mathbb{R}^2} u^2 dx = 1 \right\}. \tag{A.2}$$

By Lemma A.1, it is not difficult to know that μ_1 is simple and can be attained by a positive function $\phi_1 \in \mathcal{H}$. We now define

$$\mu_2 = \inf \left\{ \int_{\mathbb{R}^2} |\nabla u|^2 + V(x)u^2 dx : u \in Z \text{ and } \int_{\mathbb{R}^2} u^2 dx = 1 \right\}, \tag{A.3}$$

where

$$Z = \text{span}\{\phi_1\}^\perp = \left\{ u : u \in \mathcal{H}, \int_{\mathbb{R}^2} u\phi_1 dx = 0 \right\}.$$

It is known that $\mu_2 > \mu_1$ and

$$\mathcal{H} = \text{span}\{\phi_1\} \oplus Z. \tag{A.4}$$

Then, we have the following lemma, its proof is somehow standard, we omit it here.

Lemma A.2. *Under the assumption of (A.1), we have*

- (i) $\ker(-\Delta + V(x) - \mu_1) = \text{span}\{\phi_1\}$;
- (ii) $\phi_1 \notin (-\Delta + V(x) - \mu_1)Z$;
- (iii) $\text{Im}(-\Delta + V(x) - \mu_1) = (-\Delta + V(x) - \mu_1)Z$ is closed in \mathcal{H}^* ;
- (iv) $\text{codim Im}(-\Delta + V(x) - \mu_1) = 1$,

where \mathcal{H}^* denotes the dual space of \mathcal{H} . \square

Motivated by Theorem 3.2 in [5], we have the following lemma. For the sake of completeness, we give a short proof here.

Lemma A.3. Define the following C^1 functional $F : \mathcal{H} \times \mathbb{R}^2 \mapsto \mathcal{H}^*$

$$F(u, \mu, a) = (-\Delta + V(x) - \mu)u - au^3. \quad (\text{A.5})$$

Then, there exist $\delta > 0$ and a unique function $(u(a), \mu(a)) \in C^1(B_\delta(0); B_\delta(\mu_1, \phi_1))$ such that

$$\begin{cases} \mu(0) = \mu_1, & u(0) = \phi_1; \\ F(u(a), \mu(a), a) = 0; \\ \|u(a)\|_2^2 = 1. \end{cases} \quad (\text{A.6})$$

Proof. Let $g : Z \times \mathbb{R}^3 \mapsto \mathcal{H}^*$ be defined by

$$g(z, \tau, s, a) := F((1+s)\phi_1 + z, \mu_1 + \tau, a).$$

Then $g \in C^1(Z \times \mathbb{R}^3, \mathcal{H}^*)$ and

$$g(0, 0, 0, 0) = F(\phi_1, \mu_1, 0) = 0 \quad \text{and} \quad g_s(0, 0, 0, 0) = F_u(\phi_1, \mu_1, 0)\phi_1 = (-\Delta + V(x) - \mu_1)\phi_1 = 0. \quad (\text{A.7})$$

Moreover, for any $(\hat{z}, \hat{\tau}) \in Z \times \mathbb{R}$, we have

$$g_{(z,\tau)}(0, 0, 0, 0)(\hat{z}, \hat{\tau}) = F_u(\phi_1, \mu_1, 0)\hat{z} + F_\mu(\phi_1, \mu_1, 0)\hat{\tau} = (-\Delta + V(x) - \mu_1)\hat{z} - \hat{\tau}\phi_1. \quad (\text{A.8})$$

Then, by Lemma A.2, $g_{(z,\tau)}(0, 0, 0, 0) : Z \times \mathbb{R} \mapsto \mathcal{H}^*$ is an isomorphism. Therefore, by the implicit function theorem, there exist $\delta_1 > 0$ and a unique function $(z(s, a), \tau(s, a)) \in C^1(B_{\delta_1}(0, 0); B_{\delta_1}(0, 0))$ such that

$$\begin{cases} g(z(s, a), \tau(s, a), s, a) = F((1+s)\phi_1 + z(s, a), & \mu_1 + \tau(s, a), a) = 0, \\ z(0, 0) = 0, & \tau(0, 0) = 0, \\ z_s(0, 0) = -g_{(z,\tau)}^{-1}(0, 0, 0, 0) \cdot g_s(0, 0, 0, 0) = 0. \end{cases} \quad (\text{A.9})$$

Now, let

$$u(s, a) = (1+s)\phi_1 + z(s, a), \quad (s, a) \in B_{\delta_1}(0, 0),$$

and define

$$f(s, a) = \|u(s, a)\|_2^2 = (1+s)^2 + \int_{\mathbb{R}^2} z(s, a)^2 dx, \quad (s, a) \in B_{\delta_1}(0, 0).$$

It follows from (A.9) that

$$f(0, 0) = 1, \quad f_s(0, 0) = 2 + 2 \int_{\mathbb{R}^2} z_s(0, 0)z(0, 0) dx = 2.$$

Then, by applying implicit function theorem again, there exist $\delta \in (0, \delta_1)$ and a unique function $s = s(a) \in C^1(B_\delta(0); B_\delta(0))$ such that

$$f(s(a), a) = \|u(s(a), a)\|_2^2 = f(0, 0) = 1, \quad a \in B_\delta(0).$$

This and (A.9) show that, for $a \in B_\delta(0)$, there exists a unique function:

$$(u(a) := u(s(a), a), \mu(a) := \mu_1 + \tau(s(a), a)) \in C^1(B_\delta(0); B_\delta(\phi_1, \mu_1))$$

such that (A.6) holds, and the proof is therefore complete. \square

Proof of Theorem 1.1(ii). Let $u_a(x) > 0$ be a minimizer of $e(a)$ with $a \in [0, a^*)$. It is easy to see that

$$e(0) = \mu_1 \quad \text{and} \quad e(a) \leq e(0) = \mu_1, \quad (\text{A.10})$$

where μ_1 is defined by (A.2). Moreover, $e(a)$ is a concave function of a and then

$$e(a) \in C([0, a^*), \mathbb{R}^+). \tag{A.11}$$

For any $a_0 \in [0, a^*)$, it follows from (1.13) that

$$\int_{\mathbb{R}^2} u_a^4 dx \leq \frac{2e(a)}{a^* - a} \leq \frac{4\mu_1}{a^*} \quad \text{for } 0 \leq a \leq \frac{a^*}{2}. \tag{A.12}$$

Since u_a is a minimizer of (1.4), it satisfies the following Euler–Lagrange equation

$$-\Delta u_a(x) + V(x)u_a(x) - \mu_a u_a(x) - a u_a^3(x) = 0 \quad \text{in } \mathbb{R}^2,$$

where $\mu_a \in \mathbb{R}$ is a suitable Lagrange multiplier, i.e.,

$$F(u_a, \mu_a, a) = 0, \quad \text{where } F(\cdot) \text{ is defined by (A.5)}. \tag{A.13}$$

Since

$$\mu_a = e(a) - \frac{a}{2} \int_{\mathbb{R}^2} |u_a|^4 dx,$$

it then follows from (A.10)–(A.12) that there exists $a_1 > 0$ small such that

$$|\mu_a - \mu_1| \leq |e(a) - \mu_1| + \frac{a}{2} \int_{\mathbb{R}^2} |u_a|^4 dx \leq \delta \quad \text{for } 0 \leq a < a_1, \tag{A.14}$$

where $\delta > 0$ is as in Lemma A.3. On the other hand, since

$$E_0(u_a) = e(a) + \frac{a}{2} \int_{\mathbb{R}^2} |u_a|^4 dx \rightarrow e(0) = \mu_1 \quad \text{as } a \searrow 0,$$

i.e., $\{u_a \geq 0\}$ is a minimizing sequence of $e(0) = \mu_1$ as $a \searrow 0$. Noting that μ_1 is simple, we can easily deduce from Lemma A.1 that

$$u_a \rightarrow \phi_1 \quad \text{in } \mathcal{H} \text{ for all } a \searrow 0.$$

This implies that there exists $a_2 > 0$ such that

$$\|u_a - \phi_1\|_{\mathcal{H}} < \delta \quad \text{for } 0 \leq a < a_2. \tag{A.15}$$

Then using (A.13)–(A.15) and Lemma A.3, we obtain that

$$\mu_a = \mu(a); \quad u_a = u(a) \quad \text{for } 0 \leq a < \min\{a_1, a_2\},$$

i.e., $e(a)$ has a unique non-negative minimizer $u(a)$ if $a > 0$ is small. \square

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