A Dirichlet problem involving the divergence operator

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Abstract

We consider the problem

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

We show that if $\operatorname{curl} a(x_0) \neq 0$ for some $x_0 \in \Omega$, then the problem is solvable without restriction on f. We also discuss the regularity of the solution.

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1. Introduction

In this paper we study the existence of solutions for the problem

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

where a is a vector field and $\langle .; . \rangle$ stands for the scalar product in $\mathbb{R}^n$. The problem with $a \equiv 0$ has attracted considerable attention, notably by Bogovski [2], Borchers–Sohr [3], Csató–Dacorogna–Kneuss [4], Dacorogna [5,6], Dacorogna–Moser [7], Dautray–Lions [8], Galdi [9], Girault–Raviart [11], Kapitanskii–Pileckas [12], Ladyzhenskaya [13], Ladyzhenskaya–Solonnikov [14], Necas [15], Tartar [16], Von Wahl [17,18]. The following theorem is standard (cf. Theorem 9.2 in [4]).

Theorem 1. *Let $n \geq 1$, $r \geq 0$ be integers and $0 < s < 1$. Let $\Omega \subset \mathbb{R}^n$ be a bounded connected open $C^{r+2,s}$ set with outward unit normal ν. The following conditions are then equivalent.*

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(i) $f \in C^{r,s}(\overline{\Omega})$ *and* $u_0 \in C^{r+1,s}(\overline{\Omega}; \mathbb{R}^n)$ *satisfy*

$$\int_\Omega f = \int_\Omega \operatorname{div} u_0 = \int_{\partial\Omega} \langle u_0; v \rangle.$$

(ii) *There exists* $u \in C^{r+1,s}(\overline{\Omega}; \mathbb{R}^n)$ *verifying*

$$\begin{cases} \operatorname{div} u = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

If the vector field a is a gradient of a potential A, then the same result holds (just replace u by $e^A u$, u_0 by $e^A u_0$ and f by $e^A f$) and the problem

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases}$$

is solvable (with optimal regularity for a, f, u_0 and u) if and only if

$$\int_\Omega e^A f = \int_\Omega \operatorname{div}(e^A u_0) = \int_{\partial\Omega} e^A \langle u_0; v \rangle.$$

Our result will show that if

$$\operatorname{curl} a(x_0) \neq 0 \quad \text{for some } x_0 \in \Omega,$$

then (cf. Theorem 2) the problem is solvable without any integral restriction on f and u_0. Under slightly strengthening this last condition, namely $\operatorname{curl} a \neq 0$ on $\partial\Omega$, we will also provide (cf. Theorem 3) a solution with optimal regularity. We should also point out an interesting point related to the topology of the domain Ω. In a special case (cf. Theorem 5) we will see that the right condition is not $\operatorname{curl} a \not\equiv 0$ but that a is not a gradient.

In Section 4 we will also study the kernel of the operator (this kernel will be used in a significant way in Theorem 3) $L_a(u) = \operatorname{div} u + \langle a; u \rangle$. When $a \equiv 0$ (or more generally when $a = \operatorname{grad} A$), the kernel is classically given (cf. Section 4) by $\operatorname{curl}^* w$ so that

$$\operatorname{div} \operatorname{curl}^* w = 0 \quad \text{for every } w \in C^2(\mathbb{R}^n; \mathbb{R}^{n(n-1)/2}).$$

When $\operatorname{curl} a \neq 0$, it is easily seen that the kernel operator cannot be a first order operator. We provide (cf. Proposition 10) the most general second order operator (which can be reduced to the operator curl^* when $a \equiv 0$) denoted by $N_{a,2}^{\alpha,\beta,\gamma}$, so that

$$L_a\big(N_{a,2}^{\alpha,\beta,\gamma}(w)\big) = 0 \quad \text{for every } w \in C^2(\mathbb{R}^n; \mathbb{R}^{n(n-1)/2}).$$

Finally in Sections 5 and 7 we discuss a Poincaré type lemma, both on the boundary (cf. Theorem 14) and in the interior (cf. Theorems 16 and 22), of the operator L_a. For example (cf. Theorem 22), we will find, if $L_a(u) = 0$, that there exists w such that

$$u = N_{a,2}^{\alpha,\beta,\gamma}(w).$$

This is the exact analogue of the classical theorem which says that, if $\operatorname{div} u = 0$, there exists w such that $u = \operatorname{curl}^* w$.

A more detailed version of the present article can be found on the website http://caa.epfl.ch/articles.html.

2. The main theorems

In this paper we will adopt the notation $L_a(u) = \operatorname{div} u + \langle a; u \rangle$. Our first theorem is constructive and uses only elementary tools, notably the method of characteristics. It is sharp from the point of view of the condition on a (namely $\operatorname{curl} a \not\equiv 0$). However it is not sharp from the point of view of regularity (although it can be slightly improved, see Theorem 9 below).

Theorem 2. *Let $n \geq 2$, $r \geq 0$ be integers and $\Omega \subset \mathbb{R}^n$ a bounded open set, with $\overline{\Omega}$ C^{r+4} diffeomorphic to the unit closed ball. Let $f \in C^{r+3}(\overline{\Omega})$, $u_0 \in C^{r+4}(\overline{\Omega}; \mathbb{R}^n)$ and $a \in C^{r+3}(\overline{\Omega}; \mathbb{R}^n)$ be such that*

$$\operatorname{curl} a(x_0) \neq 0 \quad \text{for some } x_0 \in \Omega.$$

Then there exists $u \in C^{r+1}(\overline{\Omega}; \mathbb{R}^n)$ satisfying

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Our second theorem is sharp from the point of view of regularity (except for the regularity of a, which should be $a \in C^{r,s}$), but the hypothesis $\operatorname{curl} a \not\equiv 0$ has to be strengthened.

Theorem 3. *Let $n \geq 2$, $r \geq 0$ be integers, $0 < s < 1$ and $\Omega \subset \mathbb{R}^n$ a bounded open smooth set. Let $f \in C^{r,s}(\overline{\Omega})$, $u_0 \in C^{r+1,s}(\overline{\Omega}; \mathbb{R}^n)$ and $a \in C^{r+4,s}(\overline{\Omega}; \mathbb{R}^n)$ be such that*

$$\inf_{x \in \partial\Omega} \left\{ \left| \operatorname{curl} a(x) \right| \right\} \geq \delta > 0.$$

Then there exists $u \in C^{r+1,s}(\overline{\Omega}; \mathbb{R}^n)$ satisfying

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Moreover the correspondence $(f, u_0) \to u$ can be chosen linear and there exists $K = K(r, s, \|a\|_{C^{r+4,s}}, \delta, \Omega)$ such that

$$\|u\|_{C^{r+1,s}} \leq K \left(\|f\|_{C^{r,s}} + \|u_0\|_{C^{r+1,s}} \right).$$

Remark 4. As already said the natural hypothesis is, in view of Theorem 2, $\operatorname{curl} a(x_0) \neq 0$ for some $x_0 \in \Omega$, but, at the moment, in the present theorem we need a stronger hypothesis. It will be clear from the proof that we can replace the hypothesis $\operatorname{curl} a \neq 0$ on $\partial\Omega$, by other hypotheses such as, for example, in addition to the natural condition, $a = \operatorname{grad} A$ near $\partial\Omega$.

Finally when Ω is not simply connected we see, in a special case (including the case of harmonic fields), that the right condition is not $\operatorname{curl} a \not\equiv 0$ but that a is not a gradient.

Theorem 5. *Let $n \geq 2$, $r \geq 0$ be integers, $0 < s < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Let $f \in C^{r,s}(\overline{\Omega})$, $u_0 \in C^{r+1,s}(\overline{\Omega}; \mathbb{R}^n)$ and $a \in C^{r,s}(\overline{\Omega}; \mathbb{R}^n)$ be such that*

$$\operatorname{curl} a \equiv 0 \quad \text{in } \overline{\Omega} \quad \text{but} \quad \nexists A \in C^{r+1,s}(\overline{\Omega}) \quad \text{with } a = \operatorname{grad} A.$$

Then there exists $u \in C^{r+1,s}(\overline{\Omega}; \mathbb{R}^n)$ satisfying

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

Moreover the correspondence $(f, u_0) \to u$ can be chosen linear and there exists $K = K(r, s, \|a\|_{C^{r,s}}, \Omega)$ such that

$$\|u\|_{C^{r+1,s}} \leq K \left(\|f\|_{C^{r,s}} + \|u_0\|_{C^{r+1,s}} \right).$$

Remark 6.

(i) The hypothesis on a implies that Ω is not simply connected and $a \not\equiv 0$. Conversely if Ω is not simply connected, there exists such an a. Note that in Theorem 2 the set Ω is simply connected.

(ii) Observe that the regularity in the theorem is optimal for all the data.

We now show how a general operator of the form

$$L_{a,b}(u) = \sum_{1 \le i,j \le n} b_j^i u_{x_j}^i + \langle a; u \rangle = \langle B; \nabla u \rangle + \langle a; u \rangle$$

can be brought back to our analysis. But let us first introduce some notations. Let

$$B = \left(b_j^i\right)_{1 \le j \le n}^{1 \le i \le n} \in C^1\left(\overline{\Omega}; \mathbb{R}^{n \times n}\right)$$

and set

$$\operatorname{div} B = \left(\operatorname{div} B^1, \cdots, \operatorname{div} B^n\right) \quad \text{where } \operatorname{div} B^i = \sum_{1 \le j \le n} \left(b_j^i\right)_{x_j}.$$

Let B be invertible and define $\widetilde{a} = B^{-t}(a - \operatorname{div} B^t)$. Assume that Ω, f, u_0 and $\widetilde{a}$ verify the hypotheses of either Theorem 2 or Theorem 3. We then claim that there exists u (with the corresponding regularity) satisfying

$$\begin{cases} L_{a,b}(u) = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega. \end{cases}$$

This is easily seen by setting $u = B^{-1}v$ where

$$\begin{cases} \operatorname{div} v + \langle \widetilde{a}; v \rangle = f & \text{in } \Omega \\ v = Bu_0 & \text{on } \partial\Omega. \end{cases}$$

Indeed it suffices to observe that $\operatorname{div} v + \langle \widetilde{a}; v \rangle = L_{a,b}(u)$.

3. Proof of Theorem 2

Although we will be dealing only with vector fields and the divergence and curl operators, it will be, sometimes, simpler to use the notations of differential geometry (we adopt the notations in [4]). We will denote, when convenient, differential forms as vector fields. For example a vector field $u = (u^1, \cdots, u^n)$ is also written as $u = \sum_{i=1}^n u^i \, dx^i$, the curl and the divergence operators as

$$du = \sum_{1 \le i < j \le n} \left[u_{x_i}^j - u_{x_j}^i\right] dx^i \wedge dx^j \sim \operatorname{curl} u \quad \text{and} \quad \delta u = \sum_{i=1}^n u_{x_i}^i \sim \operatorname{div} u.$$

The operator curl* is seen as the δ operator acting on 2-forms $u = \sum u^{ij} \, dx^i \wedge dx^j$ namely (where we set $u^{ij} = -u^{ji}$ if $i \ge j$)

$$\delta u = \sum_{j=1}^n \sum_{i=1}^n u_{x_i}^{ij} \, dx^j \sim \operatorname{curl}^* u = \left(\left(\operatorname{curl}^* u\right)^1, \cdots, \left(\operatorname{curl}^* u\right)^n\right) \in \mathbb{R}^n$$

where

$$\left(\operatorname{curl}^* u\right)^i = \sum_{j=1}^{i-1} \frac{\partial u^{ji}}{\partial x_j} - \sum_{j=i+1}^n \frac{\partial u^{ij}}{\partial x_j}.$$

The Hodge * operator as well as the exterior and the interior product are defined as usual; for example the interior product of a 1-form u with a 2-form v is defined as

$$u \lrcorner v = \sum_{j=1}^n \sum_{i=1}^n u^i v^{ij} \, dx^j.$$

Before proceeding with the proof of Theorem 2. We will need two lemmas. The first one allows us to reduce the problem to the case where Ω is the unit ball B_1.

Lemma 7. *Let $r \geq 1$ be an integer, $\Omega; O \subset \mathbb{R}^n$ be two bounded open smooth sets. Let $\varphi \in \mathrm{Diff}^{r+1}(\overline{\Omega}; \overline{O})$, $a \in C^0(\overline{O}; \mathbb{R}^n)$ and $f \in C^0(\overline{O})$. Define*

$$b = \varphi^*(a) \quad and \quad g = (-1)^{n-1} * \big(\varphi^*(*f)\big).$$

Then $u \in C^r(\overline{O}; \mathbb{R}^n)$ solves the problem

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & in\ O \\ u = 0 & on\ \partial O \end{cases} \tag{1}$$

if and only if $v = *(\varphi^*(*u)) \in C^r(\overline{\Omega}; \mathbb{R}^n)$ solves

$$\begin{cases} \operatorname{div} v + \langle b; v \rangle = g & in\ \Omega \\ v = 0 & on\ \partial\Omega. \end{cases} \tag{2}$$

Proof. We rewrite (1), after composition with the $*$ operation as

$$*\delta u + *(a \lrcorner u) = *f \quad \Leftrightarrow \quad d(*u) + \big(a \wedge (*u)\big) = (*f).$$

Composing with φ, we get

$$*d(*v) + *\big(\varphi^*(a) \wedge (*v)\big) = (-1)^{n-1} * \varphi^*(*f) \quad \Leftrightarrow \quad \delta v + b \lrcorner v = g$$

which is exactly (2). $\quad \square$

The proof of the following lemma is just done by straight computation.

Lemma 8. *Let $r \geq 1$ be an integer. Let $f, g \in C^r(\overline{B}_1 \setminus \{0\})$ and $v_0 \in C^r(\partial B_1)$. Let $v \in C^r(\overline{B}_1 \setminus \{0\})$ be defined by*

$$v(x) = G(x) v_0\left(\frac{x}{|x|}\right) + V(x)$$

where $G(x) = \exp[\int_{|x|}^1 g(\frac{sx}{|x|})\frac{ds}{s}]$ and

$$V(x) = \int_1^{|x|} \left\{ \exp\left[\int_{|x|}^r g\left(\frac{sx}{|x|}\right)\frac{ds}{s}\right] f\left(\frac{rx}{|x|}\right)\frac{dr}{r} \right\}.$$

Then v satisfies

$$\begin{cases} \big\langle \nabla v(x); x \big\rangle + g(x) v(x) = f(x) & if\ x \in \overline{B}_1 \setminus \{0\} \\ v(x) = v_0(x) & if\ x \in \partial B_1. \end{cases}$$

We are now in a position to prove Theorem 2.

Proof of Theorem 2. *Step 1.* We start with some simplifications.

(i) Without loss of generality, we may assume that $u_0 = 0$; replacing u by $u - u_0$ and f by $f - \operatorname{div} u_0 - \langle a; u_0 \rangle$.

(ii) Using Lemma 7, we find that we can take Ω to be the unit ball B_1. Combining Lemma 11.13 in [4] and Lemma 7 again, we can assume, without loss of generality, that $x_0 = 0$.

(iii) Finally we choose $\epsilon > 0$ sufficiently small so that $\operatorname{curl} a \neq 0$ in $B_{2\epsilon}$.

Step 2. We then search for solutions of the form $u = v + a \lrcorner w + \delta w$, where $v \in C^{r+3}(\overline{B}_1; \mathbb{R}^n)$ (constructed in Step 3) and $w \in C^{r+2}(\overline{B}_1; \Lambda^2)$ (given in Step 4). The advantage of this decomposition is that it transforms the problem into (invoking Theorem 3.5 in [4])

$$L_a(u) = \operatorname{div} u + \langle a; u \rangle = \delta u + \langle a; u \rangle = \delta v + \langle a; v \rangle - da \lrcorner w = f.$$

So if, in addition to the above equation, we find that $v = 0$ on ∂B_1 and $w = 0$ in a neighborhood of ∂B_1, we will have established the theorem.

Step 3. We first construct v on $\overline{B}_1 \setminus B_\epsilon$ as $v(x) = xV(x)$, where $g(x) = n + \langle a(x); x \rangle$ and V is as in Lemma 8. Observe that $v \in C^{r+3}(\overline{B}_1 \setminus B_\epsilon; \mathbb{R}^n)$. Applying Lemma 8, we find that V satisfies

$$\begin{cases} \langle \nabla V(x); x \rangle + g(x)V(x) = f(x) & \text{if } x \in \overline{B}_1 \setminus B_\epsilon \\ V(x) = 0 & \text{if } x \in \partial B_1 \end{cases}$$

and thus v verifies

$$\begin{cases} \operatorname{div} v + \langle a; v \rangle = \delta v + \langle a; v \rangle = f & \text{in } \overline{B}_1 \setminus B_\epsilon \\ v = 0 & \text{on } \partial B_1. \end{cases}$$

We then extend v in any C^{r+3} way to B_ϵ.

Step 4. We finally construct $w \in C^{r+2}(\overline{B}_1; \Lambda^2)$ to be identically 0 in $\overline{B}_1 \setminus B_{2\epsilon}$ and, in $B_{2\epsilon}$, through the formula

$$w = \frac{\operatorname{curl} a}{|\operatorname{curl} a|^2} \big(\operatorname{div} v + \langle a; v \rangle - f \big).$$

The v and w have all the claimed properties. $\square$

Using standard elliptic estimates, we can slightly improve the regularity hypotheses.

Theorem 9. *Let $n \geq 2$, $r \geq 0$ be integers, $0 < s < 1$ and $\Omega \subset \mathbb{R}^n$ a bounded open set, with $\overline{\Omega}$ $C^{r+4,s}$ diffeomorphic to the closed unit ball. Let $f \in C^{r+2,s}(\overline{\Omega})$, $u_0 \in C^{r+3,s}(\overline{\Omega}; \mathbb{R}^n)$ and $a \in C^{r+3,s}(\overline{\Omega}; \mathbb{R}^n)$ be such that*

$$\operatorname{curl} a(x_0) \neq 0 \quad \text{for some } x_0 \in \Omega.$$

Then there exists $u \in C^{r+1,s}(\overline{\Omega}; \mathbb{R}^n)$ satisfying

$$\begin{cases} \operatorname{div} u + \langle a; u \rangle = f & \text{in } \Omega \\ u = u_0 & \text{on } \partial \Omega. \end{cases}$$

Proof. *Step 1.* As in the proof of Theorem 2, we can assume, without loss of generality, that $u_0 = 0$, $\Omega = B_1$ and $x_0 = 0$. Moreover $\epsilon > 0$ is chosen sufficiently small so that $\operatorname{curl} a \neq 0$ in $B_{2\epsilon}$. We then find $\alpha \in C^{r+4,s}(\overline{\Omega})$ a solution of

$$\begin{cases} \Delta \alpha + \langle a; \nabla \alpha \rangle = f & \text{in } B_1 \\ \alpha = 0 & \text{on } \partial B_1. \end{cases}$$

Since $\alpha = 0$ on ∂B_1, we find that there exists $c = \langle v; \nabla \alpha \rangle \in C^{r+3,s}(\overline{B}_1)$ such that $\nabla \alpha = cv$ on ∂B_1, where $v = v(x) = x$ is the outward unit normal to ∂B_1. We will then look for solutions u of the form $u = \nabla \alpha + \beta$, where β satisfies

$$\begin{cases} \operatorname{div} \beta + \langle a; \beta \rangle = 0 & \text{in } B_1 \\ \beta = -cv & \text{on } \partial B_1. \end{cases}$$

Step 2. We then continue exactly as in the proof of Theorem 2 where we write $\beta = v + a \lrcorner w + \delta w$. The only difference is that we require $v = -cv$ instead of $v = 0$ on ∂B_1. It is easy to see that v and w have all the appropriate properties. $\square$

4. The kernel of the operator

4.1. Definition of the kernel

We now study the kernel of $L_a(u) = \operatorname{div} u + \langle a; u \rangle$. Let us first examine the case $a \equiv 0$. We recall that for a C^1 vector field $w : \mathbb{R}^n \to \mathbb{R}^{n(n-1)/2}$ the kernel is given by $\operatorname{curl}^* w \in \mathbb{R}^n$ so that

$$\operatorname{div} \operatorname{curl}^* w = 0 \quad \text{for every } w \in C^2\big(\mathbb{R}^n; \mathbb{R}^{n(n-1)/2} \big).$$

We know (by Poincaré lemma) that if $\operatorname{div} u = 0$, then there exists w such that

$$u = \operatorname{curl}^* w.$$

We now turn to the general case where $a \not\equiv 0$. We will first define the most general second order kernel acting on functions and then extend this definition to kernels acting on 2-forms. This extension to 2-forms is motivated by extending the above result (when $a \equiv 0$). It will moreover turn out to be a crucial point in the proof of Theorem 3. Examples are discussed in Section 4.3.

Case 1: kernels acting on functions. Let $\alpha_{kl}^m, \beta_l^m, \gamma^m \in C^0(\overline{\Omega})$. We define the operator, acting on C^2 functions w, $N_a^{\alpha,\beta,\gamma} : C^2(\overline{\Omega}) \to C^0(\overline{\Omega}; \mathbb{R}^n)$ as, for $w \in C^2(\overline{\Omega})$,

$$N_a^{\alpha,\beta,\gamma}(w) = \left(\left(N_a^{\alpha,\beta,\gamma}(w)\right)^1, \cdots, \left(N_a^{\alpha,\beta,\gamma}(w)\right)^n\right) = \sum_{m=1}^n \left(N_a^{\alpha,\beta,\gamma}(w)\right)^m dx^m$$

where

$$\left(N_a^{\alpha,\beta,\gamma}(w)\right)^m = \sum_{k \leq l} \alpha_{kl}^m w_{x_k x_l} + \sum_l \beta_l^m w_{x_l} + \gamma^m w.$$

Note that $\alpha = (\alpha_{kl}^m)_{1 \leq k \leq l \leq n}^{1 \leq m \leq n} \in \mathbb{R}^{\left(\binom{n}{2}+n\right)n}$, $\beta = (\beta_l^m)_{1 \leq l \leq n}^{1 \leq m \leq n} \in \mathbb{R}^{n^2}$ and $\gamma = (\gamma^m)^{1 \leq m \leq n} \in \mathbb{R}^n$. It will also be convenient to write

$$\alpha_{kl} = \sum_m \alpha_{kl}^m dx^m, \qquad \beta_l = \sum_m \beta_l^m dx^m \quad \text{and} \quad \gamma = \sum_m \gamma^m dx^m.$$

We should observe that the dependence on a in the kernel is only implicit. It is only when we want to determine α, β and γ so that $L_a(N_a^{\alpha,\beta,\gamma}(w)) = 0$, for every w, that the a plays a role. So, sometimes, when we study the properties of the operator $N_a^{\alpha,\beta,\gamma}$ independently of the fact that $L_a(N_a^{\alpha,\beta,\gamma}(w)) = 0$, we will write only $N^{\alpha,\beta,\gamma}$.

Case 2: kernels acting on 2-forms. For $w = \sum_{i<j} w^{ij} dx^i \wedge dx^j \in C^2(\overline{\Omega}; \Lambda^2)$ we let

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \sum_m \left[\sum_{i<j}\left(N_a^{\alpha_{ij},\beta_{ij},\gamma_{ij}}\left(w^{ij}\right)\right)^m\right] dx^m.$$

Note that $\alpha = (\alpha_{ijkl}^m)_{1 \leq i < j \leq n, \, 1 \leq k \leq l \leq n}^{1 \leq m \leq n} \in \mathbb{R}^{\binom{n}{2}\left(\binom{n}{2}+n\right)n}$, while

$$\beta = \left(\beta_{ijl}^m\right)_{1 \leq i < j \leq n, \, 1 \leq l \leq n}^{1 \leq m \leq n} \in \mathbb{R}^{\binom{n}{2}n^2} \quad \text{and} \quad \gamma = \left(\gamma_{ij}^m\right)_{1 \leq i < j \leq n}^{1 \leq m \leq n} \in \mathbb{R}^{\binom{n}{2}n}.$$

4.2. The necessary and sufficient condition

We then have the following proposition.

Proposition 10. *Let $\alpha_{kl}, \beta_l, \gamma \in C^1(\overline{\Omega}; \mathbb{R}^n)$. Then $L_a(N_a^{\alpha,\beta,\gamma}(w)) = 0$ for every $w \in C^3(\overline{\Omega})$ if and only if*

$$\begin{cases} \alpha_{mm}^m = 0 & \forall m \\ \alpha_{ll}^m + \alpha_{lm}^l = \alpha_{mm}^l + \alpha_{lm}^m = 0 & \forall l < m \\ \alpha_{lm}^k + \alpha_{km}^l + \alpha_{kl}^m = 0 & \forall k < l < m \end{cases} \tag{3}$$

$$\begin{cases} L_a(\alpha_{ll}) + \beta_l^l = 0 & \forall l \\ L_a(\alpha_{lm}) + \beta_l^m + \beta_m^l = 0 & \forall l < m \end{cases} \tag{4}$$

$$\begin{cases} L_a(\beta_l) + \gamma^l = 0 & \forall l \\ L_a(\gamma) = 0. \end{cases} \tag{5}$$

Proof. We have

$$L_a\left(N_a^{\alpha,\beta,\gamma}(w)\right) = \sum_m \sum_{k \leq l} \left(\alpha_{kl}^m w_{x_k x_l}\right)_{x_m} + \sum_m \sum_l \left(\beta_l^m w_{x_l}\right)_{x_m} + \sum_m \left(\gamma^m w\right)_{x_m}$$

$$+ \sum_m \sum_{k \leq l} a^m \left(\alpha_{kl}^m w_{x_k x_l}\right) + \sum_m \sum_l a^m \left(\beta_l^m w_{x_l}\right) + \sum_m a^m \left(\gamma^m w\right)$$

and thus, since $L_a(N_a^{\alpha,\beta,\gamma}(w)) = 0$, we find

$$0 = \sum_m \sum_{k \leq l} \left(\alpha_{kl}^m w_{x_k x_l x_m} \right) + \sum_{k \leq l} \left(L_a(\alpha_{kl}) \right) w_{x_k x_l} + \sum_m \sum_l \left(\beta_l^m w_{x_l x_m} \right)$$
$$+ \sum_l \left(L_a(\beta_l) \right) w_{x_l} + \sum_m \left(\gamma^m w_{x_m} \right) + \left(L_a(\gamma) \right) w.$$

Let us now examine all the terms depending on the order of derivatives of w.

Terms of order 0 *and* 1. For the first one we immediately get that $L_a(\gamma) = 0$. For the terms of order 1, we group all the terms with the same indices

$$\sum_l \left(L_a(\beta_l) + \gamma^l \right) w_{x_l} = 0$$

and hence $L_a(\beta_l) + \gamma^l = 0$ for every l.

Terms of order 2. We find

$$\sum_{k \leq l} + \sum_m \sum_l = \left[\sum_{l\,(k=l)} + \sum_{k<l} \right] + \left[\sum_{l<m} + \sum_{l\,(m=l)} + \sum_{m<l} \right]$$

and thus rewriting with the same indices, we get

$$\sum_{k \leq l} \left(L_a(\alpha_{kl}) \right) w_{x_k x_l} + \sum_m \sum_l \left(\beta_l^m w_{x_l x_m} \right) = \sum_l \left(L_a(\alpha_{ll}) + \beta_l^l \right) w_{x_l x_l} + \sum_{l<m} \left(L_a(\alpha_{lm}) + \beta_l^m + \beta_m^l \right) w_{x_l x_m}.$$

We hence found exactly (4).

Terms of order 3. Writing the same indices we obtain

$$\sum_m \sum_{k \leq l} = \left[\sum_{m\,(k=l=m)} \right] + \left[\sum_{m<l\,(k=l)} + \sum_{m>l\,(k=l)} + \sum_{m<l\,(k=m)} + \sum_{k<m\,(l=m)} \right] + \left[\sum_{m<k<l} + \sum_{k<m<l} + \sum_{k<l<m} \right].$$

We hence find, after uniforming the indices,

$$0 = \left[\sum_m \alpha_{mm}^m w_{x_m x_m x_m} \right] + \sum_{l<m} \left(\alpha_{ll}^m + \alpha_{lm}^l \right) w_{x_l x_l x_m}$$
$$+ \sum_{l<m} \left(\alpha_{mm}^l + \alpha_{lm}^m \right) w_{x_l x_m x_m} + \sum_{k<l<m} \left[\alpha_{lm}^k + \alpha_{km}^l + \alpha_{kl}^m \right] w_{x_k x_l x_m}.$$

Therefore (3) is established and the proof is thus complete. $\quad\square$

4.3. Some examples

We now give three examples.

Example 11. When $a = \operatorname{grad} A$, we can choose, for every $w \in C^2(\overline{\Omega}; \Lambda^2)$,

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = e^{-A} \operatorname{curl}^* w$$

Proof. Fix $i < j$ and define

$$\beta_{ijl}^m = e^{-A} \begin{cases} 1 & \text{if } l = i \text{ and } m = j \\ -1 & \text{if } m = i \text{ and } l = j \\ 0 & \text{otherwise.} \end{cases}$$

Choose $\alpha_{lm} = \gamma = 0$ so that $N_a^{\alpha,\beta_{ij},\gamma}(w) = e^{-A}[-w_{x_j}\, dx^i + w_{x_i}\, dx^j]$. When applied to 2-forms $w = \sum w^{ij}\, dx^i \wedge dx^j$ we can choose a linear combination of the above $N_a^{\alpha,\beta_{ij},\gamma}(w^{ij})$. We thus find

$$e^A \left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right)^m = e^A \sum_{i<j} \left(N_a^{\alpha,\beta_{ij},\gamma}(w^{ij})\right)^m = \sum_{i<j} \sum_l \beta_{ijl}^m w_{x_l}^{ij}$$

$$= \sum_{l<m} \beta_{lml}^m w_{x_l}^{lm} + \sum_{l>m} \beta_{mll}^m w_{x_l}^{ml} = \sum_{l<m} w_{x_l}^{lm} - \sum_{l>m} w_{x_l}^{ml}$$

$$= \left(\mathrm{curl}^* w\right)^m$$

which is what had to be proved. □

The next example is in some sense generic and will be used in Theorem 22.

Example 12. Let $0 \le 2p \le n$ and $\bar{a} = \sum_{i=1}^{p} x_{2i}\, dx^{2i-1}$.

Case of 0-forms (for $\bar{a}$). We can then choose $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in C^\infty$ such that, for every $w \in C^2(\overline{\Omega})$,

$$N_{\bar{a}}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w) = e^{-x_1 x_2}(-w_{x_1 x_2} + x_1 w_{x_1} - w)dx^1 + e^{-x_1 x_2}(w_{x_1 x_1})dx^2$$

and verifying $L_{\bar{a}}(N_{\bar{a}}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w)) = 0$, for every $w \in C^3(\overline{\Omega})$.

Case of 2-forms (for $\bar{a}$). For $w = \sum_{i<j} w^{ij} dx^i \wedge dx^j \in C^2(\overline{\Omega}; \Lambda^2)$, we can find $\bar{\alpha}, \bar{\beta}, \bar{\gamma} \in C^\infty$ such that

$$N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w) = e^{-x_1 x_2}\left(-\sum_{j=2}^{n} w_{x_1 x_j}^{1j} + x_1 w_{x_1}^{12} - \sum_{i=2}^{p} x_{2i} w_{x_1}^{1(2i-1)} - w^{12}\right)dx^1 + e^{-x_1 x_2}\sum_{j=2}^{n} w_{x_1 x_1}^{1j}\, dx^j \tag{6}$$

and satisfying

$$L_{\bar{a}}\left(N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w)\right) = 0, \quad \forall w \in C^3\left(\overline{\Omega}; \Lambda^2\right). \tag{7}$$

It can, moreover, be rewritten as (where we let $\sigma = e^{-x_1 x_2} dx^1$)

$$N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w) = -(\sigma \lrcorner d\bar{a}) \lrcorner w + \bar{a} \lrcorner (\sigma \wedge \delta w) + \delta(\sigma \wedge \delta w)$$

for every w of the form $w = \sum_{j=2}^{n} w^{1j} dx^1 \wedge dx^j$.

Case of 2-forms (for generic a). If we now consider $a = \bar{a} + dS$, where S is a function, we see that if

$$(\alpha, \beta, \gamma) = e^{-S}(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \quad \Leftrightarrow \quad N_{a,2}^{\alpha,\beta,\gamma}(w) = e^{-S} N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w), \tag{8}$$

then

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right) = 0, \quad \forall w \in C^3\left(\overline{\Omega}; \Lambda^2\right).$$

Proof. *Case of 0-forms (for $\bar{a}$).* We just have to set

$$\bar{\alpha}_{kl}^m = e^{-x_1 x_2} \begin{cases} 1 & \text{if } k = l = 1 \text{ and } m = 2 \\ -1 & \text{if } k = m = 1 \text{ and } l = 2 \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{\beta}_l^m = e^{-x_1 x_2} \begin{cases} x_1 & \text{if } l = m = 1 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \bar{\gamma}^m = e^{-x_1 x_2} \begin{cases} -1 & \text{if } m = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Case of 2-forms (for $\bar{a}$). More generally we let, for $i < j$,

$$\bar{\alpha}_{ijkl}^m = e^{-x_1 x_2} \begin{cases} 1 & \text{if } i = k = l = 1 \text{ and } j = m = 2, \cdots, n \\ -1 & \text{if } i = k = m = 1 \text{ and } j = l = 2, \cdots, n \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{\beta}_{ijl}^m = e^{-x_1 x_2} \begin{cases} x_1 & \text{if } i = l = m = 1 \text{ and } j = 2 \\ -x_{2s} & \text{if } i = l = m = 1 \text{ and } j = 2s - 1 \text{ with } s = 2, \cdots, p \\ 0 & \text{otherwise} \end{cases}$$

$$\bar{\gamma}_{ij}^m = e^{-x_1 x_2} \begin{cases} -1 & \text{if } i = m = 1 \text{ and } j = 2 \\ 0 & \text{otherwise.} \end{cases}$$

We still have to prove the extra statement. For $w \in \Lambda^2$, we let $\pi^1(w) = \sum_{j=2}^n w^{1j} dx^1 \wedge dx^j$ and observe that, for every $w \in \Lambda^2$,

$$N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w) = N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(\pi^1(w)).$$

We then set, for $\sigma = e^{-x_1 x_2} dx^1$,

$$M(w) = -(\sigma \lrcorner d\bar{a}) \lrcorner w + \bar{a} \lrcorner (\sigma \wedge \delta w) + \delta(\sigma \wedge \delta w).$$

The claim is that

$$N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w) = N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(\pi^1(w)) = M(\pi^1(w)) \tag{9}$$

and we will thus have $L_{\bar{a}}(M(\pi^1(w))) = 0$, for every $w \in C^3(\overline{\Omega}; \Lambda^2)$. So let us prove (9). Let $w = \pi^1(w)$ and $\lambda = e^{-x_1 x_2}$ so that $\sigma = \lambda \, dx^1$. A direct computation gives

$$
\begin{aligned}
M(w) &= -(\sigma \lrcorner d\bar{a}) \lrcorner w + \bar{a} \lrcorner (\sigma \wedge \delta w) + \delta(\sigma \wedge \delta w) \\
&= \left[-\lambda w^{12} dx^1 \right] + \left[-\lambda \left(\sum_{i=2}^p w_{x_1}^{1(2i-1)} x_{2i} \right) dx^1 + \lambda \sum_{j=2}^n (x_2 w_{x_1}^{1j}) dx^j \right] \\
&\quad + \left[\left(-\sum_{i=2}^n (\lambda w_{x_1}^{1i})_{x_i} \right) dx^1 + \sum_{j=2}^n ((\lambda w_{x_1}^{1j})_{x_1}) dx^j \right].
\end{aligned}
$$

We have therefore established (9), namely

$$M(w) = e^{-x_1 x_2} \left(-\sum_{j=2}^n w_{x_1 x_j}^{1j} + x_1 w_{x_1}^{12} - \sum_{i=2}^p x_{2i} w_{x_1}^{1(2i-1)} - w^{12} \right) dx^1 + e^{-x_1 x_2} \sum_{j=2}^n w_{x_1 x_1}^{1j} \, dx^j.$$

Case of 2-forms (for generic a). It now remains to show the last statement of the example. We have

$$
\begin{aligned}
L_a\big(N_{a,2}^{\alpha,\beta,\gamma}(w)\big) &= \operatorname{div}\big(e^{-S} N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w)\big) + \big\langle a; e^{-S} N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w)\big\rangle \\
&= e^{-S}\big[\operatorname{div}\big(N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w)\big) + \big\langle \bar{a}; N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w)\big\rangle\big]
\end{aligned}
$$

and hence, for every $w \in C^3(\overline{\Omega}; \Lambda^2)$,

$$L_a\big(N_{a,2}^{\alpha,\beta,\gamma}(w)\big) = e^{-S} L_{\bar{a}}\big(N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w)\big) = 0.$$

The proof is therefore complete. $\quad\square$

4.4. A sufficient condition

We now show that if $\operatorname{curl} a \neq 0$, we can then find an operator $N_a^{\alpha,\beta,\gamma}$ satisfying the conditions of Proposition 10, with arbitrary α_{kl}.

Proposition 13. *Let $r \geq 1$ be an integer, $O \subset \mathbb{R}^n$ be a bounded open set, $\alpha_{kl}^m \in C^{r+4}(\overline{O})$ be arbitrary and $a \in C^{r+3}(\overline{O}; \Lambda^1)$ be such that*

$$\inf_{x \in \overline{O}} \big\{ \big|\operatorname{curl} a(x)\big| \big\} \geq \delta > 0.$$

Then there exist $\beta_l^m \in C^{r+1}(\overline{O})$ and $\gamma^m \in C^r(\overline{O})$ verifying

$$
\begin{cases}
L_a(\alpha_{ll}) + \beta_l^l = 0 & \forall l \\
L_a(\alpha_{lm}) + \beta_l^m + \beta_m^l = 0 & \forall l < m
\end{cases}
\quad \text{and} \quad
\begin{cases}
L_a(\beta_l) + \gamma^l = 0 & \forall l \\
L_a(\gamma) = 0.
\end{cases}
$$

Moreover it can be assumed that, for every l, m,

$$\operatorname{supp}[\beta_l^m], \operatorname{supp}[\gamma^m] \subset \bigcup_m \bigcup_{k \leq l} \operatorname{supp}[\alpha_{kl}^m]$$

and, for every $0 \leq s \leq 1$, there exists $K = K(\|a\|_{C^{r+3,s}}, \delta)$ such that

$$\left\|\beta_l^m\right\|_{C^{r+1,s}} + \left\|\gamma^m\right\|_{C^{r,s}} \leq K\|\alpha\|_{C^{r+4,s}}.$$

Proof. (i) Define

$$\beta_l = -\sum_{s \geq l} L_a(\alpha_{ls})dx^s + \sum_s \lambda^{ls}dx^s = b_l + dx^l \lrcorner \lambda$$

where $\lambda^{ls} = -\lambda^{sl}$ (i.e. $\lambda \in \Lambda^2$) will be determined later and where we have let

$$b_l = -\sum_{s \geq l} L_a(\alpha_{ls})dx^s.$$

We therefore have $\beta_l^l = -L_a(\alpha_{ll})$ and if $l < m$

$$\beta_m^l = \lambda^{ml} \quad \text{and} \quad \beta_l^m = -L_a(\alpha_{lm}) + \lambda^{lm}$$

which leads to $\beta_m^l + \beta_l^m = -L_a(\alpha_{lm})$.

(ii) We let γ be defined by

$$\gamma = -\sum_l L_a(\beta_l)dx^l = -\sum_l L_a(b_l)dx^l - \sum_l L_a(dx^l \lrcorner \lambda)dx^l = c - e.$$

The condition $L_a(\gamma) = 0$ therefore becomes $L_a(e) = L_a(c)$. An easy calculation shows that

$$L_a(e) = \sum_{l < s}\left[\left(a_{x_l}^s - a_{x_s}^l\right)\lambda^{ls}\right] = \operatorname{curl} a \lrcorner \lambda.$$

Since $\operatorname{curl} a \neq 0$, we can choose

$$\lambda^{ls} = \frac{(a_{x_l}^s - a_{x_s}^l)}{|\operatorname{curl} a|^2}L_a(c) \tag{10}$$

and thus the result.

(iii) The claim on the support and the estimate are obvious. Note that the dependence of K on δ follows from (10) and Proposition 16.29 in [4]. $\quad\square$

5. A Poincaré type lemma on the boundary

We now consider our operator $N_{a,2}^{\alpha,\beta,\gamma}$ as acting on 2-forms. We therefore have $\alpha_{ijkl}^m, \beta_{ijl}^m, \gamma_{ij}^m \in C^r(\overline{\Omega})$ and $w \in C^{r+2}(\overline{\Omega}; \Lambda^2)$. The operator is then given by

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \sum_m \left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right)^m dx^m$$

where

$$\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right)^m = \left(\sum_{k \leq l}\sum_{i<j} \alpha_{ijkl}^m w_{x_k x_l}^{ij}\right) + \left(\sum_l \sum_{i<j} \beta_{ijl}^m w_{x_l}^{ij}\right) + \left(\sum_{i<j} \gamma_{ij}^m w^{ij}\right)$$

and the α, β, γ are chosen (see Propositions 10 and 13) so that

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right) = 0, \quad \forall w \in C^3\left(\overline{\Omega}; \Lambda^2\right).$$

The first case that we study is a result on the boundary (in Section 7, we will study the case in the interior). More precisely given $c \in C^{r,s}(\overline{\Omega}; \mathbb{R}^n)$ with $\nu \lrcorner c = \langle \nu; c \rangle = 0$ on $\partial\Omega$ (where ν is the outward unit normal to $\partial\Omega$) we will find $w \in C^{r+2,s}(\overline{\Omega}; \Lambda^2)$ (and $\alpha_{ijkl}^m, \beta_{ijl}^m, \gamma_{ij}^m$) such that $N_{a,2}^{\alpha,\beta,\gamma}(w) = c$ on $\partial\Omega$. This is the exact analogue of solving, when $a \equiv 0$, $\operatorname{curl}^* w = c$ on $\partial\Omega$ (cf. Lemma 8.11 (ii) in [4]).

Theorem 14. *Let $r \geq 1$ be an integer, $0 < s < 1$ and $\Omega \subset \mathbb{R}^n$ be a bounded open smooth set. Let $a \in C^{r+3,s}(\overline{\Omega}; \mathbb{R}^n)$ be such that*

$$\inf_{x \in \partial\Omega} \left\{ \left| \operatorname{curl} a(x) \right| \right\} \geq \delta > 0.$$

There exist $\alpha_{ijkl}^m \in C^\infty(\overline{\Omega})$, $\beta_{ijl}^m \in C^{r+1,s}(\overline{\Omega})$, $\gamma_{ij}^m \in C^{r,s}(\overline{\Omega})$ satisfying

$$L_a\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right) = 0, \quad \forall w \in C^3(\overline{\Omega}; \Lambda^2)$$

with the additional property that for any $c \in C^{r,s}(\overline{\Omega}; \mathbb{R}^n)$ with $\nu \lrcorner c = \langle \nu; c \rangle = 0$ on $\partial\Omega$, then there exists $w \in C^{r+2,s}(\overline{\Omega}; \Lambda^2)$ verifying

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = c \quad \text{on } \partial\Omega.$$

Furthermore there exists $K = K(r, s, \|a\|_{C^{r+3,s}}, \delta, \Omega)$ such that

$$\|w\|_{C^{r+2,s}} + \left\| N_{a,2}^{\alpha,\beta,\gamma}(w) \right\|_{C^{r,s}} \leq K \|c\|_{C^{r,s}}.$$

Proof. *Step 1.* We first find (using Lemma 15 below) $\alpha_{ijkl}^m \in C^\infty(\overline{\Omega})$ satisfying (11) such that

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \nu \lrcorner \frac{\partial^2 w}{\partial \nu^2} \quad \text{on } \partial\Omega$$

for every $\beta_{ijl}^m, \gamma_{ij}^m \in C^0(\overline{\Omega})$ and $w \in C^{r+3}(\overline{\Omega}; \Lambda^2)$ verifying

$$\frac{\partial w}{\partial \nu} = w = 0 \quad \text{on } \partial\Omega.$$

It can also be ensured that α_{ijkl}^m has its support in a small neighborhood O of $\partial\Omega$ and $\operatorname{curl} a \neq 0$ in $\overline{O}$. Applying Proposition 13 on this O and extending β_{ijl}^m and γ_{ij}^m by 0 in $\Omega \setminus \overline{O}$, we then find (cf. Proposition 10) $\beta_{ijl}^m \in C^{r+1,s}(\overline{\Omega})$ and $\gamma_{ij}^m \in C^{r,s}(\overline{\Omega})$ so that $L_a(N_a^{\alpha,\beta,\gamma}(w)) = 0$ is verified for every $w \in C^3(\overline{\Omega}; \Lambda^2)$.

Step 2. Observe that, since $\nu \lrcorner c = 0$ on $\partial\Omega$, we have $c = \nu \lrcorner (\nu \wedge c)$ on $\partial\Omega$. We then let $w = \sum w^{ij} dx^i \wedge dx^j$ satisfying (component by component)

$$\begin{cases} \Delta^3 w = 0 & \text{in } \Omega \\ \dfrac{\partial^2 w}{\partial \nu^2} = \nu \wedge c \quad \text{and} \quad \dfrac{\partial w}{\partial \nu} = w = 0 & \text{on } \partial\Omega. \end{cases}$$

The solution w is (cf. Theorems 7.3 and 12.10 in [1]) in $C^\infty(\Omega; \Lambda^2) \cap C^{r+2,s}(\overline{\Omega}; \Lambda^2)$ and there exists $K = K(r, s, \Omega)$ such that

$$\|w\|_{C^{r+2,s}} \leq K \|c\|_{C^{r,s}}.$$

According to Lemma 15 w satisfies on $\partial\Omega$

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \nu \lrcorner \frac{\partial^2 w}{\partial \nu^2} = \nu \lrcorner (\nu \wedge c) = c$$

as wished. $\square$

In the proof of the theorem we used the following lemma.

Lemma 15. *Let $\Omega \subset \mathbb{R}^n$ a bounded open smooth set. Then there exists $\alpha_{ijkl}^m \in C^\infty(\overline{\Omega})$ satisfying*

$$\begin{cases} \alpha_{ijmm}^m = 0 & \forall m \; \forall i < j \\ \alpha_{ijll}^m + \alpha_{ijlm}^l = \alpha_{ijmm}^l + \alpha_{ijlm}^m = 0 & \forall l < m \; \forall i < j \\ \alpha_{ijlm}^k + \alpha_{ijkm}^l + \alpha_{ijkl}^m = 0 & \forall k < l < m \; \forall i < j \end{cases} \tag{11}$$

such that (for any $\beta_{ijl}^m, \gamma_{ij}^m \in C^0(\overline{\Omega})$)

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \nu \lrcorner \frac{\partial^2 w}{\partial \nu^2} \quad on \ \partial\Omega$$

for any $w \in C^{r+2}(\overline{\Omega}; \Lambda^2)$ verifying $w = \partial w / \partial \nu = 0$ on $\partial\Omega$, where ν is the outward unit normal to $\partial\Omega$. Furthermore α_{ijkl}^m can be chosen identically 0 outside an arbitrary small neighborhood of $\partial\Omega$.

Proof. *Step 1.* We first observe that if $\Omega \subset \mathbb{R}^n$ is a bounded open smooth set and $w \in C^2(\overline{\Omega})$ is such that $w = \partial w / \partial \nu = 0$ on $\partial\Omega$, then, for every i, j,

$$\frac{\partial^2 w}{\partial x_i \partial x_j} = \frac{\partial^2 w}{\partial \nu^2} \nu_i \nu_j \quad \text{on } \partial\Omega.$$

Step 2. In the sequel we have extended ν in a $C^\infty(\overline{\Omega}; \mathbb{R}^n)$ way so that it is identically 0 outside an arbitrary small neighborhood of $\partial\Omega$. We define the α_{ijkl}^m as functions of ν implicitly by the equation

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \delta(\nu \lrcorner \delta w) + \text{lower order terms in derivatives of } w.$$

Since $\delta\delta(\nu \lrcorner \delta w) = 0$ for all $w \in C^3(\overline{\Omega}; \mathbb{R}^{n(n-1)/2})$, we obtain that $L_a(N_{a,2}^{\alpha,\beta,\gamma}(w))$ contains only derivatives of w of order less or equal to 2. Therefore, as in the proof of [Proposition 10](#), we must have that [(11)](#) is satisfied. Note that for any $w \in C^{r+2}(\overline{\Omega}; \Lambda^2)$ verifying $w = \partial w / \partial \nu = 0$ on $\partial\Omega$, we have

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \delta \left(\sum_{i,j,k=1}^n \nu_k w_{x_i}^{ij} dx^k \wedge dx^j \right) = \sum_{i,j,k=1}^n \nu_k w_{x_k x_i}^{ij} dx^j \quad \text{on } \partial\Omega.$$

From Step 1 and the fact that $|\nu|^2 = 1$ we get that

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = \sum_{i,j,k=1}^n \nu_i \nu_k^2 \frac{\partial^2 w^{ij}}{\partial \nu^2} dx^j = \nu \lrcorner \frac{\partial^2 w}{\partial \nu^2} \quad \text{on } \partial\Omega.$$

This proves the lemma. $\quad \square$

6. Proof of [Theorems 3 and 5](#)

We now turn to the proof of [Theorem 3](#).

Proof of Theorem 3. *Step 1.* Let us first simplify the problem. We can assume, without loss of generality, that $u_0 = 0$. Indeed it suffices to set $v = u - u_0$ and we find (since $L_a(u_0) \in C^{r,s}(\overline{\Omega})$)

$$\begin{cases} L_a(v) = L_a(u) - L_a(u_0) = f - L_a(u_0) & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

So from now on we will assume that $u_0 = 0$. We next show that we can, without loss of generality, assume that

$$\text{div } a = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle a; \nu \rangle = 0 \quad \text{on } \partial\Omega \tag{12}$$

where ν is the outward unit normal to $\partial\Omega$. Indeed Theorem 6.12 (ii) in [4] implies that we can find $b \in C_N^{r+5,s}(\overline{\Omega})$, $c \in C_N^{r+5,s}(\overline{\Omega}; \Lambda^2)$ and $g \in \mathcal{H}_N(\Omega; \Lambda^1)$ so that $a = db + \delta c + g$ and hence, in particular (recall that $\nu \lrcorner c = 0$ implies $\nu \lrcorner \delta c = 0$),

$$\text{div}(\delta c + g) = \delta(\delta c + g) = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle \delta c + g; \nu \rangle = \nu \lrcorner (\delta c + g) = 0 \quad \text{on } \partial\Omega.$$

Setting $v = e^b u$, we have $v = 0$ on $\partial\Omega$ and $\delta v + (\delta c + g) \lrcorner v = e^b f$. Thus, up to replacing a by $\delta c + g$, f by $e^b f$ and u by $e^b u$, we are lead to solving our problem under the further hypotheses [(12)](#).

Step 2. Consider then the Neumann problem

$$\begin{cases} \Delta v - |a|^2 v = f & \text{in } \Omega \\ \dfrac{\partial v}{\partial \nu} = \langle \operatorname{grad} v; \nu \rangle = 0 & \text{on } \partial \Omega. \end{cases}$$

Since $\operatorname{curl} a \not\equiv 0$ (here we do not need the full hypothesis on $\operatorname{curl} a$), we have that $a \not\equiv 0$. We can then find (cf. [1] or [10]) a solution $v \in C^{r+2,s}(\overline{\Omega})$, without any restriction on f (note that here we only need $a \in C^{r,s}$). Moreover the solution is unique and there exists $K = K(r, s, \|a\|_{C^{r,s}}, \Omega)$ such that

$$\|v\|_{C^{r+2,s}} \leq K \|f\|_{C^{r,s}}.$$

Note that Step 2 is optimal both from the point of view of the regularity of a and for the condition $\operatorname{curl} a \not\equiv 0$.

Step 3. We then apply Theorem 14 to find $\alpha^m_{ijkl} \in C^\infty(\overline{\Omega})$, $\beta^m_{ijl} \in C^{r+2,s}(\overline{\Omega})$, $\gamma^m_{ij} \in C^{r+1,s}(\overline{\Omega})$ satisfying

$$L_a\big(N^{\alpha,\beta,\gamma}_{a,2}(w)\big) = 0, \quad \forall w \in C^3(\overline{\Omega}; \Lambda^2)$$

and $w \in C^{r+3,s}(\overline{\Omega}; \Lambda^2)$ verifying $N^{\alpha,\beta,\gamma}_{a,2}(w) = -\operatorname{grad} v + av$ on $\partial \Omega$. Note that this can be done, since $-\operatorname{grad} v + av \in C^{r+1,s}(\overline{\Omega}; \mathbb{R}^n)$ and $\langle v; -\operatorname{grad} v + av \rangle = 0$. We finally let

$$u = \operatorname{grad} v - av + N^{\alpha,\beta,\gamma}_{a,2}(w).$$

Clearly this is a solution of our problem. Indeed we have $u \in C^{r+1,s}(\overline{\Omega}; \mathbb{R}^n)$ and, by construction, $u = 0$ on $\partial \Omega$. Moreover (since $\operatorname{div} a = 0$) we have, in Ω, that

$$\begin{aligned} L_a(u) &= L_a(\operatorname{grad} v - av) + L_a\big(N^{\alpha,\beta,\gamma}_{a,2}(w)\big) \\ &= \operatorname{div}(\operatorname{grad} v - av) + \langle a; \operatorname{grad} v - av \rangle = \Delta v - |a|^2 v = f. \end{aligned}$$

The estimate easily follows and this concludes the proof of the theorem. □

We now prove Theorem 5.

Proof of Theorem 5. The proof is almost identical to the preceding one.

Step 1. As before we can assume that $u_0 = 0$. We next show that we can, without loss of generality, assume that $a \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$,

$$\operatorname{curl} a = 0 \quad \text{in } \Omega, \qquad \operatorname{div} a = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle a; \nu \rangle = 0 \quad \text{on } \partial \Omega$$

where ν is the outward unit normal to $\partial \Omega$. As before we can find $b \in C^{r+1,s}_N(\overline{\Omega})$, $c \in C^{r+1,s}_N(\overline{\Omega}; \Lambda^2)$ and $g \in \mathcal{H}_N(\Omega; \Lambda^1)$ so that $a = db + \delta c + g$. Observe that in the present case $\delta c \in \mathcal{H}_N(\Omega; \Lambda^1)$ and thus can be taken 0 (by incorporating it into g). Indeed since $da = 0$, we have $d(\delta c) = 0$, $\delta(\delta c) = 0$ and $\nu \lrcorner \delta c = 0$ (since $\nu \lrcorner c = 0$). Hence, in particular,

$$\operatorname{div} g = \delta g = 0 \quad \text{in } \Omega \quad \text{and} \quad \langle g; \nu \rangle = \nu \lrcorner g = 0 \quad \text{on } \partial \Omega.$$

The remaining part of the proof is then as in Step 1 of Theorem 3. Note that now, according to Theorem 6.3 in [4], $a \in C^\infty(\overline{\Omega}; \mathbb{R}^n)$.

Step 2. This step is identical to that of Theorem 3.

Step 3. We next observe that $N^{\alpha,\beta,\gamma}_{a,2}(w) = \delta w + a \lrcorner w$ indeed satisfies

$$L_a\big(N^{\alpha,\beta,\gamma}_{a,2}(w)\big) = 0, \quad \forall w \in C^2(\overline{\Omega}; \Lambda^2).$$

We then invoke Lemma 8.11 (ii) in [4] to find $w \in C^{r+2,s}(\overline{\Omega}; \Lambda^2)$ verifying

$$N^{\alpha,\beta,\gamma}_{a,2}(w) = \delta w + a \lrcorner w = -\operatorname{grad} v + av \quad \text{on } \partial \Omega.$$

Since $-\operatorname{grad} v + av \in C^{r+1,s}(\overline{\Omega}; \mathbb{R}^n)$ and $\langle v; -\operatorname{grad} v + av \rangle = 0$, this can indeed be done. We finally have that a solution is given by

$$u = \operatorname{grad} v - a v + N_{a,2}^{\alpha,\beta,\gamma}(w) = \operatorname{grad} v - a v + \delta w + a \lrcorner w.$$

The remaining part of the proof is as in Theorem 3. $\square$

7. Poincaré type lemma in the interior

The results of the present section are not used elsewhere in the article, but we give them for the sake of completeness. We have already studied the problem on the boundary, cf. Theorem 14. The second case that we want to discuss is the problem in the interior. Given u with $L_a(u) = \operatorname{div} u + \langle a; u \rangle = 0$ in Ω we will find w (and α, β, γ so that $L_a(N_{a,2}^{\alpha,\beta,\gamma}(v)) = 0$ for every v) such that

$$N_{a,2}^{\alpha,\beta,\gamma}(w) = u \quad \text{in } \Omega.$$

This is the exact analogue of finding, provided $\operatorname{div} u = 0$, w with $\operatorname{curl}^* w = u$.

7.1. The case of the standard symplectic form

We start with a global result (for the notations see Example 12).

Theorem 16. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded open smooth set. Let* $a = \bar{a} + dS \in C^\infty(\overline{\Omega}; \Lambda^1)$ *and let* α, β, γ *be defined as in (8). Let* $u \in C^\infty(\overline{\Omega}; \Lambda^1)$ *satisfy* $L_a(u) = 0$. *Then there exists* $w \in C^\infty(\overline{\Omega}; \Lambda^2)$ *verifying*

$$u = N_{a,2}^{\alpha,\beta,\gamma}(w).$$

Proof. *Step 1.* Let us first show that we can assume that $S = 0$. Indeed suppose that we already proved the lemma when $a = \bar{a}$, i.e. for every $U \in C^\infty(\overline{\Omega}; \Lambda^1)$ satisfying $L_{\bar{a}}(U) = 0$, we can find $w \in C^\infty(\overline{\Omega}; \Lambda^2)$ verifying $U = N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w)$. We then set $u = e^{-S} U$ and find that

$$u = e^{-S} N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w) = N_{a,2}^{\alpha,\beta,\gamma}(w).$$

Observe also that a direct computation shows that $L_{\bar{a}}(U) = 0$ if and only if $L_a(u) = 0$. So from now on we will assume that $a = \bar{a}$. Thus we may invoke (6) and (7).

Step 2. It remains to verify that with $a = \bar{a}$, we have the theorem. Namely if $L_{\bar{a}}(u) = 0$ in Ω, we have to find w such that $u = N_{\bar{a}}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w)$ in Ω.

(i) We introduce the following notation $x = (x_1, \cdots, x_n) = (x_1, \widehat{x})$. We first extend u to $\mathbb{R}^n$ in an arbitrary way and then define

$$\overline{w}^{1j} = \int\limits_0^{x_1} \left(\int\limits_0^s e^{t x_2} u^j(t, \widehat{x}) dt \right) ds \quad \text{for } j = 2, \cdots, n.$$

Observe that

$$\overline{w}_{x_1 x_1}^{1j} = e^{x_1 x_2} u^j \quad \text{in } \mathbb{R}^n \text{ for } j = 2, \cdots, n. \tag{13}$$

(ii) We next let

$$g(x) = -e^{x_1 x_2} u^1 - \sum_{j=2}^n \overline{w}_{x_1 x_j}^{1j} + x_1 \overline{w}_{x_1}^{12} - \sum_{i=2}^p x_{2i} \overline{w}_{x_1}^{1(2i-1)} - \overline{w}^{12}.$$

We claim that in Ω the function g is independent of x_1, that is $g(x) = g(\widehat{x})$. Indeed, using (13) and that $L_{\bar{a}}(u) = 0$ in Ω, we get that $g_{x_1} = 0$.

(iii) We finally choose $w^{12} = \overline{w}^{12} + g$ and $w^{1j} = \overline{w}^{1j}$ for $j = 3, \cdots, n$. We therefore obtain from (13) and the definition of w that, in Ω,

$$u^j = e^{-x_1 x_2} \overline{w}_{x_1 x_1}^{1j} = e^{-x_1 x_2} w_{x_1 x_1}^{1j} = \left(N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(w) \right)^j \quad \text{for } j = 2, \cdots, n.$$

It follows from the definition of g and using that $g(x) = g(\widehat{x})$ in Ω, that we also have, in Ω, $u^1 = (N^{\overline{\alpha},\overline{\beta},\overline{\gamma}}_{\overline{a},2}(w))^1$, which is what had to be established. $\quad\square$

7.2. Some intermediate results

We begin with a definition.

Definition 17. Let $A \in \mathbb{R}^{n \times n}$ be a matrix with entries a^i_j, where the upper index i stands for the row and the lower index j stands for the column. Then we define $A^\sharp$ as the matrix

$$A^\sharp = \left\{ \left(A^\sharp\right)^{kl}_{ij} \right\}^{1 \le k \le l \le n}_{1 \le i \le j \le n} \in \mathbb{R}^{\left(\binom{n}{2}+n\right) \times \left(\binom{n}{2}+n\right)}$$

by ordering the indices (k, l) standing for the rows, respectively (i, j) standing for the columns, in lexicographic order and

$$\left(A^\sharp\right)^{kl}_{ij} = a^k_i a^k_j \quad \text{if } k = l \quad \text{and} \quad \left(A^\sharp\right)^{kl}_{ij} = a^k_i a^l_j + a^k_j a^l_i \quad \text{if } k < l.$$

Lemma 18. *The following identities hold true*

$$(AB)^\sharp = A^\sharp B^\sharp, \qquad \left(A^\sharp\right)^{-1} = \left(A^{-1}\right)^\sharp \quad \text{and} \quad \det A^\sharp = (\det A)^{n+1}.$$

Proof. *Step 1.* We start with the first statement. Let $r \le t$ and $p \le q$ be given. We have to show that

$$\left((AB)^\sharp\right)^{rt}_{pq} = \left(A^\sharp B^\sharp\right)^{rt}_{pq}. \tag{14}$$

We discuss only the case $r < t$ (the case $r = t$ being handled similarly). In this case we get

$$\left((AB)^\sharp\right)^{rt}_{pq} = (AB)^r_p (AB)^t_q + (AB)^r_q (AB)^t_p = \sum_{i,j=1}^n a^r_i a^t_j b^i_p b^j_q + \sum_{i,j=1}^n a^r_i a^t_j b^i_q b^j_p. \tag{15}$$

On the other hand we have that the right hand side of (14) is

$$\left(A^\sharp B^\sharp\right)^{rt}_{pq} = \sum_{i<j} \left(a^r_i a^t_j + a^r_j a^t_i\right) \left(b^i_p b^j_q + b^i_q b^j_p\right) + \sum_{i=1}^n \left(a^r_i a^t_i + a^r_i a^t_i\right) b^i_p b^i_q.$$

If we expand the products in this last equation and split the sum in (15) as $\sum_{i,j} = \sum_{i<j} + \sum_{i>j} + \sum_{i=j}$, one easily confirms that (14) holds true.

Step 2. The second statement of the lemma follows from the first statement and the fact that $(I_n)^\sharp = I_{\binom{n}{2}+n}$, where $I_n \in \mathbb{R}^{n \times n}$ is the identity matrix.

Step 3. We next prove the statement on the determinant.

Step 3.1. Let us show that if A is an upper triangular matrix, meaning that $a^p_q = 0$ if $q < p$, then $A^\sharp$ is also an upper triangular matrix. We have to show that for every $i \le j$ and $k \le l$ $(A^\sharp)^{kl}_{ij} = 0$ if $(ij) < (kl)$. In view of the lexicographic ordering, the inequality $(ij) < (kl)$ is equivalent to: either $i < k$ or $\{i = k$ and $j < l\}$.

Case 1: $i < k$. By assumption we get that $a^k_i = 0$. Moreover if $l > k$, then also $a^l_i = 0$. We therefore have

$$\left(A^\sharp\right)^{kl}_{ij} = \begin{cases} a^k_i a^k_j = 0 & \text{if } k = l \\ a^k_i a^l_j + a^k_j a^l_i = 0 & \text{if } k < l. \end{cases}$$

Case 2: $i = k$ and $j < l$. If $k = l$, then we have by assumption that $a^k_j = a^l_j = 0$. Whereas if $k < l$, then $a^l_i = a^l_k = a^l_j = 0$. We can therefore conclude as in Case 1.

Step 3.2. Let us first assume that A is an upper triangular matrix. From Step 1 we therefore obtain

$$\det A^\sharp = \prod_{i \le j} \left(A^\sharp\right)^{ij}_{ij} = \prod_{i<j} \left(A^\sharp\right)^{ij}_{ij} \prod_{i=1}^n \left(A^\sharp\right)^{ii}_{ii} = \prod_{i<j} \left(a^i_i a^j_j\right) \prod_{i=1}^n a^i_i a^i_i.$$

It can be easily shown by induction that $\prod_{i<j} a_i^i a_j^j = (\prod_{i=1}^n a_i^i)^{n-1}$. This leads to

$$\det A^\sharp = \left(\prod_{i=1}^n a_i^i\right)^{n-1} \left(\prod_{i=1}^n a_i^i\right)^2 = \left(\prod_{i=1}^n a_i^i\right)^{n+1} = (\det A)^{n+1}.$$

This shows the lemma for upper triangular matrices. If A is an arbitrary matrix, then $PAP^{-1} = U$, where U is upper triangular. From Step 1 we obtain that $P^\sharp A^\sharp (P^\sharp)^{-1} = U^\sharp$ which leads to $\det A^\sharp = \det U^\sharp = (\det U)^{n+1} = (\det A)^{n+1}$, which was what had to be shown. □

Below we will use the following lemma. Since in the statement of the lemma there is no dependence on a, we have dropped the subindex a in $N_a^{\alpha,\beta,\gamma}$.

Lemma 19. *Let $O, \Omega \subset \mathbb{R}^n$ be two open sets and $\psi \in \mathrm{Diff}^\infty(\Omega; O)$. Then there exists a unique invertible map $\sigma = \sigma_\psi$, such that*

$$\sigma : C^\infty\big(\Omega; \mathbb{R}^{(\binom{n}{2}+n)n} \times \mathbb{R}^{n^2} \times \mathbb{R}^n\big) \to C^\infty\big(O; \mathbb{R}^{(\binom{n}{2}+n)n} \times \mathbb{R}^{n^2} \times \mathbb{R}^n\big),$$

satisfies, for every $f \in C^\infty(O)$ and $(\alpha, \beta, \gamma) \in C^\infty(\Omega; \mathbb{R}^{(\binom{n}{2}+n)n} \times \mathbb{R}^{n^2} \times \mathbb{R}^n)$,

$$*\psi^*\big(*N^{\sigma(\alpha,\beta,\gamma)}(f)\big) = N^{\alpha,\beta,\gamma}(f \circ \psi) \quad \text{in } \Omega. \tag{16}$$

Remark 20. It follows from the proof that σ is linear and can be written in the form

$$\sigma(\alpha, \beta, \gamma)(x) = A_\psi(x) \cdot \big(\alpha\big(\psi^{-1}(x)\big), \beta\big(\psi^{-1}(x)\big), \gamma\big(\psi^{-1}(x)\big)\big),$$

where A_ψ is an invertible matrix of dimension $(\binom{n}{2}+n)n + n^2 + n$ and $\cdot$ denotes the multiplication between matrices and vectors.

Proof of Lemma 19. *Step 1.* Let us write explicitly Eq. (16) in terms of $f(\psi)$, $f_{x_s}(\psi)$ and $f_{x_s x_t}(\psi)$. The right hand side of (16), namely

$$N^{\alpha,\beta,\gamma}\big(f(\psi)\big) = \sum_{m=1}^n \left[\sum_{k \leq l} \alpha_{kl}^m \big(f(\psi)\big)_{x_k x_l} + \sum_{l=1}^n \beta_l^m \big(f(\psi)\big)_{x_l} + \gamma^m f(\psi) \right] dx^m,$$

becomes after a straightforward calculation

$$\big(N^{\alpha,\beta,\gamma}\big(f(\psi)\big)\big)^m = \sum_{s<t} f_{x_s x_t}(\psi) \sum_{k \leq l} \alpha_{kl}^m \big(\psi_{x_l}^s \psi_{x_k}^t + \psi_{x_l}^t \psi_{x_k}^s\big) + \sum_{s=1}^n f_{x_s x_s}(\psi) \sum_{k \leq l} \alpha_{kl}^m \psi_{x_l}^s \psi_{x_k}^s$$

$$+ \sum_{s=1}^n f_{x_s}(\psi) \left(\sum_{k \leq l} \alpha_{kl}^m \psi_{x_l x_k}^s + \sum_{l=1}^n \beta_l^m \psi_{x_l}^s \right) + \gamma^m f(\psi). \tag{17}$$

Using the notation $\sigma(\alpha, \beta, \gamma) = (\overline{\alpha}, \overline{\beta}, \overline{\gamma})$, we get by a direct calculation that the left hand side of (16) is of the form

$$\big(*\psi^*\big(*N^{\overline{\alpha},\overline{\beta},\overline{\gamma}}(f)\big)\big)^m = \sum_{s \leq t} A_{st}^m f_{x_s x_t}(\psi) + \sum_{s=1}^n B_s^m f_{x_s}(\psi) + C^m f(\psi),$$

where A_{st}^m, B_s^m and C^m are abbreviations for

$$A_{st}^m = \sum_{q=1}^n (-1)^{q-1+n-m} \overline{\alpha}_{st}^q(\psi) \det(\nabla\psi)_{1\cdots\widehat{m}\cdots n}^{1\cdots\widehat{q}\cdots n}$$

$$B_s^m = \sum_{q=1}^n (-1)^{q-1+n-m} \overline{\beta}_s^q(\psi) \det(\nabla\psi)_{1\cdots\widehat{m}\cdots n}^{1\cdots\widehat{q}\cdots n}$$

$$C^m = \sum_{q=1}^{n} (-1)^{q-1+n-m} \overline{\gamma}^q(\psi) \det(\nabla\psi)_{1\cdots\widehat{m}\cdots n}^{1\cdots\widehat{q}\cdots n},$$

where $1\cdots\widehat{q}\cdots n$ means that the row q has been omitted in $\nabla\psi$, and $1\cdots\widehat{m}\cdots n$ means that the column m has been omitted. In view of (17), Eq. (16) is valid for every f if and only if the following set of equations is satisfied for every $m = 1, \cdots, n$

$$
\begin{cases}
\displaystyle\sum_{k\leq l} \alpha_{kl}^m (\psi_{x_l}^s \psi_{x_k}^t + \psi_{x_l}^t \psi_{x_k}^s) = A_{st}^m & \text{for every } s < t \\[2mm]
\displaystyle\sum_{k\leq l} \alpha_{kl}^m \psi_{x_l}^s \psi_{x_k}^s = A_{ss}^m & \text{for every } s = 1, \cdots, n \\[2mm]
\displaystyle\sum_{k\leq l} \alpha_{kl}^m \psi_{x_k x_l}^s + \sum_{l=1}^{n} \beta_l^m \psi_{x_l}^s = B_s^m & \text{for every } s = 1, \cdots, n \\[2mm]
\gamma^m = C^m
\end{cases}
\tag{18}
$$

Step 2. In view of Step 1, we have to show that the linear system of equations (18) can be solved uniquely for $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$ in terms of (α, β, γ), and conversely.

Step 2.1. Let us first show that we can solve (18) for $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$. Note that entries of the adjoint matrix $\mathrm{adj}(\nabla\psi)$ of $\nabla\psi$ are precisely given by

$$(\mathrm{adj}(\nabla\psi))_m^q = (-1)^{q-m} \det(\nabla\psi)_{1\cdots\widehat{m}\cdots n}^{1\cdots\widehat{q}\cdots n} \quad \text{for every } q, m = 1, \cdots, n.$$

Moreover let us denote, for $s < t$, $\overline{\alpha}_{st} = (\overline{\alpha}_{st}^1, \cdots, \overline{\alpha}_{st}^n)$ and $A_{st} = (A_{st}^1, \cdots, A_{st}^n)$. From the definition of A_{st}^m we therefore obtain that

$$A_{st} = (-1)^{n-1} (\mathrm{adj}(\nabla\psi))^t \overline{\alpha}_{st} = (-1)^{n-1} \det \nabla\psi (\nabla\psi)^{-t} \overline{\alpha}_{st}(\psi).$$

We therefore fix $s < t$ and obtain from the first system of equations in (18), that

$$\overline{\alpha}_{st}(\psi) = \frac{(-1)^{n-1}}{\det\nabla\psi} (\nabla\psi)^t A_{st} = \frac{(-1)^{n-1}}{\det\nabla\psi} (\nabla\psi)^t \left\{ \sum_{k\leq l} \alpha_{kl}^m (\psi_{x_l}^s \psi_{x_k}^t + \psi_{x_l}^t \psi_{x_k}^s) \right\}_{m=1,\cdots,n}.$$

We can proceed exactly in the same way to solve for $\overline{\alpha}_{ss}$, $\overline{\beta}$ and γ. This proves the existence of σ.

Step 2.2. Let us now show that the system (18) can be solved for (α, β, γ) in terms of $(\overline{\alpha}, \overline{\beta}, \overline{\gamma})$. Let us denote $\tau = \sigma^{-1}$ and write $\tau = \{(\tau_1^m, \tau_2^m, \tau_3^m)\}_{m=1,\cdots,n}$, meaning that $\tau_1^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = \alpha^m$, $\tau_2^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = \beta^m$ and $\tau_3^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = \gamma^m$. From the last equation in (18) we immediately get that $\gamma^m = \tau_3^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma}) = \tau_3^m(\overline{\gamma}) = C^m(\overline{\gamma})$ is well defined. It remains to show that one can solve the first three equations in (18) for α and β. Note that for each fixed $m \in \{1, \cdots, n\}$ there are exactly $\binom{n}{2} + 2n$ unknowns $(\alpha_{kl}^m, \beta_l^m)$. This is also exactly the number of linear equations. Thus one can write the first three equations of (18) in matrix form with some square matrix $M_\psi = M(\nabla\psi, \nabla^2\psi)$ as $M_\psi \cdot (\alpha^m, \beta^m) = G^m(\overline{\alpha}(\psi), \overline{\beta}(\psi), \nabla\psi)$, for some vector G^m with $\binom{n}{2} + 2n$ entries depending on A_{st}^m and B_s^m. We will show that M_ψ is invertible. Using the lexicographic order to enumerate the $\binom{n}{2} + n$ entries of α^m and then the n entries of β^m, one observes that M can be written in block-matrix form

$$M_\psi = \left(\begin{array}{c|c} A & 0 \\ \hline B & C \end{array} \right),$$

where A, C are square matrices and (see Definition 17) $A = (\nabla\psi)^\sharp$, $C = \nabla\psi$ and $B = B(\nabla^2\psi)$ is a function of the second derivatives of ψ. From the statement on the determinant in Lemma 18 we have $\det A = \det(\nabla\psi)^\sharp = (\det\nabla\psi)^{n+1}$. We thus get that $\det M_\psi = (\det\nabla\psi)^{n+2}$. We can therefore define τ_1^m and τ_2^m as

$$(\alpha^m, \beta^m) = (\tau_1^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma}), \tau_2^m(\overline{\alpha}, \overline{\beta}, \overline{\gamma})) = M_\psi^{-1} \cdot (G^m(\overline{\alpha}(\psi), \overline{\beta}(\psi), \nabla\psi)).$$

This shows the existence of τ. By construction, we obviously have that $\tau = \sigma^{-1}$. The lemma is therefore proved. $\quad\square$

We use below the following notation $a_n = \binom{n}{2}(\binom{n}{2} + n)n$, $b_n = \binom{n}{2}n^2$ and $c_n = \binom{n}{2}n$.

Proposition 21. *Let $O, \Omega \subset \mathbb{R}^n$ be open, $\psi \in \mathrm{Diff}^\infty(\Omega; O)$, $\bar{a} \in C^\infty(O; \Lambda^1)$ and $a = \psi^*(\bar{a})$. Then there exists $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) \in C^\infty(O; \mathbb{R}^{a_n} \times \mathbb{R}^{b_n} \times \mathbb{R}^{c_n})$ such that $\mathrm{Ker}\, L_{\bar{a}} = \mathrm{Range}\, N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}$ if and only if there exists $(\alpha, \beta, \gamma) \in C^\infty(\Omega; \mathbb{R}^{a_n} \times \mathbb{R}^{b_n} \times \mathbb{R}^{c_n})$ such that $\mathrm{Ker}\, L_a = \mathrm{Range}\, N_{a,2}^{\alpha,\beta,\gamma}$.*

Proof. We assume without loss of generality that (α, β, γ) exists such that $\mathrm{Ker}\, L_a = \mathrm{Range}\, N_{a,2}^{\alpha,\beta,\gamma}$. We have to show the existence of $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ such that $\mathrm{Ker}\, L_{\bar{a}} = \mathrm{Range}\, N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}$. The reverse direction follows in the same way, by repeating the argument with ψ^{-1} instead of ψ.

Step 1. From Lemma 19, we know that for every $1 \le i < j \le n$ there exists a map σ_{ij} such that $\sigma_{ij} : C^\infty(\Omega; \mathbb{R}^{(\binom{n}{2}+n)n} \times \mathbb{R}^{n^2} \times \mathbb{R}^n) \to C^\infty(O; \mathbb{R}^{(\binom{n}{2}+n)n} \times \mathbb{R}^{n^2} \times \mathbb{R}^n)$ satisfies, setting $(\bar{\alpha}_{ij}, \bar{\beta}_{ij}, \bar{\gamma}_{ij}) = \sigma_{ij}(\alpha_{ij}, \beta_{ij}, \gamma_{ij})$,

$$*\psi^*\!\left(*N_{\bar{a}}^{\bar{\alpha}_{ij},\bar{\beta}_{ij},\bar{\gamma}_{ij}}(f)\right) = N_a^{\alpha_{ij},\beta_{ij},\gamma_{ij}}(f \circ \psi), \quad \text{for every } f \in C^\infty(O). \tag{19}$$

We define $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = \{\bar{\alpha}_{ij}, \bar{\beta}_{ij}, \bar{\gamma}_{ij}\}_{1 \le i < j \le n}$. Let $\overline{w} = \sum \overline{w}^{ij} dx^i \wedge dx^j \in C^\infty$, when we write $\overline{w} \circ \psi$ we mean that $\overline{w} \circ \psi(x) = \sum \overline{w}^{ij}(\psi(x)) dx^i \wedge dx^j$. It follows from (19) and the definition of $N_{a,2}^{\alpha,\beta,\gamma}$, respectively $N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}$, that

$$*\psi^*\!\left(*N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(\overline{w})\right) = N_{a,2}^{\alpha,\beta,\gamma}(\overline{w} \circ \psi) \quad \text{for every } \overline{w} \in C^\infty(O; \Lambda^2). \tag{20}$$

Step 2. We now show that with the above choice of $(\bar{\alpha}, \bar{\beta}, \bar{\gamma})$ we have $\mathrm{Ker}\, L_{\bar{a}} = \mathrm{Range}\, N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}$. We start with one of the inclusions. So let $u \in \mathrm{Ker}\, L_{\bar{a}}$. Define then v by $v = *\psi^*(*u) \in C^\infty(\Omega; \Lambda^1)$. By Lemma 7 we have that $v \in \mathrm{Ker}\, L_{\psi^*(\bar{a})} = \mathrm{Ker}\, L_a$ and so by assumption there exists a $w \in C^\infty(\Omega; \Lambda^2)$ such that $v = N_{a,2}^{\alpha,\beta,\gamma}(w)$. Define $\overline{w} = w \circ \psi^{-1} \in C^\infty(O; \Lambda^2)$, this implies that $v = N_{a,2}^{\alpha,\beta,\gamma}(\overline{w} \circ \psi)$. We then get from (20) that $v = N_{a,2}^{\alpha,\beta,\gamma}(\overline{w} \circ \psi) = *\psi^*(*N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(\overline{w}))$, which implies that $u = N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(\overline{w})$ and thus shows that $u \in \mathrm{Range}\, N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}$. We next prove the other inclusion and let $u \in \mathrm{Range}\, N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}$. Thus there exists $\overline{w} \in C^\infty(O; \Lambda^2)$ such that $u = N_{\bar{a},2}^{\bar{\alpha},\bar{\beta},\bar{\gamma}}(\overline{w})$. Define then $v = *\psi^*(*u) \in C^\infty(\Omega; \Lambda^1)$. We obtain from (20) that $v = N_{a,2}^{\alpha,\beta,\gamma}(\overline{w} \circ \psi)$. Therefore $v \in \mathrm{Range}\, N_{a,2}^{\alpha,\beta,\gamma}$ and it follows from the assumption that $L_a(v) = L_{\psi^*(\bar{a})}(v) = 0$. Lemma 7 implies then that $L_{\bar{a}}(u) = 0$ which proves the proposition. $\square$

7.3. The local theorem

We now obtain a local result (in Theorem 16 we obtained a global result for the particular vector field $\bar{a}$ or $\bar{a} + dS$) for a general vector field a (recall that da is identified with $\mathrm{curl}\, a$).

Theorem 22. *Let $x_0 \in \mathbb{R}^n$, $0 \le 2p \le n$ and, in a neighborhood of x_0, $a \in C^\infty$ with $\mathrm{rank}[da] = 2p$. Then there exist $\alpha, \beta, \gamma \in C^\infty$ such that, in a neighborhood of x_0,*

$$L_a\!\left(N_{a,2}^{\alpha,\beta,\gamma}(w)\right) = 0, \quad \forall w \in C^3.$$

If furthermore $u \in C^\infty$ satisfies $L_a(u) = 0$, then there exists $w \in C^\infty$ such that $u = N_{a,2}^{\alpha,\beta,\gamma}(w)$, in a neighborhood of x_0, or equivalently $\mathrm{Ker}\, L_a = \mathrm{Range}\, N_{a,2}^{\alpha,\beta,\gamma}$.

Proof. From Theorem 13.6 in [4], we find a diffeomorphism ψ such that $\psi^*(a) = \bar{a} + dS$. The theorem is then a consequence of Theorem 16 and Proposition 21. $\square$

Conflict of interest statement

The authors declare there is no conflict of interest.

References

[1] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I, Commun. Pure Appl. Math. 12 (1959) 623–727.

[2] M.E. Bogovski, Solution of the first boundary value problem for the equation of continuity of an incompressible medium, Sov. Math. Dokl. 20 (1979) 1094–1098.
[3] W. Borchers, H. Sohr, On the equations rot $v = g$ and div $u = f$ with zero boundary conditions, Hokkaido Math. J. 19 (1990) 67–87.
[4] G. Csató, B. Dacorogna, O. Kneuss, The Pullback Equation for Differential Forms, Birkhäuser/Springer, New York, 2012.
[5] B. Dacorogna, Existence and regularity of solutions of $dw = f$ with Dirichlet boundary conditions, in: Nonlinear Problems in Mathematical Physics and Related Topics, Kluwer/Plenum, New York, 2002, pp. 67–82.
[6] B. Dacorogna, Direct Methods in the Calculus of Variations, Springer, 2007.
[7] B. Dacorogna, J. Moser, On a partial differential equation involving the Jacobian determinant, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 7 (1990) 1–26.
[8] R. Dautray, J.L. Lions, Analyse mathématique et calcul numérique, Masson, Paris, 1988.
[9] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations, Springer, New York, 1994.
[10] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, Berlin, 1977.
[11] V. Girault, P.A. Raviart, Finite Element Approximation of the Navier–Stokes Equations, Lect. Notes Math., vol. 749, Springer, Berlin, 1979.
[12] L.V. Kapitanskii, K. Pileckas, Certain problems of vector analysis, J. Sov. Math. 32 (1986) 469–483.
[13] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Gordon and Breach, New York, 1969.
[14] O.A. Ladyzhenskaya, V.A. Solonnikov, Some problems of vector analysis and generalized formulations of boundary value problems for the Navier–Stokes equations, J. Sov. Math. 10 (1978) 257–286.
[15] J. Necas, Les méthodes directes en théorie des équations elliptiques, Masson, Paris, 1967.
[16] L. Tartar, Topics in Nonlinear Analysis, Univ. of Wisconsin, Madison, 1975.
[17] W. Von Wahl, On necessary and sufficient conditions for the solvability of the equations rot $u = \gamma$ and div $u = \epsilon$ with u vanishing on the boundary, in: Lect. Notes Math., vol. 1431, Springer, Berlin, 1990, pp. 152–157.
[18] W. Von Wahl, Vorlesung über das Aussenraumproblem für die instationären Gleichungen von Navier–Stokes, Bonn, 1989.