Available online at www.sciencedirect.com
SCIENCE DIRECT®
Ann. I. H. Poincaré – AN 22 (2005) 187–206
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

# Nonlinear eigenvalues and bifurcation problems for Pucci's operators ☆

Jérôme Busca [a], Maria J. Esteban [a,*], Alexander Quaas [b]

[a] Ceremade UMR CNRS 7534, Université Paris IX–Dauphine, 75775 Paris Cedex 16, France
[b] Departamento de Matemática, Universidad Santa María, Casilla: V-110, Avda. Espanã 1680, Valparaíso, Chile

**Abstract**

In this paper we extend existing results concerning generalized eigenvalues of Pucci's extremal operators. In the radial case, we also give a complete description of their spectrum, together with an equivalent of Rabinowitz's Global Bifurcation Theorem. This allows us to solve nonlinear equations involving Pucci's operators.


*MSC:* primary 35J60, 35P05, 35B32; secondary 34B18

## 1. Introduction

If the solvability of fully nonlinear elliptic equations of the form

$$F(x, u, Du, D^2u) = 0 \tag{1.1}$$

has been extensively investigated for *coercive* uniformly elliptic operators $F$, comparatively little is known when the assumption on coercivity (that is, monotonicity in $u$) is dropped. In this paper, we want to focus on the model problem

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

☆ This work was partially supported by ECOS grant No. C02E08. The third author is supported by Fondecyt Grant #1040794.
* Corresponding author.
*E-mail addresses:* jerome.busca@normalesup.org (J. Busca), esteban@ceremade.dauphine.fr (M.J. Esteban), alexander.quaas@usm.cl (A. Quaas).

0294-1449/$ – see front matter 
doi:10.1016/j.anihpc.2004.05.004

(resp. $\mathcal{M}^-_{\lambda,\Lambda}$) where $\Omega$ is a bounded regular domain, and $\mathcal{M}^\pm_{\lambda,\Lambda}$ are the extremal Pucci's operators [28] with parameters $0 < \lambda \leqslant \Lambda$ defined by

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \Lambda \sum_{e_i>0} e_i + \lambda \sum_{e_i<0} e_i$$

and

$$\mathcal{M}^-_{\lambda,\Lambda}(M) = \lambda \sum_{e_i>0} e_i + \Lambda \sum_{e_i<0} e_i,$$

for any symmetric $N \times N$ matrix $M$; here $e_i = e_i(M)$, $i = 1, \ldots, N$, denote the eigenvalues of $M$. We intend to study (1.2) as a bifurcation problem from the trivial solution. Since $\mathcal{M}^\pm_{\lambda,\Lambda}$ are homogeneous of degree one, it is natural to investigate the associated "eigenvalue problem"

$$\begin{cases} -\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

(resp. $\mathcal{M}^-_{\lambda,\Lambda}$) Pucci's extremal operators appear in the context of stochastic control when the diffusion coefficient is a control variable, see the book of Bensoussan and J.L. Lions [2] or the papers of P.L. Lions [22–24] for the relation between a general Hamilton–Jacobi–Bellman and stochastic control. They also provide natural extremal equations in the sense that if $F$ in (1.1) is uniformly elliptic, with ellipticity constants $\lambda$, $\Lambda$, and depends only on the Hessian $D^2u$, then

$$\mathcal{M}^-_{\lambda,\Lambda}(M) \leqslant F(M) \leqslant \mathcal{M}^+_{\lambda,\Lambda}(M) \tag{1.4}$$

for any symmetric matrix $M$.

When $\lambda = \Lambda = 1$, $\mathcal{M}^\pm_{\lambda,\Lambda}$ coincide with the Laplace operator, so that (1.2) reads

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

whereas (1.3) simply reduces to

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

It is a very well known fact that there exists a sequence of solutions

$$\{(\mu_n, \varphi_n)\}_{n\geqslant 1}$$

to (1.6) such that:

(i) the eigenvalues $\{\mu_n\}_{n\geqslant 1}$ are real, with $\mu_n > 0$ and $\mu_n \to \infty$ as $n \to \infty$;
(ii) the set of all eigenfunctions $\{\varphi_n\}_{n\geqslant 1}$ is a basis of $L^2(\Omega)$.

Building on these eigenvalues, the classical Rabinowitz bifurcation theory [32,33] then provides a comprehensive answer to the existence of solutions of (1.5).

When $\lambda < \Lambda$, problems (1.2), (1.3) are fully nonlinear. It is our purpose to investigate to which extent the results about the Laplace operator can be generalized to this context. A few partial results in this direction have been established in the recent years and will be recalled shortly. However, they are all concerned with the first eigenvalue and special nonlinearities $f$. We provide here a bifurcation result for general nonlinearities from the first two "half-eigenvalues" in general bounded domains. And in the radial case a complete description of the spectrum and the bifurcation branches for a general nonlinearity from any point in the spectrum.

Let us mention that besides the fact that (1.2)–(1.3) appears to be a favorable case from which one might hope to address general problems like (1.1), there are other reasons why one should be interested in Pucci's extremal operators or, more generally, in Hamilton–Jacobi–Bellman operators, which are envelopes of linear operators. As a matter of fact, the problem under study has some relation to the Fučík spectrum. To explain this, let $u$ be a solution of the following problem

$$-\Delta u = \mu u^+ - \alpha\mu u^-,$$

where $\alpha$ is a fixed positive number. One easily checks that if $\alpha \geqslant 1$, then $u$ satisfies

$$\max\left\{-\Delta u, \frac{-1}{\alpha}\Delta u\right\} = \mu u,$$

whereas if $\alpha \leqslant 1$, $u$ satisfies

$$\min\left\{-\Delta u, \frac{-1}{\alpha}\Delta u\right\} = \mu u.$$

These relations mean that the Fučík spectrum can be seen as the spectrum of the maximum or minimum of two linear operators, whereas (1.2), (1.3) deal with an infinite family of operators.

We observe that understanding all the "spectrum" of the above problem is essentially the same as determining the Fučík spectrum, which in dimension $N \geqslant 2$ is still largely an open question, for which only partial results are known and, in general, they refer to a region near the usual spectrum, (that is for $\alpha$ near 1). For a further discussion on this topic, we refer the interested reader to the works of de Figueiredo and Gossez [17], H. Berestycki [3], E.N. Dancer [10], S. Fučík [18], P. Drábek [13], T. Gallouet and O. Kavian [19], M. Schechter [36] and the references therein.

Our first result deals with the existence and characterizations of the two first "half-eigenvalues". Some parts of it are already known (see below), but some are new.

**Proposition 1.1.** *Let $\Omega$ be a regular domain. There exist two positive constants $\mu_1^+$, $\mu_1^-$, that we call first half-eigenvalues such that*:

(i) *There exist two functions $\varphi_1^+, \varphi_1^- \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $(\mu_1^+, \varphi_1^+)$, $(\mu_1^-, \varphi_1^-)$ are solutions to* (1.3) *and $\varphi_1^+ > 0, \varphi_1^- < 0$ in $\Omega$. Moreover, these two half-eigenvalues are simple, that is, all positive solutions to* (1.3) *are of the form $(\mu_1^+, \alpha\varphi_1^+)$, with $\alpha > 0$. The same holds for the negative solution.*

(ii) *The two first half-eigenvalues satisfy*

$$\mu_1^+ = \inf_{A\in\mathcal{A}} \mu_1(A), \qquad \mu_1^- = \sup_{A\in\mathcal{A}} \mu_1(A),$$

*where $\mathcal{A}$ is the set of all symmetric measurable matrices such that $0 < \lambda I \leqslant A(x) \leqslant \Lambda I$ and $\mu_1(A)$ is the principal eigenvalue of the nondivergent second order linear elliptic operator associated to $A$.*

(iii) *The two half-eigenvalues have the following characterization*

$$\mu_1^+ = \sup_{u>0} \operatorname*{essinf}_{\Omega}\left(-\frac{\mathcal{M}^+_{\lambda,\Lambda}(D^2u)}{u}\right), \qquad \mu_1^- = \sup_{u<0} \operatorname*{essinf}_{\Omega}\left(-\frac{\mathcal{M}^+_{\lambda,\Lambda}(D^2u)}{u}\right).$$

*The supremum is taken over all functions $u \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$.*

(iv) *The first half-eigenvalues can be also characterized by*

$$\mu_1^+ = \sup\{\mu \mid \text{there exists } \phi > 0 \text{ in } \Omega \text{ satisfying } \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + \mu\phi \leqslant 0\},$$
$$\mu_1^- = \sup\{\mu \mid \text{there exists } \phi < 0 \text{ in } \Omega \text{ satisfying } \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + \mu\phi \geqslant 0\}.$$

**Remark 1.1.** Here and in the sequel, unless otherwise stated, it is implicitly understood that any solution (resp. sub-, super-solution) satisfies the corresponding equation (inequation) pointwise a.e. This is the framework of *strong solutions* [20].

The above existence result, that is part (i) of Proposition 1.1, can been easily proved using an adaptation, for convex (or concave) operators, of Krein–Rutman's Theorem in positive cones (see [16] in the radial symmetric case and see [30] in regular bounded domains).

This existence result, has been proved recently in the case of general positive homogeneous fully nonlinear elliptic operators, see the paper of Rouy [34]. The method used there is due to P.L. Lions who proved the result (i) of Proposition 1.1 for the Bellman operator (see [21]) and for the Monge-Ampère operator (see [26]). Moreover, the definition of $\mu_1^+$ there translates in our case as:

$$\mu_1^+ = \sup\{\mu \mid \mu \in \mathcal{I}\}, \tag{1.7}$$

where

$$\mathcal{I} = \left\{\mu \mid \exists\phi > 0 \text{ s.t. } \phi = 0 \text{ on } \partial\Omega,\ \mathcal{M}^+_{\lambda,\Lambda}(D^2\phi) + \mu\phi = -1 \text{ in } \Omega\right\}.$$

Properties (ii) of Proposition 1.1 can be generalized to any fully nonlinear elliptic operator $F$ that is positively homogeneous of degree one, with ellipticity constants $\lambda, \Lambda$. This follows by the proof of (ii) and (1.4). These properties were established by C. Pucci in [29], for other kind of extremal operators, see the comments in Section 2.

The characterization of the form (iii) and (iv) for the first eigenvalue, were introduced by Berestycki, Nirenberg and Varadhan for second order linear elliptic operators (see [5]).

From the characterization iv) it follows that

$$\mu_1^+(\Omega) \leqslant \mu_1^+(\Omega') \quad \text{and} \quad \mu_1^-(\Omega) \leqslant \mu_1^-(\Omega') \quad \text{if} \quad \Omega' \subset \Omega.$$

For the two first half-eigenvalues, many other properties will be deduced from the previous proposition (see Section 2). For example, whenever $\lambda \neq \Lambda$, we have $\mu_1^+ < \mu_1^-$, since $\mu_1^+ \leqslant \lambda\mu_1(-\Delta) \leqslant \Lambda\mu_1(-\Delta) \leqslant \mu_1^-$.

Another interesting and useful property is the following maximum principle.

**Theorem 1.1.** *The next two maximum principles hold:*

(a) *Let* $u \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ *satisfy*

$$\begin{aligned} &\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu u \geqslant 0 \quad \text{in } \Omega,\\ &u \leqslant 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.8}$$

*If* $\mu < \mu_1^+$, *then* $u \leqslant 0$ *in* $\Omega$.

(b) *Let* $u \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ *satisfy*

$$\begin{aligned} &\mathcal{M}^+_{\lambda,\Lambda}(D^2u) + \mu u \leqslant 0 \quad \text{in } \Omega,\\ &u \geqslant 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.9}$$

*If* $\mu < \mu_1^-$, *then* $u \geqslant 0$ *in* $\Omega$.

**Remark 1.2.** These maximum principles are still valid for continuous solutions in $\overline{\Omega}$ that satisfy the respective inequalities in the viscosity sense (see [8]).

Results like Proposition 1.1 and Theorem 1.1 can be obtained for $\mathcal{M}^-_{\lambda,\Lambda}$ and can be deduced just by noting that $\mathcal{M}^+_{\lambda,\Lambda}(-M) = -\mathcal{M}^-_{\lambda,\Lambda}(M)$, for any symmetric matrix $M$.

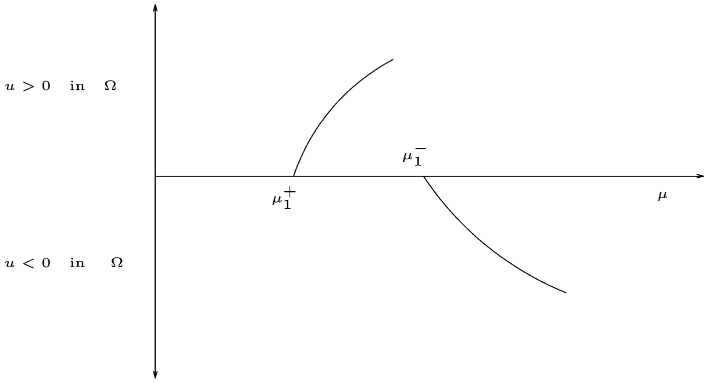


Fig. 1. Bifurcation diagram for the first half-eigenvalues in a general bounded domain.

Next, we want to look at the higher eigenvalues of Pucci's extremal operators. For that purpose we restrict ourselves to the radial case. In this case we have a precise description of the whole "spectrum" and we expect that the result below will shed some light on the general case. More precisely, we have the following theorem.

**Theorem 1.2.** *Let $\Omega = B_1$. The set of all the scalars $\mu$ such that* (1.3) *admits a nontrivial radial solution, consists of two unbounded increasing sequences*

$$0 < \mu_1^+ < \mu_2^+ < \cdots < \mu_k^+ < \cdots,$$
$$0 < \mu_1^- < \mu_2^- < \cdots < \mu_k^- < \cdots.$$

*Moreover, the set of radial solutions of* (1.3) *for $\mu = \mu_k^+$ is positively spanned by a function $\varphi_k^+$, which is positive at the origin and has exactly $k-1$ zeros in $(0,1)$, all these zeros being simple. The same holds for $\mu = \mu_k^-$, but considering $\varphi_k^-$ negative at the origin.*

Finally, we want to address our original motivation, that is, we want to prove existence results for an equation of the type (1.2). For this purpose we consider the nonlinear bifurcation problem associated with the extremal Pucci's operator, that is

$$\begin{aligned} &-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \mu u + f(u,\mu) \quad \text{in } \Omega, \\ &u = 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.10}$$

where $f$ is continuous, $f(s,\mu) = o(|s|)$ near $s = 0$, uniformly for $\mu \in \mathbb{R}$ and $\Omega$ is a general bounded domain. Concerning this problem we have the following theorem

**Theorem 1.3.** *The pair $(\mu_1^+, 0)$ (resp. $(\mu_1^-, 0)$) is a bifurcation point of positive (resp. negative) solutions to* (1.10). *Moreover, the set of nontrivial solutions of* (1.10) *whose closure contains $(\mu_1^+, 0)$ (resp. $(\mu_1^-, 0)$), is either unbounded or contains a pair $(\bar\mu, 0)$ for some $\bar\mu$, eigenvalue of* (1.3) *with $\bar\mu \neq \mu_1^+$ (resp. $\bar\mu \neq \mu_1^-$).*

Notice that a similar theorem can be proved in the case of $\mathcal{M}^-_{\lambda,\Lambda}$. The difference with Theorem 1.3 is that $(\mu_1^+, 0)$ will be a bifurcation point for the negative solutions and $(\mu_1^-, 0)$ will be a bifurcation point for the positive solutions.

**Remark 1.3.** Fig. 1 allows to visualize the above result in which the bifurcation generates only "half-branches": $u > 0$ or $u < 0$ in $\Omega$.

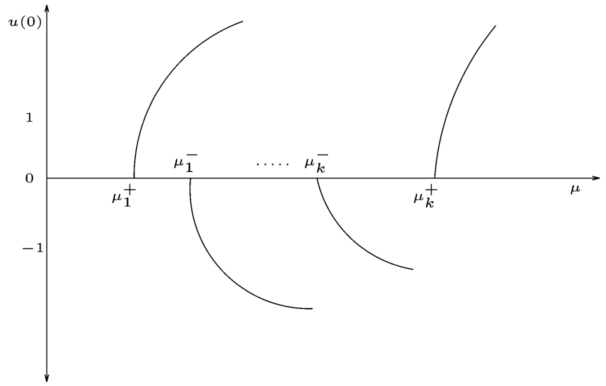


Fig. 2. Bifurcation diagram in the radially symmetric case (note that $\mu_1^+ \leqslant \mu_1^-$, but for $k \geqslant 2$ the ordering between $\mu_k^+$ and $\mu_k^-$ is not known).

For the Laplacian the result is well known, see [32,33,31]. In this case the "half-branches" become connected. Therefore, we observe a symmetry breaking phenomena when $\lambda < \Lambda$.

For the $p$-Laplacian the result is known, in the general case, see the paper of del Pino and Manásevich [12]. See also the paper of del Pino, Elgueta and Manásevich [11], for the case $N = 1$. In this case the branches are also connected. The proof of these results uses an invariance under homotopy with respect to $p$ for the Leray–Schauder degree. In our proof of Theorem 1.3 we use instead homotopy invariance with respect to $\lambda$ (the ellipticity constant), having to deal with a delicate region in which the degree is equal to zero.

A bifurcation result in the particular case $f(u,\mu) = -\mu|u|^{p-1}u$ can be found in the paper by P.L. Lions for the Bellman equation [21]. For the problem

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2u) = \mu g(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega$$

with the following assumption on $g$:

(i) $u \to g(x,u)$ is nondecreasing and $g(x,0) = 0$,
(ii) $u \to \frac{g(x,u)}{u}$ decreasing, and
(iii) $\lim_{u\to 0} \frac{g(x,u)}{u} = 1$, $\lim_{u\to\infty} \frac{g(x,u)}{u} = 0$

a similar result was proved by E. Rouy [34].

In [21] and [34] the assumptions made were used in a crucial way to construct sub and super solutions. By contrast, we use a Leray–Schauder degree argument which allows us to treat general nonlinearities.

Other kind of existence results for positive solution of (1.2), can be found in [15,14,16] and [30].

In the radially symmetric case we obtain a more complete result.

**Theorem 1.4.** *Let $\Omega = B_1$. For each $k \in \mathbb{N}$, $k \geqslant 1$ there are two connected components $S_k^\pm$ of nontrivial solutions to* (1.10)*, whose closures contains $(\mu_k^\pm, 0)$. Moreover, $S_k^\pm$ are unbounded and $(\mu, u) \in S_k^\pm$ implies that u possesses exactly $k-1$ zeros in* $(0,1)$.

**Remark 1.4.** (1) $S_k^+$ (resp. $S_k^-$) denotes the set of solutions that are positive (resp. negative) at the origin.

(2) Fig. 2, allows to visualize the above result in which the bifurcation generates only "half-branches": $u(0) > 0$ or $u(0) < 0$.

For the Laplacian this result is well known. In this case, for all $k \geqslant 1$, $\mu_k^+ = \mu_k^-$ and the "half-branches" now become connected.

Our proof is based on the invariance of the Leray–Schauder degree under homotopy. It also uses some non-existence results.

The paper is organized as follows. In Section 2 we study the problem in a general regular bounded domain, there we prove Theorems 1.1 and 1.3. In section 3 we study the radial symmetric case, and we prove Theorems 1.2 and 1.4.

## 2. First "eigenvalues" in a general domain and nonlinear bifurcation

We shall need the following version of Hopf's boundary lemma.

**Lemma 2.1.** *Let $\Omega$ be a regular domain and let $u \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ be a non-negative solution to*

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) + \mu u \leqslant 0 \quad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{2.11}$$

*with $\mu \in \mathbb{R}$. Then $u(x) > 0$ for all $x \in \Omega$. Moreover,*

$$\limsup_{x \to x_0} \frac{u(x_0) - u(x)}{|x - x_0|} < 0,$$

*where $x_0 \in \partial\Omega$ and the limit is non-tangential, that is, taken over the set of $x$ for which the angle between $x - x_0$ and the outer normal at $x_0$ is less than $\pi/2 - \delta$ for some fixed $\delta > 0$.*

**Remark 2.1.** (1) For a general strong maximum principle for degenerate convex elliptic operators, see the paper of M. Bardi, F. Da Lio [1].

(2) This lemma holds also for $u \in C(\overline{\Omega})$ that satisfies Eq. (2.11) in the viscosity sense.

(3) A solution of (1.3) in a regular domain is necessarily $C^{2,\alpha}$ up to the boundary, see [35]. Thus, if $u$ is a positive solution to (1.3), then we have $\frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$ (resp. $u < 0$, $\frac{\partial u}{\partial \nu} > 0$) if $\nu$ denotes the outer normal.

**Proof.** We use the classical Hopf barrier function, see for instance Lemma 3.4 in [20]. The rest of the proof follows the lines of this lemma by using the weak maximum principle, of P.L. Lions [25] for solutions in $W^{2,N}_{\mathrm{loc}}(\Omega)$. □

Now we are in position to prove Proposition 1.1.

**Proof of Proposition 1.1.** (i) The existence and simplicity follow by using a Krein–Rutman's Theorem in positive cones, see [30]. For alternative methods see [21] and [34]. Notice that by above Remark part (3), the two first half-eigenfunction are $C^{2,\alpha}(\overline{\Omega})$.

(ii) First notice that for a fixed function $v \in W^{2,N}_{\mathrm{loc}}(\Omega)$ there exists a symmetric measurable matrix $A(x) \in \mathcal{A}$, such that

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2v) = L_A v,$$

where $L_A$ is the second order elliptic operator associated to $A$, see [28].

That is

$$L_A = \sum A_{i,j}(x)\partial_{i,j}.$$

Since $\varphi_1^+ \in C^2(\Omega)$, $\mu_1^+ \geqslant \inf_{A\in\mathcal{A}} \mu_1(A)$. Suppose now for contradiction that

$$\mu_1^+ > \inf_{A\in\mathcal{A}} \mu_1(A).$$

Hence, there exists $\bar{A} \in \mathcal{A}$ such that $\mu_1^+ > \mu_1(\bar{A})$. The corresponding eigenfunction $u_1$ satisfies

$$-L_{\bar{A}} u_1 = \mu_1(\bar{A}) u_1.$$

Moreover, $u_1 \in C^{2,\alpha}(\overline{\Omega})$ and $\frac{\partial u_1}{\partial \nu} < 0$ on $\partial\Omega$. Same holds for $\varphi_1^+$. Thus there exists $K > 0$ such that $u_1 < K\varphi_1^+$. Notice that $u_1$ is a sub-solution and $\varepsilon\varphi_1^+$ is a super-solution to

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 v) + \mu v = 0 \quad \text{in } \Omega.$$

Hence using Perron's method we find a positive solution to (1.3), which is in contradiction with part (i). Perron's method in this setting can be found for example in [21].

(iii) We only need to prove that for any positive function $\phi \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ we have

$$\mu_1^+ \geqslant \inf_{\Omega} \frac{-\mathcal{M}^+_{\lambda,\Lambda}(D^2\phi)}{\phi}.$$

Suppose the contrary, then there exists a positive function $u \in W^{2,N}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ and $\delta > 0$ such that

$$\mu_1^+ + \delta < \inf_{\Omega} \frac{-\mathcal{M}^+_{\lambda,\Lambda}(D^2 u)}{u}.$$

So, $u$ satisfies

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u) + (\mu_1^+ + \delta) u \leqslant 0 \quad \text{in } \Omega.$$

On the other hand, $\varphi_1^+$ satisfies

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 \varphi_1^+) + (\mu_1^+ + \delta)\varphi_1^+ \geqslant 0 \quad \text{in } \Omega.$$

Using Lemma 2.1 and Remark 2.1(3), we can find $\varepsilon > 0$ such that

$$\varepsilon\varphi_1 \leqslant u \quad \text{in } \Omega.$$

Then, using Perron's method we find a positive solution to the problem

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 v) + (\mu_1^+ + \delta) v = 0 \quad \text{in } \Omega,$$

contradicting the uniqueness of the positive solution to (1.3), part (i).

(iv) follows directly from (iii). □

**Proof of Theorem 1.1.** Let $u$ by a solution (1.8) and $\hat{A} \in \mathcal{A}$ by such that $L_{\hat{A}}(u) = \mathcal{M}^+_{\lambda,\Lambda}(D^2 u)$. Define $\hat{L}(v) := L_{\hat{A}}(v) + \mu v$. Using (ii) of Proposition 1.1, $\mu_1(\hat{A}) \geqslant \mu_1^+ > \mu$. Then, clearly, the first eigenvalue of $\hat{L}$ is positive. So the maximum principle holds for $\hat{L}$ see [5]. That is if $v$ satisfies

$$\begin{aligned} &\hat{L}(v) \geqslant 0 \quad \text{in } \Omega, \\ &v \leqslant 0 \quad \text{on } \partial\Omega \end{aligned} \tag{2.12}$$

implies $v \leqslant 0$. Since $u$ satisfies (2.12), $u \leqslant 0$.

The same kind of argument can be used in case (b). □

Now we will recall the following compactness results for the Pucci's extremal operator, whose proof can be found for instance in [6].

**Proposition 2.1.** *Let $\{f_n\}_{n>0} \subset C(\Omega)$ be a bounded sequence and $\{u_n\}_{n>0} \subset C(\overline{\Omega}) \cap W^{2,N}_{\mathrm{loc}}(\Omega)$ be a sequence of solutions to*

$$\mathcal{M}^+_{\lambda,\Lambda}(D^2 u_n) \geqslant f_n \quad \text{and} \quad \mathcal{M}^-_{\lambda,\Lambda}(D^2 u_n) \leqslant f_n \quad \text{in } \Omega, \quad u_n = 0 \quad \text{on } \partial\Omega. \tag{2.13}$$

*Then, there exists $u \in C(\overline{\Omega})$ such that, up to a subsequence, $u_n \to u$ uniformly in $\Omega$.*

*Let now $\{F_n\}_{n>0}$ be a sequence of uniformly elliptic concave (or convex) operators with ellipticity constants $\lambda$ and $\Lambda$ such that $F_n \to F$ uniformly in compact sets of $\mathcal{S}_n \times \Omega$ ($\mathcal{S}_n$ is the set of symmetric matrices). Suppose in addition that $u_n$ satisfies*

$$F_n(D^2 u_n, x) = 0 \quad \text{in } \Omega, \qquad u_n = 0 \quad \text{on } \partial\Omega$$

*and that $u_n$ converges uniformly to $u$. Then, $u \in C(\overline{\Omega})$ is a solution to*

$$F(D^2 u, x) = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

**Remark 2.2.** Actually, the above proposition is proved in [6] in a more general case, when $\{f_n\}_{n>0} \subset L^\infty(\Omega)$ and $\{u_n\}_{n>0} \subset C(\overline{\Omega})$ is a sequence of viscosity solutions to (2.13).

So, to prove Proposition 2.1 we need to use the following fact. If $u \in C(\overline{\Omega}) \cap W^{2,N}_{\text{loc}}(\Omega)$ is a sub-solution (resp. super-solution) of $\mathcal{M}^+(D^2 u) = g$ with $g$ continuous, then $u$ is also viscosity sub-solution (resp. super-solution) of the same equation, see [9]. We also use the regularity to prove that the limit of $u_n$, $u$, belongs to $W^{2,N}_{\text{loc}}(\Omega)$.

Now we want to study the nonlinear bifurcation problem. We will first prove the following.

**Proposition 2.2.** *If $(\bar{\mu}, 0)$ is a bifurcation point of problem (1.10), then $\bar{\mu}$ is an eigenvalue of $\mathcal{M}^+_{\lambda,\Lambda}$.*

**Proof.** Since $(\bar{\mu}, 0)$ is a nonlinear bifurcation point, there is a sequence $\{(\mu_n, u_n)\}_{n\in\mathbb{N}}$ of nontrivial solutions of the problem (1.10) such that $\mu_n \to \bar{\mu}$ and $u_n \to 0$ in uniformly in $\Omega$. Let us define

$$\hat{u}_n = \frac{u_n}{\|u_n\|_{C(\Omega)}}$$

then $\hat{u}_n$ satisfies

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2 \hat{u}_n) = \mu_n \hat{u} + \frac{f(u_n, \mu_n)}{u_n} \hat{u}_n \quad \text{in } \Omega.$$

So, the right-hand side of the equation is bounded. Then by Proposition 2.1 we can extract a subsequence such that $\hat{u}_n \to \hat{u}$. Clearly $\hat{u}$ is a solution to (1.3). □

Before proving Theorem 1.3, we need some preliminaries in order to compute the Leray–Schauder degree of a related function.

To start, let us recall some basic properties of the matrix operators $\mathcal{M}^+_{\lambda,\Lambda}$, whose proof follows directly from the equivalent definition for $\mathcal{M}^+_{\lambda,\Lambda}$:

$$\mathcal{M}^+_{\lambda,\Lambda}(M) = \sup_{A\in\mathcal{A}} \operatorname{tr}(AM),$$

for any symmetric matrix $M$ (see [6]). Notice that the original definition of C. Pucci [28] is of this type, but $\mathcal{A}$ is a different family of symmetric matrices.

**Lemma 2.2.** *Let $M$ and $N$ be two symmetric matrices. Then:*

$$\mathcal{M}^+_{\lambda,\Lambda}(M+N) \leqslant \mathcal{M}^+_{\lambda,\Lambda}(M) + \mathcal{M}^+_{\lambda,\Lambda}(N).$$

Next we recall a very well known fact about Pucci's operator, namely that is the Alexandroff–Bakelman–Pucci estimate holds. The proof can be found for example in [6].

**Theorem 2.1** [ABP]. *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, such that* $\operatorname{diam}(\Omega) \leqslant d$ *and $f \in L^N(\Omega)$. Suppose that $u$ is continuous in $\overline{\Omega}$ and satisfies $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) \leqslant f(x)$ in $\Omega$ and $u \leqslant 0$ on $\partial\Omega$. Then,*

$$\sup u^- \leqslant C\|f^+\|_{L^N(\Omega)}.$$

*Here $C = C(\operatorname{meas}(\Omega), \lambda, \Lambda, N, d)$ is a constant and* $\operatorname{meas}(\Omega)$ *denotes Lebesgue measure of $\Omega$.*

The next corollary is a maximum principle for small domains, that was first noted by Bakelman and extensively used in [4].

**Corollary 2.1.** *Let $\Omega$ be a bounded domain in $\mathbb{R}^N$, such that* $\operatorname{diam}(\Omega) \leqslant d$. *Suppose that $u$ is continuous in $\overline{\Omega}$, satisfies $\mathcal{M}^-_{\lambda,\Lambda}(D^2u) + c(x)u(x) \leqslant 0$ in $\Omega$, $u \geqslant 0$ on $\partial\Omega$ and $c \in L^\infty(\Omega)$ with $c(x) \leqslant b$ a.e. There exists $\delta = \delta(\lambda, \Lambda, N, d, b)$ such that* $\operatorname{meas}(\Omega) < \delta$ *implies $u \geqslant 0$ in $\Omega$.*

The proof is standard and uses in a crucial way Theorem 2.1. For details see [4]. Next corollary is crucial to prove that the eigenvalue $\mu_1^-$ is isolated.

**Corollary 2.2.** *Let $\Omega_n$ be a sequence of domains such that* $\operatorname{meas}(\Omega_n) \to 0$ *as $n \to \infty$ and* $\operatorname{diam}(\Omega_n) \leqslant d$. *If $(\mu_n, u_n)$ is a positive solution to* (1.3) *with $\Omega = \Omega_n$, then $\mu_n \to \infty$ as $n \to \infty$.*

**Proof.** Suppose by contradiction that there exists $C > 0$ such that $\mu_n < C$. Then $u_n$ satisfies the equation $\mathcal{M}^+_{\lambda,\Lambda}(D^2u_n) + Cu_n \geqslant 0$. Since the measure of $\Omega_n$ is small for $n$ large, we can use the previous corollary with $-u_n$ concluding that $-u_n \geqslant 0$, which is a contradiction. □

**Remark 2.3.** (1) In the sequel, we will vary the parameter $\lambda$ while keeping $\Lambda$ fixed in the operator $\mathcal{M}^+_{\lambda,\Lambda}$. We will denote the half eigenvalues $\mu_1^+(\lambda)$, $\mu_1^-(\lambda)$ to make explicit the dependence on the parameter $\lambda \in (0, \Lambda]$.

(2) From the characterization (ii) of Proposition 1.1 it follows that if $\lambda_1 < \lambda_2$, then $\mu_1^+(\lambda_1) \leqslant \mu_1^+(\lambda_2)$ and $\mu_1^-(\lambda_1) \geqslant \mu_1^-(\lambda_2)$.

**Lemma 2.3.** *The two first half eigenvalues functions $\mu_1^+ : (0, \Lambda] \to \mathbb{R}$ and $\mu_1^- : (0, \Lambda] \to \mathbb{R}$, are continuous on $\lambda$.*

**Proof.** Let $\{\lambda_j\}_{j\in\mathbb{N}}$ be sequence in $(0, \Lambda]$ converging to $\lambda \in (0, \Lambda]$. We will show that

$$\lim_{j\to\infty} \mu_1^+(\lambda_j) = \mu_1^+(\lambda).$$

Since $\lambda_j \to \lambda$ there exists $\varepsilon > 0$ such that $\bar{\lambda} := \lambda + \varepsilon \geqslant \lambda_j \geqslant \lambda - \varepsilon =: \lambda^* > 0$, for $j$ large. From the previous Remark we have

$$\mu_1^+(\lambda^*) \leqslant \mu_1^+(\lambda_j) \leqslant \mu_1^+(\bar{\lambda}),$$

for large $j$. Therefore, up to subsequences $\mu_1^+(\lambda_j) \to \mu$.

Let $u_j$ be the corresponding eigenfunction for the eigenvalue $\mu_1^+(\lambda_j)$. We can suppose that $\|u_j\|_{C(\Omega)} = 1$, then $u_j$ satisfies

$$\mathcal{M}^+_{\lambda_*,\Lambda}(D^2u_j) \geqslant -\mu_1^+(\bar{\lambda})u_j \quad \text{and} \quad \mathcal{M}^-_{\lambda_*,\Lambda}(D^2u_j) \leqslant -\mu_1^+(\lambda^*)u_j.$$

So by Proposition 2.1 up to subsequences, $u_j \to u$ uniformly in $\Omega$. Moreover, $(\mu, u)$ is a solution to (1.3) and $\|u\|_{C(\Omega)} = 1$.

Since $u_j$ is positive in $\Omega$, we have that $u_j$ is non-negative in $\Omega$ and by the strong maximum principle, $u$ is positive in $\Omega$. Hence, by the uniqueness of the positive eigenfunction, Proposition 1.1(i), $\mu = \mu_1^+(\lambda)$, which ends the proof in this case. The same proof holds in the case of $\mu_1^-$. □

The next lemma proves that the first half-eigenvalue $\mu_1^-$ is isolated.

**Lemma 2.4.** *For every interval* $[a,b]\subset(0,\Lambda)$ *there exists a* $\delta>0$ *such that for all* $\lambda\in[a,b]$ *there is no eigenvalue of* (1.3) *in* $(\mu_1^-(\lambda),\mu_1^-(\lambda)+\delta]$.

**Proof.** Suppose that the lemma is not true. Then, there are sequences $\{\lambda_j\}_{j\in\mathbb{N}}\subset(0,\Lambda]$, $\{\mu_j\}_{j\in\mathbb{N}}\subset\mathbb{R}^+$, and $\{u_j\}_{j\in\mathbb{N}}\subset C(\Omega)\setminus\{0\}$ such that $\lambda_j\to\bar\lambda\in(0,\Lambda)$, $\mu_j>\mu_1^-(\lambda_j)$, $\lim_{j\to\infty}(\mu_j-\mu_1^-(\lambda_j))=0$, and

$$-\mathcal{M}^+_{\lambda_n,\Lambda}(D^2u_n)=\mu_nu_n.$$

Using Proposition 2.1 we have that, up to a subsequence, $u_n\to u$ uniformly in $\Omega$ and $u$ is a solution of the problem

$$-\mathcal{M}^+_{\lambda,\Lambda}(D^2u)=\mu_1^-(\bar\lambda)u\quad\text{in }\Omega.$$

Therefore by Proposition 1.1(i), $u$ is negative in $\Omega$.

On the other hand, by (i) of Proposition 1.1 $u_n$ changes sign in $\Omega$, then there exists $\Omega_n$, a connected component of $\{x\in\Omega\mid u_n(x)>0\}$, with $\text{meas}(\Omega_n)>0$. Since $u_n\to u$, $\text{meas}(\Omega_n)\to0$. Then by Corollary 2.2 $\mu(\Omega_n,\lambda^*)\to\infty$, where $\lambda^*>0$ is such that $\lambda_j>\lambda^*$. But $\mu_n=\mu_1^+(\Omega_n,\lambda_n)\geqslant\mu(\Omega_n,\lambda^*)$, contradicting the fact that $\mu_n$ converges to $\mu_1^-(\bar\lambda)$. □

Let us define

$$\mu_2(\lambda)=\inf\bigl\{\mu>\mu_1^-(\lambda)\mid\mu\text{ is an eigenvalue of (1.10)}\bigr\}.$$

Then by the previous lemma $\mu_2>\mu_1^-$. We notice that $\mu_2$ may be equal to $+\infty$. Define now $\mathcal{L}^+_\lambda$ as the inverse of $-\mathcal{M}^+_{\lambda,\Lambda}$. It is well known that $\mathcal{L}^+_\lambda$ is well defined in $\mathcal{C}:=\{u\in C(\overline{\Omega})\mid u=0\text{ on }\partial\Omega\}$ (see for example [7]) and, by Proposition 2.1, $\mathcal{L}^+_\lambda$ is compact.

Now we are in position to compute the Leray–Schauder degree and prove the following proposition.

**Proposition 2.3.** *Let* $r>0$, $\bar\lambda>0$, $\mu\in\mathbb{R}$. *Then*

$$\deg_{\mathcal{C}}\bigl(I-\mu\mathcal{L}^+_{\bar\lambda},B(0,r),0\bigr)=\begin{cases}1 & \text{if }\mu<\mu_1^+(\bar\lambda),\\ 0 & \text{if }\mu_1^+(\bar\lambda)<\mu<\mu_1^-(\bar\lambda),\\ -1 & \text{if }\mu_1^-(\bar\lambda)<\mu<\mu_2(\bar\lambda),\end{cases}$$

*here* $\mathcal{C}:=\{u\in C(\overline{\Omega})\mid u=0$ *on* $\partial\Omega\}$.

**Remark 2.4.** Since $\mathcal{L}^+_\lambda$ is compact, the degree is well defined if $0\notin(I-\mu\mathcal{L}^+_{\bar\lambda})(\partial B(0,r))$.

**Proof of Proposition 2.3.** We have that the degree

$$\deg_{\mathcal{C}}\bigl(I-s\mu\mathcal{L}^+_{\bar\lambda},B(0,r),0\bigr)$$

is well defined for any $s\in[0,1]$ and $\mu<\mu_1^+(\bar\lambda)$, since $\mathcal{M}^+_{\lambda,\Lambda}$ does not have eigenvalues below $\mu_1^+$, that is, $0\notin(I-s\mu\mathcal{L}^+_{\bar\lambda})(\partial B(0,r))$. Using the invariance of the degree under homotopy, we conclude that this degree is equal to 1, its value at $s=0$.

In the case $\mu_1^+(\bar\lambda)<\mu<\mu_1^-(\bar\lambda)$ we will use the following property of the degree to prove that the degree is zero. If $\deg_{\mathcal{C}}(I-\mu\mathcal{L}^+_{\bar\lambda},B(0,r),0)\neq0$, then $(I-\mu\mathcal{L}^+_{\bar\lambda})(B(0,r))$ is a neighborhood of zero. So we claim that if $\mu_1^+(\bar\lambda)<\mu<\mu_1^-(\bar\lambda)$, then $(I-\mu\mathcal{L}^+_{\bar\lambda})(B(0,r))$ is not a neighborhood of zero. Suppose by contradiction that

$(I - \mu\mathcal{L}^+_{\bar\lambda})(B(0,r))$ is a neighborhood of zero. Then for any smooth $h$ with $\|h\|_{C(\Omega)}$ small, there exists $u$ a solution to

$$u - \mu\mathcal{L}^+_{\bar\lambda}u = h.$$

In particular, we can take $h$ to be a solution of

$$\mathcal{M}^+_{\bar\lambda,\Lambda}(D^2h) = -\delta \quad \text{in } \Omega \quad \text{and} \quad h = 0 \quad \text{on } \partial\Omega,$$

where $\delta > 0$ is small enough.

Then, by Lemma 2.2 and the definition of $\mathcal{L}^+_{\bar\lambda}$, it follows that $u$ satisfies

$$\mathcal{M}^+_{\bar\lambda,\Lambda}(D^2u) + \mu u \leqslant -\delta \quad \text{in } \Omega.$$

On the other hand, by Lemma 2.1 and Remark 2.1(3), there exists $\varepsilon > 0$, such that $\varepsilon(-\varphi)^-_1 < u$, and $\varepsilon(-\varphi^-_1)$ satisfies

$$\mathcal{M}^+_{\bar\lambda,\Lambda}\big(D^2\varepsilon(-\varphi^-_1)\big) + \mu\varepsilon(-\varphi^-_1) \geqslant -\delta \quad \text{in } \Omega.$$

Then using Perron's method we find a positive solution $w$ to

$$\mathcal{M}^+_{\bar\lambda,\Lambda}(D^2w) + \mu w = -\delta \quad \text{in } \Omega, \qquad w = 0 \quad \text{on } \partial\Omega.$$

This leads to a contradiction with Theorem 1.1 and with the characterization for the first eigenvalue (1.7)(1). So, $\deg_{\mathcal{C}}(I - \mu\mathcal{L}^+_{\bar\lambda}, B(0,r), 0) = 0$ for $\mu^+_1(\bar\lambda) < \mu < \mu^-_1(\bar\lambda)$.

Finally, suppose that $\mu^-_1(\bar\lambda) < \mu < \mu_2(\bar\lambda)$. The continuity of $\mu^-_1(\cdot)$ and Lemma 2.4 imply the existence of a continuous function $\nu : (0,\Lambda] \to \mathbb{R}$ such that $\mu^-_1(\lambda) < \nu(\lambda) < \mu_2(\lambda)$ for all $\lambda \in (0,\Lambda]$ and $\nu(\bar\lambda) = \mu$.

The result will follow by showing that the well-defined, integer-valued function

$$d(\lambda) = \deg_{\mathcal{C}}\big(I - \nu(\lambda)\mathcal{L}^+_{\bar\lambda}, B(0,r), 0\big)$$

is constant in $[\bar\lambda, \Lambda]$. This follows by the invariance of the Leray–Schauder degree under a compact homotopy. Recall that $d(\Lambda) = -1$, hence the proposition follows. □

**Proof of Theorem 1.3.** Let us set

$$H_\mu(u) = \mathcal{L}^+_\lambda\big(\mu u + f(\mu,u)\big).$$

Suppose that $(\mu^+_1, 0)$ is not a bifurcation point of problem (1.10). Then there exist $\varepsilon$, $\delta_0 > 0$ such that for all $|\mu - \mu^+_1| \leqslant \varepsilon$ and $\delta < \delta_0$ there is no nontrivial solution of the equation

$$u - H_\mu(u) = 0$$

with $\|u\| = \delta$. From the invariance of the degree under compact homotopy we obtain that

$$\deg_{\mathcal{C}}\big(I - H_\mu, B(0,\delta), 0\big) \equiv \text{constant} \quad \text{for } \mu \in [\mu^+_1 - \varepsilon, \mu^+_1 + \varepsilon]. \tag{2.14}$$

By taking $\varepsilon$ smaller if necessary, we can assume that $\mu^+_1 + \varepsilon < \mu^-_1$. Fix now $\mu \in (\mu^+_1, \mu^+_1 + \varepsilon]$. It is easy to see that if we choose $\delta$ sufficiently small, then the equation

$$u - \mathcal{L}^+_\lambda\big(\mu u + sf(\mu,u)\big) = 0$$

has no solution $u$ with $\|u\| = \delta$ for every $s \in [0,1]$. Indeed, assuming the contrary and reasoning as in the proof of Proposition 2.2, we would find that $\mu$ is an eigenvalue of (1.3). From the invariance of the degree under homotopies and Proposition 2.3 we obtain

$$\deg_{\mathcal{C}}\big(I - H_\mu, B(0,\delta), 0\big) = \deg_{\mathcal{C}}\big(I - \mu\mathcal{L}^+_\lambda, B(0,\delta), 0\big) = 0. \tag{2.15}$$

Similarly, for $\mu \in [\mu_1^+ - \varepsilon, \mu_1^+)$ we find that

$$\deg_{\mathcal{C}}\big(I - H_\mu, B(0,\delta), 0\big) = 1. \tag{2.16}$$

Equalities (2.15) and (2.16) contradict (2.14) and hence $(\mu_1^+, 0)$ is a bifurcation point for the problem (1.10). Let define $u_\mu$ a solution to (1.10) for $\mu > \mu_1^+$, with $\|u_\mu\|_\infty \to 0$ as $\mu \to \mu_1^+$. Using the same argument of Proposition 2.3,

$$u_\mu/\|u_\mu\|_\infty \to \varphi_1^+ \quad \text{as } \mu \to \mu_1^+.$$

This shows that $u_\mu$ is positive for $\mu$ close to $\mu_1^+$.

The rest of the proof is entirely similar to that of the Rabinowitz's Global Bifurcation Theorem, see [32,33] or [31], so we omit it here. □

## 3. "Spectrum" in the radial case and nonlinear bifurcation from all "eigenvalues"

Let us first recall that the value of the Pucci's operator applied to a radially symmetric function can be computed explicitly; namely if $u(x) = \varphi(|x|)$ one has

$$D^2u(x) = \frac{\varphi'(|x|)}{|x|} I + \left[\frac{\varphi''(|x|)}{|x|^2} - \frac{\varphi'(|x|)}{|x|^3}\right] x \otimes x,$$

where $I$ is the $N \times N$ identity matrix and $x \otimes x$ is the matrix whose entries are $x_i x_j$. Then the eigenvalues of $D^2u$ are $\varphi''(|x|)$, which is simple, and $\varphi'(|x|)/|x|$, which has multiplicity $N-1$.

In view of this, we can give a more explicit definition of Pucci's operator. In the case of $\mathcal{M}^+_{\lambda,\Lambda}$ we define the functions

$$M(s) = \begin{cases} s/\Lambda, & s > 0, \\ s/\lambda, & s \leqslant 0, \end{cases} \quad \text{and} \quad m(s) = \begin{cases} \Lambda s, & s > 0, \\ \lambda s, & s \leqslant 0. \end{cases}$$

Then, we see that $u$ satisfies (1.3) with $\Omega = B_1$ and is radially symmetric if and only if $u(x) = v(|x|)$, $r = |x|$ satisfies

$$v'' = M\left(-\frac{(N-1)}{r} m(v') - \mu v\right), \tag{3.17}$$

$$v'(0) = 0, \qquad v(1) = 0. \tag{3.18}$$

Next we briefly study the existence, uniqueness, global existence, and oscillation of the solutions to the related initial value problem

$$w'' = M\left(-\frac{(N-1)}{r} m(w') - w\right), \tag{3.19}$$

$$w'(0) = 0, \qquad w(0) = 1. \tag{3.20}$$

Then we will come back to (3.17), (3.18) and to the proof of Theorem 1.2. First using a standard Schauder fixed point argument as used by Ni and Nussbaum in [27], we can prove the existence of $w \in C^2$ solution to

$$\{w' r^{N-1}\}' = -r^{N-1}\frac{w}{\lambda}, \qquad w'(0) = 0, \quad w(0) = 1.$$

Moreover, this solution is unique and for $r$ small, $w'(r)$ and $w''(r)$ are negative. Then, for some $\delta > 0$, $w$ satisfies

$$w'' = M\left(-\frac{(N-1)}{r} m(w') - w\right), \quad \text{in } (0,\delta].$$

Next we consider (3.19) with initial values $w(\delta)$ and $w'(\delta)$ at $r=\delta$. From the standard theory of ordinary differential equations we find a unique $C^2$-solution of this problem for $r\in[\delta,a)$, for $a>\delta$. Using Gronwall's inequality we can extend the local solution to $[0,+\infty)$.

In the following lemma we will show that the solution $w$ is oscillatory.

**Lemma 3.1.** *The unique solution $w$ to* (3.19), (3.20)*, $w$, is oscillatory, that is, given any $r>0$, there is a $\tau>r$ such that $w(\tau)=0$.*

The proof uses standard arguments of oscillation theory for ordinary differential equation.

**Proof.** Suppose that $w$ is not oscillatory, that is, for some $r_0$ , $w$ does not vanish on $(r_0,\infty)$. Assume that $w>0$ in $(r_0,\infty)$. Let $\phi$ be a solution to (3.19), (3.20) with $\lambda=\Lambda$, then it is known that $\phi$ is oscillatory. So we can take $r_0<r_1,r_2$ such that $\phi(r)>0$ if $r\in(r_1,r_2)$ and $\phi(r_1)=\phi(r_2)=0$. We have that $w$ and $\phi$ satisfy

$$\{w'r^{N-1}\}'\leqslant -r^{N-1}\frac{w}{\lambda},$$
$$\{\phi'r^{N-1}\}'=-r^{N-1}\frac{\phi}{\lambda}.$$

If we multiply the first equation by $\phi$ and the second by $w$, subtract them and then integrate, we get

$$r_1^{N-1}\phi'(r_1)w(r_1)-r_2^{N-1}\phi'(r_2)w(r_2)\leqslant 0,$$

getting a contradiction.

Suppose now that $w<0$ in $(r_0,\infty)$. In that case we claim that $w'>0$ in $(r_0,\infty)$, taking if necessary a larger $r_0$. If there exists a $r^*$ such that $w'(r^*)=0$, then using the equation we have that $w'>0$ in $(r^*,\infty)$. So we only need to discard the case $w'<0$ in $(r_0,\infty)$. In that case $w$ satisfies

$$\{w'r^{\tilde N^+-1}\}'=-r^{\tilde N^+-1}\frac{w}{\lambda}\quad \text{in } (r_0,\infty)$$

where $\widetilde N^+=(\lambda(N-1))/\Lambda+1$. Let denote by $g(r)=\{w'r^{\tilde N^+-1}\}$ we have that $g$ is monotone, then there exists a finite $c_1<0$ such that $\lim_{r\to\infty}g(r)=c_1$.

On the other hand, since $w'<0$, there exists $c_2\in[-\infty,0)$ such that $\lim_{r\to\infty}w(r)=c_2$, then from the equation satisfied by $w$, we get that

$$\lim_{r\to\infty}g'(r)=+\infty.$$

That is a contradiction with $\lim_{r\to\infty}g(r)=c_1$.

Define now

$$b(r)=r^{\tilde N^--1}\frac{w'(r)}{w(r)},\quad r\in(r_0,\infty),$$

here $\widetilde N^-=(\Lambda(N-1))/\lambda+1$.

Then we claim that $b$ satisfies,

$$b'+\frac{b^2}{r^{\tilde N^--1}}+\frac{r^{\tilde N^--1}}{\Lambda}\leqslant 0. \tag{3.21}$$

If $w''>0$ then $b$ satisfies

$$b'+\frac{b^2}{r^{\tilde N^--1}}+\frac{r^{\tilde N^--1}}{\lambda}=0.$$

Since $\frac{1}{\Lambda} \leqslant \frac{1}{\lambda}$, the claim follows in this case. If $w'' < 0$ then $b$ satisfies

$$b' + \frac{b^2}{r^{\tilde{N}^- -1}} + \frac{r^{\tilde{N}^- -1}}{\Lambda} = (\widetilde{N}^- - N)b.$$

Finally, since $\widetilde{N}^- - N \geqslant 0$ and $b < 0$, the claim follows also in this second case.

Integrating (3.21) from $r_0$ to $t > r_0$ we get

$$b(t) - b(r_0) + \frac{t^{\tilde{N}^-}}{\widetilde{N}^- \Lambda} - \frac{r_0^{\tilde{N}^-}}{\widetilde{N}^- \Lambda} + \int_{r_0}^{t} \frac{b^2}{r^{\tilde{N}^- -1}} \leqslant 0. \tag{3.22}$$

In particular we have

$$-b(t) \geqslant C t^{\tilde{N}^-}.$$

For some $C > 0$ and $t$ large. Define now

$$k(t) = \int_{r_0}^{t} \frac{b^2}{r^{\tilde{N}^- -1}}.$$

Then, by the previous fact, we have

$$k(t) \geqslant c t^{\tilde{N}^- +2} \quad \text{for } t \text{ and some } c > 0. \tag{3.23}$$

On the other hand from (3.22) and $b < 0$ we get

$$k(t) < -w(t),$$

or

$$k(t) < k'(t) t^{\tilde{N}^- +1}, \quad \text{for } t \text{ large.}$$

The latter inequality implies

$$C\left(\frac{1}{k(t)} - \frac{1}{k(s)}\right) \geqslant \frac{1}{t^{N^- -2}} - \frac{1}{s^{N^- -2}} \tag{3.24}$$

for some $C > 0$ and $t, s$ large with $t < s$. Letting $s \to \infty$ and noting that $k(s) \to +\infty$, we find

$$k(t) \leqslant A t^{N^- -2}. \tag{3.25}$$

However (3.23) and (3.25) are not compatible. This contradiction shows that $w$ must be oscillatory. □

Notice that the same proof holds when the initial conditions to the problem (3.19) are $w(0) = -1$, $w'(0) = 0$.

With these preliminaries we are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** Let denote $w^\nu$ the above solutions of (3.19) with initial conditions $w^\nu(0) = \pm 1$ (here and in the rest of the proof $\nu \in \{+, -\}$). From the previous lemma, $w^\nu$ has infinitely many zeros:

$$0 < \beta_1^\nu < \beta_2^\nu < \cdots < \beta_k^\nu < \cdots.$$

A standard Hopf type argument shows that they are all simple. Next we define $\mu_k^\nu = (\beta_k^\nu)^2$ of Theorem 1.2. Clearly $\mu = \mu_k^\nu$ is an eigenvalue of (1.3), with $w^\nu(\beta_k^\nu \cdot)$, $r \in [0, 1]$, being the corresponding eigenfunction with $k - 1$ zeros in $(0, 1)$. We claim that there is no radial eigenvalue of (1.3) other than these $\mu_k^\nu$'s.

Let $\mu$ be an eigenvalue of (1.3). Clearly $\mu > 0$. Let $z(r)$ be the corresponding eigenfunction and suppose that $z(0) > 0$, the uniqueness of solution to (3.19) implies that $z(r) = z(0)\,w^+(\mu^{1/2}r)$. Moreover, since $z(1) = 0$, $\mu = (\beta_k^+)^2$ for some $k \in \mathbb{N}$, and $z = z(0)\,w^+$. The same holds for $z(0) < 0$. □

Below we will exhibit some properties of the eigenvalues distribution.

**Lemma 3.2.** *For $k \in \mathbb{N}, k > 1$ we have $\mu_k^- < \mu_{k+1}^+$ and $\mu_k^+ < \mu_{k+1}^-$.*

**Proof.** We will prove the lemma in terms of the functions $w^+$ and $w^-$ defined above.

We claim that if $w^+$ has to change sign between two consecutive zeros of $w^-$, if $w^+$ has the same sign of $w^-$. Notice that this is weaker then the usual Sturm's comparison result, since there is a additional sign restriction.

Suppose first by contradiction that $w^-(r_1) = w^-(r_2) = 0$, $w^-(r) > 0$ for all $r \in (r_1, r_2)$ and $w^+(r) > 0$ for all $r \in [r_1, r_2]$. Let $r_3 < r_1 < r_2 < r_4$ be the next zeros of $w^+$, that is, $w^+(r_3) = w^+(r_4) = 0$, $w^+(r) > 0$ for all $r \in (r_3, r_4)$. Then, the first half-eigenvalue in $A_1 := \{r_1 < |x| < r_2\}$ is $\mu^+(A_1) = 1$ and first half-eigenvalue in $A_2 := \{r_3 < |x| < r_4\}$ is $\mu^+(A_2) = 1$. Define now $u(r) = w^+(\beta r)$, with $\beta > 1$ such that $r_4/\beta > r_2$. So, $u$ is a positive eigenfunction in $A_3 := \{r_3/\beta < |x| < r_4/\beta\}$ with eigenvalue $\mu^+(A_3) = \beta^2$. But $A_1 \subset A_3$, therefore $\mu^+(A_1) = 1 \geqslant \mu^+(A_3) = \beta^2$ getting a contradiction. The same kind of argument can be used in the case when $w^-$ negative in $(r_1, r_2)$ and $w^+$ negative in $[r_1, r_2]$. Hence, the claim follows.

In the two cases above we can invert the role of $w^-$ and $w^+$.

As a consequence of the previous facts, the lemma follows by examining the distribution of zeroes of $w^+$ and $w^-$. □

**Remark 3.1.** The above lemma implies that in the case $\beta_k^+ < \beta_k^-$, $w^+(r)\,w^-(r) > 0$ for all $r \in (\beta_k^+, \beta_k^-)$. The same holds true in the case $\beta_k^+ > \beta_k^-$.

**Lemma 3.3.** *The gap between the two first half-eigenvalues is larger than that between the second ones*:

$$\frac{\mu_1^-}{\mu_1^+} \geqslant \frac{\mu_2^-}{\mu_2^+}.$$

**Proof.** Let $\varphi_2^+$ and $\varphi_2^-$ the radial eigenfunctions of $\mathcal{M}_{\lambda,\Lambda}^+$ in $B_1$, with corresponding eigenvalues $\mu_2^+$ and $\mu_2^-$. Define $r^+$ (resp. $r-$) as the first zeros of $\varphi_2^+$ (resp. $\varphi_2^-$). We claim that $r^- \geqslant r^+$. Suppose by contradiction that $r^- < r^+$. Define now $A^+ = \{x \mid r^+ < |x| < 1\}$ and $A^- = \{x \mid r^- < |x| < 1\}$, then $A^+ \subset A^-$. Using the monotonicity with respect the domain of the first half-eigenvalues and Proposition 1.1(ii) we get

$$\mu_1^-(A^+) = \mu_2^+ \geqslant \mu_1^+(A^+) \geqslant \mu_1^+(A^-) = \mu_2^-.$$

On the other hand $B_{r^-} \subset B_{r^+}$, thus by the same kind of argument

$$\mu_1^-(B_{r^-}) = \mu_2^- > \mu_1^-(B_{r^+}) \geqslant \mu_1^+(B_{r^+}) = \mu_2^+.$$

Hence, we get a contradiction. So, the claim follows. Making a rescaling argument, so as in the proof of Theorem 1.2, it follows that

$$(r^+)^2\mu_2^+ = \mu_1^+ \quad \text{and} \quad (r^-)^2\mu_2^- = \mu_1^-,$$

which ends the proof. □

Next, we prove some preliminary results to prepare the proof of Theorem 1.4.

**Lemma 3.4.** *Assume that $\mu_k^+ \neq \mu_k^-$ and that there exists $r_0 \in (0,1)$ such that $\phi_\pm(r) > 0$ for all $r \in (r_0, 1]$. Then, there exists a continuous function $g$ such that there is no solution to the problem*

$$u'' = M\left(-\frac{(N-1)}{r}m(u') - \mu u + g\right) \quad \textit{in } [0, r_0], \tag{3.26}$$

*and*

$$u'' \geqslant M\left(-\frac{(N-1)}{r}m(u') - \mu u + g\right) \quad \textit{in } (r_0, 1], \tag{3.27}$$

$$u'(0) = 0, \qquad u(1) = 0 \tag{3.28}$$

*for $\mu$ between $\mu_k^+$ and $\mu_k^-$.*

**Remark 3.2.** (1) Some ideas of the proof are in the book of P. Drábek [13].

(2) There is a similar non-existence result in the case when there exists $r_0 \in (0,1)$ such that $\phi_\pm(r) < 0$ for all $r \in (r_0, 1]$ , replacing (3.27) by

$$u'' \leqslant M\left(-\frac{(N-1)}{r}m(u') - \mu u + g\right) \quad \text{in } (r_0, 1], \tag{3.29}$$

in the previous lemma.

(3) Let us denote by $\phi_+$ and $\phi_-$ the solutions of (3.26) with $r_0 = 1$ and $g = 0$ and respective initial conditions $u'(0) = 0$, $u(0) = 1$ and $u'(0) = 0$, $u(0) = -1$. Let us suppose that $\mu$ is between $\mu_k^+$ and $\mu_k^-$, then by Remark 3.1 we deduce that $\phi_+(1)\phi_-(1) > 0$.

**Proof.** Consider then the particular case

$$\phi_\pm(r) > 0, \quad \phi'_\pm(r) \leqslant 0 \quad \text{for all } r \in (r_0, 1].$$

All other cases can be treated similarly.

Let $g : [0,1] \to \mathbb{R}$ be a continuous function such that $g(r) = 0$ for all $r \in [0, r_0]$ and $g(r) > 0$ for all $r \in (r_0, 1]$.

For $\alpha \in \mathbb{R}$, let $\varphi_\alpha$ be the solution to (3.26), (3.27) and (3.28) with $\varphi_\alpha(0) = \alpha$. For $\alpha > 0$, we have

$$\varphi_\alpha(r) = \alpha\phi_+(r) \quad \text{for all } r \in [0, r_0],$$

since uniqueness holds when $g = 0$. Put $r_1 = \inf\{r \in (r_0, 1);\ \varphi_\alpha(r) = 0\}$. The interval $(r_0, r_1)$ contains a point $\tau_1$ such that

$$\left[\frac{\varphi_\alpha}{\phi_+}\right]'(\tau_1) < 0.$$

If this is not the case,

$$\frac{\varphi_\alpha(\tau)}{\phi_+(\tau)} \geqslant \frac{\varphi_\alpha(r_0)}{\phi_+(r_0)} = \alpha > 0, \qquad \tau \in (r_0, r_1),$$

which is impossible. So, we obtain

$$(\varphi'_\alpha\phi_+ - \varphi_\alpha\phi'_+)(\tau_1) < 0.$$

Define

$$G_i(r) = r^{\tilde{N}_i - 1}(\varphi'_\alpha\phi_+ - \varphi_\alpha\phi'_+), \quad i = 1, 2,$$

where $\tilde{N}_1 = N$ and $\tilde{N}_2 = \tilde{N}^+$.

Now we claim that there exists $\tau_2$, $r_0 \leqslant \tau_2 < \tau_1$ such that

$$\varphi'_\alpha(r) < 0 \quad \text{for all } r \in (\tau_2, \tau_1) \quad \text{and} \quad G_i(\tau_2) \geqslant 0, \quad i = 1, 2.$$

If $\phi'_\alpha(r) < 0$ for all $r \in (t_0, \tau_1)$, since $G_i(r_0) = 0$, we conclude in this case by taking $\tau_2 = r_0$. If not, we define $\tau_2 = \sup\{\tau \in [r_0, \tau_1), \mid \varphi'_\alpha(\tau) = 0\}$. Notice that $\tau_2 < \tau_1$ and $\varphi'_\alpha(\tau_1) < 0$, so $\phi'_\alpha(r) < 0$ for all $r \in (\tau_2, \tau_1)$. By the definition of $\tau_2$, $\varphi'_\alpha(\tau_2) = 0$. Thus, $G_i(\tau_2) > 0$ and the claim follows. From the equation satisfied by $\phi_+$ we get

$$\{r^{N-1}\phi'_+\}' \leqslant \frac{r^{N-1}}{\lambda}[-\mu\phi_+] \quad \text{in } (\tau_2, \tau_1), \tag{3.30}$$

and

$$\{r^{\tilde{N}^+-1}\phi'_+\}' \leqslant \frac{r^{\tilde{N}^+-1}}{\Lambda}[-\mu\phi_+] \quad \text{in } (\tau_2, \tau_1). \tag{3.31}$$

Since $\varphi_\alpha$ is positive in $(\tau_2, \tau_1)$, we obtain $G'_1(r) \geqslant (r^{N-1}/\lambda)g(r)\phi_+(r) > 0$, if $\varphi''_\alpha(r) < 0$ and $G'_2(r) \geqslant (r^{\tilde{N}^+-1}/\Lambda)g(r)\phi_+(r) > 0$, if $\varphi''_\alpha(r) \geqslant 0$ for all $r \in (\tau_2, \tau_1)$.

The interval $(\tau_2, \tau_1)$ can be splitted in subintervals $(s, t)$ such that $G_i(s) - G_i(t) = \int_t^s G'_i(\tau)\,d\tau > 0$, where $i$ is well chosen. Using that if $G_i(t) < 0$, then $G_j(t) < 0$ for $i \neq j$, we get a contradiction.

For $\alpha = 0$, $\varphi(r) = 0$, $r \in [0, r_0]$. Then, we find an appropriate interval to argue as in the above case. For $\alpha < 0$ we have $\varphi_\alpha(r) = |\alpha|\phi_1$ for all $r \in [0, r_0]$ and the proof is quite analogous as for $\alpha > 0$. All the above shows that there is no solution for (3.26), (3.27) and (3.28). □

**Proposition 3.1.** *Let $r > 0$, $\bar\lambda > 0$, $\mu \in \mathbb{R}$. Then*

$$\deg_{\mathcal{C}}\big(I - \mu\mathcal{L}^+_{\bar\lambda}, B(0,r), 0\big) = \begin{cases} 1 & \text{if } \mu < \mu_1^+(\bar\lambda), \\ 0 & \text{if } \mu_k^+(\bar\lambda) < \mu < \mu_k^-(\bar\lambda) \text{ or } \mu_k^-(\bar\lambda) < \mu < \mu_k^+(\bar\lambda), \\ (-1)^k & \text{if } \mu_k^+(\bar\lambda) < \mu < \mu_{k+1}^-(\bar\lambda) \text{ or } \mu_k^-(\bar\lambda) < \mu < \mu_{k+1}^+(\bar\lambda), \end{cases}$$

*here $\mathcal{C} := \{u \in C([0,1]) \mid u(1) = 0,\ u'(0) = 0\}$.*

**Remark 3.3.** (1) For $k \in \mathbb{N}$, $k > 1$, we do not expect that in general

$$\mu_k^+ \leqslant \mu_k^-,$$

but this is an open problem.

(2) If $\mu_k^+ = \mu_k^-$, the case $\deg_{\mathcal{C}}(I - \mu\mathcal{L}^+_{\bar\lambda}, B(0,r), 0) = 0$ is not present in Proposition 3.1.

**Proof.** Assume first that $\mu_k^+(\bar\lambda) < \mu < \mu_{k+1}^-(\bar\lambda)$ or $\mu_k^-(\bar\lambda) < \mu < \mu_{k+1}^+(\bar\lambda)$. The arguments used in the proof of Lemma 2.3 imply $\mu_j^\pm(\lambda)$ is a continuous function of $\lambda$. Using Lemma 3.2 we find a continuous function $\nu : (0, \Lambda] \to \mathbb{R}$ such that $\max\{\mu_k^+(\lambda), \mu_k^-(\lambda)\} < \nu(\lambda) < \min\{\mu_{k+1}^+(\lambda), \mu_{k+1}^-(\lambda)\}$ and $\nu(\bar\lambda) = \mu$. The invariance of the Leray–Schauder's degree under compact homotopies implies

$$d(\lambda) = \deg_{\mathcal{C}}\big(I - \nu(\lambda)\mathcal{L}^+_{\bar\lambda}, B(0,r), 0\big) = \text{constant},$$

for $\lambda \in (0, \Lambda]$. In particular $d(\bar\lambda) = d(\Lambda) = (-1)^k$ and the result follows. The case $\mu < \mu_1^+(\bar\lambda)$ is proved in Proposition 2.3. In the case $\mu_k^+(\bar\lambda) < \mu < \mu_k^-(\bar\lambda)$ or $\mu_k^-(\bar\lambda) < \mu < \mu_k^+(\bar\lambda)$ we will prove, as in Proposition 2.3, that $(I - \mu\mathcal{L}^+_{\bar\lambda})(B(0,r))$ is not a neighborhood of zero.

Suppose by contradiction that $(I - \mu\mathcal{L}^+_{\bar\lambda})(B(0,r))$ is a neighborhood of zero. Then, for any smooth $h$ with $\|h\|_{C([0,1])}$ small, there exists a solution $u$ to

$$u - \mu\mathcal{L}^+_{\bar\lambda}u = h.$$

In particular we can take $h$ being a solution to

$$\mathcal{M}^+_{\bar\lambda,\Lambda}(D^2h) = \psi \quad \text{in } \Omega \quad \text{and} \quad h = 0 \quad \text{on } \partial\Omega,$$

where $\|\psi\|_{C([0,1])} > 0$ is small enough. Then, by Lemma 2.2 and the definition of $\mathcal{L}^+_{\bar\lambda}$, it follows that $u$ satisfies

$$\mathcal{M}^+_{\bar\lambda,\Lambda}(D^2u) + \mu u \leqslant \psi \quad \text{in } \Omega.$$

Taking $\psi = -g$ (resp. $\psi = g$), where $g$ is a function of the type used in Lemma 3.4, we will get that $-u$ (resp. $u$) satisfies (3.26), (3.27) (resp. (3.29)) and (3.28). Thus, we get a contradiction with lemma 3.4 or Remark 3.2(2). So, $\deg_{\mathcal{C}}(I - \mu\mathcal{L}^+_{\bar\lambda}, B(0,r), 0) = 0$ in this case, and the proof is finished. □

**Proof of Theorem 1.4.** Using the same argument as in Theorem 1.3, we obtain the existence of a "half-component" $\mathcal{C}^+_k \subset \mathbb{R} \times C([0,1])$ of radially symmetric solutions to (1.10), whose closure $\bar{\mathcal{C}}^+_k$ contains $(\mu^+_k, 0)$ and is either unbounded or contains a point $(\mu^\pm_j, 0)$, with $j \neq k$ in the case of $\mu^+_j$.

Let us first prove that if $(\mu, v) \in \mathcal{C}^+_k$, it implies that $v$ is positive at the origin and possesses $k-1$ zeros in $(0,1)$. Arguing as in the proof of Theorem 1.3, we find a neighborhood $\mathcal{N}$ of $(\mu^+_k, 0)$ such that $\mathcal{N} \cap \mathcal{C}^+_k \subset S^+_k$.

Moreover, if $u \in C^1[0,1]$ is a solution to

$$u'' = M\left(-\frac{(N-1)}{r}m(u') - \mu u + f(u,\mu)\right) \quad \text{in } (0,1) \tag{3.32}$$

and there exists $r_0 \in [0,1]$ such that $u(r_0) = u'(r_0) = 0$, then $u \equiv 0$.

Using this fact we can extend the previous local properties of $\mathcal{C}^+_k$ to all of it. Hence, $\mathcal{C}^+_k$ must be unbounded. □

## References


[1] M. Bardi, F. Da Lio, Propagation of maxima and strong maximum principle for viscosity solution of degenerate elliptic equation I: convex operators, Nonlinear Anal. 44 (2001) 991–1006.

[2] A. Bensoussan, J.L. Lions, Applications of Variational Inequalities in Stochastic Control, Stud. Math. Appl. 12, North-Holland, Amsterdam, 1982, Translated from the French.

[3] H. Berestycki, On some nonlinear Sturm–Liouville problems, J. Differential Equations 26 (3) (1977) 375–390.

[4] H. Berestycki, L. Nirenberg, On the method of moving planes and the sliding method, Boll. Soc. Brasil Mat. (N.S.) 22 (1991) 237–275.

[5] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, Comm. Pure Appl. Math. 47 (1) (1994) 47–92.

[6] X. Cabré, L.A. Caffarelli, Fully Nonlinear Elliptic Equation, Amer. Math. Soc. Colloq. Publ., vol. 43, Amer. Math. Soc., 1995.

[7] L. Caffarelli, J.J. Kohn, L. Nirenberg, J. Spruck, The Dirichlet problem for nonlinear second order elliptic equations II, Comm. Pure Appl. Math. 38 (1985) 209–252.

[8] M. Crandall, H. Ishi, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1) (1992).

[9] M.G. Crandall, M. Kocan, A. Świech, $L^p$ theory for fully nonlinear uniformly parabolic equations, Comm. Partial Differential Equations 25 (11&12) (2000) 1997–2053.

[10] E.N. Dancer, On the Dirichlet problem for weakly non-linear elliptic partial differential equations, Proc. Roy. Soc. Edinburgh Sect. A 76 (4) (1976/77) 283–300.

[11] M. del Pino, M. Elgueta, R. Manásevich, A homotopic deformation along $p$ of a Leray–Schauder degree result and existence for $(|u'|^{p-2}u')' + f(t,u) = 0$, $u(0) = u(T) = 0$, $p > 1$, J. Differential Equations 80 (1) (1989) 1–13.

[12] M. del Pino, R. Manásevich, F. Global bifurcation from the eigenvalues of the $p$-Laplacian, J. Differential Equations 92 (2) (1991) 226–251.

[13] P. Drábek, Solvability and Bifurcations of Nonlinear Equations, Pitman Res. Notes Math. Ser., vol. 264, Longman Scientific and Technical, Harlow, 1992, copublished in the United States with John Wiley and Sons, New York, 1992.

[14] P. Felmer, A. Quaas, Critical exponents for the Pucci's extremal operators, C. R. Acad. Sci. Paris, Ser. I 335 (2002) 909–914.

[15] P. Felmer, A. Quaas, On critical exponents for the Pucci's extremal operators, Ann Inst. H. Poicaré Anal. Non Linéaire 20 (5) (2003) 843–865.

[16] P. Felmer, A. Quaas, Positive solutions to 'semilinear' equation involving the Pucci's operator, J. Differential Equations 199 (2004) 376–393.
[17] D. de Figueiredo, J.P. Gossez, On the first curve of the Fučik spectrum of an elliptic operator, Differential Integral Equations 7 (5–6) (1994) 1285–1302.
[18] S. Fučík, Solvability of Nonlinear Equations and Boundary Value Problems, Math. Appl., vol. 4, Reidel, 1980, With a foreword by Jean Mawhin.
[19] T. Gallouet, O. Kavian, Résultats d'existence et de non-existence pour certains problemes demi-linéaires a l'infini (in French. English summary), (Existence and nonexistence results for certain semilinear problems at infinity), Ann. Fac. Sci. Toulouse Math. 5 (3–4) (1981) 201–246.
[20] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equation of Second Order, second ed., Springer-Verlag, 1983.
[21] P.L. Lions, Bifurcation and optimal stochastic control, Nonlinear Anal. 2 (1983) 177–207.
[22] P.L. Lions, Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. I. The dynamic programming principle and applications, Comm. Partial Differential Equations 8 (10) (1983) 1101–1174.
[23] P.L. Lions, Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. II. Viscosity solutions and uniqueness, Comm. Partial Differential Equations 8 (11) (1983) 1229–1276.
[24] P.L. Lions, Optimal control of diffusion processes and Hamilton–Jacobi–Bellman equations. III. Regularity of the optimal cost function, in: Nonlinear Partial Differential Equations and Their Applications. Collège de France seminar, vol. V (Paris, 1981/1982), 1983, pp. 95–205.
[25] P.L. Lions, A remark on Bony maximum principle, Proc. Amer. Math. Soc. 88 (3) (1983) 503–508.
[26] P.L. Lions, Two remarks on Monge–Ampère equations, Ann. Mat. Pura Appl. 142 (4) (1985) 263–275.
[27] W.M. Ni, R. Nussbaum, Uniqueness and nonuniqueness for positive radial solutions of $\Delta u + f(u,r) = 0$, Comm. Pure Appl. Math. 38 (1985) 67–108.
[28] C. Pucci, Operatori ellittici estremanti, Ann. Mat. Pure Appl. 72 (1966) 141–170.
[29] C. Pucci, Maximum and minimum first eigenvalue for a class of elliptic operators, Proc. Amer. Math. Soc. 17 (1966) 788–795.
[30] A. Quaas, Existence of positive solutions to a 'semilinear' equation involving the Pucci's operator in a convex domain, Differential Integral Equations 17 (2004) 481–494.
[31] P.H. Rabinowitz, Some aspect of nonlinear eigenvalue problem, Rocky Moutain J. Math. 74 (3) (1973) 161–202.
[32] P.H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Funct. Anal. 7 (1971) 487–513.
[33] P.H. Rabinowitz, Théorie du degré topologique et applications à des problèmes aux limites non linéaires, Lectures Notes Lab. Analyse Numérique Université PARIS VI, 1975.
[34] E. Rouy, First semi-eigenvalue for nonlinear elliptic operator, preprint.
[35] M.V. Safonov, On the classical solution of Bellman elliptic equation, Soviet Math. Dokl. 30 (1984).
[36] M. Schechter, The Fučík spectrum, Indiana Univ. Math. J. 43 (4) (1994) 1139–1157.