

The universal Hopf-cyclic theory

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Abstract. We define a Hopf cyclic (co)homology theory in an arbitrary symmetric strict monoidal category. Thus we unify all different types of Hopf cyclic (co)homologies under one single universal theory. We recover Hopf cyclic (co)homology of module algebras, comodule algebras and module coalgebras along with Hopf–Hochschild (co)homology of module algebras, and describe the missing theory for comodule coalgebras.

Mathematics Subject Classification (2000). 16E40, 16W30, 46M18.

Keywords. Hopf-cyclic cohomology, transpositive algebras, monads.

1. Introduction

In noncommutative geometry, as the category of algebras of various flavors replaced the category of spaces of various flavors, Hopf algebras arose as the natural candidate to study the symmetries of a noncommutative space. Unlike the classical notion of symmetry, the notion of noncommutative symmetry has four different types: module coalgebra (MC), comodule algebra (CA), module algebra (MA) and comodule coalgebra (CC). These symmetry conditions can be expressed concisely as the (co)multiplication structure morphism of the corresponding (co)algebra being a B -(co)module morphism where B is our base Hopf algebra [11]. We are interested in such symmetries in the context of cyclic (co)homology. In the following, the term “cyclic theory” is used for functors from a suitable category of (co)algebras into the category of (co)cyclic objects over a small category \mathcal{C} , namely the category of functors from Connes’ cyclic category Λ [5] to the category \mathcal{C} . On the other hand, the term “cyclic (co)homology” is used for suitable (co)homology functors from the category of (co)cyclic objects over a small category \mathcal{C} into a category of modules over a fixed base ring.

Any Hopf algebra H is a module coalgebra and a comodule algebra over itself via its regular representations. Khalkhali and Rangipour [16] showed that the cyclic dual of the canonical cocyclic object of the (MC)-type symmetry evaluated at H is functorially isomorphic to the canonical cyclic object of the (CA)-type symmetry evaluated at the same Hopf algebra H . Then the author and Khalkhali [14] successfully unified

the cyclic theories for the (MA) and (MC)-type symmetries and their cyclic duals under the banner of bivariant Hopf-cyclic cohomology. These results suggest that there is a deep meta-symmetry lurking behind, connecting all these cyclic theories in the presence of a Hopf symmetry.

In this paper, we aim to unravel this meta-symmetry further and construct a new universal cyclic theory covering all types of symmetries we stated above and recover all of the Hopf-cyclic and equivariant cyclic (co)homologies of (co)module (co)algebras previously defined in the literature. Our universal theory relies on categorical approximation (Definition 4.1) results we obtain in Theorem 4.7 and Remark 4.9. Then each individual cohomology theory is obtained by modifying certain parameters. These parameters are (i) a symmetric monoidal category which will replace the category modules over a ground ring k , (ii) a class of morphisms called *transpositions* which will play the role of a coefficient, (iii) an arbitrary exact comonad which will replace a k -flat Hopf algebra and finally (iv) a suitable category of (co)algebras called *transpositive (co)algebras* which will play the role of (co)module (co)algebras.

One *practical* consequence of this formal exercise in category theory is that we no longer need to define a different theory for each type of symmetry and then prove that it really is cyclic, which is quite technical and involved [8], [12], [14]. The recipe we provide in this paper ensures that the end object is not only equivariantly (co)cyclic but also the right object for all known cases. The results of this article give us the license to ignore the technical problems of existence of a right kind of cyclic theory and to engage with more pressing questions such as excision, Morita invariance and homotopy invariance in the presence of a Hopf symmetry. Now that these cyclic theories are defined by universal properties, we expect such questions to become more accessible for further investigation.

Here is a plan of this paper. In Section 2 we give definitions of transpositions and transpositive (co)algebras in an arbitrary symmetric strict monoidal category \mathcal{C} . In the same section we also describe ordinary B -(co)module (co)algebras over an arbitrary bialgebra B as transpositive algebras in a specific monoidal category with respect to certain classes of transpositions. In Section 3 we construct the universal para-(co)cyclic theory for the category of transpositive (co)algebras. Next in Section 4, we incorporate an arbitrary exact comonad \mathbf{B} into our machinery. In this section, specifically in Theorem 4.7, we show that every pseudo-para-(co)cyclic \mathbf{B} -coalgebra admits an *approximation* (Definition 4.1) in the category of (co)cyclic \mathbf{B} -coalgebras. For an arbitrary bialgebra B , in Section 5 we recover the Hopf-cyclic and equivariant cyclic theories of B -module (co)algebras [14] and bialgebra cyclic theory of B -comodule algebras [12]. The key observation we use is that the universal para-cyclic theory actually takes values in the category of pseudo-para-(co)cyclic B -modules in these cases. As a side result, we recover the Hopf–Hochschild homology [13] by using the techniques developed in this paper. We end the paper by defining the missing cyclic theory for comodule coalgebras as a natural extension of the cyclic theories defined hitherto.

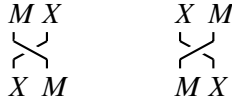
Throughout this article, we assume \mathcal{C} is a small category. If we require \mathcal{C} to be monoidal \otimes will denote the monoidal product of \mathcal{C} and we will assume (\mathcal{C}, \otimes) is a symmetric strict monoidal category with a unit object I .

Acknowledgements. We trace the genesis of this article back to a series of discussions the author had with Krzysztof Worytkiewicz on the formal properties of the Hopf-cyclic (co)homology. We would like to thank the referee for his/her constructive review of the article.

2. Transpositions and transpositive (co)algebras

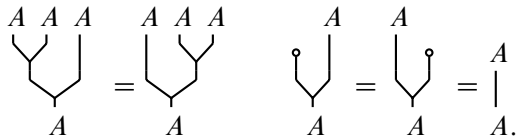
In this section we will use a rudimentary version of “calculus of braid diagrams” in monoidal categories as developed in [18].

Definition 2.1. For each object X chosen from a subset \mathcal{T} of $\text{Ob}(\mathcal{C})$, fix an object M in \mathcal{C} and a unique morphism $w_{M,X}: M \otimes X \rightarrow X \otimes M$. The datum $(M, \mathcal{T}, \{w_{M,X}\}_{X \in \mathcal{T}})$ is called a *class of transpositions* and is denoted by w . For such an object $X \in \mathcal{T}$, the morphism $w_{M,X}$ and its inverse $w_{M,X}^{-1}$ if it exists, are going to be denoted by



respectively. We do not require transpositions to be invertible, nor do we require the inverses to be transpositions themselves even if they exist. But the reader may assume so for convenience.

Recall that an object A in \mathcal{C} is called an algebra if there exist morphisms $\mu_A: A^{\otimes 2} \rightarrow A$ and $e: I \rightarrow A$ which satisfy associativity and unitality axioms. These conditions for an algebra A will also be denoted by the following diagrams:



Definition 2.2. An algebra (A, μ_A, e) in \mathcal{C} is called a *w-transpositive algebra* if there exists a morphism $w_{M,A}: M \otimes A \rightarrow A \otimes M$ in w such that the following diagram

commutes:

$$\begin{array}{ccc}
 M \otimes A \otimes A & \xrightarrow{w_{M,A} \otimes A} & A \otimes M \otimes A \xrightarrow{A \otimes w_{M,A}} A \otimes A \otimes M \\
 \downarrow M \otimes \mu_A & & \downarrow \mu_A \otimes M \\
 M \otimes A & \xrightarrow{w_{M,A}} & A \otimes M \\
 \swarrow M \otimes e & & \searrow e \otimes M \\
 & M &
 \end{array}$$

A w -transpositive coalgebra $(C, \delta_C, \varepsilon)$ is a w^{op} -transpositive algebra in the opposite monoidal category $(\mathcal{C}^{\text{op}}, \otimes^{\text{op}})$ where $U \otimes^{\text{op}} V := V \otimes U$ for any two objects $U, V \in \text{Ob}(\mathcal{C}^{\text{op}})$. These axioms are reminiscent of the half of the axioms for the distributivity laws of [1] or the entwining conditions of [4].

The interaction between the multiplication morphism and the transposition $w_{M,A}$ and its inverse will be denoted by

Similarly, the interaction between the unit morphism and the transposition $w_{M,A}$ and its inverse will be denoted by

For the examples we are going to consider below, we fix a commutative associative unital ring k . Our base symmetric monoidal category is $\text{Mod}(k)$ the category of k -modules with \otimes_k the ordinary tensor product over k taken as the monoidal product \otimes . We also assume B is an associative/coassociative unital/counital bialgebra, or a Hopf algebra with an invertible antipode whenever it is necessary. We refer the reader to [11] for the definitions of (co)module (co)algebras.

Example 2.3. Fix a left B -comodule M . We let $w_{M,X}: M \otimes X \rightarrow X \otimes M$ be a transposition if (i) X is a left B -module and (ii) $w_{M,X}$ is defined by the formula

$$w_{M,X}(m \otimes x) := m_{(-1)x} \otimes m_{(0)}$$

for any $m \otimes x \in M \otimes X$. For this class of transpositions w , an algebra (A, μ_A, e) is w -transpositive if A is a left B -module algebra. Similarly $(C, \delta_C, \varepsilon)$ is a w -transpositive coalgebra, if C is a right B -module coalgebra.

Example 2.4. Fix a right B -module M . We let $w_{M,X}: M \otimes X \rightarrow X \otimes M$ be a transposition if (i) X is a right B -comodule and (ii) $w_{M,X}$ is defined by the formula

$$w_{M,X}(m \otimes x) := x_{(0)} \otimes mx_{(1)}$$

for any $m \otimes x \in M \otimes X$. For this class of transpositions w , an algebra (A, μ_A, e) is w -transpositive if A is a right B -comodule algebra. Similarly $(C, \delta_C, \varepsilon)$ is a w -transpositive coalgebra if C is a left B -comodule coalgebra.

Example 2.5 ([19]). Fix a left B -comodule M and a right B -module N . We let $w_{M \otimes N, X}: M \otimes N \otimes X \rightarrow X \otimes M \otimes N$ be a transposition if X is a left B -comodule and (ii) $w_{M \otimes N, X}$ is defined by the formula

$$w_{M \otimes N, X}(m \otimes n \otimes x) = m_{(-1)x(0)} \otimes m_{(0)} \otimes nx_{(-1)}.$$

In this case, a coalgebra C is a w -transpositive coalgebra if C is both a B -module coalgebra and a B -comodule coalgebra. Similarly, an algebra A is a w -transpositive algebra if A is both a B -module algebra and a B -comodule algebra.

3. The universal para-(co)cyclic theory

Let Λ be Connes' cyclic category [5] and $\Lambda_{\mathbb{N}}$ be the variation of Λ as defined in [14]. A functor $F: \Lambda \rightarrow \mathcal{C}$ will be referred to as a *cocyclic module* in \mathcal{C} , while any functor of the form $F: \Lambda_{\mathbb{N}} \rightarrow \mathcal{C}$ will be referred to as a *para-cocyclic module* in \mathcal{C} . A (para-)cyclic module F in \mathcal{C} is defined to be a (para-)cocyclic module in \mathcal{C}^{op} . A morphism between (para-)cocyclic modules $h: F \rightarrow G$ in \mathcal{C} is just a natural transformation of functors.

Another category we find useful for the purposes of this paper is the category S with objects $\{0, 1\}$ where there is one unique non-trivial morphism $i \rightarrow j$ between any two distinct objects $i, j \in \{0, 1\}$. Any functor of the form $F: S \rightarrow \mathcal{C}$ will be referred to as an S -module.

Suppose that C and M are two objects in \mathcal{C} such that we have a transposition $w_{M,C}: M \otimes C \rightarrow C \otimes M$. For every $n \geq 0$, we define an S -module $P_n(C, M)$ in \mathcal{C} as follows: let $P_n(C, M)$ is the functor from S to \mathcal{C} given on the objects by

$$\begin{aligned} P_n(C, M)(0) &:= C^{\otimes n} \otimes M \otimes C, \\ P_n(C, M)(1) &:= C^{\otimes n+1} \otimes M. \end{aligned} \tag{3.1}$$

Moreover, we let $P_n(C, M)(0 \rightarrow 1) := t_{n+2}$ where $t_{n+2}: C^{\otimes n} \otimes M \otimes C \rightarrow C^{\otimes n+1} \otimes M$ is the cyclic permutation coming from the symmetric monoidal structure of \mathcal{C} . Thus, the inverse of t_{n+2} provides $P_n(C, M)(1 \rightarrow 0)$.

Theorem 3.1. *Let $(C, \delta_C, \varepsilon)$ be a w -transpositive coalgebra. Let $\text{colim}_S P_\bullet(C, M)$ be level-wise colimit of $P_\bullet(C, M)$. Then $\text{colim}_S P_\bullet(C, M)$ carries a para-cocyclic module structure.*

Proof. The cosimplicial structure morphisms are given by

$$\partial_i := C^{\otimes i} \otimes \delta_C \otimes C^{\otimes n-i} \otimes M \quad \text{and} \quad \sigma_j := C^{\otimes j+1} \otimes \varepsilon \otimes C^{\otimes n-1-j} \otimes M,$$

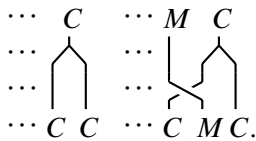
which are defined only for $0 \leq i \leq n$ and $0 \leq j \leq n - 1$ and on $C^{\otimes n+1} \otimes M$. We also let

$$\partial_{n+1} := (C^{\otimes n} \otimes w_{M,C} \otimes C) \circ (C^{\otimes n} \otimes M \otimes \delta_C),$$

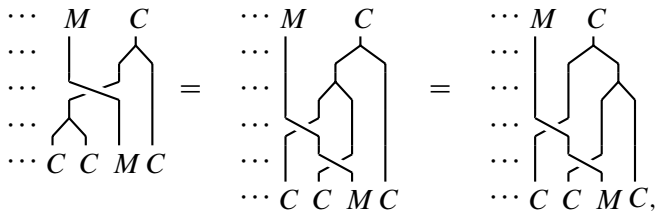
which is a morphism defined on $C^{\otimes n} \otimes M \otimes C$. One can see that we have well-defined morphisms ∂_i and σ_j for $0 \leq i \leq n + 1$ and $0 \leq j \leq n$ on the level-wise colimits. The para-cocyclic structure morphisms are defined as

$$\tau_n : P_n(C, M)(0) \rightarrow P_n(C, M)(1), \quad \tau_n := C^{\otimes n} \otimes w_{M,C},$$

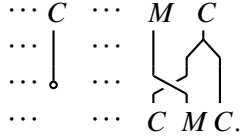
for any $n \geq 0$. Again, one can lift τ_n to the $\text{colim}_S P_n(A, M)$ for any $n \geq 0$. The verification of the cosimplicial identities between ∂_i and σ_j for the range $0 \leq i \leq n$ and $0 \leq j \leq n$ is standard and follows from the fact that C is a coassociative counital coalgebra in \mathcal{C} . As the first non-trivial case we will consider $\partial_j \partial_{n+1}$. If $0 \leq j \leq n$, one can describe the composition by



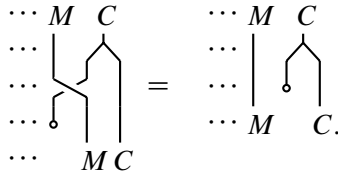
This shows that $\partial_j \partial_{n+1} = \partial_{n+2} \partial_j$ for $0 \leq j \leq n$. For $j = n + 1$, by using the fact that C is a w -transpositive coalgebra, we see that $\partial_{n+1} \partial_{n+1}$ can be described as



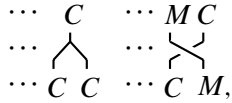
which is equivalent to saying that $\partial_{n+1}\partial_{n+1} = \partial_{n+2}\partial_{n+1}$. This finishes the proof that the level-wise colimit $\text{colim}_S P_\bullet(C, M)$ is pre-cosimplicial. Now we consider $\sigma_i\partial_{n+1}$. If $0 \leq i < n$, the composition can be described by



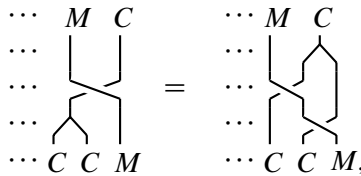
Then one can easily see that $\sigma_i\partial_{n+1} = \partial_n\sigma_i$ for $0 \leq i < n$. We also observe that $\sigma_n\partial_{n+1} = \text{id}$ since



This finishes the proof that $\text{colim}_S P_\bullet(C, M)$ is a cosimplicial object in \mathcal{C} . Now we must check the para-cocyclic identities. First we observe that $\tau_{n+1}\partial_0 = \partial_{n+1}$ by definition. Next we consider $\tau_{n+1}\partial_i$. For the range $0 < i < n$ we represent the composition by



which means that one has $\tau_{n+1}\partial_i = \partial_{i-1}\tau_n$ for the range $0 < i < n$. For $i = n$ we consider $\partial_n\tau_n$, which is represented by



which is equivalent to saying $\partial_n\tau_n = \tau_{n+1}^2\partial_0 = \tau_{n+1}\partial_{n+1}$. So far we have the following relations

$$\partial_i\tau_n = \tau_{n+1}\partial_{i+1} \text{ for } 0 \leq i < n \quad \text{and} \quad \partial_n\tau_n = \tau_{n+1}^2\partial_0.$$

Using these relations one can show that

$$\partial_i\tau_n^j = \tau_{n+1}^{j+p}\partial_q \quad \text{where } (i + j) = (n + 1)p + q$$

for any $n \geq 0, 0 \leq i \leq n + 1$ and $j \geq 0$, i.e., $\text{colim}_S P_\bullet(C, M)$ is a pre-para-cocyclic object in \mathcal{C} . We leave the verification of the identities

$$\tau_n^j \sigma_i = \sigma_q \tau_{n+1}^{i+p} \quad \text{where } (i - j) = (n + 1)(-p) + q$$

involving para-cyclic operators and the codegeneracy operators to the reader. □

For simplicity, the para-cocyclic module $\text{colim}_S P_\bullet(C, M)$ will be denoted by $T_\bullet(C, M)$.

4. Approximation theorems for pseudo-para-(co)cyclic objects

Definition 4.1. Assume \mathcal{C} is an arbitrary small category and \mathcal{D} is a subcategory. For an arbitrary object X of \mathcal{C} a morphism $u_X: \text{App}_{\mathcal{D}}(X) \rightarrow X$ is called the approximation of X within \mathcal{D} if (i) $\text{App}_{\mathcal{D}}(X)$ is an object in \mathcal{D} and (ii) every morphism $v: D \rightarrow X$ with $D \in \text{Ob}(\mathcal{D})$ factors *uniquely* through u_X , i.e., there exists a *unique* morphism $v': D \rightarrow \text{App}_{\mathcal{D}}(X)$ such that $v = u_X \circ v'$. Similarly, the co-approximation $\text{CoApp}_{\mathcal{D}}(X)$ is the approximation of X within \mathcal{D}^{op} viewed as an object of \mathcal{C}^{op} . We do not make any assumptions about the existence of (co)approximations.

Theorem 4.2. *Assume \mathcal{C} is a small category with equalizers. Then the approximation $\text{App}_{\Lambda}(X_\bullet)$ of any para-(co)cyclic module X_\bullet within the category of (co)cyclic modules in \mathcal{C} exists.*

Proof. Every para-(co)cyclic object has a canonical endomorphism ω_\bullet defined at each degree $n \geq 0$ by $\omega_n := \tau_n^{n+1}$, which commutes with all the structure morphisms. The cyclic approximation $\text{App}_{\Lambda}(X_\bullet)$ of a para-(co)cyclic object X_\bullet is defined degree-wise as the equalizer of the pair $(\omega_\bullet, \text{id}_\bullet)$ of para-(co)cyclic modules in \mathcal{C} . Since both ω_\bullet and id_\bullet are morphisms of para-(co)cyclic module in \mathcal{C} , their equalizer $\text{App}_{\Lambda}(X_\bullet) \rightarrow X_\bullet$ is a morphism of para-(co)cyclic modules in \mathcal{C} . Moreover, $\tau_n^{n+1} = \text{id}_n$ on $\text{App}_{\Lambda}(X_n)$, i.e., $\text{App}_{\Lambda}(X_\bullet)$ is a (co)cyclic module in \mathcal{C} . Suppose that we have a morphism $f_\bullet: Y_\bullet \rightarrow X_\bullet$ of para-(co)cyclic modules in \mathcal{C} where Y_\bullet is a (co)cyclic module in \mathcal{C} . Since $\omega_n f_n = \tau_n^{n+1} f_n = f_n \tau_n^{n+1} = f_n$ for any $n \geq 0$, f_\bullet factors through the equalizer $\text{App}_{\Lambda}(X_\bullet)$. □

Recall from [17] that an endo-functor $B: \mathcal{C} \rightarrow \mathcal{C}$ is called a comonad if there exist natural transformations $\Delta: B \rightarrow B^2$ and $\varepsilon: B \rightarrow \text{id}_{\mathcal{C}}$ which satisfy associativity and unitality axioms. We will refer to an object X as a B -coalgebra if there exists a

morphism $\rho_X : X \rightarrow \mathbf{B}(X)$ such that the following diagrams commute:

$$\begin{array}{ccc}
 \mathbf{B}(X) & \xrightarrow{\Delta_X} & \mathbf{B}^2(X) \\
 \rho_X \uparrow & & \uparrow \mathbf{B}(\rho_X) \\
 X & \xrightarrow{\rho_X} & \mathbf{B}(X),
 \end{array}
 \qquad
 \begin{array}{ccc}
 \mathbf{B}(X) & \xrightarrow{\varepsilon_X} & X \\
 \rho_X \swarrow & & \uparrow \text{id}_X \\
 & & X.
 \end{array}$$

A morphism $f : X \rightarrow Y$ between two \mathbf{B} -coalgebras is called a morphism of \mathbf{B} -coalgebras if one has a commuting diagram of the form

$$\begin{array}{ccccc}
 X & \xrightarrow{\rho_X} & \mathbf{B}(X) & \xrightarrow{\varepsilon_X} & X \\
 f \downarrow & & \downarrow \mathbf{B}(f) & & \downarrow f \\
 Y & \xrightarrow{\rho_Y} & \mathbf{B}(Y) & \xrightarrow{\varepsilon_Y} & Y.
 \end{array}$$

The full subcategory of \mathbf{B} -coalgebras in \mathcal{C} is denoted by $\text{CoAlg}(\mathbf{B})$ and the category of \mathbf{B} -coalgebras and their morphisms is denoted by $\mathcal{C}^{\mathbf{B}}$.

Example 4.3. Let (\mathcal{C}, \otimes) be the category of k -modules with ordinary tensor product of modules as the monoidal product. Then any k -coalgebra (C, Δ, ε) defines two comonads $(\cdot \otimes C)$ and $(C \otimes \cdot)$. Moreover, the category of coalgebras in these cases are the same as the category of right and left C -comodules respectively.

Example 4.4. Let (\mathcal{C}, \otimes) be the opposite category of k -modules with ordinary tensor product of modules as the monoidal product. Then any k -algebra (A, μ, e) determines two comonads $(\cdot \otimes A)$ and $(A \otimes \cdot)$. Moreover, the category of coalgebras with respect to these comonads are the same as the category of right and left A -modules respectively.

Definition 4.5. A comonad \mathbf{B} is called left exact (resp. right exact) if \mathbf{B} commutes with arbitrary small limits (resp. colimits). In other words for any functor $F : \mathcal{I} \rightarrow \mathcal{C}$ one has canonical isomorphisms

$$\lim_{\mathcal{I}}(\mathbf{B} \circ F) \cong \mathbf{B}(\lim_{\mathcal{I}} F) \quad (\text{resp. } \text{colim}_{\mathcal{I}}(\mathbf{B} \circ F) \cong \mathbf{B}(\text{colim}_{\mathcal{I}} F)).$$

And a comonad is called exact if it is both left and right exact.

Let Λ_+ be the subcategory of $\Lambda_{\mathbb{N}}$ generated by ∂_j^n and σ_i^n with only $0 \leq i \leq n$ and $0 \leq j \leq n$. In other words, Λ_+ is the subcategory of the category Λ leaving out the cyclic morphisms and the last face morphisms ∂_{n+1}^n at each degree $n \geq 0$.

Definition 4.6. Let \mathbf{B} be a comonad on a category \mathcal{C} . A para-(co)cyclic object $T_{\bullet} : \Lambda_{\mathbb{N}} \rightarrow \text{CoAlg}(\mathbf{B})$ is called a pseudo-para-(co)cyclic \mathbf{B} -coalgebra if its restriction to the subcategory Λ_+ factors through $\mathcal{C}^{\mathbf{B}}$.

Theorem 4.7. *Let \mathbf{B} be a left exact comonad on a complete category \mathcal{C} . Then every pseudo-para-cyclic \mathbf{B} -coalgebra $T_\bullet: \Lambda_{\mathbb{N}}^{\text{op}} \rightarrow \text{CoAlg}(\mathbf{B})$ admits an approximation of the form $\text{App}_\Delta(T_\bullet^{\mathbf{B}}): \Lambda_{\mathbb{N}}^{\text{op}} \rightarrow \mathcal{C}^{\mathbf{B}}$ within the category of cyclic \mathbf{B} -coalgebras.*

Proof. We are going to abuse the notation and use ∂_j, σ_i and τ_n^ℓ to denote $T(\partial_j^n), T(\sigma_i^n)$ and $T(\tau_n^\ell)$ respectively. For every $n \geq 0$, denote the \mathbf{B} -coalgebra structure morphisms $T_n \rightarrow \mathbf{B}(T_n)$ by ρ_n . For any $n \geq 0$, define $\eta_{n,m}: T_n^m \rightarrow T_n$ as the equalizer of the pair of morphisms $\mathbf{B}(\tau_n^m)\rho_n$ and $\rho_n\tau_n^m$ for every $m \in \mathbb{N}$. Now define

$$T_n^{\mathbf{B}} := \lim_{m \in \mathbb{N}} T_n^m \xrightarrow{\eta_{n,m}} T_n,$$

where $\eta_n: T_n^{\mathbf{B}} \rightarrow T_n$ is the canonical morphism into T_n for any $n \geq 0$. Consider the following non-commutative diagram in \mathcal{C} :

$$\begin{array}{ccccc} \mathbf{B}(T_n) & \xrightarrow{\mathbf{B}(\tau_n^j)} & \mathbf{B}(T_n) & \xrightarrow{\mathbf{B}(\tau_n^i)} & \mathbf{B}(T_n) \\ \rho_n \uparrow & & \uparrow \rho_n & & \uparrow \rho_n \\ T_n & \xrightarrow{\tau_n^j} & T_n & \xrightarrow{\tau_n^i} & T_n \\ \eta_n \uparrow & & \uparrow \eta_n & & \\ T_n^{\mathbf{B}} & & T_n^{\mathbf{B}} & & \end{array}$$

Since η_n is the equalizer of the pairs of morphisms $(\rho_n\tau_n^i, \mathbf{B}(\tau_n^i)\rho_n)$ for all $i \in \mathbb{N}$, if we can show that

$$\rho_n\tau_n^i\tau_n^j\eta_n = \mathbf{B}(\tau_n^i)\rho_n\tau_n^j\eta_n \tag{4.1}$$

for all $i \in \mathbb{N}$ we will obtain a functorial ‘restriction’ of τ_n^j to $T_n^{\mathbf{B}}$, which will be denoted by $(\tau_n^j)^{\mathbf{B}}$ for any $j \in \mathbb{N}$. Consider the left-hand side of equation 4.1, which is

$$\rho_n\tau_n^{i+j}\eta_n = \mathbf{B}(\tau_n^{i+j})\rho_n\eta_n = \mathbf{B}(\tau_n^i)\mathbf{B}(\tau_n^j)\rho_n\eta_n = \mathbf{B}(\tau_n^i)\rho_n\tau_n^j\eta_n,$$

as we wanted to show.

Now, for $0 \leq j \leq n + 1$ consider the following diagram in \mathcal{C} :

$$\begin{array}{ccccc} \mathbf{B}(T_{n+1}) & \xrightarrow{\mathbf{B}(\partial_j)} & \mathbf{B}(T_n) & \xrightarrow{\mathbf{B}(\tau_n^i)} & \mathbf{B}(T_n) \\ \rho_{n+1} \uparrow & & \uparrow \rho_n & & \uparrow \rho_n \\ T_{n+1} & \xrightarrow{\partial_j} & T_n & \xrightarrow{\tau_n^i} & T_n \\ \eta_{n+1} \uparrow & & \uparrow \eta_n & & \\ T_{n+1}^{\mathbf{B}} & & T_n^{\mathbf{B}} & & \end{array}$$

Here the square on top right does not commute and the square on top left commutes as long as $0 \leq j \leq n$. However, since $\partial_{n+1} = \partial_0 \tau_{n+1}$ (recall that T_\bullet is cyclic, not cocyclic) and τ_{n+1} has a restriction to T_{n+1}^B , one can assume without loss of generality that $0 \leq j \leq n$. If we can show that

$$\rho_n \tau_n^i \partial_j \eta_{n+1} = \mathbf{B}(\tau_n^i) \rho_n \partial_j \eta_{n+1} \tag{4.2}$$

for any $i \in \mathbb{N}$, one obtains a unique morphism $T_{n+1}^B \rightarrow T_n^B$, which is going to be denoted as $(\partial_j)^B$. The uniqueness of this morphism implies its functoriality. Consider the left-hand side of the equation 4.2,

$$\rho_n \tau_n^i \partial_j \eta_{n+1} = \rho_n \partial_q \tau_{n+1}^{i+p} \eta_{n+1},$$

where $(i + j) = (n + 1)p + q$ and $0 \leq q \leq n$. Now use the fact that $0 \leq j \leq n$ and T_\bullet is a pseudo-para-cyclic to deduce

$$\begin{aligned} \rho_n \partial_q \tau_{n+1}^{i+p} \eta_{n+1} &= \mathbf{B}(\partial_q) \rho_n \tau_{n+1}^{i+p} \eta_{n+1} = \mathbf{B}(\partial_q) \mathbf{B}(\tau_{n+1}^{i+p}) \rho_n \eta_{n+1} \\ &= \mathbf{B}(\tau_{n+1}^i) \mathbf{B}(\partial_j) \rho_n \eta_{n+1} = \mathbf{B}(\tau_{n+1}^i) \rho_n \partial_j \eta_{n+1}, \end{aligned}$$

as we wanted to show. One can similarly prove that the relevant diagrams commute for the degeneracy morphisms. This finishes the proof that T_\bullet^B is a para-cyclic module in \mathcal{C} .

Now, for an arbitrary $j \in \mathbb{N}$ consider the non-commutative diagram

$$\begin{array}{ccccc} T_n & \xrightarrow{\rho_n} & \mathbf{B}(T_n) & \xrightarrow{\mathbf{B}(\rho_n)} & \mathbf{B}^2(T_n) \\ \tau_n^j \uparrow & & \uparrow \mathbf{B}(\tau_n^j) & & \uparrow \mathbf{B}^2(\tau_n^j) \\ T_n & \xrightarrow{\rho_n} & \mathbf{B}(T_n) & \xrightarrow{\mathbf{B}(\rho_n)} & \mathbf{B}^2(T_n) \\ \eta_n \uparrow & & \uparrow \mathbf{B}(\eta_n) & & \\ T_n^B & & \mathbf{B}(T_n^B) & & \end{array}$$

and the composition

$$\begin{aligned} \mathbf{B}^2(\tau_n^j) \mathbf{B}(\rho_n) \rho_n \eta_n &= \mathbf{B}^2(\tau_n^j) \Delta_{T_n} \rho_n \eta_n = \Delta_{T_n} \mathbf{B}(\tau_n^j) \rho_n \eta_n \\ &= \Delta_{T_n} \rho_n \tau_n^j \eta_n = \mathbf{B}(\rho_n) \rho_n \tau_n^j \eta_n \\ &= \mathbf{B}(\rho_n) \mathbf{B}(\tau_n^j) \rho_n \eta_n. \end{aligned}$$

The equality of the first and the last terms implies that $\rho_n \eta_n$ factors through the limit of the equalizers of the pairs $\mathbf{B}^2(\tau_n^j) \mathbf{B}(\rho_n)$ and $\mathbf{B}(\rho_n) \mathbf{B}(\tau_n^j)$ as j runs through the set of all natural numbers. But \mathbf{B} is a left exact comonad, which means that this limit is

exactly $B(T_n^B)$. Thus we get the B -coalgebra structure on T_n^B , which implies T_\bullet^B is a para-cyclic module in $\text{CoAlg}(B)$.

Now we need to show that given any morphism $\phi: [n] \rightarrow [m]$ in $\Lambda_{\mathbb{N}}$, the morphism $\phi^B: T_m^B \rightarrow T_n^B$ is a morphism of B -coalgebras. In order to prove this we need the following diagram to commute for any $i \geq 0$ and $0 \leq j \leq n$:

$$\begin{array}{ccc} B(T_n^B) & \xleftarrow{B(\phi)} & B(T_m^B) \\ \rho_n \uparrow & & \uparrow \rho_m \\ T_n^B & \xleftarrow{\phi^B} & T_m^B \end{array}$$

To achieve this, first we need to show that the larger squares in the following diagrams commute:

$$\begin{array}{ccc} B(T_n) & \xleftarrow{B(\partial_j)} & B(T_{n+1}) \\ \rho_n \uparrow & & \uparrow \rho_n \\ T_n & \xleftarrow{\partial_j} & T_{n+1} \\ \eta_n \uparrow & & \uparrow \eta_n \\ T_n^B & \xleftarrow{(\partial_j)^B} & T_{n+1}^B \end{array} \qquad \begin{array}{ccc} B(T_n) & \xrightarrow{B(\sigma_j)} & B(T_{n+1}) \\ \rho_n \uparrow & & \uparrow \rho_m \\ T_n & \xrightarrow{\sigma_j} & T_{n+1} \\ \eta_n \uparrow & & \uparrow \eta_n \\ T_n^B & \xrightarrow{(\sigma_j)^B} & T_{n+1}^B \end{array}$$

In these diagrams the top squares commute since T_\bullet is pseudo-para-cyclic. We have already shown that the bottom squares commute. Thus both diagrams commute for the prescribed range. Then we must show that the larger square in the following diagram commutes:

$$\begin{array}{ccc} B(T_n) & \xrightarrow{B(\tau_n^i)} & B(T_n) \\ \rho_n \uparrow & & \uparrow \rho_n \\ T_n & \xrightarrow{\tau_n^i} & T_n \\ \eta_n \uparrow & & \uparrow \eta_n \\ T_n^B & \xrightarrow{(\tau_n^i)^B} & T_n^B \end{array}$$

The bottom square commutes, while the top square does not. However, η_n equalizes $\rho_n \tau_n^i$ and $B(\tau_n^i) \rho_n$. Therefore the larger diagram commutes. This finally finishes the proof that T_\bullet^B is a para-cyclic B -coalgebra. Now we use Theorem 4.2 to complete the proof. □

Theorem 4.8. *Let \mathbf{B} be a left exact comonad on a complete category \mathcal{C} . Then every pseudo-para-cocyclic \mathbf{B} -coalgebra $T_\bullet: \Lambda_{\mathbb{N}} \rightarrow \text{CoAlg}(\mathbf{B})$ admits an approximation of the form $\text{App}_\Lambda(T_\bullet^{\mathbf{B}}): \Lambda_{\mathbb{N}} \rightarrow \mathcal{C}^{\mathbf{B}}$ within the category of cocyclic \mathbf{B} -coalgebras.*

Proof. As before let ρ_n denote the \mathbf{B} -coalgebra structure morphism on T_n for any $n \geq 0$. Let $\Gamma(n)$ be the set of pairs of morphism of the form

$$(\rho_n \tau_n^i, \mathbf{B}(\tau_n^i) \rho_n) \text{ for } i \geq 0 \quad \text{or} \quad (\rho_n \partial_{n+1}, \mathbf{B}(\partial_{n+1}) \rho_n),$$

and we define $\eta(\gamma): T_n^\gamma \rightarrow T_n$ as the equalizer of a pair $\gamma \in \Gamma(n)$. Next we define the approximation $T_n^{\mathbf{B}}$ for each $n \geq 0$ as

$$T_n^{\mathbf{B}} := \lim_{\gamma} T_n^\gamma \xrightarrow{\eta(\gamma)} T_n,$$

where we use $\eta_n: T_n^{\mathbf{B}} \rightarrow T_n$ to denote the canonical morphism into T_n . The rest of the proof is very similar to that of Theorem 4.7, and we leave it to the reader. \square

Remark 4.9. There are eight versions of Theorem 4.7

Assume \mathbf{B} is an exact (co)monad on \mathcal{C} . Then any pseudo-para-(co)cyclic \mathbf{B} -(co)algebra in \mathcal{C} admits an(a) (co)approximation in the category of (co)cyclic \mathbf{B} -(co)algebras.

By assuming \mathcal{C} is both complete and cocomplete, one can use \mathcal{C} and \mathcal{C}^{op} interchangeably. This reduces the number of versions to four:

Assume \mathbf{B} is an exact (co)monad on \mathcal{C} . Then any pseudo-para-(co)cyclic \mathbf{B} -(co)algebra in \mathcal{C} admits an approximation in the category of (co)cyclic \mathbf{B} -(co)algebras.

We gave proofs for two of these statements in Theorem 4.7 and Theorem 4.8 above. The proofs of the remaining 2 statements are very similar and therefore will be omitted.

Definition 4.10. Assume that \mathbf{B} is a (co)monad on \mathcal{C} . The (co)cyclic \mathbf{B} -(co)algebra $\text{App}_\Lambda(T_\bullet^{\mathbf{B}})$ corresponding to a pseudo-para-(co)cyclic \mathbf{B} -(co)algebra T_\bullet is called the universal (co)cyclic \mathbf{B} -(co)algebra of T_\bullet . Moreover, given a functor of the form $\mathcal{F}: \text{CoAlg}(\mathbf{B}) \rightarrow \text{Mod}(k)$ (resp. $\mathcal{F}: \text{Alg}(\mathbf{B}) \rightarrow \text{Mod}(k)$) and a (co)homology functor \mathcal{H}_* on the category of (co)cyclic k -modules, one can compute

$$\mathcal{H}_* \mathcal{F}(\text{App}_\Lambda(T_\bullet^{\mathbf{B}})).$$

We will call this (co)homology the \mathbf{B} -equivariant \mathcal{H} -(co)homology of T_\bullet with coefficients in \mathcal{F} .

5. The universal cyclic theory of (co)module (co)algebras

Cyclic cohomology of Hopf algebras is defined by Connes and Moscovici in their work on the transverse index theorem [6], [7]. This theory evolved to include algebras admitting Hopf symmetries [8]. However, a (co)cyclic theory and basic tools of cyclic cohomology had to be built from scratch for each type of symmetry separately [8], [12], [14]. In the following, we give the construction of Hopf-cyclic theories for (co)module (co)algebras including the missing case for comodule coalgebras using the universal Hopf-cyclic theory we developed above.

5.1. Hopf and equivariant cyclic theory of module coalgebras. Fix a commutative unital ring k and an associative/coassociative unital/counital k -bialgebra B . Our base category is the category of k -modules with the ordinary tensor product over k , i.e., $(\mathcal{C}, \otimes) := (\text{Mod}(k), \otimes_k)$. Our base monad in \mathcal{C} is going to be $\mathbf{B} := (B \otimes \cdot)$. Since we use B as a monad, we will use the algebra structure on B .

The category of \mathbf{B} -algebras in \mathcal{C} (i.e., left B -modules) is a monoidal category with respect to the ordinary tensor product of k -modules with the diagonal action of B on the left. Explicitly, given a pair of B -modules X and Y , the B -module structure on the product is given by

$$b(x \otimes y) := b_{(1)}x \otimes b_{(2)}y$$

for any $x \otimes y \in X \otimes Y$. However, the product is not symmetric unless B is co-commutative, but there is a braided monoidal structure if one restricts oneself to use Yetter–Drinfeld modules. If we denote the full subcategory of left B -modules of $\text{Mod}(k)$ by $\mathcal{L}(B)$ then one can see that $\text{Alg}(\mathbf{B}) = \mathcal{L}(B)$. Therefore, the subcategory $\mathcal{C}^{\mathbf{B}}$ of \mathcal{C} which consists of \mathbf{B} -algebras and their morphisms is the monoidal category of left B -modules $\text{Mod}(B)$.

Fix a left/left B -module/comodule M and for each $X \in \text{Ob}(\mathcal{C}^{\mathbf{B}})$ define a transposition $w_{M,X} : M \otimes X \rightarrow X \otimes M$ by

$$w_{M,X}(m \otimes x) := m_{(-1)}x \otimes m_{(0)}$$

for any $m \otimes x \in M \otimes X$, as in Example 2.3. Any coalgebra $(C, \delta_C, \varepsilon)$ in $\mathcal{C}^{\mathbf{B}}$ is a B -module coalgebra and therefore is automatically w -transpositive. We form the objects $P_{\bullet}(C, M)$ and $T_{\bullet}(C, M) := \text{colim}_S P_{\bullet}(C, M) \cong \bigoplus_{n \geq 0} C^{\otimes n+1} \otimes M$ in \mathcal{C} and consider the latter as a para-cocyclic module in \mathcal{C} . The structure maps are defined as

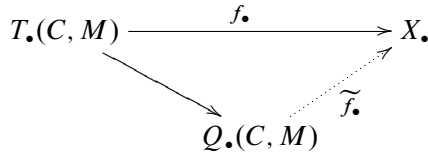
$$\begin{aligned} \partial_j^n(c^0 \otimes \cdots \otimes c^n \otimes m) &= \begin{cases} (\cdots \otimes c_{(1)}^j \otimes c_{(2)}^j \otimes \cdots \otimes m) & \text{if } 0 \leq j \leq n, \\ (c_{(1)}^0 \otimes c^1 \otimes \cdots \otimes c^n \otimes m_{(-1)}c_{(2)}^0 \otimes m_{(0)}) & \text{if } j = n + 1; \end{cases} \end{aligned}$$

$$\begin{aligned} & \sigma_i^n(c^0 \otimes \cdots \otimes c^n \otimes m) \\ &= \begin{cases} \varepsilon(c^{j+1})(c^0 \otimes \cdots \otimes c^j \otimes c^{j+2} \otimes \cdots \otimes m) & \text{if } 0 \leq j \leq n-1, \\ \varepsilon(c^0)(c^1 \otimes \cdots \otimes c^n \otimes m) & \text{if } j = n; \end{cases} \\ & \tau_n(c^0 \otimes \cdots \otimes c^n \otimes m) = (c^1 \otimes \cdots \otimes c^n \otimes m_{(-1)}c^0 \otimes m). \end{aligned}$$

In fact $T_\bullet(C, M)$ carries a pseudo-para-cocyclic B -algebra structure. In this context this means that (i) $T_\bullet(C, M)$ is a graded B -module, (ii) $T_\bullet(C, M)$ is a para-cocyclic k -module, (iii) every $\partial_j^n: T_n(C, M) \rightarrow T_{n+1}(C, M)$ for $0 \leq j \leq n$ is a B -module map. Note that as a part of the definition, we exclude ∂_{n+1}^n and τ_n from being B -module maps for any $n \geq 0$. Then we define

$$Q_\bullet(C, M) := \text{App}_\Lambda(T_\bullet(C, M)^B),$$

which is the quotient of the $T_\bullet(C, M)$ by the smallest B -submodule and cocyclic k -submodule generated by the images of the commutators $[L_b, \tau_n^i]$ of the linear operators τ_n^i and L_b (left action by $b \in B$) for any $n \geq 0, i \in \mathbb{N}$ and $b \in B$. Then $Q_\bullet(C, M)$ is the largest quotient of $T_\bullet(C, M)$ which is a cocyclic B -module. In other words, if we have a commutative diagram



where X_\bullet is a cocyclic B -module and f_\bullet is a morphism of graded B -modules and para-cocyclic k -modules, then one has a unique morphism \tilde{f}_\bullet of cocyclic B -modules. This is the Hopf-equivariant cocyclic object defined in [14] for a B -module coalgebra C and an arbitrary B -module/comodule M . Therefore, the Hopf cyclic cohomology of the triple (C, B, M) is defined as the cyclic cohomology of the cocyclic k -module $C_\bullet(C, M) := k \otimes_B Q_\bullet(C, M)$.

Similarly, any algebra $(A, \mu_A, 1)$ in \mathcal{C}^B is a B -module algebra and therefore is automatically w -transpositive. We form the objects $P_\bullet(A, M)$ and $T_\bullet(A, M) := \text{colim}_S P_\bullet(A, M) \cong \bigoplus_{n \geq 0} A^{\otimes n+1} \otimes M$ in \mathcal{C} and consider the latter as a para-cyclic module in \mathcal{C} . In fact $T_\bullet(A, M)$ carries a pseudo-para-cyclic B -algebra structure. In this context this means that (i) $T_\bullet(A, M)$ is a graded B -module, (ii) $T_\bullet(A, M)$ is a para-cyclic k -module and (iii) every $\partial_j^{n-1}: T_n(C, M) \rightarrow T_{n-1}(C, M)$ for $0 \leq j \leq n-1$ is a B -module. As a part of the definition, we again exclude $\partial_n^{n-1}: T_n(A, M) \rightarrow T_{n-1}(A, M)$ and τ_n from being B -module maps for any $n \geq 1$. If we let

$$Q_\bullet(A, M) := \text{CoApp}_\Lambda(T_\bullet(A, M)^B)$$

then $Q_\bullet(A, M)$ is the largest pseudo-para-cyclic submodule of $T_\bullet(A, M)$ which is a cyclic B -module. This is the Hopf-equivariant cyclic object defined in [14] for a B -module algebra A and an arbitrary B -module/comodule M . Therefore, the Hopf-cyclic (co)homology of the triple (A, B, M) is defined as the cyclic (co)homology of the cyclic k -module $C_\bullet(A, M) := k \otimes_B Q_\bullet(A, M)$.

5.2. Hopf–Hochschild homology. Let $(\mathcal{C}, \otimes), M, w, B$ and \mathbf{B} be as before. Assume that A is a B -module algebra and construct $P_\bullet(A, M)$ in \mathcal{C} . Again, we define $T_n(A, M)$ as the colimit of $P_n(A, M): S \rightarrow \mathcal{C}$ for each n . Then $T_\bullet(A, M) \cong \bigoplus_{n \geq 0} A^{\otimes n+1} \otimes M$ is a pseudo-para-cyclic object in $\mathcal{C}^{\mathbf{B}}$. Observe that any para-cyclic object is also a simplicial object. Now define $T'_\bullet(A, M)$ as $\text{App}_\Delta T_\bullet(A, M)^{\mathbf{B}}$ replacing the cyclic category Λ by its simplicial subcategory Δ . This is the largest quotient of $T_\bullet(A, M)$ which is a simplicial B -module. The singular homology of the simplicial module $k \otimes_B T'_\bullet(A, M)$ is the Hopf–Hochschild homology of the triple (A, B, M) as constructed in [13].

5.3. Hopf and equivariant cyclic theory of comodule (co)algebras. Fix a commutative unital ring k and a Hopf algebra B . Our base category is the category of k -modules with the ordinary tensor product over k , i.e., $(\mathcal{C}, \otimes) := (\text{Mod}(k), \otimes_k)$. Our base comonad in \mathcal{C} is going to be $\mathbf{B} := (B \otimes \cdot)$, thus we will use the coalgebra structure on B .

The category of left B -comodules (i.e., \mathbf{B} -coalgebras in \mathcal{C}) is a monoidal category with respect to the ordinary tensor product of k -modules, with the diagonal coaction of B on the left. Explicitly, given a pair of B -modules X and Y , the B -comodule structure on the product is given by

$$\rho(x \otimes y) := x_{[-1]}y_{[-1]} \otimes (x_{[0]} \otimes y_{[0]})$$

for any $x \otimes y \in X \otimes Y$. However, the product is not symmetric unless B is commutative, but there is a braided monoidal structure if one restricts oneself to use Yetter–Drinfeld modules.

Fix a left/left B -module/comodule M and for each $X \in \text{Ob}(\mathcal{C}^{\mathbf{B}})$ define a transposition $w_{M,X}: M \otimes X \rightarrow X \otimes M$ by

$$w_{M,X}(m \otimes x) := x_{[0]} \otimes x_{[-1]}m$$

for any $m \otimes x \in M \otimes X$, as in Example 2.4.

Any coalgebra $(C, \delta_C, \varepsilon)$ in $\text{CoAlg}(\mathbf{B})$ is a B -comodule coalgebra and therefore is automatically w -transpositive. We form the objects $P_\bullet(C, M)$ and $T_\bullet(C, M) := \text{colim}_S P_\bullet(C, M) \cong \bigoplus_{n \geq 0} C^{\otimes n+1} \otimes M$ in \mathcal{C} and consider the latter as a para-

cocyclic module in \mathcal{C} . The structure maps are defined as

$$\begin{aligned} \partial_0^n(c^0 \otimes \cdots \otimes c^n \otimes m) &= (c_{(1)}^0 \otimes c_{(2)}^0 \otimes c^1 \otimes \cdots \otimes c^n \otimes m), \\ \sigma_0^n(c^0 \otimes \cdots \otimes c^n \otimes m) &= \varepsilon(c^1)(c^0 \otimes c^2 \otimes \cdots \otimes m), \\ \tau_n(c^0 \otimes \cdots \otimes c^n \otimes m) &= (c^1 \otimes \cdots \otimes c^n \otimes c_{[0]}^0 \otimes c_{[-1]}^0 m), \end{aligned}$$

with $\partial_j^n := \tau_{n+1}^{-j} \partial_0^n \tau_n^j$ and $\sigma_j^n = \tau_{n-1}^{-j} \sigma_0^n \tau_n^j$ for appropriate j . Here we distinguish the B -comodule structure and coalgebra structures on C by using $\lambda_C(c) = c_{[-1]} \otimes c_{[0]}$ for the former and $\Delta_C(c) = c_{(1)} \otimes c_{(2)}$ for the latter. In fact $T_\bullet(C, M)$ carries a pseudo-para-cocyclic B -comodule structure. In this context this means that $T_\bullet(C, M)$ is a graded B -module and the structure maps ∂_j^n and σ_i^n are all B -module maps except ∂_{n+1}^n and τ_n^i for any $n \geq 0$ and $i \in \mathbb{Z}$. Then we define

$$Q_\bullet(C, M) := \text{App}_\Lambda(T_\bullet(C, M)^B),$$

which is the quotient of $T_\bullet(C, M)$ by the smallest graded B -submodule and cocyclic k -submodule generated by the images of the commutators $[L_b, \tau_n^i]$ of the linear operators τ_n^i and L_b with $b \in B$ and $i \in \mathbb{Z}$. Then $Q_\bullet(C, M)$ is a cocyclic B -coalgebra, i.e., a cocyclic B -comodule, and we define the Hopf-cyclic cohomology of C with coefficients in M as the cyclic cohomology of the cocyclic k -module $k \otimes_B Q_\bullet(C, M)$.

Similarly, any algebra $(A, \mu_A, 1)$ in the category $\text{CoAlg}(B)$ is a B -comodule algebra and therefore is automatically w -transpositive. We form the objects $P_\bullet(A, M)$ and $T_\bullet(A, M) := \text{colim}_S P_\bullet(A, M)$ in \mathcal{C} and consider the latter as a para-cyclic module in \mathcal{C} . In fact $T_\bullet(A, M)$ carries a pseudo-para-cyclic B -coalgebra structure and we see that

$$Q_\bullet(A, M) := \text{CoApp}_\Lambda(T_\bullet(A, M)^B)$$

is the largest pseudo-para-cyclic submodule of $T_\bullet(A, M)$ which is a cyclic B -comodule. Moreover, the cyclic cohomology of the cyclic k -module $k \otimes_B Q_\bullet(A, M)$ is the bialgebra cyclic homology of a module coalgebra as defined in [12].

Remark 5.1. Based on Connes and Moscovici’s seminal work [6], [7] on cyclic cohomology of Hopf algebras, we see two parallel yet different families of (co)cyclic theories for (co)module (co)algebras in the literature. One starts with [8], [9] by Hajac, Khalkhali, Rangipour and Sommerhäuser, which lead to [12] by the author and [14] by Khalkhali and the author, which in turn forms the basis of this article. The parallel family starts with [15] by Khalkhali and Rangipour and then [10] by Jara and Ştefan and evolves into [2], [3] by Böhm and Ştefan, where the authors expand the cyclic theories we develop in this paper for arbitrary monoidal categories in the dual direction. Namely, the authors construct a cyclic object for a coalgebra and a cocyclic object for an algebra. We now know that the (co)cyclic modules developed in [10] and [8] are cyclic duals of each other (in the sense of Connes’ [5]) thanks to

[16] by Khalkhali and Rangipour. We conjecture that (co)cyclic theories developed here and in [2] are cyclic duals for symmetric strict monoidal categories, provided that the class of transpositions satisfy the analogue of the SAYD condition. Nonetheless, we do recover the dual Hopf-cyclic cohomology of (co)module (co)algebras with arbitrary coefficients via the bivariant Hopf-cyclic cohomology developed in [14] and the (co)cyclic theories developed in this paper using the classical version of Khalkhali–Rangipour duality isomorphism [16].

References

- [1] J. Beck, Distributive laws. In *Seminar on triples and categorical homology theory* (ETH, Zürich, 1966/67), Lecture Notes in Math. 80, Springer, Berlin 1969, 119–140. [Zbl 0186.02902](#) [MR 0241502](#)
- [2] G. Böhm and D. Ştefan, (Co)cyclic (co)homology of bialgebroids: an approach via (co)monads. Preprint 2007. [arXiv:0705.3190](#).
- [3] G. Böhm and D. Ştefan, Examples of para-cocyclic objects induced by BD-laws. Preprint 2007. [arXiv:0801.0033](#).
- [4] T. Brzezinski and R. Wisbauer, *Corings and comodules*. London Math. Soc. Lecture Note Ser. 309, Cambridge University Press, Cambridge 2003. [Zbl 1035.16030](#) [MR 2012570](#)
- [5] A. Connes, Cohomologie cyclique et foncteurs Ext^n . *C. R. Acad. Sci. Paris Sér. I Math.* **296** (1983), 953–958. [Zbl 0534.18009](#) [MR 777584](#)
- [6] A. Connes and H. Moscovici, Hopf algebras, cyclic cohomology and the transverse index theorem. *Comm. Math. Phys.* **198** (1998), 199–246. [Zbl 0940.58005](#) [MR 1657389](#)
- [7] A. Connes and H. Moscovici, Cyclic cohomology and Hopf algebra symmetry. *Lett. Math. Phys.* **52** (2000), 1–28. [Zbl 0974.58006](#) [MR 1800488](#)
- [8] P. M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser, Hopf-cyclic homology and cohomology with coefficients. *C. R. Math. Acad. Sci. Paris* **338** (2004), 667–672. [Zbl 1064.16006](#) [MR 2065371](#)
- [9] P. M. Hajac, M. Khalkhali, B. Rangipour, and Y. Sommerhäuser, Stable anti-Yetter–Drinfeld modules. *C. R. Math. Acad. Sci. Paris* **338** (2004), 587–590. [Zbl 1060.16037](#) [MR 2056464](#)
- [10] P. Jara and D. Ştefan, Hopf-cyclic homology and relative cyclic homology of Hopf–Galois extensions. *Proc. London Math. Soc.* (3) **93** (2006), 138–174. [Zbl 05044847](#) [MR 2235945](#)
- [11] C. Kassel, *Quantum groups*. Grad. Texts in Math. 155, Springer-Verlag, New York 1995. [Zbl 0808.17003](#) [MR 1321145](#)
- [12] A. Kaygun, Bialgebra cyclic homology with coefficients. *K-Theory* **34** (2005), 151–194. [Zbl 1114.16011](#) [MR 2180111](#)
- [13] A. Kaygun, Hopf-Hochschild (co)homology of module algebras. *Homology, Homotopy Appl.* **9** (2007), 451–472. [Zbl 1130.16009](#) [MR 2366958](#)

- [14] A. Kaygun and M. Khalkhali, Bivariant Hopf cyclic cohomology. Preprint 2006. [arXiv:math/0606341](https://arxiv.org/abs/math/0606341).
- [15] M. Khalkhali and B. Rangipour, A new cyclic module for Hopf algebras. *K-Theory* **27** (2002), 111–131. [Zbl 1020.16029](https://zbmath.org/?q=1020.16029) [MR 1942182](https://mathscinet.org/mr/1942182)
- [16] M. Khalkhali and B. Rangipour, A note on cyclic duality and Hopf algebras. *Comm. Algebra* **33** (2005), 763–773. [Zbl 1089.16011](https://zbmath.org/?q=1089.16011) [MR 2128410](https://mathscinet.org/mr/2128410)
- [17] S. Mac Lane, *Categories for the working mathematician*. 2nd ed., Grad. Texts in Math. 5, Springer-Verlag, New York 1998. [Zbl 0906.18001](https://zbmath.org/?q=0906.18001) [MR 1712872](https://mathscinet.org/mr/1712872)
- [18] S. Majid, Cross products by braided groups and bosonization. *J. Algebra* **163** (1994), 165–190. [Zbl 0807.16036](https://zbmath.org/?q=0807.16036) [MR 1257312](https://mathscinet.org/mr/1257312)
- [19] M. Moscovici and B. Rangipour, Hopf algebras of primitive Lie pseudogroups and Hopf cyclic cohomology. Preprint 2008. [arXiv:0803.1320](https://arxiv.org/abs/0803.1320).

Received March 4, 2008; revised April 1, 2008

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