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Intrinsic Lipschitz graphs in Heisenberg groups and continuous solutions of a balance equation

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Abstract

We provide a characterization of intrinsic Lipschitz graphs in the sub-Riemannian Heisenberg groups in terms of their distributional gradients. Moreover, we prove the equivalence of different notions of continuous weak solutions to the equation $\frac{\partial\phi}{\partial z} + \frac{\partial}{\partial t}[\phi^2/2] = w$, where w is a bounded measurable function.

Keywords: Intrinsic Lipschitz graphs; Heisenberg groups; Lagrangian formulation; Scalar balance laws; Continuous solutions; Peano phenomenon

1. Introduction

In the last years it has been largely developed the study of intrinsic submanifolds inside the Heisenberg groups $\mathbb{H}^n$ or more general Carnot groups, endowed with their Carnot–Carathéodory metric structure, also named sub-Riemannian. By an intrinsic regular (or intrinsic Lipschitz) hypersurfaces we mean a submanifold which has, in the intrinsic geometry of $\mathbb{H}^n$, the same role like a C^1 (or Lipschitz) regular graph has in the Euclidean geometry. Intrinsic regular graphs had several applications within the theory of rectifiable sets and minimal surfaces in CC geometry, in theoretical computer science, geometry of Banach spaces and mathematical models in neurosciences, see [8,9] and the references therein.

We postpone complete definitions of $\mathbb{H}^n$ to Section 2. We only remind that the Heisenberg group $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R} \equiv \mathbb{R}^{2n+1}$ is the simplest example of Carnot group, endowed with a left-invariant metric d_∞ (equivalent to its Carnot–Carathéodory metric), not equivalent to the Euclidean metric. $\mathbb{H}^n$ is a (connected, simply connected and

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stratified) Lie group and has a sufficiently rich compatible underlying structure, due to the existence of intrinsic families of left translations and dilations and depending on the horizontal vector fields $X_1, \dots, X_{2n}$. We call intrinsic any notion depending directly by the structure and geometry of $\mathbb{H}^n$. For a complete description of Carnot groups [8,21,25,26] are recommended.

As we said, we will study intrinsic submanifolds in $\mathbb{H}^n$. An intrinsic regular hypersurface $S \subset \mathbb{H}^n$ is locally defined as the non-critical level set of a horizontal differentiable function, more precisely there exists locally a continuous function $f : \mathbb{H}^n \to \mathbb{R}$ such that $S = \{f = 0\}$ and the intrinsic gradient $\nabla_{\mathbb{H}} f = (X_1 f, \dots, X_{2n} f)$ exists in the sense of distributions and it is continuous and non-vanishing on S. Intrinsic regular hypersurfaces can be locally represented as X_1-graph by a function $\phi : \omega \subset \mathbb{W} \equiv \mathbb{R}^{2n} \to \mathbb{R}$, where $\mathbb{W} = \{x_1 = 0\}$, through an implicit function theorem (see [17]). In [4,6,7] the parameterization ϕ has been characterized as weak solution of a system of non-linear first order PDEs $\nabla^\phi \phi = w$, where $w : \omega \to \mathbb{R}^{2n-1}$ and $\nabla^\phi = (X_2, \dots, X_n, \frac{\partial}{\partial x_{n+1}} + \phi \frac{\partial}{\partial t}, X_{n+2}, \dots, X_{2n})$ (see Theorem 2.9). By an intrinsic point of view, the operator $\nabla^\phi \phi$ acts as the intrinsic gradient of the function $\phi : \mathbb{W} \to \mathbb{R}$. In particular it can be proved that ϕ is a continuous distributional solution of the problem $\nabla^\phi \phi = w$ with $w \in C^0(\omega; \mathbb{R}^{2n-1})$ if and only if ϕ induces an intrinsic regular graph (see [7]).

Let us point out that an intrinsic regular graph can be very irregular from the Euclidean point of view: indeed, there are examples of intrinsic regular graphs in $\mathbb{H}^1$ which are fractal sets in the Euclidean sense [24].

The aim of our work is to characterize intrinsic Lipschitz graphs in terms of the intrinsic distributional gradient. It is well-know that in the Euclidean setting a Lipschitz graph $S = \{(A, \phi(A)) : A \in \omega\}$, with $\phi : \omega \subset \mathbb{R}^m \to \mathbb{R}$ can be equivalently defined

- by means of cones: there exists $L > 0$ such that
$$C\left(\left(A_0, \phi(A_0)\right); \frac{1}{L}\right) \cap S = \left\{\left(A_0, \phi(A_0)\right)\right\}$$
for each $A_0 \in \omega$, where $C((A_0, \phi(A_0)); \frac{1}{L}) = \{(A, s) \in \mathbb{R}^m \times \mathbb{R} : |A - A_0|_{\mathbb{R}^m} \le \frac{1}{L}|s - \phi(A_0)|\}$;
- in a metric way: there exists $L > 0$ such that $|\phi(A) - \phi(B)| \le L|A - B|$ for every $A, B \in \omega$;
- by the distributional derivatives: there exist the distributional derivatives
$$\frac{\partial \phi}{\partial x_i} \in L^\infty(\omega) \quad \forall i = 1, \dots, m$$
provided that ω is a regular connected open bounded set.

Intrinsic Lipschitz graphs in $\mathbb{H}^n$ have been introduced in [19,20], by means of a suitable notion of intrinsic cone in $\mathbb{H}^n$. As a consequence, the metric definition (see Definition 2.13) is given with respect to the graph quasidistance d_ϕ, (see (2.6)) i.e the function $\phi : (\omega, d_\phi) \to \mathbb{R}$ is meant Lipschitz in classical metric sense. This notion turns out to be the right one in the setting of the intrinsic rectifiability in $\mathbb{H}^n$. Indeed, for instance, it was proved in [20] that the notion of rectifiable set in terms of an intrinsic regular hypersurface is equivalent to the one in terms of intrinsic Lipschitz graphs.

We will denote by $\mathrm{Lip}_{\mathbb{W}}(\omega)$ the class of all intrinsic Lipschitz function $\phi : \omega \to \mathbb{R}$ and by $\mathrm{Lip}_{\mathbb{W},\mathrm{loc}}(\omega)$ the one of locally intrinsic Lipschitz functions. Notice that $\mathrm{Lip}_{\mathbb{W}}(\omega)$ is not a vector space and that

$$\mathrm{Lip}(\omega) \subsetneq \mathrm{Lip}_{\mathbb{W},\mathrm{loc}}(\omega) \subsetneq C^{0,1/2}_{\mathrm{loc}}(\omega),$$

where $\mathrm{Lip}(\omega)$ and $C^{0,1/2}_{\mathrm{loc}}(\omega)$ denote respectively the classes of Euclidean Lipschitz and 1/2-Hölder functions in ω. For a complete presentation of intrinsic Lipschitz graphs [11,20] are recommended.

The first main result of this paper is the characterization of a parameterization $\phi : \omega \to \mathbb{R}$ of an intrinsic Lipschitz graph as a continuous distributional solution of $\nabla^\phi \phi = w$, where $w \in L^\infty(\omega; \mathbb{R}^{2n-1})$.

Theorem 1.1. *Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be an open set, $\phi : \omega \to \mathbb{R}$ be a continuous function. $\phi \in \mathrm{Lip}_{\mathbb{W},\mathrm{loc}}(\omega; \mathbb{R})$ if and only if there exists $w \in L^\infty_{\mathrm{loc}}(\omega; \mathbb{R}^{2n-1})$ such that ϕ is a distributional solution of the system $\nabla^\phi \phi = w$ in ω.*

We stress that this is indeed different from proving a Rademacher theorem, which is more related to a pointwise rather than distributional characterization for the derivative, see [20]. Nevertheless, we find that the density of the (intrinsic) distributional derivative is indeed given by the function one finds by Rademacher theorem. We also stress that there are a priori different notions of *continuous* solutions $\phi : \omega \to \mathbb{R}$ to $\nabla^\phi \phi = w$, which express the Lagrangian and Eulerian viewpoints. They will turn out to be equivalent descriptions of intrinsic Lipschitz graphs, when the source w belongs to $L^\infty(\omega; \mathbb{R}^{2n-1})$. This is proved in Section 6 and it is summarized in the following theorem.

We introduce first a notation: $\mathfrak{L}^\infty(\omega; \mathbb{R}^{2n-1})$ denotes the set of measurable bounded functions from ω to $\mathbb{R}^{2n-1}$, while $L^\infty(\omega; \mathbb{R}^{2n-1})$ denotes the equivalence classes of Lebesgue measurable functions in $\mathfrak{L}^\infty(\omega; \mathbb{R}^{2n-1})$ which are identified when differing on a Lebesgue negligible set of ω. We will keep this notation throughout the paper: its relevance is explained by Examples 1.3, 1.4 below.

Theorem 1.2. *Let $\phi : \omega \to \mathbb{R}$ be a continuous function. The following conditions are equivalent*

(i) *ϕ is a distributional solution of the system $\nabla^\phi \phi = w$ with $w \in L^\infty(\omega; \mathbb{R}^{2n-1})$;*
(ii) *ϕ is a broad solution of $\nabla^\phi \phi = w$, i.e. there exists a Borel function $\hat{w} \in \mathfrak{L}^\infty(\omega; \mathbb{R}^{2n-1})$ s.t.*
  (B.1) *$w(A) = \hat{w}(A)$ $\mathcal{L}^{2n}$-a.e. $A \in \omega$;*
  (B.2) *for every continuous vector field ∇_i^ϕ $(i = 1, \dots, 2n-1)$ having an integral curve $\Gamma \in C^1((-\delta, \delta); \omega)$, $\phi \circ \Gamma$ is absolutely continuous and it satisfies at a.e.-s*

$$\frac{d}{ds}\phi\big(\Gamma(s)\big) = \hat{w}_i\big(\Gamma(s)\big).$$

1.1. Outline of the proofs

With the intention of focusing on the nonlinear field, we fix the attention on the case $n = 1$.

Given a continuous distributional solution $\phi \in C^0(\omega)$ of the PDE

$$\nabla^\phi \phi(z,t) = \frac{\partial \phi}{\partial z}(z,t) + \frac{\partial}{\partial t}\left[\frac{\phi^2(z,t)}{2}\right] = w(z,t) \quad \text{for } (z,t) \in \omega,$$

we first prove that it is Lipschitz when restricted along any characteristic curve $\Gamma : [a,b] \to \omega$ $\Gamma(z) = (z, \gamma(z))$, where $\dot{\gamma} = \phi \circ \Gamma$. The proof follows a previous argument by Dafermos (see Lemma 5.1). By a construction based on the classical existence theory of ODEs with continuous coefficients, we can then define a change of variable $(z, \chi(z,\tau))$ which straightens characteristics. This change of variables does not enjoy SBV or Lipschitz regularity, it fails injectivity in an essential way, though it is continuous and we impose an important monotonicity property. This monotonicity, relying on the fact that we basically work in dimension 2, is the regularity property which allows us the change of variables. As we exemplify below, we indeed have an approach different from providing a regular Lagrangian flow of Ambrosio–Di Perna–Lions's theory, and it is essentially two dimensional. After the change of variables, the PDE is, roughly, linear, and we indeed find a family of ODEs for ϕ on the family of characteristics composing χ, with coefficients which now are not anymore continuous, but which are however bounded. By generalizing a lemma on ODEs already present in [6], we prove the $1/2$-Hölder continuity of ϕ on the vertical direction (z constant), and a posteriori in the whole domain. These are the main ingredients for establishing that ϕ defines indeed a Lipschitz graph: given two points, we connect them by a curve made first by a characteristic curve which joins the two vertical lines through the points, then by the remaining vertical segment. We manage this way to control the variation of ϕ between the two points with their graph distance d_ϕ, checking therefore the metric definition of intrinsic Lipschitz graphs.

The other implication of Theorem 1.1 is based on the possibility of suitably approximating an intrinsic Lipschitz graph with intrinsic regular graphs. A geometric approximation is provided by [11]. We also provide a more analytic, and weaker, approximation as a byproduct of the change of variable χ which straightens characteristics, by mollification (see the proof of Theorem 6.10).

We stop now for a while in order to clarify the features of the statement in Theorem 1.2, and why it is so important here to distinguish among $\mathfrak{L}^\infty(\omega; \mathbb{R}^{2n-1})$, which are pointwise defined functions, and $L^\infty(\omega; \mathbb{R}^{2n-1})$, which are equivalence classes. Lagrangian formulations are affected by altering the representative, as the following example stresses.

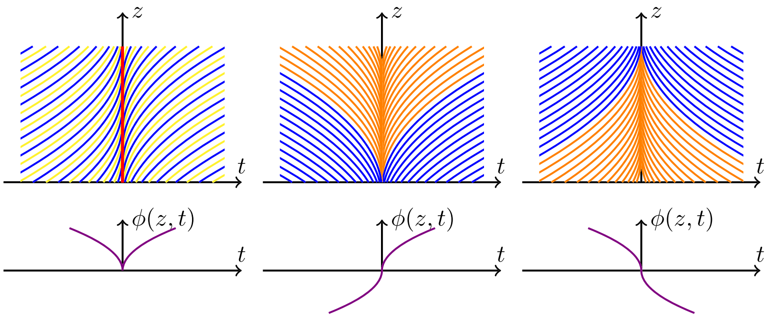


Fig. 1. Illustration of Examples 1.3, 1.4. Characteristics are drown for three particular continuous distributional solution to the equation $\frac{\partial\phi}{\partial z}+\frac{\partial}{\partial t}[\phi^2/2]=\mathrm{sgn}(t)/2$. Below the graphs of the corresponding functions ϕ are depicted.

Example 1.3. (Fig. 1.) Let $\omega=(0,1)\times(-1,1)$ and let $\phi, w:\omega\to\mathbb{R}$ be the functions defined as

$$\phi(z,t):=\sqrt{|t|},\qquad w(z,t):=\begin{cases}1/2 & \text{if } t\geq 0\\ -1/2 & \text{if } t<0\end{cases}$$

Then it is easy to verify that ϕ is a continuous distributional solution of

$$\nabla^\phi\phi=\frac{\partial\phi}{\partial z}+\frac{\partial}{\partial t}\frac{\phi^2}{2}=w.$$

Consider the specific characteristic curve $(z,\gamma(z)):=(z,0)$. Even if $\dot\gamma(z)=\phi(z,\gamma(z))=0$, the derivative of ϕ along this characteristic curve is not the right one:

$$\frac{\partial\phi}{\partial z}(z,0)=0\neq\frac{1}{2}=w(z,0).$$

Eq. (3.9) holds however on every characteristic curves provided we choose correctly an L^∞-representative $\hat w\in\mathfrak{L}^\infty(\omega)$ of the source w: it is enough to consider

$$\hat w(z,t):=\begin{cases}1/2 & \text{if } t>0\\ 0 & \text{if } t=0\\ -1/2 & \text{if } t<0\end{cases}$$

Notice that $w(z,t)=\hat w(z,t)$ for $\mathcal{L}^2$-a.e. $(z,t)\in\omega$. $\square$

Before outlining Theorem 1.2 we exemplify other features mentioned above by similar examples.

Example 1.4. (Fig. 1.) Let $\omega=(0,1)\times(-1,1)$, and choose

$$\phi(z,t)=-\,\mathrm{sgn}\,t\sqrt{|t|}.$$

Again

$$\nabla^\phi\phi=\frac{\partial\phi}{\partial z}+\frac{\partial}{\partial t}\frac{\phi^2}{2}=\begin{cases}1/2 & \text{if } t>0\\ -1/2 & \text{if } t\leq 0\end{cases}=:w(z,t).$$

One easily sees that characteristics do collapse in an essential way. Considering instead

$$\phi(z,t)=\mathrm{sgn}\,t\sqrt{|t|}$$

characteristics do split in an unavoidable way. Therefore, while it is proved in [14] that in Example 1.3 one can choose for changing variable a flux which is better than a generic other—*the regular Lagrangian flow*—we are not always in this case. This is the main reason why we refer in our change of variables to a monotonicity property.

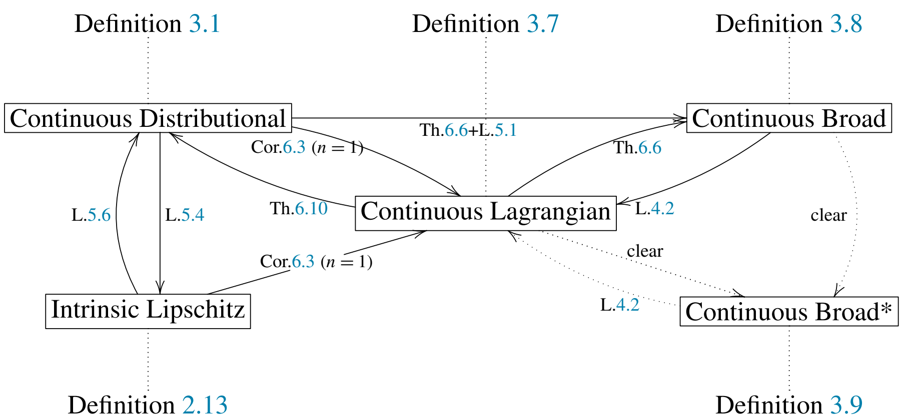
We have now motivated the further study for the stronger statement of Theorem 1.2. In order to prove it, we consider also the weaker concept of *Lagrangian solution*: the idea is that the reduction on characteristics is not required on *any* characteristic, but on a set of characteristics $\{\chi(z,\tau)\}_{\tau\in\mathbb{R}}$ composing the change of variables χ that one has chosen. Exhibiting a suitable set of characteristics for the change of variables $\chi(z,\tau)$ is part of the proof. Roughly, an $\mathfrak{L}^\infty$-representative w_χ for the source of the ODEs related to χ is provided by taking the z-derivative of $\phi(z,\chi(z,\tau))$, which is ϕ evaluated along the characteristics $\{\chi(z,\tau)\}_{\tau\in\mathbb{R}}$ of the Lagrangian parameterization; by construction, it coincides with the second z-derivative of $\chi(z,\tau)$. Here we have hidden the fact that we need to come back from (z,τ) to (z,t), a change of variable that is not single valued, not surjective because the second derivative was defined only almost everywhere, and which may map negligible sets into positive measure sets [1]. We overcome the difficulty showing that it is enough selecting any value of the second derivative when present. However, if one changes the set of characteristics in general one arrives to a different function $w_{\chi'}\in\mathfrak{L}^\infty(\omega;\mathbb{R})$, which however identifies the same distribution as w_χ. This issue is sensibly more complex for scalar balance laws with non-convex flux [1].

We defined *broad solution* a function which satisfies the reduction on every characteristic curve. In order to have this stronger characterization, we give a different argument borrowed from [1]. We define a universal source term $\hat{w}$ in an abstract way, by a selection theorem, at each point where there exists a characteristic curve with second derivative, without restricting anymore to a fixed family of characteristics providing a change of variables. After showing that this is well defined, we have provided a universal representative of the intrinsic gradient of ϕ. In cases as Examples 1.3, 1.4 it extends the one, defined only almost everywhere, provided by Rademacher theorem.

1.2. *Outline of the paper*

The paper is organized as follows. In Section 2 we recall basic notions about the Heinsenberg groups. In Section 3 we fix instead notations relative to the PDE, mainly specifying the different notions of solutions we will consider. One of them will involve a change of variables, for passing to the Lagrangian formulation, which is mainly matter of Section 4 and it is basically concerned with classical theory on ODEs. Then Appendix A also explains how to extend a partial change of variables of that kind to become surjective, and it provides a counterexample to its local Lipschitz regularity; it is improved in [1] showing that it may map negligible sets into positive measure set. In Section 5 we prove the equivalence among the facts that either a continuous function ϕ describes a Lipschitz graph, with intrinsic gradient w, or it is a distributional solution to the PDE $\nabla^\phi\phi=w$, where $w\in L^\infty$. The further equivalencies are finally matter of Section 6.

With some simplification, we can illustrate the main connections by the following papillon. As mentioned above, there is also a connection with the existence of smooth approximations (see the proof of Theorem 6.10 by mollification and [11] with a more geometric procedure).



Description of the identification of the source terms. The intrinsic gradient is unique. Suppose ϕ is both intrinsic Lipschitz and a broad solution: then by definition at points of intrinsic differentiability the intrinsic gradient must coincide with $\hat{w}$, where $\hat{w}$ is either the Souslin or Borel selection representative of w constructed in Section 6.2. As a broad solution is also a Lagrangian solution, at those points it must coincide also with any Lagrangian source $\bar{w}\in\mathfrak{L}^\infty(\omega;\mathbb{R})$. By Rademacher theorem, these functions are therefore identified Lebesgue almost everywhere, and by Lemma 5.6 they are representative of the $L^\infty(\omega;\mathbb{R}^{2n-1})$-function which is the source term of the balance law. A different proof of this compatibility is given in [1].

We conclude remarking again that this identification Lebesgue almost everywhere is not enough for fixing all of them: both Lagrangian solutions and broad solutions require the right representative of the source also on some of the points of non-intrinsic differentiability, not captured by the source term in the distributional formulation (Fig. 1). A source term in the Lagrangian formulation instead fixes directly the source term of the distributional formulation, and it is fixed in turn by a source term in the broad formulation. This chain of implications fails instead for $1D$-scalar balance laws

$$\frac{\partial}{\partial z}\big[\phi(z,t)\big]+\frac{\partial}{\partial t}\big[f\big(\phi(z,t)\big)\big]=w(z,t)$$

when f is smooth but not *uniformly* convex, as shown in [1], because Rademacher theorem fails.

2. Sub-Riemannian Heisenberg group

We first remind the definition of the Heisenberg groups $\mathbb{H}^n\equiv\mathbb{C}^n\times\mathbb{R}\equiv\mathbb{R}^{2n+1}$ as noncommutative Lie groups. We later remind notions of intrinsic differential calculus and the main topic of this paper, which are intrinsic Lipschitz functions and graphs. We denote the points of $\mathbb{H}^n$ by

$$P=(x,t),\quad x\in\mathbb{R}^{2n},\ t\in\mathbb{R}.$$

If $P=(x,t)$, $Q=(x',t')\in\mathbb{H}^n$ and $r>0$, the group operation reads as

$$P\cdot Q:=\left(x+x',t+t'+\frac{1}{2}\sum_{j=1}^{n}\big(x_jx'_{j+n}-x_{j+n}x'_j\big)\right). \tag{2.1}$$

The group identity is the origin 0 and one has $(x,t)^{-1}=(-x,-t)$. In $\mathbb{H}^n$ there is a natural one parameter group of non-isotropic dilations $\delta_r(P):=(rx,r^2t)$, $r>0$.

The group $\mathbb{H}^n$ can be endowed with the homogeneous norm

$$\|P\|_\infty:=\max\big\{|x|,|t|^{1/2}\big\}$$

and with the left-invariant and homogeneous distance

$$d_\infty(P,Q):=\big\|P^{-1}\cdot Q\big\|_\infty.$$

The metric d_∞ is equivalent to the standard Carnot–Carathéodory distance. It follows that the Hausdorff dimension of $(\mathbb{H}^n,d_\infty)$ is $2n+2$, whereas its topological dimension is $2n+1$.

The Lie algebra $\mathfrak{h}_n$ of left invariant vector fields is (linearly) generated by

$$X_j=\frac{\partial}{\partial x_j}-\frac{1}{2}x_{j+n}\frac{\partial}{\partial t},\qquad X_{j+n}=\frac{\partial}{\partial x_{j+n}}+\frac{1}{2}x_j\frac{\partial}{\partial t},\quad j=1,\dots,n,\qquad T=\frac{\partial}{\partial t} \tag{2.2}$$

and the only nonvanishing commutators are

$$[X_j,X_{j+n}]=T,\quad j=1,\dots,n.$$

2.1. Horizontal fields and differential calculus

We shall identify vector fields and associated first order differential operators; thus the vector fields $X_1,\dots,X_{2n}$ generate a vector bundle on $\mathbb{H}^n$, the so called *horizontal* vector bundle $\mathrm{H}\mathbb{H}^n$ according to the notation of Gromov (see [21]), that is a vector subbundle of $\mathrm{T}\mathbb{H}^n$, the tangent vector bundle of $\mathbb{H}^n$. Since each fiber of $\mathrm{H}\mathbb{H}^n$ can be canonically identified with a vector subspace of $\mathbb{R}^{2n+1}$, each section φ of $\mathrm{H}\mathbb{H}^n$ can be identified with a map $\varphi:\mathbb{H}^n\to\mathbb{R}^{2n+1}$. At each point $P\in\mathbb{H}^n$ the horizontal fiber is indicated as $\mathrm{H}\mathbb{H}^n{}_P$ and each fiber can be endowed with the scalar product $\langle\cdot,\cdot\rangle_P$ and the associated norm $\|\cdot\|_P$ that make the vector fields $X_1,\dots,X_{2n}$ orthonormal.

Definition 2.1. A real valued function f, defined on an open set $\Omega \subset \mathbb{H}^n$, is said to be of class $C^1_{\mathbb{H}}(\Omega)$ if $f \in C^0(\Omega)$ and the distribution

$$\nabla_{\mathbb{H}} f := (X_1 f, \dots, X_{2n} f)$$

is represented by a continuous function.

Definition 2.2. We shall say that $S \subset \mathbb{H}^n$ is an $\mathbb{H}$-*regular hypersurface* if for every $P \in S$ there exist an open ball $U_\infty(P,r)$ and a function $f \in C^1_{\mathbb{H}}(U_\infty(P,r))$ such that

(i) $S \cap U_\infty(P,r) = \{Q \in U_\infty(P,r) : f(Q) = 0\}$;
(ii) $\nabla_{\mathbb{H}} f(P) \neq 0$.

The *horizontal normal* to S at P is $\nu_S(P) := -\frac{\nabla_{\mathbb{H}} f(P)}{|\nabla_{\mathbb{H}} f(P)|}$.

Notation 2.3. Recalling that we denote a point of $\mathbb{H}^n$ as $P = (x,t) \in \mathbb{H}^n$, where $x = (x_1, \dots, x_{2n}) \in \mathbb{R}^{2n}$, $t \in \mathbb{R}$, it is convenient to introduce the following notations:

- We will denote a point of $\mathbb{R}^{2n} \equiv \mathbb{R}^{2n-1} \times \mathbb{R}$ as $A = (z,t) \in \mathbb{R}^{2n}$, where $z \in \mathbb{R}^{2n-1}$, $t \in \mathbb{R}$. If $z = (z_1, \dots, z_{2n-1}) \in \mathbb{R}^{2n-1}$, for given $i = 1, \dots, 2n-1$, we denote $\hat{z}_i = (z_1, \dots, z_{i-1}, z_{i+1}, \dots, z_{2n}) \in \mathbb{R}^{2n-2}$. When we will use this notation, we also denote a point $A = (z,t) \in \mathbb{R}^{2n}$ as $A = (z_i, \hat{z}_i, t) \in \mathbb{R} \times \mathbb{R}^{2n-2} \times \mathbb{R}$ if $n \geq 2$.
- Let $B \subset \mathbb{R}^{2n}$, for given $\hat{z}_i \in \mathbb{R}^{2n-2}$ and $t \in \mathbb{R}$ we will denote $B_{\hat{z}_i,t} := \{s \in \mathbb{R} : (s, \hat{z}_i, t) \in B\}$;
- Let $B \subset \mathbb{R}^{2n}$, for given $z \in \mathbb{R}^{2n-1}$ we will denote $B_z := \{t \in \mathbb{R} : (z,t) \in B\}$.
- Let $B \subset \mathbb{R}^{2n}$, for given $\hat{z}_i \in \mathbb{R}^{2n-2}$ we will denote $B_{\hat{z}_i} := \{(s,t) \in \mathbb{R}^2 : (s, \hat{z}_i, t) \in B\}$;
- $\mathbb{W} = \{(x,t) \in \mathbb{H}^n : x_1 = 0\}$. A point $A = (0, z_1, \dots, z_{2n-1}, t) \in \mathbb{W}$ will be identified with the point $(z,t) \in \mathbb{R}^{2n}$.
- Let $\omega \subset \mathbb{W}$, for given $z_n \in \mathbb{R}$ we will denote $\omega_{z_n} := \{(\hat{z}_n, t) \in \mathbb{R}^{2n-1} : (z_n, \hat{z}_n, t) \in \omega\}$.

Definition 2.4. A set $S \subset \mathbb{H}^n$ is an X_1-graph if there is a function $\phi : \omega \subset \mathbb{W} \to \mathbb{R}$ such that $S = G^1_{\mathbb{H},\phi}(\omega) := \{A \cdot \phi(A) e_1 : A \in \omega\}$. When not otherwise stated, by intrinsic graph we will mean X_1-graph.

Let us recall the following results proved in [17] (see also [10,18]).

Theorem 2.5 *(Implicit function theorem). Let Ω be an open set in $\mathbb{H}^n$, $0 \in \Omega$, and let $f \in C^1_{\mathbb{H}}(\Omega)$ be such that $X_1 f > 0$. Let $S := \{(x,t) \in \Omega : f(x,t) = 0\}$; then there exist a connected open neighborhood $\mathcal{U}$ of 0 and a unique continuous function $\phi : \omega \subset \mathbb{W} \to [-h,h]$ such that $S \cap \overline{\mathcal{U}} = \Phi(\omega)$, where $h > 0$ and Φ is the map defined as*

$$\omega \ni (z,t) \mapsto \Phi(z,t) = (z,t) \cdot \phi(z,t) e_1$$

and given explicitly by

$$\Phi(z,t) = \left(\phi(z,t), z_1, \dots, z_{2n-1}, t - \frac{z_n}{2}\phi(z,t)\right).$$

Let $n \geq 2$, $A_0 = (z^0_1, \dots, z^0_{2n-1}, t^0) \in \mathbb{R}^{2n}$ and define

$$I_r(A_0) := \left\{(z,t) \in \mathbb{R}^{2n} : \left|z_n - z^0_n\right| < r, \sum_{\substack{i=1 \\ i \neq n}}^{2n-1} \left[\left(z_i - z^0_i\right)^2\right] < r^2, \left|t - t^0\right| < r\right\}.$$

When $n = 1$ and $A_0 = (z^0, t^0) \in \mathbb{R}^2$ let

$$I_r(A_0) := \left\{(z,t) \in \mathbb{R}^2 : \left|z - z^0\right| < r, \left|t - t^0\right| < r\right\}.$$

Following [4,5,28] we define the graph quasidistance d_ϕ on ω. We set $\mathbb{O}_1 := \{(x,t) \in \mathbb{H}^n : x_1 = 0, t = 0\}$, $\mathbb{T} := \{(x,t) \in \mathbb{H}^n : x_1 = 0, \dots, x_{2n} = 0\}$.

Definition 2.6. For $A=(z_n,\hat{z}_n,t),\ A'=(z'_n,\hat{z}'_n,t')\in\omega$ we define

$$d_\phi(A,A') := \big\|\pi_{\mathbb{O}_1}\big(\Phi(A)^{-1}\cdot\Phi(A')\big)\big\|_\infty + \big\|\pi_{\mathbb{T}}\big(\Phi(A)^{-1}\cdot\Phi(A')\big)\big\|_\infty \tag{2.3}$$

Following Notation 2.3 we have explicitly if $n\geq 2$

$$d_\phi(A,A') = \big|(z'_n,\hat{z}'_n)-(z_n,\hat{z}_n)\big| + \left|t'-t-\frac{1}{2}\big(\phi(A)+\phi(A')\big)(z'_n-z_n)+\sigma(\hat{z}_n,\hat{z}'_n)\right|^{1/2};$$

where $\sigma(\hat{z}_n,\hat{z}'_n)=\frac{1}{2}\sum_{j=1}^{n-1}(z_{j+n}z'_j-z_jz'_{j+n})$. If $n=1$ and $A=(z_2,t),\ A'=(z'_1,t')\in\omega$ we have

$$d_\phi(A,A') = |z'_1-z_1| + \left|t'-t-\frac{1}{2}\big(\phi(A)+\phi(A')\big)(z'_1-z_1)\right|^{1/2}.$$

An intrinsic differentiable structure can be induced on $\mathbb{W}$ by means of d_ϕ, see [4,5,11,28]. We remind that a map $L:\mathbb{W}\to\mathbb{R}$ is $\mathbb{W}$-linear if it is a group homomorphism and $L(rz,r^2t)=rL(z,t)$ for all $r>0$ and $(z,t)\in\mathbb{W}$. We remind then the notion of ∇^ϕ-differentiablility.

Definition 2.7. Let $\phi:\omega\subset\mathbb{W}\to\mathbb{R}$ be a fixed continuous function, and let $A_0\in\omega$ and $\psi:\omega\to\mathbb{R}$ be given.

- We say that ψ is ∇^ϕ-differentiable at A_0 if there is a $\mathbb{W}$-linear functional $L:\mathbb{W}\to\mathbb{R}$ such that

$$\lim_{\substack{A\to A_0\\ A\neq A_0}} \frac{\psi(A)-\psi(A_0)-L(A_0^{-1}\cdot A)}{d_\phi(A_0,A)} = 0. \tag{2.4}$$

- We say that ψ is uniformly ∇^ϕ-differentiable at A_0 if there is a $\mathbb{W}$-linear functional $L:\mathbb{W}\to\mathbb{R}$ such that

$$\lim_{r\to 0}\sup_{\substack{A,B\in I_r(A_0)\\ A\neq B}}\left\{\frac{|\psi(B)-\psi(A)-L(B^{-1}\cdot A)|}{d_\phi(A,B)}\right\}=0 \tag{2.5}$$

We will denote $L=d_{\mathbb{W}}\phi(A_0)$. If ϕ is uniformly ∇^ϕ-differentiable at A_0, then ϕ is ∇^ϕ-differentiable at A_0.

Remark 2.8. If $\phi:\omega\to\mathbb{R}$ is ∇^ϕ-differentiable at A_0 then its differential $d_{\mathbb{W}}\phi(A_0)$ is unique (see [4]).

In [4] it has been proved that each $\mathbb{H}$-regular X_1-graph $\Phi(\omega)$ admits an intrinsic gradient $\nabla^\phi\phi\in C^0(\omega;\mathbb{R}^{2n})$, in sense of distributions, which shares a lot of properties with the Euclidean gradient. Notice that $\mathbb{W}=\exp(\mathrm{span}\{X_2,\dots,X_{2n},T\})$, where the vector fields $X_2,\dots,X_{2n},T$ must be understood as restricted to $\mathbb{W}$. It is possible to define the differential operators $\nabla^\phi:=(\nabla_1^\phi,\dots,\nabla_{2n-1}^\phi)$ by

$$\nabla_j^\phi := \begin{cases} X_{j+1}=\frac{\partial}{\partial z_j}-\frac{z_{j+n}}{2}\frac{\partial}{\partial t} & j=1,\dots,n-1\\ W^\phi = X_{n+1}+\phi T=\frac{\partial}{\partial z_n}+\phi\frac{\partial}{\partial t} & j=n\\ X_{j+1}=\frac{\partial}{\partial z_j}+\frac{z_{j-n}}{2}\frac{\partial}{\partial t} & j=n+1,\dots,2n-1\end{cases}$$

We identify them with a family of $(2n-1)$-vector fields on ω. Each ∇_j^ϕ is well defined from $C^1(\omega)$ with values in $C^0(\omega)$, and the fields ∇_j^ϕ for $j\neq n$ are more regular. Even if ϕ is not an admissible function for the action of these differential operators, we can define it in sense of distributions by

$$W^\phi\phi := X_{n+1}\phi+\frac{1}{2}T(\phi^2)=\frac{\partial\phi}{\partial x_{n+1}}+\frac{1}{2}\frac{\partial\phi^2}{\partial t},$$

$$\nabla^\phi\phi := \begin{cases}(X_2\phi,\dots,X_n\phi,W^\phi\phi,X_{2+n}\phi,\dots,X_{2n}\phi) & \text{if } n\geq 2\\ W^\phi\phi & \text{if } n=1.\end{cases} \tag{2.6}$$

The following characterizations were proved in [4,5,7]. The definitions of broad* and distributional solution of the system $\nabla^\phi\phi=w$ are recalled in Section 3.

Theorem 2.9. *Let $\omega \subset \mathbb{W} \equiv \mathbb{R}^{2n}$ be an open set and let $\phi : \omega \to \mathbb{R}$ be a continuous function. Then*

(i) *The set $S := \Phi(\omega)$ is an $\mathbb{H}$-regular surface and $\nu_S^1(P) < 0$ for all $P \in S$, where $\nu_S(P) = (\nu_S^1(P), \dots, \nu_S^{2n}(P))$ is the horizontal normal to S at P.*

is equivalent to each one of the following conditions:

(ii) *There exists $w \in C^0(\omega; \mathbb{R}^{2n-1})$ and a family $(\phi_\epsilon)_{\epsilon>0} \subset C^1(\omega)$ such that, as $\epsilon \to 0^+$,*

$$\phi_\epsilon \to \phi \quad \text{and} \quad \nabla^{\phi_\epsilon}\phi_\epsilon \to w \quad \text{in } L^\infty_{\text{loc}}(\omega),$$

and $\nabla^\phi \phi = w$ in ω, in sense of distributions, where $\nabla^\phi \phi$ is defined in (2.6).

(iii) *There exists $w \in C^0(\omega; \mathbb{R}^{2n-1})$ such that ϕ is a broad* solution of the system $\nabla^\phi \phi = w$.*

(iv) *There exists $w \in C^0(\omega; \mathbb{R}^{2n-1})$ such that ϕ is a distributional solution of $\nabla^\phi \phi = w$.*

(v) *ϕ is uniformly ∇^ϕ-differentiable at A for all $A \in \omega$.*

In particular, $\nu_S(P) = \left(-\frac{1}{\sqrt{1+|\nabla^\phi\phi|^2}}, \frac{\nabla^\phi\phi}{\sqrt{1+|\nabla^\phi\phi|^2}}\right)(\Phi^{-1}(P))$.

Remark 2.10. For given $\phi : \omega \to \mathbb{R}$ ∇^ϕ-differentiable at $A_0 \in \omega$, then the differential $d_\mathbb{W}\phi(A_0) : \mathbb{W} \equiv \mathbb{R}^{2n} \to \mathbb{R}$ can be represented in terms of the intrinsic gradient $\nabla^\phi\phi(A_0)$. More precisely (see [4,11])

$$d_\mathbb{W}\phi(A_0)(z) = \begin{cases} \nabla^\phi\phi(A_0)z & n = 1 \\ \sum_{i=1}^{2n-1} \nabla_i^\phi \phi(A_0) z_i & n \geq 2 \end{cases}$$

Definition 2.11. Let $\phi : \omega \to \mathbb{R}$ be a continuous function and let $A \in \omega$ be given. We say that ϕ admits ∇_i^ϕ-derivative at A if there exists $\alpha \in \mathbb{R}$ such that for each integral curve $\Gamma : (-\delta, \delta) \to \omega$ of ∇_i^ϕ with $\Gamma(0) = A$

$$\exists \frac{d}{ds}\phi\big(\Gamma(s)\big)\big|_{s=0} = \lim_{s\to 0} \frac{\phi(\Gamma(s)) - \phi(\Gamma(0))}{s} = \alpha,$$

and we will write $\alpha = \nabla_i^\phi \phi(A)$.

Remark 2.12. Notice that if $S = \Phi(\omega)$ is an $\mathbb{H}$-regular hypersurface then ϕ admits ∇_i^ϕ-derivative at A for every $A \in \omega$ for $i = 1, \dots, 2n-1$ (see [4,28]) and $\nabla_i^\phi\phi(A) = d_\mathbb{W}\phi(A)(e_i)$, where $\{e_1, e_2, \dots, e_{2n-1}\}$ denotes the standard basis of $\mathbb{R}^{2n-1}$.

2.2. Intrinsic Lipschitz functions and intrinsic Lipschitz graphs

Let us now introduce the concept of intrinsic Lipschitz function and intrinsic Lipschitz graph.

Definition 2.13. Let $\phi : \omega \subset \mathbb{W} \to \mathbb{R}$. We say that ϕ is an intrinsic Lipschitz continuous function in ω and write $\phi \in \text{Lip}_\mathbb{W}(\omega)$, if there is a constant $L > 0$ such that

$$\big|\phi(A) - \phi(B)\big| \leq L d_\phi(A, B) \quad \forall A, B \in \omega \tag{2.7}$$

Moreover we say that ϕ is a locally intrinsic Lipschitz function in ω and we write $\phi \in \text{Lip}_{\mathbb{W},\text{loc}}(\omega)$ if $\phi \in \text{Lip}_\mathbb{W}(\omega')$ for every $\omega' \Subset \omega$.

Remark 2.14. When $\phi : \omega \subset \mathbb{W} \equiv \mathbb{R}^2 \to \mathbb{R}$ is intrinsic Lipschitz and $\gamma : [a, b] \to \omega$ is an integral curve of the vector field W^ϕ, then $[a, b] \ni s \mapsto \phi(\gamma(s))$ is Lipschitz continuous. Indeed, from the estimate (52) in [11], it follows that there exists a positive constant C, depending only on the Lipschitz constant of ϕ, such that

$$d_\phi\big(\gamma(s), \gamma(t)\big) \leq C|t - s| \quad \forall t, s \in [a, b].$$

The inequality $d_\phi(\gamma(s), \gamma(t)) \geq |t-s|$ is true instead by Definition 2.6. Moreover we will see in Lemmas 4.2, 5.3 that ϕ is $1/2$-Hölder continuous on the lines where t is fixed, and then globally in the two variables.

In [11] is proved the following characterization for intrinsic Lipschitz functions.

Theorem 2.15. *Let $\omega \subset \mathbb{W}$ be open and bounded, let $\phi : \omega \to \mathbb{R}$. Then the following are equivalent:*

(i) $\phi \in \mathrm{Lip}_{\mathbb{W},\mathrm{loc}}(\omega)$
(ii) *there exist $\{\phi_k\}_{k\in\mathbb{N}} \subset C^\infty(\omega)$ and $w \in (L^\infty_{\mathrm{loc}}(\omega))^{2n-1}$ such that $\forall \omega' \Subset \omega$ there exists $C = C(\omega') > 0$ such that*
  (ii1) *$\{\phi_k\}_{k\in\mathbb{N}}$ uniformly converges to ϕ on the compact sets of ω;*
  (ii2) *$|\nabla^{\phi_k}\phi_k(A)| \leq C$ $\mathcal{L}^{2n}$-a.e. $x \in \omega'$, $k \in \mathbb{N}$;*
  (ii3) *$\nabla^{\phi_k}\phi_k(A) \to w(A)$ $\mathcal{L}^{2n}$-a.e. $A \in \omega$.*

Moreover if (ii) holds, then $\nabla^\phi\phi(A) = w(A)$ $\mathcal{L}^{2n}$-a.e. $A \in \omega$.

Let us finally recall the following Rademacher type theorem, proved in [20], see also [11].

Theorem 2.16. *If $\phi \in \mathrm{Lip}_{\mathbb{W}}(\omega)$ then ϕ is ∇^ϕ-differentiable for $\mathcal{L}^{2n}$-a.e $A \in \omega$.*

Remark 2.17. If $\phi \in \mathrm{Lip}_{\mathbb{W}}(\omega)$, then, from Theorem 2.16, ϕ admits ∇^ϕ_i-derivative which by Remark 2.14 must be the derivative of ϕ restricted along any integral curve of the vector field ∇^ϕ_i.

3. Solutions of the intrinsic gradient differential equation

Even when $w \in C^0(\omega)$, where ω is an open subset of $\mathbb{W} \equiv \mathbb{R}^2$, the equation

$$\frac{\partial \phi}{\partial z}(z,t) + \frac{\partial}{\partial t}\left[\frac{\phi^2(z,t)}{2}\right] = w(z,t) \quad \text{in } \omega \tag{3.1}$$

allows in general for discontinuous solution. However, it is the case $n = 1$ of the system (2.6)

$$\nabla^\phi\phi = w \quad \Leftrightarrow \quad \begin{cases} X_{j+1}\phi = w_j & j = 1,\dots,n-1,n+1,\dots,2n-1 \\ W^\phi\phi = w_n \end{cases} \tag{3.2}$$

where $w \in C^0(\omega, \mathbb{R}^{2n-1})$ and this system, by Theorem 2.9, describes an $\mathbb{H}$-regular surface $S := \Phi(\omega)$ which is an X_1-graph and has intrinsic gradient identified by $w(z,t)$. When describing intrinsic Lipschitz graphs, matter of the present paper, it is then natural to assume that $w \in L^\infty(\omega; \mathbb{R}^{2n-1})$ rather than being continuous, but the continuity of ϕ remains natural.

There are a priori different notions of *continuous* solutions $\phi : \omega \to \mathbb{R}$ to (3.1). We recall some of them in this section: distributional, Lagrangian, broad, broad*. All of them will finally coincide.

After giving in the present sections the definitions for all n, we will focus in the next one the analysis on the non-linear equation in the case $n = 1$, which conveys the attention on the planar case (3.1). The generalization to other cases $n \geq 2$ of most of the lemmas is straightforward, because the fields X_j and Y_j are linear, with the exception of Lemma 6.2, where we prefer taking advantage of the continuity of χ; however, we have no reason to prove it in full generality.

We recall that in general solutions are not smooth, even if we assume the continuity—see e.g. Example A.2 below. Improved local Lipschitz regularity, in the open set where $\phi(s,t)$ is non-vanishing, holds indeed only in the case of autonomous sources $w = w(t)$ [2]. The equation is then generally interpreted in a distributional way.

Definition 3.1 *(Distributional solution).* A continuous function $\phi : \omega \to \mathbb{R}$ is a *distributional solution* to (3.2) if for each $\varphi \in C^\infty_c(\omega)$

$$\int_\omega \phi\, \nabla^\phi_j \varphi\, d\mathcal{L}^{2n} = -\int_\omega w_j \varphi\, d\mathcal{L}^{2n}, \quad j = 1,\dots,n-1,n+1,\dots,2n-1 \tag{3.3}$$

and

$$\int_\omega \left(\phi \frac{\partial\varphi}{\partial z_n} + \frac{1}{2}\phi^2 \frac{\partial\varphi}{\partial t}\right) d\mathcal{L}^{2n} = -\int_\omega w_n \varphi \, d\mathcal{L}^{2n}.$$

We consider now different versions for the Lagrangian formulation of the PDE. The first one somehow englobes a choice of trajectories for passing from Lagrangian to Eulerian variables, and imposes the evolution equation on these trajectories.

Definition 3.2 *(Lagrangian parameterization).* A family of *partial Lagrangian parameterizations* associated to a continuous function $\phi : \omega \to \mathbb{R}$ and to the system (3.2) is a family of couples $(\tilde\omega_i, \chi_i)$ $(i = 1, \dots, 2n-1)$ with $\tilde\omega_i \subset \mathbb{R}^{2n}$ open sets and $\chi_i = \chi_i(\xi, t) = \chi_i(\xi_i, \hat\xi_i, t) : \tilde\omega_i \to \mathbb{R}$ Borel functions such that for each $i = 1, \dots, 2n-1$

(L.1) the map $\Upsilon_i : \tilde\omega_i \to \mathbb{R}^{2n}$, $\Upsilon_i(\xi, t) = (\xi, \chi_i(\xi, t))$ is valued in ω;
(L.2) for every $\xi \in \mathbb{R}^{2n-1}$, the function $\tilde\omega_{i,\xi} \ni t \mapsto \chi_i(\xi, t)$ is nondecreasing;
(L.3) for every interval $\hat\xi_i \in \mathbb{R}^{2n-2}$, $t \in \mathbb{R}$, for every $(s_1, s_2) \subset \tilde\omega_{i,\hat\xi_i,t}$, the function $(s_1, s_2) \ni s \mapsto \Upsilon_i(s, \hat\xi_i, t)$ is absolutely continuous and Υ_i is an integral curve of the vector field ∇_i^ϕ:

$$\frac{\partial \Upsilon_i}{\partial s}(s, \hat\xi_i, t) = \nabla_i^\phi\big(\Upsilon_i(s, \hat\xi_i, t)\big) \quad \text{a.e. } s \in (s_1, s_2). \tag{3.4}$$

We call it a family of *(full) Lagrangian parameterizations* if $\chi_i : (\tilde\omega_i)_\xi \to \omega_\xi$ is onto the section ω_ξ for all ξ.

We emphasize in this definition the nonlinear PDE of the system: a Lagrangian parameterization provides a covering of ω by characteristic lines for that equation. Indeed, a covering by characteristic lines of the other equations is immediately given by an expression like

$$\chi_i(z_1, \dots, z_{2n-1}, t) = \begin{cases} t - \frac{z_{i+n}}{2} z_i & i = 1, \dots, n-1 \\ t + \frac{z_{i-n}}{2} z_i & i = n+1, \dots, 2n-1 \end{cases}$$

Moreover, the reduction along characteristics for the linear equations, and thus the equivalence between Lagrangian and distributional solution, holds with less technicality.

Definition 3.3. A family of (partial) parameterizations $(\tilde\omega_i, \chi_i)$ extends the family of (partial) parameterizations $(\tilde\omega_i', \tilde\chi_i')$, we denote $(\tilde\omega_i', \tilde\chi_i') \preceq (\tilde\omega_i, \chi_i)$, if there exists a family of Borel injective maps

$$J_i : \tilde\omega_i' \ni (z, \tau) \mapsto \big(z, j(z, \tau)\big) \in \tilde\omega_i \quad \text{such that} \quad \chi_i \circ J_i = \tilde\chi_i' \quad \forall i = 1, \dots, 2n-1.$$

When $(\tilde\omega_i', \tilde\chi_i') \preceq (\tilde\omega_i, \chi_i)$ and $(\tilde\omega_i, \chi_i) \preceq (\tilde\omega_i', \tilde\chi_i')$ they are called equivalent.

Remark 3.4. The notion of Lagrangian parameterization given above does not consist in a different formulation for the notion of regular Lagrangian flow in the sense by Ambrosio–Di Perna–Lions (see [13] for an effective presentation). Particles are really allowed both to split and to join, therefore in particular the compressibility condition here is not required, while instead we have a monotonicity property.

Notation 3.5. When we need to distinguish sources, we denote with $\bar{\cdot}$ functions defined on ω but possibly related to a parameterization, with $\tilde{\cdot}$ functions defined on $\tilde\omega$, and with $\hat{\cdot}$ functions defined on ω not related to specific parameterizations.

Remark 3.6. Notice that, given a full Lagrangian parameterization $\Psi : \tilde\omega_n \to \omega$ of a continuous function $\phi : \omega \to \mathbb{R}$, then for a given $\hat z_n \in \mathbb{R}^{2n-2}$, the function $\Psi_n = \Psi_n(\cdot, \hat z_n, \cdot) : \tilde\omega_{n,\hat z_n} \to \omega$ is continuous. Indeed assume $n = 1$, then $\Psi_1 \equiv \Psi : \tilde\omega \to \omega$, $\Psi(s, \tau) = (s, \chi(s, \tau))$ with $(s, \tau) \in \tilde\omega$ and $\chi(s, \cdot) : \tilde\omega_s \to \omega_s$ is onto $\forall s \in \mathbb{R}$. Fix $(s_0, \tau_0) \in \tilde\omega$ and let $\delta_0 > 0$ such that

$$Q = [s_0 - \delta_0, s_0 + \delta_0] \times [\tau_0 - \delta_0, \tau_0 + \delta_0] \subset \tilde\omega.$$

Observe now that $\tilde\omega_{s_0} \supseteq [\tau_0 - \delta_0, \tau_0 + \delta_0]$ and, since ω is open, there exists $\sigma_0 = \sigma_0(s)$ such that

$$\omega_{s_0} \supseteq \big[\chi(s_0, \tau_0), \chi(s_0, \tau_0) + \sigma_0\big]. \tag{3.5}$$

Since $\chi(s_0, \cdot) : \tilde\omega_{s_0} \to \omega_{s_0}$ is nondecreasing and onto, it follows that

$$\exists \lim_{\tau\to\tau_0^-} \chi(s_0, \tau) = \chi(s_0, \tau_0) \quad \forall s \in [s_0 - \delta_0, s_0 + \delta_0], \tag{3.6}$$

because otherwise one of the inequalities below would be strict

$$\lim_{\tau\to\tau_0^-} \chi(s_0, \tau) \le \chi(s_0, \tau_0) \le \lim_{\tau\to\tau_0^+} \chi(s_0, \tau) \tag{3.7}$$

and a contradiction to surjectivity would follow. On the other hand, notice that

$$\chi(\cdot, \tau) : [s_0 - \delta_0, s_0 + \delta_0] \to \mathbb{R} \quad \text{is Lipschitz continuous} \tag{3.8}$$

uniformly with respect to $\tau \in [\tau_0 - \delta_0, \tau_0 + \delta_0]$. In fact, because the function $[s_0 - \delta_0, s_0 + \delta_0] \ni s \mapsto \chi(s, \tau_0 \pm \delta_0)$ is continuous and $[\tau_0 - \delta_0, \tau_0 + \delta_0] \to \chi(s, \tau)$ is non-decreasing $\forall s \in [s_0 - \delta_0, s_0 + \delta_0]$, it follows that the map $\Psi : Q \to \omega$ is bounded. Then by (L.3) (3.8) follows. Thus, (3.6) and (3.8) imply the continuity of $\chi : \tilde\omega \to \mathbb{R}$ at (s_0, τ_0).

When $n > 1$ we can repeat the same arguments of $n = 1$. Observe that, taking the proof in the case $n = 1$ into account, we could get the (global) continuity of the map $\Psi_n : \tilde\omega_n \to \omega$ if for each $A_0 = (s_0, \hat z_{0,n}, \tau_0) \in \tilde\omega_n$

- $\tilde\omega_{n,s_0} \ni (\hat z_n, \tau) \mapsto \chi_n(s_0, \hat z_n, \tau)$ is continuous.

We do not mind about this global continuity when $n > 1$.

Before giving the notion of Lagrangian solution, we recall that a set $A \subset \mathbb{R}^n$ is *universally measurable* if it is measurable w.r.t. every Borel measure, see [27, Section 5.5], see also [12, Chapter 8, Section 4]. Universally measurable sets constitute a σ-algebra, which includes analytic sets. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said universally measurable if it is measurable w.r.t. this σ-algebra. In particular, it will be measurable w.r.t. any Borel measure.

Notice that Borel counterimages of universally measurable sets are universally measurable. Then the composition $\varphi \circ \psi$ of any universally measurable function φ with a Borel function ψ is universally measurable. This composition would be nasty with φ just Lebesgue measurable.

Since restrictions of Borel functions on Borel sets are Borel, all the terms in the following definition are thus meaningful.

Definition 3.7 *(Lagrangian solution).* A continuous function $\phi : \omega \to \mathbb{R}$ is a *Lagrangian solution* of (3.2) if there exists a family of Lagrangian parameterizations $(\tilde\omega_i, \chi_i)$ $(i = 1, \dots, 2n - 1)$, associated to ϕ and (3.2), and a family of universally measurable functions $\bar w_{\chi_i} \in \mathfrak{L}^\infty(\omega)$ $(i = 1, \dots, 2n - 1)$, such that $\forall \hat\xi_i \in \mathbb{R}^{2n-2}$, $\forall t \in \mathbb{R}$, $\forall (s_1, s_2) \subset \tilde\omega_{\xi_i,t}$, it holds that

(LS1) the function $(s_1, s_2) \ni s \mapsto \phi(\Upsilon_i(s, \xi_i, t))$ is absolutely continuous and

$$\frac{d}{ds}\phi\big(\Upsilon_i(s, \hat\xi_i, t)\big) = \bar w_{\chi_i}\big(\Upsilon_i(s, \hat\xi_i, t)\big) \quad \mathcal{L}^1\text{-a.e. } s \in (s_1, s_2) \tag{3.9}$$

where Υ_i is given in (L.1) of Definition 3.2.

(LS2) if ϕ admits ∇_i^ϕ-derivative at $A \in \omega$, then $\bar w_{\chi_i}(A) = \nabla_i^\phi \phi(A)$ $\forall i = 1, \dots, 2n - 1$.

(LS3) $\bar w_{\chi_i} = w_i$ $\mathcal{L}^{2n}$-a.e. in ω, for every $i = 1, 2, \dots, 2n - 1$.

We are going to prove in Section 6.3 that ϕ above is a distributional solution to (3.2).

We give now the strongest notion of solution: the evolution equation is imposed on every trajectories.

Definition 3.8 *(Broad solution).* A continuous function $\phi : \omega \to \mathbb{R}$ is a *broad solution* of (3.2) if there exists a universally measurable function $\hat w = (\hat w_1, \dots, \hat w_{2n-1}) \in \mathfrak{L}^\infty(\omega; \mathbb{R}^{2n-1})$ such that

(B.1) $\hat{w} = w$ $\mathcal{L}^{2n}$-a.e. in ω;

(B.2) $\forall i = 1, \dots, 2n-1$, for every integral curve $\Gamma_i \in C^1((-\delta, \delta); \omega)$ of the vector field ∇_i^ϕ it holds that $(-\delta, \delta) \ni s \mapsto \phi(\Gamma_i(s))$ is absolutely continuous and

$$\frac{d}{ds}\phi\big(\Gamma_i(s)\big) = \hat{w}\big(\Gamma_i(s)\big) \quad \text{a.e. } s \in (-\delta, \delta);$$

(B.3) if ϕ admits ∇_i^ϕ-derivative at $A \in \omega$, then $\hat{w}_i(A) = \nabla_i^\phi \phi(A)$ $\forall i = 1, \dots, 2n-1$.

We also remind the intermediate notion of broad* solution introduced in [6].

Definition 3.9 *(Broad* solution).* A continuous function ϕ is a *broad* solution* of (3.2) if for every $A_0 \in \omega$ there exists a map, that we will call exponential map,

$$\exp_{A_0}\big(\nabla_i^\phi\big)(\cdot) : [-\delta_2, \delta_2] \times I_{\delta_2}(A_0) \to I_{\delta_1}(A_0) \Subset \omega, \tag{3.10}$$

where $0 < \delta_2 < \delta_1$ such that, if $\Gamma_i^A(s) = \exp_{A_0}(s\nabla_i^\phi)(A)$ for $i = 1, \dots, 2n-1$, then

(B*.1) $\Gamma_i^A \in C^1([-\delta_2, \delta_2]; \mathbb{R}^{2n})$,

(B*.2) $\begin{cases} \dot{\Gamma}_i^A = \nabla_i^\phi \circ \Gamma_i^A \\ \Gamma_i^A(0) = A, \end{cases}$

(B*.3) there exists a universally measurable function $\bar{w} \in \mathfrak{L}^\infty(\omega; \mathbb{R}^{2n-1})$ such that $\bar{w} = w$ $\mathcal{L}^{2n}$-a.e. in ω, $\forall i = 1, \dots, 2n-1$ the map $(-\delta_2, \delta_2) \ni s \mapsto \phi(\Gamma_i^A(s))$ is absolutely continuous and

$$\frac{d}{ds}\phi\big(\Gamma_i^A(s)\big) = \bar{w}\big(\Gamma_i^A(s)\big) \quad \text{a.e. } s \in (-\delta_2, \delta_2) \quad \forall A \in I_{\delta_2}(A_0);$$

(B*.4) if ϕ admits ∇_i^ϕ-derivative at $A \in \omega$, then $\bar{w}_i(A) = \nabla_i^\phi \phi(A)$ $\forall i = 1, \dots, 2n-1$.

Being a distributional solution to the PDE, a Lagrangian solution will be in particular a broad* solution with $w := \bar{w}$. Viceversa, if the exponential map related to a broad solution satisfies the relative monotonicity property, then one can prove that the broad* solution is also a Lagrangian solution, with the same $\bar{w}$. One can moreover derive a procedure for constructing a Lagrangian parameterization in Lemma A.1, where the curves $\gamma(s; \bar{s}, \bar{t})$ should be replaced by the ones of the exponential map.

4. Existence of Lagrangian parameterizations

We focus in this section on the case $n = 1$. We discuss the generalization to the case $n \geq 2$ in Remark 4.3.

Given a continuous function ϕ on $\mathrm{clos}(\omega) \subset \mathbb{W}$, we defined what we mean by 'Lagrangian parameterization'. Here we show that the definition is non-empty. It is part of the very classical theory on ODEs with continuous coefficients covering ω with integral curves of the continuous vector field $(1, \phi)$, see for example the first chapters of [22]. In the definition of Lagrangian parameterization it is however essential that we cover the region with curves which are ordered and not with generic ones: we therefore briefly remind below one way of doing it, with also the aim being self-contained.

Firstly, in Lemma 4.1 we show, for familiarizing with notations, that if one takes the integral curves through an s-section of ω which are minimal, in the sense that any other curve through that point lies on its right side, then we get a partial Lagrangian parameterization. The same happens when selecting the maximal ones. Extending it to a full one will be matter of Appendix A. A full Lagrangian parameterization basically amounts to an order preserving parameterization, with a *real* valued parameter, of non-crossing curves through each point of the plane. We immediately give an example of a full one (Lemma 4.2 below).

Notice that the definition of Lagrangian parameterization concerns only classical theory on ODEs with continuous coefficients, as therefore Lemmas 4.1, 4.2, 6.2, 5.3, A.1.

Lemma 4.1. *Let $\phi : \mathrm{clos}(\omega) \to \mathbb{R}$ be a continuous function, $(0,0) \in \omega$. Then there are domains $\tilde{\omega}_m$, $\tilde{\omega}_M$ associated to the functions*

$$\chi_m(s,\tau) := \min\{\gamma(s) : (r,\gamma(r)) \in \mathrm{clos}(\omega),\ \dot{\gamma}(r) = \phi(r,\gamma(r)),\ \gamma(0) = \tau\}$$
$$\chi_M(s,\tau) := \max\{\gamma(s) : (r,\gamma(r)) \in \mathrm{clos}(\omega),\ \dot{\gamma}(r) = \phi(r,\gamma(r)),\ \gamma(0) = \tau\}$$

for which $(\tilde{\omega}_m, \chi_m)$, $(\tilde{\omega}_M, \chi_M)$ are partial Lagrangian parameterization relative to ϕ.

Proof. For every $(\bar{s}, \bar{\tau}) \in \omega$ one could consider the minimal and maximal curves satisfying on $\mathrm{clos}(\omega)$ the ODE for characteristics (3.4) and passing through that point: indeed the functions

$$\gamma_{(\bar{s},\bar{\tau})}(s) := \min\{\gamma(s) : (r,\gamma(r)) \in \mathrm{clos}(\omega),\ \dot{\gamma}(r) = \phi(r,\gamma(r)),\ \gamma(\bar{s}) = \bar{\tau}\} \tag{4.1a}$$
$$\gamma^{(\bar{s},\bar{\tau})}(s) := \max\{\gamma(s) : (r,\gamma(r)) \in \mathrm{clos}(\omega),\ \dot{\gamma}(r) = \phi(r,\gamma(r)),\ \gamma(\bar{s}) = \bar{\tau}\} \tag{4.1b}$$

are well defined, Lipschitz, and because of the continuity of ϕ they are still integral curves [22, Section 3.2].

Denoting by r.i. the relative interior of a set, we define the domain

$$\tilde{\omega}_m = \mathrm{r.i.}\{(s,\tau) \in \mathbb{R} \times \omega_0 : \tau \in \omega_0,\ (s, \gamma_{(0,\tau)}(s)) \in \omega\}$$

where recall that $\omega_o = \{t \in \mathbb{R} : (0,t) \in \omega\}$. Since we are assuming that ω contains the origin, $\tilde{\omega}_m$ is nonempty. The domain $\tilde{\omega}_M$ is analogous.

Being $\gamma_{(\bar{s},\bar{\tau})}(s)$, $\gamma^{(\bar{s},\bar{\tau})}(s)$ Lipschitz solutions to the ODE with continuous coefficients, the functions $\chi_m(s,\tau)$, $\chi_M(s,\tau)$ in the statement are $C^1(\omega \cap \{t = \tau\})$ in the s variable for every τ fixed. χ_m, χ^M are jointly Borel in (s,τ) by continuity in s and monotonicity in τ, monotonicity that now we show.

Notice the semigroup property: for $t, h > 0$, for example for (4.1a)

$$\gamma_{(0,\tau)}(s) =: \bar{\tau}, \quad \gamma_{(s,\bar{\tau})}(h) =: \bar{\bar{\tau}} \quad \Longrightarrow \quad \gamma_{(0,\tau)}(s+h) = \bar{\bar{\tau}}.$$

This yields the monotonicity of $\chi_m(s,\tau) = \gamma_{(0,\tau)}(s)$ for each s fixed. Indeed, if $\tau_1 < \tau_2$ and $\gamma_{(0,\tau_1)}(\hat{s}) \ge \gamma_{(0,\tau_2)}(\hat{s})$ at a certain $\hat{s} > 0$, by the continuity of the curves there exists $0 < \bar{s} \le \hat{s}$ when $\gamma_{(0,\tau_1)}(\bar{s}) = \gamma_{(0,\tau_2)}(\bar{s})$. But then the curve

$$\gamma(s) = \begin{cases} \gamma_{(0,\tau_1)}(s) & \text{for } s \le \bar{s} \\ \gamma_{(0,\tau_2)}(s) & \text{for } s \ge \bar{s} \end{cases}$$

is a good competitor for the definition of $\gamma_{(0,\tau_1)}$, which implies

$$\gamma_{(0,\tau_2)}(\hat{s}) = \gamma(\hat{s}) \ge \gamma_{(0,\tau_1)}(\hat{s}),$$

and therefore equality. For $\gamma^{(0,\tau)}$ the argument is similar. □

In Lemma A.1 we show how to make a partial parameterization χ surjective: thus we cover ω by a family of characteristic curves which includes the ones of χ. In the following lemma, w.l.o.g. in a simpler setting, we provide instead a full Lagrangian parameterization, defined at once instead of extending a given one.

Lemma 4.2. *There exists a (full) Lagrangian parameterization associated to a continuous function $\phi : (0,1)^2 \to \mathbb{R}$ which is also continuous on the closure $[0,1]^2$.*

Proof. Step 1. Let ϕ be a continuous function on $[0,1]^2$, we want to give a Lagrangian parameterization for his restriction to $(0,1)^2$ as we defined it on an open set. One can assume ϕ is compactly supported in $(s,\tau) \in [0,1] \times (0,1)$. If not, one can extend it to a compactly supported function $\bar{\phi}$ on $[0,1] \times (-1,2)$: restricting the Lagrangian parameterization $(\tilde{\omega}^{\bar{\phi}}, \chi^{\bar{\phi}})$ for $\bar{\phi}$, defined as described below, to the open set

$$\tilde{\omega} := \{(s,\tau) \in \tilde{\omega}^{\bar{\phi}} : \chi^{\bar{\phi}}(s,\tau) \in (0,1)\},$$

one will get a Lagrangian parameterization for ϕ. The assumption of τ compactly supported in $(0,1)$ implies that there are two characteristics, one starting from $(0,0)$ and one from $(0,1)$, which satisfy $\dot{\gamma}(s) = \phi(s,\gamma(s)) \equiv 0$. This means respectively $\gamma(s) \equiv 1$ and $\gamma(s) \equiv 0$ for each $s \in [0,1]$.

Step 2. After this simplification, we associate to each point $(\bar{s}, \bar{t}) \in [0,1]^2$ a curve $\gamma(s; \bar{s}, \bar{t})$ which is minimal forward in s, maximal backward:

$$\gamma(s; \bar{s}, \bar{t}) = \begin{cases} \gamma_{(\bar{s},\bar{t})}(s) & s \geq \bar{s}, \\ \gamma^{(\bar{s},\bar{t})}(s) & s < \bar{s}. \end{cases}$$

where $\gamma_{(\bar{s},\bar{t})}(s)$, $\gamma^{(\bar{s},\bar{t})}(s)$ were defined in (4.1). Notice that the curves $\gamma(\cdot; \bar{s}, \bar{t})$ are defined on the whole $[0,1]$. Let us denote by

$$\mathcal{C} := \left\{ \gamma(\cdot; \bar{s}, \bar{t}) : [0,1] \to [0,1],\ (\bar{s}, \bar{t}) \in [0,1]^2 \right\}.$$

We will endow $\mathcal{C}$ by the topology of uniform convergence on $[0,1]$ and the following total order relation

$$\gamma(\cdot; s_1, t_1) \preceq \gamma(\cdot; s_2, t_2) \quad \Longleftrightarrow \quad \gamma(s; s_1, t_1) \leq \gamma(s; s_2, t_2) \quad \forall s \in [0,1]. \tag{4.2}$$

Let us denote by $\mathcal{C}^*$ the closure of $\mathcal{C} \subset C^0([0,1])$ endowed with the topology of uniform convergence. We prove in the rest of this step the following claims:

$$\mathcal{C}^* \text{ is compact;} \tag{4.3}$$

$$\text{the total order relation (4.2) still applies in } \mathcal{C}^*; \tag{4.4}$$

$$\mathcal{C}^* \text{ is connected;} \tag{4.5}$$

$$\mathcal{C}^* \text{ is still a family of characteristic curves for } \phi, \tag{4.6}$$

i.e. for each $\gamma \in \mathcal{C}^*$ $\dot{\gamma}(s) = \phi(s, \gamma(s))$ $\forall s \in [0,1]$.

Proof of (4.3). Since $\mathcal{C}$ is a family of equi-Lipschitz continuous and bounded functions of $C^0([0,1])$, then, from Arzelà–Ascoli's theorem $\mathcal{C}^*$ is compact.

Proof of (4.4). Let $\gamma, \tilde{\gamma} \in \mathcal{C}^*$ and let us prove that $\gamma \preceq \tilde{\gamma}$ or $\tilde{\gamma} \preceq \gamma$. By definition there are two sequences $\{\gamma_h\}_h, \{\tilde{\gamma}\}_h \subset \mathcal{C}$ such that $\gamma_h \to \gamma$ and $\tilde{\gamma}_h \to \tilde{\gamma}$ uniformly on $[0,1]$. Assume $\gamma \neq \tilde{\gamma}$, then there is $s_0 \in [0,1]$ such that

$$\tilde{\gamma}(s_0) < \gamma(s_0) \quad \text{or} \quad \gamma(s_0) < \tilde{\gamma}(s_0).$$

For instance, let $\tilde{\gamma}(s_0) < \gamma(s_0)$ and let us prove that

$$\tilde{\gamma}(s) \leq \gamma(s) \quad \forall s \in [0,1]. \tag{4.7}$$

Let $0 < \epsilon < \frac{\gamma(s_0) - \tilde{\gamma}(s_0)}{2}$, there exists $\bar{h} = \bar{h}(\epsilon)$ such that

$$\left|\gamma(s) - \gamma_h(s)\right| < \epsilon \quad \text{and} \quad \left|\tilde{\gamma}(s) - \tilde{\gamma}_h(s)\right| < \epsilon \quad \forall s \in [0,1],\ \forall h > \bar{h}. \tag{4.8}$$

From (4.8) it follows that

$$\tilde{\gamma}_h(s_0) < \tilde{\gamma}(s_0) + \epsilon < \gamma(s_0) - \epsilon < \gamma_h(s_0), \quad \forall h > \bar{h}.$$

Because $\tilde{\gamma}_h$ and γ_h are ordered, it follows that $\tilde{\gamma}_h(s) \leq \gamma_h(s)$ $\forall s \in [0,1]$, $\forall h > \bar{h}$. Passing to the limit as $h \to \infty$ in the previous inequality, we get (4.7).

Proof of (4.5). By contradiction, suppose that $\mathcal{C}^* = \mathcal{C}_1 \cup \mathcal{C}_2$, with $\mathcal{C}_1$ and $\mathcal{C}_2$ non-empty, closed sets in $C^0([0,1])$. It is well-known that, from (4.3) and (4.4), for each subset $\mathcal{A} \subset \mathcal{C}^*$ there exists the least upper bound (or supremum) and greatest lower bound (or infimum) of $\mathcal{A}$; we will denote respectively by $\sup \mathcal{A}$, $\inf \mathcal{A}$. Thus let

$$l_1 := \inf \mathcal{C}_1 \ \preceq\ L_1 := \sup \mathcal{C}_1, \qquad l_2 := \inf \mathcal{C}_2 \ \preceq\ L_2 := \sup \mathcal{C}_2.$$

Because $\mathcal{C}_1$ and $\mathcal{C}_2$ are closed, $l_1, L_1 \in \mathcal{C}_1$ and $l_2, L_2 \in \mathcal{C}_2$. Since $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, we have $L_1 \preceq l_2$ or $L_2 \preceq l_1$. Assume, for instance, that $L_1 \preceq l_2$. Then $L_1(s) \leq l_2(s)$ $\forall s \in [0,1]$ and $L_1(s_0) < l_2(s_0)$ for a suitable $s_0 \in [0,1]$. Let

$$\bar{s} := s_0, \qquad \bar{t} := \frac{L_1(s_0) + l_2(s_0)}{2}, \qquad \gamma(s) := \gamma(s; \bar{s}, \bar{t}) \quad s \in [0,1].$$

By definition, $\gamma \in \mathcal{C} \subseteq \mathcal{C}^*$, but $L_1 \preceq \gamma \preceq l_2$ and therefore we have a contradiction since $\gamma \notin \mathcal{C}_1 \cup \mathcal{C}_2 = \mathcal{C}^*$.

Proof of (4.6). Let $\gamma \in \mathcal{C}^*$, then by definition there exists a sequence $\{\gamma_h\}_h \subset \mathcal{C}$ such that $\gamma_h \to \gamma$ uniformly in $[0, 1]$. Because

$$\gamma_h(s) - \gamma_h(0) = \int_0^s \phi\big(\sigma, \gamma_h(\sigma)\big)\, d\sigma \quad \forall \sigma \in [0,1],\ \forall h,$$

passing to the limit as $h \to \infty$ in the previous identity we get (4.6).

Step 3. Let us now consider the map $\theta : \mathcal{C}^* \to \mathbb{R}$ defined by

$$\theta(\gamma) := \sum_{k=0}^{\infty} \frac{1}{2^k} \gamma(r_k)$$

where $\{r_k\}_{k\in\mathbb{N}}$ is an enumeration of $\mathbb{Q} \cap [0, 1]$. Notice that θ satisfies the following properties:

$$\theta \text{ is continuous;} \tag{4.9}$$

$$\theta \text{ is strictly order preserving, that is } \theta(\gamma_1) < \theta(\gamma_2) \text{ if } \gamma_1 \prec \gamma_2; \tag{4.10}$$

$$\theta\big(\mathcal{C}^*\big) = [0, 2]; \tag{4.11}$$

$$\text{there exists } \theta^{-1} : [0, 2] \to \mathcal{C}^* \text{ continuous.} \tag{4.12}$$

Indeed (4.9) and (4.10) immediately follow by the definition of θ. Equality (4.11) follows by (4.5) and (4.9) noticing that $\theta(\gamma(\cdot; 0, 0)) = 0$, $\theta(\gamma(\cdot; 0, 1)) = 2$; (4.12) is a consequence of (4.3), (4.10) and (4.11). Eventually let us consider the map $\chi : [0, 1] \times [0, 2] \to [0, 1]$

$$\chi(s, \tau) := \theta^{-1}(\tau)(s) \quad \text{if } (s, \tau) \in [0, 1] \times [0, 2].$$

Then, from (4.12), χ is continuous. Moreover the map $\Upsilon : (0, 1) \times (0, 2) \to (0, 1)^2$, defined by $\Upsilon(s, \tau) := (s, \chi(s, \tau))$, turns out to be a full Lagrangian parameterization associated to $\phi : (0, 1)^2 \to \mathbb{R}$. □

Remark 4.3. Notice that the same construction works with more variables, considering analogous characteristic curves $\gamma(s; \bar{s}, \bar{\hat{z}}_n, \bar{t})$—having the same order relation at $\bar{\hat{z}}_n$ frozen—and the relative parameterization given by $\chi(s, \hat{z}_n, \tau) := \gamma(s; \bar{s}, \hat{z}_n, \bar{t})$ with $\tau = \theta(\gamma(s; \bar{s}, \hat{z}_n, \bar{t}))$ defined as above. The continuity in $\hat{z}_n$ however is not guaranteed.

5. Distributional solutions = Intrinsic Lipschitz graphs

In this section we prove Theorem 1.1 about the distributional characterization of intrinsic Lipschitz graphs, without dimensional restrictions.

5.1. Some properties of distributional solutions

Preliminary we highlight here two properties of continuous distributional solutions $\phi(z, t)$ to the problem $\nabla^\phi \phi = w$. In particular we need regularity results of the solution along the characteristics lines γ of the fields ∇^ϕ_j. In the case of the non-linear field W^ϕ, the integral curves of $\dot{\gamma}(s) = \phi(s, \hat{z}_n, \gamma(s))$ exist by the continuity and boundedness of ϕ.

Lemma 5.1. *Let $\omega \subset \mathbb{W}$ be an open set. A continuous distributional solution $\phi : \omega \to \mathbb{R}$ to $\nabla^\phi \phi = w$ in ω is $\|w_n\|_{L^\infty(\omega)}$-Lipschitz along any characteristic line $\gamma : [-\delta, \delta] \to \mathbb{R}$ satisfying*

$$\dot{\gamma}(s) = \phi\big(s, \hat{z}_n, \gamma(s)\big) \quad s \in [-\delta, \delta],\ \hat{z}_n \textit{ fixed.}$$

Proof. In the same way in Dafermos [15] for $n = 1$ and as in [7, Theorem 1.2], for $n \geq 2$, we obtain for $a, b \in (-\delta, \delta)$ and $\hat{z}_n$ fixed

$$\int_{\gamma(b)}^{\gamma(b)+\epsilon} \phi(b, \hat{z}_n, t)dt - \int_{\gamma(a)}^{\gamma(a)+\epsilon} \phi(a, \hat{z}_n, t)dt - \int_a^b \int_{\gamma(s)}^{\gamma(s)+\epsilon} w_n(s, \hat{z}_n, t)\, dt\, ds$$
$$= -\int_a^b \big[\phi\big(s, \hat{z}_n, \gamma(s)+\epsilon\big) - \phi\big(s, \hat{z}_n, \gamma(s)\big)\big]^2 ds \leq 0 \tag{5.1}$$

and then

$$\int_{\gamma(b)}^{\gamma(b)+\epsilon} \phi(b, \hat{z}_n, t)dt - \int_{\gamma(a)}^{\gamma(a)+\epsilon} \phi(a, \hat{z}_n, t)dt \leq \int_a^b \int_{\gamma(s)}^{\gamma(s)+\epsilon} w_n(s, \hat{z}_n, t)\, dt\, ds$$
$$\leq \|w_n\|_{L^\infty(\omega)}(b-a)\epsilon.$$

Dividing by ϵ and getting to the limit to $\epsilon \to 0$ we obtain

$$\phi\big(b, \hat{z}_n, \gamma(b)\big) - \phi\big(a, \hat{z}_n, \gamma(a)\big) \leq \|w_n\|_{L^\infty(\omega)}(b-a).$$

The opposite inequality is obtained in a similar way integrating on the left of the characteristic. □

We obtain the same result of Lemma 5.1 for the linear fields X_j, $j = 2, \dots, n, n+2, \dots, 2n$, following the same proof.

Lemma 5.2. *Let $\omega \subset \mathbb{W}$ be an open set. A continuous distributional solution $\phi : \omega \to \mathbb{R}$ to $\nabla^\phi \phi = w$ is $\|w_j\|_{L^\infty(\omega)}$-Lipschitz along any characteristic line $\Gamma : [-\delta, \delta] \to \omega$ satisfying*

$$\dot{\Gamma}(s) = \nabla_j^\phi\big(\Gamma(s)\big) \quad j = 1, \dots, n-1, n+1, \dots, 2n-1.$$

We pass now to the Hölder continuity in the other variable.

Lemma 5.3. *Let $Q_1 = [-\delta, \delta] \times [\tau_0 - \delta_1, \tau_0 + \delta_1]$ and $Q_2 = [-\delta, \delta] \times [\tau_0 - \delta, \tau_0 + \delta]$ with $0 < \delta < \delta_1$. Let $f \in C^0(Q_1)$ and $\gamma : Q_2 \to [\tau_0 - \delta_1, \tau_0 + \delta_1]$ be given such that*

(i) $\gamma(\cdot, \tau) \in C^1([-\delta, \delta])$ $\forall \tau \in [\tau_0 - \delta, \tau_0 + \delta]$.

(ii) $\begin{cases} \frac{d\gamma}{ds}(s, \tau) = f(s, \gamma(s)) \\ \gamma(0, \tau) = \tau \end{cases}$ $\forall s \in [-\delta, \delta]$, $\forall \tau \in [\tau_0 - \delta, \tau_0 + \delta]$.

(iii) $\exists L > 0$ *such that*

$$\big|f\big(s, \gamma(s, \tau)\big) - f\big(s', \gamma(s', \tau)\big)\big| \leq L|s - s'| \quad \forall s, s' \in [-\delta, \delta],\ \forall \tau \in [\tau_0 - \delta, \tau_0 + \delta].$$

Then $\forall \tau_1, \tau \in [\tau_0 - \delta, \tau_0 + \delta]$, $|\tau_1 - \tau| < \frac{\delta^4}{16}$,

$$\big|f(0, \tau_1) - f(0, \tau)\big| \leq 2\sqrt{2L}\sqrt{|\tau_1 - \tau_2|}.$$

Proof. Let $0 < r_0 < \frac{\delta^4}{16}$ and let us denote

$$\beta := L, \qquad \alpha := \max\big\{2\sqrt{2L}, r_0^{1/4}\big\}, \qquad f_0(\tau) = f(0, \tau).$$

Let us observe that $\frac{\beta}{\alpha^2} \leq \frac{1}{8}$.

By contradiction, let us assume there exist $-\delta \leq \tau_2 < \tau_1 \leq \delta$, $0 < \bar{r} < r_0 < \delta$ such that

$$0 < |\tau_1 - \tau_2| \leq \bar{r} \tag{5.2}$$

$$\frac{|f_0(\tau_1) - f_0(\tau_2)|}{\sqrt{\tau_1 - \tau_2}} > \alpha. \tag{5.3}$$

The idea of the proof is the following: the Lipschitz condition in the hypothesis is an upper bound on the second derivative of the mentioned curves γ. Therefore, if their first derivative wants to change it takes some time in s. If we assume that at $s = 0$ the first derivative differs at two points τ_1, τ_2 at least of the ratio (5.3), then the relative curves γ_1, γ_2 starting from those points must meet soon. However, at the time they meet the first derivative did not have the time to change enough to become equal, providing an absurd.

Let us introduce curves γ_1, γ_2 such that for $i = 1, 2$, $s \in (-\delta, \delta)$

$$\dot{\gamma}_i(s) = f\big(s, \gamma_i(s)\big),$$

$$\gamma_i(0) := \tau_i.$$

We observe that, by our Lipschitz assumption, $\frac{d}{ds} f(s, \gamma_i(s)) := h_i(s)$ is a function bounded by L. Therefore we can represent each $\gamma_i(s)$ for each $s \in [-\delta, \delta]$ as

$$\begin{aligned}\gamma_i(s) &= \tau_i + \int_0^s f\big(\sigma, \gamma_i(\sigma)\big)\, d\sigma \\ &= \tau_i + f_0(\tau_i)\, s + \int_0^s \int_0^\sigma h_i(z)\, dz\, d\sigma. \end{aligned} \tag{5.4}$$

In particular by the second equality in (5.4), for $-\delta \le s \le \delta$,

$$\gamma_1(s) - \gamma_2(s) \le \tau_1 - \tau_2 + \big(f_0(\tau_1) - f_0(\tau_2)\big)s + 2\beta s^2. \tag{5.5}$$

By (5.3) we get either

$$f_0(\tau_1) - f_0(\tau_2) < -\alpha\sqrt{\tau_1 - \tau_2} \tag{5.6}$$

or

$$f_0(\tau_1) - f_0(\tau_2) > \alpha\sqrt{\tau_1 - \tau_2}. \tag{5.7}$$

Let us prove now that if (5.6) holds then there exists $0 < s^* < \delta$ such that

$$\gamma_1\big(s^*\big) = \gamma_2\big(s^*\big). \tag{5.8}$$

Let $\bar{s} := 2\frac{\sqrt{\tau_1 - \tau_2}}{\alpha}$ then

$$\bar{s} \in [0, \delta], \quad \gamma_1(\bar{s}) \le \gamma_2(\bar{s}). \tag{5.9}$$

Indeed by (5.2) and the definition of α, $\bar{s} = 2\frac{\sqrt{\tau_1 - \tau_2}}{\alpha} \le 2(\tau_1 - \tau_2)^{1/4} \le 2\bar{r}^{1/4} \le \delta$. On the other hand by (5.5) (with $s = \bar{s}$), (5.6) gives

$$\begin{aligned}\gamma_1(\bar{s}) - \gamma_2(\bar{s}) &\le \tau_1 - \tau_2 - 2(\tau_1 - \tau_2) + \frac{8\beta}{\alpha^2}(\tau_1 - \tau_2) \\ &= (\tau_1 - \tau_2)\left(-1 + \frac{8\beta}{\alpha^2}\right) \le 0\end{aligned}$$

Then (5.9) follows. Let

$$s^* := \sup\big\{s \in [0, \delta] : \gamma_1(s) > \gamma_2(s)\big\}$$

then by (5.8) $0 < s^* < \bar{s} \le \delta$ and it satisfies (5.8).

If (5.7) holds we can repeat the argument reversing the s-axis getting that there exist $-\delta < s^* < 0$ such that (5.8) still holds.

Let us prove now that

$$f\big(s^*, \gamma_1(s^*)\big) \neq f\big(s^*, \gamma_2(s^*)\big), \tag{5.10}$$

then a contradiction and the thesis will follow. Indeed, for instance, let us assume (5.6). Then by (5.4) and the bound on h_i

$$\begin{aligned} f\big(s^*, \gamma_1(s^*)\big) - f\big(s^*, \gamma_2(s^*)\big) &= f_0(\tau_1) - f_0(\tau_2) + \int_0^{s^*} h_1(\sigma) - h_2(\sigma) d\sigma \\ &\le f_0(\tau_1) - f_0(\tau_2) + 2\beta s^* \le f_0(\tau_1) - f_0(\tau_2) + 2\beta \bar{s} \\ &\le -\alpha \sqrt{\tau_1 - \tau_2} + 2\frac{2\beta}{\alpha}\sqrt{\tau_1 - \tau_2} \\ &\le 2\alpha\sqrt{\tau_1 - \tau_2}\left[-\frac{1}{2} + \frac{2\beta}{\alpha^2}\right]. \end{aligned}$$

Therefore we get that

$$f\big(s^*, \gamma_1(s^*)\big) - f\big(s^*, \gamma_2(s^*)\big) < 0$$

and (5.10) follows. □

5.2. *Proof of the equivalence*

We are now able to give the proof of Theorem 1.1. We distinguish the two different implications in the following two lemmas.

Lemma 5.4. *Let $\omega \subset \mathbb{W}$ be an open set and $w \in L^\infty_{\mathrm{loc}}(\omega; \mathbb{R}^{2n-1})$. If ϕ is a continuous distributional solution of $\nabla^\phi \phi = w$ in ω, then $\phi \in \mathrm{Lip}_{\mathbb{W},\mathrm{loc}}(\omega)$.*

Proof. Let $B = (z_n, \hat{z}_n, t)$, $B' = (z'_n, \hat{z}'_n, t') \in I_{\delta_0}(A)$ for a sufficiently small δ_0. For $n \ge 2$ let $\overline{X}$, W^ϕ be the vector fields given by

$$\overline{X} := \sum_{\substack{j=1 \\ j\neq n}}^{2n-1} \big(z'_j - z_j\big)\nabla^\phi_j + \frac{1}{2}\sum_{j=1}^{n-1}\big(z'_j - z_j\big)\big(z'_{n+j} - z_{n+j}\big)T, \qquad W^\phi := \frac{\partial}{\partial z_n} + \phi\frac{\partial}{\partial t}.$$

We apologize for the following abuse of notations. Taking the classical notion into account, we denote by

$$[-\delta_0, \delta_0] \ni t \mapsto \exp\big(tW^\phi\big)(B) \in \omega$$

a solution $\Gamma : [-\delta_0, \delta_0] \to \omega$ of the problem

$$\begin{cases} \dot{\Gamma}(t) = W^\phi\big(\Gamma(t)\big) \\ \Gamma(0) = B \end{cases}.$$

Define

$$\begin{aligned} B^* &:= \exp(\overline{X})(B) \\ &= B \cdot \left(0, \big(z'_1 - z_1, \dots, z'_{n-1} - z_{n-1}, z'_{n+1} - z_{n+1}, \dots, z'_{2n-1} - z_{2n-1}\big), t + \frac{1}{2}\sum_{j=1}^{n-1}\big(z'_j - z_j\big)\big(z'_{n+j} - z_{n+j}\big)\right) \\ &= \left(z_n, \hat{z}'_n, t + \frac{1}{2}\sum_{j=1}^{n-1}\big(z'_j - z_j\big)\big(z'_{n+j} - z_{n+j}\big) - \sigma\big(\hat{z}', \hat{z}\big)\right) \end{aligned}$$

$$B'' := \exp\big(\big(z'_n - z_n\big)W^\phi\big)\big(B^*\big) = \big(z'_n, \hat{z}'_n, t''\big) \quad \big(\text{for a certain } t''\big).$$

Observe that B^* and B'' are well defined, if $B, B' \in I_{\delta_0}(A)$ for a sufficiently small δ_0. Moreover, by the Baker–Campbell–Hausdorff formula, since the only non-vanishing commutators are $[\nabla_j^\phi, \nabla_{n+j}^\phi] = T$, we get

$$\exp(\overline{X})(B) = \exp\big((z'_{2n-1} - z_{2n-1})\nabla_{2n-1}^\phi\big) \circ \cdots \circ \exp\big((z'_{n+1} - z_{n+1})\nabla_{n+1}^\phi\big) \circ \exp\big((z'_{n-1} - z_{n-1})\nabla_{n-1}^\phi\big) \circ \cdots \circ \exp\big((z'_1 - z_1)\nabla_1^\phi\big)(B). \tag{5.11}$$

For $n = 1$, $\overline{X}$ is not defined and we set $B^* = B$ and $B'' := \exp((z'_n - z_n)W^\phi)(B) = (z'_n, t'')$.

We have to show that there exists $L > 0$ such that

$$\big|\phi(B) - \phi(B')\big| \le L d_\phi\big(B, B'\big) \quad \forall B, B' \in I_{\delta_0}(A). \tag{5.12}$$

We have

$$\big|\phi(B) - \phi(B')\big| \le \big|\phi(B) - \phi(B^*)\big| + \big|\phi(B^*) - \phi(B'')\big| + \big|\phi(B'') - \phi(B')\big| \tag{5.13}$$

Let us now prove that, if $n \ge 2$,

$$\big|\phi(B) - \phi(B^*)\big| \le \left(\sum_{\substack{j=1\\ j\neq n}}^{2n-1} \|w_j\|_{L^\infty(\omega)}\right)\big|\hat{z}'_n - \hat{z}_n\big| \tag{5.14}$$

from which we will get

$$\big|\phi(B) - \phi(B^*)\big| \le L_1 d_\phi\big(B, B'\big) \tag{5.15}$$

with $L_1 := \sum_{\substack{j=1\\ j\neq n+1}}^{2n-1} \|w_j\|_{L^\infty(\omega)}$. Define $B_0 = B$, $B_j = \exp((z'_j - z_j)\nabla_j^\phi)(B_{j-1})$ if $1 \le j \le n-1$ and $B_j = \exp((z'_{j+1} - z_{j+1})\nabla_{j+1}^\phi)(B_{j-1})$ if $n \le j \le 2n-2$. From Lemma 5.2 and (5.11) it follows that

$$\begin{aligned}\big|\phi(B^*) - \phi(B)\big| &= \big|\phi(B_{2n-2}) - \phi(B_0)\big| \le \sum_{j=1}^{2n-2} \big|\phi(B_j) - \phi(B_{j-1})\big| \\ &\le \sum_{\substack{j=1\\ j\neq n}}^{2n-1} \|w_j\|_{L^\infty(\omega)}\big|z'_j - z_j\big|\end{aligned}$$

and (5.14) follows. From Lemma 5.1

$$\big|\phi(B'') - \phi(B^*)\big| \le \|w_n\|_{L^\infty(\omega)}\big|z'_n - z_n\big| \tag{5.16}$$

and then $|\phi(B'') - \phi(B^*)| \le L_2 d_\phi(B, B')$ with $L_2 := \|w_n\|_{L^\infty(\omega)}$. By using Lemmas 5.1 and 5.3 and arguing as in the proof of [6, Theorem 3.2], it can be proved that there exists a positive constant $L_3 = L_3(\|w_n\|_{L^\infty(\omega)}) > 0$ such that

$$\big|\phi(B') - \phi(B'')\big| \le L_3\sqrt{\big|t' - t''\big|} \tag{5.17}$$

Let us observe that

$$\begin{aligned}\big|t' - t''\big| &= \Bigg|t' - t - \frac{1}{2}\sum_{j=1}^{n-1}\big(z'_j - z_j\big)\big(z'_{n+j} - z_{n+j}\big) + \sigma\big(\hat{z}'_n, \hat{z}_n\big) \\ &\quad - \int_0^{z'_n - z_n} \phi\big(\exp\big(sW^\phi\big)\big(B^*\big)\big)\,ds\Bigg| \\ &\le \bigg|t' - t - \frac{1}{2}\big(\phi(B') + \phi(B)\big)\big(z'_n - z_n\big) + \sigma\big(\hat{z}'_n, \hat{z}_n\big)\bigg| + \big|\hat{z}'_n - \hat{z}_n\big|^2\end{aligned}$$

$$+\frac{1}{2}\left|\big(\phi(B')+\phi(B)\big)(z_n'-z_n)-2\int_0^{z_n'-z_n}\phi\big(\exp(sW^\phi)(B^*)\big)\,ds\right|$$

$$\le d_\phi(B',B)^2+\frac{1}{2}\left|\phi(B')-\phi(B'')\right|\left|z_n'-z_n\right|+\frac{1}{2}\left|\phi(B^*)-\phi(B)\right|\left|z_n'-z_n\right|$$

$$+\frac{1}{2}\left|\big[\phi(B'')+\phi(B^*)\big](z_n'-z_n)-2\int_0^{z_n'-z_n}\phi\big(\exp(sW^\phi)(B^*)\big)\,ds\right|$$

$$=: d_\phi(B',B)^2+R_1(B',B)+R_2(B',B)+R_3(B',B). \tag{5.18}$$

For the case $n=1$ we arrive to (5.18) with the same line (it is sufficient to follow the same steps "erasing" the term $\sigma(\hat{z}_n',\hat{z}_n)$).

Now we want to prove that for all $\epsilon>0$ there exists $C_1=C_1(\epsilon)>0$ such that

$$R_1(B',B)\le C_1\left|z_n'-z_n\right|^2+\epsilon\left|t'-t''\right| \tag{5.19}$$

for all $B',B\in I_{\delta_0}(A)$ and that there exist $C_2,C_3>0$ such that

$$R_2(B',B)\le C_2 d_\phi(B',B)^2 \tag{5.20}$$

$$R_3(B',B)\le C_3\left|z_n'-z_n\right|^2 \tag{5.21}$$

for all $B',B\in I_{\delta_0}(A)$,

These estimates are sufficient to conclude: in fact, choosing $\epsilon:=1/2$ and using (5.18), (5.21), (5.19) and (5.20), we get

$$\left|t'-t''\right|\le d_\phi(B',B)^2+C_1\left|z_n'-z_n\right|^2+\left|z_n'-z_n\right|^2+\left|t'-t''\right|/2+C_2 d_\phi(B',B)^2$$

whence there exists $C_4>0$ such that

$$\left|t'-t''\right|^{1/2}\le C_4\, d_\phi(B,B').$$

Then by (5.17) the desired estimate (5.12) follows.

By (5.17) we obtain

$$R_1(B',B)\le 2L_3\sqrt{\left|t'-t''\right|}\left|z_n'-z_n\right|\le\epsilon\left|t'-t''\right|+\frac{L_3^2}{\epsilon}\left|z_n'-z_n\right|^2,$$

whence (5.19) follows.

Observe that (5.20) follows from $R_2(B,B')=0$ if $n=1$, and from

$$\begin{aligned}R_2(B',B)&=\frac{1}{2}\left|z_n'-z_n\right|\left|\phi(B)-\phi(B^*)\right|\\&\le 2C_2\left|z_n'-z_n\right|\left|\hat{z}_n'-\hat{z}_n\right|\le C_2\left|z'-z\right|^2\le C_2 d_\phi(B',B)^2\end{aligned}$$

if $n\ge 2$. Finally, for $s\in[-\delta_0,\delta_0]$ we can define

$$g(s):=2\int_0^s\phi\big(\exp(rW^\phi)(B^*)\big)\,dr-\big[\phi\big(\exp(sW^\phi)(B^*)\big)+\phi(B^*)\big]s; \tag{5.22}$$

We have

$$g(s)=2\int_0^s\big[\phi\big(\exp(rW^\phi)(B^*)\big)-\phi(B^*)\big]\,dr-\big[\phi\big(\exp(sW^\phi)(B^*)\big)-\phi(B^*)\big]s=\mathrm{O}(s^2)$$

because $(-\delta_0, \delta_0) \ni s \mapsto \phi(\exp(sW^\phi)(B^*))$ is Lipschitz by Lemma 5.1. Therefore (5.21) follows with $s = z_n' - z_n$. □

Corollary 5.5. *Let $\omega \subset \mathbb{W}$ be an open set and $w \in L^\infty(\omega; \mathbb{R}^{2n-1})$. If $\phi \in C^0(\omega)$ is a distributional solution of $\nabla^\phi \phi = w$ in ω, then for each $\omega' \Subset \omega$ there exists a positive constant $C = C(\omega') > 0$ such that*

$$\left|\phi(A) - \phi(B)r\right| \le C\sqrt{|A - B|} \quad \forall A, B \in \omega' \tag{5.23}$$

Lemma 5.6. *If $\phi \in \mathrm{Lip}_{\mathbb{W},\mathrm{loc}}(\omega)$, then ϕ is a distributional solution of $\nabla^\phi \phi = w$ in ω for a suitable $w \in L^\infty_{\mathrm{loc}}(\omega, \mathbb{R}^{2n-1})$ such that $w(A) = \nabla^\phi \phi(A)$ $\mathcal{L}^{2n}$-a.e. $A \in \omega$.*

Proof. By Theorem 2.15 there exist $\{\phi_k\}_{k\in\mathbb{N}} \subset C^\infty(\omega)$, such that $\{\phi_k\}_k$ uniformly converges to ϕ on the compact sets of ω, $|\nabla^{\phi_k}\phi_k(A)| \le C$ $\mathcal{L}^{2n}$-a.e. $A \in \omega$ for every $k \in \mathbb{N}$ and $\nabla^{\phi_k}\phi_k(A) \to w(A)$ $\mathcal{L}^{2n}$-a.e. $A \in \omega$. Therefore, denoting $w_k := \nabla^{\phi_k}\phi_k$, we have for every $k \in \mathbb{N}$ and for every $\varphi \in C_c^\infty(\omega)$

$$\int_\omega \phi_k X_j \varphi \, d\mathcal{L}^{2n} = -\int_\omega w_{j,k}\varphi \, d\mathcal{L}^{2n}, \quad j = 1, \dots, n-1, n+1, \dots, 2n-1$$

$$\int_\omega \left(\phi_k \frac{\partial \varphi}{\partial z_n} + \frac{1}{2}\phi_k^2 \frac{\partial \varphi}{\partial t}\right) d\mathcal{L}^{2n} = -\int_\omega w_{n,k}\varphi \, d\mathcal{L}^{2n}.$$

Getting to the limit for $k \to \infty$ we obtain

$$\int_\omega \phi X_j \varphi \, d\mathcal{L}^{2n} = -\int_\omega w_j \varphi \, d\mathcal{L}^{2n}, \quad j = 1, \dots, n-1, n+1, \dots, 2n-1$$

$$\int_\omega \left(\phi \frac{\partial \varphi}{\partial z_n} + \frac{1}{2}\phi^2 \frac{\partial \varphi}{\partial t}\right) d\mathcal{L}^{2n} = -\int_\omega w_n \varphi \, d\mathcal{L}^{2n}$$

i.e. ϕ is a distributional solution of the problem $\nabla^\phi \phi = w$ in ω. □

From Lemmas 5.4 and 5.6, Theorem 2.16 and Remarks 2.10 and 2.17, it follows that

Corollary 5.7. *Let $\omega \subset \mathbb{W}$ be an open set. If $\phi \in C^0(\omega)$ is a distributional solution of $\nabla^\phi \phi = w$ in ω for a suitable $w \in L^\infty_{\mathrm{loc}}(\omega; \mathbb{R}^{2n-1})$ then ϕ is ∇^ϕ-differentiable $\mathcal{L}^{2n}$-a.e. in ω and $\nabla^\phi \phi(A) = w(A)$ $\mathcal{L}^{2n}$-a.e. $A \in \omega$.*

6. Further equivalences

In the previous section we established the equivalence between

'$\phi : \omega \subset \mathbb{W} \equiv\subset \mathbb{R}^2 \to \mathbb{R}$ is intrinsic Lipschitz continuous with intrinsic gradient $\tilde{w}$'

and, denoting by w the distribution identified by the bounded, a.e defined function $\tilde{w}$,

'$\phi \in C^0(\omega)$ and there exists $w \in L^\infty\big(\omega; \mathbb{R}^{2n-1}\big)$ such that $\nabla^\phi \phi = w$ in $\mathcal{D}'(\omega)$'.

We establish now in Section 6.2 the following characterization: one can reduce the PDE

$$\nabla^\phi \phi = w \quad \text{in } \omega$$

along *any* integral line of the vector fields ∇_i^ϕ, $i = 1, \dots, 2n-1$. This involves as an essential point a suitable choice of the $L^\infty(\omega; \mathbb{R}^{2n-1})$ representative $\hat{w}$ of the distribution identified by w.

As an introduction, we first prove in Section 6.1 the following weaker statement: one can reduce the PDE to ODEs along a selected family of characteristics constituting a Lagrangian parameterization. As well, the converse holds: if the ODEs on characteristics are satisfied, one has a continuous distributional solution to the PDE. The sources of the two formulations can be identified, but the explicit proof is not contained in this section. Once proved that Lagrangian solutions are intrinsic Lipschitz continuous functions, the latter implication is known from [11], as recalled in Lemma 5.6. A different proof is in [1].

The conclusion of this last section will be the following.

Corollary 6.1. *The various notions of continuous solutions to* (3.1), (3.2) *we defined are equivalent.*

6.1. Distributional solutions are Lagrangian solutions

The present section is an introduction to the next one, which proves the stronger statement that distributional solutions are broad solutions. Being technically simpler, this section provides a guideline for some ideas implemented next. One can in particular notice that the proof concerns only ODEs.

In order to avoid technicalities we focus on $n = 1$ and we do proofs for $\omega = [0, 1]^2$ and $\tilde{\omega} = [0, 1]^2$. We maintain the different notations for ω and $\tilde{\omega}$ in order to distinguish better domain and codomain of χ. We refer to Section 6.2 for the stronger statement with the universal source term $\hat{w}$.

Lemma 6.2. *Let* $\phi : [0, 1]^2 \to \mathbb{R}$ *be a continuous function. Consider a Lagrangian parameterization* $([0, 1]^2, \chi(s, \tau))$ *and assume that* $[0, 1] \ni s \mapsto \phi(s, \chi(s, \tau))$ *is Lipschitz continuous for all* $\tau \in [0, 1]$. *Then there exists a Borel function* $\bar{w} : [0, 1]^2 \to \mathbb{R}$ *such that for all* τ

$$\frac{\partial \phi}{\partial s}\big(s, \chi(s, \tau)\big) = \frac{\partial^2 \chi}{\partial s^2}(s, \tau) = \bar{w}\big(s, \chi(s, \tau)\big) \quad \text{for } \mathcal{L}^1\text{-a.e. } s \in [0, 1].$$

Corollary 6.3. *If a function* $\phi : [0, 1]^2 \to \mathbb{R}$ *is*

- *either an intrinsic Lipschitz continuous function with intrinsic gradient* w
- *or a continuous distributional solutions* ϕ *of the balance law* (3.1)

then it is a Lagrangian solution to the equation $\nabla^\phi \phi = \bar{w}$ *in* $[0, 1]^2$ *for a bounded function* $\bar{w} : [0, 1]^2 \to \mathbb{R}$ *which identifies the* L^1*-function* w.

Proof of Corollary 6.3. The existence of a Lagrangian parameterization has been given in Lemma 4.2. By Remark 2.14 an intrinsic Lipschitz continuous function is continuous and Lipschitz along characteristics. Also continuous distributional solutions to the balance law (3.1) are Lipschitz continuous along characteristics by Lemma 5.1. Then from Lemma 6.2, properties (LS1) and (LS2) immediately follow. The identification of w and $\bar{w}$ is not the point here, but we mention how can be deduced.

Eventually, if $\nabla^\phi \phi = w$ in $[0, 1]^2$ in sense of distributions, then by Corollary 5.7

$$d_{\mathbb{W}}\phi(A)(e_1) = \nabla^\phi \phi(A) = w(A) \quad \mathcal{L}^2\text{-a.e. } A \in \omega. \tag{6.1}$$

On the other hand, if

$$\frac{\partial \phi}{\partial s}\big(s, \chi(s, \tau)\big) = \bar{w}\big(s, \chi(s, \tau)\big) \quad \mathcal{L}^1\text{-a.e. } s \in [0, 1],\ \forall \tau \in [0, 1]$$

and $\chi(s, \cdot) : [0, 1] \to [0, 1]$ is onto, by Remark 2.14, we also get that

$$d_{\mathbb{W}}\phi(A)(e_1) = \nabla^\phi \phi(A) = \bar{w}(A) \quad \mathcal{L}^2\text{-a.e. } A \in [0, 1]^2. \tag{6.2}$$

Thus by (6.1) and (6.2), (LS3) follows. □

Proof of Lemma 6.2. We prefer to distinguish the notations for the domains $\tilde{\omega}$ and ω, even if both are $[0, 1]^2$. This is important to make clear in which space we are defining the functions. We remind the notation

$$\Upsilon : \tilde{\omega} \to \omega \qquad \Upsilon(s, \tau) := \big(s, \chi(s, \tau)\big).$$

Step 1: Second derivative in $\tilde{\omega}$.

Claim. *The subset $B \subset \tilde{\omega}$ of those (s,τ) where $\chi(s,\tau)$ is twice s-differentiable is a full measure, $F_{\sigma\delta}$ set. $\frac{\partial^2\chi}{\partial s^2}(s,\tau)$ is a Borel function on it.*

Proof. By assumption $\frac{\partial\chi}{\partial s}(s,\tau) = \phi(s,\chi(s,\tau))$ is Lipschitz continuous in s, therefore $\frac{\partial^2\chi}{\partial s^2}(s,\tau)$ is a Borel function on its domain. Moreover, the subset $B \subset \tilde{\omega}$ of those (s,τ) where $\chi(s,\tau)$ is twice s-differentiable must have full measure by Tonelli theorem, because each τ-section has full measure by the Lipschitz continuity of $\phi(s,\chi(s,\tau))$. Being ϕ continuous, one can see that B is an $F_{\sigma\delta}$ set: just write it as

$$\bigcap_{\varepsilon\downarrow 0}\bigcup_{r\in\mathbb{Q}}\bigcup_{\delta>0}\left\{(s,\tau) : \frac{\phi(B_\delta(s,\tau)) - \phi(s,\tau)}{\delta} \subset B_\varepsilon(r)\right\}.$$

The second differentiability of $\chi(s,\tau)$ however does not answer the question: the function $\frac{\partial^2\chi}{\partial s^2}(s,\tau)$ is defined on $\tilde{\omega}$, while we are looking for a function defined on ω.

Step 2: A preliminary comment. In order to check that Υ lifts $\frac{\partial^2\chi}{\partial s^2}$ to a map $\bar{w}$ a.e. defined on ω, which would provide our thesis, it would be natural to show that

- $\Upsilon(B)$ is a Lebesgue measurable subset of ω with full measure;
- $\frac{\partial^2\chi}{\partial s^2}(s,\tau)$ is constant on the level set of Υ intersected with B.

Since χ is not Lipschitz, we do not manage to prove at this point that $\Upsilon(B)$ has full measure. We instead assign a specific value, 0, to the function out of $\Upsilon(B)$. This choice of the extension does not affect our claim. We notice moreover that the second point is not true in that strong form, but we show that the points of the level set of Υ corresponding to more values of $\frac{\partial^2\chi}{\partial s^2}$ are not relevant.

Step 3: Analysis of $\Upsilon(B)$ and partial inverse of Υ. The proof of the Borel measurability of $\Upsilon(B)$ requires some technicality: we apply a theorem due to Srivastava ([27], Theorem 5.9.2) deriving that there exists a Borel restriction which is one-to-one with image $\Upsilon(B)$. Precisely we mean that there exists a Borel set S and a Borel injective function

$$\Xi^{-1} : S \subset \tilde{\omega} \to \Upsilon(B) \subset \omega \quad \text{s.t.} \quad \Xi^{-1}(s,\tau) \equiv \Upsilon(s,\tau) \quad \forall (s,\tau) \in S, \quad \operatorname{Im}(\Xi^{-1}) = \Upsilon(B).$$

This implies that $\Upsilon(B)$ is a Borel image by a one-to-one map. Theorem 4.12.4 in [27], due to Lusin, asserts then on one hand that $\Upsilon(B)$ is Borel, and moreover that this restriction Ξ^{-1} has a Borel inverse

$$\Xi : \Upsilon(\tilde{\omega}) \subset \omega \to \tilde{\omega}.$$

We are left now with checking that we can apply Srivastava's theorem. We partition $\tilde{\omega}$ into the level sets of Υ, which are G_δ. Indeed it is easy to see that, if $(s_0,t_0) \in \Upsilon(\tilde{\omega})$, then

$$\Upsilon^{-1}\big((s_0,t_0)\big) = \{s_0\} \times I_{t_0}$$

where I_{t_0} is a closed bounded interval. In order to prove the Borel measurability of the partition $\{\Upsilon^{-1}((s_0,t_0)) : (s_0,t_0) \in \Upsilon(\tilde{\omega})\}$, see [27, Section 5.1], let us prove that $\Upsilon^{-1}(\Upsilon(O))$ is a Borel set for each open set $O \subset \mathbb{R}^2$. For simplicity, consider the case when χ is already a full parameterization and thus it is continuous (see Remark 3.6). Every open set O is σ-compact: thus, by continuity, $\Upsilon(O)$ is σ-compact, and finally $\Upsilon^{-1}(\Upsilon(O))$ is σ-compact. Therefore by Srivastava's theorem there is a Borel cross section S for the partition: defining the function Ξ^{-1} as the restriction of Υ to $S \cap B$, then Ξ^{-1} is Borel, injective and onto $\Upsilon(B)$.

Step 4: Analysis of $\frac{\partial^2\chi}{\partial s^2}(s,\tau)$ for $(s,\tau) \in B$. We can define $\bar{w}$ as

$$\bar{w}(s,\tau) = \begin{cases} \frac{\partial^2\chi}{\partial s^2}(\Xi(s,\tau)) & (s,\tau) \in \Upsilon(B), \\ 0 & (s,\tau) \in \omega \setminus \Upsilon(B) \end{cases}$$

This map satisfies the claim by the next analysis of the set where $(\frac{\partial^2\chi}{\partial s^2}) \circ \Upsilon^{-1}$ is multivalued.

Step 5: Analysis of multivalued points of $(\frac{\partial^2\chi}{\partial s^2}) \circ \Upsilon^{-1}$. The set of points $(s,\tau) \in B$ such that there exists $(\bar{s},\bar{\tau}) \in B$ satisfying $\Upsilon(s,\tau) = \Upsilon(\bar{s},\bar{\tau})$ and $\frac{\partial^2\chi}{\partial s^2}(s,\tau) \neq \frac{\partial^2\chi}{\partial s^2}(\bar{s},\bar{\tau})$ is Borel: just see it as

$$M := \Upsilon^{-1} \circ \Upsilon\left(\left\{(s,\tau) \in B : \left|\frac{\partial^2\chi}{\partial s^2}(s,\tau) - \frac{\partial^2\chi}{\partial s^2}\big(\Xi\big(\Upsilon(s,\tau)\big)\big)\right| > 0\right\}\right).$$

The value of $\bar{w}$ on this set is not at all relevant: we show below that for any fixed $\tau \in [0,1]$ the intersection of $\Upsilon(M)$ with the characteristics curve $\{\chi(s,\tau)\}_{s\in[0,1]}$ is at most countable.

Fix $m \in \mathbb{N}$ and let

$$M_m^* := \Upsilon\left(\left\{(s,\tau) \in B : \frac{1}{m} \leq \left|\frac{\partial^2\chi}{\partial s^2}(s,\tau) - \frac{\partial^2\chi}{\partial s^2}\big(\Xi\big(\Upsilon(s,\tau)\big)\big)\right| \leq m\right\}\right).$$

Let us prove that the set $\{s \in [0,1] : (s,\tau) \in B \text{ and } \Upsilon((s,\tau)) \in M_m^* \cap \{\chi(s,\tau)\}_{s\in[0,1]}\}$ is composed by isolated points. From which it follows that

$$\Upsilon(M) \cap \{\chi(s,\tau)\}_{s\in[0,1]} = \bigcup_{m=1}^{\infty} \big(M_m^* \cap \{\chi(s,\tau)\}_{s\in[0,1]}\big)$$

is at most countable.

Indeed, fix a curve $\gamma(s) := \gamma(s,\tau)$, for simplicity assume that $\gamma(s) = 0$. Let $(\bar{s},\tau) \in B$ be such that $\Upsilon(\bar{s},\tau) \in M_m^* \cap \{\gamma(s)\}_{s\in[0,1]}$, let $(\bar{s},\tau') := \Xi(\Upsilon(\bar{s},\tau))$, let $\gamma' : [0,1] \to \mathbb{R}$ be the function defined by $\gamma'(s) := \chi(s,\tau')$. Notice that, in particular,

$$\gamma'(\bar{s}) = \gamma(\bar{s}) = 0, \quad \dot{\gamma}'(\bar{s}) = \phi\big(\bar{s},\gamma'(\bar{s})\big) = \phi\big(\bar{s},\gamma(\bar{s})\big) = \dot{\gamma}(\bar{s}) = 0$$

and we can assume, for instance, that

$$\frac{1}{m} \leq \ddot{\gamma}'(\bar{s}) \leq m.$$

By Taylor's expansion, there exists $\bar{\delta} = \delta(\bar{s}) > 0$ such that

$$\frac{(s-\bar{s})^2}{3m} \leq \gamma'(s) \leq \frac{3}{2}m(s-\bar{s})^2 \quad \forall s \in (\bar{s}-\bar{\delta},\bar{s}+\bar{\delta}). \tag{6.3}$$

By contradiction, assume there exists a different $(\bar{s}_2,\tau) \in B$ such that $\Upsilon(\bar{s}_2,\tau) \in M_m^* \cap \{\gamma(s)\}_{s\in[0,1]}$ and $|\bar{s}-\bar{s}_2| < \bar{\delta}$. Let

$$\big(\bar{s}_2,\tau_2'\big) := \Xi\big(\Upsilon(\bar{s}_2,\tau)\big), \quad \gamma_2'(s) := \chi\big(s,\tau_2'\big)\ s \in [0,1], \quad \bar{\delta}_2 = \delta(\bar{s}_2).$$

Then, for instance, let $\bar{s} < \bar{s}_2$, by applying (6.3) with γ' and then with $\gamma' \equiv \gamma_2'$, it follows that there exists $\xi \in (\bar{s},\bar{s}_2)$ such that the curves $\{\Upsilon(s,\tau') : s \in [0,1]\}$ and $\{\Upsilon(s,\tau_2') : s \in [0,1]\}$ intersect at $s = \xi$ and cross each other. Then a contradiction for (L.2) of Definition 3.2. □

6.2. Distributional solutions are broad solutions

Below we show Theorem 6.6: there exists Borel functions $\hat{w}_i(z,t)$, $i = 1,\dots,2n-1$, such that every curve $\Gamma_i(s)$ satisfying the ODE with continuous coefficient

$$\dot{\Gamma}_i(s) = \nabla_i^{\phi}\big(\Gamma_i(s)\big)$$

has first derivative Lipschitz, and it satisfies in the sense of distributions

$$\frac{d}{ds}\phi\big(\Gamma_i(s)\big) = \hat{w}_i\big(\Gamma_i(s)\big).$$

For the nonlinear field $i = n$ this is a statement on the second derivative of $\Gamma_n(s)$. The remarkable fact is that $\hat{w}_n(z,t)$ could be defined independently of any set of characteristic curves. This is thus different, and stronger, from what we already proved for $n = 1$, which is that there exists a Lagrangian parameterization χ and an associated function w_χ satisfying $\frac{\partial^2\chi}{\partial s^2}(s,\tau) = w_\chi(s,\chi(s,\tau))$. The new point is indeed that $\hat{w}$ is a universal representative for the source term.

Due to its nature this proof works for any n with no further complications. The only difference is that ω will be a subset of $\mathbb{R}^{2n}$ instead of $\mathbb{R}^2$. We write it with $n = 1$ only for notational convenience. In particular we generalize here the previous Lemma 6.2.

Since the argument is more intuitive, we mention first how to construct such a Souslin function $\hat{w}(z,t)$. We proceed then with the Borel construction because it gives a better result.

6.2.1. Souslin selection

The first step is to define pointwise, but in a measurable way, by von Neumann's selection principle, a representative function $\hat{w} : \omega \to \mathbb{R}$ which will replace the original source w. More precisely, let $(z,t) \in \omega$ be a given point and consider the family of $C^{1,1}$ curves $[z-\delta, z+\delta] \ni s \mapsto (s,\gamma(s)) \in \omega$ with $\gamma(z) = t$ satisfying

$$\exists \lim_{\sigma\downarrow 0} \frac{1}{\pm\sigma} \int_z^{z\pm\sigma} \ddot{\gamma}(s)ds = \ddot{\gamma}(z) \in \mathbb{R} \tag{6.4}$$

and

$$\dot{\gamma}(s) = \phi\big(s,\gamma(s)\big) \quad \forall s \in [z-\delta, z+\delta]. \tag{6.5}$$

Then, by the afore-mentioned selection principle, we will define $\hat{w}(z,t) := \ddot{\gamma}_{(z,t)}(z)$ for a suitable curve $\gamma_{(z,t)}$ chosen among those satisfying (6.4) and (6.5).

Let $\phi : \mathrm{clos}(\omega) \to \mathbb{R}$ be continuous and assume that it is uniformly Lipschitz along characteristic curves, that is, there exists $L > 0$ such that for each curve $[a,b] \ni s \mapsto (s,\bar{\gamma}(s)) \in \omega$ satisfying $\dot{\bar{\gamma}}(s) = \phi(s,\bar{\gamma}(s))$ $\forall s \in [a,b]$, it holds

$$\big|\dot{\bar{\gamma}}(s) - \dot{\bar{\gamma}}(s')\big| \le L\big|s - s'\big| \quad \forall s,s' \in [a,b]. \tag{6.6}$$

Let $\delta > 0$,

$$X = \omega_\delta := \big\{(z,t) \in \omega : \ \mathrm{dist}\big((z,t),\partial\omega\big) \ge \delta\|\phi\|_\infty\big\},$$

and

$$Y = C^1\big([-\delta,\delta]\big) \times [-L,L]$$

where, in the following, $\|\cdot\|_\infty := \|\cdot\|_{L^\infty(\omega)}$. We will denote by $\mathcal{G} = \mathcal{G}(\phi, L, \delta)$ the subset of

$$\omega_\delta \times C^1\big([-\delta,\delta]\big) \times [-L,L] \supset \mathcal{G} \ni \big((z,t),\gamma,\zeta\big)$$

defined by $((z,t),\gamma,\zeta) \in \mathcal{G}$ if and only if

(G1) $(z,t) \in \omega_\delta$

(G2) $\gamma \in C^1([-\delta,\delta])$, $(-\delta,\delta) \ni s \mapsto (s+z,\gamma(s)) \in \omega$,

$$\begin{cases} \dot{\gamma}(s) = \phi\big(s+z,\gamma(s)\big) & \forall s \in [-\delta,\delta] \\ \gamma(0) = t \end{cases}$$

(G3) $$\big|\dot{\gamma}(s) - \dot{\gamma}(s')\big| \le L\big|s-s'\big| \quad \forall s,s' \in [-\delta,\delta]$$

(G4) $$\zeta \in [-L,L] \quad \text{and} \quad \zeta = \lim_{\sigma\downarrow 0} \frac{1}{\pm\sigma} \int_z^{z\pm\sigma} \ddot{\gamma}(s)ds.$$

We are going to apply Von Neumann's selection theorem ([27, Section 5.5], [16]) to the subset $\mathcal{G}$, within the environment of the space $\omega_\delta \times C^1([-\delta,\delta]) \times [-L,L]$ understood, with the same notation of [27], as a product of Polish spaces $X=\omega_\delta$ and $Y=C^1([-\delta,\delta])\times[-L,L]$ endowed by their standard topologies. In order to apply Von Neumann's result, we need that $\mathcal{G}$ is, at least, universally measurable, even though we prove a better regularity from the measurability viewpoint in the lemma below.

Let $\Pi_X : X\times Y\to X$ denote the projection on $X=\omega_\delta$ in the following.

Lemma 6.4. *$\mathcal{G}$ is a Borel set.*

Proof. Components $((z,t),\gamma)$. Consider the closed subset

$$\big((z,t),\gamma,\zeta\big)\in C\subset\omega_\delta\times C^1\big([-\delta,\delta]\big)\times[-L,L] \tag{6.7}$$

identified by the constraints (G2) and (G3). Its projection is $\Pi_X(C)=X=\omega_\delta$. Indeed one can find through every point a characteristic curve by Peano Theorem, and Lemma 5.1 ensures that ϕ is Lipschitz continuous along it.

Component ζ, discretization. In order to establish the existence (and the value) of the limit in (G4) for the second derivative of γ_z at z, it suffices to prove the existence of the limit for the sequence

$$h_{n+1}=h_n-h_n^2,\qquad h_1=1/2.$$

Indeed, then for $h\in(h_{n+1},h_n]$, for example at $z=0$

$$\left|\frac{1}{h}\int_0^h\ddot{\gamma}_z-\frac{1}{h_n}\int_0^{h_n}\ddot{\gamma}_z\right|=\left|\left(\frac{1}{h}-\frac{1}{h_n}\right)\int_0^h\ddot{\gamma}_z-\frac{1}{h_n}\int_h^{h_n}\ddot{\gamma}_z\right|\le 2L\,\frac{h_n-h}{h_n}.$$

By construction however

$$|h_n-h|\le|h_n-h_{n+1}|=h_n^2,$$

yielding that the existence of the limit along $\{h_n\}_n$ implies the existence of the limit for any $h\downarrow 0$. Notice that this would not hold choosing a generic $\tilde{h}_n\downarrow 0$ instead of $\{h_n\}_n$.

Measurability of $\mathcal{G}$. The further condition (G3) can be written in terms of suitable sets $C_{k,n}$ $(k,n\in\mathbb{N})$, defined as the points $((z,t),\gamma,\zeta)$ such that

$$(z,t)\in\omega_\delta,\quad \gamma_z(z)=t,\qquad \left|\zeta-\frac{\dot{\gamma}_z(z\pm h_n)-\dot{\gamma}_z(z)}{\pm h_n}\right|\le 2^{-k}.$$

Therefore, we can represent $\mathcal{G}$ as the set

$$\mathcal{G}=C\cap\left(\bigcap_{k\in\mathbb{N}}\left(\bigcup_{\bar{n}\in\mathbb{N}}\left(\bigcap_{n\ge\bar{n}}C_{k,n}\right)\right)\right). \tag{6.8}$$

Since $C_{k,n}$ is closed for each $k,n\in\mathbb{N}$, $\mathcal{G}$ is Borel. □

Remark 6.5. Let $\mathcal{D}:=\Pi_X(\mathcal{G})\subseteq X=\omega_\delta$, then it has full measure, that is $\mathcal{L}^2(\mathcal{D})=\mathcal{L}^2(\omega_\delta)$. This property is proved in [1] only by assuming the uniform Lipschitz continuity of ϕ along its characteristic curves (that is (6.6)). Here we point out it easily follows by assuming that ϕ is intrinsic Lipschitz. Indeed, from Theorem 2.16, ϕ admits ∇^ϕ-derivative for $\mathcal{L}^{2n}$-a.e. $A\in\omega$. Thus, for $\mathcal{L}^{2n}$-a.e. $A=(z,t)\in\omega$ and for each characteristic curve $[a,b]\ni s\mapsto(s,\gamma(s))\in\omega$ passing through A, it holds that $((z,t),\gamma,\nabla^\phi\phi(A))\in\mathcal{G}$ by Remark 2.14.

By Lemma 6.4, we can apply Von Neumann theorem [27, Theorem 5.5.2] to define a Souslin selection of $\mathcal{G}$ in $X\times Y$, that is a function

$$X=\omega_\delta\supseteq\mathcal{D}:=\Pi_X(\mathcal{G})\ni(z,t)\mapsto\big(\gamma_{(z,t)},\ddot{\gamma}_{(z,t)}(z)\big)\in Y=C^1\big([-\delta,\delta]\big)\times[-L,L]$$

such that $((z,t),\gamma_{(z,t)},\ddot{\gamma}_{(z,t)}(z))\in\mathcal{G}$. This map is measurable with respect to the σ-algebra of analytic subset of ω_δ and, as a byproduct, it is also universally measurable. As well, one can define a Souslin function $\hat{w}_\delta\to\mathbb{R}$ such that

$$\hat{w}_\delta(z,t):=\ddot{\gamma}_{(z,t)}(z)=w_{(z,t)}\quad\forall(z,t)\in\mathcal{D}\subseteq\omega_\delta$$

which we can extend to the whole ω in the following way. Let $(\delta_m)_m \downarrow 0$ and define $\omega_1 := \omega_{\delta_1}$ and $\omega_m := \omega_{\delta_m} \setminus \omega_{\delta_{m-1}}$ if $m \geq 2$. Since the sequence of sets $(\omega_m)_m$ are pairwise disjoint and $\omega = \bigcup_{m=1}^{\infty} \omega_m$, we can define a universally measurable function $\hat{w}$, defined $\mathcal{L}^2$-a.e. in ω, by

$$\hat{w}(z,t) := \hat{w}_{\delta_m}(z,t) \quad \text{if } (z,t) \in \omega_m. \tag{6.9}$$

The importance of this selection is due to the following theorem.

Theorem 6.6. *Let $\phi : \mathrm{clos}(\omega) \to \mathbb{R}$ be a continuous function which is uniformly Lipschitz along characteristic curves, that is (6.6) holds. Then, for each characteristic curve $[a,b] \ni s \mapsto (s, \bar{\gamma}(s)) \in \omega$, one has*

$$\frac{d}{ds}\phi\big(s, \bar{\gamma}(s)\big) = \hat{w}\big(s, \bar{\gamma}(s)\big) \quad \mathcal{L}^1\text{-a.e. } s$$

where $\hat{w} : \omega \to \mathbb{R}$ is the function defined in (6.9)

Corollary 6.7. *A continuous distributional solution $\phi : \omega \to \mathbb{R}$ of $\nabla^{\phi}\phi = w$ in ω, with $w \in L^{\infty}(\omega)$, is also a broad solution with a source term $\hat{w}$ satisfying the same uniform bound.*

By the identification of the source, discussed e.g. in Corollary 6.3, we also know that $\hat{w}$ identifies w.

Proof of Corollary 6.7. Step 1. Assume that $\phi \in C^0(\mathrm{clos}(\omega))$. Observe that (6.6) holds from Lemma 5.1 with $L = \|w\|_{\infty}$. Thus, from Theorem 6.6, conditions (B.2) and (B.3) of Definition 3.8 immediately follow. Condition (B.1) follows from Corollary 5.7 and Remark 2.17.

Step 2. In the general case, let $\{\omega_j\}_{j\in\mathbb{N}}$ be an increasing sequence of open sets such that

$$\omega_j \subset \mathrm{clos}(\omega_j) \subset \omega \quad \forall j \in \mathbb{N}, \quad \bigcup_{j=1}^{\infty} \omega_j = \omega.$$

Let us observe that, for a given j, $\phi \in C^0(\mathrm{close}(\omega_j))$ and ϕ is still a continuous distributional solution of $\nabla^{\phi}\phi = w$ in ω_j. From the previous step, ϕ is also a broad solution. On the other hand, by a standard argument, it is easy to see that ϕ is a broad solution of $\nabla^{\phi}\phi = w$ in ω. □

6.2.2. Borel selection

Before proving Theorem 6.6, for the sake of completeness, we show that one can define as well a Borel function, that we still denote as $\hat{w}(z,t)$, for which Theorem 6.6 still holds. This requires a bit more work than the previous argument, and it is conceptually a little more involved: we do not associate immediately to each point (where it is possible) an eligible curve and its second derivative, but something which must be close to it. We will find then with the proof of Theorem 6.6 that we end up basically with the same selection.

Lemma 6.8. *For every $\delta, \varepsilon > 0$, there are $\bar{h} = h(\varepsilon) > 0$ and a Borel function defined on the projection $\mathcal{D} := \Pi_X(\mathcal{G}) \subseteq X = \omega_\delta$*

$$\mathcal{D} \ni (z,t) \mapsto (\gamma_{\varepsilon,(z,t)}, w_{\varepsilon,(z,t)}) \in C^1\big([-\delta,\delta]\big) \times [-L,L]$$

such that $((z,t), \gamma_{\varepsilon,(z,t)}, w_{\varepsilon,(z,t)}) \in C$ of (6.7) and

$$\left| w_{\varepsilon,(z,t)} - \frac{1}{h}\int_z^{z+h} \ddot{\gamma}_{\varepsilon,(z,t)}\, ds \right| < \varepsilon \quad \forall 0 < |h| < \bar{h}.$$

Proof. We apply Arsenin–Kunugui selection theorem ([23], or Th. 5.12.1 of [27]) to the set

$$\tilde{C} := \bigcup_{n\in\mathbb{N}} \bigcap_{m>n} \left\{ (z,t,\gamma,\zeta) \in C : \left| \zeta - \frac{\dot{\gamma}(z \pm h_m) - \dot{\gamma}(z)}{\pm h_m} \right| \leq \varepsilon \right\},$$

where C was defined in (6.7) and $\{h_n\}_{n\in\mathbb{N}}$ immediately below that, in the same proof. More precisely, for every $(\bar{y},\bar{t})\in\omega_\delta$ the subset of $C^1([-\delta,\delta])\times[-L,L]$ given by the section

$$C_{(\bar{z},\bar{t})}=C\cap\big\{(z,t,\gamma,\zeta)\ :\ z=\bar{z},t=\bar{t}\big\},$$

which is made by $C^1([-\delta,\delta])$ curves through $(\bar{z},\bar{t})$ with L-Lipschitz continuous second derivative, is compact by Ascoli–Arzelà theorem, because ϕ is continuous and bounded. Moreover

$$\left\{(z,t,\gamma,\zeta):\left|\zeta-\frac{\dot{\gamma}(z+h)-\dot{\gamma}(z)}{h}\right|\le\varepsilon,\ z=\bar{z},t=\bar{t}\right\}$$

is closed: therefore each $(\bar{z},\bar{t})$-section of $\tilde{C}$ is σ-compact. Then the hypothesis of the theorem are satisfied: it assures there exists a Borel section of $\tilde{C}$, that is $\Pi_X(\tilde{C})$ is a Borel set and there exists a Borel map

$$X=\omega_\delta\supseteq\Pi_X(\tilde{C})\ni(z,t)\mapsto(\gamma_{\varepsilon,(z,t)},w_{\varepsilon,(z,t)})\in Y=C^1\big([-\delta,\delta]\big)\times[-L,L]$$

such that $((z,t),\gamma_{\varepsilon,(z,t)},w_{\varepsilon,(z,t)})\in\tilde{C}$. □

From Lemma 6.9, it follows, for every $\delta,\varepsilon>0$, the existence of a Borel function $\hat{w}_{\delta,\varepsilon}:\omega_\delta\to[-L,L]$ defined by

$$\hat{w}_{\delta,\varepsilon}(z,t):=\chi_{\{(z,t)\in\mathcal{D}\}}w_{\varepsilon,(z,t)}\quad\text{ if }(z,t)\in\omega_\delta.$$

Now, arguing as in the construction of the Souslin representative $\hat{w}:\omega\to\mathbb{R}$ (see (6.9)), we can extend it to a Borel function $\hat{w}_\varepsilon:\omega\to[-L,L]$ defined by

$$\hat{w}_\varepsilon(z,t):=\hat{w}_{\delta_m,\varepsilon}(z,t)\quad\text{ if }(z,t)\in\omega_m.\tag{6.10}$$

Definition 6.9. We define the Borel representative of w, $\hat{w}:\omega\to[-L,L]$, as the function

$$\hat{w}(z,t)=\liminf_{\varepsilon\downarrow0}w_\varepsilon(z,t).$$

6.2.3. Proof of Theorem 6.6

We provide now the proof of Theorem 6.6 with $\hat{w}:\omega\to[-L,L]$ either the Borel function in Definition 6.9 or the Souslin one in (6.9): we consider any characteristic $[a,b]\ni s\mapsto\Gamma(s):=(s,\bar{\gamma}(s))\in\omega$ and remind that, by (6.6), $[a,b]\ni s\mapsto\dot{\bar{\gamma}}(s)$ is Lipschitz. We prove that

$$\ddot{\bar{\gamma}}(s)=\hat{w}\big(s,\bar{\gamma}(s)\big)\quad\text{ a.e. }s\in[a,b].\tag{6.11}$$

Notice also that, if $(\delta_m)_m\downarrow0$, there is $\bar{m}\in\mathbb{N}$ such that $\Gamma([a,b])\subset\omega_{\delta_{\bar{m}}}\subset\omega$. As a consequence, both Borel and Souslin representative functions $\hat{w}$ can be meant as defined in $\omega_{\delta_{\bar{m}}}$.

Step 1: Countable decomposition. Let D denote the differentiability points of the function $(a,b)\ni s\mapsto\dot{\bar{\gamma}}(s)$. By Rademacher theorem, D has full measure in $[a,b]$ and $(z,\bar{\gamma}(z))\in\mathcal{D}_{\delta_{\bar{m}}}$ for every $z\in D$, since $\exists\,\ddot{\bar{\gamma}}(z)$, where $\mathcal{D}_{\delta_{\bar{m}}}\equiv\mathcal{D}$ is the set in Lemma 6.4 with $\delta=\delta_{\bar{m}}$.

Let

$$S_k:=\left\{z\in D:\ \big|\hat{w}\big(z,\bar{\gamma}(z)\big)-\ddot{\bar{\gamma}}(z)\big|\ge\frac{1}{k}\right\}\quad k\in\mathbb{N},$$

then

$$\big\{z\in D:\hat{w}\big(z,\bar{\gamma}(z)\big)\ne\ddot{\bar{\gamma}}(z)\big\}=\bigcup_{k=1}^{\infty}S_k.\tag{6.12}$$

Let us observe that, from the definition of Borel and Souslin representative functions, for each $z\in D$ and $k\in\mathbb{N}$ there exist $\bar{n}\in\mathbb{N}$ and a curve $[z-\delta_{\bar{m}},z+\delta_{\bar{m}}]\ni s\mapsto(s,\gamma_{k,(z,\bar{\gamma}(z))}(s))$ such that,

$$\left|\hat{w}\big(z,\bar{\gamma}(z)\big)-\frac{1}{\pm\sigma}\int_z^{z\pm\sigma}\gamma_{k,(z,\bar{\gamma}(z))}\right|<\frac{1}{k},\quad\forall0<\sigma<\delta_{\bar{m}},n>\bar{n},\tag{6.13}$$

where, respectively, if one considering the Borel representative function in Definition 6.9, $\gamma_{k,(z,\bar\gamma(z))}(s) := \gamma_{\varepsilon_k,(z,\bar\gamma(z))}(s-z)$ and

$$[-\delta_{\bar m}, \delta_{\bar m}] \ni s \mapsto \big(z+s, \gamma_{\varepsilon_k,(z,\bar\gamma(z))}(s)\big) \in \omega$$

is a curve selected in Lemma 6.8 for a suitable $\varepsilon_k < 1/k$, or, if one considering the Souslin representative function in (6.9), $\gamma_{k,(z,\bar\gamma(z))}(s) := \gamma_{(z,\bar\gamma(z))}(s-z)$.

Define for $k, n \in \mathbb{N}$

$$S_{k,n} := \left\{ z \in D : \forall 0 < \sigma < \min\{\delta_{\bar m}, 2^{-n}\}, \left| \hat w\big(z, \bar\gamma(z)\big) - \frac{1}{\sigma}\int_z^{z+\sigma} \ddot{\bar\gamma} \right| \geq \frac{1}{k}, \left| \hat w\big(z, \bar\gamma(z)\big) - \frac{1}{\sigma}\int_{z-\sigma}^{z} \ddot{\bar\gamma} \right| \geq \frac{1}{k} \right\}$$

and, respectively,

$$S^*_{k,n} := \left\{ z \in D : \forall 0 < \sigma < \min\{\delta_{\bar m}, 2^{-n}\}, \left| \hat w\big(z, \bar\gamma(z)\big) - \frac{1}{\pm\sigma}\int_z^{z\pm\sigma} \ddot{\bar\gamma} \right| \geq \frac{3}{k}, \right.$$
$$\left. \left| \hat w\big(z, \bar\gamma(z)\big) - \frac{1}{\pm\sigma}\int_z^{z\pm\sigma} \ddot\gamma_{k,(z,\bar\gamma(z))} \right| < \frac{1}{k} \right\}.$$

By (6.13),

$$S_k \subseteq \bigcup_{n=1}^{\infty} S_{k,n} \subseteq \bigcup_{n=1}^{\infty} S^*_{k,n}. \tag{6.14}$$

Step 2: Reduction argument. Notice that for $k, n \in \mathbb{N}$

$$S^*_{k,n} \subseteq \left\{ z \in D : \forall 0 < \sigma < \min\{\delta_{\bar m}, 2^{-n}\},\ \hat w\big(z, \bar\gamma(z)\big) > \frac{1}{\pm\sigma}\int_z^{z\pm\sigma} \ddot{\bar\gamma} + \frac{3}{k}, \right.$$
$$\left. \left| \hat w\big(z, \bar\gamma(z)\big) - \frac{1}{\pm\sigma}\int_z^{z\pm\sigma} \ddot\gamma_{k,(z,\bar\gamma(z))} \right| < \frac{1}{k} \right\}. \tag{6.15}$$

We prove that, for given $k, n \in \mathbb{N}$, the set

$$\left\{ z \in D : \forall 0 < \sigma < \min\{\delta_{\bar m}, 2^{-n}\},\ \hat w\big(z, \bar\gamma(z)\big) > \frac{1}{\pm\sigma}\int_z^{z\pm\sigma} \ddot{\bar\gamma} + \frac{3}{k}, \right.$$
$$\left. \left| \hat w\big(z, \bar\gamma(z)\big) - \frac{1}{\pm\sigma}\int_z^{z+\sigma} \ddot\gamma_{k,(z,\bar\gamma(z))} \right| < \frac{1}{k} \right\} \tag{6.16}$$

cannot contain points z_1, z_2 with $|z_1 - z_2| \leq \delta_n := \min\{\delta_{\bar m}, 2^{-n}\}$. Then the thesis will follow: by (6.12), (6.14) and (6.15), the set of $z \in (a,b)$ where the second derivative of $\bar\gamma(z)$ exists and it is different from $w_{t,\bar\gamma(z)}$ will be countable. Therefore the second derivative of $\bar\gamma(z)$ will be almost everywhere precisely $\hat w(z, \bar\gamma(z))$.

Step 3: Analysis of the single sets. By contradiction, assume that (6.16) contains two such points, z_1, z_2 with $z_1 < z_2$. Let us prove that the curves through $(z_1, \bar\gamma(z_1))$, $(z_2, \bar\gamma(z_2))$,

$$\gamma_1 := \gamma_{k,(z_1,\bar\gamma(z_1))}, \qquad \gamma_2 := \gamma_{k,(z_2,\bar\gamma(z_2))}$$

must intersect in the time interval $[z_1, z_2]$, say at time y'. Indeed, denote $w_i := \hat w(z_i, \bar\gamma(z_i))$ $(i = 1, 2)$, then from the first and second inequality defining set (6.16) it respectively follows that, for each $i = 1, 2$, $\forall s \in [z_i - \delta_n, z_i + \delta_n] \setminus \{z_i\}$

$$\bar{\gamma}(s) < \bar{\gamma}(z_i) + \dot{\bar{\gamma}}(z_i)(s - z_i) + \frac{w_i - 3/k}{2}(s - z_i)^2, \tag{6.17}$$

$$\begin{aligned}\bar{\gamma}(z_i) + \dot{\bar{\gamma}}(z_i)(s - z_i) + \frac{w_i - 1/k}{2}(s - z_i)^2 < \gamma_i(s) \\ < \bar{\gamma}(z_i) + \dot{\bar{\gamma}}(z_i)(s - z_i) + \frac{w_i + 1/k}{2}(s - z_i)^2.\end{aligned} \tag{6.18}$$

By (6.17) and (6.18), we get that

$$\gamma_1(z_1) = \bar{\gamma}(z_1) < \gamma_2(z_1), \qquad \gamma_1(z_2) > \gamma_2(z_2) = \bar{\gamma}(z_2),$$

then the desired conclusion. Since curves γ_i, $i = 1, 2$ satisfy the ODE for characteristics, where they intersect they have the same derivative. Being all of them Lipschitz, we have then

$$\dot{\bar{\gamma}}(z_1) + \int_{z_1}^{z'} \ddot{\gamma}_1 = \dot{\gamma}_1(z') = \dot{\gamma}_2(z') = \dot{\bar{\gamma}}(z_2) - \int_{z'}^{z_2} \ddot{\gamma}_2.$$

Comparing the LHS and the RHS, one arrives to

$$\dot{\bar{\gamma}}(z_2) - \dot{\bar{\gamma}}(z_1) = \int_{z_1}^{z'} \ddot{\gamma}_1 + \int_{z'}^{z_2} \ddot{\gamma}_2.$$

However, since the times z_1, z_2 belong by construction to the set (6.16) one has

$$\int_{z_1}^{z'} \ddot{\gamma}_1 + \int_{z'}^{z} \ddot{\gamma}_2 > w_1(z' - z_1) + w_2(z_2 - z') - 2/k > \int_{z_1}^{z_2} \ddot{\bar{\gamma}} + 1/k = \dot{\bar{\gamma}}(z_2) - \dot{\bar{\gamma}}(z_1) + 1/k$$

reaching a contradiction. □

6.3. Lagrangian solutions are distributional solutions

In this section we prove, without passing through the implicit function theorem, that if a continuous function ϕ satisfies the Lagrangian formulation (see Definition 3.7) of the balance law (3.1), then the distribution

$$\frac{\partial \phi}{\partial z}(z, t) + \frac{\partial}{\partial t}\left[\frac{\phi^2(z, t)}{2}\right]$$

corresponds to a bounded function in $L^\infty(\omega)$. We rely here on a mollification procedure in the Lagrangian variables. We do not discuss here the fact that this function identifies the same distribution as $w(z, t)$. An explicit proof based on the computation in Lemma 5.1 can be found in [1], and here it is discussed at page 929 according to the previous literature. See also [11], where a different, pointwise approximation of the distributional solution is provided, starting from a broad* solution. This basically shows the converse of Dafermos' statement in [15].

We already motivated why focusing on the case $n = 1$, whereas the equation reduces to

$$\frac{\partial \phi}{\partial z}(z, t) + \frac{\partial}{\partial t}\left[\frac{\phi^2(z, t)}{2}\right] = w(z, t). \tag{3.1}$$

We give generalizations to the case $n \geq 2$ in Remark 6.12

Theorem 6.10. *Every continuous Lagrangian solution to (3.1) is also a distributional solution.*

Corollary 6.11. *Any continuous broad solution ϕ to (3.1) is also a distributional solution.*

Proof of Theorem 6.10. Let $\Upsilon : \tilde{\omega} \to \omega$, $\Upsilon(z, \tau) = (z, \chi(z, \tau))$, be a Lagrangian parameterization associated to ϕ, and $\bar{w} : \omega \to \mathbb{R}$ (see Definition 3.7) such that, for each $\tau \in \mathbb{R}$, for each interval $(z_1, z_2) \subset \tilde{\omega}_\tau$,

$$\phi\big(z, \chi(z, \tau)\big) - \phi\big(z_1, \chi(z_1, \tau)\big) = \int_{z_1}^{z} \bar{w}\big(r, \chi(r, \tau)\big) dr \quad \forall z \in (z_1, z_2).$$

We prove then that ϕ is a distributional solution of the balance equation

$$\frac{\partial \phi}{\partial z}(z, t) + \frac{\partial}{\partial t}\left[\frac{\phi^2(z, t)}{2}\right] = \bar{w}(z, t) \quad \text{in } \omega.$$

As this is a local argument, one can assume in order to avoid technicalities that χ is constant out of a compact: just modify χ, and consequently ϕ and $\bar{w}$, out of an open ball where one wants to prove the statement. By a partition of unity the statement will hold then on the desired domain. By a change of variables one can as well assume that χ is valued in $(0, 1)$ and that the support of ϕ is compactly contained in $(0, 1)^2$. In particular w.l.o.g. we can assume that $\Upsilon : \tilde{\omega} := (0, 1) \times (a, b) \to \omega := (0, 1) \times (t_1, t_2)$ with $-\infty < a < b < +\infty$, $0 < t_1 < t_2 < 1$ and that χ can be meant as a continuous function $(0, 1) \times \mathbb{R} \to (0, 1)$ by defining it $\chi(z, \tau) = t_1$ for each $z \in (0, 1)$ and $\tau \le a$, and $\chi(z, \tau) = t_2$ for each $z \in (0, 1)$ and $\tau \ge b$.

Step 1: Smoothing of χ in the τ-variable. Consider a suitable convolution kernel $\rho_\varepsilon(\tau)$ compactly supported in $\{|\tau| < \varepsilon\}$ and define the τ-regularized function $\chi^\varepsilon : (0, 1) \times \mathbb{R} \to \mathbb{R}$ given by

$$\chi^\varepsilon(z, \tau) = (1 + \varepsilon\tau)\big(\chi(z, \cdot) * \rho_\varepsilon\big)(\tau) = (1 + \varepsilon\tau) \int_{\mathbb{R}} \chi(z, \xi)\rho_\varepsilon(\tau - \xi) d\xi.$$

Claim. *The function χ^ε is (locally) Lipschitz continuous and, for each $z \in (0, 1)$, $\chi^\varepsilon(z, \cdot) : \mathbb{R} \to \mathbb{R}$ is strictly increasing, smooth and, for ε small enough and each $(z, \tau) \in (0, 1) \times \mathbb{R}$,*

$$\frac{\partial \chi^\varepsilon}{\partial \tau}(z, \tau) = \varepsilon \big(\chi(z, \cdot) * \rho_\varepsilon\big)(\tau) + (1 + \varepsilon\tau)\left(\frac{\partial \chi}{\partial \tau}(z, \cdot) * \rho_\varepsilon\right)(\tau) > 0, \tag{6.19}$$

where $\frac{\partial \chi}{\partial \tau}(z, \cdot) * \rho_\varepsilon$ denotes the convolution between the (finite) non-negative measure $\frac{\partial \chi}{\partial \tau}(z, \cdot)$ and the kernel $\rho_\varepsilon(\tau)$ (see, for instance, [3, Theorem 2.2]). Moreover χ^ε converges locally uniformly to χ for $\varepsilon \downarrow 0$.

Proof. The Lipschitz regularity in the variable z remains clearly unchanged. The smoothness in τ and (6.19) follow by general properties of the convolution and conditions $0 < t_1 \le \chi(z, \tau) \le t_2 < 1$ for each $z \in (0, 1)$, $\tau \in [a, b]$, $\chi \equiv t_1$ in $(0, 1) \times (-\infty, a)$, $\chi \equiv t_2$ in $(0, 1) \times (b, \infty)$. The locally uniform convergence of χ^ε to χ is also a well-known convolution's feature. The monotonicity is a direct consequence of the monotonicity of $\chi(z, \tau)$ and it becomes strict because of the scaling factor $(1 + \varepsilon\tau)$: from

$$\tau_1 < \tau_2 \quad \Rightarrow \quad \chi(z, \tau_1) \le \chi(z, \tau_2)$$

one can compute that

$$\begin{aligned}
\chi^\varepsilon(z, \tau_1) &= (1 + \varepsilon\tau_1) \int_{-\varepsilon}^{\varepsilon} \chi(z, \tau_1 + \xi)\rho_\varepsilon(\xi) d\xi \\
&\le (1 + \varepsilon\tau_1) \int_{-\varepsilon}^{\varepsilon} \chi(z, \tau_2 + \xi)\rho_\varepsilon(\xi) d\xi \\
&< (1 + \varepsilon\tau_2) \int_{-\varepsilon}^{\varepsilon} \chi(z, \tau_2 + \xi)\rho_\varepsilon(\xi) d\xi = \chi^\varepsilon(z, \tau_2).
\end{aligned}$$

Step 2: Smoothing of ϕ in the τ-variable. Let $\Upsilon_\varepsilon : (0, 1) \times \mathbb{R} \to (0, 1) \times \mathbb{R}$, $\Upsilon_\varepsilon(z, \tau) := (z, \chi^\varepsilon(z, \tau))$.

Claim. *Υ_ε is a homeomorphism, locally Lipschitz continuous with its inverse.*

Proof. By definition Υ_ε is 1-to-1, onto and continuous (see Remark 3.6). Then its inverse $\Upsilon_\varepsilon^{-1} : (0,1)\times\mathbb{R} \to (0,1)\times\mathbb{R}$, $\Upsilon_\varepsilon^{-1}(y,t) := (y, \theta^\varepsilon(y,t))$ is also continuous, being the inverse of an injective and onto continuous map between two open sets of $\mathbb{R}^2$. Eventually, by standard arguments about the regularity of an inverse function and (6.19), it follows that $\theta^\varepsilon \in W^{1,\infty}_{\mathrm{loc}}((0,1)\times\mathbb{R})$ and

$$\frac{\partial\theta^\varepsilon}{\partial y}(y,t) = -\frac{\frac{\partial\chi^\varepsilon}{\partial z}}{\frac{\partial\chi^\varepsilon}{\partial\tau}}\big(\Upsilon_\varepsilon^{-1}(y,t)\big), \qquad \frac{\partial\theta^\varepsilon}{\partial t}(y,t) = -\frac{1}{\frac{\partial\chi^\varepsilon}{\partial\tau}}\big(\Upsilon_\varepsilon^{-1}(y,t)\big).$$

Let $\tilde\phi_\varepsilon : (0,1)\times\mathbb{R} \to \mathbb{R}$ be the function defined by $\tilde\phi_\varepsilon(z,\tau) := (1+\varepsilon\tau)(\phi(z,\chi(z,\cdot)) * \rho_\varepsilon)(\tau)$ and define the approximation $\phi^\varepsilon : (0,1)\times\mathbb{R}\to\mathbb{R}$ by $\phi^\varepsilon(z,t) := \tilde\phi_\varepsilon(\Upsilon_\varepsilon^{-1}(z,t))$. Observe that

$$\phi^\varepsilon\big(z,\chi^\varepsilon(z,\tau)\big) = \tilde\phi_\varepsilon(z,\tau) = (1+\varepsilon\tau)\left(\frac{\partial\chi}{\partial z}(z,\cdot) * \rho_\varepsilon\right)(\tau) = \frac{\partial\chi^\varepsilon}{\partial z}(z,\tau). \tag{6.20}$$

Claim. *ϕ^ε is a (locally) Lipschitz smooth function.*

Proof. By definition, the function $\tilde\phi_\varepsilon \in W^{1,\infty}_{\mathrm{loc}}((0,1)\times\mathbb{R})$ and, by the previous claim,

$$\Upsilon_\varepsilon^{-1} \in W^{1,\infty}_{\mathrm{loc}}\big((0,1)\times\mathbb{R};\mathbb{R}^2\big).$$

Thus the composition is still Lipschitz regular.

Claim. *$\phi^\varepsilon(z,t)$ converges in $L^1((0,1)^2)$ to $\phi(z,t)$ as $\varepsilon\downarrow 0$.*

Proof. By the properties of convolutions one has that $\phi^\varepsilon(z,\chi^\varepsilon(z,\tau))$ is converging uniformly to $\phi(z,\chi(z,\tau))$ as $\varepsilon\downarrow 0$. Because ϕ^ε and ϕ are compactly supported in $(0,1)^2$, by a change of variables through the map Υ_ε, one has

$$\begin{aligned}
\int_{(0,1)^2} \big|\phi^\varepsilon(y,t) - \phi(y,t)\big|dydt &= \int_{(0,1)\times\mathbb{R}} \big|\phi^\varepsilon(z,t) - \phi(z,t)\big|dzdt \\
&= \int_{(0,1)\times\mathbb{R}} \big|\phi^\varepsilon\big(z,\chi^\varepsilon(z,\tau)\big) - \phi\big(z,\chi^\varepsilon(z,\tau)\big)\big|\frac{\partial\chi^\varepsilon}{\partial\tau}(z,\tau)dzd\tau \\
&\le \int_{(0,1)\times\mathbb{R}} \big|\phi^\varepsilon\big(z,\chi^\varepsilon(z,\tau)\big) - \phi\big(z,\chi(z,\tau)\big)\big|\frac{\partial\chi^\varepsilon}{\partial\tau}(z,\tau)dzd\tau \\
&\quad + \int_{(0,1)\times\mathbb{R}} \big|\phi\big(z,\chi(z,\tau)\big) - \phi\big(z,\chi^\varepsilon(z,\tau)\big)\big|\frac{\partial\chi^\varepsilon}{\partial\tau}(z,\tau)dzd\tau.
\end{aligned}$$

The first factor of both the integrals in the RHS is uniformly convergent to zero, while $\frac{\partial\chi^\varepsilon}{\partial\tau}(z,\tau)dzd\tau$ weakly* converges, as measure, to the nonnegative measure $\frac{\partial\chi}{\partial\tau}(dz,d\tau)$.

Step 3: Approximation of the source. Define an approximate source $w^\varepsilon : \omega\to\mathbb{R}$ by

$$\frac{\partial\phi^\varepsilon}{\partial z}(z,t) + \frac{\partial}{\partial t}\frac{(\phi^\varepsilon)^2}{2}(z,t) = w^\varepsilon(z,t) \quad \mathcal{L}^2\text{-a.e. } (z,t)\in\omega. \tag{6.21}$$

Being ϕ^ε a Lipschitz function, we can assume that w^ε is a Borel function. From the smoothness and the definition (6.20) of the function $\phi^\varepsilon(s,t)$, the expression (6.21) is immediately equivalent to

$$w^\varepsilon\big(s,\chi^\varepsilon(s,\tau)\big) = \frac{\partial^2\chi^\varepsilon}{\partial s^2}(s,\tau) = \frac{\partial\phi^\varepsilon}{\partial s}\big(s,\chi^\varepsilon(s,\tau)\big).$$

Since we started from a Lagrangian parameterization, the further regularity in the s variable

$$\frac{\partial^2 \chi}{\partial s^2}(s,\tau) = \frac{\partial \phi}{\partial s}\big(s, \chi(s,\tau)\big) = \bar{w}\big(s, \chi(s,\tau)\big)$$

for the relative pointwise representative $\bar{w}$ implies the relation

$$\begin{aligned} w^\varepsilon\big(s, \chi^\varepsilon(s,\tau)\big) &= \frac{\partial^2 \chi^\varepsilon}{\partial s^2}(s,\tau) = (1+\varepsilon\tau)\frac{\partial^2 \chi}{\partial s^2}(s,\tau) * \rho_\varepsilon(\tau) \\ &= (1+\varepsilon\tau)\bar{w}\big(s, \chi(s,\tau)\big) * \rho_\varepsilon(\tau). \end{aligned}$$

In particular, the sources w^ε are uniformly bounded by $(1+\varepsilon)$ times the uniform bound for $\bar{w}$. Moreover, for each s fixed $w^\varepsilon(s, \chi^\varepsilon(s,\tau))$ converges in all $L^p(d\tau)$ to $\bar{w}(s, \chi(s,\tau))$, and thus in $L^p(dsd\tau)$; the convergence is clearly uniform when $\bar{w}$ is continuous.

Step 4: Limiting argument. The LHS of equation (6.21) passes to the weak limit by the L^1-convergence of $\phi^\varepsilon(z,t)$ to $\phi(z,t)$ established above. The same holds as well for the RHS, since $w^\varepsilon(z,t)dzdt$ converges $w^* - L^\infty$ to a function $\bar{\bar{w}}(z,t)dzdt$. Thus ϕ is a distributional solution of the equation

$$\frac{\partial \phi}{\partial z}(z,t) + \frac{\partial}{\partial t}\left[\frac{\phi^2(z,t)}{2}\right] = \bar{\bar{w}}(z,t) \quad \text{in } \omega.$$

By Corollary 5.7 the function $\bar{\bar{w}}(z,t)$ coincides $\mathcal{L}^2$-a.e. with the intrinsic gradient $\nabla^\phi \phi$ of ϕ. However, by Remark 2.14 and the definition of intrinsic gradient, this is $\mathcal{L}^2$-a.e. equal to the Lagrangian source $\bar{w}(z,t)$. □

Remark 6.12. Theorem 6.10 works immediately also in higher dimensions, because for almost every $\hat{z}_1 \in \mathbb{R}^{2n-2}$ the restriction to the plane $\omega_{\hat{z}_1}$ of ϕ is still a Lagrangian solution, with source term the restriction $\tilde{w} \restriction_{\omega_{\hat{z}_1}}$. For every test function $\varphi \in C_c^\infty(\omega)$ we have then by Fubini–Tonelli Theorem

$$\begin{aligned} \iint_\omega \left[\phi \frac{\partial \varphi}{\partial z_1} + \frac{\phi^2}{2}\frac{\partial \varphi}{\partial t}\right] dzdt &= \iint_{\mathbb{R}^{2n-2}} d\hat{z}_1 \int_{\omega_{\hat{z}_1}} dz_1 dt \left[\phi \frac{\partial \varphi}{\partial z_1} + \frac{\phi^2}{2}\frac{\partial \varphi}{\partial t}\right] \restriction_{\omega_{\hat{z}_1}} \\ &\overset{\text{Th. 6.10}}{=} -\iint_{\mathbb{R}^{2n-2}} d\hat{z}_1 \int_{\omega_{\hat{z}_1}} dz_1 dt [\varphi \tilde{w} \restriction_{\omega_{\hat{z}_1}}] = -\iint_\omega \varphi \tilde{w}\, dzdt. \end{aligned}$$

Considering also the linear fields we gain the implication from (ii) to (i) in Theorem 1.2, and more generally that a Lagrangian solution is also a distributional solution.

Conflict of interest statement

None declared.

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Appendix A. Filling in Lagrangian parameterizations

In the present section we deal with the issue of extending a partial Lagrangian parameterization to a 'full' one. We construct a function $\chi(s,\tau)$ satisfying the ODE (3.4), both monotone and surjective in the τ variable, which extends a given one $\tilde{\chi}$. This is the matter of Lemma A.1: we recall below how to extend a solution to an ODE with continuous coefficients, whose existence is a classical result.

The procedure can be first understood considering Example A.2 below, illustrated in Fig. 2. This deals with the simpler case of an s-independent ϕ, but it has all the ingredients of the general construction of Lemma A.1.

Moreover, Example A.2 provides a counterexample for the following fact: even if characteristics are C^1 in s with Lipschitz derivative, *it is not possible in general to extend a partial, monotone Lagrangian parameterization to a full one which is locally Lipschitz continuous*.

The reduction of the balance law along characteristics, which is Eq. (3.9), has been inserted in the text (Lemma 4.2 and Theorem 6.6). Here we just notice that it holds, with some $\bar{w}_\chi$ pointwise defined in ω, for a particular Lagrangian parameterization $(\tilde{\omega}, \chi)$. The argument is 2-dimensional.

Lemma A.1. *Any partial Lagrangian parameterization can be extended to a full one.*

Proof. Let $\tilde{\chi}(s, \tau)$ be a partial Lagrangian parameterization. Focus e.g. the attention on $s, \tau \in [0, 1]$ and $\tilde{\chi}(s, \tau)$ valued in $[0, 1]$ and right continuous, the general case being similar. We fix also $\|\phi\|_\infty \leq 1$.

We construct an extension $\tilde{\chi}'$ by a recursive procedure. For convenience, the induction index is given by couples (h, n) with $n \in \mathbb{N}$ and $h = 0, \dots, 2^{n-1} - 1$. The ordering is lexicographic, starting from the second variable: $(h_1, n_1) \leq (h_2, n_2)$ iff either $n_1 < n_2$ or $n_1 = n_2$ and $h_1 \leq h_2$.

The starting point is $\chi^0 = \tilde{\chi}$ defined for $s \in [0, 1]$, $\tau \in T_0 = [0, 1]$. Consider the dichotomous points $s^{h,n} = 2^{-n} + 2^{-n+1}h$, which go from 2^{-n} to $1 - 2^{-n}$ at step 2^{-n+1}, associated to the indexes (h, n) with $n \in \mathbb{N}$ and $h = 0, \dots, 2^{n-1} - 1$.

Induction step (h, n), $n \geq 1$: General description. Assume you have been given χ defined on $(s, \tau) \in [0, 1] \times T$ by a previous step. If at $s = s^{h,n}$ the map $\tau \mapsto \chi(s^{h,n}, \tau)$ is not onto $[\chi(s^{h,n}, 0), \chi(s^{h,n}, 1)]$ we construct an extension $\chi^{h,n}$ such that

- $\tau \mapsto \chi^{h,n}(s^{h,n}, \tau)$ is onto $[\chi(s^{h,n}, 0), \chi(s^{h,n}, 1)]$ for $h = 0, \dots, 2^{n-1} - 1$;
- there exists a strictly increasing map $j^{h,n}$, with $\mathcal{L}^1(j^{h,n}(T)) - \mathcal{L}^1(T) \leq 2^{1-2n}$, such that

$$\chi^{h,n}\bigl(s, j^{h,n}(\tau)\bigr) = \chi(s, \tau).$$

These properties of the new partial Lagrangian parameterization will determine that we get at the end a limit which is a full parameterization extending $\tilde{\chi}$.

Induction step (h, n), $n \geq 1$: Change of parameter set. Because of monotonicity the complementary of the image of $\tau \mapsto \chi(s^{h,n}, \tau)$ is the at most countable union of disjoint intervals $\{I_k\}_k$, which correspond to the discontinuities $\{\tau_k\}_k$ of this real valued map. At those parameters $\{\tau_k\}_k$ the two characteristics $s \mapsto \chi(s, \tau_k^-)$ and $s \mapsto \chi(s, \tau_k^+)$ bifurcate, and at time $s^{h,n}$ their opening is an interval I_k:

$$\min I_k = \chi\bigl(s^{h,n}, (\tau_k)^-\bigr) \qquad \sup I_k = \chi\bigl(s^{h,n}, \tau_k\bigr).$$

Define consequently the strictly increasing map opening each of those parameters τ_k into an interval proportional (with factor $1/2^{2n-1}$) to the hole I_k that the relative characteristics leave at $s^{h,n}$:

$$\begin{aligned} T \ &\rightarrow\ T_{h,n} = \bigl[0, j^{h,n}(1)\bigr] \\ j^{h,n} : \tau \ &\mapsto\ \tau + \mathcal{L}^1\Biggl(\bigcup_{\tau_k \leq \tau} I_k\Biggr)\Big/ 2^{2n-1}. \end{aligned}$$

The inequality $|T_{h,n}| - |T| \leq 2^{1-2n}$ holds because we are assuming that χ is valued in $[0, 1]$, thus $\mathcal{L}^1(\bigcup_k I_k) \leq 1$. The set $T_{h,n}$ will be the new space of parameters, and the injection $j^{h,n}$ will bring from the old set T to the new one.

Induction step (h, n), $n \geq 1$: Extension of the parameterization. We just constructed a new parameter set $T_{h,n}$ and an immersion $j^{h,n}$ from the old one T. In particular, the Lagrangian parameterization is fixed on $j^{h,n}(T)$, but in order to conclude the induction step we need to define the new Lagrangian parameterization on $T_{h,n} \setminus j^{h,n}(T)$. We clearly need to respect also the monotonicity property, taking into account that part of the parameterization is already fixed: the new characteristics that we are going to define must not cross the old ones.

For each $z \in I_k$, consider the C^1 maximal curve through $(s^{h,n}, z)$ defined at (4.1) until it touches either $s \mapsto \chi(s, \tau_k^+)$ or $s \mapsto \chi(s, \tau_k^-)$, in which case it goes on with the curve which has been touched. It is a little complicated to write formally, but it is just that. The times when this touching happens, if it ever happens, are

$$s_+^{\pm}(s^{h,n},z)=\inf\bigl\{1,\ \bigl\{s>s^{h,n}:\ \gamma^{(s^{h,n},z)}(s)=\chi\bigl(s,\tau_k^{\pm}\bigr)\bigr\}\bigr\}$$
$$s_-^{\pm}(s^{h,n},z)=\sup\bigl\{0,\ \bigl\{s>s^{h,n}:\ \gamma^{(s^{h,n},z)}(s)=\chi\bigl(s,\tau_k^{\pm}\bigr)\bigr\}\bigr\}.$$

With the notation that $\{s : a \le s \le b\}$ is empty if $b < a$, a possible right continuous extension is then give by $\gamma(s; s^{h,n}, z)$ defined as

$$\begin{cases} \gamma^{(s^{h,n},z)}(s) & \text{for } s_-^-(s^{h,n},z)\vee s_-^+(s^{h,n},z)\le s\le s_+^-(s^{h,n},z)\wedge s_+^+(s^{h,n},z),\\ \chi\bigl(s,\tau_k^+\bigr) & \text{for } s_+^+(s^{h,n},z)\le s\le s_+^-(s^{h,n},z) \text{ and } s_-^-(s^{h,n},z)\le s\le s_-^+(s^{h,n},z)\\ \chi\bigl(s,\tau_k^-\bigr) & \text{otherwise.}\end{cases}$$

Define finally on $[0,1]\times T_{h,n}$ the Lagrangian parameterization which coincides with the previous one on the image of the old parameter set, and which is extended as described above elsewhere: $\chi^{h,n}(s,\sigma)$ defined as

$$\begin{cases} \text{if } \sigma = j^{h,n}(\tau) & \chi(s,\tau),\\ \text{if } \sigma\in\bigl[j^{h,n}\bigl(\tau_k^-\bigr), j^{h,n}(\tau_k)\bigr),\ z:=\chi\bigl(s^{h,k},\tau_k^+\bigr)-2^{2n-1}\bigl(j^{h,n}\bigl(\tau_k^+\bigr)-\sigma\bigr) & \gamma\bigl(s;s^{h,n},z\bigr).\end{cases}$$

For the second line, $j^{h,n}(\tau_k^+)-\sigma$ is the length of the segment $[\![\sigma, j^{h,n}(\tau_k^+)]\!]$, which was added to the parameter set precisely at the (h,n)-th step. Rescaled by 2^{2n-1}, it gives the length of the segment $[\![z,\chi(s^{h,k},\tau_k^+)]\!]$: the point $(s^{h,n},z)$ is where we start for defining the characteristic $\gamma(s;s^{h,n},z)$ that we are inserting for extending the Lagrangian parameterization. Notice that also the surjectivity property at $s=s^{h,n}$ is satisfied.

Conclusion. Let us first look at how much the domain T_0 grows in the extension process. Since 2^{n-1} couples of indices have second variable n, then the total size of the intervals added by those couples altogether is at most 2^{-n}: thus, setting $T_n=\bigcup_h T_{h,n}$,

$$T_0=[0,1],\quad T_1\subset[0,3/2],\quad \ldots,\quad T_n\subset\bigl[0,2-2^{-n}\bigr],\quad \ldots.$$

The maps $j^{\bar h,\bar n}\circ\cdots\circ j^{0,1}$ are

- strictly increasing;
- valued in $[0,2-2^{-\bar n}]$;
- with 1-Lipschitz inverses, provided that at discontinuity points of $j^{\bar h,\bar n}$ one fills the graph.

By Ascoli–Arzelà theorem the inverses converge uniformly, to a monotone map which is the inverse of a right continuous function $j:[0,1]\to[0,2]$. By construction j is strictly increasing.

By construction each $\chi^{\bar h,\bar n}$ is surjective at each $s=s^{h,n}$ with $(h,n)\le(\bar h,\bar n)$.

The Lagrangian parameterizations $\chi^{\bar h,\bar n}$ and $\chi^{h,n}$ have different domains for the second components, the space of parameters $T_{h,n}$, $T_{\bar h,\bar n}$. However, as seen in the previous step there are strictly monotone injections from each of them to the interval $[\![0,2]\!]$ given by $\circ_{(\ell,m)\ge(h,n)} j^{\ell,m}$ and $\circ_{(\ell,m)\ge(\bar h,\bar n)} j^{\ell,m}$. Being strictly monotone, they are invertible if we fill the graphs at discontinuity points: we can compare these compositions, with the same second component in $[0,2]$, and we find for them

$$\bigl\|\chi^{\bar h,\bar n}-\chi^{h,n}\bigr\|_\infty\le 2^{-n}.$$

The sequence of these compositions therefore converges uniformly. One verifies that the limit is a monotone Lagrangian parameterization χ which extends $\tilde\chi$, with injection map j.

Being surjective and monotone, each $\tau\to\chi(s^{h,m},\tau)$ is continuous. By the continuity in s we deduce then surjectivity also at the remaining times: indeed if by absurd we had $\chi(\bar s,\bar\tau^-)\ne\chi(\bar s,\bar\tau^+)$, we could not have $\chi(s^{h,m},\bar\tau^-)=\chi(s^{h,m},\bar\tau^+)$ at $s^{h,m}$ arbitrarily close to $\bar s$. □

The following example introduces the extension of a Lagrangian parameterization. It shows moreover that it is not possible in general to get a full one which is Lipschitz continuous, even though characteristics below are twice continuously differentiable.

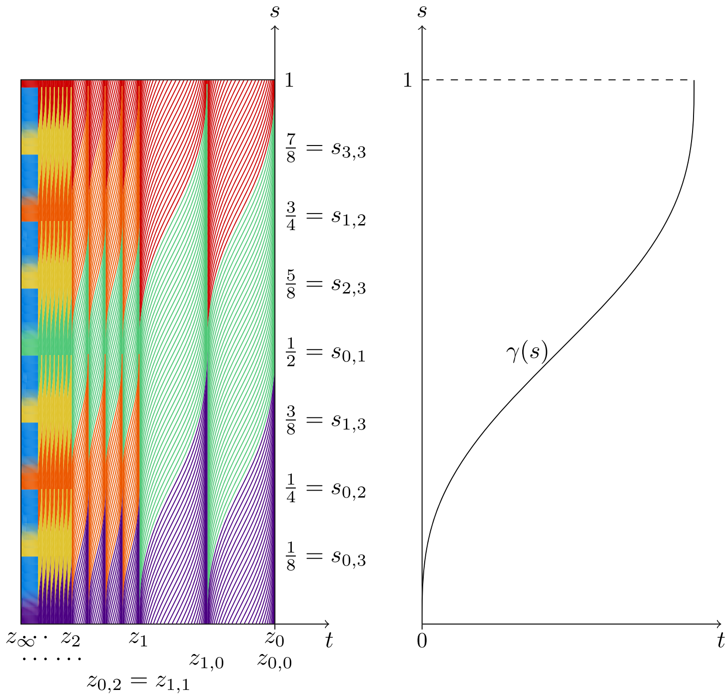

Fig. 2. A partial monotone Lagrangian parameterization is extended to a 'full' one. Different colors denote different steps in the extension, which are countably many. Each step corresponds to a dichotomous value of s: the relative extension of the parameterization must cover the relative s-section. In this example the full parameterization cannot be locally Lipschitz. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Example A.2. Consider the very simple equation for (z, t) in the rectangle $[z_\infty, 0] \times [0, 1]$

$$\frac{\partial}{\partial t}\left[\frac{\phi^2(z,t)}{2}\right] = w(z,t), \qquad \phi_z(z,t) = 0.$$

Being ϕ dependent on one variable, we change notation and we write $\phi(z,t) = \phi(t)$.

We consider the partial Lagrangian parameterization χ_m defined in Lemma 4.1. This would need to specify the function ϕ: since the construction is involved, we specify it below and the reader can immediately visualize it in Fig. 2 (LHS), where a family of integral curves $\dot{\gamma}(s) = \phi(\gamma(s))$ is drown.

We now describe in details the construction.

Step 1: Building block (Fig. 2, RHS). Define first a smooth function $\gamma(s)$, for $s \in [0, 1]$, which increases continuously from 0 to 1. Let $\dot{\gamma}(s - 1/2)$ be even, strictly increasing in the first half interval from 0 to its maximum. Let $\ddot{\gamma}$ vanish at 0; 1/2; 1 and be positive in $[0, 1/2]$. For instance, consider

$$\gamma(s) = s + 1/(2\pi)\sin(2\pi s - \pi), \qquad \gamma'(s) = 1 + \cos(2\pi s - \pi).$$

Step 2: Iteration step. We define now a first sequence of points $z_0 \geq z_1 \geq \dots z_i \downarrow z_\infty$ and intermediate ones $z_i = z_{2^i,i} \leq \dots \leq z_{0,i} = z_{i-1}$ where $\phi(z)$ and $w(z)$ will vanish. The first ones are approximatively 0, −0.455, −0.635, −0.713, −0.748, −0.764, −0.772, ..., and each interval $[z_{i+1}, z_i]$ is divided into 2^i equal subintervals for determining the points $z_{h,i}$. For $i \in \mathbb{N}$, $h = 0, \dots, 2^i$

$$z_0 := 0, \qquad z_{i-1} - z_i = \frac{2^{-i}}{\ln(i+2)}, \qquad z_\infty := -\sum_{j=1}^{\infty} \frac{2^{-j}}{\ln(j+2)},$$

$$z_i = -\sum_{j=1}^{i} \frac{2^{-j}}{\ln(j+2)} = z_{2^i,i} = z_{0,i+1}, \qquad z_{h,i} := z_{i-1} - \frac{2^{-2i}h}{\ln(i+2)}.$$

Step 3: Iteration. Now we define the functions $\phi(z)$, $w(z)$ on each subinterval $[z_{h+1,i}, z_{h,i}]$. As a preliminary half-step consider the rescaled smooth functions $\gamma_{h,i}(s)$ given by

$$\gamma_{h,i}(s) = z_{h+1,i} + \frac{\gamma(2^i s)}{2^{2i}\ln(i+2)}, \quad s \in [0, 2^{-i}].$$

Notice that each increases monotonically from $z_{h+1,i}$ to $z_{h,i}$, and the first two derivatives vanish at the endpoints. In particular, we can associate to each $z \in [z_\infty, 0]$ a curve $\gamma_{h,i}(s)$ such that

$$\gamma_{h,i}(s) = z.$$

It will be unique out of the points $\{z_{h,i}\}_{h,i}$, while at these junctions there will be two such curves, with however vanishing first two derivatives. Observing Fig. 2, this curve $\gamma_{h,i}(s)$ is just the first segment of what will be part of χ_m

$$[0, 2^i] \ni s \mapsto \gamma_{h,i}(s) \equiv \chi_m(s, z^+_{h+1,i}).$$

Define then,

$$\text{for } s\text{: } \gamma_{h,i}(s) = z, \qquad \phi(z) = \dot\gamma_{h,i}(s) \quad \text{and} \quad w(z) = \ddot\gamma_{h,i}(s).$$

These functions are clearly continuous out of the nodes $\{z_{h,i}\}_{h,i}$, and they vanish there, but the continuity at the limit point z_∞ should be checked. It holds because $|\dot\gamma_{h,i}| \le 2^{-i}/\ln(i+2)$ and $|\ddot\gamma_{h,i}| \le 1/\log(i+2)$, which implies they vanish for $i \uparrow \infty$.

Step 4: Non-Lipschitz Lagrangian parameterization. Having defined the function ϕ, we have already defined the partial Lagrangian parameterization χ_m of Lemma 4.1. We now extend it to a surjective one, but there is no way of having it Lipschitz continuous, as we compute now. Different colors in Fig. 2 show different steps of the extension process.

1. Minimal characteristics starting from $z = 0$ do not cover almost all the interval at $z = 1$. Adding those starting at $z = 1$, it remains to cover at $z = 1/2$ open intervals of total length

$$\sum_{i=1}^{\infty}\sum_{h=0}^{2^i}(z_{h,i} - z_{h+1,i}) = \sum_{i=1}^{\infty} 2^i \frac{2^{-2i}}{\ln(i+2)} = z_0 - z_\infty.$$

2. Including at the second step all those minimal characteristics which intersect the line $z = 1/2$, similarly, one does not cover the whole line $z = 1/4; 3/4$: a length

$$\sum_{i=2}^{\infty} \frac{2^{-i}}{\ln(i+2)} = z_1 - z_\infty$$

remains to cover at both those two values of z.

(i) At the subsequent i-th step, at each of the 2^{i-1} values of

$$s_{h,i} = 2^{-i} + h2^{-i+1}, \quad h = 0, \dots, 2^{i-1} - 1$$

one needs covering a length $\sum_{j=i}^{\infty} 2^{-j}/\ln(j+2)$. In the whole process, it must be covered a total length equal to

$$\sum_{j=1}^{\infty}(1 + \dots + 2^{j-1})\frac{2^{-j}}{\ln(j+2)} = \sum_{j=1}^{\infty}(2^j - 1)\frac{2^{-j}}{\ln(j+2)}$$
$$= \sum_{j=1}^{\infty}\frac{1}{\ln(j+2)} - \sum_{j=1}^{\infty}\frac{2^{-j}}{\ln(j+2)} = +\infty.$$

Any monotone, Lagrangian parameterization χ must map a disjoint family of real intervals $\{I_{h,i}\}_{h,i}$ with $\sum_{h,i}|I_{h,i}| < 1$ to the intervals $\{[z_\infty, z_{i+1}]\}_i$, respectively at the above values $\{s_{h,i}\}_{h,i}$. However, we have just computed that for all constants C

$$\begin{aligned}\infty &= \sum_{h=0,\dots,2^{i-1}-1,\ i\in\mathbb{N}} |z_{i+1} - z_\infty| \\ &= \sum_{h=0,\dots,2^{i-1}-1,\ i\in\mathbb{N}} \chi\big(z_{h,i}, [I_{h,i}]\big) \\ &> C \sum_{h=0,\dots,2^{i-1}-1,\ i\in\mathbb{N}} |I_{h,i}|,\end{aligned}$$

preventing any Lipschitz regularity. Indeed, the map $\tau \mapsto \chi(s_{h,i}, \tau)$ can be Lipschitz with some constant C_i, but C_i must blow up as $i \to \infty$. At other values of s, this map is just $W^{1,1}(\mathbb{R})$.

Notice finally that the fact that the blow-up of the Lipschitz constant is caused here by a behavior approaching the boundary $z = z_\infty$ is incidental. Indeed, one can extend the present construction also on $\{z < z_\infty\}$ for example reflecting the characteristics w.r.t. the axis $\{z = z_\infty\}$.

One can as well construct examples where χ has a Cantor part (see [1]).

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