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Phases of unimodular complex valued maps: optimal estimates, the factorization method, and the sum-intersection property of Sobolev spaces

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Abstract

We address and answer the question of optimal lifting estimates for unimodular complex valued maps: given $s > 0$ and $1 \leq p < \infty$, find the best possible estimate of the form $|\varphi|_{W^{s,p}} \lesssim F(|e^{\imath\varphi}|_{W^{s,p}})$.

The most delicate case is $sp < 1$. In this case, we extend the results obtained in [3,4] for $p = 2$ (using L^2 Fourier analysis and optimal constants in the Sobolev embeddings) by developing non-L^2 estimates and an approach based on symmetrization. Following an idea of Bourgain (presented in [3]), our proof also relies on averaged estimates for martingales. As a byproduct of our arguments, we obtain a characterization of fractional Sobolev spaces with $0 < s < 1$ involving averaged martingale estimates.

Also when $sp < 1$, we propose a new phase construction method, based on oscillations detection, and discuss existence of a bounded phase.

When $sp \geq 1$, we extend to higher dimensions a result on optimal estimates of Merlet [20], based on one-dimensional arguments. This extension requires new ingredients (factorization techniques, duality methods).

Keywords: Unimodular maps; Lifting; Sobolev spaces

Contents

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1. Introduction

Our first motivation is provided by the following problem.

Lifting estimate question. Let $\Omega \subset \mathbb{R}^n$ be smooth bounded simply connected. Let $0 < s < \infty$, $1 \le p < \infty$. Assume that $W^{s,p}(\Omega; \mathbb{S}^1)$ has the *lifting property*, i.e., that every $u \in W^{s,p}(\Omega; \mathbb{S}^1)$ has a phase $\varphi \in W^{s,p}(\Omega; \mathbb{R})$. Which is the best possible estimate of the form

$$|\varphi|_{W^{s,p}} \lesssim F\big(|u|_{W^{s,p}}\big)? \tag{1.1}$$

Here, $A \lesssim B$ means $A \le CB$, with C possibly depending on p and on the space dimension n, but not on s or u.

Estimate (1.1) can be seen as a reverse estimate for superposition operators. Superposition operators are mappings of the form

$$T_\Phi(\varphi) = \Phi \circ \varphi, \quad \forall \varphi \in X,$$

with X a function space. Classical questions concerning such operators are: under which regularity assumptions on Φ we have $T_\Phi : X \to X$, and existence of estimates of the form

$$\big\|T_\Phi(\varphi)\big\|_X \le G\big(\|\varphi\|_X\big); \tag{1.2}$$

see e.g. [27] for a detailed account of these topics. The questions we discuss in the present paper are related to a sort of converse of (1.2), namely existence of estimates of the form

$$\|\varphi\|_X \le F\big(\|T_\Phi(\varphi)\|_X\big) \tag{1.3}$$

(or of a similar estimate where the full norm $\| \ \|_X$ is replaced by a semi-norm $| \ |_X$). Clearly, (1.3) cannot hold for every Φ, even smooth (take $\Phi = 0$). A hint is given by the analysis of the case where $X = W^{1,p}$. The fact that

$$\big\|\nabla(\Phi \circ \varphi)\big\|_{L^p} = \big\|\Phi'(\varphi)\nabla\varphi\big\|_{L^p}$$

suggests that, in order to have both (1.2) and (1.3), a reasonable condition is that

$$0 < a \le \big|\Phi'\big| \le b < \infty.$$

This suggests considering the *model nonlinearity* $\Phi(t) = e^{\imath t}$, which satisfies $|\Phi'| = 1$, and then the corresponding problem is given by (1.1).

For simplicity, we consider only periodic maps $u : \mathbb{T}^n \to \mathbb{S}^1$, where $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (but it will be transparent from the proofs that the constructions and arguments we present extend to maps defined on Lipschitz bounded domains). We set $C = [0, 1)^n$. If $u : \mathbb{T}^n \to \mathbb{S}^1$ is smooth, then u has a smooth phase $\varphi : C \to \mathbb{R}$. Of course, such a phase need not be $\mathbb{Z}^n$-periodic and thus cannot be identified with a smooth map on $\mathbb{T}^n$. However, for notational simplicity, we still write most of the times $\varphi : \mathbb{T}^n \to \mathbb{R}$. When periodicity may play a role, we turn back to the notation $\varphi : C = [0, 1)^n \to \mathbb{R}$.

The maps we consider are normed in the standard way (over a period); e.g., we let $\|f\|_{L^p} := \|f\|_{L^p(C)}$.

Before presenting our contribution, let us briefly recall some previously known results concerning the existence of phases $\varphi : C \to \mathbb{R}$ of maps $u : \mathbb{T}^n \to \mathbb{S}^1$, and the corresponding estimates. First, the characterization of s and p such that $W^{s,p}(\Omega; \mathbb{S}^1)$ has the lifting property was obtained in [3] and is the following.

1.1. Theorem. *(See [3].) The space $W^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$ has the lifting property precisely in the following cases:*

1. $sp < 1$.
2. $sp \ge n$.
3. $s \ge 1$ *and* $sp \ge 2$.

Concerning optimal estimates of the form (1.1), two qualitatively different situations are to be considered. As an illustration, let us assume that we have an estimate of the form (1.1) at our disposal, and also that the equality $|\varphi_0|_{W^{s,p}} = F(|u|_{W^{s,p}})$ holds for some $\varphi_0 \in W^{s,p}$, with $u := e^{\imath\varphi_0}$. Starting from this, we would like to assert that (1.1) is optimal. This is easily obtained when $sp \geq 1$. Indeed, in this case, if $u = e^{\imath\varphi_1} = e^{\imath\varphi_2}$ with $\varphi_1, \varphi_2 \in W^{s,p}$, then $\varphi_1 = \varphi_2 \pmod{2\pi}$ [3, Theorem B.1]; thus the phase (if it exists) is unique. Consequently, there is *no phase* $\varphi \in W^{s,p}$ of u such that $|\varphi|_{W^{s,p}} < F(|u|_{W^{s,p}})$, and thus (1.1) is optimal. We will present in Section 5 the optimal estimates corresponding to the range $sp \geq 1$; for the time being let us only mention the strategy. First, an inspection of the construction of phases in [3] and [21] leads to estimates of the form (1.1). Next, we test these estimates on typical $W^{s,p}$ functions (like $x \mapsto |x|^{-\alpha}$, with $(\alpha + s)p < n$) and conclude to their optimality.[1]

Much more involved is the case where $sp < 1$. Indeed, assume that we have established an estimate of the type (1.1) and that we want to prove its optimality. This time, if φ is a $W^{s,p}$ phase of u, then so is $\varphi + 2\pi\mathbb{1}_A$, with A a smooth compact subset of Ω. Thus even if the estimate (1.1) cannot be improved for a *specific* φ, it could be possible to obtain *another phase* of u satisfying a *better estimate*.

Optimality when $sp < 1$ and $p = 2$ was investigated in [3] and [4]; the corresponding optimal estimates have implications in the analysis of the Ginzburg–Landau equation [5] and were part of the original motivation in studying (1.1). In order to explain the results obtained in [3,4], we first recall a phase construction method due to Bourgain and presented in [3]. Assume that $sp < 1$ and let $u \in W^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$. For $j \in \mathbb{N}$, we let $\mathscr{P}_j$ denote the set of the (dyadic) cubes of the form $2^{-j}\prod_{l=1}^n [m_l, m_l + 1)$, with $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$. Thus each $x \in \mathbb{R}^n$ belongs to exactly one cube $Q_j(x) \in \mathscr{P}_j$, and we have $Q_j(x) \subset Q_{j-1}(x)$ if $j \geq 1$. If $u \in L^1_{loc}(\mathbb{R}^n)$, then we let

$$u_j(x) = E_j u(x) \quad \text{denote the average of } u \text{ on } Q_j(x). \tag{1.4}$$

We let $\mathscr{E}_j$ denote the set of functions which are constant on every cube of $\mathscr{P}_j$. For a given $u \in W^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$, the construction of a phase φ goes as follows. Let u_j be as in (1.4), and set $U^j := \frac{u_j}{|u_j|} \in \mathscr{E}_j$, with the convention $\frac{0}{0} = 1$. We then let φ^0 be any real number such that $U^0 = e^{\imath\varphi^0}$ and next construct inductively a phase $\varphi^j \in \mathscr{E}_j$ of U^j such that

$$\left|\varphi^j - \varphi^{j-1}\right| \lesssim \left|U^j - U^{j-1}\right|. \tag{1.5}$$

The arguments developed in [3] imply that the sequence (φ^j) converges in L^p to a phase φ of u satisfying the estimate (1.6) below.

1.2. Theorem. *(See [3].) Assume that $sp < 1$. Then every $u \in W^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$ has a phase $\varphi \in W^{s,p}$ satisfying*

$$|\varphi|^p_{W^{s,p}} \lesssim \frac{1}{s^p(1-sp)^p}|u|^p_{W^{s,p}}. \tag{1.6}$$

Here, $|\cdot|_{W^{s,p}}$ is the standard Gagliardo semi-norm,

$$|f|^p_{W^{s,p}} = \int\int \frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy.$$

As explained above, when $sp < 1$ the phase is not unique, and this raises the question of the optimality of (1.6). It turns out that (1.6) is not optimal.[2] When $p > 1$, an improved estimate is provided by the following result.

1.3. Theorem. *Let $0 < s < 1$ and $1 \leq p < \infty$ be such that $sp < 1$. Let $u \in W^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$. Then there exists a phase φ of u satisfying the estimate*

$$|\varphi|^p_{W^{s,p}} \lesssim \frac{1}{s^p(1-sp)}|u|^p_{W^{s,p}}. \tag{1.7}$$

[1] Special cases of the results in Section 5 were obtained by Merlet [20].

[2] It is proved in [3, Section 5 and Appendix A] that the estimates used in the proof of Theorem 1.2 are essentially optimal and thus cannot lead to an estimate better than (1.6). However, this does not imply that the phase obtained via the iterative construction in formula (1.5) does not satisfy an improved estimate. We do not have an example of u such that the corresponding φ does not satisfy (1.7).

When $p = 2$, the above result is due to Bourgain [3, Theorem 3.1]. Bourgain's proof relies on an averaging method, reminiscent of Garnett and Jones [13]. The idea is to perform the dyadic construction explained above starting from $u^y := u(\cdot - y)$ instead of u, and obtain a corresponding phase φ^y. Then prove that, for some $y \in \mathbb{T}^n$, $\varphi^y(\cdot + y)$ (which is clearly a phase of u) satisfies the improved estimate (1.7). While the first part of the proof (construction of φ^y) does not depend on p, the argument leading to the last part (existence of an appropriate y) in [3] is based on L^2 Fourier analysis. Thus, in the proof of Theorem 1.3, our main task was to develop new, non-L^2, arguments.

We continue with a digression related to the use of the averaging method. In [3], the proof of (1.6) (and of the corresponding phase existence result) is based on the semi-norm equivalence [3, Theorem A.1]

$$|f|^p_{W^{s,p}} \sim \sum_{j\ge 1} 2^{sjp}\|f_j - f_{j-1}\|^p_{L^p}. \tag{1.8}$$

Here, the averages f_j are as in (1.4) (with u replaced by f), and $|\ |_{W^{s,p}}$ is any standard semi-norm on $W^{s,p}$, e.g. the Gagliardo one.[3] It is easy to see that the above semi-norm equivalence *cannot hold* when $sp \ge 1$. Indeed, let $0 < s < 1$ and $1 \le p < \infty$ be such that $sp \ge 1$. Let f be (the periodic extension of) the characteristic function of $[0, 1/2)^n$. Then the right-hand side of (1.8) is finite,[4] but $f \notin W^{s,p}$, as one may easily check. However, we have the following result, proving that the semi-norm equivalence (1.8) is valid *in average* when $0 < s < 1$, irrespective of the assumption $sp < 1$.

1.4. Theorem. *Let $0 < s < 1$ and $1 \le p < \infty$. Let $f^y(x) := f(x - y)$. Then we have*

$$|f|^p_{W^{s,p}} \sim \int_{\mathbb{T}^n} \sum_{j\ge 1} 2^{sjp}\left\|\left(f^y\right)_j - \left(f^y\right)_{j-1}\right\|^p_{L^p} dy. \tag{1.9}$$

This leads to the following picture, reminiscent of the connection discovered in [13] between BMO and dyadic BMO semi-norms:

1. The dyadic semi-norm $(\sum_{j\ge 1} 2^{sjp}\|f_j - f_{j-1}\|^p_{L^p})^{1/p}$ is equivalent to the standard semi-norm $|\ |_{W^{s,p}}$ precisely when $sp < 1$. This is Bourdaud's result [2, Théorème 5]. We note that this equivalence requires $0 < s < 1$, and for such s it holds for only *for some* p's in the range $[1, \infty)$.
2. However, in average, the two semi-norms are equivalent in the *full range* $0 < s < 1$, $1 \le p < \infty$.

We next turn to the question of the optimality of the estimate (1.6), settled in [3, Remark 7] for $p = 2$ and $n \ge 2$, and in [4, Theorem 2] for $p = 2$ and $n = 1$.

1.5. Theorem. *Assume that $1 < p < \infty$. Then estimate (1.7) is optimal.*

Here, optimality means that (1.7) cannot be improved to

$$|\varphi|^p_{W^{s,p}} \le \frac{\varepsilon(s)}{s^p(1 - sp)}|u|^p_{W^{s,p}},$$

with $\varepsilon(s) \to 0$ as $sp \nearrow 1$.

The original argument in [4, Theorem 2] relies on an involved result: the behavior of the best constant in the embedding $W^{1-\varepsilon,1}((0,1)) \hookrightarrow L^{1/\varepsilon}((0,1))$. We develop here a related, but simpler, argument, whose main ingredient is the fact that the nonincreasing rearrangement on an interval does not increase the fractional Sobolev norms. This is well-known on the real line, and goes back to Riesz when $p = 2$ [17, Lemma 3.6]; on an interval, the corresponding result is more recent and is due to Garsia and Rodemich [14].

As it turns out, the proofs of Theorems 1.3 and 1.5 we present below are slightly simpler than the original ones even when $p = 2$.

[3] See formula (3.1) below.
[4] Since $f_j = f$, $\forall j \ge 1$.

The reader may wonder about the role of the assumption $p > 1$ in Theorem 1.5. It turns out that this result is wrong when $p = 1$. Instead, we have the following improved estimate.

1.6. Proposition. *Let* $0 < s < 1$. *Then every map* $u \in W^{s,1}(\mathbb{T}^n; \mathbb{S}^1)$ *has a phase* φ *such that*

$$|\varphi|_{W^{s,1}} \le 2|u|_{W^{s,1}}. \tag{1.10}$$

Estimate (1.10) is essentially optimal, since we clearly have $|u|_{W^{s,1}} \le |\varphi|_{W^{s,1}}$. The proof of Proposition 1.6 follows the approach of Dávila and Ignat [12], who established, for BV maps $u : \mathbb{T}^n \to \mathbb{S}^1$, the existence of a BV phase φ satisfying the (optimal) estimate $|\varphi|_{BV} \le 2|u|_{BV}$.

Our paper is organized as follows. Sections 3, 4 and 5 are devoted to optimal estimates. In Section 3, we prove Theorem 1.3, which leads to an optimal estimate when $sp < 1$ and $p > 1$, and Proposition 1.6, giving an optimal estimate when $s < 1$ and $p = 1$. Section 4 contains the proof of Theorem 1.5, which asserts the optimality of the estimate in Theorem 1.3. In Section 5, we examine optimal estimates when $sp \ge 1$.

Sections 6 and 7 are devoted to further developments. In Section 6.1 we discuss the existence of a bounded phase when $sp < 1$. In Section 6.2, we describe a new method for constructing phases when $sp < 1$. This construction combines a factorization technique developed by the first author [22,23] with an averaging idea due to Dávila and Ignat [12]. Section 7 is devoted to the proof of Theorem 1.4.

The final Section 8 gathers various useful auxiliary estimates.

2. Notation

We present here some notation that we use throughout the paper.

1. $|x| = |(x_1, \dots, x_n)| := \max_{j \in [\![1,n]\!]} |x_j|$.
2. If $r \le 1$ and $x \in \mathbb{T}^n$, then $B(x, r) = \{y \in \mathbb{T}^n; |y - x| < r\}$.
3. $\{e_i\}_{i=1}^n$ is the canonical basis of $\mathbb{R}^n$.
4. $\mathscr{P}_j$, $j \ge 0$, is the family of dyadic cubes of side length 2^{-j} of $\mathbb{T}^n$. Thus an element of $\mathscr{P}_j$ is of the form $Q_j = 2^{-j} \prod_{\ell=1}^n [m_\ell, m_\ell + 1)$, with $m = (m_1, \dots, m_n) \in [\![0, 2^j - 1]\!]^n$.
5. If $x \in \mathbb{T}^n$, then $Q_j(x)$ is the (one and only one) cube $Q_j \in \mathscr{P}_j$ such that $x \in Q_j$.
6. $\mathscr{E}_j := \{f : \mathbb{T}^n \to \mathbb{C};\ f$ is constant on each $Q_j \in \mathscr{P}_j\}$.
7. The average of f on $Q_j(x)$ is denoted either $f_j(x)$ or $E_j f(x)$. Thus $f_j(x) = E_j f(x) := \fint_{Q_j(x)} f$.
8. $\tau_h f(x) := f(x - h)$.
   In the next four items, we let $0 < s < 1$ and $1 \le p < \infty$.
9. $|f|^p_{W^{s,p}(\mathbb{T}^n)} := \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\, dx\, dy = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \frac{|f(x)-\tau_h f(x)|^p}{|h|^{n+sp}}\, dx\, dh$.
10. We also sometimes denote $X(f) := |f|^p_{W^{s,p}}$.
11. $Y(f) := \sum_{j \ge 1} 2^{spj} \|f_j - f_{j-1}\|^p_{L^p}$.
12. $Z(f) := \sum_{j \ge 0} 2^{spj} \|f - f_j\|^p_{L^p}$.
13. The characteristic function of A is denoted $\mathbb{1}_A$.
14. ${}^c A$ is the complement of A.
15. $\sqcup$ denotes a disjoint union.
16. If $u = (f, g) \in C^1(\Omega; \mathbb{R}^2)$, with $\Omega \subset \mathbb{R}^2$, then the $\operatorname{Jac} u := \det(\nabla f, \nabla g)$ is the Jacobian determinant of u.
17. $A(f) \lesssim B(f)$ stands for $A(f) \le C B(f)$, with C a constant independent of f. When $f \in W^{s,p}$, we will further specify whether C depends on the parameters n, s and p.
18. $A(f) \approx B(f)$ stands for $B(f) \lesssim A(f) \lesssim B(f)$.
19. "$\wedge$" is used for the vector product of complex numbers: $(u_1 + \imath u_2) \wedge (v_1 + \imath v_2) = u_1 v_2 - u_2 v_1$. Similarly, $(u_1 + \imath u_2) \wedge \nabla(v_1 + \imath v_2) = u_1 \nabla v_2 - u_2 \nabla v_1$.

3. Optimal estimates when $sp < 1$. Proof of Theorem 1.3

We start with some preliminary results. We recall that $Q_j(x)$ is the unique cube in $\mathscr{P}_j$ such that $x \in Q_j(x)$. We set $f_j(x) := \fint_{Q_j(x)} f$, $\tau_h f(x) := f(x - h)$, and we associate with f, s and p the following quantities:

$$X(f) := |f|^p_{W^{s,p}} = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+sp}}\,dx\,dy = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{|f(x)-\tau_h f(x)|^p}{|h|^{n+sp}}\,dx\,dh, \tag{3.1}$$

$$Y(f) := \sum_{j\ge 1} 2^{spj}\|f_j - f_{j-1}\|^p_{L^p}, \tag{3.2}$$

$$Z(f) := \sum_{j\ge 0} 2^{spj}\|f - f_j\|^p_{L^p}. \tag{3.3}$$

When $sp<1$, we have that $X(f)$, $Y(f)$ and $Z(f)$ are equivalent semi-norms in $W^{s,p}(\mathbb{T}^n)$. This fact was established by Bourdaud [2]; see [3, Theorem A.1] for a quantitative form of this equivalence. For the convenience of the reader, we briefly recall in Section 8.1 the result in [3] with a slightly different proof; see Lemma 8.3.

It can be easily shown that the phases φ^j given by (1.5) satisfy the following inequality [3, (1.5)]:

$$|\varphi^j - \varphi^{j-1}| \lesssim |u-u_j| + |u-u_{j-1}|, \quad \forall j\ge 1. \tag{3.4}$$

In [3], estimate (1.6) is obtained by combining (3.4) with the (quantitative form of) the equivalence between $X(u)$, $Y(u)$ and $Z(u)$ (with X, Y and Z as in (3.1)–(3.3)).

The proof of the improved estimate (1.7) is more subtle. In order to obtain (1.7), we follow the approach in [3], which is itself inspired by a result of Garnett and Jones [13] showing that one can recover the standard BMO norm of a function u from the dyadic BMO norm of a suitable translation $\tau_h u$ of u. More specifically, the argument goes as follows. Let $u^y := \tau_y u$ and let φ^y be the phase of u^y obtained via Bourgain's construction, i.e., $\varphi^y := \lim_{j\to\infty}\varphi^{y,j}$. Here, $\varphi^{y,j}\in\mathscr{E}_j$ is a phase of $u^y_j/|u^y_j|$ satisfying

$$|\varphi^{y,j} - \varphi^{y,j-1}| \lesssim |u^y - u^y_j| + |u^y - u^y_{j-1}|, \quad \forall j\ge 1. \tag{3.5}$$

In the spirit of [3], we will prove that

$$\int_{\mathbb{T}^n} |\varphi^y|^p_{W^{s,p}}\,dy \lesssim \frac{1}{s^p(1-sp)}|u|^p_{W^{s,p}}. \tag{3.6}$$

Indeed, for every measurable function $f:\mathbb{T}^n\to\mathbb{C}$ we clearly have

$$|f|^p_{W^{s,p}} = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{|(\tau_h - \mathrm{id})f(x)|^p}{|h|^{n+sp}}\,dx\,dh \le \sum_{j\ge 0} 2^{(n+sp)(j+1)} \int_{|h|\in I_j}\int_{\mathbb{T}^n} |(\tau_h-\mathrm{id})f(x)|^p\,dx\,dh,$$

where $I_j := [2^{-j-1}, 2^{-j})$. We find that the average of $|\varphi^y|^p_{W^{s,p}}$ can be estimated by

$$\int_{\mathbb{T}^n} |\varphi^y|^p_{W^{s,p}}\,dy \le \int_{\mathbb{T}^n}\sum_{j\ge 0} 2^{(n+sp)(j+1)} \int_{|h|\in I_j}\int_{\mathbb{T}^n} |(\tau_h-\mathrm{id})\varphi^y|^p\,dx\,dh\,dy. \tag{3.7}$$

In order to estimate the right-hand side of (3.7), we start from

$$|(\tau_h-\mathrm{id})\varphi^y| \le |(\tau_h-\mathrm{id})\varphi^{y,j}| + |(\tau_h-\mathrm{id})(\varphi^y - \varphi^{y,j})|, \quad \forall j\ge 0. \tag{3.8}$$

Consider now $\rho = \mathbb{1}_{(-1/2,1/2)^n}$, and set $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(x/\varepsilon)$, $\forall\varepsilon>0$, $\forall x$. We define

$$A_{k,j} := \{x\in\mathbb{T}^n;\ \mathrm{dist}(x,\partial Q)\le 2^{-j} \text{ for some } Q\in\mathscr{P}_k\}.$$

By Lemma 8.6 in Section 8.2, when $|h|\in I_j$ we have

$$\begin{aligned}|(\tau_h-\mathrm{id})\varphi^{y,j}| &= |(\tau_h-\mathrm{id})(\varphi^{y,j}-\varphi^{y,0})| = \Big|\sum_{1\le k\le j}(\tau_h-\mathrm{id})(\varphi^{y,k}-\varphi^{y,k-1})\Big| \\ &\le \sum_{1\le k\le j}|(\tau_h-\mathrm{id})(\varphi^{y,k}-\varphi^{y,k-1})| \lesssim \sum_{1\le k\le j}|\varphi^{y,k}-\varphi^{y,k-1}| * \rho_{2^{2-k}}\mathbb{1}_{A_{k,j}}.\end{aligned} \tag{3.9}$$

Before going further, let us note that

$$\rho_{2^{2-k}} \lesssim \rho_{2^{3-k}} \quad \text{and} \quad A_{k+1,j} \subset A_{k,j}. \tag{3.10}$$

By (3.5), (3.9) and (3.10), we thus have

$$\left|(\tau_h - \mathrm{id})\varphi^{y,j}\right| \lesssim \sum_{0\le k\le j} \left|u^y - u_k^y\right| * \rho_{2^{3-k}} \mathbb{1}_{A_{k+1,j}}.$$

Thus

$$\left|(\tau_h - \mathrm{id})\varphi^{y,j}(x)\right|^p \lesssim \left(\sum_{0\le k\le j} \left|u^y - u_k^y\right| * \rho_{2^{3-k}} \mathbb{1}_{A_{k+1,j}}(x)\right)^p =: J_{1,j}(x,y). \tag{3.11}$$

On the other hand, (3.5) implies

$$\begin{aligned}\left\|(\tau_h - \mathrm{id})\left(\varphi^y - \varphi^{y,j}\right)\right\|_{L^p}^p \lesssim \left\|\varphi^y - \varphi^{y,j}\right\|_{L^p}^p &\le \int_{\mathbb{T}^n} \left(\sum_{k\ge j+1} \left|\varphi^{y,k}(x) - \varphi^{y,k-1}(x)\right|\right)^p dx \\ &\lesssim \int_{\mathbb{T}^n} \left(\sum_{k\ge j} \left|u^y(x) - u_k^y(x)\right|\right)^p dx =: J_{2,j}(y).\end{aligned} \tag{3.12}$$

By combining the estimates (3.11) and (3.12) with (3.7) and (3.8), we find that

$$\int_{\mathbb{T}^n} |\varphi^y|_{W^{s,p}}^p \, dy \lesssim \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \sum_{j\ge 0} 2^{spj} J_{1,j}(x,y)\, dy\, dx + \int_{\mathbb{T}^n} \sum_{j\ge 0} 2^{spj} J_{2,j}(y)\, dy =: L_1 + L_2.$$

We first estimate the term L_2, *via* a Schur type estimate (Corollary 8.2) and Lemma 8.3:

$$\begin{aligned}\sum_{j\ge 0} 2^{spj} J_{2,j}(y) &= \int_{\mathbb{T}^n} \sum_{j\ge 0} \left(\sum_{k\ge j} 2^{s(j-k)} \left(2^{sk} \left|u^y(x) - u_k^y(x)\right|\right)\right)^p dx \lesssim \frac{1}{s^p} \sum_{k\ge 0} 2^{skp} \left\|u^y - u_k^y\right\|_{L^p}^p \\ &= \frac{1}{s^p} Z\left(u^y\right) \lesssim \frac{1}{s^p} X\left(u^y\right) = \frac{1}{s^p} |u|_{W^{s,p}}^p, \quad \forall y \in \mathbb{T}^n.\end{aligned}$$

Consequently,

$$L_2 \lesssim \frac{1}{s^p} |u|_{W^{s,p}}^p.\text{[5]} \tag{3.13}$$

We now turn to L_1. We decompose the sets $A_{k,j}$, which are increasing with k, as a finite disjoint union of sets by defining

$$B_{k,j} := A_{k,j} \setminus A_{k-1,j}, \quad \forall k \ge 2 \quad \text{and} \quad B_{1,j} := A_{1,j}.$$

Thus, $A_{k,j} = \bigsqcup_{1\le t\le k} B_{t,j}$ and we have

$$\begin{aligned}L_1 &= \sum_{j\ge 0} 2^{spj} \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \left(\sum_{k=0}^{j} \sum_{t=1}^{k+1} \left|u^y - u_k^y\right| * \rho_{2^{4-k}} \mathbb{1}_{B_{t,j}}(x)\right)^p dy\, dx \\ &= \sum_{j\ge 0} 2^{spj} \sum_{1\le t\le j+1} \int_{B_{t,j}} \left\| \sum_{t-1\le k\le j} \left|u^y - u_k^y\right| * \rho_{2^{4-k}}(x)\right\|_{L_y^p(\mathbb{T}^n)}^p dx.\end{aligned}$$

Using Minkowski's inequality and noting that $|B_{t,j}| \le |A_{t,j}| \lesssim 2^{t-j}$, we find

[5] As in [3], the integration with respect to y does not play any role in the estimate satisfied by L_2.

$$L_1 \lesssim \sum_{j\ge 0} 2^{spj} \sum_{1\le t\le j+1} 2^{t-j}\Bigg(\sum_{t-1\le k\le j} \sup_x \big\| |u^y - u_k^y| * \rho_{2^{4-k}}(x)\big\|_{L_y^p(\mathbb{T}^n)}\Bigg)^p.$$

Now comes the key estimate.

3.1. Lemma. *Assume that* $0<s<1$. *Let* $u \in W^{s,p}(\mathbb{T}^n)$, *and define* $g_k : \mathbb{T}^n \times \mathbb{T}^n \to \mathbb{R}$ *by*

$$g_k(x,y) := |u^y - u_k^y| * \rho_{2^{4-k}}(x).$$

Consider also the quantity

$$a_k := 2^{sk} \sup_x \big\|g_k(x,\cdot)\big\|_{L^p}, \quad \forall k \ge 0.$$

Then $\sum_{k\ge 0} a_k^p \le 2|u|_{W^{s,p}}^p$.

Proof. Hölder's inequality combined with the fact that the integral of ρ equals 1 gives

$$\int_{\mathbb{T}^n} \big|g_k(x,y)\big|^p\,dy \le \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} |u^y - u_k^y|^p(x-z)\rho_{2^{4-k}}(z)\,dy\,dz. \tag{3.14}$$

We next note that

$$|u^y - u_k^y|^p(x-z) = \Bigg| \fint_{Q_k(x-z)} \big(u^y(x-z) - u^y(w)\big)\,dw\Bigg|^p \le 2^{nk} \int_{B(x-z,2^{-k})} \big|u^y(x-z) - u^y(w)\big|^p\,dw; \tag{3.15}$$

here, we use Hölder's inequality together with the fact that $Q_k(x-z) \subset B(x-z,2^{-k})$.

Integration of (3.15) over y leads to

$$\begin{aligned}\int_{\mathbb{T}^n} |u^y - u_k^y|^p(x-z)\,dy &\le 2^{nk} \int_{\mathbb{T}^n} \int_{B(x-z,2^{-k})} \big|u^y(x-z) - u^y(w)\big|^p\,dy\,dw \\ &= 2^{nk} \int_{\mathbb{T}^n}\int_{|h|\le 2^{-k}} \big|u(t) - u(t-h)\big|^p\,dh\,dt, \quad \forall x,z \in \mathbb{T}^n. \end{aligned}\tag{3.16}$$

Using (3.14), we obtain

$$\begin{aligned}\sum_{k\ge 0} a_k^p &\le \sum_{k\ge 0} \int_{\mathbb{T}^n}\int_{|h|\le 2^{-k}} 2^{(n+sp)k}\big|u(t)-u(t-h)\big|^p\,dh\,dt \\ &= \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \sum_{2^k\le 1/|h|} 2^{(n+sp)k}\big|u(t)-u(t-h)\big|^p\,dh\,dt \le c\int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{|u(t)-u(t-h)|^p}{|h|^{n+sp}}\,dh\,dt,\end{aligned}$$

with

$$c = c(n,s,p) := \sup_{|h|\le 1} |h|^{n+sp} \sum_{2^k \le 1/|h|} 2^{(n+sp)k} \le 2.$$

Therefore, we have $\sum_{k\ge 0} a_k^p \le 2|u|_{W^{s,p}}^p$. □

Proof of Theorem 1.3 completed. By the above lemma and Corollary 8.2 we have

$$\begin{aligned} L_1 &\lesssim \sum_{j\ge 0} 2^{(sp-1)j} \sum_{1\le t\le j+1} 2^t \Bigg(\sum_{t-1\le k\le j} 2^{-sk} a_k\Bigg)^p = \sum_{t\ge 1}\sum_{j\ge t-1} 2^{(sp-1)(j-t)}\Bigg(\sum_{t-1\le k\le j} 2^{-s(k-t)} a_k\Bigg)^p \\ &\lesssim \frac{1}{1-sp}\sum_{t\ge 1}\Bigg(\sum_{k\ge t-1} 2^{-s(k-t)} a_k\Bigg)^p \lesssim \frac{1}{s^p(1-sp)}\sum_{k\ge 0} a_k^p \lesssim \frac{1}{s^p(1-sp)} |u|_{W^{s,p}}^p. \end{aligned}$$

By combining this with the estimate (3.13) of L_2, we find that

$$\int_{\mathbb{T}^n} |\varphi^y|^p_{W^{s,p}}\, dy \lesssim \frac{1}{s^p(1-sp)} |u|^p_{W^{s,p}}. \qquad \square$$

Proof of Proposition 1.6. As mentioned in the introduction, we rely on an argument devised, for BV maps, by Dávila and Ignat [12]. Let $u \in W^{s,1}(\mathbb{T}^n; \mathbb{S}^1)$. For every $\alpha \in \mathbb{S}^1$ define $\varphi_\alpha := \theta_\alpha(u)$, where $\theta_\alpha(z)$ represents the unique argument of $z \in \mathbb{S}^1$ in the interval $(\alpha - 2\pi, \alpha]$. The functions φ_α are clearly measurable phases of u. We claim that there exists $\alpha \in \mathbb{S}^1$ such that $|\varphi_\alpha|_{W^{s,1}} \le 2|u|_{W^{s,1}}$. For this purpose, we estimate the average of $|\varphi_\alpha|_{W^{s,1}}$ over $\mathbb{S}^1$:

$$\begin{aligned} \fint_{\mathbb{S}^1} |\varphi_\alpha|_{W^{s,1}}\, d\alpha &= \fint_{\mathbb{S}^1} \left(\int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \frac{|\varphi_\alpha(x) - \varphi_\alpha(y)|}{|x-y|^{n+s}}\, dx\, dy \right) d\alpha \\ &= \frac{1}{2\pi} \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \frac{1}{|x-y|^{n+s}} \left(\int_{\mathbb{S}^1} \left| \theta_\alpha\big(u(x)\big) - \theta_\alpha\big(u(y)\big) \right| d\alpha \right) dx\, dy. \end{aligned} \tag{3.17}$$

Applying Lemma 8.12 and using (3.17), we obtain $\fint_{\mathbb{S}^1} |\varphi_\alpha|_{W^{s,1}}\, d\alpha \le 2|u|_{W^{s,1}}$, which proves the claim and completes the proof of the proposition. $\square$

4. Optimality when $sp < 1$. Proof of Theorem 1.5

The next result quantifies the asymptotic optimality of Theorem 1.3 in the special case where $n = 1$, $1 < p < \infty$ and $s = (1-\varepsilon)/p$, with $\varepsilon \to 0$. As we will see, the general case is an easy consequence of Proposition 4.1.

4.1. Proposition. *For every $\varepsilon \in (0, 1/2)$, there exists $u_\varepsilon \in W^{(1-\varepsilon)/p,p}(\mathbb{T}; \mathbb{S}^1)$ such that any phase $\varphi \in W^{(1-\varepsilon)/p,p}((0,1); \mathbb{R})$ of u_ε satisfies*

$$|\varphi|_{W^{(1-\varepsilon)/p,p}} \gtrsim \varepsilon^{-1/p} |u|_{W^{(1-\varepsilon)/p,p}}.$$

The above proposition is a variant of [4, Theorem 2]. In turn, [4, Theorem 2] relies on a very involved result [4, Theorem 1] providing the asymptotic behavior of the best Sobolev constant in the embedding $W^{1-\varepsilon,1}((0,1)) \hookrightarrow L^{1/\varepsilon}((0,1))$. We present below a cousin argument, based on an inequality involving non-increasing rearrangements of functions, obtained by Garsia and Rodemich [14].

Proof of Proposition 4.1. As in [4, Proof of Theorem 2], the key step consists in establishing the following estimate

$$|A|\left|{}^cA\right| \le \left(C\varepsilon \int_A \int_{{}^cA} \frac{dxdy}{|x-y|^{2-\varepsilon}} \right)^{1/\varepsilon}, \tag{4.1}$$

for every $\varepsilon \in (0, 1/2)$ and every measurable set $A \subset (0,1)$. Here, ${}^cA$ is the complement of A, and C is an absolute constant.

Step 1. Proof of (4.1).

Recall that, if $f : (0,1) \to \mathbb{R}_+$ is a measurable function, then its non-increasing rearrangement $f^* : (0,1) \to \mathbb{R}_+$ is defined by

$$f^*(x) = \inf\left\{\lambda \in \mathbb{R};\ \left|\left\{t \in [0,1);\ f(t) > \lambda\right\}\right| \le x\right\}, \quad \forall x \in (0,1).$$

It is easy to see that, when $A \subset (0,1)$ is a measurable set, we have $(\mathbb{1}_A)^* = \mathbb{1}_{A^*}$, with $A^* = (0, |A|)$. Thus

$$\int_{A^*} \int_{{}^c(A^*)} \frac{dxdy}{|x-y|^{2-\varepsilon}} = \int_0^{|A|} \int_{|A|}^1 \frac{dxdy}{|x-y|^{2-\varepsilon}} = \frac{(1-|A|)^\varepsilon + |A|^\varepsilon - 1}{\varepsilon(1-\varepsilon)} = \frac{|A|^\varepsilon + |{}^cA|^\varepsilon - 1}{\varepsilon(1-\varepsilon)}. \tag{4.2}$$

On the other hand, we have

$$|A|^\varepsilon |{}^c A|^\varepsilon \lesssim (1-|A|)^\varepsilon + |A|^\varepsilon - 1 \tag{4.3}$$

(see Lemma 8.13 in Section 8.4).

In view of (4.2) and (4.3), in order to establish (4.1) it suffices to prove that

$$\int_{A^*}\int_{{}^c(A^*)} \frac{dx\,dy}{|x-y|^{2-\varepsilon}} \le \int_A\int_{{}^cA} \frac{dx\,dy}{|x-y|^{2-\varepsilon}}.$$

This is precisely the rearrangement inequality of Garsia and Rodemich [14, Theorem I.1]

$$\int_0^1\int_0^1 \Psi\left(\frac{f^*(x)-f^*(y)}{p(x-y)}\right)dx\,dy \le \int_0^1\int_0^1 \Psi\left(\frac{f(x)-f(y)}{p(x-y)}\right)dx\,dy,$$

applied with $f := \mathbb{1}_A$, $p(t) := |t|^{2-\varepsilon}$ and $\Psi(t) := |t|$.

Step 2. Proof of Proposition 4.1 completed.

This part follows closely [4, Proof of Theorem 2], with some slight simplifications. For the convenience of the reader, we also detail some arguments which are only sketched in [4].

For $\delta \in (0, 1/2)$, we define the phase

$$\varphi_\delta(x) := \begin{cases} 0, & \text{if } x < 1/2, \\ (2x-1)\pi/\delta, & \text{if } 1/2 < x < 1/2+\delta, \\ 2\pi, & \text{if } 1/2+\delta < x. \end{cases} \tag{4.4}$$

We next choose $\delta = \delta(\varepsilon) := e^{-1/\varepsilon}$. For this choice of δ, the map $u_\varepsilon(x) := e^{\imath\varphi_\delta(x)}$, for $x \in (0,1)$, satisfies

$$|u|_{W^{(1-\varepsilon)/p,p}((0,1))} \approx 1 \quad \text{when } \varepsilon \to 0 \tag{4.5}$$

(see Lemma 8.14 in Section 8.4).

In order to prove Proposition 4.1, it suffices to show that any lifting φ of u_ε satisfies

$$|\varphi|_{W^{(1-\varepsilon)/p,p}} \gtrsim \varepsilon^{-1/p},$$

for $\varepsilon \in (0, 1/2)$. Arguing by contradiction, we assume that, for every $\eta > 0$, there are some $\varepsilon \in (0, 1/2)$ and $\varphi \in W^{(1-\varepsilon)/p,p}((0,1);\mathbb{R})$ such that $u_\varepsilon \equiv e^{\imath\varphi}$ and

$$|\varphi|^p_{W^{(1-\varepsilon)/p,p}} < \frac{\eta}{\varepsilon}. \tag{4.6}$$

We set $\psi := \frac{\varphi-\varphi_\delta}{2\pi}$. Since both φ and φ_δ are liftings of u_ε, the function ψ takes its values into $\mathbb{Z}$. Straightforward calculations (see Lemma 8.15) show that

$$|\psi(x)-\psi(y)| \le |\varphi(x)-\varphi(y)| \quad \text{if } x, y \in I_1 := \left(0, \frac{1}{2}+\frac{2\delta}{3}\right), \text{ or if } x, y \in I_2 := \left(\frac{1}{2}+\frac{\delta}{3}, 1\right). \tag{4.7}$$

We next invoke the following result, whose proof is postponed to Section 8.4.

4.2. Lemma. *Let $I \subset \mathbb{R}$ be an interval and let $\psi : I \to \mathbb{Z}$ be any measurable function. Then there exists some $k \in \mathbb{Z}$ such that*

$$\left|\{x \in I;\ \psi(x) \ne k\}\right| \le 4\left(C\varepsilon \int_I\int_I \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{2-\varepsilon}}\,dx\,dy\right)^{1/\varepsilon},$$

for all $\varepsilon \in (0, 1/2)$, where C is the absolute constant in (4.1).

Step 2 continued. Applying Lemma 4.2 with $I = I_1$ and with $I = I_2$, and using (4.7) together with (4.6), we obtain that there exist $m_1, m_2 \in \mathbb{Z}$ such that

$$|{}^c(A_1)| \le 4(C\eta)^{1/\varepsilon} \quad \text{and} \quad |{}^c(A_2)| \le 4(C\eta)^{1/\varepsilon}, \tag{4.8}$$

where

$$A_1 := \{x \in I_1; \psi(x) = m_1\} \quad \text{and} \quad A_2 := \{x \in I_2; \psi(x) = m_2\}.$$

We now choose $\eta > 0$ such that $\eta < \frac{1}{\sqrt{24Ce}}$. With this choice of η, we have (using (4.8))

$$|A_1 \cap A_2| = |A_1 \cap A_2 \cap I_1 \cap I_2| \ge |I_1 \cap I_2| - |{}^c(A_1)| - |{}^c(A_2)| \ge \frac{\delta(\varepsilon)}{3} - 8(C\eta)^{1/\varepsilon} > 0,$$

and thus we must have $m_1 = m_2$. We may further assume that $m_1 = m_2 = 0$.

Consider the following sets:

$$B_1 := \{x \in (0, 1/2); \psi(x) \neq 0\} \subset (0, 1/2) \quad \text{and} \quad B_2 := \{x \in (1/2 + \delta, 1); \psi(x) \neq 0\} \subset (1/2 + \delta, 1).$$

We clearly have

$$\varphi = \varphi_\delta \quad \text{on } {}^c(B_1) \quad \text{and} \quad \varphi = \varphi_\delta \quad \text{on } {}^c(B_2)$$

and, in addition, by (4.8) we also have

$$|B_1| \le \delta/6 \quad \text{and} \quad |B_2| \le \delta/6. \tag{4.9}$$

By the definition of φ_δ, we then find

$$|\varphi|^p_{W^{(1-\varepsilon)/p,p}} \ge \int_{{}^c(B_1)} \int_{{}^c(B_2)} \frac{|\varphi(x) - \varphi(y)|^p}{(x-y)^{2-\varepsilon}} \, dx\, dy = \int_{{}^c(B_1)} \int_{{}^c(B_2)} \frac{|\varphi_\delta(x) - \varphi_\delta(y)|^p}{(x-y)^{2-\varepsilon}} \, dx\, dy$$
$$\ge (2\pi)^p \int_{{}^c(B_1)} \int_{{}^c(B_2)} \frac{dxdy}{(x-y)^{2-\varepsilon}}.$$

It is easy to see that the latter quantity does not increase if the sets B_1 and B_2 are replaced respectively by the intervals $\widetilde{B}_1 := (\frac{1}{2} - |B_1|, \frac{1}{2})$, and $\widetilde{B}_2 := (\frac{1}{2} + \delta, \frac{1}{2} + \delta + |B_2|)$; see Lemma 8.16. Hence, using (4.9) and the fact that $\delta = e^{-1/\varepsilon}$, we obtain

$$|\varphi|^p_{W^{(1-\varepsilon)/p,p}} \ge (2\pi)^p \int_0^{1/2-\delta/6} \int_{1/2+7\delta/6}^{1} \frac{dxdy}{(x-y)^{2-\varepsilon}} = \frac{(2\pi)^p}{\varepsilon(1-\varepsilon)} \big(1 - 1/e + o(1)\big), \tag{4.10}$$

when $\varepsilon \to 0$. For an appropriate choice of η, (4.10) contradicts (4.6). □

Proof of Theorem 1.5. The optimality of the estimate (1.7) in Theorem 1.3 means that, for every $0 < s < 1$ with $1 - sp \ll 1$, there exists a map $u \in W^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$ such that any lifting $\varphi \in W^{s,p}((0,1)^n; \mathbb{R})$ of u satisfies

$$|\varphi|^p_{W^{s,p}} \gtrsim \frac{1}{1 - sp} |u|^p_{W^{s,p}}. \tag{4.11}$$

This is true in dimension $n = 1$ by the above Proposition 4.1. In order to prove (4.11) in arbitrary dimension, we use a dimensional reduction argument. More specifically, for every $s \in (1/(2p), 1/p)$ we define

$$u(x) = u(x_1, x_2, \ldots, x_n) = u(x_1, x') := u_s(x_1), \quad \forall x \in \mathbb{T}^n.$$

Here, u_s is a map in $W^{s,p}(\mathbb{T}; \mathbb{S}^1)$ that satisfies the property that for any lifting $\varphi \in W^{s,p}([0,1]; \mathbb{R})$ of u_s we have

$$|\varphi|^p_{W^{s,p}([0,1])} \gtrsim \frac{1}{1 - sp} |u_s|^p_{W^{s,p}(\mathbb{T})}. \tag{4.12}$$

Note that the existence of u_s follows from Proposition 4.1.

Consider an arbitrary lifting $\psi \in W^{s,p}((0,1)^n; \mathbb{R})$ of u. Clearly, for almost every $x' := (x_2, \ldots, x_n) \in (0,1)^{n-1}$, the map $x_1 \mapsto \psi(x_1, x') =: \varphi_{x'}(x_1)$ is a lifting of u_s, and thus satisfies the estimate (4.12). By combining this fact with Corollary 8.19, we find

$$|\psi|^p_{W^{s,p}(\mathbb{T}^n)} \approx |\varphi_{x'}|^p_{W^{s,p}(\mathbb{T})} \gtrsim \frac{1}{1-sp}|u_s|^p_{W^{s,p}(\mathbb{T})}. \tag{4.13}$$

On the other hand, we have, again by Corollary 8.19, that $|u|_{W^{s,p}(\mathbb{T}^n)} \approx |u_s|_{W^{s,p}(\mathbb{T})}$, which, together with (4.13), leads to

$$|\psi|^p_{W^{s,p}(\mathbb{T}^n)} \gtrsim \frac{1}{1-sp}|u|^p_{W^{s,p}(\mathbb{T}^n)}. \qquad \square$$

5. Optimal estimates when $sp \geq 1$

As we will see below, when $sp \geq 1$ two quantitatively different types of estimates occur: linear estimates of the form $|\varphi|_{W^{s,p}} \lesssim |u|_{W^{s,p}}$, and superlinear estimates of the form

$$|\varphi|_{W^{s,p}} \lesssim |u|_{W^{s,p}} + |u|^{\alpha}_{W^{s,p}}, \tag{5.1}$$

with $\alpha > 1$.

The linear regime corresponds to the case where $s \geq 1$. When $s = 1$, we actually have the identity $|\varphi|_{W^{1,p}} = |u|_{W^{1,p}}$, and optimality is irrelevant. When $s > 1$, several natural semi-norms[6] $|\cdot|_{W^{s,p}}$ can be considered, and optimal estimates do depend on the choice of such a semi-norm. Therefore, we restrict ourselves to a more modest task, which consists in proving that optimal estimates are indeed linear.

When $s < 1$, we will obtain superlinear estimates of type (5.1). In this case, we focus on the optimality of the exponent α (when $|u|_{W^{s,p}}$ is large) and of the linear term $|u|_{W^{s,p}}$ (when $|u|_{W^{s,p}}$ is small).

5.1. Theorem. *Let $s \geq 1$, $1 \leq p < \infty$ be such that $sp \geq 2$. Let $u \in W^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$ and let $\varphi \in W^{s,p}(\mathbb{T}^n; \mathbb{R})$ be a lifting of u. Then*

$$|\varphi|_{W^{s,p}} \leq C(s,p)|u|_{W^{s,p}}. \tag{5.2}$$

Moreover, the above estimate is optimal in the sense that

$$\limsup_{|\varphi|_{W^{s,p}} \to 0} \frac{|\varphi|_{W^{s,p}}}{|u|_{W^{s,p}}} > 0, \quad \textit{and} \quad \limsup_{|u|_{W^{s,p}} \to \infty} \frac{|\varphi|_{W^{s,p}}}{|u|_{W^{s,p}}} > 0.$$

Proof. Since the estimate (5.2) does not depend on the choice of the semi-norm, we can work for convenience with the semi-norm $|f|_{W^{s,p}} := \|\nabla f\|_{W^{s-1,p}}$.

Step 1. Proof of (5.2).

Since $s \geq 1$, we may differentiate once the equality $u = e^{\imath\varphi}$ and find that $\nabla\varphi = u \wedge \nabla u$.[7] Thus we have to establish the estimate $\|u \wedge \nabla u\|_{W^{s-1,p}} \lesssim |u|_{W^{s,p}}$. We first extend u to a map in $\mathbb{R}^n$ using a standard extension operator $P : W^{s,p}(\mathbb{T}^n) \to W^{s,p}(\mathbb{R}^n)$. This goes as follows. We first define $v := P(u - \fint_{\mathbb{T}^n} u)$, which belongs to $W^{s,p}(\mathbb{R}^n)$ and then $w := v + \fint_{\mathbb{T}^n} u$ is an extension of u. We next note that

$$\begin{aligned}\|u \wedge \nabla u\|_{W^{s-1,p}(\mathbb{T}^n)} &\leq \|w \wedge \nabla w\|_{W^{s-1,p}(\mathbb{R}^n)} = \left\| \left(v + \fint_{\mathbb{T}^n} u\right) \wedge \nabla v \right\|_{W^{s-1,p}(\mathbb{R}^n)} \\ &\lesssim \|v \wedge \nabla v\|_{W^{s-1,p}(\mathbb{R}^n)} + \|\nabla v\|_{W^{s-1,p}(\mathbb{R}^n)} = \|v \wedge \nabla v\|_{W^{s-1,p}(\mathbb{R}^n)} + |v|_{W^{s,p}(\mathbb{R}^n)}.\end{aligned}$$

In the last inequality we used the fact that $|\fint u| \leq 1$. Therefore, by Lemma 8.21, and by the fact that $|v| \leq 2$, we obtain

$$\begin{aligned}\|u \wedge \nabla u\|_{W^{s-1,p}(\mathbb{T}^n)} &\lesssim \|v\|_{W^{s,p}(\mathbb{R}^n)} = \left\| P\left(u - \fint_{\mathbb{T}^n} u\right) \right\|_{W^{s,p}(\mathbb{R}^n)} \\ &\lesssim \left\| u - \fint_{\mathbb{T}^n} u \right\|_{W^{s,p}(\mathbb{T}^n)} \lesssim |u|_{W^{s-1,p}(\mathbb{T}^n)}\end{aligned}$$

(the last inequality following from Poincaré's inequality).

[6] A natural semi-norm is a semi-norm modulo constant functions and equivalent to the standard norm on the quotient space $W^{s,p}/\mathbb{C}$.

[7] Recall that $(u_1 + \imath u_2) \wedge \nabla(v_1 + \imath v_2) = u_1 \nabla v_2 - u_2 \nabla v_1$.

Step 2. Optimality in dimension $n = 1$.

The optimality of (5.2) needs to be checked for $|\varphi|_{W^{s,p}} \to 0$ and for $|u|_{W^{s,p}} \to \infty$, that is we need to show that:

1. There exists $(\varphi_j)_{j\ge 1}$ in $W^{s,p}(\mathbb{T};\mathbb{R})$ such that $|\varphi_j|_{W^{s,p}} \to 0$ and $|u_j|_{W^{s,p}} \lesssim |\varphi_j|_{W^{s,p}}$, where $u_j := e^{\imath\varphi_j}$.
2. There exists $(u_j)_{j\ge 1}$ in $W^{s,p}(\mathbb{T};\mathbb{S}^1)$ such that $|u_j|_{W^{s,p}} \to \infty$ and $|u_j|_{W^{s,p}} \lesssim |\varphi_j|_{W^{s,p}}$, where φ_j is the (unique modulo 2π) lifting of u_j.

For the optimality "at zero", we let $f \in C_c^\infty((0,1);\mathbb{R})$, and define $\varphi_j := f/j$, and $u_j := e^{\imath\varphi_j}$. Clearly, we have

$$|\varphi_j|_{W^{r,p}} = \frac{|f|_{W^{r,p}}}{j} \approx \frac{1}{j} \to 0 \quad \text{as } j \to \infty,\ \forall r > 0. \tag{5.3}$$

Using (5.3) and a straightforward induction, it is easy to see that, when $k \ge 1$ is an integer, we may write

$$D^k u_j = g_{k,j} u_j, \quad \text{for some } g_{k,j} \in C_c^\infty\big((0,1);\mathbb{C}\big) \text{ such that } |g_{k,j}|_{W^{r,p}} \lesssim 1/j,\ \forall r > 0. \tag{5.4}$$

We now establish item 1 for the above choice of φ_j and u_j. Assume first that s is an integer. Then by (5.4) we have

$$|u_j|_{W^{s,p}} = \|\nabla u_j\|_{W^{s-1,p}} = \sum_{1\le k\le s} \big\|D^k u_j\big\|_{L^p} \lesssim \frac{1}{j}. \tag{5.5}$$

By (5.3) and (5.5), we find that $|u_j|_{W^{s,p}} \lesssim |\varphi_j|_{W^{s,p}}$.

Assume next that s is not an integer and set $\sigma := s - [s] \in (0,1)$. By (5.5), we have

$$\big\|D^k u_j\big\|_{L^p} \lesssim 1/j, \quad \forall 1 \le k \le [s]. \tag{5.6}$$

As a consequence of (5.6) with $k = 1$, we also have

$$|u_j|_{W^{r,p}} \lesssim 1/j, \quad \forall r \in (0,1). \tag{5.7}$$

In order to conclude, it suffices to establish the estimate $|D^{[s]}u_j|_{W^{\sigma,p}} \lesssim 1/j$. This estimate is an immediate consequence of (5.4), of (5.7) and of the inequality

$$\big|D^{[s]}u_j\big|^p_{W^{\sigma,p}} \lesssim |u_j|^p_{W^{\sigma,p}} + |g_{[s],j}|^p_{W^{s,p}} \lesssim |u_j|^p_{W^{\sigma,p}} + 1/j^p. \tag{5.8}$$

In turn, (5.8) follows from

$$\begin{aligned}\big|g_{[s],j}(x)u_j(x) - g_{[s],j}(y)u_j(y)\big| &\le \big|g_{[s],j}(x)\big|\big|u_j(x) - u_j(y)\big| + \big|g_{[s],j}(x) - g_{[s],j}(y)\big| \\ &\lesssim \big|u_j(x) - u_j(y)\big| + \big|g_{[s],j}(x) - g_{[s],j}(y)\big|.\end{aligned}$$

In order to prove the optimality of (5.2) "at infinity" we take φ_j to be a sum of j copies of a properly scaled C_c^∞ function. More precisely, we fix $f \in C_c^\infty((0,1);\mathbb{R})$ and we define the functions $\varphi_j := \sum_{k=0}^{j-1} f(xj-k)$, $\forall j \ge 1$, whose semi-norms can be estimated by

$$|\varphi_j|_{W^{s,p}} \approx j^s \tag{5.9}$$

(see Lemma 8.20).[8] Next, we take $g := e^{\imath f} - 1 \in C_c^\infty((0,1);\mathbb{C})$ and

$$u_j := \sum_{k=0}^{j-1} g(xj-k) + 1, \quad \forall j \ge 1.$$

Since $u_j(x) = \sum_{k=0}^{j-1}(e^{\imath f(xj-k)} - 1) + 1 = \prod_{k=0}^{j-1} e^{\imath f(xj-k)} = e^{\imath\varphi_j(x)}$, the function φ_j is "the" lifting of u_j. On the other hand, by Lemma 8.20 we have

$$|u_j|_{W^{s,p}} \approx j^s. \tag{5.10}$$

By (5.9) and (5.10), we have $|u_j|_{W^{s,p}} \approx |\varphi_j|_{W^{s,p}} \to \infty$ when $j \to \infty$, which proves item 2 when $n = 1$.

[8] Recall that $A \approx B$ stands for $B \lesssim A \lesssim B$.

Step 3. Optimality in higher dimension.

Let φ_j, u_j be as in Step 2. As in the proof of Proposition 4.1, we let:

$$\psi_j(x_1, x') := \varphi_j(x_1), \qquad v_j(x_1, x') := u_j(x_1) = e^{\imath\varphi_j(x_1)}, \quad \forall x_1 \in \mathbb{T}, x' \in \mathbb{T}^{n-1}. \tag{5.11}$$

Then $v_j = e^{\imath\psi_j}$ and, by Corollary 8.19, we have the equivalence of norms $|\psi_j|_{W^{s,p}} \approx |\varphi_j|_{W^{s,p}}$ and $|v_j|_{W^{s,p}} \approx |u_j|_{W^{s,p}}$. Therefore, since φ_j and u_j were chosen such that $|u_j|_{W^{s,p}} \lesssim |\varphi_j|_{W^{s,p}}$, we also have $|v_j|_{W^{s,p}} \lesssim |\psi_j|_{W^{s,p}}$. □

We next turn to the case where $0 < s < 1$. In view of [3] (see also the Introduction), when $0 < s < 1$, $W^{s,p}$ has the lifting property if and only if $sp \ge n$. We start by presenting an exceptional case, already observed in [3], where there is no possible estimate of φ in terms of u. More specifically, we have the following:

5.2. Proposition. *(See [3].) Let $1 < p < \infty$. Then there is no estimate of the form $|\varphi|_{W^{1/p,p}} \le F(|u|_{W^{1/p,p}})$.*

Let us briefly recall the argument in [3]. Let φ_δ be as in (4.4) and set $u_\delta := e^{\imath\varphi_\delta}$. Then it is easily checked that $|u_\delta|_{W^{1/p,p}} \lesssim 1$ and $|\varphi|_{W^{1/p,p}} \to \infty$ as $\delta \to 0$. Since φ_δ is the unique phase (mod 2π) of u_δ, we obtain the non-existence of an estimate of the form $|\varphi|_{W^{1/p,p}} \le F(|u|_{W^{1/p,p}})$.

As we will see below, this is the only exceptional case. In the remaining cases, we will establish several positive results. We start by recalling an elementary estimate, due to Merlet [20, Theorem 1.1], and whose proof is postponed.

5.3. Theorem. *(See [20].) Let $n = 1$. Assume that $0 < s < 1$, $1 < p < \infty$ and $sp > 1$. Let $u \in W^{s,p}(\mathbb{T}; \mathbb{S}^1)$ and let $\varphi \in W^{s,p}(\mathbb{T}; \mathbb{R})$ be a lifting of u. Then*

$$|\varphi|_{W^{s,p}} \lesssim |u|_{W^{s,p}} + |u|^{1/s}_{W^{s,p}}. \tag{5.12}$$

In higher dimensions, we obtain the same result as the one in Theorem 5.3, but the corresponding proof is much more involved.

5.4. Theorem. *Let $n \ge 2$. Assume that $0 < s < 1$, $1 < p < \infty$ and $sp \ge n$. Let $u \in W^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$ and let $\varphi \in W^{s,p}(\mathbb{T}^n; \mathbb{R})$ be a lifting of u. Then*

$$|\varphi|_{W^{s,p}} \lesssim |u|_{W^{s,p}} + |u|^{1/s}_{W^{s,p}}. \tag{5.13}$$

We start with the

Proof of Theorem 5.4. Estimate (5.13) will be obtained via the factorization method presented in [21]. More precisely, the arguments in [21], that we will detail below, lead to the existence of some $\varphi_1 \in W^{s,p}(\mathbb{T}^n; \mathbb{R})$ such that

$$|\varphi_1|_{W^{s,p}} \lesssim |u|_{W^{s,p}} \quad \text{and} \quad \left\|\nabla\left(ue^{-\imath\varphi_1}\right)\right\|_{L^{sp}} \lesssim |u|^{1/s}_{W^{s,p}}.$$

The construction of the map φ_1 goes as follows. First, by suitably extending u,[9] we may identify u with a map in $\mathbb{R}^n$, still denoted u, with the following properties:

1. $|u|_{W^{s,p}(\mathbb{R}^n)} \lesssim |u|_{W^{s,p}(\mathbb{T}^n)}$.
2. $|u| \le 2$.
3. u is $\mathbb{S}^1$-valued in $(-3, 4)^n$.
4. u is constant outside $(-4, 5)^n$.

[9] As in Step 1 in the proof of Theorem 5.1.

We next consider a mollifier $\rho \in C_c^\infty(\mathbb{R}^n)$ satisfying: $\rho \geq 0$, $\int_{\mathbb{R}^n} \rho = 1$ and $\operatorname{supp} \rho \subset B(0,2) \setminus \overline{B(0,1)}$. We then let

$$w(x,\varepsilon) := u * \rho_\varepsilon(x), \quad \forall x \in \mathbb{R}^n, \ \forall \varepsilon > 0, \tag{5.14}$$

and define

$$\varphi_1(x) := -\int_0^\infty \Pi \circ w(x,\varepsilon) \wedge \frac{\partial}{\partial \varepsilon} \Pi \circ w(x,\varepsilon)\, d\varepsilon. \tag{5.15}$$

Here,

$$\Pi \in C^\infty\big(\mathbb{R}^2; \mathbb{R}^2\big) \quad \text{and} \quad \Pi(z) = z/|z| \quad \text{when } |z| \geq 1/2. \tag{5.16}$$

We now explain the motivation behind this construction. Intuitively, φ_1 encodes the small amplitude oscillations of u, while the remainder $ue^{-\imath\varphi_1}$ encodes the large amplitude oscillations (as those contained in the topological singularities of type $z/|z|$). The reason is the following. Assume that u has only small amplitude oscillations, say around the value 1. Then the extension w of u is still close to 1, and thus the restriction of $\Pi \circ w$ to $\mathbb{T}^n \times (0,\infty)$ is a smooth $\mathbb{S}^1$-valued extension of u. It follows that $\Pi \circ w$ has a smooth phase ψ. By differentiating the identity $\Pi \circ w \equiv e^{\imath\psi}$, we find that

$$\frac{\partial}{\partial \varepsilon}\psi(x,\varepsilon) = \Pi \circ w(x,\varepsilon) \wedge \frac{\partial}{\partial \varepsilon}(\Pi \circ w)(x,\varepsilon), \quad \forall x \in \mathbb{T}^n, \ \forall \varepsilon > 0. \tag{5.17}$$

Assuming in addition that $\Pi \circ w$ converges sufficiently fast to 1 as $\varepsilon \to \infty$, we may integrate (5.17) and find that

$$u(x) = \lim_{\varepsilon \to 0} w(x,\varepsilon) = e^{\imath\varphi_1(x)} \quad \text{for a.e. } x, \text{ with } \varphi_1 \text{ given by (5.15)}.$$

Therefore, φ_1 gives (under some reasonable assumptions) a phase of u provided u has small amplitude oscillations. In general, u need not have small amplitude oscillations, and the remainder $ue^{-\imath\varphi_1}$ measures what is left, i.e., the large amplitude oscillations.

We now turn to the implications of this construction for the proof of Theorem 5.4. The next two results are from [21].

5.5. Lemma. *Let $n \geq 1$, $0 < s < 1$ and $1 \leq p < \infty$. Let $u : \mathbb{R}^n \to \mathbb{C}$ satisfy items 1, 2 and 4 above. Let φ_1 be as in (5.15). Then:*

1. *The function φ_1 is well-defined a.e. on $\mathbb{T}^n$, in the sense that the integral in (5.15) is absolutely convergent for a.e. $x \in \mathbb{T}^n$.*
2. *$\varphi_1 \in W^{s,p}(\mathbb{T}^n)$ and*

$$|\varphi_1|_{W^{s,p}} \lesssim |u|_{W^{s,p}}. \tag{5.18}$$

5.6. Lemma. *Let $n \geq 1$, $s > 0$ and $1 \leq p < \infty$ be such that $sp \geq 1$. Let $u : \mathbb{R}^n \to \mathbb{C}$ satisfy properties 1–4 above. Let φ_1 be as in (5.15).*

Then $ue^{-\imath\varphi_1} \in W^{1,sp}(\mathbb{T}^n; \mathbb{S}^1)$ and

$$\big\|\nabla\big(ue^{-\imath\varphi_1}\big)\big\|_{L^{sp}(\mathbb{T}^n)} \lesssim |u|_{W^{s,p}(\mathbb{T}^n)}^{1/s}. \tag{5.19}$$

Proof of Theorem 5.4 completed. Let φ_1 be as in (5.15). By Lemma 5.6, the map $ue^{-\imath\varphi_1}$ belongs to the space $W^{1,sp}(\mathbb{T}^n; \mathbb{S}^1)$. Since $sp \geq 2$, by Theorem 1.1 we may write $ue^{-\imath\varphi_1} = e^{\imath\varphi_2}$ with $\varphi_2 \in W^{1,sp}(\mathbb{T}^n)$. Since $sp \geq n$, we have $W^{1,sp}(\mathbb{T}^n) \hookrightarrow W^{s,p}(\mathbb{T}^n)$, and thus $u = e^{\imath\varphi}$ with $\varphi := \varphi_1 + \varphi_2 \in W^{s,p}(\mathbb{T}^n; \mathbb{R})$. Since $sp \geq 1$, φ is the unique $(\operatorname{mod} 2\pi)$ phase of u in $W^{s,p}$ [3, Theorem B.1]. Moreover, using (5.18) and (5.19), we can estimate $|\varphi|_{W^{s,p}}$ as follows.

$$\begin{aligned} |\varphi|_{W^{s,p}} &\lesssim |\varphi_1|_{W^{s,p}} + |\varphi_2|_{W^{s,p}} \lesssim |u|_{W^{s,p}} + |\varphi_2|_{W^{1,sp}} = |u|_{W^{s,p}} + \|\nabla\varphi_2\|_{L^{sp}} \\ &= |u|_{W^{s,p}} + \big\|\nabla\big(ue^{-\imath\varphi_1}\big)\big\|_{L^{sp}} \lesssim |u|_{W^{s,p}} + |u|_{W^{s,p}}^{1/s}. \qquad \square \end{aligned} \tag{5.20}$$

We now turn to Theorem 5.3 and present three different proofs, with different flavors. The first one is a variant of the proof of Theorem 5.4. The second one simplifies Merlet's original argument. The third one is non-constructive (unlike the proof of Theorem 5.4) and is inspired by an argument in Nguyen [26].

First proof of Theorem 5.3. The following argument is similar to the one in Theorem 5.4. We consider the phase φ_1 defined therein. This time, we have $ue^{-\imath\varphi_1}$ in $W^{1,sp}(\mathbb{T};\mathbb{S}^1)$ with $sp \geq 1$. Since (by Theorem 1.1) in dimension $n=1$ all the Sobolev spaces do have the lifting property, we can write $ue^{-\imath\varphi_1} = e^{\imath\varphi_2}$ with $\varphi_2 \in W^{1,sp}(\mathbb{T})$. Since $sp > 1$, we have $W^{1,sp}(\mathbb{T}) \hookrightarrow W^{s,p}(\mathbb{T})$, and thus $u = e^{\imath\varphi}$ with $\varphi := \varphi_1 + \varphi_2 \in W^{s,p}(\mathbb{T};\mathbb{R})$. We now obtain the estimate (5.12) following the argument leading to (5.20). $\square$

Second proof of Theorem 5.3. The starting point is the estimate (5.21) below, due to Merlet [20]:

$$\left|\varphi(x)-\varphi(y)\right|^p \lesssim \left|u(x)-u(y)\right|^p + (y-x)^{p-1/s}|u|^{p/s}_{W^{s,p}((x,y))}, \quad 0 \leq x < y \leq 1. \tag{5.21}$$

(For a simplification of Merlet's original argument leading to (5.21), see the proof of Lemma 8.25.)

Dividing the inequality (5.21) by $(y-x)^{1+sp}$ and then integrating in x and y, we find that

$$|\varphi|^p_{W^{s,p}(\mathbb{T})} \lesssim |u|^p_{W^{s,p}(\mathbb{T})} + \int_0^1\int_0^y \frac{|u|^{p/s}_{W^{s,p}((x,y))}}{(y-x)^\alpha}\,dx\,dy, \tag{5.22}$$

with $\alpha := 1 + sp - p + 1/s$. Next we note that, since $s < 1$, we have $p/s > p$ and therefore

$$|u|^{p/s}_{W^{s,p}((x,y))} \leq |u|^{p/s-p}_{W^{s,p}(\mathbb{T})}|u|^p_{W^{s,p}((x,y))}. \tag{5.23}$$

On the other hand, since $s < 1$ and $sp > 1$, we have $\alpha < 2$. We obtain

$$\begin{aligned}\int_0^1\int_0^y \frac{|u|^p_{W^{s,p}((x,y))}}{(y-x)^\alpha}\,dx\,dy &\approx \int_0^1\int_0^y \frac{1}{(y-x)^\alpha}\int_x^y\int_x^z \frac{|u(z)-u(t)|^p}{(z-t)^{1+sp}}\,dt\,dz\,dx\,dy \\ &= \int_0^1\int_0^z \frac{|u(z)-u(t)|^p}{(z-t)^{1+sp}}\int_z^1\int_0^t \frac{1}{(y-x)^\alpha}\,dx\,dy\,dt\,dz \\ &\leq \int_0^1\int_0^z \frac{|u(z)-u(t)|^p}{(z-t)^{1+sp}}\int_t^{t+1}\int_{t-1}^t \frac{1}{(y-x)^\alpha}\,dx\,dy\,dt\,dz \leq C|u|^p_{W^{s,p}(\mathbb{T})}.\end{aligned}$$

Here, $C := \int_{-1}^0\int_0^1 \frac{1}{(y-x)^\alpha}\,dx\,dy < \infty$ (since $\alpha < 2$). The above inequality together with (5.22) and (5.23) implies $|\varphi|^p_{W^{s,p}} \lesssim |u|^p_{W^{s,p}} + |u|^{p/s}_{W^{s,p}}$. Thus (5.13) holds. $\square$

Third proof of Theorem 5.3.

Step 1. Proof of (5.13) when u is smooth and has a smooth periodic phase.

Suppose that u belongs to $C^\infty(\mathbb{T};\mathbb{S}^1)$ and that we may write $u = e^{\imath\varphi}$, with $\varphi \in C^\infty(\mathbb{T};\mathbb{R})$.[10] In this case, we will prove the existence of two linear maps, T_1 and T_2, such that for every $\zeta \in C^\infty(\mathbb{T};\mathbb{R})$ we have

1. $\int_{\mathbb{T}} \varphi'(x)\zeta(x)\,dx = T_1(\zeta) + T_2(\zeta)$.
2. $T_1(1) = T_2(1) = 0$.
3. $|T_1(\zeta)| \lesssim \|\zeta\|_{L^{(sp)'}}|u|^{1/s}_{W^{s,p}}$.
4. $|T_2(\zeta)| \lesssim |\zeta|_{W^{1-s,p'}}|u|_{W^{s,p}}$.

[10] This is equivalent to $\deg(u;\mathbb{T}) = 0$.

Assume for the moment that items 1 to 4 are proved. Using the dualities $(L^{(sp)'})^* = L^{sp}$, respectively $(W^{1-s,p'})^* = W^{s-1,p}$, we find that there exist some $\psi_1 \in L^{sp}$ and $\psi_2 \in W^{s-1,p}$ such that

a) $\varphi' = \psi_1 + \psi_2$ in the distributional sense.
b) $\int_{\mathbb{T}} \psi_1 = 0$ and $\psi_2(1) = 0$.
c) $\|\psi_1\|_{L^{sp}} \lesssim |u|^{1/s}_{W^{s,p}}$.
d) $\|\psi_2\|_{W^{s-1,p}} \lesssim |u|_{W^{s,p}}$.[11]

By estimates c) and d) and by property b), we may find some $\varphi_1 \in W^{1,sp}$ and $\varphi_2 \in W^{s,p}$ such that $\varphi_1' = \psi_1$ and $\varphi_2' = \psi_2$. In addition, we note the estimates $|\varphi_1|_{W^{1,sp}} \lesssim |u|^{1/s}_{W^{s,p}}$ and $|\varphi_2|_{W^{s,p}} \lesssim |u|_{W^{s,p}}$. By construction, we have $\varphi = \varphi_1 + \varphi_2$ (up to an additive constant). Using the Sobolev embedding $W^{1,sp}(\mathbb{T}) \hookrightarrow W^{s,p}(\mathbb{T})$, we obtain

$$|\varphi|_{W^{s,p}} \le |\varphi_2|_{W^{s,p}} + |\varphi_1|_{W^{s,p}} \lesssim |\varphi_2|_{W^{s,p}} + |\varphi_1|_{W^{1,sp}} \lesssim |u|_{W^{s,p}} + |u|^{1/s}_{W^{s,p}},$$

which is the desired conclusion.

So let us construct T_1 and T_2 satisfying items 1–4. We identify $\mathbb{T}$ with the boundary $\mathbb{S}^1$ of the unit disc $\mathbb{D}$ and we identify the derivative on $\mathbb{T}$ with the tangential derivative on $\mathbb{S}^1$. Let ξ be the harmonic extension to $\mathbb{D}$ of ζ, and let $\widetilde{u} = \widetilde{u}_1 + \imath \widetilde{u}_2$ be a smooth extension of $u = u_1 + \imath u_2$ to $\mathbb{D}$.[12] Noting the fact that the Jacobian determinant $\operatorname{Jac}(f,g) := \det(\nabla f, \nabla g)$, $f, g : \mathbb{D} \to \mathbb{R}$, satisfies the identities

$$\int_{\mathbb{S}^1} f \frac{\partial g}{\partial \tau} = \int_{\mathbb{D}} \operatorname{Jac}(f,g) \quad \text{and} \quad \operatorname{Jac}(f, gh) = h \operatorname{Jac}(f,g) + g \operatorname{Jac}(f,h), \quad \forall f,g,h \in C^1(\overline{\mathbb{D}}; \mathbb{R}),$$

we find that

$$\begin{aligned}\int_{\mathbb{T}} \varphi' \zeta &\equiv \int_{\mathbb{S}^1} \frac{\partial \varphi}{\partial \tau} \zeta = \int_{\mathbb{S}^1} \left[(u_1 \zeta) \frac{\partial u_2}{\partial \tau} - (u_2 \zeta) \frac{\partial u_1}{\partial \tau} \right] = \int_{\mathbb{D}} \big(\operatorname{Jac}(\widetilde{u}_1 \xi, \widetilde{u}_2) - \operatorname{Jac}(\widetilde{u}_2 \xi, \widetilde{u}_1) \big) \\ &= 2 \int_{\mathbb{D}} \xi \operatorname{Jac} \widetilde{u} + \int_{\mathbb{D}} \big(\widetilde{u}_1 \operatorname{Jac}(\xi, \widetilde{u}_2) - \widetilde{u}_2 \operatorname{Jac}(\xi, \widetilde{u}_1) \big) \\ &= 2 \int_{\mathbb{D}} \xi \operatorname{Jac} \widetilde{u} + \int_{\mathbb{D}} \nabla \xi \wedge (\widetilde{u} \wedge \nabla \widetilde{u}) := T_1(\zeta) + T_2(\zeta).\end{aligned}$$

We next prove that, for an appropriate choice of $\widetilde{u}$, T_1 and T_2 satisfy items 2, 3 and 4 above.

Proof of 2. We clearly have $T_2(1) = 0$ and $\int_{\mathbb{T}} \varphi' = 0$. This leads to $T_1(1) = 0$. □

Proof of 3. Let $M\zeta$ denote the maximal function of ζ. Recall the inequality

$$\sup_{0 \le r \le 1} \big| \xi(r\omega) \big| \le M\zeta(\omega), \quad \forall \omega \in \mathbb{S}^1 \tag{5.24}$$

(see Lemma 8.26). We have

$$\begin{aligned}\frac{1}{2} \big| T_1(\zeta) \big| &\le \int_{\mathbb{D}} \big| \xi(x) \big| \big| \operatorname{Jac} \widetilde{u}(x) \big| \, dx = \int_{\mathbb{S}^1} \int_0^1 \big| \xi(r\omega) \big| \big| \operatorname{Jac} \widetilde{u}(r\omega) \big| r \, dr \, d\omega \le \int_{\mathbb{S}^1} \int_0^1 \big| \xi(r\omega) \big| \big| \operatorname{Jac} \widetilde{u}(r\omega) \big| \, dr \, d\omega \\ &\le \int_{\mathbb{S}^1} \sup_{0 \le r \le 1} \big| \xi(r\omega) \big| \int_0^1 \big| \operatorname{Jac} \widetilde{u}(r\omega) \big| \, dr \, d\omega \le \int_{\mathbb{S}^1} (M\zeta)(\omega) \int_0^1 \big| \operatorname{Jac} \widetilde{u}(r\omega) \big| \, dr \, d\omega =: \int_{\mathbb{S}^1} (M\zeta)(\omega) \varepsilon(\omega) \, d\omega.\end{aligned}$$

[11] Item d) requires a proof. In view of item 4, of the Poincaré inequality $\|u - \fint u\|_{W^{\sigma,q}} \lesssim |u|_{W^{\sigma,q}}$ (with $0 < \sigma < 1$ and $1 \le q \le \infty$) and of the fact that $T_2(1) = 0$, we find that $|T_2(\zeta)| = |T_2(\zeta - \fint \zeta)| \lesssim |\zeta|_{W^{1-s,p'}} |u|_{W^{s,p}} \lesssim \|\zeta\|_{W^{1-s,p'}} |u|_{W^{s,p}}$. This leads to item d).

[12] The choice of $\widetilde{u}$ will be specified later.

Here, $\varepsilon(\omega) := \int_0^1 |\operatorname{Jac} \widetilde{u}(r\omega)|\, dr$. Applying Hölder's inequality, we obtain

$$\frac{1}{2}\big|T_1(\zeta)\big| \le \|M\zeta\|_{L^{(sp)'}} \|\varepsilon\|_{L^{sp}} \lesssim \|\zeta\|_{L^{(sp)'}} \|\varepsilon\|_{L^{sp}}, \tag{5.25}$$

by the maximal function theorem.

We now specify $\widetilde{u}$. Let v be the harmonic extension of u to $\mathbb{D}$. Let Π be as in (5.16). Then we set

$$\widetilde{u} := \Pi \circ v. \tag{5.26}$$

The key estimate is

$$\|\varepsilon\|_{L^{sp}} \lesssim |u|_{W^{s,p}}^{1/s} \tag{5.27}$$

(see Lemma 8.27). This estimate, combined with (5.25), leads to $|T_1(\zeta)| \lesssim \|\zeta\|_{L^{(sp)'}} |u|_{W^{s,p}}^{1/s}$, i.e., item 3 holds. □

Proof of 4. We have

$$\big|T_2(\zeta)\big| \le \int_{\mathbb{D}} \big|\nabla\xi \wedge (\widetilde{u} \wedge \nabla\widetilde{u})\big|(x)\, dx \le \int_{\mathbb{D}} \big|\nabla\xi(x)\big|\big|\nabla\widetilde{u}(x)\big|\, dx = \int_{\mathbb{D}} \big(h(x)^{-1}\big|\nabla\xi(x)\big|\big)\big(h(x)\big|\nabla\widetilde{u}(x)\big|\big)\, dx,$$

where $h(x)$ will be specified afterwards. By Hölder's inequality we obtain

$$\big|T_2(\zeta)\big| \lesssim \left(\int_{\mathbb{D}} h(x)^{-p'}\big|\nabla\xi(x)\big|^{p'}\, dx\right)^{1/p'} \left(\int_{\mathbb{D}} h(x)^{p}\big|\nabla\widetilde{u}(x)\big|^{p}\, dx\right)^{1/p}. \tag{5.28}$$

In order to estimate the right-hand side of (5.28), we rely on Lemma 8.31, which implies that

$$\int_{\mathbb{D}} \big(1-|x|\big)^{p-sp-1}\big|\nabla\widetilde{u}(x)\big|^{p}\, dx \lesssim |u|_{W^{s,p}}^{p} \tag{5.29}$$

and

$$\int_{\mathbb{D}} \big(1-|x|\big)^{sp'-1}\big|\nabla\xi(x)\big|^{p'}\, dx \lesssim |\zeta|_{W^{1-s,p'}}^{p'}. \tag{5.30}$$

By combining (5.29), (5.30) and (5.28) (applied with $h(x) := (1-|x|)^{1-s-1/p}$), we obtain the desired estimate $|T_2(\zeta)| \lesssim |\zeta|_{W^{1-s,p'}} |u|_{W^{s,p}}$. □

Step 2. Proof of (5.13) in the general case.

We assume now that $u \in W^{s,p}(\mathbb{T};\mathbb{S}^1)$ and that $\varphi \in W^{s,p}((0,1);\mathbb{R})$ is a phase of u. In order to use the result from Step 1, we proceed as follows. By extending φ by reflection and 2-periodicity we obtain a function ψ which belongs to $W^{s,p}_{loc}(\mathbb{R};\mathbb{R})$ and is periodic. We define $w := e^{\imath\psi}$. We clearly have $|w|_{W^{s,p}} \approx |u|_{W^{s,p}}$. If ρ is a mollifier, then the maps $\psi_\varepsilon := \psi * \rho_\varepsilon$ and $w_\varepsilon := e^{\imath\psi_\varepsilon}$ are smooth and verify $\psi_\varepsilon \to \psi$ and $w_\varepsilon \to w$ in $W^{s,p}$, as $\varepsilon \to 0$.[13] By the previous step, we can write ψ_ε as the sum of two functions $\psi_{\varepsilon,1}$ and $\psi_{\varepsilon,2}$ in $W^{s,p}(\mathbb{T};\mathbb{R})$ that satisfy the estimates

$$|\psi_{\varepsilon,1}|_{W^{s,p}} \lesssim |w_\varepsilon|_{W^{s,p}}^{1/s} \quad \text{and} \quad |\psi_{\varepsilon,2}|_{W^{s,p}} \lesssim |w_\varepsilon|_{W^{s,p}}.$$

Since $|w_\varepsilon|_{W^{s,p}} \to |w|_{W^{s,p}}$, we can apply Fatou's lemma to find some convergent subsequences $\psi_{j,1} \to \psi_1$ and $\psi_{j,2} \to \psi_2$ in L^p such that

$$|\psi_1|_{W^{s,p}} \lesssim \liminf_j |\psi_{j,1}|_{W^{s,p}} \quad \text{and} \quad |\psi_2|_{W^{s,p}} \lesssim \liminf_j |\psi_{j,2}|_{W^{s,p}}.$$

[13] The convergence $w_\varepsilon \to w$ relies on the continuity of the map $W^{s,p}(\mathbb{T}^n;\mathbb{R}) \ni \psi \mapsto e^{\imath\psi} \in W^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$ when $0 < s < 1$ and $1 \le p < \infty$ [27, Theorem 1, Section 5.3.6].

We thus have $\psi = \psi_1 + \psi_2$. Consequently, we may write $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1 := \psi_{1|(0,1)} \in W^{s,p}(\mathbb{T};\mathbb{R})$ and $\varphi_2 := \psi_{2|(0,1)} \in W^{s,p}(\mathbb{T};\mathbb{R})$ satisfying the estimates

$$|\varphi_1|_{W^{s,p}} \lesssim |w|^{1/s}_{W^{s,p}} \approx |u|^{1/s}_{W^{s,p}} \quad \text{and} \quad |\varphi_2|_{W^{s,p}} \lesssim |w|_{W^{s,p}} \approx |u|_{W^{s,p}}. \qquad \square$$

We end this section by establishing the optimality of the estimates (5.12) and (5.13).

5.7. Proposition. *The estimates (5.12) (when $n = 1$) and (5.13) (when $n \geq 2$) are optimal in the sense that* $\limsup_{|\varphi|_{W^{s,p}} \to 0} \frac{|\varphi|_{W^{s,p}}}{|u|_{W^{s,p}}} > 0$ *and* $\limsup_{|u|_{W^{s,p}} \to \infty} \frac{|\varphi|_{W^{s,p}}}{|u|^{1/s}_{W^{s,p}}} > 0$.

Proof. When $n = 1$, the optimality of (5.12) "at ∞" was obtained by Merlet [20, Theorem 1.1]. We reproduce here its argument. Let $f \in C^\infty_c((0,1);[0,1))$ be such that $f \not\equiv 0$. Define, for $j \geq 1$, $\varphi_j := jf$ and $u_j := e^{\imath\varphi_j}$. Clearly, we have

$$|\varphi_j|_{W^{s,p}} = j|f|_{W^{s,p}} \approx j. \tag{5.31}$$

In computing $|u_j|_{W^{s,p}}$, we use the estimates

$$\big|u_j(x) - u_j(y)\big| \approx j\big|f(x) - f(y)\big| \quad \text{when } |x-y| < 1/j,$$

$$\int_0^{1-h} \big|f(x+h) - f(x)\big|^a \, dx \approx |h|^a, \quad \forall h \in (0,1/2) \text{ (with } a \in \mathbb{R} \text{ fixed)},$$

and

$$\big|u_j(x) - u_j(y)\big| \lesssim 1 \quad \text{when } |x-y| \geq 1/j.$$

Thus we have

$$\begin{aligned} j^{sp} &\approx j^p \iint\limits_{|x-y|<1/j} \frac{dx\,dy}{|x-y|^{1+(s-1)p}} \lesssim |u_j|^p_{W^{s,p}} \\ &\lesssim j^p \iint\limits_{|x-y|<1/j} \frac{dx\,dy}{|x-y|^{1+(s-1)p}} + \iint\limits_{|x-y|>1/j} \frac{dx\,dy}{|x-y|^{1+sp}} \approx j^{sp}. \end{aligned} \tag{5.32}$$

In particular, we have $|u_j|^p_{W^{s,p}} \to \infty$ when $j \to \infty$. Moreover, (5.32) together with (5.31) yield $|u_j|^{1/s}_{W^{s,p}} \approx |\varphi_j|_{W^{s,p}}$.

The above example extends to higher dimension as in the Step 3 of the proof of Theorem 5.1.

The optimality "at zero" is obvious since $|e^{\imath\varphi}|_{W^{s,p}} \leq |\varphi|_{W^{s,p}}$ for any $\varphi \in W^{s,p}(\mathbb{T}^n;\mathbb{R})$. $\square$

6. Further thoughts when $sp < 1$

6.1. Existence of bounded phases and the sum-intersection property

We address here the following question.[14]

Question (Q). Let $0 < s < 1$, $1 \leq p < \infty$ be such that $sp < 1$. Let $u \in W^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$. Is there some $\varphi \in W^{s,p} \cap L^\infty(\mathbb{T}^n;\mathbb{R})$ such that $u = e^{\imath\varphi}$?

The motivation behind this question is the following. The phase φ whose construction is described in the introduction depends only on u, not on s or p. This has the following consequence. Let $0 \leq \theta < 1$, $0 < s < 1$ and $1 \leq p < \infty$ be such that $sp < 1$. Let $u \in W^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$. Then u belongs to all the spaces $W^{\theta s,p/\theta}(\mathbb{T}^n;\mathbb{S}^1)$ (by the Gagliardo–Nirenberg embeddings) and thus $\varphi \in W^{\theta s,p/\theta}$, $\forall \theta \in (0,1]$. We find that

[14] Also discussed in [7].

$$\varphi \in \bigcap_{0<\theta\le 1} W^{\theta s, p/\theta} \subset W^{s,p} \cap \bigcap_{q<\infty} L^q.$$

It is then natural to ask whether the above conclusion can be improved to $\varphi \in W^{s,p} \cap L^\infty$.

We start by noting that the answer to (Q) is positive when $p = 1$. Indeed, an inspection of the proof of Proposition 1.6 shows that the phase constructed there is bounded.

We next turn to the relevant range $0 < s < 1$, $1 < p < \infty$, $sp < 1$. Our main result here is the reduction of (Q) to a sum-intersection property of function spaces. In order to describe this property, we start with a very simple case which requires no technology. If $f \in L^2$, then $f \in L^1 + L^\infty$ (since $[L^1, L^\infty]_{1/2} = L^2$) and thus $L^2 \subset L^1 + L^\infty$. Thus each map $f \in L^2$ splits as $f = f_1 + f_2$, with $f_1 \in L^1$ and $f_2 \in L^\infty$. But more can be said. Indeed, we have $f = f_1 + f_2$, with $f_1 := f\mathbb{1}_{\{|f|>1\}} \in L^2 \cap L^1$ and $f_2 := f\mathbb{1}_{\{|f|\le 1\}} \in L^2 \cap L^\infty$. Thus $L^2 = (L^2 \cap L^1) + (L^2 \cap L^\infty)$. This is the sum-intersection property for the triple (L^2, L^1, L^∞). This property extends to other function spaces. Here is an example [21]. If $\sigma > 1$ is not an integer and $p > \sigma$, then

$$W^{1,\sigma} = \left(W^{1,\sigma} \cap W^{\sigma/p,p}\right) + \left(W^{1,\sigma} \cap W^{\sigma,1}\right).$$

For an investigation of this property in usual function spaces, see the forthcoming work [18].

We are now ready to reformulate (Q).

6.1. Proposition. (Q) *holds if and only if* (R) *holds, where* (R) *is the property*

(R). $W^{s,p}(\mathbb{T}^n;\mathbb{R}) = (W^{s,p} \cap L^\infty)(\mathbb{T}^n;\mathbb{R}) + (W^{s,p} \cap W^{sp,1})(\mathbb{T}^n;\mathbb{R})$.

Proof. We may assume that $p > 1$, since both (Q) and (R) hold when $p = 1$.

Implication "(Q) ⇒ (R)". Let $\varphi \in W^{s,p}(\mathbb{T}^n;\mathbb{R})$. Let $u := e^{\imath\varphi}$. Consider some $\psi \in W^{s,p} \cap L^\infty(\mathbb{T}^n;\mathbb{R})$ such that $u = e^{\imath\psi}$. Then $\varphi = \psi + 2\pi f$, where $f := (\varphi - \psi)/2\pi \in W^{s,p}(\mathbb{T}^n;\mathbb{Z})$. We leave to the reader the following straightforward inequality. If $0 < s < 1$, $1 \le p < \infty$ and if $f \in W^{s,p}$ is integer-valued, then

$$|f|_{W^{sp,1}} \le |f|^p_{W^{s,p}}. \tag{6.1}$$

Using (6.1), we obtain that $\varphi = \psi + 2\pi f$, with $\psi \in W^{s,p} \cap L^\infty$ and $2\pi f \in \cap W^{s,p} \cap W^{sp,1}$. Therefore, (R) holds.

Implication "(R) ⇒ (Q)". Let $u \in W^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$. Let $\varphi \in W^{s,p}(\mathbb{T}^n;\mathbb{R})$ be such that $u = e^{\imath\varphi}$. Write $\varphi = \varphi_1 + \varphi_2$, with $\varphi_1 \in W^{s,p} \cap L^\infty$ and $\varphi_2 \in W^{s,p} \cap W^{sp,1}$. Set $v := e^{\imath\varphi_2} \in W^{sp,1}$. Then $v = e^{\imath\varphi_3}$ for some $\varphi_3 \in W^{sp,1} \cap L^\infty$ (by the proof of Proposition 1.6). By the Gagliardo–Nirenberg embeddings, we have $\varphi_3 \in W^{s,p} \cap L^\infty$. Thus $u = e^{\imath\psi}$, where $\psi := \varphi_1 + \varphi_3 \in W^{s,p} \cap L^\infty$. □

We do not know whether (R) holds. It is easy to see that a weaker form of (R), where L^∞ is replaced by the slightly larger Besov space $B^0_{\infty,\infty}$, is valid:

$$W^{s,p} = \left(W^{s,p} \cap B^0_{\infty,\infty}\right) + \left(W^{s,p} \cap W^{sp,1}\right)$$

(see Lemma 8.32).

6.2. Lifting via the factorization method

In this section, we propose a new lifting construction in the case where $sp < 1$. Our method relies on three ingredients:

1. The factorization method.[15]
2. The averaging method of Dávila and Ignat [12].[16]
3. The theory of weighted Sobolev spaces, due among others to Uspenskiĭ [30].[17]

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[15] Explained in Section 5, and used in the proof of Theorem 5.4.

[16] Which proved useful in Section 3, in the proof of Proposition 1.6.

[17] For the results we use here, see also [19, Section 10.1.1, Theorem 1, p. 512] and the comprehensive discussion in [24].

Let us explain the construction. Let $u : \mathbb{T}^n \to \mathbb{S}^1$. We first extend u to $\mathbb{R}^n$ as explained in the proof of Theorem 5.4, and define φ_1 as in (5.15). Recall that $\varphi_1 \in W^{s,p}$ (Lemma 5.5). The key is the following substitute of Lemma 5.6.

6.2. Lemma. *Let $1 \le p < \infty$ and $0 < s < 1$ be such that $sp < 1$. Let $u \in W^{s,p}(\mathbb{T}^n; \mathbb{S}^1)$ and let φ_1 be as in (5.15). Then we have $ue^{-\imath\varphi_1} \in W^{sp,1}(\mathbb{T}^n; \mathbb{S}^1)$.*

Assuming Lemma 6.2 proved for the moment, we complete the construction of a phase of u as follows. Set $v := ue^{-\imath\varphi_1}$. Since the map v belongs to $W^{sp,1}$, we find that v has a phase φ_2 in the space $W^{sp,1} \cap L^\infty$ (by the proof of Proposition 1.6). The Gagliardo–Nirenberg embeddings and the fact that φ_2 belongs to $W^{sp,1} \cap L^\infty$ imply that we also have $\varphi_2 \in W^{s,p}$. In conclusion, $\varphi := \varphi_1 + \varphi_2$ is a $W^{s,p}$ phase of u.

It remains to proceed to the

Proof of Lemma 6.2. A first ingredient of the proof is the following flat version of [6, Lemma 1.3].[18] Let w be given by (5.14). For $x \in \mathbb{R}^n$, set

$$\lambda(x) := \inf\{\varepsilon > 0; \left|w(x,\varepsilon)\right| = 1/2\}. \tag{6.2}$$

Then λ satisfies

$$\int\limits_{(-2,3)^n} \frac{1}{\lambda^{sp}(x)}\, dx \lesssim |u|^p_{W^{s,p}} + 1. \tag{6.3}$$

Estimate (6.3) is established in [21]. Alternatively, (6.3) can be obtained by adapting the proof of Lemma 8.28.

A second ingredient is provided by the following local estimate in the spirit of the theory of weighted Sobolev spaces.

6.3. Lemma. *Let $0 < \sigma < 1$. Let $U : \mathbb{T}^n \times (0,\infty) \to \mathbb{C}$ be a smooth map. Assume that*

$$f(x) := \lim_{\varepsilon \to 0} U(x,\varepsilon) \quad \textit{exists for a.e. } x \in \mathbb{T}^n. \tag{6.4}$$

Then

$$|f|_{W^{\sigma,1}(\mathbb{T}^n)} \lesssim \int\limits_{\mathbb{T}^n \times (0,1)} \varepsilon^{-\sigma} \left|\nabla U(x,\varepsilon)\right| dx\, d\varepsilon. \tag{6.5}$$

The proof of Lemma 6.3 is postponed to Section 8.6.

We will apply Lemma 6.3 with $\sigma := sp$ and $U(x,\varepsilon) := \Pi \circ w(x,\varepsilon) e^{-\imath\psi(x,\varepsilon)}$. Here, w is as in (5.14), Π satisfies (5.16), and we set

$$\psi(x,\varepsilon) := -\int\limits_{\varepsilon}^{\infty} \Pi \circ w(x,t) \wedge \frac{\partial}{\partial t} \Pi \circ w(x,t)\, dt, \quad \forall x \in \mathbb{R}^n,\ \forall \varepsilon > 0. \tag{6.6}$$

We now explain how these ingredients lead to the conclusion of Lemma 6.2.

Step 1. U is smooth and (6.4) holds with $f := ue^{-\imath\varphi_1}$.

Indeed, since u equals a constant C in the set $\mathbb{R}^n \setminus (-4,5)^n$, we have

$$w(x,\varepsilon) = C + \frac{1}{\varepsilon^n} \int\limits_{\mathbb{R}^n} \rho\left(\frac{x-y}{\varepsilon}\right)\left[u(y) - C\right] dy = C + \frac{1}{\varepsilon^n} \int\limits_{(-4,5)^n} \rho\left(\frac{x-y}{\varepsilon}\right)\left[u(y) - C\right] dy. \tag{6.7}$$

On the other hand, a straightforward induction on $|\alpha|$ leads to

[18] For a related result, see Lemma 8.28.

$$\partial^\alpha\left(\frac{1}{\varepsilon^n}\rho\left(\frac{x-y}{\varepsilon}\right)\right)=O\left(\frac{1}{\varepsilon^{n+|\alpha|}}\right),\quad \forall\alpha\in\mathbb{N}^{n+1}. \tag{6.8}$$

By combining (6.7) with (6.8) and with the fact that u is bounded, we find that

$$\partial^\alpha w(x,\varepsilon)=\int_{(-4,5)^n}\partial^\alpha\left(\frac{1}{\varepsilon^n}\rho\left(\frac{x-y}{\varepsilon}\right)\right)\big[u(y)-C\big]\,dy=O\left(\frac{1}{\varepsilon^{n+|\alpha|}}\right),\quad \forall\alpha\in\mathbb{N}^{n+1}\setminus\{0\}. \tag{6.9}$$

In view of (6.9), we obtain by induction on $|\alpha|\ge 1$ that

$$\partial^\alpha(\Pi\circ w)(x,\varepsilon)=O\left(\frac{1}{\varepsilon^{n+|\alpha|}}\right)+O\left(\frac{1}{\varepsilon^{|\alpha|n+|\alpha|}}\right),\quad \forall\alpha\in\mathbb{N}^{n+1}\setminus\{0\}. \tag{6.10}$$

This shows that ψ defined by (6.6) is smooth, and thus so is U. For further use, we also note that all derivatives of ψ are obtained by differentiating under the integral sign.

On the other hand, by Lebesgue's differentiation theorem we have $\lim_{\varepsilon\to 0}w(x,\varepsilon)=u(x)$ for a.e. $x\in\mathbb{R}^n$. In addition, Lemma 5.5 *1* implies that $\lim_{\varepsilon\to 0}\psi(x,\varepsilon)=\varphi_1(x)$ for a.e. $x\in\mathbb{T}^n$. We find that $\lim_{\varepsilon\to 0}U(x,\varepsilon)=u(x)e^{-\imath\varphi_1(x)}$ for a.e. $x\in\mathbb{T}^n$.

Step 2. Basic estimates.

Let us note the fact that the inequality $|u|\le 2$ implies that, in addition to (6.9), we have

$$\left|\partial^\alpha w(x,\varepsilon)\right|\lesssim\frac{1}{\varepsilon^{|\alpha|}},\quad \forall\alpha\in\mathbb{N}^{n+1}. \tag{6.11}$$

In turn, (6.11) and formulas (5.14) and (5.16) lead, by induction on $|\alpha|$, to

$$\left|\partial^\alpha(\Pi\circ w)(x,\varepsilon)\right|\lesssim\frac{1}{\varepsilon^{|\alpha|}},\quad \forall\alpha\in\mathbb{N}^{n+1}. \tag{6.12}$$

Finally, (6.12) combined with the definition (6.6) of ψ leads to

$$\left|\partial^\alpha U(x,\varepsilon)\right|\lesssim\frac{1}{\varepsilon^{|\alpha|}},\quad \forall\alpha\in\mathbb{N}^{n+1}. \tag{6.13}$$

Step 3. The role of $\lambda(x)$.

Let $\lambda(x)$ be as in (6.2). In this step, we establish several identities valid at a point (x,ε) with $\varepsilon<\lambda(x)$.

To start with, it follows from the definition (6.2) of $\lambda(x)$ and from (5.16) that

$$\left|\Pi\circ w(x,\varepsilon)\right|\equiv 1\quad\text{in the open set } \mathscr{V}:=\big\{(x,\varepsilon)\in\mathbb{R}^n\times(0,\infty);\,0<\varepsilon<\lambda(x)\big\}. \tag{6.14}$$

By differentiating the identity (6.14), we find that

$$\nabla(\Pi\circ w)(x,\varepsilon)\perp\Pi\circ w(x,\varepsilon),\quad \forall(x,\varepsilon)\in\mathscr{V}. \tag{6.15}$$

By combining (6.15) with the identity

$$y=\imath\omega(\omega\wedge y),\quad \forall y\in\mathbb{C},\ \forall\omega\in\mathbb{S}^1\text{ such that } y\perp\omega,$$

we find that

$$\nabla(\Pi\circ w)(x,\varepsilon)=\imath\big(\Pi\circ w(x,\varepsilon)\big)\big[\Pi\circ w(x,\varepsilon))\wedge\big(\nabla(\Pi\circ w)(x,\varepsilon)\big)\big]\quad\text{in }\mathscr{V}. \tag{6.16}$$

On the other hand, (6.15) implies that in $\mathscr{V}$ the partial derivatives of $\Pi\circ w$ are mutually parallel. This leads to

$$\left(\frac{\partial}{\partial\varepsilon}(\Pi\circ w)\right)\wedge\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)(x,\varepsilon)=0\quad\text{in }\mathscr{V},\ \forall j\in[\![1,n]\!]. \tag{6.17}$$

We are now in position to compute ∇U in $\mathscr{V}$.

First, using (6.6) and (6.16) we find that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}U(x,\varepsilon)&=\frac{\partial}{\partial\varepsilon}(\Pi\circ w)(x,\varepsilon)e^{-\imath\psi(x,\varepsilon)}-\imath\frac{\partial}{\partial\varepsilon}\psi(x,\varepsilon)\Pi\circ w(x,\varepsilon)e^{-\imath\psi(x,\varepsilon)}\\&=\left(\frac{\partial}{\partial\varepsilon}(\Pi\circ w)(x,\varepsilon)-\imath\Pi\circ w(x,\varepsilon)\left[(\Pi\circ w)\wedge\left(\frac{\partial}{\partial\varepsilon}(\Pi\circ w)\right)\right](x,\varepsilon)\right)e^{-\imath\psi(x,\varepsilon)}\\&=0\quad\text{in }\mathscr{V}.\end{aligned}\tag{6.18}$$

We next note that an integration by parts combined with (6.6) and (6.17) leads to

$$\begin{aligned}\frac{\partial}{\partial x_j}\psi(x,\varepsilon)&=-\int_\varepsilon^\infty(\Pi\circ w)\wedge\left(\frac{\partial^2}{\partial t\partial x_j}(\Pi\circ w)\right)(x,t)\,dt\\&\quad-\int_\varepsilon^\infty\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)\wedge\left(\frac{\partial}{\partial t}(\Pi\circ w)\right)(x,t)\,dt\\&=(\Pi\circ w)\wedge\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)(x,\varepsilon)-2\int_\varepsilon^\infty\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)\wedge\left(\frac{\partial}{\partial t}(\Pi\circ w)\right)(x,t)\,dt\\&=(\Pi\circ w)\wedge\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)(x,\varepsilon)\\&\quad-2\int_{\lambda(x)}^\infty\left(\frac{\partial}{\partial x_j}(\Pi\circ w)\right)\wedge\left(\frac{\partial}{\partial t}(\Pi\circ w)\right)(x,t)\,dt\quad\text{in }\mathscr{V}.\end{aligned}\tag{6.19}$$

A calculation similar to the one leading to (6.18) yields (using (6.19))

$$\nabla_x U(x,\varepsilon)=2\imath U(x,\varepsilon)\int_{\lambda(x)}^\infty\left(\nabla_x(\Pi\circ w)\right)\wedge\left(\frac{\partial}{\partial t}(\Pi\circ w)\right)(x,t)\,dt.\tag{6.20}$$

Step 4. Estimate of $\partial U/\partial\varepsilon$.

By combining (6.13) with (6.18), we find that

$$\int_{\mathbb{T}^n\times(0,\infty)}\varepsilon^{-sp}\left|\frac{\partial}{\partial\varepsilon}U(x,\varepsilon)\right|dx\,d\varepsilon\lesssim\int_{\mathbb{T}^n}\int_{\lambda(x)}^\infty\varepsilon^{-sp-1}\,d\varepsilon\,dx\lesssim\int_{\mathbb{T}^n}\frac{1}{\lambda(x)^{sp}}\,dx.\tag{6.21}$$

Step 5. Estimate of $\nabla_x U$.

This time, (6.13) combined with (6.20) and with the fact that $sp<1$ leads to

$$\begin{aligned}\int_{\mathbb{T}^n\times(0,\infty)}\varepsilon^{-sp}\left|\nabla_x U(x,\varepsilon)\right|dx\,d\varepsilon&\lesssim\int_{\mathbb{T}^n}\int_{\lambda(x)}^\infty\varepsilon^{-sp-1}\,d\varepsilon\,dx+\int_{\mathbb{T}^n}\left(\int_0^{\lambda(x)}\varepsilon^{-sp}\,d\varepsilon\right)\left(\int_{\lambda(x)}^\infty\frac{1}{t^2}\,dt\right)dx\\&\lesssim\int_{\mathbb{T}^n}\frac{1}{\lambda(x)^{sp}}\,dx.\end{aligned}\tag{6.22}$$

Step 6. Final conclusion.

By combining Step 1 and Lemma 6.3 with estimates (6.3), (6.21) and (6.22), we find that $ue^{-\imath\varphi^1}\in W^{sp,1}(\mathbb{T}^n)$, which is the conclusion of Lemma 6.2. □

7. Another application of the averaging method. Proof of Theorem 1.4

In this section, we prove a quantitative version of Theorem 1.4. For the convenience of the reader, we start with the case $n = 1$, which is easier to follow. In this case, the main ingredient is Proposition 7.1. Once this proposition is obtained, the one-dimensional case follows easily; see the proof of Theorem 7.2.

Our first result in this section is a sort of averaged "discrete" semi-norm estimate.

7.1. Proposition. *Let* $0 < s < 1$, $1 \le p < \infty$ *and* $f \in W^{s,p}(\mathbb{T})$. *Then*

$$\sum_{j\ge 1} 2^{spj} \left\| (\tau_{2^{-j}} - \mathrm{id}) f \right\|_{L^p(\mathbb{T})}^p \lesssim \left[\frac{1}{s} \left(\frac{C_p}{1-s} \right)^{1/(1-s)} \right]^p \int_{\mathbb{T}} Y\left(f^y\right) dy.$$

Proof. By Lebesgue's differentiation theorem we have, for a.e. $x \in \mathbb{T}$,

$$f(x) = \lim_{k\to\infty} \fint_x^{x+2^{-k}} f(z)\, dz = \sum_{k\ge 1} \left(\fint_x^{x+2^{-k}} f - \fint_x^{x+2^{1-k}} f \right) + \int_{\mathbb{T}} f =: \sum_{k\ge 1} \delta_{2^{-k}} f(x) + \int_{\mathbb{T}} f. \tag{7.1}$$

Here, $\delta_\varepsilon f(x) := \fint_x^{x+\varepsilon} f(z)\, dz - \fint_x^{x+2\varepsilon} f(z)\, dz$.

Let $j \ge 1$. Applying the operator $\tau_{2^{-j}} - \mathrm{id}$ to the identity (7.1), we obtain

$$(\tau_{2^{-j}} - \mathrm{id}) f(x) = (\tau_{2^{-j}} - \mathrm{id}) \left(\sum_{k\ge 1} \delta_{2^{-k}} f(x) \right), \quad \text{for a.e. } x \in \mathbb{T}.$$

By Minkowski's inequality and the above estimate, we obtain that

$$\left\| (\tau_{2^{-j}} - \mathrm{id}) f \right\|_{L^p} \le \sum_{k\ge 1} \left\| (\tau_{2^{-j}} - \mathrm{id}) \delta_{2^{-k}} f \right\|_{L^p}. \tag{7.2}$$

We split the sum in (7.2) as $\sum_{k\ge 1} \cdots = \sum_{1\le k\le j-\ell} \cdots + \sum_{k\ge j-\ell+1} \cdots =: S_1 + S_2$ (with ℓ integer to be determined later). On the one hand, we estimate S_1 via Lemma 8.10. On the other hand, we estimate S_2 using the trivial inequality

$$\left\| (\tau_h - \mathrm{id}) g \right\|_{L^p} \le 2 \|g\|_{L^p}. \tag{7.3}$$

By combining (7.2), Lemma 8.10 and (7.3), we obtain

$$\left\| (\tau_{2^{-j}} - \mathrm{id}) f \right\|_{L^p} \lesssim \sum_{1\le k\le j-\ell} 2^{k-j} \left\| (\tau_{2^{-k}} - \mathrm{id}) f \right\|_{L^p} + \sum_{k\ge j-\ell+1} \|\delta_{2^{-k}} f\|_{L^p}.$$

Hence for every $j \ge 1$ we have

$$2^{sj} \left\| (\tau_{2^{-j}} - \mathrm{id}) f \right\|_{L^p} \lesssim \sum_{1\le k\le j-\ell} 2^{k-(1-s)j} \left\| (\tau_{2^{-k}} - \mathrm{id}) f \right\|_{L^p} + \sum_{k\ge j-\ell+1} 2^{sj} \|\delta_{2^{-k}} f\|_{L^p}. \tag{7.4}$$

Raising the inequalities in (7.4) to the power p and summing over j we find

$$\sum_{j\ge 1} 2^{spj} \left\| (\tau_{2^{-j}} - \mathrm{id}) f \right\|_{L^p}^p \lesssim \sum_{j\ge 1} \left(\sum_{1\le k\le j-\ell} 2^{k-(1-s)j} \left\| (\tau_{2^{-k}} - \mathrm{id}) f \right\|_{L^p} \right)^p + \sum_{j\ge 1} \left(\sum_{k\ge j-\ell+1} 2^{sj} \|\delta_{2^{-k}} f\|_{L^p} \right)^p. \tag{7.5}$$

In what follows we use the notation $X_j := 2^{sj} \|(\tau_{2^{-j}} - \mathrm{id}) f\|_{L^p}$ and $Y_k := 2^{sk} \|\delta_{2^{-k}} f\|_{L^p}$. In terms of X_j and Y_k, (7.5) reads

$$\sum_{j\ge 1} X_j^p \lesssim \sum_{j\ge 1} \left(\sum_{1\le k\le j-\ell} 2^{-(1-s)(j-k)} X_k \right)^p + \sum_{j\ge 1} \left(\sum_{k\ge j-\ell+1} 2^{s(j-k)} Y_k \right)^p. \tag{7.6}$$

In order to estimate the sums in the right-hand side of (7.6), we apply Corollary 8.2. By combining (7.6) with Corollary 8.2, we obtain

$$\sum_{j\geq 1} X_j^p \leq C_1\left(\frac{2^{-(1-s)\ell}}{1-s}\right)^p \sum_{k\geq 1} X_k^p + C_2\left(\frac{2^{s\ell}}{s}\right)^p Y_k^p.$$

Hence

$$\left[1-C_1\left(\frac{2^{-(1-s)\ell}}{1-s}\right)^p\right]\sum_{j\geq 1} X_j^p \lesssim \left(\frac{2^{s\ell}}{s}\right)^p \sum_{k\geq 1} Y_k^p. \tag{7.7}$$

We may choose a fixed real M and an integer $\ell = \ell(s,p)$ such that

$$\ell = -\frac{\log_2(1-s)}{(1-s)} + \frac{M+o(1)}{1-s} \quad \text{and} \quad 1-C_1\left(\frac{2^{-(1-s)\ell}}{1-s}\right)^p = \frac{1}{2}+o(1) \quad \text{as } s \nearrow 1. \tag{7.8}$$

With this choice of ℓ, (7.7) and (7.8) lead to

$$\begin{aligned}\sum_{j\geq 1} 2^{spj}\left\|(\tau_{2^{-j}}-\mathrm{id})f\right\|_{L^p}^p &\leq \left(\frac{1}{2}+o(1)\right)\left(\frac{1}{s(1-s)^s}2^{M+o(1)}\right)^{p/(1-s)} \sum_{k\geq 1} 2^{spk}\|\delta_{2^{-k}} f\|_{L^p}^p \\ &\leq \left(\frac{1}{2}+o(1)\right)\left(\frac{1}{s(1-s)}2^{M+o(1)}\right)^{p/(1-s)} \sum_{k\geq 1} 2^{spk}\|\delta_{2^{-k}} f\|_{L^p}^p \\ &\leq K_p\left[\frac{1}{s}\left(\frac{C_p}{1-s}\right)^{1/(1-s)}\right]^p \sum_{k\geq 1} 2^{spk}\|\delta_{2^{-k}} f\|_{L^p}^p. \end{aligned} \tag{7.9}$$

By Lemma 8.7, we have

$$\|\delta_{2^{-k}} f\|_{L^p}^p \leq 2\int_{\mathbb{T}} \left\|\left(f^y\right)_k - \left(f^y\right)_{k-1}\right\|_{L^p}^p dy. \tag{7.10}$$

We complete the proof of Proposition 7.1 by combining (7.9) with (7.10). □

We now state and prove a quantitative form of Theorem 1.4 with $n=1$.

7.2. Theorem. *Let* $0<s<1$, $1\leq p<\infty$ *and* $f\in W^{s,p}(\mathbb{T})$. *Then*

$$X(f) \lesssim \left[\frac{1}{s^2}\left(\frac{C_p}{1-s}\right)^{1/(1-s)}\right]^p \int_{\mathbb{T}} Y\left(f^y\right) dy.$$

Proof. We first note that

$$\begin{aligned} X(f) &= \int_{\mathbb{T}} \frac{\|(\tau_h-\mathrm{id})f\|_{L^p}^p}{h^{1+sp}}\,dh = \sum_{j\geq 1}\int_{2^{-j}}^{2^{1-j}} \frac{\|(\tau_h-\mathrm{id})f\|_{L^p}^p}{h^{1+sp}}\,dh \\ &\leq \sum_{j\geq 1} 2^{j(1+sp)} \int_{2^{-j}}^{2^{1-j}} \left\|(\tau_h-\mathrm{id})f\right\|_{L^p}^p dh. \end{aligned} \tag{7.11}$$

Let $j\geq 1$. For every $h\in[1/2^j, 1/2^{j-1})$ and $k\geq j$, we denote by $\varepsilon_k(h)\in\{0,1\}$ the kth binary digit of h; thus

$$h = \sum_{k\geq j}\frac{\varepsilon_k(h)}{2^k} = \sum_{\substack{k\geq j\\ \varepsilon_k(h)=1}} \frac{1}{2^k}. \tag{7.12}$$

We also note that

$$[0,1)\ni h\mapsto \left\|(\tau_h-\mathrm{id})f\right\|_{L^p}\quad \text{is subadditive.}^{19} \tag{7.13}$$

From (7.12) and (7.13), we obtain that

$$\left\|(\tau_h-\mathrm{id})f\right\|_{L^p}\le \sum_{\substack{k\ge j\\ \varepsilon_k(h)=1}}\left\|(\tau_{2^{-k}}-\mathrm{id})f\right\|_{L^p}\le \sum_{k\ge j}\left\|(\tau_{2^{-k}}-\mathrm{id})f\right\|_{L^p},\quad \forall h\in\left[2^{-j},2^{1-j}\right). \tag{7.14}$$

Inserting (7.14) into (7.11), we find that

$$\begin{aligned}X(f)&\le \sum_{j\ge 1}2^{j(1+sp)}\int_{2^{-j}}^{2^{1-j}}\left(\sum_{k\ge j}\left\|(\tau_{2^{-k}}-\mathrm{id})f\right\|_{L^p}\right)^p dh\\ &=\sum_{j\ge 1}2^{spj}\left(\sum_{k\ge j}\left\|(\tau_{2^{-k}}-\mathrm{id})f\right\|_{L^p}\right)^p=\sum_{j\ge 1}\left[\sum_{k\ge j}2^{s(j-k)}\left(2^{sk}\left\|(\tau_{2^{-k}}-\mathrm{id})f\right\|_{L^p}\right)\right]^p.\end{aligned} \tag{7.15}$$

If we estimate the last sum in (7.15) via Corollary 8.2, we find that

$$X(f)\lesssim \frac{1}{s^p}\sum_{k\ge 1}2^{spk}\left\|(\tau_{2^{-k}}-\mathrm{id})f\right\|_{L^p}^p. \tag{7.16}$$

We complete the proof of Theorem 7.2 by combining (7.16) with Proposition 7.1. □

We now consider the case of an arbitrary n.

We start by adapting Proposition 7.1.

7.3. Proposition. *Let $0<s<1$, $1\le p<\infty$ and $f\in W^{s,p}(\mathbb{T}^n)$. Let $\{e_i\}_{i=1}^n$ be the canonical basis of $\mathbb{R}^n$. Then, for every $i\in [\![1,n]\!]$,*

$$\sum_{j\ge 1}2^{spj}\left\|(\tau_{2^{-j}e_i}-\mathrm{id})f\right\|_{L^p(\mathbb{T}^n)}^p\lesssim \left[\frac{1}{s}\left(\frac{C_p}{1-s}\right)^{1/(1-s)}\right]^p\int_{\mathbb{T}^n}Y\left(f^y\right)dy.$$

Proof. We start by noting that, for a.e. $x\in\mathbb{T}$,

$$f(x)=\lim_{k\to\infty}\fint_{x+(0,2^{-k})^n}f(z)\,dz=\sum_{k\ge 1}\left(\fint_{x+(0,2^{-k})^n}f-\fint_{x+(0,2^{1-k})^n}f\right)+\int_{\mathbb{T}}f=:\sum_{k\ge 1}\delta_{2^{-k}}f(x)+\int_{\mathbb{T}}f. \tag{7.17}$$

Here, $\delta_\varepsilon f(x):=\fint_{x+(0,\varepsilon)^n}f(z)\,dz-\fint_{x+(0,2\varepsilon)^n}f(z)\,dz$.

Let $j\ge 1$ and $i\in[\![1,n]\!]$. Applying the operator $\tau_{2^{-j}e_i}-\mathrm{id}$ to the identity (7.17), and then Minkowski's inequality, we obtain

$$\left\|(\tau_{2^{-j}e_i}-\mathrm{id})f\right\|_{L^p(\mathbb{T}^n)}\le\sum_{k\ge 1}\left\|(\tau_{2^{-j}e_i}-\mathrm{id})\delta_{2^{-k}}f\right\|_{L^p(\mathbb{T}^n)}. \tag{7.18}$$

As in the proof of Proposition 7.1, we split the sum in (7.18) as $\sum_{k\ge 1}\cdots=\sum_{1\le k\le j-\ell}\cdots+\sum_{k\ge j-\ell+1}\cdots=:S_1+S_2$, with ℓ an integer to be determined. We estimate S_1 via Lemma 8.11, and S_2 using the trivial inequality

$$\left\|(\tau_{he_i}-\mathrm{id})g\right\|_{L^p(\mathbb{T}^n)}\le 2\|g\|_{L^p(\mathbb{T}^n)}. \tag{7.19}$$

Therefore, by combining (7.18), Lemma 8.11 and (7.19), we obtain

[19] This follows from $|(\tau_{h+l}-\mathrm{id})f|\le|\tau_h(\tau_l-\mathrm{id})f|+|(\tau_h-\mathrm{id})f|$ together with Minkowski's inequality.

$$\|(\tau_{2^{-j}e_i}-\mathrm{id})f\|_{L^p(\mathbb{T}^n)}\lesssim\sum_{1\le k\le j-\ell}2^{k-j}\|(\tau_{2^{-k}e_i}-\mathrm{id})f\|_{L^p(\mathbb{T}^n)}+\sum_{k\ge j-\ell+1}\|\delta_{2^{-k}}f\|_{L^p(\mathbb{T}^n)}. \tag{7.20}$$

As in the proof of (7.5), this leads to

$$\sum_{j\ge1}2^{spj}\|(\tau_{2^{-j}e_i}-\mathrm{id})f\|^p_{L^p(\mathbb{T}^n)}\lesssim\sum_{j\ge1}\Bigl(\sum_{1\le k\le j-\ell}2^{k-(1-s)j}\|(\tau_{2^{-k}e_i}-\mathrm{id})f\|_{L^p(\mathbb{T}^n)}\Bigr)^p+\sum_{j\ge1}\Bigl(\sum_{k\ge j-\ell+1}2^{sj}\|\delta_{2^{-k}}f\|_{L^p(\mathbb{T}^n)}\Bigr)^p. \tag{7.21}$$

Using the notation $X^i_j:=2^{sj}\|(\tau_{2^{-j}e_i}-\mathrm{id})f\|_{L^p(\mathbb{T}^n)}$ and $Y_k:=2^{sk}\|\delta_{2^{-k}}f\|_{L^p(\mathbb{T}^n)}$, (7.21) reads

$$\sum_{j\ge1}(X^i_j)^p\lesssim\sum_{j\ge1}\Bigl(\sum_{1\le k\le j-\ell}2^{-(1-s)(j-k)}X^i_k\Bigr)^p+\sum_{j\ge1}\Bigl(\sum_{k\ge j-\ell+1}2^{s(j-k)}Y_k\Bigr)^p. \tag{7.22}$$

As in the proof of (7.9), Corollary 8.2 combined with (7.22) leads, for an appropriate choice of ℓ, to

$$\sum_{j\ge1}2^{spj}\|(\tau_{2^{-j}e_i}-\mathrm{id})f\|^p_{L^p(\mathbb{T}^n)}\le K_p\left[\frac{1}{s}\left(\frac{C_p}{1-s}\right)^{1/(1-s)}\right]^p\sum_{k\ge1}2^{spk}\|\delta_{2^{-k}}f\|^p_{L^p(\mathbb{T}^n)}. \tag{7.23}$$

We complete the proof of Proposition 7.3 by combining (7.23) with the inequality

$$\|\delta_{2^{-k}}f\|^p_{L^p(\mathbb{T}^n)}\le 2^n\int_{\mathbb{T}^n}\|(f^y)_k-(f^y)_{k-1}\|^p_{L^p(\mathbb{T}^n)}\,dy \tag{7.24}$$

(see Lemma 8.9). □

Proof of Theorem 1.4. Let $f\in W^{s,p}(\mathbb{T}^n)$. Since $[0,1)^n\ni v\mapsto\|(\tau_v-\mathrm{id})f\|_{L^p(\mathbb{T}^n)}$ is subadditive, we can estimate $X(f)$ by

$$\begin{aligned}X(f)&=\int_{\mathbb{T}^n}\frac{\|(\tau_v-\mathrm{id})f\|^p_{L^p(\mathbb{T}^n)}}{|v|^{n+sp}}\,dv\lesssim\sum_{i=1}^n\int_{\mathbb{T}^n}\frac{\|(\tau_{v_ie_i}-\mathrm{id})f\|^p_{L^p(\mathbb{T}^n)}}{|(v_1,\dots,v_n)|^{n+sp}}\,dv_1\dots dv_n\\&\lesssim\sum_{i=1}^n\int_{\mathbb{T}}\frac{\|(\tau_{he_i}-\mathrm{id})f\|^p_{L^p(\mathbb{T}^n)}}{h^{1+sp}}\,dh\le\sum_{i=1}^n\sum_{j\ge1}2^{j(1+sp)}\int_{2^{-j}}^{2^{1-j}}\|(\tau_{he_i}-\mathrm{id})f\|^p_{L^p(\mathbb{T}^n)}\,dh.\end{aligned} \tag{7.25}$$

In (7.25), we rely on Corollary 8.19 in order to justify the second inequality.

Following the proof of (7.16), we obtain, for every $i\in[\![1,n]\!]$, the estimate

$$\sum_{j\ge1}2^{j(1+sp)}\int_{2^{-j}}^{2^{1-j}}\|(\tau_{he_i}-\mathrm{id})f\|^p_{L^p(\mathbb{T}^n)}\,dh\lesssim\frac{1}{s^p}\sum_{k\ge1}2^{spk}\|(\tau_{2^{-k}e_i}-\mathrm{id})f\|^p_{L^p(\mathbb{T}^n)}. \tag{7.26}$$

By combining (7.25) and (7.26), we find that

$$X(f)\lesssim\frac{1}{s^p}\sum_{i=1}^n\sum_{k\ge1}2^{spk}\|(\tau_{2^{-k}e_i}-\mathrm{id})f\|^p_{L^p(\mathbb{T}^n)}. \tag{7.27}$$

Applying Proposition 7.3 to (7.27), we obtain

$$X(f)\lesssim\left[\frac{1}{s^2}\left(\frac{C_p}{1-s}\right)^{1/(1-s)}\right]^p\int_{\mathbb{T}^n}Y(f^y)\,dy, \tag{7.28}$$

hence the conclusion. □

7.4. Remark. It would be interesting to obtain the analog of Theorem 1.4 when $s \geq 1$. Here is a hint suggesting that such an analog should exist. Using Fourier series, it is easy to see that the right-hand side of (1.9) converges when $f \in W^{1,2} = H^1$, and that we have the estimate

$$|f|^2_{W^{1,2}} \lesssim \int_{\mathbb{T}^n} \sum_{j\geq 1} 2^{2j} \left\| \left(f^y\right)_j - \left(f^y\right)_{j-1} \right\|^2_{L^2} dy. \tag{7.29}$$

Here, we consider e.g. the semi-norm

$$\left| \sum_{m\in\mathbb{Z}^n} c_m e^{2\imath\pi m\cdot x} \right|^2_{W^{1,2}} = \sum_{m\in\mathbb{Z}^n} |m|^2 |c_m|^2.$$

The analog of (7.29) for other values of $s \geq 1$ and p has not been investigated.

7.5. Remark. The quantitative form of Theorem 1.4 is not optimal, at least when $p = 1$. Indeed, when $p = 1$ estimate (7.28) deteriorates exponentially fast when $s \nearrow 1$, while we know from estimate (8.4) that the growth is of the order of $1/(1-s)$. We do not know the optimal blow up rate when $1 \leq p < \infty$ and $s \nearrow 1$.

8. Toolbox

We present here the proofs of several auxiliary estimates used in the previous sections.

8.1. Schur's criterion and applications

The material presented in this section was mainly used in the proof of Theorem 1.3.

We start by recalling (a slight generalization of) Schur's condition—or Schur's criterion—on the boundedness of integral operators and by presenting some of its consequences of interest to us. For a further discussion on Schur's criterion, see e.g. [16, Appendix I].

8.1. Lemma. *Let (X, μ) and (Y, ν) be two measure spaces, and let p, q be conjugated exponents. Consider the integral operator T associated with a measurable kernel $\kappa : X \times Y \to \mathbb{C}$, defined formally by*

$$Tu(x) = \int_Y \kappa(x, y)u(y)\, d\nu(y), \quad \forall u : Y \to \mathbb{C}.$$

Let $f : X \times Y \to \mathbb{C}$ be a measurable function on X, and set $g(x) := \|f(x,\cdot)|\kappa(x,\cdot)|^{1/q}\|_{L^q(Y)}$.

If $M := \operatorname{ess\,sup}_y \|\frac{g}{f(\cdot,y)}|\kappa(\cdot, y)|^{1/p}\|_{L^p(X)}$ is finite, then T defines a bounded operator from $L^p(Y)$ into $L^p(X)$, with $\|T\| \leq M$.

In particular (with the choice $f \equiv 1$) we have

$$\|T\| \leq M_1^{1/q} M_2^{1/p}, \tag{8.1}$$

where $M_1 := \operatorname{ess\,sup}_x \int_Y |\kappa(x, y)|\, d\nu(y)$ and $M_2 := \operatorname{ess\,sup}_y \int_X |\kappa(x, y)|\, d\mu(x)$.

Proof. By a standard argument, it suffices to establish the bound $\|Tu\|_{L^p} \leq M\|u\|_{L^p}$ when κ, f and u are non-negative. We assume that $p < \infty$; the case where $p = \infty$ is similar. By a suitable application of Hölder's inequality, we find that

$$\begin{aligned}\left(\int_Y \kappa(x, y)u(y)\, d\nu(y)\right)^p &= \left(\int_Y \left(f(x, y)\kappa^{1/q}(x, y)\right)\left(\kappa^{1/p}(x, y)\frac{u(y)}{f(x, y)}\right) d\nu(y)\right)^p \\ &\leq g^p(x) \int_Y \kappa(x, y)\frac{u^p(y)}{f^p(x, y)}\, d\nu(y).\end{aligned}$$

Therefore,

$$\int_X (Tu(x))^p\,d\mu(x) \le \int_Y \left(\int_X \frac{g^p(x)}{f^p(x,y)}\kappa(x,y)\,d\mu(x)\right) u^p(y)\,d\nu(y) \le M^p \|u\|^p_{L^p(Y)}.$$

The special case is obtained by noting that, when $f \equiv 1$, we have $g \le M_1^{1/q}$, which implies that $M \le M_1^{1/q} M_2^{1/p}$. □

By taking $f \equiv 1$ in the above lemma, we obtain the following consequence.

8.2. Corollary. *Let $(\alpha_{j,k})_{j,k\ge 0}$ be an infinite matrix, and let $1 \le p < \infty$. Consider the operator T formally defined by $T(x_k)_{k\ge 0} = (\sum_{k\ge 0} \alpha_{j,k} x_k)_{j\ge 0}$.*

If the quantity $M := \sup_{i\ge 0}(\sum_{j=0}^\infty |\alpha_{j,i}|(\sum_{k=0}^\infty |\alpha_{j,k}|)^{p-1})^{1/p}$ is finite, then T is continuous from ℓ^p into ℓ^p, with $\|T\| \le M$.

In particular, we have, for $1 \le p \le \infty$,

$$\|T\| \le \left(\sup_j \sum_k |\alpha_{j,k}|\right)^{1/q} \left(\sup_k \sum_j |\alpha_{j,k}|\right)^{1/p}.$$

We continue with a quantitative form of the equivalence $X(f) \sim Y(f) \sim Z(f)$ when $sp < 1$. Here, $X(F)$, $Y(f)$ and $Z(f)$ are given by (3.1)–(3.3). The next result and its proof follow closely [3, Appendix A].

8.3. Lemma. *Let $0 < s < 1$ and $1 \le p < \infty$. Let $f \in L^p(\mathbb{T}^n)$, and let $X(f)$, $Y(f)$ and $Z(f)$ be as in (3.1)–(3.3). Then*

$$s^p Z(f) \lesssim Y(f) \le 2Z(f), \tag{8.2}$$

$$Z(f) \lesssim X(f) \tag{8.3}$$

and, if $sp < 1$,

$$X(f) \lesssim \frac{1}{s^p(1-sp)^p} Y(f). \tag{8.4}$$

Proof.

Step 1. Proof of (8.4).

We have that

$$X(f) = \int_{\mathbb{T}^n}\int_{\mathbb{T}^n} \frac{|(\tau_h - \mathrm{id})f(x)|^p}{|h|^{n+sp}}\,dx\,dh \le \sum_{j=1}^\infty 2^{(n+sp)j} \int_{|h|\in I_j} \left\|(\tau_h - \mathrm{id})f\right\|^p_{L^p}\,dh, \tag{8.5}$$

where $I_j = [2^{-j}, 2^{-(j-1)})$. Since $f_k \to f$ in $L^p(\mathbb{T}^n)$, and $f u_0 = \int_{\mathbb{T}^n} f$ is constant, we have

$$\left\|(\tau_h - \mathrm{id})f\right\|_{L^p} = \left\|\sum_{k=1}^\infty (\tau_h - \mathrm{id})(f_k - f_{k-1})\right\|_{L^p} \le \sum_{k=1}^\infty \left\|(\tau_h - \mathrm{id})(f_k - f_{k-1})\right\|_{L^p}. \tag{8.6}$$

We next invoke [3, Lemma A.2] in the following form:

8.4. Lemma. *Let $f \in \mathscr{E}_k$, $j \ge 1$ and $h \in \mathbb{T}^n$ be such that $|h| < 2^{1-j}$. Then*

$$\left\|(\tau_h - \mathrm{id})f\right\|_{L^p} \lesssim \beta_{j,k}\|f\|_{L^p}, \tag{8.7}$$

where

$$\beta_{j,k} := \begin{cases} 1, & \text{if } j \le k,\\ (2^{k-j})^{1/p}, & \text{if } j > k.\end{cases} \tag{8.8}$$

Step 1 completed.

Let $x_k := 2^{sk}\|f_k - f_{k-1}\|_{L^p}$, $\forall k \geq 1$,[20] and set $\alpha_{j,k} := 2^{s(j-k)}\beta_{j,k}$. We note that

$$\sum_j \alpha_{j,k} = \sum_k \alpha_{j,k} = \frac{1}{1-2^{-s}} + \frac{1}{2^{s-1/p}-1} \lesssim \frac{1}{s} + \frac{1}{1-sp} \lesssim \frac{1}{s(1-sp)}. \tag{8.9}$$

Next, Lemma 8.4 combined with (8.5) and (8.6) gives

$$X(f) \lesssim \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}\alpha_{j,k}x_k\right)^p. \tag{8.10}$$

We obtain (8.4) by combining (8.10) with Corollary 8.2.

Step 2. Proof of (8.2).

Since $\|f_j - f_{j-1}\|_{L^p} = \|E_j(f - f_{j-1})\|_{L^p} \leq \|f - f_{j-1}\|_{L^p}$, we find that

$$Y(f) \leq 2^{sp} Z(f) \leq 2Z(f).$$

On the other hand, we have

$$\|f - f_j\|_{L^p} = \left\|\sum_{k\geq j+1}(f_k - f_{k-1})\right\|_{L^p} \leq \sum_{k\geq j+1}\|f_k - f_{k-1}\|_{L^p},$$

and thus

$$Z(f) \leq \sum_{j\geq 0}\left(\sum_{k\geq j+1} 2^{-s(k-j)}x_k\right)^p \lesssim \frac{1}{s^p}Y(f)$$

(the last inequality following from Corollary 8.2).

Step 3. Proof of (8.3).

By Hölder's inequality we have

$$\|f - f_j\|_{L^p}^p \leq \int_{\mathbb{T}^n}\left(\fint_{Q_j(x)}|f(x) - f(y)|\,dy\right)^p dx \leq \int_{\mathbb{T}^n}\fint_{Q_j(x)}|f(x) - f(y)|^p\,dy\,dx.$$

Therefore,

$$\begin{aligned} Z(f) &\leq \sum_{j\geq 0} 2^{(n+sp)j}\int_{\mathbb{T}^n}\int_{Q_j(x)}|f(x) - f(y)|^p\,dy\,dx \\ &= \int_{\mathbb{T}^n}\int_{\mathbb{T}^n}\left(|x-y|^{n+sp}\sum_{j\geq 0}2^{(n+sp)j}\mathbb{1}_{Q_j(x)}(y)\right)\frac{|f(x) - f(y)|^p}{|x-y|^{n+sp}}\,dy\,dx. \end{aligned}$$

In order to evaluate the above expression between brackets, we fix $x \neq y$ in $\mathbb{T}^n$ and we let k be such that $|x - y| \in I_k$. Then

$$|x-y|^{n+sp}\sum_{j\geq 0}2^{(n+sp)j}\mathbb{1}_{Q_j(x)}(y) \leq |x-y|^{n+sp}\sum_{j=0}^{k-1}2^{(n+sp)j} \lesssim 1,$$

which implies (8.3). □

For further use, let us recall the following cousin of Lemma 8.3 [3, Corollary A.1].

[20] So that $Y(f) = \|(x_k)_{k\geq 1}\|_{\ell^p}^p$.

8.5. Lemma. *Let $0 < s < 1$ and $1 \le p < \infty$ be such that $sp < 1$. Let $f^j : \mathbb{T}^n \to \mathbb{R}$ be a sequence of functions such that $f^j \in \mathscr{E}_j$, $\forall j$.*[21] *If*

$$\sum_{j\ge 1} 2^{spj} \| f^j - f^{j-1} \|_{L^p}^p < \infty,$$

then f^j converges in L^p to a function $f \in W^{s,p}$. In addition, we have

$$|f|_{W^{s,p}}^p \lesssim \sum_{j\ge 1} 2^{spj} \| f^j - f^{j-1} \|_{L^p}^p.$$

8.2. *Estimates for averages*

The material in this section was used in the proofs of Theorems 1.3 and 1.4.
We start with a version of [3, (E.17)].

8.6. Lemma. *Let f belong to $\mathscr{E}_k$, $\rho := \mathbb{1}_{(-1/2,1/2)^n}$, and $\rho_\varepsilon(x) := \varepsilon^{-n}\rho(\frac{x}{\varepsilon})$, $\forall \varepsilon > 0$, $\forall x$. Let h satisfy $|h| < 2^{-j}$, where $j \ge k$. Then*

$$|\tau_h f - f| \le 2^{2n+1} |f| * \rho_{2^{2-k}} \mathbb{1}_{A_{k,j}},$$

where

$$A_{k,j} := \left\{x \in \mathbb{T}^n; \operatorname{dist}(x, \partial Q) \le 2^{-j} \text{ for some } Q \in \mathscr{P}_k\right\}.$$

Proof. Since f is constant in each cube $Q \in \mathscr{P}_k$ and $|h| < 2^{-k}$, we have

$$|\tau_h f|(x) = \fint_{Q_k(x-h)} |f| \le 2^{nk} \int_{B(x-h,2^{-k})} |f| \le 2^{nk} \int_{B(x,2^{1-k})} |f|. \tag{8.11}$$

We note also that

$$|f| * \rho_{2^{2-k}}(x) = 2^{n(k-2)} \int_{B(x,2^{1-k})} |f|. \tag{8.12}$$

By combining (8.11) with (8.12) we obtain

$$|\tau_h f| \le 2^{2n} |f| * \rho_{2^{2-k}}. \tag{8.13}$$

By letting $h \to 0$ in (8.13), we find that

$$|f| \le 2^{2n} |f| * \rho_{2^{2-k}}. \tag{8.14}$$

By (8.13) and (8.14), we obtain

$$|\tau_h f - f| \le |\tau_h f| + |f| \le 2^{2n+1} |f| * \rho_{2^{2-k}}.$$

Now the conclusion follows by noting that, when x does not belong to $A_{k,j}$ we have $Q_k(x-h) = Q_k(x)$, and thus $\tau_h f(x) = f(x)$. $\square$

We next turn to Lemmas 8.7, 8.9, 8.10 and 8.11 which were used in Section 7.

[21] We recall that $\mathscr{E}_j$ denotes the class of functions which are constant on each dyadic cube of $\mathscr{P}_j$.

8.7. Lemma. *Let* $1 \le p < \infty$ *and* $f \in L^p(\mathbb{T})$. *Let* δ_ε *be the operator given by*

$$\delta_\varepsilon f(x) := \fint_0^\varepsilon f(x+z)\,dz - \fint_0^{2\varepsilon} f(x+z)\,dz. \tag{8.15}$$

Then, for every $k \ge 1$,

$$\int_{\mathbb{T}} \|(f^y)_k - (f^y)_{k-1}\|^p_{L^p(\mathbb{T})}\,dy \ge \frac{1}{2}\|\delta_{2^{-k}} f\|^p_{L^p(\mathbb{T})}.$$

Recall that $f^y(x) = f(x-y)$.

Proof. We first note that for every $x \in \mathbb{T}$, the dyadic cube (interval) of x of order k is given by

$$Q_k(x) = 2^{-k}[2^k x] + [0, 2^{-k}).$$

Note also that if x belongs to an interval of the form $J_{k,\ell} := [2^{1-k}\ell, 2^{-k}(2\ell+1))$ with $\ell \in [\![0, 2^{k-1}-1]\!]$, then we have $2^{1-k}[2^{k-1}x] = 2^{-k}[2^k x]$. Thus, for every such x and every $y \in \mathbb{T}$, we have

$$\begin{aligned}\fint_{Q_k(x)} f^y - \fint_{Q_{k-1}(x)} f^y &= \fint_0^{2^{-k}} f^y\big(z + 2^{-k}[2^k x]\big)\,dz - \fint_0^{2^{1-k}} f^y\big(z + 2^{1-k}[2^{k-1}x]\big)\,dz \\ &= \delta_{2^{-k}} f^y\big(2^{-k}[2^k x]\big).\end{aligned} \tag{8.16}$$

We next note that $\delta_\varepsilon f^y(x) = \delta_\varepsilon f(x-y)$, $\forall x, y \in \mathbb{T}^n$. If $x \in J_{k,\ell}$, then by integrating (8.16) with respect to y we find that

$$\begin{aligned}\int_{\mathbb{T}} \left| \fint_{Q_k(x)} f^y - \fint_{Q_{k-1}(x)} f^y \right|^p dy &= \int_{\mathbb{T}} \left|\delta_{2^{-k}} f^y\big(2^{-k}[2^k x]\big)\right|^p dy \\ &= \int_{\mathbb{T}} \left|\delta_{2^{-k}} f\big(2^{-k}[2^k x] - y\big)\right|^p dy = \int_{\mathbb{T}} \left|\delta_{2^{-k}} f(y)\right|^p dy.\end{aligned} \tag{8.17}$$

We obtain the conclusion by integrating the left-hand side of (8.17) with respect to $x \in J_{k,\ell}$, $\forall \ell$. □

8.8. Remark. It is not difficult to see that the following extension of (8.16) holds for every $x \in \mathbb{T}$:

$$\left| \fint_{Q_k(x)} f^y - \fint_{Q_{k-1}(x)} f^y \right| = \left|\delta_{2^{-k}} f^y\big(2^{-k}[2^k x]\big)\right|.$$

Hence the conclusion of Lemma 8.7 can be improved to

$$\int_{\mathbb{T}} \|(f^y)_k - (f^y)_{k-1}\|^p_{L^p(\mathbb{T})}\,dy = \|\delta_{2^{-k}} f\|^p_{L^p(\mathbb{T})}.$$

However, the advantage of Lemma 8.7 stated as above is that its proof can be easily generalized to higher dimension.

Lemma 8.7 has the following n-dimensional analog.

8.9. Lemma. *Let* $1 \le p < \infty$ *and* $f \in L^p(\mathbb{T}^n)$. *Let* δ_ε *be the operator given by*

$$\delta_\varepsilon f(x) := \fint_{(0,\varepsilon)^n} f(x+z)\,dz - \fint_{(0,2\varepsilon)^n} f(x+z)\,dz. \tag{8.18}$$

Then, for every $k \ge 1$,

$$\int_{\mathbb{T}^n} \left\| \left(f^y\right)_k - \left(f^y\right)_{k-1} \right\|^p_{L^p(\mathbb{T}^n)} \, dy \ge \frac{1}{2^n} \|\delta_{2^{-k}} f\|^p_{L^p(\mathbb{T}^n)}.$$

The proof of Lemma 8.9 is identical to the proof of Lemma 8.7 and is left to the reader.

8.10. Lemma. *Let* $f \in L^p(\mathbb{T})$ *and let* δ_ε *the operator given by* (8.15)*. Then*

$$\left\|(\tau_h - \mathrm{id})\delta_\varepsilon f\right\|_{L^p} \le \frac{h}{\varepsilon}\left\|(\tau_\varepsilon - \mathrm{id}) f\right\|_{L^p}, \quad \forall h \in [0, \varepsilon].$$

Proof. Let $0 \le h \le \varepsilon$. For every $x \in [0, 1)$, by the definition of δ_ε we have

$$\begin{aligned}
(\tau_h - \mathrm{id})\delta_\varepsilon f(x) &= \frac{1}{2\varepsilon}\left(2\int_{-h}^{\varepsilon-h} f(x+z)\,dz - \int_{-h}^{2\varepsilon-h} f(x+z)\,dz\right) \\
&\quad - \frac{1}{2\varepsilon}\left(2\int_0^{\varepsilon} f(x+z)\,dz - \int_0^{2\varepsilon} f(x+z)\,dz\right) \\
&= \frac{1}{2\varepsilon}\left(\int_{-h}^{0} f(x+z)\,dz - 2\int_{\varepsilon-h}^{\varepsilon} f(x+z)\,dz + \int_{2\varepsilon-h}^{2\varepsilon} f(x+z)\,dz\right) \\
&= \frac{1}{2\varepsilon}\left(\int_{-h}^{0} f(x+z)\,dz - \int_{\varepsilon-h}^{\varepsilon} f(x+z)\,dz\right) \\
&\quad - \frac{1}{2\varepsilon}\left(\int_{\varepsilon-h}^{\varepsilon} f(x+z)\,dz - \int_{2\varepsilon-h}^{2\varepsilon} f(x+z)\,dz\right) \\
&= \frac{1}{2\varepsilon}\left[\int_{\varepsilon-h}^{\varepsilon} (\tau_\varepsilon - \mathrm{id}) f(x+z)\,dz - \int_{2\varepsilon-h}^{2\varepsilon} (\tau_\varepsilon - \mathrm{id}) f(x+z)\,dz\right].
\end{aligned}$$

Hence, for every $x \in \mathbb{T}$, we have

$$\left|(\tau_h - \mathrm{id})\delta_\varepsilon f(x)\right| \le \frac{1}{2\varepsilon}\left[\int_{\varepsilon-h}^{\varepsilon} \left|(\tau_\varepsilon - \mathrm{id}) f(x+z)\right| dz + \int_{2\varepsilon-h}^{2\varepsilon} \left|(\tau_\varepsilon - \mathrm{id}) f(x+z)\right| dz\right]. \tag{8.19}$$

We note that for any $F \in L^p(\mathbb{T})$ and for $\rho = \mathbb{1}_{(-1/2,1/2)}$ we have $F * \rho_h(t) = \frac{1}{h}\int_{t-h/2}^{t+h/2} F(z)\,dz$. Thus

$$\begin{aligned}
&\frac{1}{h}\left(\int_{\varepsilon-h}^{\varepsilon} \left|(\tau_\varepsilon - \mathrm{id}) f(x+z)\right| dz + \int_{2\varepsilon-h}^{2\varepsilon} \left|(\tau_\varepsilon - \mathrm{id}) f(x+z)\right| dz\right) \\
&\quad = \left|(\tau_\varepsilon - \mathrm{id}) f(x+\cdot)\right| * \rho_h(\varepsilon - h/2) + \left|(\tau_\varepsilon - \mathrm{id}) f(x+\cdot)\right| * \rho_h(2\varepsilon - h/2) \\
&\quad = \left|(\tau_\varepsilon - \mathrm{id}) f\right| * \rho_h(x + \varepsilon - h/2) + \left|(\tau_\varepsilon - \mathrm{id}) f\right| * \rho_h(x + 2\varepsilon - h/2).
\end{aligned} \tag{8.20}$$

Since the L^p norm on $\mathbb{T}$ is independent of translations, we obtain from (8.19) and (8.20) that

$$\left\|(\tau_h - \mathrm{id})\delta_\varepsilon f\right\|_{L^p} \le 2\frac{h}{2\varepsilon}\left\|\left|(\tau_\varepsilon - \mathrm{id}) f\right| * \rho_h\right\|_{L^p} \le \frac{h}{\varepsilon}\left\|(\tau_\varepsilon - \mathrm{id}) f\right\|_{L^p}\|\rho_h\|_{L^1} = \frac{h}{\varepsilon}\left\|(\tau_\varepsilon - \mathrm{id}) f\right\|_{L^p}. \qquad \square$$

The same argument leads to the following n-dimensional version of Lemma 8.10.

8.11. Lemma. *Let $f \in L^p(\mathbb{T}^n)$ and let δ_ε the operator given by (8.18). Then, for every $i \in [\![1, n]\!]$,*

$$\left\|(\tau_{he_i} - \mathrm{id})\delta_\varepsilon f\right\|_{L^p(\mathbb{T}^n)} \le \frac{h}{\varepsilon}\left\|(\tau_{\varepsilon e_i} - \mathrm{id})f\right\|_{L^p(\mathbb{T}^n)}, \quad \forall h \in [0, \varepsilon].$$

8.3. A lemma on cuts

The following lemma is due to Merlet [20], and was used in the proof of Proposition 1.6. For the convenience of the reader, we reproduce the argument in [20].

8.12. Lemma. *Let $\alpha \in \mathbb{S}^1$. For every $z \in \mathbb{S}^1$, we let $\theta_\alpha(z)$ be the unique $\theta \in (\alpha - 2\pi, \alpha]$ such that $z = e^{\imath\theta}$. Then, for every $z, w \in \mathbb{S}^1$,*

$$\int_{\mathbb{S}^1} \left|\theta_\alpha(w) - \theta_\alpha(z)\right| \mathrm{d}\alpha = 2|\widehat{zw}|\left(2\pi - |\widehat{zw}|\right) \le 4\pi|z - w|.$$

Here, $\widehat{zw}$ is (one of) the geodesic arc(s) that connects z and w on the circle, and $|\widehat{zw}|$ is the geodesic distance on the circle.

Proof. It is easy to see that

$$\left|\theta_\alpha(z) - \theta_\alpha(w)\right| = \begin{cases} 2\pi - |\widehat{zw}|, & \text{if } \alpha \in \widehat{zw}, \\ |\widehat{zw}|, & \text{if } \alpha \notin \widehat{zw}. \end{cases}$$

Hence

$$\int_{\mathbb{S}^1} \left|\theta_\alpha(w) - \theta_\alpha(z)\right| \mathrm{d}\alpha = \int_{\alpha \notin \widehat{zw}} |\widehat{zw}| \,\mathrm{d}\alpha + \int_{\alpha \in \widehat{zw}} \left(2\pi - |\widehat{zw}|\right) \mathrm{d}\alpha = 2|\widehat{zw}|\left(2\pi - |\widehat{zw}|\right). \tag{8.21}$$

We now use the inequality $\sin x \ge x(1 - x/\pi)$, valid for every $x \in [0, \pi/2]$, to find that

$$|z - w| = 2\left|\sin \frac{\widehat{zw}}{2}\right| \ge 2\frac{|\widehat{zw}|}{2}\left(1 - \frac{|\widehat{zw}|}{2\pi}\right) = \frac{1}{4\pi} 2|\widehat{zw}|\left(2\pi - |\widehat{zw}|\right),$$

which together with (8.21) proves the lemma. □

8.4. Toolbox for the proof of Theorem 1.5

We gather here the auxiliary results used in the proof of Theorem 1.5 in Section 4, as well as the proof of Lemma 4.2.

We start by establishing estimate (4.3), that we recall in the next statement.

8.13. Lemma. *Let $a, \varepsilon \in (0, 1)$. Then*

$$(1 - a)^\varepsilon + a^\varepsilon - 1 \ge (1 - \varepsilon)a^\varepsilon(1 - a)^\varepsilon.$$

Proof. By symmetry, we may assume that $a \le 1/2$. By the mean value theorem, we have, for some $\xi \in (0, a)$,

$$1 - (1 - a)^\varepsilon = \varepsilon a(1 - \xi)^{(\varepsilon - 1)} \le \varepsilon a^\varepsilon$$

(since $1 - \xi \ge a$ and therefore $(1 - \xi)^{\varepsilon - 1} \le a^{\varepsilon - 1}$). Thus

$$(1 - a)^\varepsilon + a^\varepsilon - 1 \ge (1 - \varepsilon)a^\varepsilon \ge (1 - \varepsilon)a^\varepsilon(1 - a)^\varepsilon. \qquad \square$$

We continue with a proof of the estimate (4.5); this is the purpose of the next lemma.

8.14. Lemma. *Let* $1 < p < \infty$. *Set* $\delta(\varepsilon) := e^{-1/\varepsilon}$, *for every* $0 < \varepsilon < 1$, *and* $u_\varepsilon := e^{\imath\varphi_{\delta(\varepsilon)}}$, *where* φ_δ *is given by* (4.4). *Then* $|u_\varepsilon|_{W^{(1-\varepsilon)/p,p}} \approx 1$.

Proof. We start with the following obvious estimate of $|u_\varepsilon|_{W^{(1-\varepsilon)/p,p}}$:

$$|u_\varepsilon|^p_{W^{(1-\varepsilon)/p,p}} \approx \int_0^{1/2}\int_{1/2}^{1/2+\delta} \frac{|e^{\imath 2\pi(x-1/2)/\delta}-1|^p}{(x-y)^{2-\varepsilon}}\,dx\,dy + \int_{1/2}^{1/2+\delta}\int_{1/2}^{1/2+\delta} \frac{|e^{\imath 2\pi(x-1/2)/\delta}-e^{\imath 2\pi(y-1/2)/\delta}|^p}{|x-y|^{2-\varepsilon}}\,dx\,dy$$
$$+ \int_{1/2+\delta}^{1}\int_{1/2}^{1/2+\delta} \frac{|e^{\imath 2\pi(x-1/2)/\delta}-1|^p}{(y-x)^{2-\varepsilon}}\,dx\,dy =: I_1 + I_2 + I_3.$$

We next estimate each of the three integrals I_1, I_2 and I_3, using simple calculations and the fact that $\delta^\varepsilon = 1/e$. To start with, we have

$$I_2 \approx \int_{1/2}^{1/2+\delta}\int_{1/2}^{1/2+\delta} \frac{|\sin\pi(x-y)/\delta|^p}{|x-y|^{2-\varepsilon}}\,dx\,dy = \left(\frac{\delta}{\pi}\right)^\varepsilon \int_{\pi/(2\delta)}^{\pi/(2\delta)+\pi}\int_{\pi/(2\delta)}^{\pi/(2\delta)+\pi} \frac{|\sin(x-y)|^p}{|x-y|^{2-\varepsilon}}\,dx\,dy$$
$$= \left(\frac{\delta}{\pi}\right)^\varepsilon \int_{\pi/(2\delta)}^{\pi/(2\delta)+\pi}\left(\int_{\pi/(2\delta)-y}^{\pi/(2\delta)+\pi-y} \frac{|\sin t|^p}{|t|^{2-\varepsilon}}\,dt\right)dy \approx \int_{-\pi}^{\pi} \frac{|\sin t|^p}{|t|^{2-\varepsilon}}\,dt \approx 1;$$

the latter conclusion uses the fact that $p > 1$.

We next estimate I_1 as follows.

$$I_1 \approx \int_0^{1/2}\int_{1/2}^{1/2+\delta} \frac{|\sin\pi(x-1/2)/\delta|^p}{(x-y)^{2-\varepsilon}}\,dx\,dy \approx \int_{-\pi/(2\delta)}^{0}\int_0^{\pi} \frac{\sin^p x}{(x-y)^{2-\varepsilon}}\,dx\,dy = \int_0^{\pi}\int_x^{x+\pi/(2\delta)} \frac{\sin^p x}{t^{2-\varepsilon}}\,dt\,dx$$
$$\approx \int_0^{\pi} \sin^p x\left(\frac{1}{x^{1-\varepsilon}} - \frac{1}{(x+\pi/(2\delta))^{1-\varepsilon}}\right)dx = \int_0^{\pi} \frac{\sin^p x}{x}\,dx + o_\varepsilon(1) - \int_0^{\pi} \frac{\sin^p x}{(x+\pi/(2\delta))^{1-\varepsilon}}\,dx$$
$$= \int_0^{\pi} \frac{\sin^p x}{x}\,dx + o_\varepsilon(1) + O(\delta) = \int_0^{\pi} \frac{\sin^p x}{x}\,dx + o_\varepsilon(1) \approx 1.$$

Similarly, we have

$$I_3 \approx \int_{1/2+\delta}^{1}\int_{1/2}^{1/2+\delta} \frac{|\sin\pi(x-1/2)/\delta|^p}{(y-x)^{2-\varepsilon}}\,dx\,dy \approx \int_{\pi}^{\pi/(2\delta)}\int_0^{\pi} \frac{\sin^p x}{(y-x)^{2-\varepsilon}}\,dx\,dy = \int_0^{\pi}\int_{\pi-x}^{\pi/(2\delta)-x} \frac{\sin^p x}{t^{2-\varepsilon}}\,dt\,dx$$
$$\approx \int_0^{\pi} \sin^p x\left(\frac{1}{(\pi-x)^{1-\varepsilon}} - \frac{1}{(\pi/(2\delta)-x)^{1-\varepsilon}}\right)dx \approx 1.$$

By the above estimates of I_1, I_2 and I_3, we conclude that $|u_\varepsilon|_{W^{(1-\varepsilon)/p,p}} \approx 1$ as $\varepsilon \to 0$. □

We next present the proof of Lemma 4.2, in the spirit of [4, Lemma 2].

Proof of Lemma 4.2. By scale invariance, we may assume that $I = (0, 1)$.

For every $\ell \in \mathbb{Z}$, we define the sets $A_\ell := \{x \in I;\ \psi(x) < \ell\}$. Since (A_ℓ) is a non-decreasing sequence with $|A_\ell| \to 0$ when $\ell \to -\infty$ and $|A_\ell| \to 1$ when $\ell \to \infty$, there exists some $k \in \mathbb{Z}$ such that $|A_k| \le 1/2$ and $|A_{k+1}| > 1/2$.

Note that

$$\left|\psi(x)-\psi(y)\right| \geq 1, \quad \forall \ell \in \mathbb{Z},\ \forall x \in A_\ell,\ \forall y \in {}^c(A_\ell). \tag{8.22}$$

Hence, by applying inequality (4.1) first to A_k and next to A_{k+1}, and by using (8.22), we get

$$1/2|A_k| \leq \left(C\varepsilon \int_{A_k}\int_{{}^c(A_k)} \frac{1}{|x-y|^{2-\varepsilon}}\,dx\,dy\right)^{1/\varepsilon} \leq \left(C\varepsilon \int_{A_k}\int_{{}^c(A_k)} \frac{|\psi(x)-\psi(y)|}{|x-y|^{2-\varepsilon}}\,dx\,dy\right)^{1/\varepsilon}$$

and

$$1/2\left|{}^c(A_{k+1})\right| \leq \left(C\varepsilon \int_{A_{k+1}}\int_{{}^c(A_{k+1})} \frac{1}{|x-y|^{2-\varepsilon}}\,dx\,dy\right)^{1/\varepsilon} \leq \left(C\varepsilon \int_{A_{k+1}}\int_{{}^c(A_{k+1})} \frac{|\psi(x)-\psi(y)|}{|x-y|^{2-\varepsilon}}\,dx\,dy\right)^{1/\varepsilon}.$$

We find that:

$$\left|\left\{x \in I;\ \psi(x) \neq k\right\}\right| = |A_k| + \left|{}^c(A_{k+1})\right| \leq 4\left(C\varepsilon \int_I\int_I \frac{|\psi(x)-\psi(y)|^p}{|x-y|^{2-\varepsilon}}\,dx\,dy\right)^{1/\varepsilon}. \qquad \square$$

8.15. Lemma. *Let φ be a lifting of $u = e^{\imath\varphi_\delta}$, where φ_δ is given by (4.4), i.e.,*

$$\varphi_\delta(x) := \begin{cases} 0, & \text{if } x < 1/2, \\ (2x-1)\pi/\delta, & \text{if } 1/2 < x < 1/2+\delta, \\ 2\pi, & \text{if } 1/2+\delta < x. \end{cases}$$

Let $\psi := \frac{\varphi-\varphi_\delta}{2\pi}$.

Then, if $x, y \in (0, \frac{1}{2}+\frac{2\delta}{3})$, or if $x, y \in (\frac{1}{2}+\frac{\delta}{3}, 1)$, we have

$$\left|\psi(x)-\psi(y)\right| \leq \left|\varphi(x)-\varphi(y)\right|. \tag{8.23}$$

Proof. We will verify (8.23) when $x, y \in (0, 1/2+2\delta/3)$, since the proof when x, y both belong to the second interval is similar. Estimate (8.23) being clear when $0 < x \leq 1/2$ and $0 < y \leq 1/2$, we may assume that $y > 1/2$. To summarize, we will establish (8.23) when $y \in (1/2, 1/2+2\delta/3)$ and $x \in (0, 1/2+2\delta/3)$. Two cases will be considered: $x \in (0, 1/2]$ and $x \in (1/2, 1/2+2\delta/3)$.

Since φ and φ_δ are liftings of the same function u, for every x there exists an integer $k(x)$ such that $\varphi(x) = \varphi_\delta(x) + 2\pi k(x)$. Same for y. We may always assume, with no loss of generality, that $k(y) = 0$.

To start with, assume that $x \in (0, 1/2]$. Then (8.23) is equivalent to

$$\left|k(x)\right| \leq \left|2\pi k(x) - (2y-1)\pi/\delta\right| = \left|2\pi k(x) - Y\right|, \tag{8.24}$$

where we let $Y := (2y-1)\pi/\delta$. Note that $0 < Y < 4\pi/3$. If $k(x) \leq 0$, then (8.24) is obviously true. In the case where $k(x) > 0$ is nonnegative, (8.24) becomes

$$(2\pi - 1)k(x) \geq Y, \tag{8.25}$$

and follows from $Y < 4\pi/3$.

Suppose next that we have $x \in (1/2, 1/2+2\delta/3)$. Then (8.23) becomes

$$\left|k(x)\right| \leq \left|2(x-y)\pi/\delta + 2\pi k(x)\right| = \left|X + 2\pi k(x)\right|, \tag{8.26}$$

where $X := 2(x-y)\pi/\delta$. Note that $-4\pi/3 < X < 4\pi/3$. We investigate the validity of (8.26) when $X \geq 0$; the case where $X < 0$ is similar and is left to the reader. When $X \geq 0$, inequality (8.26) is always true if $k(x)$ is non-negative. When $k(x) < 0$, (8.26) amounts to

$$(2\pi - 1)\left(-k(x)\right) \geq X,$$

which holds since $X < 4\pi/3$. $\square$

8.16. Lemma. *Let A, $B \subset (a,b)$ be such that "A is on the left of B", i.e., $y < x$, $\forall y \in A$, $\forall x \in B$. Define $A_\ell := (a, a+|A|)$ and $B_r := (b-|B|, b)$. Let $t > 0$.*

Then

$$\int_A \int_B \frac{1}{(x-y)^t}\,dx\,dy \geq \int_{A_\ell} \int_{B_r} \frac{1}{(x-y)^t}\,dx\,dy.$$

Proof. It suffices to establish the inequality

$$\int_A \frac{dy}{(x-y)^t} \geq \int_{A_\ell} \frac{dy}{(x-y)^t}, \quad \forall x \in (a,b) \text{ such that } y < x, \forall y \in A. \tag{8.27}$$

Indeed, assume that (8.27) holds. Then by symmetry we have

$$\int_B \frac{dx}{(x-y)^t} \geq \int_{B_r} \frac{dx}{(x-y)^t}, \quad \forall y \in (a,b) \text{ such that } x > y, \forall x \in B. \tag{8.28}$$

By (8.27) and (8.28), we have

$$\int_B \left(\int_A \frac{dy}{(x-y)^t} \right) dx \geq \int_B \left(\int_{A_\ell} \frac{dy}{(x-y)^t} \right) dx = \int_{A_\ell} \left(\int_B \frac{dx}{(x-y)^t} \right) dy \geq \int_{A_\ell} \int_{B_r} \frac{1}{(x-y)^t}\,dx\,dy.$$

It remains to prove (8.27). We first note that (8.27) is true when A is an interval.[22] By a standard argument, we find that (8.27) holds: first when A is an open set, next when A is compact, and finally for every measurable A. □

8.17. Lemma. *Let $0 < s < 1$ and $1 \leq p < \infty$. Then, for any $\varphi \in W^{s,p}(\mathbb{T}^n; \mathbb{R})$, we have*

$$|\varphi|^p_{W^{s,p}(\mathbb{T}^n)} \gtrsim \int_{\mathbb{T}^{n-1}} |\varphi(\cdot, x_2, \dots, x_n)|^p_{W^{s,p}(\mathbb{T})}\,dx_2 \dots dx_n. \tag{8.29}$$

Proof. Let A denote the integral in the right-hand side of (8.29), that is,

$$A = \int_{\mathbb{T}^{n-1}} \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|\varphi(x_1, x') - \varphi(z_1, x')|^p}{|x_1 - z_1|^{1+sp}}\,dx_1\,dz_1\,dx'.$$

We will use the notation $x := (x_1, x')$ and $z := (z_1, x')$, with $x_1, z_1 \in \mathbb{T}$ and $x' \in \mathbb{T}^{n-1}$. Integrating the inequality

$$\left|\varphi(x) - \varphi(z)\right|^p \leq 2^{p-1}\left(\left|\varphi(x) - \varphi(y)\right|^p + \left|\varphi(y) - \varphi(z)\right|^p\right), \quad \forall y \in \mathbb{T}^n,$$

with respect to $y \in B((x+z)/2, |x_1 - z_1|/4)$, we find that

$$A \lesssim \int_{\mathbb{T}^n} \int_{\mathbb{T}} \fint_{B((x+z)/2, |x_1-z_1|/4)} \frac{|\varphi(x) - \varphi(y)|^p}{|x_1 - z_1|^{1+sp}}\,dy\,dz_1\,dx.$$

Noting that $B((x+z)/2, |x_1 - z_1|/4) \subset B(x, 3|x_1 - z_1|/4)$, we find that

$$A \lesssim \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \int_{|x_1 - z_1| \geq 4|x-y|/3} \frac{dz_1}{|x_1 - z_1|^{n+1+sp}} \left|\varphi(x) - \varphi(y)\right|^p dy\,dx \lesssim \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \frac{|\varphi(x) - \varphi(y)|^p}{|x-y|^{n+sp}}\,dy\,dx. \quad \square$$

[22] By explicit calculation of both sides in (8.27).

8.18. Lemma. *Let* $0 < s < 1$ *and* $1 \le p < \infty$. *Set*

$$K^n(x_1, y_1) := \int_{\mathbb{T}^{n-1}} \int_{\mathbb{T}^{n-1}} \frac{dx'dy'}{|(x_1, x') - (y_1, y')|^{n+sp}}, \quad \forall x_1, y_1 \in \mathbb{T}.$$

Then we have

$$K^n(x_1, y_1) \lesssim \frac{1}{|x_1 - y_1|^{1+sp}}.$$

Proof. Set $t := x_1 - y_1$ and $z' := x' - y'$. Then

$$K^n(x_1, y_1) = \int_{\mathbb{T}^{n-1}} \int_{\mathbb{T}^{n-1}} \frac{dx'dy'}{|(t, x' - y')|^{n+sp}} \le \int_{|x'|\le 1} \int_{|z'|\le 2} \frac{dz'dx'}{|(t, z')|^{n+sp}}$$
$$\lesssim \int_{\mathbb{R}^{n-1}} \frac{dz'}{|(t, z')|^{n+sp}} \lesssim \frac{1}{|t|^{1+sp}}. \qquad \square$$

8.19. Corollary. *Let* $s > 0$ *and* $1 \le p < \infty$. *Let* $f \in W^{s,p}(\mathbb{T}; \mathbb{C})$ *and consider the function* $F : \mathbb{T}^n \to \mathbb{C}$ *defined by* $F(x_1, x') := f(x_1)$, $\forall x = (x_1, x') \in \mathbb{T}^n$.

Then $F \in W^{s,p}(\mathbb{T}^n; \mathbb{C})$ *and* $|F|_{W^{s,p}(\mathbb{T}^n)} \approx |f|_{W^{s,p}(\mathbb{T})}$.

Proof. If k is an integer and $k \le s$, then we clearly have

$$\|D^k F\|_{L^p(\mathbb{T}^n)} = \|D^k f\|_{L^p(\mathbb{T})}. \tag{8.30}$$

In particular, the conclusion of the lemma holds when s is an integer.

Suppose now that s is not an integer and write $s = [s] + \sigma$, with $\sigma \in (0, 1)$. By Lemma 8.17, we have

$$|D^{[s]} F|^p_{W^{\sigma,p}(\mathbb{T}^n)} \gtrsim \int_{\mathbb{T}^{n-1}} |D^{[s]} F(\cdot, x')|^p_{W^{\sigma,p}(\mathbb{T})} dx' = \int_{\mathbb{T}^{n-1}} |D^{[s]} f|^p_{W^{\sigma,p}(\mathbb{T})} dx' = |D^{[s]} f|^p_{W^{\sigma,p}(\mathbb{T})}. \tag{8.31}$$

On the other hand, using Lemma 8.18 for $s = \sigma$, we obtain

$$|D^{[s]} F|^p_{W^{\sigma,p}(\mathbb{T}^n)} = \int_{\mathbb{T}^n} \int_{\mathbb{T}^n} \frac{|D^{[s]} F(x_1, x') - D^{[s]} F(y_1, y')|^p}{|x - y|^{n+\sigma p}} dx\, dy$$
$$= \int_0^1 \int_0^1 |D^{[s]} f(x_1) - D^{[s]} f(y_1)|^p K^n(x_1, y_1)\, dx_1\, dx_2$$
$$\lesssim \int_0^1 \int_0^1 \frac{|D^{[s]} f(x_1) - D^{[s]} f(y_1)|^p}{|x_1 - y_1|^{1+\sigma p}} dx_1\, dx_2 = |D^{[s]} f|^p_{W^{\sigma,p}(\mathbb{T})}. \tag{8.32}$$

From (8.30), (8.31) and (8.32), we have $|F|_{W^{s,p}(\mathbb{T}^n)} \approx |f|_{W^{s,p}(\mathbb{T})}$. $\square$

8.5. Toolbox for optimal estimates when $sp \ge 1$

In this section, we establish the auxiliary results required in Section 5.

8.20. Lemma. *Let* $s > 0$ *and* $1 \le p < \infty$. *Let* $f \in C_c^\infty((0, 1); \mathbb{C})$, $f \not\equiv 0$. *Consider the functions* $f_j := \sum_{0\le k\le j-1} f(xj - k)$, $\forall j \ge 1$. *Then*

$$|f_j|_{W^{s,p}((0,1))} \approx j^s. \tag{8.33}$$

Proof. If s, k are integers and $k \le s$, then we have

$$\|D^k f_j\| \approx j^k. \tag{8.34}$$

In particular, (8.33) holds when s is an integer.

Suppose now that $s \notin \mathbb{N}$ and let $\sigma := s - [s] \in (0,1)$. Then we have

$$|D^{[s]} f_j|^p_{W^{\sigma,p}} = j^{[s]p} \sum_{k,\ell=0}^{j-1} I_{k,\ell}, \tag{8.35}$$

with

$$I_{k,\ell} := \int_{k/j}^{(k+1)/j} \int_{\ell/j}^{(\ell+1)/j} \frac{|f^{([s])}(xj-l) - f^{([s])}(yj-k)|^p}{|x-y|^{1+\sigma p}}\, dx\, dy = j^{1-\sigma p} \int_0^1 \int_0^1 \frac{|f^{([s])}(X) - f^{([s])}(Y)|^p}{|\ell - k + X - Y|^{1+\sigma p}}\, dX\, dY.$$

If $k \ne \ell$, then $I_{k,\ell}$ can be estimated as follows.

$$I_{k,\ell} \lesssim \sup |f^{([s])}|^p \frac{j^{\sigma p-1}}{|\ell - k|^{1+\sigma p}}. \tag{8.36}$$

(When $|\ell - k| \ge 2$, estimate (8.36) follows from the fact that $|\ell - k + X - Y| \approx |\ell - k|$. When $|\ell - k| = 1$, we rely on the fact that $f \in C_c^\infty((0,1))$, and thus there exists some $\varepsilon > 0$ such that $|f^{([s])}(X) - f^{([s])}(Y)| = 0$ when $|\ell - k + X - Y| \le \varepsilon$.)

Thus

$$\sum_{k \ne \ell} I_{k,\ell} \lesssim j^{\sigma p-1} \sum_{\ell=1}^{j-1} \sum_{k=0}^{\ell-1} \frac{1}{(l-k)^{1+\sigma p}} \lesssim j^{\sigma p-1} \left(j \sum_{k=1}^{j-1} \frac{1}{k^{1+\sigma p}} - \sum_{k=1}^{j-1} \frac{1}{k^{\sigma p}} \right) \lesssim j^{\sigma p}. \tag{8.37}$$

On the other hand, for $k = \ell$ we obtain

$$I_{k,k} = \int_{k/j}^{(k+1)/j} \int_{k/j}^{(k+1)/j} \frac{|f^{([s])}(xj-k) - f^{([s])}(yj-k)|^p}{|x-y|^{1+\sigma p}}\, dx\, dy = j^{\sigma p-1} |f^{([s])}|^p_{W^{s,p}((0,1))} \approx j^{\sigma p-1}.$$

Therefore, we have

$$\sum_{k=0}^{j-1} I_{k,k} \approx j^{\sigma p}. \tag{8.38}$$

By combining (8.34)–(8.38), we find that $|D^{[s]} f_j|_{W^{\sigma,p}((0,1))} \approx j^s$, and therefore $|f_j|_{W^{s,p}((0,1))} \approx j^s$. $\square$

The next result is a variant of [3, Lemma D.2].

8.21. Lemma. *Let $s \ge 1$, $1 \le p < \infty$ and $v \in W^{s,p}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Then*

$$\|v \wedge \nabla v\|_{W^{s-1,p}} \lesssim \|v\|_{L^\infty} \|v\|_{W^{s,p}}. \tag{8.39}$$

The proof of Lemma 8.21, as well as the one of Lemma 8.32, relies on Littlewood–Paley decompositions. For the convenience of the reader, we gather some standard properties of such decompositions.

Let $\zeta \in C_c^\infty(B(0,1); \mathbb{R}_+)$ be such that

$$\zeta \equiv 1 \quad \text{in } \overline{B}(0,3/4) \quad \text{and} \quad \operatorname{supp} \zeta \subset B(0,4/5). \tag{8.40}$$

Define φ_j, $j \ge 0$, by

$$\widehat{\varphi_0}(\xi) := \zeta(\xi) \quad \text{and, for every } j \ge 1, \quad \widehat{\varphi_j}(\xi) := \zeta(\xi/2^{j+1}) - \zeta(\xi/2^j). \tag{8.41}$$

Given $f \in \mathscr{S}'$, we let $f = \sum f_j = \sum f * \varphi_j$ be its Littlewood–Paley decomposition, and recall [29, Section 2.3.1, Definition 2, p. 45], [29, Section 2.5.7, Theorem, p. 90] that

$$\|f\|_{B^0_{\infty,\infty}} \sim \sup_j \|f_j\|_{L^\infty}, \tag{8.42}$$

$$\|f\|^p_{W^{s,p}} \sim \sum_j 2^{spj}\|f_j\|^p_{L^p}, \quad \forall s > 0,\ \forall 1 \le p < \infty,\ s \text{ non-integer}. \tag{8.43}$$

Recall also the following Nikolskiĭ type inequalities [31]. Set $\mathscr{C}_0 := B(0,2)$ and, for $j \ge 1$, $\mathscr{C}_j := B(0, 2^{j+1}) \setminus B(0, 2^{j-1})$. If $f^j \in \mathscr{S}'$ and

$$\operatorname{supp} \widehat{f^j} \subset \bigcup_{|\ell - j| \le k} \mathscr{C}_\ell \quad \text{for some fixed } k, \tag{8.44}$$

then

$$\Big\|\sum_j f^j\Big\|_{B^0_{\infty,\infty}} \lesssim \sup_j \|f^j\|_{L^\infty} \tag{8.45}$$

and

$$\Big\|\sum_j f^j\Big\|^p_{W^{s,p}} \lesssim \sum_j 2^{spj}\|f^j\|^p_{L^p}, \quad \forall s > 0,\ \forall 1 \le p < \infty,\ s \text{ non-integer}. \tag{8.46}$$

8.22. Remark. The inequality (8.46) also holds if the assumption (8.44) is weakened to $\operatorname{supp} \widehat{f^j} \subset B(0, 2^{j+k})$ for some fixed k [8, Lemma 1]; see also [31].

We next recall the following standard inequalities; see e.g. [11, Lemma 2.1.1] for the first result, and [8, Corollary 1, Lemma 2] for the next one.

8.23. Lemma. *Let $f \in \mathscr{S}'(\mathbb{R}^n)$ be such that $\operatorname{supp} \widehat{f} \subset B(0, R)$. Then, for any $1 \le q \le \infty$,*

$$\|\nabla f\|_{L^q} \le C(q) R \|f\|_{L^q}.$$

8.24. Lemma. *Let $1 \le q \le \infty$ and $f \in L^q(\mathbb{R}^n)$. Let $f = \sum f_i$ be the Littlewood–Paley decomposition of f. Then*

$$\Big\|\sum_{k \le j} f_k\Big\|_{L^q} \le C(q)\|f\|_{L^q}.$$

Proof of Lemma 8.21. Suppose first that $s \ge 1$ is an integer. Then we have

$$\|v \wedge \nabla v\|^p_{W^{s-1,p}} \lesssim \|v \wedge \nabla v\|^p_{L^p} + \big\|D^{s-1}(v \wedge \nabla v)\big\|^p_{L^p} \lesssim \|v\|^p_{L^\infty}\|\nabla v\|^p_{L^p} + \sum_{|\alpha|+|\beta|=s} \big\|D^\alpha v \wedge D^\beta v\big\|^p_{L^p}. \tag{8.47}$$

By applying the Hölder and the Gagliardo–Nirenberg inequalities, we find that, for every $m_1, m_2 \in \mathbb{N}$ with $m_1 + m_2 = s$,

$$\begin{aligned}\big\|D^{m_1} v \wedge D^{m_2} v\big\|_{L^p} &\lesssim \big\|D^{m_1} v\big\|_{L^{sp/m_1}} \big\|D^{m_2} v\big\|_{L^{sp/m_2}} \lesssim \big(\|v\|^{1-m_1/s}_{L^\infty}\big\|D^s v\big\|^{m_1/s}_{L^p}\big)\big(\|v\|^{1-m_2/s}_{L^\infty}\big\|D^s v\big\|^{m_2/s}_{L^p}\big) \\ &= \|v\|_{L^\infty}\big\|D^s v\big\|_{L^p} \lesssim \|v\|_{L^\infty}\|v\|_{W^{s,p}},\end{aligned}$$

which, together with (8.47), proves (8.39).

We next assume that $s > 1$ is not an integer. In this case, the proof uses the same idea as in [3, Lemma D.2]. We consider the Littlewood–Paley decomposition of v in $\mathscr{S}'(\mathbb{R}^n)$, $v = \sum_{j \ge 0} v_j := \sum_{j \ge 0} v * \varphi_j$, with the functions φ_j previously defined by (8.40) and (8.41).

We next define

$$r_j := v_j \wedge \nabla \sum_{k<j} v_k, \quad \forall j \geq 1,\ r_0 := 0, \quad \text{and} \quad s_j := \sum_{k\leq j} v_k \wedge \nabla v_j, \quad \forall j \geq 0.$$

Then we have $v \wedge \nabla v = \sum_{j\geq 0}(r_j + s_j)$. Note that $\operatorname{supp}(\widehat{r_j} + \widehat{s_j}) \subset B(0, 2^{j+2})$ and that $s - 1 > 0$. Hence, by (8.46) and Remark 8.22, we have that

$$\|v \wedge \nabla v\|^p_{W^{s-1,p}} = \Big\|\sum_{j\geq 0}(r_j + s_j)\Big\|^p_{W^{s-1,p}} \lesssim \sum_{j\geq 0} 2^{(s-1)pj}\|r_j + s_j\|^p_{L^p} \lesssim \sum_{j\geq 0} 2^{(s-1)pj}\big(\|r_j\|^p_{L^p} + \|s_j\|^p_{L^p}\big). \quad (8.48)$$

We will now estimate $\|r_j\|_{L^p}$ and $\|s_j\|_{L^p}$ using Lemmas 8.23 and 8.24. First, since $\operatorname{supp}\sum_{k<j}\widehat{v_k} \subset B(0, 2^{j+1})$, we have

$$\|r_j\|_{L^p} \leq \|v_j\|_{L^p}\Big\|\nabla \sum_{k<j} v_k\Big\|_{L^\infty} \lesssim 2^j\|v_j\|_{L^p}\Big\|\sum_{k<j} v_k\Big\|_{L^\infty} \lesssim 2^j\|v_j\|_{L^p}\|v\|_{L^\infty}.$$

Next, since $\operatorname{supp}\widehat{v_j} \subset B(0, 2^{j+1})$, we have

$$\|s_j\|_{L^p} \leq \Big\|\sum_{k\leq j} v_k\Big\|_{L^\infty}\|\nabla v_j\|_{L^p} \lesssim 2^j\Big\|\sum_{k\leq j} v_k\Big\|_{L^\infty}\|v_j\|_{L^p} \lesssim 2^j\|v\|_{L^\infty}\|v_j\|_{L^p}.$$

Combining the two above estimates with (8.43), (8.46) and (8.48), we find

$$\|v \wedge \nabla v\|^p_{W^{s-1,p}} \lesssim \|v\|^p_{L^\infty}\sum_{j\geq 0} 2^{spj}\|v_j\|^p_{L^p} \lesssim \|v\|^p_{L^\infty}\|v\|^p_{W^{s,p}}. \qquad \square$$

We now turn to the proof of some estimates used in the different proofs of Theorem 5.3 (Lemmas 8.25, 8.26, 8.27, 8.28, 8.30 and 8.31).

The next result appears in Merlet [20]. We present below a simplified argument.

8.25. Lemma. *Let $0 < s < 1$ and $1 < p < \infty$ be such that $sp > 1$, and let $0 \leq x \leq y \leq 1$. Let $u \in W^{s,p}(\mathbb{T}; \mathbb{S}^1)$ and let $\varphi \in W^{s,p}(\mathbb{T}; \mathbb{R})$ be a lifting of u. Then*

$$\big|\varphi(x) - \varphi(y)\big|^p \lesssim \big|u(x) - u(y)\big|^p + (y - x)^{p-1/s}|u|^{p/s}_{W^{s,p}((x,y))}.$$

Proof. We will show that

(a) $|\varphi(x) - \varphi(y)| \leq \pi \Longrightarrow |\varphi(x) - \varphi(y)| \lesssim |u(x) - u(y)|$.
(b) $|\varphi(x) - \varphi(y)| > \pi \Longrightarrow |\varphi(x) - \varphi(y)| \lesssim (y - x)^{1-1/sp}|u|^{1/s}_{W^{s,p}((x,y))}$.

The first case is obvious. Indeed, if $|\varphi(x) - \varphi(y)| \leq \pi$, then

$$\big|\varphi(x) - \varphi(y)\big| \leq \pi\left|\sin\frac{\varphi(x) - \varphi(y)}{2}\right| = \frac{\pi}{2}\big|u(x) - u(y)\big|.$$

Consider now the case where $|\varphi(x) - \varphi(y)| > \pi$. We may assume that $\varphi(x) = 0$. In addition, using the monotonicity in y of the right-hand side of (b), it suffices to establish (b) when y is replaced by $z \in [x, y]$ such that $|\varphi(z)| = \max_{[x,y]}|\varphi|$. Therefore, with no loss of generality we may assume that $\varphi(y) = \max_{[x,y]}|\varphi|$. Let α be such that $\pi < \alpha < \min\{|\varphi(y)|, 2\pi\}$, and decompose the interval $[x, y]$ as

$$[x, y] = [x_0, x_1] \cup [x_1, x_2] \cup \ldots \cup [x_J, x_{J+1}],$$

with

$$x_0 := x, \qquad J := \left[\frac{\varphi(y)}{\alpha}\right], \qquad x_j := \text{the smallest solution } t \text{ of } \varphi(t) = j\alpha, \quad \forall j \in [\![1, J]\!], \quad \text{and}$$
$$x_{J+1} := y. \quad (8.49)$$

We then have $J \ge 1$ and, by (8.49),

$$\left|\varphi(x) - \varphi(y)\right| = \varphi(y) < \alpha(J+1) \lesssim J. \tag{8.50}$$

We next note, as in [20], the following quantitative form of the continuous embedding $W^{s,p}((t,z)) \hookrightarrow C^{0,s-1/p}([t,z])$, with $sp > 1$ and $0 \le t \le z \le 1$:

$$\left|u(z_0) - u(t_0)\right| \le c(z-t)^{s-1/p}|u|_{W^{s,p}((z,t))}, \quad \forall 0 \le t < t_0 < z_0 < z \le 1, \tag{8.51}$$

where $c = c(s,p)$.[23]

When $|\varphi(z) - \varphi(t)| > \pi$ for some $t < z$, (8.51) implies that we necessarily have

$$(z-t)^{s-1/p}|u|_{W^{s,p}((t,z))} > 1/c$$

(with c as in (8.51)). Indeed, argue by contradiction. If $(z-t)^{s-1/p}|u|_{W^{s,p}((t,z))} \le 1/c$, then, by (8.51), we have

$$\left|u(z_0) - u(t_0)\right| \le 1, \quad \forall 0 \le t < t_0 < z_0 < z \le 1. \tag{8.52}$$

Using (8.52) and the continuity of u and φ, we find that $|\varphi(z_0) - \varphi(t_0)| \le \pi/3$ for every t_0 and z_0 as above. In particular, we obtain the contradiction $\varphi(z) - \varphi(t) < \pi$.

Therefore, for every $1 \le j \le J$, we have

$$(x_j - x_{j-1})^{s-1/p}|u|_{W^{s,p}((x_{j-1},x_j))} > 1/c. \tag{8.53}$$

Using (8.53), we find that

$$J = c^{1/s} \sum_{1 \le j \le J} 1/c^{1/s} \le c^{1/s} \sum_{1 \le j \le J} (x_j - x_{j-1})^{1-1/sp}|u|^{1/s}_{W^{s,p}((x_{j-1},x_j))}. \tag{8.54}$$

Applying in (8.54) Hölder's inequality with the exponents $sp/(sp-1)$ and sp, we obtain

$$J \lesssim \left(\sum_{1 \le j \le J} (x_j - x_{j-1})\right)^{1-1/sp} \left(\sum_{1 \le j \le J} |u|^p_{W^{s,p}((x_{j-1},x_j))}\right)^{1/sp} \lesssim (y-x)^{1-1/sp}|u|^{1/s}_{W^{s,p}(x,y)}.$$

This combined with (8.50) proves the assertion (b). □

We next establish several estimates used in the third proof of Theorem 5.3.

We start with the proof of (5.24). This estimate is certainly well-known to experts, but we were unable to find it in the literature and we present an argument for the sake of completeness.

8.26. Lemma. *Let $f \in L^1(\mathbb{S}^1; \mathbb{C})$ and let $\widetilde{f}$ be the harmonic extension of f. Let Mf be the maximal function of f. Then we have*

$$\left|\widetilde{f}(r\omega)\right| \le Mf(\omega), \quad \forall \omega \in \mathbb{S}^1,\ \forall r \in [0,1].$$

Proof. Let $P(x,y)$ be the Poisson kernel on the unit disc, i.e., $P(x,y) := \frac{1-r^2}{2\pi|x-y|^2}$. Here, $x = r\omega$, $\omega \in \mathbb{S}^1$, $r \in [0,1]$, and $y \in \mathbb{S}^1$. We note that $P(x,\cdot)$ is positive and "symmetric with respect to $O\omega$ and decreasing in y". More specifically, if y and y' are symmetric with respect to $O\omega$, then $P(x,y) = P(x,y')$. On the other hand, $P(x,\cdot)$ decreases with the distance from y to ω. This allows us to mimic the proof in [28, Chapter II, Section 2.1, formula (17), p. 57] and obtain the estimate

$$\left|\widetilde{f}(x)\right| \le \int_{\mathbb{S}^1} \left|f(y)\right| P(x,y)\,dy \le Mf(\omega) \int_{\mathbb{S}^1} P(x,y)\,dy = Mf(\omega). \qquad \square$$

We continue with the proof of the estimate (5.27), that we restate here for the convenience of the reader.

[23] Estimate (8.51) follows e.g. from [15, Lemma 1.1] (with $\Psi(t) := |t|^p$ and $p(t) := |t|^{s+1/p}$).

8.27. Lemma. *Let $1 \le p < \infty$ and $0 < s < 1$ be such that $sp \ge 1$. Let $u \in W^{s,p}(\mathbb{S}^1;\mathbb{S}^1)$ and let $\widetilde{u}$ be given by (5.26). Define $\varepsilon(\omega) := \int_0^1 |\operatorname{Jac}\widetilde{u}(r\omega)|\,dr$, $\forall \omega \in \mathbb{S}^1$. Then*

$$\|\varepsilon\|_{L^{sp}} \lesssim |u|_{W^{s,p}}^{1/s}. \tag{8.55}$$

In the proof of the above lemma, we will need the following cousin of [6, Lemma 1.3].

8.28. Lemma. *Let $1 \le p < \infty$ and $0 < s < 1$. Let $u \in W^{s,p}(\mathbb{T};\mathbb{S}^1)$ and let $v \in W^{s+1/p,p}(\mathbb{D};\mathbb{D})$ be its harmonic extension. Define $d(\omega) := \sup\{r \in (0,1);\ |v(r\omega)| \le 1/2\}$, $\forall \omega \in \mathbb{S}^1$ (with the convention $d(\omega) = 0$ if $|v(r\omega)| > 1/2$ for every r). Then*

$$\int_{\mathbb{S}^1} \frac{1}{(1-d(\omega))^{sp}}\,d\omega \lesssim |u|_{W^{s,p}}^p + 1. \tag{8.56}$$

Proof of Lemma 8.28. We may estimate the integral in (8.56) as follows:

$$\int_{\mathbb{S}^1} \frac{1}{(1-d(\omega))^{sp}}\,d\omega \lesssim \int_{\{d(\omega)>1/2\}} \frac{1}{(1-d(\omega))^{sp}}\,d\omega + 1.$$

Thus it suffices to consider the ω's such that $d(\omega) > 1/2$ and to prove, instead of (8.56), that

$$\int_{\{d(\omega)>1/2\}} \frac{1}{(1-d(\omega))^{sp}}\,d\omega \lesssim |u|_{W^{s,p}}^p. \tag{8.57}$$

We next note the following norm equivalence. In the domain $\mathbb{D} \setminus \overline{\mathbb{D}}_{1/2}$ (where $\overline{\mathbb{D}}_{1/2}$ is the disc $\{x \in \mathbb{C};\ |x| \le 1/2\}$) we have

$$|v|_{W^{s+1/p,p}(\mathbb{D}\setminus\mathbb{D}_{1/2})}^p \approx \int_{\mathbb{S}^1} \left|v(\cdot\omega)\right|_{W^{s+1/p,p}((1/2,1))}^p d\omega + \int_{1/2}^1 \left|v(r\cdot)\right|_{W^{s+1/p,p}(\mathbb{S}^1)}^p dr. \tag{8.58}$$

The above equivalence is standard in the flat case, where $\mathbb{D} \setminus \mathbb{D}_{1/2}$ is replaced by $\mathbb{R}^n \times (1/2,1)$, and $\mathbb{S}^1$ is replaced by $\mathbb{R}^n \times \{1\}$ [1, Theorem 7.46]. Estimate (8.58) is a straightforward variant of its "flat analog". We now note that (8.58) implies that for a.e. $\omega \in \mathbb{S}^1$, the map $(1/2,1) \ni r \mapsto v(r\omega)$ belongs to $W^{s+1/p,p}((1/2,1))$, and the latter space embeds into $C^{0,s}([1/2,1])$.[24] Therefore, we have

$$\frac{|v(\omega) - v(d(\omega)\omega)|}{(1-d(\omega))^s} \le \sup_{r,t\in[1/2,1]} \frac{|v(r\omega) - v(t\omega)|}{|r-t|^s} = \left|v(\cdot\omega)\right|_{C^{0,s}([1/2,1])} \lesssim \left|v(\cdot\omega)\right|_{W^{s+1/p,p}((1/2,1))}. \tag{8.59}$$

Since $|v(\omega) - v(d(\omega)\omega)| \ge |v(\omega)| - |v(d(\omega)\omega)| = 1/2$, we obtain from (8.59) that

$$\frac{1}{(1-d(\omega))^{sp}} \lesssim \frac{1}{|v(\omega) - v(d(\omega)\omega)|^p}\left|v(\cdot\omega)\right|_{W^{s+1/p,p}((1/2,1))}^p \lesssim \left|v(\cdot\omega)\right|_{W^{s+1/p,p}((1/2,1))}^p. \tag{8.60}$$

Integrating the above estimate, and using (8.58), we find that

$$\int_{\mathbb{S}^1} \frac{1}{(1-d(\omega))^{sp}} \lesssim \int_{\mathbb{S}^1} \left|v(\cdot\omega)\right|_{W^{s+1/p,p}((1/2,1))}^p \lesssim |v|_{W^{s+1/p,p}(\mathbb{D}\setminus\mathbb{D}_R)}^p \le |v|_{W^{s+1/p,p}(\mathbb{D})}^p. \tag{8.61}$$

Finally, since the Poisson extension operator is bounded from $W^{s,p}(\mathbb{S}^1)$ onto $W^{s+1/p,p}(\mathbb{D})$ [29, Thm. 4.3.3(i)], we have $|v|_{W^{s+1/p,p}(\mathbb{D})} \lesssim |u|_{W^{s,p}(\mathbb{S}^1)}$, which combined with (8.61) gives (8.57). □

[24] In particular, for a.e. $\omega \in \mathbb{S}^1$ we have $d(\omega) < 1$.

Proof of Lemma 8.27. We start by establishing (8.55) when $|u|_{W^{s,p}} \ll 1$. In this case, by the continuous embedding $W^{s,p}(\mathbb{S}^1) \hookrightarrow \mathrm{VMO}(\mathbb{S}^1)$ (valid when $sp \geq 1$), we also have $|u|_{\mathrm{BMO}} \ll 1$. Next we use the following property of the harmonic extension v of an $\mathbb{S}^1$-valued function:

$$\operatorname{dist}\big(v(x), \mathbb{S}^1\big) \lesssim |u|_{\mathrm{BMO}}, \quad \forall x \in \mathbb{D};$$

see Lemma 8.30 below. By combining the above estimate with the fact that $|u|_{\mathrm{BMO}} \ll 1$, we find that $\operatorname{dist}(v(x), \mathbb{S}^1) \leq 1/2$, $\forall x \in \mathbb{D}$. Therefore, $|v(x)| \geq 1/2$, $\forall x \in \mathbb{D}$. Recalling the definition of $\widetilde{u}$, this implies that $|\widetilde{u}| = 1$ in $\mathbb{D}$ and thus $|\operatorname{Jac} \widetilde{u}(x)| = 0$, $\forall x \in \mathbb{D}$. Thus the estimate (8.55) is trivially satisfied when $|u|_{W^{s,p}} \ll 1$.

Suppose now that $|u|_{W^{s,p}}$ is greater than some constant C. In this case, it suffices to prove, instead of (8.55), the following weaker estimate

$$\int_{\mathbb{S}^1} \left(\int_0^1 \big|\operatorname{Jac} \widetilde{u}(r\omega)\big|\, dr \right)^{sp} d\omega \lesssim |u|^p_{W^{s,p}} + 1. \tag{8.62}$$

Again considering the definition of $\widetilde{u}$, we note that $|\operatorname{Jac} \widetilde{u}(x)| \lesssim |\nabla v(x)|^2$, and that the Jacobian $\operatorname{Jac} \widetilde{u}(x)$ vanishes whenever $|v(x)| > 1/2$. Since the map $v : \mathbb{D} \to \mathbb{D}$ is harmonic, its gradient satisfies the estimate $|\nabla v(x)| \leq 1/\operatorname{dist}(x, \mathbb{S}^1) = (1 - |x|)^{-1}$. Consequently, using the notation $d(\omega)$ given in Lemma 8.28, we have

$$\int_0^1 \big|\operatorname{Jac} \widetilde{u}(r\omega)\big|\, dr = \int_0^{d(\omega)} \big|\operatorname{Jac} \widetilde{u}(r\omega)\big|\, dr \lesssim \int_0^{d(\omega)} \big|\nabla v(r\omega)\big|^2 dr \leq \int_0^{d(\omega)} \frac{1}{(1-r)^2}\, dr = \frac{1}{1 - d(\omega)}. \tag{8.63}$$

Using (8.63) together with Lemma 8.28, we obtain (8.62). □

8.29. Remark. By the Gagliardo–Nirenberg embedding $W^{s,p} \cap L^\infty \hookrightarrow W^{\theta s, p/\theta}$, $0 < \theta < 1$ (valid except when $s = p = 1$), it is possible to remove the condition $s < 1$ in the statement of Lemma 8.27.

In contrast, if we remove the condition $sp \geq 1$, then the first part of the proof of Lemma 8.27 does not hold anymore. However, the second part of the proof is still valid, and leads to the weaker conclusion $\|\varepsilon\|_{L^{sp}} \lesssim |u|^{1/s}_{W^{s,p}} + 1$ (valid whether for semi-norm $|u|_{W^{s,p}}$ is small or not).

The next result was used in the proof of Lemma 8.27.

8.30. Lemma. *Let* $u \in \mathrm{VMO}(\mathbb{S}^1; \mathbb{S}^1)$ *and let* $v : \mathbb{D} \to \mathbb{D}$ *be its harmonic extension to* $\mathbb{D}$. *Then*

$$\operatorname{dist}\big(v(x), \mathbb{S}^1\big) \lesssim |u|_{\mathrm{BMO}}, \quad \forall x \in \mathbb{D}.$$

Proof. Let $I(\omega, \delta) := B_\delta(\omega) \cap \mathbb{S}^1$, $\forall \omega \in \mathbb{S}^1$, $\forall 0 < \delta < 1$. By [10, Lemma A3.1], there exists an $R \in (0, 1)$ such that

$$\left| v(r\omega) - \fint_{I(\omega, 1-r)} u \right| \lesssim |u|_{\mathrm{BMO}}, \quad \forall r > R,\ \forall \omega \in \mathbb{S}^1.^{25} \tag{8.64}$$

On the other hand, from [9, Eq. (7), p. 206] we have

$$\operatorname{dist}\left(\fint_{I(\omega,\delta)} u, \mathbb{S}^1 \right) \lesssim |u|_{\mathrm{BMO}}, \quad \forall \omega \in \mathbb{S}^1,\ \forall \delta \leq 2. \tag{8.65}$$

By combining (8.64) and (8.65) we find that

$$\operatorname{dist}\big(v(r\omega), \mathbb{S}^1\big) \lesssim |u|_{\mathrm{BMO}}, \quad \forall r > R,\ \forall \omega \in \mathbb{S}^1. \tag{8.66}$$

It remains to obtain the conclusion of the lemma when $|x| \leq R$. For this purpose, we proceed as follows. We integrate the inequality $\operatorname{dist}(v(x), \mathbb{S}^1) \leq |v(x) - u(z)|$, $\forall z \in \mathbb{S}^1$, and find that

[25] A crucial point is that R does not depend on u.

$$\operatorname{dist}\big(v(x),\mathbb{S}^1\big)\le \fint_{\mathbb{S}^1}\big|v(x)-u(z)\big|\,dz=\fint_{\mathbb{S}^1}\bigg|\int_{\mathbb{S}^1}P(x,y)u(y)\,dy-u(z)\bigg|\,dz;$$

we recall that $P(x,y)$ denotes the Poisson kernel. Since $\int_{\mathbb{S}^1}P(x,y)\,dy\equiv 1$ and $|x|\le R$, we find that

$$\begin{aligned}\operatorname{dist}\big(v(x),\mathbb{S}^1\big)&\le \fint_{\mathbb{S}^1}\bigg|\int_{\mathbb{S}^1}P(x,y)\big[u(y)-u(z)\big]\,dy\bigg|\,dz\lesssim \fint_{\mathbb{S}^1}\fint_{\mathbb{S}^1}P(x,y)\big|u(y)-u(z)\big|\,dy\,dz\\&\lesssim \fint_{\mathbb{S}^1}\fint_{\mathbb{S}^1}\big|u(y)-u(z)\big|\,dy\,dz\lesssim |u|_{\mathrm{BMO}}.\end{aligned}$$ [26]

This estimate together with (8.66) concludes the proof. □

We now establish the following result, in the spirit of the theory of weighted Sobolev spaces [30], [29, Section 2.12.2, Theorem, p. 184]. For related results, see also [21,25].

8.31. Lemma. *Let $1\le p<\infty$ and $0<s<1$. Given $u\in C^\infty(\mathbb{T};\mathbb{C})$, let v be the harmonic extension of u and let $\widetilde{u}$ be given by (5.26). Let $\delta(x):=1-|x|$, $\forall x\in\mathbb{D}$. Then*

$$\int_{\mathbb{D}}\delta(x)^{p-sp-1}\big|\nabla v(x)\big|^p\,dx\lesssim |u|^p_{W^{s,p}}\quad\textit{and}\quad\int_{\mathbb{D}}\delta(x)^{p-sp-1}\big|\nabla\widetilde{u}(x)\big|^p\,dx\lesssim |u|^p_{W^{s,p}}. \tag{8.67}$$

Proof. We start by noting that it suffices to prove the first inequality in (8.67). Indeed, we have $\widetilde{u}=\Pi\circ v$, with Π smooth, and therefore $|\nabla\widetilde{u}|\lesssim|\nabla v|$. Therefore, the second estimate in (8.67) is a consequence of the first one.

Let $P(x,y)$ denote the Poisson kernel in the unit disc $\mathbb{D}$. Since v is the harmonic extension of u in $\mathbb{D}$ and $\int_{\mathbb{S}^1}\nabla_x P(x,y)\,dy=0$,[27] we have

$$\nabla v(x)=\int_{\mathbb{S}^1}\nabla_x P(x,y)u(y)\,dy=\int_{\mathbb{S}^1}\nabla_x P(x,y)\big[u(y)-u(\omega)\big]\,dy,\quad\forall\omega\in\mathbb{S}^1. \tag{8.68}$$

We next pass to polar coordinates into the first integral in (8.67). Using the fact that $r\le 1$, we obtain

$$\int_{\mathbb{D}}\delta(x)^{p-sp-1}\big|\nabla v(x)\big|^p\,dx\le\int_0^1\int_{\mathbb{S}^1}\delta(r\omega)^{p-sp-1}\big|\nabla v(r\omega)\big|^p\,d\omega\,dr. \tag{8.69}$$

We next estimate $|\nabla v(r\omega)|$. For this purpose, we rely on the following properties of $\nabla_x P(x,y)$:

$$\big|\nabla_x P(x,y)\big|\lesssim\frac{1}{\delta(x)^2},\quad\forall x\in\mathbb{D},\ \forall y\in\mathbb{S}^1 \tag{8.70}$$

and

$$\big|\nabla_x P(r\omega,y)\big|\lesssim 1/|y-\omega|^2,\quad\forall\omega\in\mathbb{S}^1,\ \forall r\in[0,1),\ \forall y\in\mathbb{S}^1\text{ such that }|y-\omega|\ge\delta(r\omega). \tag{8.71}$$

The above inequalities are obtained as follows. We start from the straightforward estimates

$$\big|\nabla_x P(x,y)\big|\lesssim\frac{1}{|x-y|^2}+\frac{1-|x|}{|x-y|^3}\lesssim\frac{1}{|x-y|^2}. \tag{8.72}$$

Then (8.70) is a consequence of (8.72) combined with $|x-y|\ge\delta(x)$. On the other hand, we have $|y-r\omega|\ge 1-r$ and therefore

[26] The next to the last inequality comes from the fact that $P(x,y)$ is uniformly bounded when $|x|\le R$ and $y\in\mathbb{S}^1$.
[27] This follows by differentiating the identity $\int_{\mathbb{S}^1}P(x,y)\,dy\equiv 1$.

$$|y-\omega| \le |y-r\omega| + |\omega - r\omega| = |y-r\omega| + 1 - r \le 2|y-r\omega|. \tag{8.73}$$

Estimate (8.71) is a consequence of (8.72) and of (8.73).

We return to (8.68) and we split the integral as follows:

$$\begin{aligned} \left|\nabla v(r\omega)\right|^p &\le \Big(\int_{\mathbb{S}^1} \left|\nabla_x P(r\omega,y)\right|\left|u(y)-u(\omega)\right| dy\Big)^p \\ &\lesssim \Big(\int_{|y-\omega|\le\delta(r\omega)} \left|\nabla_x P(r\omega,y)\right|\left|u(y)-u(\omega)\right| dy\Big)^p \\ &\quad + \Big(\int_{|y-\omega|\ge\delta(r\omega)} \left|\nabla_x P(r\omega,y)\right|\left|u(y)-u(\omega)\right| dy\Big)^p =: I_1(r,\omega) + I_2(r,\omega). \end{aligned} \tag{8.74}$$

Then estimates (8.69) and (8.74) lead to

$$\int_{\mathbb{D}} \delta(x)^{p-sp-1}\left|\nabla v(x)\right|^p dx \lesssim \int_0^1 \int_{\mathbb{S}^1} \delta(r\omega)^{p-sp-1}\big[I_1(r,\omega)+I_2(r,\omega)\big]\, d\omega\, dr =: J_1 + J_2. \tag{8.75}$$

It remains to estimate J_1 and J_2.

Using (8.70) and Hölder's inequality we find

$$\begin{aligned} I_1(r,\omega) &\lesssim \Big(\int_{|y-\omega|\le\delta(r\omega)} \frac{|u(y)-u(\omega)|}{\delta(r\omega)^2}\, dy\Big)^p = \frac{1}{\delta(r\omega)^{2p}}\Big(\int_{|y-\omega|\le\delta(r\omega)} \left|u(y)-u(\omega)\right| dy\Big)^p \\ &\lesssim \frac{1}{\delta(r\omega)^{2p}}\delta(r\omega)^{p-1} \int_{|y-\omega|\le\delta(r\omega)} \left|u(y)-u(\omega)\right|^p dy = \frac{1}{\delta(r\omega)^{p+1}} \int_{|y-\omega|\le\delta(r\omega)} \left|u(y)-u(\omega)\right|^p dy. \end{aligned}$$

Inserting the above estimate of $I_1(r,\omega)$ in the expression of J_1, we find that

$$\begin{aligned} J_1 &\lesssim \int_0^1 \int_{\mathbb{S}^1} \frac{1}{\delta(r\omega)^{sp+2}} \int_{|y-\omega|\le\delta(r\omega)} \left|u(y)-u(\omega)\right|^p dy\, d\omega\, dr \\ &= \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \Big(\int_0^{1-|y-\omega|} \frac{1}{(1-r)^{sp+2}}\, dr\Big) \left|u(y)-u(\omega)\right|^p dy\, d\omega \\ &\lesssim \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{|u(y)-u(\omega)|^p}{|y-\omega|^{sp+1}}\, dy\, d\omega = |u|^p_{W^{s,p}}. \end{aligned} \tag{8.76}$$

Similarly, for I_2 we use (8.71) and Hölder's inequality as follows:

$$\begin{aligned} I_2(r,\omega) &\lesssim \Big(\int_{|y-\omega|>\delta(r\omega)} \frac{1}{|y-\omega|^2}\left|u(y)-u(\omega)\right| dy\Big)^p = \Big(\int_{|y-\omega|>\delta(r\omega)} \frac{|u(y)-u(\omega)|}{|y-\omega|^{2-\alpha}}|y-\omega|^{-\alpha}\, dy\Big)^p \\ &\le \int_{|y-\omega|>\delta(r\omega)} \frac{|u(y)-u(\omega)|^p}{|y-\omega|^{(2-\alpha)p}}\, dy \Big(\int_{|y-\omega|>\delta(r\omega)} |y-\omega|^{-\alpha p/(p-1)}\, dy\Big)^{p-1}. \end{aligned} \tag{8.77}$$

Assuming that $\alpha > 1 - 1/p$, the last integral in (8.77) can be estimated as follows:

$$\int_{|y-\omega|>\delta(r\omega)} |y-\omega|^{-\alpha p/(p-1)}\, dy \lesssim \delta(r\omega)^{-\alpha p/(p-1)+1}.$$

Hence, returning to (8.77), we have

$$I_2(r,\omega) \lesssim \delta(r\omega)^{-\alpha p+p-1} \int\limits_{|y-\omega|>\delta(r\omega)} \frac{|u(y)-u(\omega)|^p}{|y-\omega|^{(2-\alpha)p}}\,dy.$$

Using the above estimate of $I_2(r,\omega)$ in J_2 we find

$$\begin{aligned} J_2 &\lesssim \int_0^1 \int_{\mathbb{S}^1} \delta(r\omega)^{2p-sp-2-\alpha p} \int\limits_{|y-\omega|\ge\delta(r\omega)} \frac{|u(y)-u(\omega)|^p}{|y-\omega|^{(2-\alpha)p}}\,dy\,d\omega\,dr \\ &= \int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\Bigg(\int_{1-|y-\omega|}^1 (1-r)^{2p-sp-2-\alpha p}\,dr\Bigg)\frac{|u(y)-u(\omega)|^p}{|y-\omega|^{(2-\alpha)p}}\,dy\,d\omega. \end{aligned} \tag{8.78}$$

By the above, if we choose $\alpha\in(1-1/p, 2-s-1/p)$, then we obtain $J_2 \lesssim |u|^p_{W^{s,p}}$. This estimate, together with (8.76) and (8.75), leads to $\int_{\mathbb{D}} \delta(x)^{p-sp-1}|\nabla v(x)|^p\,dx \lesssim |u|^p_{W^{s,p}}$. □

8.6. Toolbox for "further thoughts"

This section contains the lemmas needed in Section 6.

We start by proving that property (R) discussed in Section 6.1 holds in the following weaker form.

8.32. Lemma. *Let $0<s<\infty$ and $1\le p<\infty$ be such that s and sp are not integers. Then*

$$W^{s,p}\big((0,1)^n\big) = \big(W^{s,p}\big((0,1)^n\big)\cap B^0_{\infty,\infty}\big((0,1)^n\big)\big) + \big(W^{s,p}\big((0,1)^n\big)\cap W^{sp,1}\big((0,1)^n\big)\big). \tag{8.79}$$

Proof. By a standard extension argument, it suffices to prove that the above holds when $(0,1)^n$ is replaced by $\mathbb{R}^n$.

In order to obtain the analog of (8.79) in the whole $\mathbb{R}^n$, we rely on Littlewood–Paley decompositions.[28] Let $\eta,\lambda\in C^\infty_c(B(0,1);\mathbb{R}_+)$ be such that $\eta=1$ in $\overline{B}(0,4/5)$ and $\lambda\equiv1$ in $\overline{B}(0,4/5)\setminus B(0,3/8)$. Define ψ_j, $j\ge0$, by

$$\widehat{\psi_0}(\xi):=\eta(\xi) \quad\text{and, for every } j\ge1,\quad \widehat{\psi_j}(\xi):=\lambda\big(\xi/2^j\big).$$

It is easy to check that, with φ_j given by (8.40) and (8.41), we have $\widehat{\varphi_j}\widehat{\psi_j}=\widehat{\varphi_j}$, and thus

$$\varphi_j*\psi_j=\varphi_j,\quad \forall j. \tag{8.80}$$

On the other hand, we have

$$\|\psi_j\|_{L^1} = \big\|\big(\mathscr{F}^{-1}\lambda\big)_{2^{-j}}\big\|_{L^1} = \big\|\mathscr{F}^{-1}\lambda\big\|_{L^1},\quad \forall j\ge1. \tag{8.81}$$

Let $f\in W^{s,p}$. We split $f_j=g_j+h_j$, where $g_j:=f_j\mathbb{1}_{\{|f_j|\le1\}}$ and $h_j:=f_j\mathbb{1}_{\{|f_j|>1\}}$. Clearly,

$$\|g_j\|_{L^p}\le\|f_j\|_{L^p},\qquad \|g_j\|_{L^\infty}\le1,\qquad \|h_j\|_{L^1}\le\|f_j\|^p_{L^p}. \tag{8.82}$$

Using (8.80), (8.81) and (8.82), we obtain

$$f_j=f_j*\psi_j=g_j*\psi_j+h_j*\psi_j:=G_j+H_j,$$

with

$$\|G_j\|_{L^p}\lesssim\|f_j\|_{L^p},\qquad \|G_j\|_{L^\infty}\lesssim1,\qquad \|H_j\|_{L^1}\lesssim\|f_j\|^p_{L^p} \tag{8.83}$$

and

[28] Alternatively, we could use wavelets as in [18].

$$\operatorname{supp}\widehat{G_j}, \qquad \operatorname{supp}\widehat{H_j}\subset\operatorname{supp}\widehat{\psi_j}\subset\mathscr{C}_j\cup\mathscr{C}_{j-1}. \tag{8.84}$$

By (8.45), (8.46), (8.83) and (8.84), we find that $f=g+h$, where $g:=\sum G_j$ and $h:=\sum H_j$ satisfy

$$g\in W^{s,p}\cap B^0_{\infty,\infty},\ h\in W^{s,p}\cap W^{sp,1}, \quad \|g\|_{B^0_{\infty,\infty}}\lesssim 1, \qquad \|g\|^p_{W^{s,p}}+\|h\|_{W^{sp,1}}\lesssim\|f\|^p_{W^{s,p}}. \qquad \square \tag{8.85}$$

We now prove Lemma 6.3, used in Section 6.2 for constructing a lifting in $W^{s,p}(\mathbb{T}^n;\mathbb{S}^1)$ when $sp<1$.

Proof of Lemma 6.3. Assume first that U is smooth in $\mathbb{T}^n\times[0,1]$. In this case we have $f(x)=U(x,0)$. Let $x,y\in\mathbb{T}^n$ and set $r:=|y-x|\in[0,1]$ and $\omega:=(y-x)/|y-x|$, which satisfies $|\omega|=1$. We have

$$\begin{aligned}\big|f(y)-f(x)\big| &\le \big|f(y)-U(y,r)\big|+\big|f(x)-U(x,r)\big|+\big|U(y,r)-U(x,r)\big|\\ &\le \int_0^r\big|\nabla U(y,\varepsilon)\big|\,d\varepsilon+\int_0^r\big|\nabla U(x,\varepsilon)\big|\,d\varepsilon+\int_0^r\big|\nabla U(x+\varepsilon\omega,r)\big|\,d\varepsilon:=F(x,r,\omega).\end{aligned} \tag{8.86}$$

Integrating (8.86), we find that

$$\int_{\mathbb{T}^n}\big|f(x+r\omega)-f(x)\big|\,dx\le\int_{\mathbb{T}^n}F(x,r,\omega)\,dx. \tag{8.87}$$

Assume next that U is not necessarily smooth up to $\varepsilon=0$. Then we may assume that

$$\int_{\mathbb{T}^n\times(0,1)}\varepsilon^{-\sigma}\big|\nabla U(x,\varepsilon)\big|\,dx\,d\varepsilon<0,$$

for otherwise there is nothing to prove. Then $U\in W^{1,1}(\mathbb{T}^n\times(0,1))$. By a standard approximation procedure, we find that (with $f=\operatorname{tr}U$) inequality (8.87) still holds for such U.

By combining (8.87) with the formula of the $W^{\sigma,1}$ semi-norm and passing to spherical coordinates, we find that

$$\begin{aligned}|f|_{W^{\sigma,1}(\mathbb{T}^n)} &= \int_{\mathbb{T}^n\times\mathbb{T}^n}\frac{|f(y)-f(x)|}{|y-x|^{n+\sigma}}\,dx\,dy\\ &\lesssim \int_{\mathbb{T}^n\times\mathbb{T}^n}\frac{1}{|y-x|^{n+\sigma}}\left(\int_0^{|y-x|}\big|\nabla U(x,\varepsilon)\big|\,d\varepsilon\right)dx\,dy\\ &\quad+\int_{\mathbb{T}^n\times\mathbb{T}^n}\frac{1}{|y-x|^{n+\sigma}}\left(\int_0^{|y-x|}\big|\nabla U\big(x+\varepsilon(y-x)/|y-x|,|y-x|\big)\big|\,d\varepsilon\right)dx\,dy\\ &\lesssim \int_{\mathbb{T}^n}\int_0^1\frac{1}{r^{1+\sigma}}\int_0^r\big|\nabla U(x,\varepsilon)\big|\,d\varepsilon\,dr\,dx+\int_{\mathbb{T}^n}\int_0^1\int_{\mathbb{S}^{n-1}}\frac{1}{r^{1+\sigma}}\int_0^r\big|\nabla U(x+\varepsilon\omega,r)\big|\,d\varepsilon\,d\omega\,dr\,dx\\ &\lesssim \int_{\mathbb{T}^n\times(0,1)}\left(\int_\varepsilon^1\frac{1}{r^{1+\sigma}}\,dr\right)\big|\nabla U(x,\varepsilon)\big|\,dx\,d\varepsilon\\ &\quad+\int_{(0,1)}\frac{1}{r^{1+\sigma}}\left(\int_{\mathbb{S}^{n-1}\times(0,r)}\left(\int_{\mathbb{T}^n}\big|\nabla U(x+\varepsilon\omega,r)\big|\,dx\right)d\omega\,d\varepsilon\right)dr\\ &\lesssim \int_{\mathbb{T}^n\times(0,1)}\varepsilon^{-\sigma}\big|\nabla U(x,\varepsilon)\big|\,dx\,d\varepsilon,\end{aligned}$$

i.e., (6.5) holds. $\square$

Conflict of interest statement

None declared.

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