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Derivation of a homogenized von-Kármán shell theory from 3D elasticity

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Abstract

We derive homogenized von Kármán shell theories starting from three dimensional nonlinear elasticity. The original three dimensional model contains two small parameters: the period of oscillation ε of the material properties and the thickness h of the shell. Depending on the asymptotic ratio of these two parameters, we obtain different asymptotic theories. In the case $h \ll \varepsilon$ we identify two different asymptotic theories, depending on the ratio of h and ε^2. In the case of convex shells we obtain a complete picture in the whole regime $h \ll \varepsilon$.

Keywords: Elasticity; Dimension reduction; Homogenization; Shell theory; Two-scale convergence

Contents

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1. Introduction

This paper is about von Kármán's theory for thin elastic shells. There is a vast literature on shell theories in elasticity. An overview of the derivation of models for linear and nonlinear shells by the method of formal asymptotic expansions can be found in [7]. In the case of linearly elastic shells, the models thus obtained can also be justified by a rigorous convergence result, starting from three dimensional linearized elasticity.

In the last two decades, rigorous justifications of nonlinear models for rods, curved rods, plates and shells were obtained by means of Γ-convergence, starting from three dimensional nonlinear elasticity. The first papers in that direction are [20,21] for membranes (plate and shells, respectively). The rigorous derivation of the nonlinear bending theory of plates was achieved in [9]. The Föppl-von Kármán theory for plates was derived in [10]. In [11] the nonlinear bending theory for shells was derived, and in [22] the von Kármán theory for shells was derived. In [23] the authors obtain limit models, in an intermediate energy scaling regime between bending and von Kármán theories, for the special case of elliptic surfaces.

Here we are interested in an ansatz-free derivation of the homogenized von Kármán shell theory by simultaneous homogenization and dimension reduction. Our starting point is the energy functional of $3d$ nonlinear elasticity. It attributes to a deformation u of a given shell $S^h \subset \mathbb{R}^3$ of small thickness $h > 0$ around a surface $S \subset \mathbb{R}^3$ the stored elastic energy

$$\frac{1}{h^4|S^h|}\int_{S^h} W_\varepsilon\big(x, \nabla u(x)\big)\,dx, \quad u \in H^1\big(S^h, \mathbb{R}^3\big). \tag{1}$$

Here W_ε is a non-degenerate stored energy function that oscillates periodically in x, with some period $\varepsilon \ll 1$. We are interested in the effective behavior when both the thickness h and the period ε are small. The separate limits $h \to 0$ and $\varepsilon \to 0$ are reasonably well understood: In [22] it is shown that, when W_ε does not depend on ε, then the functionals (1) Γ-converge, as $h \to 0$, to a two-dimensional von Kármán shell theory. Regarding the limit $\varepsilon \to 0$, which is related to homogenization, the first rigorous results relevant in nonlinear elasticity were obtained by Braides [6] and independently by Müller [26]. They proved that, under suitable growth assumptions on W_ε, the energy (1) Γ-converges as $\varepsilon \to 0$ (for fixed h) to the functional obtained by replacing W_ε in (1) with the homogenized energy density given by an infinite-cell homogenization formula.

In this paper we study the asymptotic behavior when both the thickness h and the period ε tend to zero *simultaneously*. Such a combination of dimensional reduction and homogenization has already been studied in numerous papers; we shall mention just few of them. In [13] the authors study the effects of simultaneous homogenization and dimensional reduction for linear elasticity system without periodicity assumption introducing a variant of H-convergence adapted to dimensional reduction. In [5] the authors study the same effects for nonlinear systems (membrane plate) by means of Γ-convergence, also without periodicity assumptions. In [8] the authors study nonlinear monotone operators in the context of simultaneous homogenization and dimensional reduction, without periodicity assumption. Much earlier in [19] the authors study the same effects for the linear rod case where it was assumed that the rod is homogeneous along its central line, but the microstructure is given in the cross section. We also mention the work of Arrieta on Laplace equation in thin domains with an oscillatory boundary (see e.g. [4]). In [27,28] the author systematically combined the techniques from [9,10] with two-scale convergence to obtain a model of homogenized rods in the bending regime. Independently, the second author obtained a model of periodically wrinkled plate in the von Kármán regime, also by a combination of dimensional reduction and two-scale convergence techniques (see [33]). This work was inspired by the earlier works [1,3] on wrinkled plates derived from Koiter shell models by two-scale convergence techniques (without dimensional reduction).

In this paper, we obtain two-dimensional von Kármán shell theory with homogenized material properties as asymptotic theories (i.e., Γ-limits). Recently the plate model in the von Kárman regime (see [29]) and the bending regime (see [15,31]) was analyzed. As explained there (and before in [27,33,28]), in these cases one does not obtain an infinite-cell homogenization formula as in the membrane case (see [5]). The basic reason for that is the fact that we are in the small strain regime and that the energy is essentially convex in the strain. This is why we can use two-scale convergence techniques in all these cases. However, every case has its own peculiarities. In the von Kármán theory of plates, one obtains a limiting quadratic energy density which is continuous in the asymptotic ratio γ between h and ε, for all $\gamma \in [0, \infty]$. Moreover, the case $\gamma = 0$ corresponds to the situation when the dimensional reduction dominates and the resulting model is just the homogenized plate model in von Kármán regime. The situation $\gamma = \infty$ corresponds to the case when homogenization dominates and the resulting model is the plate model in the von Kármán regime corresponding to the homogenized functional. Recently the second author identified the limiting model of a homogenized plate in the von Kármán regime without periodicity assumption (see [32]). The nonlinear bending theory of plates is more involved in the periodic case. In [15] the authors obtain asymptotic models in the case $\gamma \in (0, \infty]$. In [31] the author obtained the asymptotic model corresponding to the regime $\gamma = 0$ under the additional assumption that $\varepsilon^2 \ll h \ll \varepsilon$.

We also emphasize the fact that bending theories of plates and von Kármán shell theories require a different and more involved approach when compared to earlier results (such as [27,33,28,29]). This is partially due to the fact that the compactness results given in [9,22] are more subtle than those for rods and for the von Kármán theory of plates, and that is more difficult to identify the oscillatory part of the limiting strain. In the case of the von Kármán theory it is not even clear how to homogenize the two-dimensional equations of shells, due to the appearance of the space $\mathcal{B}$ (the space of L^2-limits of symmetrized gradients on shells) in the compactness result, cf. Section 4. The result of our paper shows, in particular, that in order to answer questions about the homogenization of von Kármán shell theories one necessarily needs to start from the $3d$ elasticity equations and perform simultaneous homogenization and dimensional reduction. As in the case of the bending theory of plates, the regimes where dimensional reduction dominates are of particular interest. The case of the bending theory of shells is still open, and it seems likely to be even more involved.

Here we encounter two different scenarios in the regime $h \ll \varepsilon$, depending on whether $h \sim \varepsilon^2$ or $h \gg \varepsilon^2$. Our main result is presented in Theorem 3.1. We are not able to cover the case $h \ll \varepsilon^2$ in a generic way for arbitrary reference surfaces S. A stronger influence of the geometry of the reference surface S is expected in this case. In fact, in the case when S is a convex surface, we are able to derive the limiting model even in the regime $h \ll \varepsilon^2$, see Theorem 6.2 below. For a heuristic explanation why the scaling $h \sim \varepsilon^2$ is critical for shells, but not for plates, we refer to Remark 1 below.

Our analysis requires both techniques from dimension reduction, in particular, the quantitative rigidity estimate and approximation schemes developed in [9,10]; and techniques from homogenization methods, in particular, two-scale convergence [2,34,35].

Other questions about the homogenization of shells are addressed in numerous papers in the mathematical and engineering literature, see e.g. [24] and references therein, see also [12] in the context of linear piezoelectric perforated shell. The homogenization for linearly elastic shells was carried out in [25]. To our knowledge, ours is the first rigorous result combining homogenization and dimension reduction for shells in the von Kármán energy regime, and indeed the first one addressing the rigorous derivation of homogenized nonlinear shell theories from $3d$ elasticity. Along the way we develop a geometric framework for the von Kármán shell theory which is new and applicable even in the homogeneous case studied in [23].

This paper is organized as follows: after introducing the setting and basic objects in Sections 2 and 3 we state the main result in Section 3. In Section 4 we identify the two-scale limit of the strain and prove the lower bound for the Γ-limit. In Section 5 we construct the recovery sequences and thus prove the sharpness of the lower bound. All these results are obtained for general surfaces and in the scaling regimes $h \gg \varepsilon^2$ and $h \sim \varepsilon^2$. In the last section, we address the scaling regime $h \ll \varepsilon^2$ under the additional assumption that the shell be convex.

Notation. The notation $A \lesssim B$ means that $A \le CB$ with C depending only on quantities regarded as constant in the context in question; we also write $A \ll B$ to denote that $A/B \to 0$. We set $Y = [0, 1)^2$ and we denote by $\mathcal{Y}$ the Euclidean space $\mathbb{R}^2$ equipped with the torus topology, that is for all $z \in \mathbb{Z}^2$ the points $y+z$ and y are identified in $\mathcal{Y}$. We write $C^0(\mathcal{Y})$ to denote the space of continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfying $f(y + z) = f(y)$ for all $z \in \mathbb{Z}^2$. We denote by $C^k(\mathcal{Y})$ those functions in $C^k(\mathbb{R}^2) \cap C^0(\mathcal{Y})$ whose derivatives up to the k-th order belong to $C^0(\mathcal{Y})$. We denote by $L^2(\mathcal{Y})$, $H^1(\mathcal{Y})$ and $H^1(S\times\mathcal{Y})$ the Banach spaces obtained as the closure of $C^\infty(\mathcal{Y})$ and $C^\infty(\bar{S}, C^\infty(\mathcal{Y}))$ with respect to the norm in $L^2(Y)$, $H^1(Y)$ and $H^1(S\times Y)$, respectively. By $\dot{L}^2(\mathcal{Y})$, $\dot{H}^k(\mathcal{Y})$ etc. we denote the subspaces of $L^2(\mathcal{Y})$, $H^k(\mathcal{Y})$ etc. whose mid-value over $\mathcal{Y}$ is zero. For $A \subset \mathbb{R}^d$ measurable and X a Banach space, $L^2(A, X)$ is understood in the sense of Bochner. We identify the spaces $L^2(A, L^2(B))$ and $L^2(A \times B)$ in usual way. Standard basis vectors in $\mathbb{R}^2$ are denoted by e_i. By SO(3) we denote the set of rotational matrices in $\mathbb{R}^{3\times 3}$, by so(3) the space of skew symmetric matrices in $\mathbb{R}^{3\times 3}$, while by $\mathbb{R}^{3\times 3}_{\text{sym}}$ the space of symmetric matrices in $\mathbb{R}^{3\times 3}$.

2. Setting the stage

2.1. Geometry

Throughout this paper, $\omega \subset \mathbb{R}^2$ is a bounded domain with boundary of class C^3. We set $I = (-\frac{1}{2}, \frac{1}{2})$ and $\Omega^h = \omega \times (hI)$, and $\Omega = \omega \times I$. From now on, S denotes a compact connected oriented surface with boundary which is embedded in $\mathbb{R}^3$. For convenience we assume that S is parametrized by a single chart: From now on, $\psi \in C^3(\overline{\omega}; \mathbb{R}^3)$ denotes an embedding with $\psi(\omega) = S$. The inverse of ψ is denoted by $r : S \to \omega$, and we assume it to be C^3 up to the boundary. We leave it to the interested reader to verify to which extent these regularity assumptions on S can be weakened without altering our arguments.

By $g = (\nabla\psi)^T(\nabla\psi)$ we denote the Riemannian metric on ω induced by ψ. Its Christoffel symbols are denoted by $\Gamma^\gamma_{\alpha\beta}$. In what follows we recall some standard notions and set the notation. Readers who are unfamiliar with geometry may safely regard objects such as Hess f simply as short-hand notations. All notions are discussed in detail in most basic textbooks on Riemannian geometry.

- The volume element on S is denoted by $dvol_S$.
- The scalar product on a vector space V is denoted by $\langle x, y\rangle_V$, and we define $x \cdot y = \langle x, y\rangle_{\mathbb{R}^3}$.
- We denote by TS the tangent bundle over S, i.e., the collection of tangent spaces T_xS with $x \in S$. A basis of T_xS is given by $\tau_1(x)$, $\tau_2(x)$, defined by

  $$\tau_\alpha(x) = (\partial_\alpha\psi)\big(r(x)\big) \quad \text{for all } x \in S.$$

  We can regard T_xS as a subspace of $\mathbb{R}^3$; then we write $\sigma \cdot \tau = \langle\sigma, \tau\rangle_{T_xS}$.
- By T^*S we denote the cotangent bundle. A basis of T^*_xS is given by $(\tau^1(x), \tau^2(x))$ dual to (τ_1, τ_2). It is uniquely determined by the condition

  $$\tau^\alpha(\tau_\beta) = \delta_{\alpha\beta},$$

  where $\delta_{\alpha\beta}$ is the Kronecker symbol. Observe that τ^α is more commonly denoted by dx^α, but we will not use that notation. We can identify T_xS with T^*_xS via $\tau \mapsto \langle\tau, \cdot\rangle$. Via this identification we can identify $\tau \in T^*S$ with the unique vector $v \in T_xS \subset \mathbb{R}^3$ with the property that $\tau = \langle v, \cdot\rangle_{\mathbb{R}^3}$ on T_xS.
- By $n : S \to \mathbb{S}^2$ we denote the unit normal to S, i.e.,

  $$n(x) = \frac{\tau_1(x) \wedge \tau_2(x)}{|\tau_1(x) \wedge \tau_2(x)|} \quad \text{for all } x \in S.$$

  We define $\tau^3 = \tau_3 = n$. The normal bundle of S is denoted by NS and by definition has fibers N_xS given by the span of $n(x)$. We denote by

  $$T_S(x) = I - n(x) \otimes n(x)$$

  the orthogonal projection from $\mathbb{R}^3$ onto T_xS. We will frequently deal with vector fields $V : S \to \mathbb{R}^3$ on the surface. By $V_{\tan}$ we denote the projection of a vector field V onto the tangent space, i.e., $V_{\tan} = T_SV$.
- The tensor product bundles $TS \otimes TS$ etc. are defined fiberwise. If T_xS is regarded as a subspace of $\mathbb{R}^3$, then $T^*_xS \otimes T^*_xS$ can be regarded as a subspace of $\mathbb{R}^{3\times 3}$.

  The symmetric product $E \odot F$ of two vector spaces (or bundles) E and F by definition consists of elements of the form

  $$a \odot b := \frac{1}{2}(a \otimes b + b \otimes a)$$

  with $a \in E$ and $b \in F$.

Sections of the bundle $T^*S \odot T^*S$ are called quadratic forms on S. We define the pull-back of a quadratic form q by

$$\psi^* q = q(\psi)(\nabla\psi, \nabla\psi).$$

Sections B of $T^*S \otimes T^*S$ can be regarded as maps from S into $\mathbb{R}^{3\times 3}$ via the embedding ι defined by $\iota(B) = B(T_S, T_S)$. (On the right-hand side and elsewhere we identify $(\mathbb{R}^3)^* \otimes (\mathbb{R}^3)^*$ with $\mathbb{R}^{3\times 3}$.) By definition, $B(T_S, T_S) : S \to \mathbb{R}^{3\times 3}$ takes the vector fields $v, w : S \to \mathbb{R}^3$ into the function $x \mapsto B(x)(T_S(x)v(x), T_S(x)w(x))$.

- For any vector bundle E over S we denote by $L^2(S; E)$ the space of all L^2-sections of E. The spaces $H^1(S; E)$ etc. are defined similarly. Explicitly, e.g. for $E = TS$ we have

$$L^2(S; TS) = \{ f \in L^2(S; \mathbb{R}^3) : f(x) \in T_x S \text{ for a.e. } x \in S \}.$$

- For any vector bundle E over S with fibers E_x, we denote by $L^2(\mathcal{Y}, E)$ the vector bundle over S with fibers $L^2(\mathcal{Y}, E_x)$. The bundles $H^1(\mathcal{Y}, E)$ etc. are defined similarly. For example, L^2-sections of the bundle $H^1(\mathcal{Y}, TS)$ are given by

$$L^2(S; H^1(\mathcal{Y}, TS)) = \{ Z \in L^2(S; H^1(\mathcal{Y}, \mathbb{R}^3)) : Z(x) \in H^1(\mathcal{Y}; T_x S) \text{ for a.e. } x \in S \}.$$

- For a scalar function $f : S \to \mathbb{R}$ its gradient field along S will be denoted by df, which is also the notation for the corresponding 1-form. In other words,

$$df(x)(\tau) = \nabla_\tau f(x) \quad \text{for all } \tau \in T_x S.$$

Here and elsewhere ∇ denotes the usual gradient on $\mathbb{R}^3$ (or on $\mathbb{R}^2$) of the extension of f, and $\nabla_\tau f = \tau \cdot \nabla f = \sum_i \tau_i \partial_i f$. We extend these definitions componentwise to maps into $\mathbb{R}^3$.

- For tangent vectorfields τ, σ we define the covariant derivative $D_\sigma \tau$ of τ in direction σ as usual by the formula $D_\sigma \tau = T_S(\nabla_\sigma \tau)$ The covariant derivative extends naturally to cotangent vectorfields and to tensor fields.
- For a scalar function $f : S \to \mathbb{R}$ the Hessian Hess f is defined as usual to be the section of $T^*S \odot T^*S$ given by the covariant derivative of the gradient field of f, i.e., Hess $f = Ddf$. In local coordinates we have

$$(\text{Hess } f)(\psi) = \big(\partial_\alpha \partial_\beta (f \circ \psi) - \Gamma^\gamma_{\alpha\beta} \partial_\gamma (f \circ \psi)\big) \tau^\alpha(\psi) \otimes \tau^\beta(\psi). \tag{2}$$

Here and in what follows we tacitly sum over repeated greek indices from 1 to 2. We extend the definition of Hess componentwise to maps $f : S \to \mathbb{R}^k$.

- For functions $f \in L^2(S, H^2(\mathcal{Y}))$ we define $\text{Hess}_\mathcal{Y} f$ to be the section of the bundle $L^2(\mathcal{Y}; T^*S \odot T^*S)$ over S given by

$$(\text{Hess}_\mathcal{Y} f)(x, y) = (\nabla_y^2 f)_{\alpha\beta}(x, y) \tau^\alpha(x) \otimes \tau^\beta(x),$$

where $(\nabla_y^2 f)_{\alpha\beta} = \partial_{y_\alpha} \partial_{y_\beta} f$.

- For $v \in L^2(S; H^1(\mathcal{Y}; \mathbb{R}^2))$ we define the section $\text{Def}_\mathcal{Y} v$ of the bundle $L^2(\mathcal{Y}, T^*S \odot T^*S)$ by

$$(\text{Def}_\mathcal{Y} v)(x, y) = \big(\text{sym} \nabla_y v(x, y)\big)_{\alpha\beta} \tau^\alpha(x) \otimes \tau^\beta(x).$$

Here and elsewhere ∇_y is gradient in $\mathcal{Y}$ with respect to the variable y.

- The Weingarten map $\mathbf{S}$ of S is given by $\mathbf{S} = dn$, i.e.,

$$\mathbf{S}(x)\tau = (\nabla_\tau n)(x) \quad \text{for all } x \in S, \ \tau \in T_x M.$$

We extend $\mathbf{S}$ to a linear map on $TS \oplus NS \stackrel{\sim}{=} \mathbb{R}^3$ by setting $\mathbf{S} = \mathbf{S} \circ T_S$, i.e., we define $\mathbf{S}(x)n(x) = 0$.

Using the Weingarten map, the covariant derivative of a tangent vector field τ along another tangent vector field σ is given by

$$D_\sigma \tau = \nabla_\sigma \tau + \langle \mathbf{S}\tau, \sigma \rangle n, \tag{3}$$

or briefly: $D\tau = \nabla\tau + n \otimes \mathbf{S}\tau$.

2.2. Displacements and infinitesimal bendings

With a given displacement $V : S \to \mathbb{R}^3$ one associates the following quantities:

- The quadratic form $(dV)^2$ given by $(dV)^2(\tau, \eta) = \nabla_\tau V \cdot \nabla_\eta V$ for all tangent vectorfields τ, η along S.
- The quadratic form q_V given by

$$q_V(\tau, \eta) = \frac{1}{2}(\eta \cdot \nabla_\tau V + \tau \cdot \nabla_\eta V),$$

for all tangent vectorfields τ, η along S. Setting $\widetilde{V} = V \circ \psi$, we the following expression for the pull-back of q_V:

$$\psi^* q_V = \operatorname{sym}\big((\nabla\psi)^T \nabla\widetilde{V}\big). \tag{4}$$

It is well-known that the quadratic form q_V typically arises in the context of thin elastic shells, because it is just the first variation of the metric of S under the displacement V. For example, in [14] its pullback is denoted (in coordinates) by $\gamma_{\alpha\beta}$ and in [22] q_V is denoted by $\operatorname{sym}\nabla V$.
- The cotangent vectorfield μ_V given by

$$\mu_V = -n \cdot dV \equiv -\sum_{i=1}^{3} n_i dV_i. \tag{5}$$

- The map $\Omega_V : S \to \mathbb{R}^{3\times 3}$ given by

$$\Omega_V = dV \circ T_S + \mu_V \otimes n. \tag{6}$$

- The quadratic form b_V (called linearized Weingarten map) given by

$$b_V = n \cdot \operatorname{Hess} V \equiv \sum_{i=1}^{3} n_i \cdot \operatorname{Hess} V_i. \tag{7}$$

Following common notation, for *tangent* vector fields v along S the quadratic form corresponding to q_v is denoted by $\operatorname{Def}_S v$ and called deformation tensor of v. It is given by the Lie-derivative of the metric in direction v, i.e.,

$$(\operatorname{Def}_S v)(\tau, \eta) = \frac{1}{2}(\eta \cdot D_\tau v + \tau \cdot D_\eta v) = \frac{1}{2}(\eta \cdot \nabla_\tau v + \tau \cdot \nabla_\eta v),$$

for all tangent vectorfields τ and σ.

A displacement $V : S \to \mathbb{R}^3$ is called an *infinitesimal bending* of S provided that $q_V = 0$, i.e.,

$$\tau \cdot \nabla_\sigma V + \sigma \cdot \nabla_\tau V = 0 \quad \text{on } S \tag{8}$$

for all tangent vector fields τ and σ. Infinitesimal bendings have been studied extensively both in the applied literature (see e.g. [7]) and in the geometry literature (see e.g. the references in [17]). Recently, they have been found to be relevant as well to fully nonlinear bending theories, cf. [16]. In the next lemma we collect some useful identities.

Lemma 2.1. *Let $V \in H^1(S; \mathbb{R}^3)$. Then we have, almost everywhere on S,*

$$q_V = \operatorname{Def}_S V_{\tan} + (V \cdot n)\mathbf{S} \tag{9}$$

$$\operatorname{sym}\Omega_V = q_V(T_S, T_S) \tag{10}$$

$$\mu_V = T_S \frac{\nabla_{\tau_1} V \wedge \tau_2 + \tau_1 \wedge \nabla_{\tau_2} V}{|\tau_1 \wedge \tau_2|}. \tag{11}$$

If, moreover, V is an infinitesimal bending, then we have

$$\Omega_V^2(T_S, T_S) = -(dV)^2(T_S, T_S), \tag{12}$$

that is, $\Omega_V^2(\tau, \sigma) = -\partial_\tau V \cdot \partial_\sigma V$ for all tangent vector fields τ, σ along S.

Proof. The identities (9) and (11) can be verified by direct computations. To prove (10), note that $n\cdot\Omega_V n = n\cdot\mu_V = 0$ and for any tangent vector field τ along S we have

$$\tau\cdot\Omega_V n + n\cdot\Omega_V\tau = \tau\cdot\mu_V + n\cdot\partial_\tau V = 0$$

by (5). For any tangent vector field σ we have $\tau\cdot\Omega_V\sigma + \sigma\cdot\Omega_V\tau = 2q_V(\sigma,\tau)$.

To prove (12) note that, by skew symmetry (cf. (8)), $\Omega_V^2(\tau,\sigma) = -\Omega_V\tau\cdot\Omega_V\sigma$, and $\partial_\tau V = \Omega_V\tau$. □

We denote by $\mathcal{B}$ the L^2-closure of the set $\{q_V : V\in H^1(S;\mathbb{R}^3)\}$. As this is a linear space, its strong and its weak L^2-closure coincide. The set $\mathcal{B}$ is a closed linear subspace of $L^2(S; T^*S\odot T^*S)$. The space $\mathcal{B}$ is also encountered in the context of shell models derived from linearized elasticity; see [14] for details.

2.3. *The nearest point retraction*

The nearest point retraction of a tubular neighbourhood of S onto S will be denoted by π. Hence

$$\pi\big(x+tn(x)\big) = x \quad \text{whenever } |t| \text{ is small enough.}$$

After rescaling the ambient space, we may assume that the curvature of S is as small as we please. In particular, we may assume without loss of generality that π is well-defined on a domain containing the closure of $\{x+tn(x);\ x\in S,\ -1<t<1\}$, and that $|Id + t\mathbf{S}(x)|\in(1/2,3/2)$ for all $t\in(-\frac{1}{2},\frac{1}{2})$ and all $x\in S$.

For a subset $A\subset S$ and $h\in(0,1]$ we define $A^h = \{x+tn(x):\ x\in S,\ -h/2<t<h/2\}$. In particular, the shell is given by

$$S^h = \big\{x+tn(x):\ x\in S,\ t\in(-h/2,h/2)\big\}.$$

We introduce the function $t: S^1\to\mathbb{R}$ by

$$t(x) = \big(x-\pi(x)\big)\cdot n(x) \quad \text{for all } x\in S^1. \tag{13}$$

We extend all maps $f: S\to\mathbb{R}^k$ trivially from S to S^1, simply by defining

$$f(x) = f\big(\pi(x)\big) \quad \text{for all } x\in S^1. \tag{14}$$

In particular, we extend r, T_S and $\mathbf{S}$ trivially to S^1, i.e., we have $\mathbf{S}(x) = \mathbf{S}(\pi(x))$ and $T_S(x) = T_S(\pi(x))$ and $r(x) = r(\pi(x))$ for all $x\in S^1$.

Lemma 2.2. *For all $x\in S^1$ we have $(\nabla\pi)(x) = T_S(x)(I+t(x)\mathbf{S}(x))^{-1}$.*

Proof. Let $t\in[-1,1]$, let $x\in S$, let $\tau\in T_xS$ and let $\gamma\in C^1((-1,1),M)$ with $\gamma(0)=x$ and $\dot\gamma(0)=\tau$. Then $\pi(\gamma + tn(\gamma)) = \gamma$ on $(-1,1)$. Taking the derivative in zero this implies $(\nabla\pi)(\gamma+tn(\gamma))(\tau + t\mathbf{S}(\gamma)\tau) = \tau$. As $x\in S$ and $\tau\in T_xS$ were arbitrary, we conclude that

$$(\nabla\pi)\big(x+tn(x)\big)\big(I+t\mathbf{S}(x)\big) = T_S(x), \tag{15}$$

on T_xS. But by definition $\mathbf{S}(x)n(x)=0$, and clearly $(\nabla\pi)(x+tn(x))n(x)=0$, too. Hence both sides of (15) agree on all of $\mathbb{R}^3$. □

The easy proof of the following lemma is left to the reader.

Lemma 2.3. *Let $f: S\to\mathbb{R}^k$. Then the following formula holds for the full derivative of its trivial extension $f\circ\pi$ in terms of the derivative of f:*

$$\nabla(f\circ\pi)(x) = (df)\big(\pi(x)\big)T_S\big(\pi(x)\big)\big(I+t(x)\mathbf{S}\big(\pi(x)\big)\big)^{-1} \quad \textit{for all } x\in S^1. \tag{16}$$

The following formula links the Hessian of f to that of its trivial extension:

$$(\mathrm{Hess}\, f)(\tau,\sigma) = \nabla^2(f\circ\pi):(\tau\otimes\sigma) \quad \textit{on } S \textit{ for all tangent vector fields } \tau,\sigma.$$

The following lemma summarizes a computation that will later be used for the generic type of ansatz functions.

Lemma 2.4. *Let $h \in (0, 1/2)$, let $V \in H^2(S; \mathbb{R}^3)$, and for $x \in S^h$ define the displacement $\rho : S^h \to \mathbb{R}^3$ by setting $\rho(x) = V(\pi(x)) + t(x)\mu_V(\pi(x))$ for all $x \in S^h$. Then the following equality holds on S^h:*

$$\nabla\rho = \Omega_V - tb_V(T_S, T_S) - 2tq_V(\mathbf{S} \circ T_S, T_S) - t^2\nabla\mu_V \circ \mathbf{S} + (dV + t\nabla\mu_V) \circ T_S \circ Q, \tag{17}$$

where Q is as in (18) below, and where we extend V, μ_V, Ω_V, b_V, dV etc. trivially from S to S^h.

Proof. For all $x \in S^h$ define

$$Q(x) = \big(I + t(x)\mathbf{S}(x)\big)^{-1} - \big(I - t(x)\mathbf{S}(x)\big). \tag{18}$$

Since clearly $\nabla t = n$, formulae (16) and (3) show that on S^h:

$$\begin{aligned}\nabla\rho &= (dV + t\nabla\mu_V)T_S(I + t\mathbf{S})^{-1} + \mu_V \otimes n \\ &= (dV + tD\mu_V - tn \otimes \mathbf{S}\mu_V)T_S(I - t\mathbf{S}) + \mu_V \otimes n + (dV + t\nabla\mu_V)T_S Q.\end{aligned}$$

By the definition of Ω_V it remains to show that

$$(D\mu_V - n \otimes \mathbf{S}\mu - dV \circ \mathbf{S}) \circ T_S = 2q_V(\mathbf{S} \circ T_S, T_S) - b_V(T_S, T_S).$$

But in fact, recalling (11), the action of the left-hand side on a tangent vector τ is given by

$$\begin{aligned}&-D_\tau(n \cdot dV) + (n \cdot \nabla_{\mathbf{S}\tau} V)n - \nabla_{\mathbf{S}\tau} V \\ &\quad = -n \cdot D_\tau dV - (\mathbf{S}\tau)_i\, dV_i \circ T_S - T_S(\nabla_{\mathbf{S}\tau} V).\end{aligned}$$

The first term on the right gives rise to $-b_V$, and the last two are readily seen to give rise to $-2q_V(\mathbf{S} \circ T_S, T_S)$. □

2.4. *Thin films*

To deal with thin films, we introduce the map $\Psi : \omega \times \mathbb{R} \to \mathbb{R}^3$ by setting

$$\Psi\big(z', z_3\big) = \psi\big(z'\big) + z_3 n\big(\psi\big(z'\big)\big) \quad \text{for all } z' \in \omega \text{ and } z_3 \in \mathbb{R}.$$

As in [9] we will use the diffeomorphism $\widetilde{\Phi}^h : \Omega_h \to \Omega$ given by $\widetilde{\Phi}^h(z_1, z_2, z_3) = (z_1, z_2, z_3/h)$, and for a map $\widetilde{y} : \Omega \to \mathbb{R}^3$ we introduce the scaled gradient $\widetilde{\nabla}_h y = (\partial_1 y, \partial_2 y, \frac{1}{h}\partial_3 y)$. The counterpart of $\widetilde{\Phi}^h$ on the shell is the diffeomorphism $\Phi^h : S^h \to S^1$ given by

$$\Phi^h(x) = \pi(x) + \frac{t(x)}{h} n(x). \tag{19}$$

It is easy to see that

$$\Phi^h \circ \Psi = \Psi \circ \widetilde{\Phi}^h \quad \text{on } \Omega_h. \tag{20}$$

For given $u : S^h \to \mathbb{R}^3$ we define its pulled back version $\widetilde{u} : \Omega_h \to \mathbb{R}^3$ by $\widetilde{u} = u \circ \Psi$. We also define its rescaled version $y : S^1 \to \mathbb{R}^3$ by $y(\Phi^h) = u$ on S^h and we define the pulled back version $\widetilde{y}$ of this map by $\widetilde{y} = y \circ \Psi$. Then it is easy to see that

$$(\widetilde{\nabla}_h \widetilde{y}) \circ \widetilde{\Phi}^h = \nabla\widetilde{u} \quad \text{on } \Omega_h. \tag{21}$$

We define the rescaled gradient $\nabla_h y$ of y by the condition

$$(\nabla_h y) \circ \Phi^h = \nabla u \quad \text{on } S^h. \tag{22}$$

Using (21) and (20) it is easy to see that

$$\widetilde{\nabla}_h \widetilde{y} = \nabla u(\Psi)\big((\nabla\Psi) \circ \big(\widetilde{\Phi}^h\big)^{-1}\big). \tag{23}$$

Since $\nabla t = n$, Lemma 2.2 and formula (16) show (recall that n is extended trivially to S^1):

$$\nabla\Phi^h = \nabla\pi + \frac{t}{h}\nabla n + \frac{1}{h} n\otimes n = T_S(I+t\mathbf{S})^{-1} + \frac{t}{h}\mathbf{S}T_S(I+t\mathbf{S})^{-1} + \frac{1}{h}n\otimes n.$$

Since T_S clearly commutes with $\mathbf{S}$, we see that T_S commutes with $(I+t\mathbf{S})^{-1}$ as well. Hence

$$\nabla\Phi^h = \left(I_h + \frac{t}{h}\mathbf{S}\right)(I+t\mathbf{S})^{-1} \quad \text{on } S^h, \tag{24}$$

where $I_h = T_S + \frac{1}{h} n\otimes n$. To express $\nabla_h y$ in terms of ∇y, insert the definition of y into (22) and use (24) to find

$$\nabla_h y = \nabla y(I_h + t\mathbf{S})(I+ht\mathbf{S})^{-1} \quad \text{on } S^1. \tag{25}$$

2.5. Two-scale convergence

Recall that we extend the chart r trivially from S to S^1. We make the following definitions:

- A sequence $(f^h)\subset L^2(S^1)$ is said to converge weakly two-scale on S^1 to the function $f\in L^2(S^1, L^2(\mathcal{Y}))$ as $h\to 0$, provided that the sequence (f^h) is bounded in $L^2(S^1)$ and

$$\lim_{h\to 0}\int_{S^1} f^h(x)\rho\big(x, r(x)/\varepsilon\big)\,dx = \int_{S^1}\int_{\mathcal{Y}} f(x,y)\rho(x,y)\,dy\,dx, \tag{26}$$

  for all $\rho\in C_c^0(S^1, C^0(\mathcal{Y}))$.
- We say that f^h strongly two-scale converges to f if, in addition,

$$\lim_{h\to 0}\|f^h\|_{L^2(S^1)} = \|f\|_{L^2(S^1\times Y)}.$$

- For a sequence $(f^h)\subset L^2(S^1)$ and for $f_1\in L^2(S^1\times\mathcal{Y})$ with $\int_{\mathcal{Y}} f_1(x,y)\,dy = 0$ for almost every $x\in S^1$, we write $f^h \xrightarrow{osc} f_1$ provided that

$$\int_{S^1} f^h(x)\varphi(x)\rho\big(r(x)/\varepsilon\big)\,dx \to \int_{S^1}\int_{\mathcal{Y}} f_1(x,y)\varphi(x)\rho(y)\,dy\,dx \tag{27}$$

  for all $\varphi\in C_0^\infty(S^1)$ and all $\rho\in C^\infty(\mathcal{Y})$ with $\int_{\mathcal{Y}}\rho\,dy = 0$.

We write $f^h \overset{2}{\rightharpoonup} f$ to denote weak two-scale convergence and $f^h \overset{2}{\to} f$ to denote strong two-scale convergence. If $f^h \overset{2}{\rightharpoonup} f$ then $f^h \rightharpoonup \int_{\mathcal{Y}} f(\cdot,y)\,dy$ weakly in L^2. If f^h is bounded in $L^2(S^1)$ then it has a subsequence which converges weakly two-scale to some $f\in L^2(S^1; L^2(\mathcal{Y}))$. These and other facts can be deduced from the corresponding results on planar domains (cf. [2,34]) by means of the following simple observations.

Defining $\widetilde{f^h} = f^h\circ\Psi$ and $\widetilde{f}(z,y) = f(\Psi(z), y)$, and taking

$$\widetilde{\rho}(z,y) = \rho\big(\Psi(z), y\big)\big(\det\nabla\Psi^T(z)\nabla\Psi(z)\big)^{1/2},$$

a change of variables shows that (26) is equivalent to

$$\int_{\Omega}\widetilde{f^h}(z)\widetilde{\rho}\big(z, z'/\varepsilon\big)\,dz \to \int_{\Omega}\int_{\mathcal{Y}}\widetilde{f}(z,y)\widetilde{\rho}(z,y)\,dy\,dz, \tag{28}$$

where z' is the projection of z onto $\mathbb{R}^2$. Hence $f^h \overset{2}{\rightharpoonup} f$ on S^1 if and only if $\widetilde{f^h} \overset{2}{\rightharpoonup} \widetilde{f}$ on Ω in the usual sense.

When $f^h : S\to\mathbb{R}$, then $f^h \overset{2}{\rightharpoonup} f$ on S means, by definition, that the trivial extensions converge weakly two-scale on S^1. In particular, $f^h \overset{2}{\rightharpoonup} f$ on S if and only if $\widetilde{f^h} \overset{2}{\rightharpoonup} \widetilde{f}$ on ω. All these definitions are extended componentwise to vector-valued maps. For quadratic forms q, q^h on S we say $q^h \overset{2}{\rightharpoonup} q$ if $q^h(\tau,\sigma) \overset{2}{\rightharpoonup} q(\tau,\sigma)$ for all $\tau,\sigma\in C^1(S, TS)$. A similar definition applies to other bundles. Using the pull-back $\psi^* q$ of the quadratic form q, it is easy to see that

$$q^h \xrightharpoonup{2} q \quad \text{on } S \quad \text{if and only if} \quad \psi^* q^h \xrightharpoonup{2} \psi^* q \quad \text{on } \omega. \tag{29}$$

It is also easily seen that if $f^h \xrightharpoonup{2} f$ then $f^h \xrightharpoonup{osc} f - \int_{\mathcal{Y}} f(\cdot, y)\, dy$.

Lemma 2.5. *Let (w^h) be a bounded sequence in $H^2(S, \mathbb{R}^k)$. Then there exist $w_0 \in H^2(S, \mathbb{R}^k)$ and $w_1 \in L^2(S; \dot{H}^2(\mathcal{Y}; \mathbb{R}^k))$ such that (after passing to a subsequence)*

$$\operatorname{Hess} w^h \xrightharpoonup{2} \operatorname{Hess} w_0 + \operatorname{Hess}_{\mathcal{Y}} w_1 \quad \text{on } S \tag{30}$$

as sections of $\mathbb{R}^k \otimes T^*S \otimes T^*S$. Moreover, if $k = 3$ and if we define $\widehat{w}_1 \in L^2(S, \dot{H}^2(\mathcal{Y}, \mathbb{R}^2))$ by setting $(\widehat{w}_1)_\alpha = w_1 \cdot \tau_\alpha$ for $\alpha = 1, 2$, then $q_{w^h}/\varepsilon \xrightharpoonup{osc} \operatorname{Def}_{\mathcal{Y}} \widehat{w}_1$.

Proof. We assume without loss of generality that $k = 1$ and we set $\widetilde{w}^h = w^h(\psi)$. Clearly $\nabla^2 \widetilde{w}^h$ is bounded in $L^2(\omega)$, so by classical results about two-scale convergence there exist $\widetilde{w}_0 \in H^2(\omega)$ and $\widetilde{w}_1 \in L^2(\omega; H^2(\mathcal{Y}))$ such that

$$\nabla^2 \widetilde{w}^h \xrightharpoonup{2} \nabla^2 \widetilde{w}_0 + \nabla_y^2 \widetilde{w}_1 \quad \text{on } \omega. \tag{31}$$

Define w_0 and w_1 by $w_0(\psi) = \widetilde{w}_0$ and $w_1(\psi(z'), y) = \widetilde{w}_1(z', y)$. Since the lower order term in (2) converges strongly, (31) implies that

$$\psi^*\big(\operatorname{Hess} w^h\big) \xrightharpoonup{2} \psi^*(\operatorname{Hess} w_0) + \psi^*(\operatorname{Hess}_{\mathcal{Y}} w_1) \quad \text{on } \omega.$$

By (29) this is equivalent to (30).

To prove the last assertion, recall (31) and apply Lemma 2.6 with $f^h = \nabla \widetilde{w}^h$ (componentwise). It implies that

$$(\nabla\psi)^T \nabla \widetilde{w}^h \xrightharpoonup{osc} (\nabla\psi)^T \nabla_y \widetilde{w}_1 = \nabla_y\big((\nabla\psi)^T \widetilde{w}_1\big).$$

But by definition $\widehat{w}_1(\psi(z'), y) = (\nabla\psi(z))^T \widetilde{w}_1(z', y)$, and

$$\psi^*(\operatorname{Def}_{\mathcal{Y}} \widehat{w}_1)\big(z', y\big) = \operatorname{sym} \nabla_y\big(\widehat{w}_1\big(\psi\big(z'\big), y\big)\big).$$

Hence the claim follows from (4) and (29). □

Results such as the following one (which was used as it stands in the proof of Lemma 2.5) can be also adapted to the curved setting following the same pattern as above (writing just S^1 instead of Ω):

Lemma 2.6.

(i) *Let f_0 and $f^h \in H^1(\Omega)$ be such that $f^h \rightharpoonup f_0$ weakly in $H^1(\Omega)$ and assume that $\nabla f^h \xrightharpoonup{2} \nabla f_0 + \nabla_y \phi$ for some $\phi \in L^2(\Omega; \dot{H}^1(\mathcal{Y}))$. Then $\frac{f^h}{\varepsilon} \xrightharpoonup{osc} \phi$.*

(ii) *Let f_0 and $f^h \in H^1(\Omega)$ be such that $f^h \rightharpoonup f_0$ weakly in $H^2(\Omega)$ and assume that $\nabla f^h \xrightharpoonup{2} \nabla f_0 + \nabla_y \phi$ for some $\phi \in L^2(\Omega; \dot{H}^1(\mathcal{Y}))$. $\nabla^2 f^h \xrightharpoonup{2} \nabla^2 f_0 + \nabla_y^2 \phi$ for some $\phi \in L^2(\Omega; \dot{H}^2(\mathcal{Y}))$. Then $\frac{f^h}{\varepsilon^2} \xrightharpoonup{osc} \phi$.*

Proof. The proof of (i) can be found in [15, Lemma 3.7], and that of (ii) is similar. □

3. Elasticity framework and main result

Throughout this paper we assume that the limit

$$\gamma := \lim_{h \to 0} \frac{h}{\varepsilon(h)}$$

exists in $[0, \infty]$. We will frequently write ε instead of $\varepsilon(h)$, but always with the understanding that ε depends on h.

From now on we fix a Borel measurable energy density

$$W : S^1 \times \mathbb{R}^2 \times \mathbb{R}^{3\times 3} \to \mathbb{R}^+ \cup \{+\infty\}$$

with the following properties:

- $W(\cdot, y, \boldsymbol{F})$ is continuous for almost every $y \in \mathbb{R}^2$ and $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$.
- $W(x, \cdot, \boldsymbol{F})$ is $\mathcal{Y}$-periodic for all $x \in S^1$ and almost every $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$.
- For all $(x, y) \in S^1 \times \mathcal{Y}$ we have $W(x, y, I) = 0$ and $W(x, y, \boldsymbol{R}\boldsymbol{F}) = W(x, y, \boldsymbol{F})$ for all $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$, $\boldsymbol{R} \in \mathrm{SO}(3)$.
- There exist constants $0 < \alpha \le \beta$ and $\rho > 0$ such that for all $(x, y) \in S^1 \times \mathcal{Y}$ we have

$$W(x, y, F) \ge \alpha \operatorname{dist}^2\big(\boldsymbol{F}, \mathrm{SO}(3)\big) \quad \text{for all } F \in \mathbb{R}^{3\times 3}$$
$$W(x, y, F) \le \beta \operatorname{dist}^2\big(\boldsymbol{F}, \mathrm{SO}(3)\big) \quad \text{for all } F \in \mathbb{R}^{3\times 3} \text{ with } \operatorname{dist}^2\big(\boldsymbol{F}, \mathrm{SO}(3)\big) \le \rho.$$

- For each $(x, y) \in S^1 \times \mathcal{Y}$ there exists a quadratic form $\mathcal{Q}(x, y, \cdot) : \mathbb{R}^{3\times 3} \to \mathbb{R}$ such that

$$\operatorname*{ess\,sup}_{(x,y)\in S^1\times\mathcal{Y}} \frac{|W(x, y, I + G) - \mathcal{Q}(x, y, G)|}{|G|^2} \to 0 \quad \text{as } G \to 0. \tag{32}$$

Clearly $\mathcal{Q}(\cdot, y, \cdot)$ is continuous for almost every $y \in \mathbb{R}^2$ and $\mathcal{Q}(x, \cdot, \boldsymbol{G})$ is Y-periodic for all $x \in S^1$ and all $\boldsymbol{F} \in \mathbb{R}^{3\times 3}$.

The elastic energy per unit thickness of a deformation $u^h \in H^1(S^h; \mathbb{R}^3)$ of the shell S^h is given by

$$J^h\big(u^h\big) = \frac{1}{h} \int_{S^h} W\big(\Phi^h(x), r(x)/\varepsilon, \nabla u^h(x)\big)\, dx.$$

In order to express the elastic energy in terms of the new variables, we associate with $y : S^1 \to \mathbb{R}^3$ the energy

$$I^h(y) = \int_{S^1} W\big(x, r(x)/\varepsilon, \nabla_h y(x)\big) \det\big(I + t(x)\mathbf{S}(x)\big)^{-1}\, dx$$
$$= \int_S \int_I W\big(x + tn(x), r(x)/\varepsilon, \nabla_h y\big(x + tn(x)\big)\big)\, dt\, dvol_S(x).$$

By a change of variables we have

$$J^h\big(u^h\big) = \frac{1}{h} \int_{S^1} W\big(x, r(x)/\varepsilon, \nabla_h y^h(x)\big) \big|\det \nabla\big(\Phi^h\big)^{-1}(x)\big|\, dx.$$

Using (24) it is easy to see that

$$J^h\big(u^h\big) = I^h\big(y^h\big)\big(1 + O(h)\big) \quad \text{as } h \to 0, \tag{33}$$

where $|O(h)| \le Ch$.

3.1. Asymptotic energy functionals and main result

Next we will introduce the asymptotic energy functionals. In order to do so, we need the definition of the relaxation fields and the cell formulas.

Recall that $a \odot b = \frac{1}{2}(a \otimes b + b \otimes a)$. We now define relaxation operators with range in the space of L^2-sections of the vector bundle over S with fibers given for each $x \in S$ by

$$L^2\big(I \times \mathcal{Y}; \big(T_x^*S \odot T_x^*S\big) \oplus \big(T_x^*S \odot N_x^*S\big) \oplus \big(N_x^*S \odot N_x^*S\big)\big). \tag{34}$$

Of course each of these fibers is isomorphic to $L^2(I \times \mathcal{Y}; \mathbb{R}^{3\times 3}_{\mathrm{sym}})$. We now make the following definitions:

Set $D(\mathcal{U}_0) = \dot{H}^1(\mathcal{Y};\mathbb{R}^2) \times L^2(I \times \mathcal{Y};\mathbb{R}^3)$ and define

$$\mathcal{U}_0(\zeta, g) = \mathrm{Def}_{\mathcal{Y}}\,\zeta + 2g_\alpha \tau^\alpha \odot n + g_3 n \otimes n \quad \text{for all } (\zeta, g) \in L^2\big(S, D(\mathcal{U}_0)\big).$$

Set $D(\mathcal{U}_0^0) = D(\mathcal{U}_{0,\gamma_1}^1) = \dot{H}^1(\mathcal{Y};\mathbb{R}^2) \times \dot{H}^2(\mathcal{Y}) \times L^2(I \times \mathcal{Y};\mathbb{R}^3)$ and define

$$\mathcal{U}_0^0(\boldsymbol{\zeta}, \varphi, g) = \mathcal{U}_0(\zeta, g) - t\,\mathrm{Hess}_{\mathcal{Y}}\,\varphi \quad \text{for all } (\zeta, \varphi, g) \in L^2\big(S, D\big(\mathcal{U}_0^0\big)\big),$$

$$\mathcal{U}_{0,\gamma_1}^1(\zeta, \varphi, g) = \mathcal{U}_0(\zeta, g) - t\,\mathrm{Hess}_{\mathcal{Y}}\,\varphi + \frac{1}{\gamma_1}\varphi\mathbf{S} \quad \text{for all } (\zeta, \varphi, g) \in L^2\big(S, D\big(\mathcal{U}_{0,\gamma_1}^1\big)\big).$$

Set $D(\mathcal{U}_\infty) = L^2(I;\dot{H}^1(\mathcal{Y};\mathbb{R}^2)) \times L^2(I;\dot{H}^1(\mathcal{Y})) \times L^2(I;\mathbb{R}^3)$ and define

$$\mathcal{U}_\infty(\zeta, \rho, c) = \mathrm{Def}_{\mathcal{Y}}\,\zeta + 2(\partial_{y_\alpha}\rho + c_\alpha)\tau^\alpha \odot n + c_3 n \otimes n \quad \text{for all } (\zeta, \rho, c) \in L^2\big(S, D(\mathcal{U}_\infty)\big).$$

Set $D(\mathcal{U}_\gamma) = \dot{H}^1(I \times \mathcal{Y};\mathbb{R}^2) \times \dot{H}^1(I \times \mathcal{Y})$ and define

$$\mathcal{U}_\gamma(\zeta, \rho) = \mathrm{Def}_{\mathcal{Y}}\,\zeta + \left(\partial_{y_\alpha}\rho + \frac{1}{\gamma}\partial_3\zeta_\alpha\right)\tau^\alpha \odot n + \left(\frac{1}{\gamma}\partial_3\rho\right) n \otimes n$$

for all $(\zeta, \rho) \in L^2(S, D(\mathcal{U}_\gamma))$.

By trivially embedding $D(\mathcal{U}_0)$ as constant maps into $L^2(S, D(\mathcal{U}_0))$, we can regard $\mathcal{U}_0$ also as a map from $D(\mathcal{U}_0)$ itself into (34). For each $x \in S$ the fiberwise action $\mathcal{U}_0^{(x)}$ of $\mathcal{U}_0$ is

$$\mathcal{U}_0^{(x)}(\zeta, g) = (\mathrm{Def}_{\mathcal{Y}}\,\zeta)(x) + 2g_\alpha \tau^\alpha(x) \odot n(x) + g_3 n(x) \otimes n(x) \quad \text{for all } (\zeta, g) \in D(\mathcal{U}_0).$$

For each $x \in S$ we define $L_0^{(x)}(I \times \mathcal{Y}) = \mathcal{U}_0^{(x)}(D(\mathcal{U}_0))$, i.e.,

$$L_0^{(x)}(I \times \mathcal{Y}) = \big\{\mathcal{U}_0^{(x)}(\zeta, g) : (\zeta, g) \in D(\mathcal{U}_0)\big\}.$$

This is a subspace of (34), i.e., of $L^2(I \times \mathcal{Y};\mathbb{R}^{3\times3}_{\mathrm{sym}})$. We denote by $L_0(I \times \mathcal{Y})$ the vector bundle over S with fibers $L_0^{(x)}(I \times \mathcal{Y})$; in what follows we will frequently omit the index (x) for the fibers. The bundles $L_0^0(I \times \mathcal{Y})$, $L_{0,\gamma_1}^1(I \times \mathcal{Y})$, $L_\infty(I \times \mathcal{Y})$ and $L_\gamma(I \times \mathcal{Y})$ are defined analogously. The elements of these spaces are the *relaxation fields*.

For $\gamma \in (0, \infty]$ and $x \in S$ we define $\mathcal{Q}_\gamma(x) : (T_x^* S \otimes T_x^* S)^2 \to \mathbb{R}$ by setting

$$\mathcal{Q}_\gamma\big(x, q^1, q^2\big) = \inf_{U \in L_\gamma^{(x)}(I \times \mathcal{Y})} \int_I \int_{\mathcal{Y}} \mathcal{Q}\big(x + tn(x), y, q^1 + tq^2 + U(t, y)\big)\, dy\, dt \tag{35}$$

for each $x \in S$ and $q^1, q^2 \in T_x^* S \otimes T_x^* S$. For $\gamma_1 \in (0, \infty)$ we define $\mathcal{Q}_0^0(x)$, $\mathcal{Q}_{0,\gamma_1}^1(x)$ analogously, replacing L_γ by L_0^0 (resp. L_{0,γ_1}^1).

For $\gamma \in (0, \infty]$ define the functionals $I_\gamma : H^2(S;\mathbb{R}^3) \times L^2(S; T^*S \otimes T^*S) \to \mathbb{R}$ by setting

$$I_\gamma(V, B_w) = \int_S \mathcal{Q}_\gamma\left(x, B_w(x) + \frac{1}{2}(dV)^2(x), -b_V(x)\right) dvol_S(x). \tag{36}$$

For $\gamma_1 \in (0, \infty)$, the functionals $I_{0,\gamma_1}^1, I_0^0 : H^2(S;\mathbb{R}^3) \times L^2(S; T^*S \otimes T^*S) \to \mathbb{R}$ are defined analogously, by replacing $\mathcal{Q}_\gamma$ by $\mathcal{Q}_{0,\gamma_1}$ (resp. $\mathcal{Q}_0^0$).

This is our main result:

Theorem 3.1. *Let W be as above and assume that $u^h \in H^1(S^h;\mathbb{R}^3)$ satisfy*

$$\limsup_{h\to 0} h^{-4} J^h\big(u^h\big) < \infty. \tag{37}$$

Define $\bar{y}^h : S^1 \to \mathbb{R}^3$ by $\bar{y}^h(\Phi^h) = u^h$. Then the following are true:

(i) *(compactness) There exists a subsequence, still denoted by* $(\bar{y}^h)$*, and there exist* $Q^h \in \mathrm{SO}(3)$ *and* $c^h \in \mathbb{R}^3$ *such that the sequences* y^h *and* V^h *defined by* $y^h = (Q^h)^T \bar{y}^h - c^h$ *and*

$$V^h(x) = \frac{1}{h}\left(\int_I y^h\big(x + tn(x)\big)\,dt - x\right) \quad \textit{for all } x \in S,$$

satisfy the following:
(a) *We have* $y^h \to \pi$ *strongly in* $H^1(S^1;\mathbb{R}^3)$.
(b) *There exists an infinitesimal bending* $V \in H^2(S;\mathbb{R}^3)$ *of* S *such that* $V^h \to V$*, strongly in* $H^1(S;\mathbb{R}^3)$.
(c) *There exists* $B_w \in L^2(S; T^*S \odot T^*S)$ *such that* $\frac{1}{h} q_{V^h} \rightharpoonup B_w$ *weakly in* $L^2(S)$.

(ii) *(lower bound) We have*

$$\liminf_{h\to 0} h^{-4} J^h\big(u^h\big) \geq \begin{cases} I_\gamma(V, B_w) & \text{if } h/\varepsilon \to \gamma \in (0,\infty], \\ I_0^0(V, B_w) & \text{if } \varepsilon \gg h \gg \varepsilon^2, \\ I_{0,\gamma_1}^1(V, B_w) & \text{if } \varepsilon^2/h \to \frac{1}{\gamma_1} \in (0,\infty). \end{cases}$$

(iii) *(recovery sequence) For any infinitesimal bending* $V \in H^2(S,\mathbb{R}^3)$ *of* S *and any* $B_w \in \mathcal{B}$*, there exist* $u^h \in H^1(S^h;\mathbb{R}^3)$ *satisfying (37), and such that the conclusions of part* (i) *are true with* $Q^h = I$ *and* $c^h = 0$*. Moreover,*

$$\lim_{h\to 0} h^{-4} J^h\big(u^h\big) = \begin{cases} I_\gamma(V, B_w) & \text{if } h/\varepsilon \to \gamma \in (0,\infty], \\ I_0^0(V, B_w) & \text{if } \varepsilon \gg h \gg \varepsilon^2, \\ I_{0,\gamma_1}^1(V, B_w) & \text{if } \varepsilon^2/h \to \frac{1}{\gamma_1} \in (0,\infty). \end{cases}$$

We will prove the lower bound in Section 4. The upper bound will be an immediate consequence of (33) and of Proposition 5.5 below.

For $x \in S$ and $q^1, q^2 \in T_x^*S \odot T_x^*S$ define the homogeneous relaxation (cf. [22]):

$$\mathcal{Q}_2\big(x, t, q^1, q^2\big) = \min_{M \in \mathbb{R}^{3\times 3}_{\mathrm{sym}}} \big\{\mathcal{Q}\big(x + tn(x), M\big) : M(T_S, T_S) = \big(q^1 + tq^2\big)(T_S, T_S)\big\}. \tag{38}$$

Then it is easy to see that

$$\mathcal{Q}_0^0\big(x, q^1, q^2\big) = \inf \int_{I\times\mathcal{Y}} \mathcal{Q}_2\big(x + tn(x), y, q^1 + tq^2 + \mathrm{Def}_{\mathcal{Y}}\,\zeta - t\,\mathrm{Hess}_{\mathcal{Y}}\,\varphi\big)\,dt\,dy,$$

$$\mathcal{Q}_{0,\gamma_1}^1\big(x, q^1, q^2\big) = \inf \int_{I\times\mathcal{Y}} \mathcal{Q}_2\big(x + tn(x), y, q^1 + tq^2 + \mathrm{Def}_{\mathcal{Y}}\,\zeta - t\,\mathrm{Hess}_{\mathcal{Y}}\,\varphi + \gamma_1^{-1}\varphi\mathbf{S}\big)\,dt\,dy,$$

where both infima are taken over all $\zeta \in \dot{H}^1(\mathcal{Y},\mathbb{R}^2)$ and all $\varphi \in \dot{H}^2(\mathcal{Y})$. In the case when $\mathcal{Q}$ does not depend on t, i.e., the material is homogeneous in the thickness direction, the relaxation decouples and we find:

$$\mathcal{Q}_0^0\big(x, q^1, q^2\big) = \inf_{\zeta \in \dot{H}^1(\mathcal{Y},\mathbb{R}^2)} \int_{\mathcal{Y}} \mathcal{Q}_2\big(x, y, q^1 + \mathrm{Def}_{\mathcal{Y}}\,\zeta\big)\,dy + \frac{1}{12} \inf_{\varphi \in \dot{H}^2(\mathcal{Y})} \int_{\mathcal{Y}} \mathcal{Q}_2\big(x, y, q^2 + \mathrm{Hess}_{\mathcal{Y}}\,\varphi\big)\,dy.$$

The analogous formula holds for $\mathcal{Q}_{0,\gamma_1}^1$, too.

As in [29], one can prove also here that for all $q^1, q^2 \in T_x^*S \odot T_x^*S$ we have

$$\lim_{\gamma\to\infty} \mathcal{Q}_\gamma\big(x, q^1, q^2\big) = \mathcal{Q}_\infty\big(x, q^1, q^2\big) \quad \text{and} \quad \lim_{\gamma\to 0} \mathcal{Q}_\gamma\big(x, q^1, q^2\big) = \mathcal{Q}_0^0\big(x, q^1, q^2\big).$$

Note that in the particular case of a plate (i.e. $\mathbf{S} = 0$), for $\gamma_1 \in (0,\infty)$ all spaces L_{0,γ_1}^1 coincide with L_0^0. This corresponds to the fact that in the von Kármán plate theory for $\gamma = 0$ one obtains only one relaxation space, cf. [29] for details.

3.2. FJM-compactness

From now on $u^h \in H^1(S^h; \mathbb{R}^3)$ will always denote a sequence satisfying (37). The following lemma proves the first part of Theorem 3.1. It is a direct consequence of [9, Theorem 3.1] and of arguments in [10]. We refer to [22] for the extension to the present setting.

Lemma 3.2. *There exist a constant $C > 0$, independent of h, and a sequence of matrix fields $(R^h) \subset H^1(S; \mathrm{SO}(3))$ (extended trivially to S^h) and there exists a sequence of matrices $(Q^h) \subset \mathrm{SO}(3)$ such that:*

(i) $\limsup_{h\to 0} h^{-5/2}\|\nabla u^h - R^h\|_{L^2(S^h)} < \infty$.
(ii) $\limsup_{h\to 0} h^{-1}\|\nabla R^h\|_{L^2(S)} < \infty$.
(iii) $\limsup_{h\to 0} h^{-1}\|(Q^h)^T R^h - I\|_{L^p(S)} < \infty$, *for all* $p \in [1, \infty)$.
(iv) $(Q^h)^T R^h \to I$ *strongly in* H^1.

Moreover, there exists a matrix field $A \in H^1(S, \mathrm{so}(3))$ taking values in the space of skew symmetric matrices, such that (after passing to subsequences)

(v) $\frac{1}{h}((Q^h)^T R^h - I) \rightharpoonup A$, *weakly in* $H^1(S)$.
(vi) $\frac{1}{h^2}\,\mathrm{sym}((Q^h)^T R^h - I) \to \frac{1}{2}A^2$, *strongly in* $L^p(S)$, *for all* $p \in [1, \infty)$.

Moreover, if we define $\bar{y}^h : S^1 \to \mathbb{R}^3$ by $\bar{y}^h(\Phi^h) = u^h$, the following are true:

(i) $\limsup_{h\to 0} \frac{1}{h^2}\|\nabla_h \bar{y}^h - R^h\|_{L^2(S^1)} < \infty$.
(ii) $\frac{1}{h}((Q^h)^T \nabla_h \bar{y}^h - I) \rightharpoonup A$, *weakly in* H^1 *up to a subsequence.*

Define $y^h \in H^1(S^1; \mathbb{R}^3)$ by $y^h = (Q^h)^T \bar{y}^h - c^h$, where $c^h = \fint_S \int_I ((Q^h)^T \bar{y}^h(x + tn(x)) - x)\,dt\,d\mathrm{vol}_S(x)$. Introduce the (average) midplane displacements $V^h : S \to \mathbb{R}^3$ by setting

$$V^h(x) := \frac{1}{h}\left(\int_I y^h\big(x + tn(x)\big)\,dt - x\right) \quad \text{for all } x \in S. \tag{39}$$

Then $\fint_S V^h = 0$ and (after passing to a subsequence)

(iii) $y^h \to \pi$ *strongly in* $H^1(S^1)$.
(iv) *There exists an infinitesimal bending $V \in H^2(S; \mathbb{R}^3)$ of S with $\Omega_V = A$ and such that $V^h \to V$ strongly in $H^1(S; \mathbb{R}^3)$.*
(v) $\frac{1}{h} q_{V^h}$ *is bounded in* $L^2(S)$.

In what follows we replace the sequence R^h by $(Q^h)^T R^h$ and the sequence y^h by $\bar{y}^h$, so we assume without loss of generality that $Q^h = Id$. Expressed in unrescaled variables, we have

$$V^h(x) = \frac{1}{h^2}\left(\int_{I^h} u^h\big(x + tn(x)\big)\,dt - x\right),$$

i.e. $x + hV^h(x) = \fint_{I^h} u^h(x + tn(x))\,dt$.

We begin by modifying the displacement fields V^h into more regular fields V_*^h enjoying a similar compactness. (The asterisk in the notation does not denote any push-forward operation.)

Lemma 3.3. *There exist $V_*^h \in H^2(S; \mathbb{R}^3)$ with $\fint_S V_*^h = 0$ satisfying*

$$\limsup_{h\to 0} h^{-1}\left\|V_*^h - V^h\right\|_{H^1(S)} < \infty \tag{40}$$

and

$$\left\| \left(dV_*^h - \frac{R^h - I}{h} \right) \circ T_S \right\|_{L^2(S)} \le \left\| \left(dV^h - \frac{R^h - I}{h} \right) \circ T_S \right\|_{L^2(S)}. \tag{41}$$

Moreover, (V_*^h) *is uniformly bounded in* $H^2(S)$ *and*

$$V_*^h \rightharpoonup V \quad \text{weakly in } H^2(S;\mathbb{R}^3). \tag{42}$$

Proof. We follow [29, Proposition 3.1]. For $i = 1, 2, 3$ denote by p_i the i-th row of the matrix $\frac{R^h(\psi)-I}{h}\nabla\psi$. We define $\tilde{V}_*^h \in H^2(\omega;\mathbb{R}^3)$ such that $(\tilde{V}_*^h)_i$ is a minimizer of the functional $v \mapsto \int_\omega |\nabla v - p_i|^2\,dx$ among all $v \in H^1(\omega)$ with average zero, and we define V_*^h via $V_*^h(\psi) = \tilde{V}_*^h$. The bound (41) follows from the minimality of V_*^h. Combining the uniform bounds on the tangential components of F^h and F_*^h introduced in Lemma 3.4 below (note that these bounds follow from (41) alone), we obtain $\|dV_*^h - dV^h\|_{L^2(S)} \le Ch$. Hence (40) follows from Poincaré's inequality on S. Standard estimates for minimizers imply that $V_*^h \in H^2(S)$ with bounds

$$\|V_*^h\|_{H^2(S)} \le C\big(\|\operatorname{div} p\|_{L^2(\omega)} + \|p\|_{L^2(\omega)}\big).$$

Hence Lemma 3.2 (v) ensures that the V_*^h are uniformly bounded in $H^2(S)$. Since $V^h \to V$ in H^1, the bound (40) therefore implies (42). □

Lemma 3.4. *Let* V_*^h *be as defined in Lemma 3.3. Then the maps* F_*^h, $F^h \in L^2(S;\mathbb{R}^{3\times 3})$ *defined by* $R^h = I + h\Omega_{V^h} + h^2F^h$ *and* $R^h = I + h\Omega_{V_*^h} + h^2F_*^h$ *satisfy*

$$\limsup_{h\to 0}\big(\|F_*^h\|_{L^2(S)} + \|F^h\|_{L^2(S)}\big) < \infty.$$

Moreover,

$$\operatorname{sym} F^h + \frac{1}{h} q_{V^h}(T_S, T_S) \to \frac{1}{2}\Omega_V^2 \quad \text{in all } L^p \text{ with } p \in [1,\infty). \tag{43}$$

Proof. Note that (43) follows from Lemma 3.2 and (10). From (43) and from Lemma 3.2 we deduce that $\operatorname{sym} F^h$ is uniformly bounded in L^2.

For brevity, we set $\mu_*^h = \mu_{V_*^h}$ and $\mu^h = \mu_{V^h}$. In order to verify the L^2-bound on F^hT_S, let τ be a C^1 tangent vector field along S. Then by the definition of V^h, we see that $(I + hdV^hT_S)\tau$ equals

$$\nabla_\tau\big(id + hV^h\big) = \frac{1}{h}\nabla_\tau\left(\int_{hI} u^h\big(x + tn(x)\big)\,dt\right) = \frac{1}{h}\int_{hI} \nabla u^h\big(x + tn(x)\big)\big(I + t\mathbf{S}(x)\big)\,dt\,\tau(x).$$

Introducing

$$M^h(x) = -\frac{1}{h}\int_{hI} \big(\nabla u^h\big(x + tn(x)\big) - R^h(x)\big)\big(I + t\mathbf{S}(x)\big)\,dt$$

and using $\int_{hI} R^h(x)t\mathbf{S}(x)\,dt = 0$, we conclude that $F^h\tau = h^{-2}M^h$. And

$$\begin{aligned}\int_S |M^h|^2\,dvol_S &= \frac{1}{h}\int_{S\times hI} \big|\nabla u^h\big(x + tn(x)\big) - R^h(x)\big|^2\big|I + t\mathbf{S}(x)\big|^2\,dvol_S(x)\,dt\\ &\le \frac{C}{h}\int_{S^h} \big|\nabla u^h(x) - R^h(x)\big|^2\,dx \le Ch^4.\end{aligned}$$

This proves that $F^h\tau$ is L^2-bounded for every regular tangent vector field τ along S. But since $\operatorname{sym} F^h$ is bounded in L^2, this already implies that F^h itself is bounded in L^2.

To prove the bound on F_*^h note that (41) simply reads $\|F_*^h T_S\|_{L^2(S)} \le \|F^h T_S\|_{L^2(S)}$, so $F_*^h T_S$ is clearly bounded in L^2. As we saw in Lemma 3.3, this implies (40), which in turn shows that $\|\Omega_{V^h} - \Omega_{V_*^h}\|_{L^2(S)} \le Ch$. Since by definition $F_*^h - F^h = (\Omega_{V^h} - \Omega_{V_*^h})/h$, the boundedness of F_*^h follows from that of F^h. □

4. Two-scale compactness and lower bound

Lemma 4.1. *Let $v^h \in H^1(S, TS)$ be such that $v^h \xrightarrow{osc} 0$, and assume that there is $B \in L^2(S, L^2(\mathcal{Y}, T^*S \odot T^*S))$ such that $\mathrm{Def}_S\, v^h \xrightarrow{osc} B$. Then there exists $v \in L^2(S, \dot{H}^1(\mathcal{Y}, \mathbb{R}^2))$ such that $B = \mathrm{Def}_{\mathcal{Y}}\, v$.*

Proof. For $\alpha, \beta = 1, 2$, let $\widetilde{G}^{\alpha\beta} \in C^\infty(\omega)$ satisfy $\widetilde{G}^{12} = \widetilde{G}^{21}$, and define G by setting $G(x) = \widetilde{G}^{\alpha\beta}(r(x))\tau_\alpha(x) \otimes \tau_\beta(x)$. Then we have (with the usual definition of the divergence, see e.g. [18] for details)

$$(\mathrm{div}_S\, G)(x) = \partial_\beta \widetilde{G}^{\alpha\beta}\big(r(x)\big)\tau_\alpha(x) + \widetilde{G}^{\alpha\beta}\big(r(x)\big)X_{(\alpha\beta)}(x), \tag{44}$$

where each $X_{(\alpha,\beta)}$ is a continuous tangent vector field on $\bar{S}$ (which involves the Christoffel symbols). Now let $\rho^{\alpha\beta} \in \dot{C}^\infty(\mathcal{Y})$ be symmetric and divergence-free in $\mathbb{R}^2$, i.e., $\partial_\beta \rho^{\alpha\beta} = 0$, and let $\varphi \in C_0^\infty(S)$. Apply (44) to G_h defined like G above, but with $\widetilde{G}_h^{\alpha\beta}(z) = \rho^{\alpha\beta}(z/\varepsilon)\varphi(\psi(z))$ for $z \in \omega$ instead of $\widetilde{G}^{\alpha\beta}$. That shows:

$$(\mathrm{div}_S\, G_h)(x) = \rho^{\alpha\beta}\big(r(x)/\varepsilon\big)(\nabla_{\tau_\beta}\varphi)(x) + \varphi(x)\rho^{\alpha\beta}\big(r(x)/\varepsilon\big)X_{(\alpha\beta)}(x), \tag{45}$$

because the term involving ε^{-1} is zero since $\rho^{\alpha\beta}$ is divergence-free.

Hence, since the formal adjoint of Def_S is $-\mathrm{div}_S$ and since G_h has compact support in S, we have (identifying T^*S with TS as usual)

$$-\int_S \big\langle \mathrm{Def}_S\, v^h, G_h\big\rangle_{TS\otimes TS} = \int_S \big\langle v^h, \mathrm{div}_S(G_h)\big\rangle_{TS}$$
$$= \int_S \big\langle v^h(x), \rho^{\alpha\beta}\big(r(x)/\varepsilon\big)(\nabla_{\tau_\beta}\varphi)(x) + \varphi(x)\rho^{\alpha\beta}\big(r(x)/\varepsilon\big)X_{(\alpha\beta)}(x)\big\rangle_{T_xS}\, dvol_S(x).$$

As $h \to 0$, this converges to zero because $v^h \xrightarrow{osc} 0$. Hence by the definition of B and G_h, writing $B(x, y) = B_{\alpha\beta}(x, y)\tau^\alpha(x) \otimes \tau^\beta(x)$ and then using arbitrariness of φ, we conclude that

$$\int_{\mathcal{Y}} B_{\alpha\beta}(x, y)\rho^{\alpha\beta}(y)\, dy = 0 \quad \text{for almost every } x \in S$$

and for all $\rho^{\alpha\beta}$ as above. This implies the claim because in $\mathcal{Y}$ the L^2-orthogonal complement of divergence free maps is the space of symmetrized gradients. □

Proposition 4.2. *Let (w^h) be a bounded sequence in $H^2(S; \mathbb{R}^3)$ such that $\frac{1}{h}q_{w^h}$ is bounded in $L^2(S; T^*S \otimes T^*S)$. Then there exist $w_0 \in H^2(S)$, $w_1 \in L^2(S; \dot{H}^2(\mathcal{Y}; \mathbb{R}^3))$ and $B \in L^2(S, \dot{L}^2(\mathcal{Y}; T^*S \odot T^*S))$ such that, after passing to a subsequence, $q_{w^h}/h \xrightharpoonup{2} B$ and $\mathrm{Hess}\, w^h \xrightharpoonup{2} \mathrm{Hess}\, w_0 + \mathrm{Hess}_{\mathcal{Y}}\, w_1$. Set $B_w = \int_Y B(\cdot, y)\, dy$. Then the following are true:*

(i) *If $h \gg \varepsilon^2$ then there exists a unique $v \in L^2(S; \dot{H}^1(\mathcal{Y}; \mathbb{R}^2))$ such that*

$$B = B_w + \mathrm{Def}_{\mathcal{Y}}\, v.$$

(ii) *If $h \sim \varepsilon^2$ and if we set $\frac{1}{\gamma_1} = \lim_{h\to 0} \frac{\varepsilon^2}{h}$, then there exists a unique $v \in L^2(S; \dot{H}^1(\mathcal{Y}; \mathbb{R}^2))$ such that*

$$B = B_w + \mathrm{Def}_{\mathcal{Y}}\, v + \frac{1}{\gamma_1}(w_1 \cdot n)\mathbf{S}.$$

(iii) *If $h \ll \varepsilon^2$, then there exists a unique $v \in L^2(S; \dot{H}^1(\mathcal{Y}; \mathbb{R}^2))$ such that*

$$\mathrm{Def}_{\mathcal{Y}}\, v + (w_1 \cdot n)\mathbf{S} = 0.$$

Remark 1. This proposition explains why the scaling $h \sim \varepsilon^2$ is critical. In fact, the (corrected) displacements V_*^h satisfy the hypotheses, and thus (heuristically) $V_*^h = V + \varepsilon^2 V_o(x, \frac{r(x)}{\varepsilon})$, where $V_o \in L^2(S, \dot{H}^2(\mathcal{Y}, \mathbb{R}^3))$. Since $q_{V_*^h} = \mathrm{Def}_S\, V_{*,\tan}^h + (V_*^h \cdot n)\mathbf{S}$ and since the corrector $V_o \cdot n$ is of order ε^2 when $h \sim \varepsilon^2$, the normal component of the corrector arises in the relaxation field of $q_{V_*^h}/h$. This phenomenon clearly does not occur in the case of plates, where $\mathbf{S} = 0$. This normal component, with its second gradient, also appears in the two-scale limit of the strain, when one looks at the displacement across the thickness, see (52) below.

Proof. The existence of w_0 and w_1 is ensured by Lemma 2.5. Applied to w^h and to $w_{\tan}^h$ it also implies that

$$\frac{q_{w^h}}{\varepsilon} \xrightarrow{osc} \mathrm{Def}_{\mathcal{Y}}\, \widehat{w}_1 \quad \text{and} \quad \frac{\mathrm{Def}_S\, w_{\tan}^h}{\varepsilon} \xrightarrow{osc} \mathrm{Def}_{\mathcal{Y}}\, \widehat{w}_1 \quad \text{on } S, \tag{46}$$

where the $\mathbb{R}^2$-valued map $\widehat{w}_1$ is defined by $(\widehat{w}_1)_\alpha = \tau_\alpha \cdot w_1$. By Lemma 2.6 we have

$$\frac{w^h}{\varepsilon^2} \xrightarrow{osc} w_1 \quad \text{on } S. \tag{47}$$

Claim 1. *If $h \ll \varepsilon$ then $(w_1)_{\tan} = 0$.*

In fact, since q_{w^h}/h is bounded in L^2, we have $q_{w^h}/\varepsilon \to 0$ in L^2. So (46) implies that $\mathrm{Def}_{\mathcal{Y}}\, \widehat{w}_1 = 0$. Since $\widehat{w}_1(x, \cdot)$ is $\mathcal{Y}$-periodic with average zero, Korn's inequality applied in $\mathcal{Y}$ shows that $\widehat{w}_1 = 0$, so indeed $(w_1)_{\tan} = 0$.

Claim 2. *If $\varepsilon^2 \lesssim h$ then there exists $v \in L^2(S, \dot{H}^1(\mathcal{Y}, \mathbb{R}^2))$ such that $\frac{\mathrm{Def}_S\, w_{\tan}^h}{h} \xrightarrow{osc} \mathrm{Def}_{\mathcal{Y}}\, v$.*

In fact, if $\varepsilon^2 \lesssim h \ll \varepsilon$, then Claim 1 and (47) imply that $v^h := w_{\tan}^h/h \xrightarrow{osc} 0$. Hence Lemma 4.1 implies the claim in this case. If $\varepsilon \lesssim h$ then Claim 2 follows from (46), simply by taking $v = (\lim \varepsilon/h)\widehat{w}_1$.

Now we prove the proposition in the case $\varepsilon^2 \lesssim h$. In this case, $(w^h \cdot n)/h \xrightarrow{osc} \gamma_1^{-1} w_1 \cdot n$ (with the understanding that the right-hand side is zero if $\gamma_1 = \infty$), by (47). Hence Claim 2 shows that there exists v such that

$$\frac{q_{w^h}}{h} = \frac{\mathrm{Def}_S\, w_{\tan}^h}{h} + \frac{w^h \cdot n}{h}\mathbf{S} \xrightarrow{osc} \mathrm{Def}_{\mathcal{Y}}\, v + \frac{1}{\gamma_1}(w_1 \cdot n)\mathbf{S}.$$

This proves the first two parts of the proposition.

Finally consider the supercritical case $h \ll \varepsilon^2$. Claim 1 implies that $(w_1)_{\tan} = 0$. Hence the tangential part of (47) implies that $\varepsilon^{-2} w_{\tan}^h \xrightarrow{osc} 0$. Lemma 4.1 shows that $\varepsilon^{-2}\, \mathrm{Def}_S\, w_{\tan}^h \xrightarrow{osc} \mathrm{Def}_{\mathcal{Y}}\, v$ for some v. On the other hand,

$$\frac{\mathrm{Def}_S\, w_{\tan}^h}{\varepsilon^2} = \frac{q_{w^h}}{\varepsilon^2} - \frac{w^h \cdot n}{\varepsilon^2}\mathbf{S} \xrightarrow{osc} -(w_1 \cdot n)\mathbf{S},$$

because $\varepsilon^{-2} q_{w^h} \to 0$ (by the boundedness of q_{w^h}/h) and because of the normal part of (47).

In all cases uniqueness of v follows from the fact that zero is the only $\mathcal{Y}$-periodic skew affine map. □

Lemma 4.3. *Let $w^h \in H^1(S^1; \mathbb{R}^3)$ be such that*

$$\limsup_{h \to 0} \left(\|w^h\|_{L^2(S^1)} + \|\nabla_h w^h\|_{L^2(S^1)} \right) < \infty.$$

Then there exists a map $w_0 \in H^1(S; \mathbb{R}^3)$ and a field $H_\gamma \in L^2(S \times I \times \mathcal{Y}; \mathbb{R}^{3\times 3})$ of the form

$$H_\gamma = \begin{cases} (\nabla_y w_1, w_2) & \textit{for some} \begin{Bmatrix} w_1 \in L^2(S; \dot{H}^1(\mathcal{Y}; \mathbb{R}^3)) \\ w_2 \in L^2(S \times \mathcal{Y} \times I; \mathbb{R}^3) \end{Bmatrix} \\ & \textit{if } \gamma = 0, \\ (\nabla_y w_1, \frac{1}{\gamma}\partial_3 w_1) & \textit{for some } w_1 \in L^2(S, \dot{H}^1(I \times \mathcal{Y}; \mathbb{R}^3)) \\ & \textit{if } \gamma \in (0, \infty), \\ (\nabla_y w_1, w_2) & \textit{for some} \begin{Bmatrix} w_1 \in L^2(S \times I; \dot{H}^1(\mathcal{Y}; \mathbb{R}^3)) \\ w_2 \in L^2(S \times I; \mathbb{R}^3) \end{Bmatrix} \\ & \textit{if } \gamma = \infty, \end{cases} \tag{48}$$

such that, up to a subsequence, $w^h \to w_0$ in L^2 and

$$\nabla_h w^h \overset{2}{\rightharpoonup} dw_0 \circ T_S + \sum_{i,j=1}^{3} (\hat{H}_\gamma)_{ij} \tau^i \otimes \tau^j \quad \textit{weakly two-scale on } S^1.$$

Here, w_0 is the weak limit in $H^1(S)$ of $\int_I w^h(x + tn(x))\,dt$ and $\hat{H}_\gamma \in L^2(S^1 \times \mathcal{Y}; \mathbb{R}^{3\times 3})$ is defined by $\hat{H}_\gamma(x, y) = H_\gamma(\pi(x), t(x), y)$.

Proof. The hypotheses imply, e.g. by (25), that the w^h are uniformly bounded in $H^1(S^1)$, so up to a subsequence $w^h \rightharpoonup: w_0$ in $H^1(S^1)$. Set $\widetilde{w}^h = w^h \circ \Psi$, so clearly $\widetilde{w}^h$ is uniformly bounded in $L^2(\Omega)$. From the uniform L^2-bound on $\nabla_h w^h$ and from (23) we deduce that $\widetilde{\nabla}_h \widetilde{w}^h$ is uniformly bounded in $L^2(\Omega)$. Hence there is $\widetilde{w}_0 \in H^1(\Omega; \mathbb{R}^3)$ with $\partial_3 \widetilde{w}_0 = 0$ such that $\widetilde{w}^h \rightharpoonup \widetilde{w}_0$ weakly in $H^1(\Omega; \mathbb{R}^3)$; clearly $\widetilde{w}_0 = w_0 \circ \Psi$, so (since $\partial_3 \widetilde{w}_0 = 0$) in particular w_0 is the trivial extension of a map defined on S.

By uniform boundedness in $L^2(\Omega)$, in the case $\gamma \in (0, \infty)$ there exists (see [27, Proposition 6.3.5]) $\bar{w}_1 \in L^2(\omega; \dot{H}^1(I \times \mathcal{Y}; \mathbb{R}^3))$ such that

$$\widetilde{\nabla}_h \widetilde{w}^h \overset{2}{\rightharpoonup} (\partial_1 \widetilde{w}_0, \partial_2 \widetilde{w}_0, 0) + \left(\partial_{y_1} \bar{w}_1, \partial_{y_2} \bar{w}_1, \frac{1}{\gamma}\partial_3 \bar{w}_1\right) \quad \text{in } \Omega.$$

By (23) the left-hand side equals $(\nabla_h w)(\Psi)\nabla\Psi(\widetilde{\Phi}_h^{-1})$. As $\nabla\Psi(\widetilde{\Phi}_h^{-1})$ converges uniformly on S^1 to $(\partial_1\psi, \partial_2\psi, n(\psi))$ (extended trivially in the x_3-direction), we conclude:

$$\nabla_h w^h(\Psi) \overset{2}{\rightharpoonup} \left((\partial_1 \widetilde{w}_0, \partial_2 \widetilde{w}_0, 0) + \left(\partial_{y_1} \bar{w}_1, \partial_{y_2} \bar{w}_1, \frac{1}{\gamma}\partial_3 \bar{w}_1\right)\right)\big(\partial_1\psi, \partial_2\psi, n(\psi)\big)^{-1}.$$

On the right-hand side we use

$$\big(\partial_1\psi, \partial_2\psi, n(\psi)\big)^{-1} \circ \Psi^{-1} = (\tau_1, \tau_2, n)^{-1} = \big(\tau^1, \tau^2, n\big)^T$$

and $(\partial_\alpha \widetilde{w}_0) \circ \Psi^{-1} = dw_0(\tau_\alpha)$ to obtain the claim when $\gamma \in (0, \infty)$, after defining $(w_1)_i = (\bar{w}_1 \circ r) \cdot \tau_i$, for $i = 1, 2, 3$. The other two cases are proven similarly. □

The following lemma goes by a truncation argument already exploited in [9,10]; we refer to [27, Corollary 2.3.4] for a proof.

Lemma 4.4. *Let $(E^h_{\text{app}}) \subset L^2(\Omega; \mathbb{R}^{3\times 3})$ be such that $E^h_{\text{app}} \overset{2}{\rightharpoonup} E_{\text{app}}$ in $L^2(\Omega \times \mathcal{Y}; \mathbb{R}^{3\times 3})$. Then*

$$h^{-2}\left(\sqrt{\big(I + h^2 E^h_{\text{app}}\big)^T\big(I + h^2 E^h_{\text{app}}\big)} - I\right) \overset{2}{\rightharpoonup} \operatorname{sym} E_{\text{app}} \quad \textit{in } L^2\big(\Omega \times \mathcal{Y}; \mathbb{R}^{3\times 3}\big). \tag{49}$$

We continue to use the notation from Lemmas 3.2 and 3.3.

Proposition 4.5. *After passing to subsequences, there exists $B \in L^2(S; L^2(\mathcal{Y}; T^*S \odot T^*S))$ such that $\frac{1}{h} q_{V^h_*} \overset{2}{\rightharpoonup} B$ in $L^2(S, T^*S \otimes T^*S)$, there exists $B_w \in L^2(S, T^*S \odot T^*S)$ such that $\frac{1}{h} q_{V^h} \rightharpoonup B_w$ weakly in $L^2(S; T^*S \otimes T^*S)$, and there exists $\varphi \in L^2(S; \dot{H}^2(\mathcal{Y}))$ such that*

$$n \cdot \operatorname{Hess} V_*^h \stackrel{2}{\rightharpoonup} n \cdot \operatorname{Hess} V + \operatorname{Hess}_{\mathcal{Y}} \varphi. \tag{50}$$

Set

$$E^h = h^{-2}\big(\big(\big(\nabla_h y^h\big)^T \nabla_h y^h\big)^{1/2} - I\big) \tag{51}$$

and $\dot{B} = B - \int_Y B(\cdot, y)\,dy$. *Then there exists* $U_\gamma \in L^2(S; L_\gamma(I \times \mathcal{Y}))$ *such that, after passing to a subsequence,* $E^h \stackrel{2}{\rightharpoonup} E$ *on* S^1*, with* E *given by*

$$E = \left(B_w + \dot{B} + \frac{1}{2}(dV)^2 - t b_V\right)(T_S, T_S) - t \operatorname{Hess}_{\mathcal{Y}} \varphi + \hat{U}_\gamma. \tag{52}$$

In particular, the following are true:

(i) *If* $\gamma \in (0, \infty]$ *then there exists* $U_\gamma \in L^2(S; L_\gamma(I \times \mathcal{Y}))$ *such that*

$$E = B_w + \frac{1}{2}(dV)^2 - t b_V + \hat{U}_\gamma. \tag{53}$$

(ii) *If* $\varepsilon^2 \ll h \ll \varepsilon$*, there exists* $U \in L^2(S; L_0^0(I \times \mathcal{Y}))$ *such that*

$$E = B_w + \frac{1}{2}(dV)^2 - t b_V + \hat{U}. \tag{54}$$

(iii) *If* $h \sim \varepsilon^2$*, with* $\lim_{h\to 0} \frac{\varepsilon^2}{h} = \frac{1}{\gamma_1} \in (0, \infty)$*, there exists* $U \in L^2(S; L^1_{0,\gamma_1}(I \times \mathcal{Y}))$ *such that*

$$E = B_w + \frac{1}{2}(dV)^2 - t b_V + \hat{U}. \tag{55}$$

Here we have denoted by $\hat{U}_\gamma$ *(i.e.* $\hat{U}$*) the appropriate mapping defined on* $S^1 \times \mathcal{Y}$ *by* $\hat{U}_\gamma(x, y) = U_\gamma(\pi(x), t(x), y)$*.*

Proof. First we note that Lemma 3.2 combined with Lemmas 2.5 and 3.3 ensure the existence of B, B_w and of φ. By L^2-boundedness of E^h (which follows from (37) and the properties of the energy density W), there exist E and a subsequence such that $E^h \stackrel{2}{\rightharpoonup} E$ on S^1. Denote by E^h_{app} the approximate strain

$$E^h_{\mathrm{app}} := \frac{(R^h)^T \nabla_h y^h - I}{h^2}, \tag{56}$$

and note that E^h agrees with the left-hand side of (49). Hence Lemma 4.4 implies that $E = \operatorname{sym} E_{\mathrm{app}}$, where E_{app} is the weak two-scale limit of E^h_{app}, which exists by the properties of R^h stated in Lemma 3.2. So it is enough to identify the two-scale limit of $\operatorname{sym} E^h_{\mathrm{app}}$. Clearly

$$R^h E^h_{\mathrm{app}} = \frac{\nabla_h y^h - I}{h^2} - \frac{R^h - I}{h^2}. \tag{57}$$

We have $\operatorname{sym}(R^h E^h_{\mathrm{app}}) \stackrel{2}{\rightharpoonup} E$, because $R^h \to I$ boundedly in measure. By property (vi) of Lemma 3.2 the symmetric part of the second term converges strongly in L^2 (and thus two-scale) to $\Omega_V^2/2$. So we need to identify the two-scale limit of $\operatorname{sym}(\frac{\nabla_h y^h - I}{h^2})$.

For brevity we set $\mu_*^h = \mu_{V_*^h}$. As usual, we extend V_*^h, n and μ_*^h trivially to S^h. In what follows we abuse notation using t also as an independent variable. We define the maps $z^h : S^h \to \mathbb{R}^3$ by setting

$$z^h(x) = x + h\big(V_*^h(x) + t(x)\mu_*^h(x)\big) \quad \text{for all } x \in S^h.$$

Define $Q(x)$ as in (18) and define $b^h(x) = b_{V_*^h}(x) \equiv n \cdot \operatorname{Hess} V_*^h$ and (compare (6))

$$\Omega^h(x) = \Omega_{V_*^h}(x) \equiv dV_*^h(x) \circ T_S(x) + \mu_*^h(x) \otimes n(x).$$

Then Lemma 2.4 shows that

$$\nabla z^h = I + h\Omega^h - htb^h - 2htq_{V_*^h}(\mathbf{S}\circ T_S, T_S) - ht^2\nabla\mu_*^h\circ\mathbf{S} + h\big(dV_*^h + t\nabla\mu_*^h\big)\circ T_S Q. \tag{58}$$

Note that $|Q|\le Ct^2\le Ch^2$ on S^h, so $\|Q\|_{L^2(S^h)}\le Ch^{5/2}$. In what follows $\Theta^h\in L^2(S^h)$ denote maps which may change from expression to expression, but which always satisfy $\|\Theta^h\|_{L^2(S^h)}\le Ch^{5/2}$. We see from (58) that

$$\nabla z^h = I + h\Omega^h + \Theta^h = R^h + \Theta^h,$$

by Lemma 3.4. On the other hand, Lemma 3.2 shows that $\nabla u^h = R^h + \Theta^h$. Hence

$$\|\nabla u^h - \nabla z^h\|_{L^2(S^h)} \le Ch^{5/2}. \tag{59}$$

However, by the definition of V^h and of z^h we have, for $x\in S$,

$$\frac{1}{h}\int_{I^h} z^h\big(x+tn(x)\big) - u^h\big(x+tn(x)\big)\,dt = h\big(V_*^h(x) - V^h(x)\big).$$

Hence Poincaré's inequality implies that

$$\begin{aligned}\|u^h - z^h\|_{L^2(S^h)} &\le \|u^h - z^h - h\big(V_*^h - V^h\big)\|_{L^2(S^h)} + h\|V_*^h - V^h\|_{L^2(S^h)}\\ &\le \|\nabla u^h - \nabla z^h\|_{L^2(S^h)} + h^{3/2}\|V_*^h - V^h\|_{L^2(S)} \le Ch^{5/2},\end{aligned}$$

by (59) and (40). Defining $Z^h : S^1\to\mathbb{R}^3$ by setting $Z^h(\Phi^h) = z^h$ on S^h, we have the equivalent bounds

$$\|y^h - Z^h\|_{L^2(S^1)} + \|\nabla_h\big(y^h - Z^h\big)\|_{L^2(S^1)} \le Ch^2.$$

Thus, using Lemma 4.3, we conclude that there exists $H_\gamma\in L^2(S\times I\times\mathcal{Y};\mathbb{R}^{3\times 3})$ of the form given in Lemma 4.3 and $w\in H^1(S;\mathbb{R}^3)$ such that (after passing to a subsequence)

$$\frac{1}{h^2}\nabla_h\big(y^h - Z^h\big) \xrightarrow{2} dw\circ T_S + \sum_{i,j=1}^{3}(\hat{H}_\gamma)_{ij}\tau^i\otimes\tau^j. \tag{60}$$

Here w is the weak $H^1(S;\mathbb{R}^3)$-limit of $(V^h - V_*^h)/h$, which exists by Lemma 3.3. We will now identify the two-scale limit on S^1 of the quantity $\operatorname{sym}(\frac{\nabla_h Z^h - I}{h^2})$. By (58) there is M^h such that for all $x\in S^h$:

$$\frac{\nabla z^h(x) - I}{h^2} = \frac{1}{h}\Omega^h(x) - \frac{t(x)}{h}b^h(x) + M^h(x), \tag{61}$$

where $\|M^h\|_{L^2(S^h)}\le Ch^{3/2}$ because $q_{V_*^h}/h$ is bounded in $L^2(S)$. We must therefore identify the two-scale limits on S of the symmetric part of the first two terms in (61). But formula (50) just asserts that the two-scale limit of b^h is $b_V + \operatorname{Hess}_{\mathcal{Y}}\varphi$. And by definition of B, Lemma 2.1 implies that

$$\frac{1}{h}\operatorname{sym}\Omega^h = \frac{1}{h}q_{V_*^h}(T_S, T_S) \xrightarrow{2} B(T_S, T_S) \quad \text{on } S. \tag{62}$$

By (61), these convergence results on S imply that

$$\operatorname{sym}\left(\frac{\nabla_h Z^h - I}{h^2}\right) \xrightarrow{2} (B - tb_V)(T_S, T_S) - t\operatorname{Hess}_{\mathcal{Y}}\varphi \tag{63}$$

weakly two-scale on S^1. We conclude from (60) and (63) that

$$E = B + \operatorname{sym}(dw\circ T_S) - \frac{1}{2}\big(\Omega_V^2\big) - t\operatorname{Hess}_{\mathcal{Y}}\varphi - tb_V + \hat{\tilde{U}}_\gamma, \tag{64}$$

for some $\tilde{U}_\gamma\in L^2(S; L_\gamma(I\times\mathcal{Y}))$. It is not difficult to see that

$$B + \operatorname{sym}(dw\circ T_S) - (B_w + \dot{B}) \in L^2\big(S; L_\gamma(I\times\mathcal{Y})\big),$$

after identification of S^1 with $S \times I$. Hence by (64) and using (12), formula (52) follows. The remaining claims now follow from Proposition 4.2. Indeed, parts (ii) and (iii) follow at once, and (i) is a consequence of the fact that, setting $\zeta(t, y) = t\nabla_y\varphi$, we have $t\,\mathrm{Hess}_{\mathcal{Y}}\,\varphi = \mathcal{U}_\gamma(\zeta, -\varphi/\gamma)$ for all $\gamma \in (0, \infty)$. □

Lemma 4.6. *Let $(y^h) \subset H^1(S^1; \mathbb{R}^3)$, define $E^h : S^1 \to \mathbb{R}^{3\times 3}$ by (51) and let E be such that $E^h \xrightharpoonup{2} E$. Then we have*

$$\liminf_{h\to 0} \int_S \int_I \mathcal{Q}\big(x + tn(x), r(x)/\varepsilon, E^h\big(x + tn(x)\big)\big)\,dt\,dvol_S(x)$$
$$\geq \int_S \int_I \int_{\mathcal{Y}} \mathcal{Q}\big(x + tn(x), y, E\big(x + tn(x), y\big)\big)\,dy\,dt\,dvol_S(x),$$

and

$$\liminf_{h\to 0} \frac{1}{h^4} \int_S \int_I W\big(x + tn(x), r(x)/\varepsilon, I + h^2 E^h\big(x + tn(x)\big)\big)\,dt\,dvol_S(x)$$
$$\geq \int_S \int_I \int_{\mathcal{Y}} \mathcal{Q}\big(x + tn(x), y, E\big(x + tn(x), y\big)\big)\,dy\,dt\,dvol_S(x).$$

Proof. For the first claim we refer to [34,35]. The second claim then follows from a standard truncation argument already exploited in [9,10] (see [27] for details). □

The lower bound parts of Theorems 3.1 and 6.2 are now direct consequences of Proposition 4.5 and of Lemma 4.6.

5. Upper bound

The proof of upper bound uses the construction in [22], but is more complex due to the fact that we need to add the additional oscillations. It is easy to see, by using Korn's inequality, that each fiber of $L_\gamma(I \times \mathcal{Y})$ as well as $L_0^0(I \times \mathcal{Y})$, $L^1_{0,\gamma_1}(I \times \mathcal{Y})$ for $\gamma \in [0, \infty]$ and $\gamma_1 \in (0, \infty)$ is a closed subspace of $L^2(I \times \mathcal{Y}, \mathbb{R}^{3\times 3}_{\mathrm{sym}})$. Also by Korn's inequality it is easy to see (see also [27,28]) that the following coercivity bound is satisfied:

$$\big\|\mathcal{U}_0(\zeta, g)\big\|^2_{L^2} \geq C\big(\|\zeta\|^2_{H^1} + \|g\|^2_{L^2}\big) \quad \text{for all } (\zeta, g) \in D(\mathcal{U}_0),$$

where the constant C depends on the embedding ψ. Analogous bounds are satisfied by $\mathcal{U}_0^0$, $\mathcal{U}^1_{0,\gamma_1}$ and $\mathcal{U}_\infty$, with the obvious norms on their respective domains of definition.

The following two lemmas and remark are analogous to [29, Lemma 2.10, 2.11].

Lemma 5.1. *For $\gamma \in (0, \infty]$ there exists a bounded linear operator*

$$\Pi_\gamma : L^2\big(S, T^*S \otimes T^*S\big) \times L^2\big(S, T^*S \otimes T^*S\big) \to L^2\big(S, L_\gamma(I \times \mathcal{Y})\big),$$

such that for almost every $x \in S$ we have

$$\mathcal{Q}_\gamma\big(x, q^1(x), q^2(x)\big) = \int_I \int_{\mathcal{Y}} \mathcal{Q}\big(x + tn(x), y, q^1(x) + tq^2(x) + \Pi_\gamma\big[q^1, q^2\big](x, t, y)\big)\,dy\,dt.$$

Moreover, if $q^1, q^2 \in C^0(S, T^*S \otimes T^*S)$ then $\Pi_\gamma[q^1, q^2] \in C^0(S \times I \times \mathcal{Y})$. Similar maps Π_0^0 resp. Π^1_{0,γ_1} enjoying similar properties exist for $\mathcal{Q}_0^0$ resp. $\mathcal{Q}^1_{0,\gamma_1}$.

Before stating the next lemma we observe that the definition (35) of $\mathcal{Q}_\gamma$ makes sense even for arbitrary $q^1, q^2 \in \mathbb{R}^{3\times 3}$. The same is true for $\mathcal{Q}_{0,\gamma_1}$ and $\mathcal{Q}_0^0$.

Lemma 5.2. *For every $\gamma \in (0, \infty)$ the function $\mathcal{Q}_\gamma : S \times (\mathbb{R}^{3\times 3})^2 \to \mathbb{R}^+$ is continuous, and there exists a constant $C > 0$ depending only on the energy density W and the surface S such that, for all $x \in S$ and all $q^1, q^2 \in T_x^* S \odot T_x^* S$,*

$$C^{-1}\left(\left|q^1\right|^2 + \left|q^2\right|^2\right) \le \mathcal{Q}_\gamma\left(x, q^1, q^2\right) \le C\left(\left|q^1\right|^2 + \left|q^2\right|^2\right).$$

Similar statements apply to $\mathcal{Q}_0^0$ and $\mathcal{Q}_{0,\gamma_1}^1$.

We will need the following fact about the linearization of the square root of a matrix.

Lemma 5.3. *There exists $\eta > 0$ and a modulus of continuity m, i.e., a function $m \in C^0([0, \infty), \mathbb{R}^+)$ with $m(0) = 0$, such that the following is true for all $M \in \mathbb{R}$ and all $\delta \in (0, \eta)$:*

Assume that $G^h \in L^2(S^1, \mathbb{R}^{3\times 3})$ and $K^h \in L^4(S^1, \mathbb{R}^{3\times 3})$ satisfy

$$\limsup_{h\to 0}\left(\left\|\operatorname{sym} G^h\right\|_{L^2(S^1)} + \left\|K^h\right\|_{L^4(S^1)}\right) \le M$$

and $\limsup_{h\to 0} h\|K^h\|_{L^\infty(S^1)} \le \delta$ as well as $\limsup_{h\to 0} h^2\|G^h\|_{L^\infty(S^1)} \le \delta$, and that $h^{-1} \operatorname{sym} K^h \to 0$ strongly in $L^2(S^1)$ and $hG^h \to 0$ strongly in $L^4(S^1)$ as $h \to 0$. Set

$$E^h = h^{-2}\left(\left(\left(I + hK^h + h^2 G^h\right)^T\left(I + hK^h + h^2 G^h\right)\right)^{1/2} - I\right)$$

and $E_{\mathrm{app}}^h = \operatorname{sym} G^h - \frac{1}{2}(K^h)^2$. Then we have $\limsup_{h\to 0} \|E^h - E_{\mathrm{app}}^h\|_{L^2(S^1)} \le M^2 m(\delta)$, and

$$\limsup_{h\to 0}\left|\frac{1}{h^4}\int_S\int_I W\left(x + tn(x), r(x)/\varepsilon, I + h^2 E^h\left(x + tn(x)\right)\right) dt\, dvol_S(x)\right.$$
$$\left. - \int_S\int_I \mathcal{Q}\left(x + tn(x), r(x)/\varepsilon, E_{\mathrm{app}}^h\left(x + tn(x)\right)\right) dt\, dvol_S(x)\right| \le (M+1)^4 m(\delta). \quad (65)$$

If, moreover, $E_{\mathrm{app}}^h \xrightarrow{2} E$ strongly two-scale, then

$$\limsup_{h\to 0}\left|\frac{1}{h^4}\int_S\int_I W\left(x + tn(x), r(x)/\varepsilon, I + h^2 E^h\left(x + tn(x)\right)\right) dt\, dvol_S(x)\right.$$
$$\left. - \int_S\int_I\int_{\mathcal{Y}} \mathcal{Q}\left(x + tn(x), y, E\left(x + tn(x), y\right)\right) dy\, dt\, dvol_S(x)\right| \le (M+1)^4 m(\delta). \quad (66)$$

Proof. We will only sketch the proof. By Taylor expansion there exists $\eta_1 > 0$ and a modulus of continuity m_1 such that for every $A \in \mathbb{R}^{3\times 3}$ with $|A - I| < \eta_1$ we have

$$\left|\sqrt{\left(I + A^T\right)(I + A)} - \left(I + \operatorname{sym} A + \frac{1}{2}A^T A\right)\right| \le m_1\left(|A - I|\right)\left(\operatorname{sym} A + \frac{1}{2}A^T A\right). \quad (67)$$

Applying this to $A = hK^h + h^2 G^h$, we obtain

$$\limsup_{h\to 0}\left\|E^h - \tilde{E}_{\mathrm{app}}^h\right\|_{L^2} \le m_1(\delta)\left\|\tilde{E}_{\mathrm{app}}^h\right\|_{L^2},$$

where

$$\tilde{E}_{\mathrm{app}}^h = \frac{\operatorname{sym} K^h}{h} + \operatorname{sym} G^h + \frac{1}{2}\left(K^h\right)^T K^h + h \operatorname{sym}\left(\left(G^h\right)^T K^h\right) + \frac{1}{2}h^2\left(G^h\right)^T G^h.$$

It is easy to deduce from the hypotheses that $\|\tilde{E}_{\mathrm{app}}^h - E_{\mathrm{app}}^h\|_{L^2} \to 0$. Now the L^2-bound on $E^h - E_{\mathrm{app}}^h$ follows at once. Formula (65) follows from (32) and the triangle inequality, while (66) follows from the fact that

$$\int_S \int_I \mathcal{Q}\big(x + tn(x), r(x)/\varepsilon, E^h_{\mathrm{app}}\big(x + tn(x)\big)\big)\, dt\, dvol_S$$
$$\to \quad \int_S \int_I \int_{\mathcal{Y}} \mathcal{Q}\big(x + tn(x), y, E\big(x + tn(x), y\big)\big)\, dy\, dt\, dvol_S(x).$$

The latter is a consequence of the continuity of integral functionals with respect to strong two-scale convergence, cf. [34,35]. □

We provide here a general computation that will be needed in the proof of the following result. Let $P \in C^1(S^1; C^1(\mathcal{Y}; \mathbb{R}^3))$ define $P^h : S^1 \to \mathbb{R}^3$ by $P^h = P(\cdot, r/\varepsilon)$, where, as usual, r is extended trivially from S to S^1. Then by (25)

$$\begin{aligned}
\nabla_h P^h &= \frac{1}{h}\partial_n P^h \otimes n + \nabla P^h T_S(I + t\mathbf{S})(I + ht\mathbf{S})^{-1} T_S \\
&= \frac{1}{h}\partial_n P(\cdot, r/\varepsilon) \otimes n \\
&\quad + \left(\nabla P(\cdot, r/\varepsilon) + \frac{1}{\varepsilon}\nabla_y P(\cdot, r/\varepsilon)\nabla r \circ T_S(I + t\mathbf{S})^{-1}\right)(I + t\mathbf{S})(I + ht\mathbf{S})^{-1} T_S,
\end{aligned}$$

because having extended r trivially to S^1, we have $\partial_n r = 0$ and (16) applies. We use the notation $\partial_n P$ to denote the n-derivative with respect to the first argument only. Since $(I + ht\mathbf{S})^{-1}$ agrees with I up to a term that on S^1 is uniformly bounded by h, and if $h \le C\varepsilon$, then we conclude that

$$\left\| \nabla_h P^h - \frac{1}{\varepsilon}\mathcal{F}_\gamma(P)(\cdot, r/\varepsilon) \right\|_{L^\infty(S^1)} \le C, \tag{68}$$

where for $x \in S^1$ and $y \in \mathcal{Y}$ we have introduced

$$\mathcal{F}_\gamma(P)(x, y) = \frac{1}{\gamma}\partial_n P(x, y) \otimes n(x) + (\partial_{y_\alpha} P)(x, y) \otimes \tau^\alpha(x).$$

We have used that the restriction of the linear operator ∇r to the tangent space can be expressed as

$$\nabla r \circ T_S = e_1 \otimes \tau^1 + e_2 \otimes \tau^2,$$

which is just the pullback operator ψ^* acting on tangent vector fields. A short computation shows that setting

$$\zeta_\alpha(x, t, y) = P\big(x + tn(x), y\big) \cdot \tau_\alpha(x) \tag{69}$$
$$\rho(x, t, y) = P\big(x + tn(x), y\big) \cdot n(x) \tag{70}$$

for all $x \in S$, $t \in I$ and $y \in \mathcal{Y}$, we have for $\gamma \in (0, \infty)$:

$$\operatorname{sym} \mathcal{F}_\gamma(P)\big(x + tn(x), y\big) = \mathcal{U}_\gamma(\zeta, \rho)(x, t, y).$$

Lemma 5.4. *There exists a modulus of continuity m and for every $M > 0$ there exists a constant $C(M)$ such that for every $\delta \in (0, 1)$ the following is true:*

- *Let $V \in H^2(S; \mathbb{R}^3)$ be an infinitesimal bending with $\|V\|_{H^2(S)} \le M$ and let $v^h \in W^{2,\infty}(S, \mathbb{R}^3)$ be such that $v^h \to V$ strongly in $H^2(S)$ and $\mathcal{H}^2(\{v^h \neq V\}) \ll h^2$, and such that*

$$\limsup_{h \to 0} h \|v^h\|_{W^{2,\infty}(S)} \le \delta. \tag{71}$$

- *Let $B, b \in L^2(S, L^2(\mathcal{Y}, T^*S \odot T^*S))$ be such that*

$$\|B\|_{L^2(S \times \mathcal{Y})} + \|b\|_{L^2(S \times \mathcal{Y})} \le M. \tag{72}$$

Let $w^h \in H^2(S, \mathbb{R}^3)$ *and be such that*

$$q_{w^h} \xrightarrow{2} B \quad \text{and} \quad hb_{w^h} \xrightarrow{2} b \quad \text{strongly two-scale on } S \tag{73}$$

and such that

$$h\|dw^h\|_{L^4(S)} + h^2\|dw^h\|_{L^\infty(S)} + h^2\|\mathrm{Hess}\, w^h\|_{L^2(S)} + h^3\|\mathrm{Hess}\, w^h\|_{L^\infty(S)} \to 0. \tag{74}$$

- *Let* $p, o \in C^1(S^1, C^1(\mathcal{Y}, \mathbb{R}^3))$. *Moreover, if* $h \ll \varepsilon$ *then assume that* $o = o \circ \pi$, *while if* $h \gg \varepsilon$ *then assume that* p *does not depend on* y. *(If* $h \sim \varepsilon$ *then we make no extra assumption.) Set* $p^h = p(\cdot, r/\varepsilon)$ *and* $o^h = o(\cdot, r/\varepsilon)$ *and define* $z^h : S^h \to \mathbb{R}^3$ *by*

$$z^h = id + h\big(v^h + hw^h + t\mu_{v^h+hw^h}\big) + h^3 p^h\big(\Phi^h\big) + \varepsilon h^2 o^h\big(\Phi^h\big),$$

and define $y^h \in H^1(S^1, \mathbb{R}^3)$ *by* $y^h(\Phi^h) = z^h$, *and define* V^h *by* (39). *Assume also that*

$$\begin{aligned}&\|\check{p}\|_{L^2(S;H^1(I\times\mathcal{Y}))} + \|\check{o}\|_{L^2(S;H^1(I\times\mathcal{Y}))} \le M, \quad \text{if } \gamma \in (0,\infty),\\ &\|\partial_n p\|_{L^2(S^1\times\mathcal{Y})} + \|o\|_{L^2(S^1;H^1(\mathcal{Y}))} \le M, \quad \text{otherwise.}\end{aligned} \tag{75}$$

Here $\check{p}(x,t,y) = p(x+tn(x), y)$ *and* $\check{o}(x,t,y) = o(x+tn(x), y)$.

- *Define* $\Xi : S^1 \times \mathcal{Y} \to \mathbb{R}^{3\times3}$ *by*

$$\Xi = \begin{cases} \mathcal{F}_\gamma(o) + \gamma\mathcal{F}_\gamma(p) & \text{if } \gamma \in (0,\infty)\\ \partial_n p \otimes n + \nabla_y o \nabla r \circ T_S & \text{otherwise,}\end{cases}$$

- *Define* $E^h : S^1 \to \mathbb{R}^{3\times3}$ *by* $E^h = h^{-2}(((\nabla_h y^h)^T(\nabla_h y^h))^{1/2} - I)$ *and define* $E : S^1 \times \mathcal{Y} \to \mathbb{R}^{3\times3}$ *by setting*

$$\begin{aligned}E(x,y) = {}& B(x,y)\big(T_S(x), T_S(x)\big) - t(x)\big(b(x,y) + b_V(x)\big)\big(T_S(x), T_S(x)\big)\\ &- \frac{1}{2}\Omega_V^2(x) + \mathrm{sym}\,\Xi(x,y).\end{aligned}$$

Then $y^h \to \pi$ *strongly in* $H^1(S^1)$ *and* $V^h \to V$ *strongly in* $H^1(S)$ *and*

$$\frac{1}{h}q_{V^h} \rightharpoonup \int_{\mathcal{Y}} B(\cdot, y)\,dy \quad \text{weakly in } L^2(S), \tag{76}$$

and

$$\limsup_{h\to0}\left\|\frac{1}{h}q_{V^h}\right\|_{L^2(S)} \le C(M), \tag{77}$$

and

$$\begin{aligned}&\Bigg|h^{-4}\int_S\int_I W\big(x+tn(x), r(x)/\varepsilon, I + h^2E^h\big(x+tn(x)\big)\big)\,dt\,dvol_S(x)\\ &\quad - \int_S\int_I\int_{\mathcal{Y}} \mathcal{Q}\big(x+tn(x), y, E\big(x+tn(x)\big)\big)\,dy\,dt\,dvol_S(x)\Bigg| \le C(M)m(\delta).\end{aligned} \tag{78}$$

Proof. Notice that (73) and (72) imply that

$$\limsup_{h\to0}\big(h\|b_{w^h}\|_{L^2} + \|q_{w^h}\|_{L^2}\big) \le M \tag{79}$$

We claim that there is a constant C_1 depending only on S such that

$$\limsup_{h\to0}\|q_{v^h}\|_{L^\infty(S)} \le C_1. \tag{80}$$

In fact, note that by (71) the Lipschitz constants of all q_{v^h} are bounded in particular by $1/h$. Since $q_{v^h} = q_V = 0$ almost everywhere on $\{v^h = V\}$, there exists a constant C depending only on S such that for $\mathcal{H}^2$ a.e. $x \in S$ we have

$$\left|q_{v^h}(x)\right| \le C\frac{1}{h}\operatorname{dist}\left(x, \left\{v^h = V\right\}\right). \tag{81}$$

But due to the hypothesis on the measure of $\{v^h \neq V\}$ and to the bounded curvature of S, for small h the set $\{v^h \neq V\}$ cannot contain a disk of radius h. Hence (80) follows from (81).

Next we claim that

$$\frac{q_{v^h}}{h} \to 0 \quad \text{strongly in } L^2(S). \tag{82}$$

In fact, since $q_{v^h} = q_V = 0$ almost everywhere on $\{v^h = V\}$,

$$\frac{1}{h}\|q_{v^h}\|_{L^2(S)} \le \frac{1}{h}\left(\mathcal{H}^2\left(\left\{v^h \neq V\right\}\right)\right)^{1/2} \cdot \|q_{v^h}\|_{L^\infty(S)}, \tag{83}$$

and this converges to zero by (80) and by the hypothesis on $\{v^h \neq V\}$.

Clearly we have $y^h \to \pi$ strongly in $H^1(S^1)$. Moreover, defining $\widetilde{p}^h, \widetilde{o}^h : S \to \mathbb{R}^3$ by $\widetilde{o}^h(x) = \int_I o^h(x + tn(x))\,dt$ and $\widetilde{p}^h(x) = \int_I o^h(x + tn(x))\,dt$, we see that

$$V^h = v^h + hw^h + \varepsilon h\widetilde{o}^h + h^2\widetilde{p}^h. \tag{84}$$

From (73) and (82) we deduce that $q_{v^h}/h + q_{w^h} \overset{2}{\to} B$ strongly two-scale on S. It is easy to see that the weak L^2-limit of $\varepsilon q_{\widetilde{o}^h}$ and of $hq_{\widetilde{p}^h}$ is zero. Hence the convergence (76) follows from (84). From (84) and from (82), (79), (75) we deduce the bound (77), because clearly $\varepsilon hq_{\widetilde{o}^h}$ and $h^2 q_{\widetilde{p}^h}$ converge strongly to zero in L^2 – this is certainly true if $h \lesssim \varepsilon$, and if $h \gg \varepsilon$ then it follows from the hypothesis that p be independent of y.

Next we address the limiting behaviour of E^h. Define $G^h : S^1 \to \mathbb{R}^{3\times 3}$ by the formula

$$\nabla_h y^h\left(\Phi^h\right) = \nabla z^h = I + h\Omega_{v^h} + h^2 G^h\left(\Phi^h\right).$$

Defining $Q : S^h \to \mathbb{R}^{3\times 3}$ by (18) and using Lemma 2.4 we see that the following equation is satisfied on S^h:

$$\begin{aligned} G^h\left(\Phi^h\right) = {} & \Omega_{w^h} - \frac{t}{h} b_{v^h}(T_S, T_S) - t b_{w^h}(T_S, T_S) - \frac{t^2}{h}\nabla\mu_{v^h + hw^h}\mathbf{S} \\ & + \frac{1}{h}\left(dv^h + h\,dw^h + t\nabla\mu_{v^h+hw^h}\right)T_S Q \\ & + h\nabla_h p^h\left(\Phi^h\right) + \varepsilon\nabla_h o^h\left(\Phi^h\right). \end{aligned} \tag{85}$$

We deduce from (68) (applied with $P = p$ and $P = o$) that

$$h\nabla_h p^h + \varepsilon\nabla_h o^h \overset{2}{\to} \Xi. \tag{86}$$

Notice that by (68) the convergence (86) is indeed also true for $\gamma = 0$ because in that case we assume that $\partial_n o \equiv 0$, and for $\gamma = \infty$ because in that case we assume that p^h does not depend on y.

The hypotheses on v^h imply that $b_{v^h}(T_S, T_S) \to b_V(T_S, T_S)$ strongly in L^2 and those on w^h imply that

$$\left(tb_{w^h}(T_S, T_S)\right) \circ \left(\Phi^h\right)^{-1} = thb_{w^h}(T_S, T_S) \overset{2}{\to} tb(T_S, T_S) \quad \text{on } S^1.$$

Lemma 2.1 and (73) show that

$$\operatorname{sym}\Omega_{w^h} = q_{w^h}(T_S, T_S) \overset{2}{\to} B(T_S, T_S) \quad \text{on } S. \tag{87}$$

We conclude that

$$\operatorname{sym} G^h \overset{2}{\to} B(T_S, T_S) - t(b_V + b)(T_S, T_S) + \operatorname{sym}\Xi \tag{88}$$

strongly two-scale in S^1. By (88) the map $E^h_{\text{app}} : S^1 \to \mathbb{R}^{3\times 3}$ defined by

$$E^h_{\text{app}} = \operatorname{sym} G^h - \frac{1}{2}(\Omega_{v^h})^2$$

converges strongly two-scale on S^1 to E.

In order to deduce (78) we apply Lemma 5.3 to E^h and E^h_{app}. This finishes the proof once we have verified that the hypotheses of Lemma 5.3 are satisfied (with $K^h = \Omega_{v^h}$).

Regarding the bounds on Ω_{v^h}, notice that $v^h \to V$ in H^2 implies that $\Omega_{v^h} \to \Omega_V$ in H^1. Hence the L^4-bound on Ω_{v^h} follows from the H^2-bound on V and Sobolev embedding. Since $h^{-1}\operatorname{sym}\Omega_{v^h} = h^{-1}q_{v^h}(T_S, T_S)$ by (10), we have $h^{-1}\operatorname{sym}\Omega_{v^h} \to 0$ in L^2 by (82). And (71) implies that $\|h\Omega_{v^h}\|_{L^\infty} \le C\delta$.

Regarding the bounds on G^h, first note that (88) implies convergence of norms, so the bounds on V, B, b, o, p imply that $\lim_{h\to 0}\|\operatorname{sym} G^h\|_{L^2(S^1)}$ is bounded by a multiple of M. It remains to show that $hG^h \to 0$ in L^4 and that $\limsup_{h\to 0}\|h^2 G^h\|_{L^\infty}$ is dominated by δ (we will show that it is equal to zero). We see from (85) that G^h is dominated pointwise on S^1 by

$$|\Omega_{w^h}| + |b_{v^h}| + h|b_{w^h}| + h^2|\operatorname{Hess} w^h| + h|\operatorname{Hess} v^h| + |h\nabla_h p^h + \varepsilon\nabla_h o^h| \tag{89}$$

plus terms of lower order. By (75) the last term in (89) remains bounded in L^2 as $h \to 0$. Hence $h(h\nabla_h p^h + \varepsilon\nabla_h o^h)$ converges strongly to zero L^2. As it is uniformly bounded in L^∞ by (68), we see that $h^2(h\nabla_h p^h + \varepsilon\nabla_h o^h)$ converges to zero in L^∞.

The analogous claim is valid for the forth and fifth term by (71) and (74). Regarding the first term in (89) note that $h\|\Omega_{w^h}\|_{L^4}$ and $h^2\|\Omega_{w^h}\|_{L^\infty}$ converge to zero by (74) because $|\Omega_{w^h}| \le C|dw^h|$.

Regarding the second term in (89) note that $|b_{v^h}| \le |\operatorname{Hess} v^h|$. And $h\operatorname{Hess} v^h$ is bounded by 1 in L^∞ by (71), so in particular $h^2\operatorname{Hess} v^h \to 0$ in L^∞.

Finally, $h^3 b_{w^h} \to 0$ in L^∞ by (74) and hb_{w^h} is L^2-bounded by (79). So by interpolation $h^2 b_{w^h} \to 0$ in L^4. □

Proposition 5.5. *Assume that $\varepsilon^2 \lesssim h$, let $B_w \in \mathcal{B}$ and let $V \in H^2(S;\mathbb{R}^3)$ be an infinitesimal bending of S. Then there exists a sequence $(y^h) \subset H^1(S^1,\mathbb{R}^3)$ such that $y^h \to \pi$ strongly in $H^1(S^1;\mathbb{R}^3)$ and such that V^h given by (39) satisfy $V^h \to V$ strongly in $H^1(S)$ and $q_{V^h}/h \rightharpoonup B_w$ weakly in $L^2(S)$. Moreover,*

$$\lim_{h\to 0} h^{-4} I^h(y^h) = \begin{cases} I_\gamma(V, B_w) & \text{if } \lim h/\varepsilon = \gamma \in (0,\infty], \\ I_0^0(V, B_w) & \text{if } \varepsilon \gg h \gg \varepsilon^2, \\ I^1_{0,\gamma_1}(V, B_w) & \text{if } h \sim \varepsilon^2 \text{ with } \lim \varepsilon^2/h = 1/\gamma_1. \end{cases}$$

Proof. By definition and by density, since $B_w \in \mathcal{B}$, there exist $w_n \in C_0^\infty(S;\mathbb{R}^3)$ such that $q_{w_n} \to B_w$ strongly in $L^2(S)$. By choosing a suitable index sequence $n_h \to \infty$, we may assume that the maps $\widetilde{w}^h = w_{n_h}$ satisfy

$$h\widetilde{w}^h \to 0 \quad \text{strongly in } W^{2,\infty}(S) \tag{90}$$

and

$$q_{\widetilde{w}^h} \to B_w \quad \text{strongly in } L^2(S). \tag{91}$$

For each $\delta \in (0,1)$ let $\varphi^\delta \in C^1(S; \dot{C}^2(\mathcal{Y}))$ and define

$$w^{\delta,h} = \widetilde{w}^h + \frac{\varepsilon^2}{h}\varphi^\delta(\cdot, r/\varepsilon)n. \tag{92}$$

If $\varepsilon \lesssim h$ then we choose $\varphi^\delta \equiv 0$ for all δ.

We claim that $w^{\delta,h}$ satisfies the hypotheses of Lemma 5.4 for each fixed δ (with $w^{\delta,h}$ playing the role of the w^h from Lemma 5.4) but with δ-independent M, provided that

$$\limsup_{\delta\to 0}\left(\|\varphi^\delta\|_{L^2(S\times\mathcal{Y})} + \|\nabla_y^2\varphi^\delta\|_{L^2(S\times\mathcal{Y})}\right) < \infty. \tag{93}$$

In fact, the bounds (74) follow from (90) (recall that $\varepsilon^2/h \lesssim 1$).

It is easy to see that

$$\varepsilon^2 \operatorname{Hess}\big(\varphi(\cdot, r/\varepsilon)\big) \xrightarrow{2} \operatorname{Hess}_{\mathcal{Y}}\varphi \quad \text{on } S.$$

Since moreover $q_{w^{\delta,h}} = q_{w^h} + \frac{\varepsilon^2}{h}\varphi^\delta(\cdot, r/\varepsilon)\mathbf{S}$, we see that (73) is satisfied with $b = \operatorname{Hess}_{\mathcal{Y}}\varphi^\delta$ and with

$$B = B_w + \frac{1}{\gamma_1}\varphi^\delta(x,y)\mathbf{S} \quad \big(\text{with } 1/\gamma_1 := 0 \text{ when } h \gg \varepsilon^2\big).$$

In particular, the bound (72) is satisfied for some δ-independent M provided that (93) holds.

We will also choose δ-dependent p^δ and o^δ. Defining Ξ^δ in analogy to Ξ in Lemma 5.4, we need to choose p^δ and o^δ in such a way that they satisfy (75). This will follow from the construction of p^δ, o^δ below.

Next, for each $\delta \in (0,1)$, we approximate V by a sequence $v^{\delta,h} \in W^{2,\infty}(S;\mathbb{R}^3)$ such that the map $v^{\delta,h}$ satisfies the same hypotheses as v^h in Lemma 5.4 (where the index in $v^{\delta,h}$ is the δ from (71)). The existence of such $v^{\delta,h}$ follows from [10, Proposition 2].

Let $A_\delta \in C^2(S;\mathrm{so}(3))$ be such that $A_\delta \to \Omega_V$ strongly in $H^1(S)$ as $\delta \to 0$. Notice that

$$\begin{aligned} A_\delta^2 - A_\delta^2(T_S,T_S) &= (n\otimes n)A_\delta^2 - (n\otimes n)A_\delta^2(n\otimes n) + A_\delta^2(n\otimes n) \\ &= 2\,\mathrm{sym}\left(A_\delta^2 n\otimes n + \frac{|A_\delta n|^2}{2} n\otimes n\right). \end{aligned} \tag{94}$$

because A_δ is skew symmetric. With A_δ at hand, we can now construct the oscillations p^δ and o^δ. We have to distinguish the three basic cases; observe that for $\gamma \neq 0$ the bound (93) is trivial because here we chose $\varphi^\delta \equiv 0$.

The case $\gamma \in (0,\infty)$ (i.e. $h \sim \varepsilon$). Define $p^\delta : S^1 \times \mathcal{Y} \to \mathbb{R}^3$ by

$$p^\delta(x,y) = t(x)\left(\frac{|A_\delta(x)n(x)|^2}{2} I + A_\delta^2(x)\right)n(x);$$

note that p^δ does not depend on y. Since the right-hand side of (94) equals $2\,\mathrm{sym}(\partial_n p^\delta \otimes n)$ because clearly $\partial_n p^\delta = \frac{|A_\delta n|^2}{2} n + A_\delta^2 n$, we conclude that

$$\gamma\,\mathrm{sym}\,\mathcal{F}_\gamma\big(p^\delta\big) = \mathrm{sym}\big(\partial_n p^\delta \otimes n\big) = \frac{1}{2}\big(A_\delta^2 - A_\delta^2(T_S,T_S)\big). \tag{95}$$

Now we choose $\zeta^\delta \in C^1(S,\dot{C}^1(I\times\mathcal{Y};\mathbb{R}^2))$ and $\rho^\delta \in C^1(S,\dot{C}^1(I\times\mathcal{Y}))$ in such a way that

$$\mathcal{U}_\gamma\big(\zeta^\delta,\rho^\delta\big) \to \Pi_\gamma\left(B_w + \frac{(dV)^2}{2}, -b_V\right) \quad \text{strongly in } L^2. \tag{96}$$

as $\delta \to 0$. (Here, the operator Π_γ is as in Lemma 5.1.)

Defining o^δ via (69), (70) (with o^δ playing the role of P), we have

$$\hat{\mathcal{U}}_\gamma\big(\zeta^\delta,\rho^\delta\big) = \mathrm{sym}\,\mathcal{F}_\gamma\big(o^\delta\big). \tag{97}$$

Here $\hat{\mathcal{U}}_\gamma(x,y) = \hat{\mathcal{U}}_\gamma(\pi(x),t(x),y)$. From (95), (97) and from the definition of E^δ and Ξ^δ in Lemma 5.4 (with obvious notational changes involving the index δ), we conclude:

$$E^\delta = \left(B_w - tb_V - \frac{1}{2}A_\delta^2\right)(T_S,T_S) + \hat{\mathcal{U}}_\gamma\big(\zeta^\delta,\rho^\delta\big) + \frac{1}{2}\big(A_\delta^2 - \Omega_V^2\big).$$

Then

$$E^\delta \to \left(B_w - tb_V + \frac{1}{2}(dV)^2\right)(T_S,T_S) + \hat{\Pi}_\gamma\left(B_w + \frac{(dV)^2}{2}, -b_V\right), \quad \text{strongly in } L^2(S), \tag{98}$$

as $\delta \to 0$ by the choice of A_δ and because $-A_\delta^2(T_S,T_S) \to (dV)^2$ by (12). Here we define $\hat{\Pi}_\gamma$ with respect to Π_γ in the analogous way as $\hat{\mathcal{U}}_\gamma$ with respect to $\mathcal{U}_\gamma$. By (98) and by Lemma 5.4, we see that

$$k(\delta,h) = \big\|y^{\delta,h} - \pi\big\|_{H^1(S^1)} + \big\|V^{\delta,h} - V\big\|_{H^1(S)} + \tilde{d}^K\left(\frac{1}{h}q_{V^{\delta,h}}, B_w\right) + \left|\frac{1}{h^4}I^h\big(y^h\big) - I_\gamma(V,B_w)\right|,$$

satisfies $\limsup_{\delta\to 0}\limsup_{h\to 0} k(\delta,h) = 0$. Here $\tilde{d}^K : L^2(S;\mathbb{R}^{3\times 3}) \times L^2(S;\mathbb{R}^{3\times 3}) \to \mathbb{R}$ is defined as follows: For $K > 0$ there exists a metric d^K which defines the weak topology on the ball of radius K. We define:

$$\tilde{d}^K(M_1,M_2) = \begin{cases} d^K(M_1,M_2), & \text{if } \|M_1\|_{L^2} < K \text{ and } \|M_2\|_{L^2} < K, \\ +\infty, & \text{otherwise.} \end{cases}$$

By (77) the constant K can indeed be chosen independently of δ. Finally, a standard diagonalization procedure then yields a sequence $\delta_h \to 0$ such that $k(\delta_h, h) \to 0$ as $h \to 0$.

The case $\gamma = \infty$ (i.e. $h \gg \varepsilon$). Define $\boldsymbol{\zeta}^\delta \in C^1(S; \dot{C}^1(I \times \mathcal{Y}; \mathbb{R}^2))$, $\psi^\delta \in C^1(S; \dot{C}^1(I \times \mathcal{Y}))$, $c^\delta \in C^1(S; \dot{C}^1(I; \mathbb{R}^3))$ such that

$$\mathcal{U}_\infty(\boldsymbol{\zeta}^\delta, \psi^\delta, c^\delta) \to \Pi_\infty\left(B_w + \frac{(dV)^2}{2}, -b_V\right) \quad \text{strongly in } L^2(S)$$

as $\delta \to 0$. We will use the following fact: if $f : I \times \mathcal{Y} \to \mathbb{R}^3$ then $F(x, y) = \int_0^{t(x)} f(s, y)\, ds$ satisfies $\partial_n F(x, y) = f(t(x), y)$. We wish to have p^δ independent of y, in order to satisfy the hypotheses of Lemma 5.4. We define

$$p^\delta(x, y) = t(x)\left(\frac{|A_\delta(x)n(x)|^2}{2} I + A_\delta^2(x)\right)n(x) + 2\int_0^{t(x)} c_\alpha(s)\, ds\, \tau^\alpha + \int_0^{t(x)} c_3(s)\, ds\, \tau^3.$$

Then $\partial_n p^\delta = \frac{|A_\delta n|^2}{2} n + A_\delta^2 n + 2c_\alpha(t)\tau^\alpha + c_3(t)\tau^3$. For $x \in S^1$ and $y \in \mathcal{Y}$ set

$$o^\delta(x, y) = \zeta_\alpha^\delta\big(\pi(x), t(x), y\big)\tau^\alpha(x) + 2\psi^\delta\big(\pi(x), t(x), y\big)n(x).$$

Then $\nabla_y o^\delta = \tau^\alpha \otimes \nabla_y \zeta_\alpha^\delta + 2n \otimes \nabla_y \psi^\delta$. Since $\gamma = \infty$, we have

$$\operatorname{sym} \Xi^\delta = \operatorname{sym}\big(\partial_n p^\delta \otimes n + \nabla_y o^\delta \nabla r \circ T_S\big) = \hat{\mathcal{U}}_\infty(\boldsymbol{\zeta}^\delta, \psi^\delta, c^\delta) + \frac{1}{2}\big(A_\delta^2 - A_\delta^2(T_S, T_S)\big).$$

From now on the proof is analogous to the case $\gamma \in (0, \infty)$.

Construction for $\gamma = 0$ (i.e. $h \ll \varepsilon$). If $\varepsilon \gg h \gg \varepsilon^2$ then choose φ^δ in (92) and $\boldsymbol{\zeta}^\delta \in C^1(S; \dot{C}^1(\mathcal{Y}; \mathbb{R}^2))$ as well as $g^\delta \in C^1(S; \dot{C}^1(I \times \mathcal{Y}; \mathbb{R}^3))$ such that

$$\mathcal{U}_0^0(\boldsymbol{\zeta}^\delta, \varphi^\delta, g^\delta) \to \Pi_0^0\left(B_w + \frac{(dV)^2}{2}, -b_V\right) \quad \text{strongly in } L^2(S), \tag{99}$$

as $\delta \to 0$. If $h \sim \varepsilon^2$ with $\varepsilon^2/h \to 1/\gamma_1$, choose them such that

$$\mathcal{U}_{0,\gamma_1}^1(\boldsymbol{\zeta}^\delta, \varphi^\delta, g^\delta) \to \Pi_{0,\gamma_1}^1\left(B_w + \frac{(dV)^2}{2}, -b_V\right) \quad \text{strongly in } L^2(S). \tag{100}$$

Extend ζ^δ trivially to S^1 and define

$$o^\delta(x, y) = \zeta_\alpha^\delta(x, y)\tau^\alpha(x).$$

Then $\partial_n o^\delta \equiv 0$, so the hypotheses of Lemma 5.4 are satisfied. We define

$$p^\delta(x, y) = t(x)\left(\frac{|A_\delta(x)n(x)|^2}{2} I + A_\delta^2(x)\right)n(x) + 2\int_0^{t(x)} g_\alpha^\delta\big(\pi(x), s, y\big)\, ds\, \tau^\alpha(x)$$
$$+ \int_0^{t(x)} g_3^\delta\big(\pi(x), s, y\big)\, ds\, \tau^3(x).$$

Arguing as in the case $\gamma \in (0, \infty)$, we conclude the proof of the proposition. Finally, note that (99) and (100) together with the coercivity properties of $\mathcal{U}_0^0$ and of $\mathcal{U}_{0,\gamma_1}^1$ ensure that (75) and (93) are indeed satisfied. □

6. Convex shells

In this chapter we shall identify the Γ-limit for convex shells in the remaining case, i.e. $h \ll \varepsilon^2$. We wish to illustrate the stronger influence of the geometry in this case. We work under the assumption that S is uniformly convex, i.e., there exists $C > 0$ such that

$$\mathbf{S}(x)\tau\cdot\tau \geq C|\tau|^2_{T_xS}, \quad \forall x\in S,\ \tau\in T_xS. \tag{101}$$

For $x\in S$ we define a relaxation operator with values in the bundle (34) as follows: Set $D(\mathcal{U}_0^{2,c}) = \dot{L}^2(\mathcal{Y};\mathbb{R}^{2\times 2}_{\mathrm{sym}})\times L^2(I\times\mathcal{Y};\mathbb{R}^3)$ and for all $(\dot{B},g)\in D(\mathcal{U}_0^{2,c})$ define

$$\mathcal{U}_0^{2,c}(\dot{B},g) = \sum_{i,j=1}^{3}\begin{pmatrix} & \dot{B} & & g_1\\ & & & g_2\\ g_1 & & g_2 & g_3\end{pmatrix}_{ij}\tau^i\otimes\tau^j.$$

As usual, we introduce the vector bundle $L_0^{2,c}(I\times\mathcal{Y})$ of *relaxation fields* to be the range of $\mathcal{U}_0^{2,c}$ similarly to the bundles $L_0(I\times\mathcal{Y})$ introduced earlier. As in the case of general surfaces, each fiber of $L_0^{2,c}(I\times\mathcal{Y})$ is a closed subspace of $L^2(I\times\mathcal{Y};\mathbb{R}^{3\times3}_{\mathrm{sym}})$. We also define the functional $I_0^{2,c}: H^2(S;\mathbb{R}^3)\times L^2(S;\mathbb{S})\to\mathbb{R}$ by

$$I_0^{2,c}(V,B_w) = \int_S \mathcal{Q}_0^{2,c}\left(x, B_w + \frac{1}{2}(dV)^2, -b_V\right)dx, \tag{102}$$

with the quadratic form $\mathcal{Q}_0^{2,c}(x):(T^*S\odot T^*S)^2\to\mathbb{R}$ given by

$$\mathcal{Q}_0^{2,c}(x,q^1,q^2) = \inf_{U\in L_0^{2,c}(I\times\mathcal{Y})}\iint_{I\times\mathcal{Y}}\mathcal{Q}\big(x+tn(x), y, q^1+tq^2+U\big)\,dt\,dy. \tag{103}$$

As before, one can relax slicewise, and when the energy is homogeneous in the thickness direction then we have a decoupling similar to the one described for general shells.

Under the assumption (101) it is well-known that $\mathcal{B} = L^2(S, T^*S\odot T^*S)$, cf. e.g. [7]. Thus if one wants additionally to relax the functional $I_0^{2,c}$ with respect to B_w, one obtains the functional $\tilde{I}_0^{2,c}: H^2(S;\mathbb{R}^3)\to\mathbb{R}$ given by

$$\tilde{I}_0^{2,c}(V) = \frac{1}{12}\int_S\int_{\mathcal{Y}}\mathcal{Q}_2\big(x,y,-b_V(x)\big)\,dy\,dvol_S(x). \tag{104}$$

This functional is the same as the one arising in the ordinary von Kármán model. For the form $\mathcal{Q}_0^{2,c}$ one can make assertions analogous to Lemma 5.1 with the appropriate operator $\Pi_0^{2,c}$ and Lemma 5.2. We introduce the space

$$FL\big(S;\dot{C}^\infty(\mathcal{Y})\big) = \left\{(x,y)\mapsto\sum_{k\in\mathbb{Z}^2,\ |k|\leq n,\ k\neq 0} c^k(x)e^{2\pi i k\cdot y} : n\in\mathbf{N} \text{ and } c^k\in C_0^1(S;\mathbb{C}) \text{ with } \bar{c}^k = c^{-k}\right\}.$$

By Fourier transform it can be easily seen that $FL(S;C^\infty(\mathcal{Y}))$ is dense in $L^2(S;\dot{H}^m(\mathcal{Y}))$, for any $m\in\mathbb{N}_0$.

The following lemma resembles Lemma 3.3 in [30]. The same system arises in [23, Chapter 3]. However, here we deal with a linear PDE system with constant coefficients, which is of course much easier than the linear system with variable coefficients in [23, Chapter 3].

Lemma 6.1. *Assume that (101) is satisfied and let $\dot{B}\in L^2(S;\dot{L}^2(\mathcal{Y};T^*S\otimes T^*S))$. Then there exist unique $w\in L^2(S;\dot{H}^1(\mathcal{Y};\mathbb{R}^2))$ and $\varphi\in L^2(S;\dot{L}^2(Y))$ such that*

$$\mathrm{Def}_{\mathcal{Y}}\, w + \varphi\mathbf{S} = \dot{B}. \tag{105}$$

Moreover, if $\dot{B}_{ij}\in FL(S;\dot{C}^\infty(\mathcal{Y}))$ for every $i,j=1,2$ then $w_i\in FL(S;\dot{H}^1(\mathcal{Y}))$, for $i=1,2$ and $\varphi\in FL(S;\dot{H}^1(\mathcal{Y}))$.

Proof. There exist b_{ij}^k such that for all $i,j=1,2$:

$$\dot{B}(x,y)_{ij} = \sum_{k\in\mathbb{Z}^2} b_{ij}^k(x)e^{2\pi i k\cdot y}, \quad \text{where } \sum_{k\in\mathbb{Z}^2}\|b_{ij}^k\|_{L^2}<\infty,\ \bar{b}_{ij}^k = b_{ij}^{-k},\ b_{ij}^0 = 0.$$

We assume that for $i=1,2$:

$$w_i = \sum_{k\in\mathbb{Z}^2} c_i^k(x)e^{2\pi i k\cdot y}, \quad \bar{c}_i^k = c_i^{-k},\ c_i^0 = 0,\ \varphi = \sum_{k\in\mathbb{Z}^2} d^k(x)e^{2\pi i k\cdot y},\ \bar{d}^k = d^{-k},\ d^0 = 0.$$

Equation (105) is equivalent to the following problem for every $(k_1, k_2) \in \mathbb{Z}^2$ find complex coefficients c_j^k, b_{ij}^k, d^k such that

$$\begin{aligned} k_1 c_1^k + d^k \mathbf{S}_{11} &= b_{11}^k, \\ \frac{1}{2}\big(k_2 c_1^k + k_1 c_2^k\big) + d^k \mathbf{S}_{12} &= b_{12}^k, \\ k_2 c_2^k + d^k \mathbf{S}_{22} &= b_{22}^k. \end{aligned}$$

By the hypotheses on the embedding ψ and by (101) it is easy to see that there exists $C > 0$ such that the determinant of the system is bounded from below by $C(k_1^2 + k_2^2)$. Using this it follows that there exists $C > 0$ such that

$$\big|d^k(x)\big|^2 + \sum_{i=1}^{2} |k|^2 \big|c_i^k(x)\big|^2 \le C\left(\sum_{i,j=1}^{2} \big|b_{ij}^k(x)\big|^2\right), \quad \forall x \in S.$$

Now all claims follow easily. □

Theorem 6.2. *Under the hypotheses and with the notation of Theorem 3.1 and assuming, in addition, that S is uniformly convex and that $h \ll \varepsilon^2$, the conclusion of Theorem 3.1(i) is satisfied and, moreover, the following are true:*

- *We have*

$$\liminf_{h\to 0} h^{-4} J^h\big(u^h\big) \ge I_0^{2,c}(V, \tilde{B}_w).$$

- *For any infinitesimal bending $V \in H^2(S, \mathbb{R}^3)$ of S and any $B_w \in L^2(S, T^*S \odot T^*S)$ there exist $u^h \in H^1(S^h; \mathbb{R}^3)$ satisfying (37), and such that the conclusions of Theorem 3.1(i) are satisfied with $Q^h = I$ and $c^h = 0$. Moreover,*

$$\lim_{h\to 0} h^{-4} J^h\big(u^h\big) = I_0^{2,c}(V, B_w).$$

Proof. We will only sketch the proof as it is similar to the previous cases. As in Proposition 4.5 there exists $\varphi \in L^2(S; \dot{H}^2(\mathcal{Y}))$ such that (50) is satisfied. Using Proposition 4.2(iii) as well as Lemma 6.1, we conclude that $\varphi = 0$. Thus by Proposition 4.5 there exist $U \in L^2(S; L_0(I \times \mathcal{Y}))$ and $\dot{B} \in L^2(S; \dot{L}^2(\mathcal{Y}; T^*S \otimes T^*S))$ such that the maps E^h defined as in (51) converge weakly two-scale to

$$E = B_w + \dot{B} + \frac{1}{2}(dV)^2 - t b_V + \hat{U},$$

Hence lower bound part follows readily from Lemma 4.6 and the definition of the functional $I_0^{2,c}$.

To prove the upper bound, we follow the proof of Proposition 5.5 in the case $\gamma = 0$. Let A^δ be as in that proof. Let $\dot{B}^\delta$ with $(\dot{B}^\delta)_{ij} \in FL(S; \dot{C}^\infty(\mathcal{Y}))$ for $i, j = 1, 2$ and $g^\delta \in C^1(S; C^1(I \times \mathcal{Y}; \mathbb{R}^3))$ be such that

$$\mathcal{U}_0^{2,c}\big(\dot{B}^\delta, g^\delta\big) \to \Pi_0^{2,c}\left(B_w + \frac{1}{2}(dV)^2, -b_V\right) \quad \text{strongly in } L^2(S)$$

as $\delta \to 0$. By Lemma 6.1 there exist $z^\delta \in (FL(S; \dot{C}^\infty(\mathcal{Y})))^2$ and $\varphi^\delta \in FL(S; \dot{C}^\infty(\mathcal{Y}))$ solving the system $\mathrm{Def}_{\mathcal{Y}} z^\delta + \varphi^\delta \mathbf{S} = \dot{B}^\delta$. We choose $v^{\delta,h}$ and $\widetilde{w}^h$ as in the proof of Proposition 5.5, and we define

$$w^{\delta,h} = \widetilde{w}^h + \varphi^\delta(\cdot, r/\varepsilon)n + \varepsilon\big(z_1^\delta(\cdot, r/\varepsilon)\tau^1 + z_2^\delta(\cdot, r/\varepsilon)\tau^2\big).$$

We define p^δ by

$$p^\delta(x,y) = t(x)\left(\frac{|A_\delta(x)n(x)|^2}{2} I + A_\delta^2(x)\right)n(x) + 2\int_0^{t(x)} g_\alpha^\delta\big(\pi(x),s,y\big)\,ds\,\tau^\alpha(x)$$
$$+ \int_0^{t(x)} g_3^\delta\big(\pi(x),s,y\big)\,ds\,\tau^3(x).$$

Now we can argue as in the proof of Proposition 5.5. □

Conflict of interest statement

Manuscript title: Derivation of a homogenized von-Karman shell theory from 3D elasticity

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