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Harnack inequalities, exponential separation, and perturbations of principal Floquet bundles for linear parabolic equations

Inégalités de Harnack, séparation exponentielle, et fibré principal de Floquet pour des équations linéaires paraboliques

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Abstract

We consider the Dirichlet problem for linear nonautonomous second order parabolic equations with bounded measurable coefficients on bounded Lipschitz domains. Using a new Harnack-type inequality for quotients of positive solutions, we show that each positive solution exponentially dominates any solution which changes sign for all times. We then examine continuity and robustness properties of a principal Floquet bundle and the associated exponential separation under perturbations of the coefficients and the spatial domain.
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Résumé

On considère le problème de Dirichlet pour des équations paraboliques linéaires non autonomes du second ordre avec coefficients bornés mesurables sur un domaine borné de Lipschitz. Utilisant une nouvelle inégalité du type Harnack pour les quotients de solutions strictement positives, on montre que toute solution positive domine exponentiellement toute solution qui change de signe en tout temps. On examine ensuite les propriétés de continuité et de robustesse pour un fibré principal de Floquet et la séparation exponentielle associée à des perturbations des coefficients et du domaine spatial.
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1. Introduction

Consider the following Dirichlet problem for a linear nonautonomous parabolic equation

$$\begin{aligned} &u_t + Lu = 0 \quad \text{in } \Omega \times J, \\ &u = 0 \quad \text{on } \partial\Omega \times J. \end{aligned} \tag{1.1}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain, J is an open interval in $\mathbb{R}$, and L is a time-dependent second-order elliptic operator of either the divergence form

$$Lu = -\partial_i\big(a_{ij}(x,t)\partial_j u + a_i(x,t)u\big) + b_i(x,t)\partial_i u + c_0(x,t)u \tag{D}$$

or the non-divergence form

$$Lu = -a_{ij}(x,t)\partial_i\partial_j u + b_i(x,t)\partial_i u + c_0(x,t)u \tag{ND}$$

(we use the summation convention and the notation $\partial_i = \partial/\partial x_i$). We assume that the coefficients are real valued and bounded:

$$a_{ij}, a_i, b_i, c_0 \in \mathcal{B}_{d_0} \quad (i, j = 1, \ldots, N), \tag{1.2}$$

where $d_0 > 0$ is a fixed constant and

$$\mathcal{B}_{d_0} := \big\{ f \in L^\infty(\Omega \times \mathbb{R}) \colon \|f\|_{L^\infty(\Omega\times\mathbb{R})} \leqslant d_0 \big\}. \tag{1.3}$$

For some results in the case (ND), when explicitly indicated, we in addition assume that $a_{ij} \in C(\Omega \times \mathbb{R})$, $i, j = 1, \ldots, N$. We always consider uniformly parabolic equations: there exists $\alpha_0 > 0$ such that

$$a_{ij}(x,t)\xi_i\xi_j \geqslant \alpha_0|\xi|^2 \quad \big((x,t) \in \Omega \times \mathbb{R},\ \xi \in \mathbb{R}^N\big). \tag{1.4}$$

Concerning Ω, our assumption is that it is a bounded *Lipschitz* domain in $\mathbb{R}^N$. This means that there are positive constants r_Ω and m_Ω such that for each $y \in \partial\Omega$, there is an orthonormal coordinate system centered at y in which

$$\Omega \cap B_{r_\Omega}(y) = \big\{ x = (x', x_N) \colon x' \in \mathbb{R}^{N-1}, x_N > \phi(x'), |x| < r_\Omega \big\}, \tag{1.5}$$

and

$$\|\nabla\phi\|_{L^\infty} \leqslant m_\Omega. \tag{1.6}$$

Here and below $B_r(x)$ denotes the ball in $\mathbb{R}^N$ of radius $r > 0$ and center x. For the remainder of this paper, if not stated otherwise, we shall assume that Ω is a domain as above with

$$r_\Omega \geqslant r_0, \quad m_\Omega \leqslant M_0, \quad \operatorname{diam}\Omega \leqslant R_0, \tag{1.7}$$

where r_0, M_0, R_0 are fixed positive constants.

The main goal of our paper is to examine, in this general context, properties of solutions of (1.1) that are analogous to properties of principal eigenfunctions of time-independent (elliptic) or time-periodic parabolic problems. The analogy is best seen in one of our main results, Theorem 2.6, where we establish the existence of two time-dependent spaces $X^1(t)$, $X^2(t)$. For each $t \in \mathbb{R}$, these are subspaces of a suitable Banach space X in which the initial conditions for (1.1) are taken ($X = L^2(\Omega)$ in the divergence case and $X = C_0(\overline{\Omega})$, the space of continuous functions vanishing on the boundary, in the nondivergence case); $X^1(t)$ is the (one-dimensional) span of a positive function, $X^2(t) \setminus \{0\}$ does not contain any nonnegative function and the subspaces are complementary to one another:

$$X = X^1(t) \oplus X^2(t) \quad (t \in \mathbb{R}).$$

The subspace $X^2(s)$ is characterized by the property that if a nontrivial solution of (1.1) has its initial condition at time s contained in $X^2(s)$, then it changes sign for all $t > s$. On the other hand, the solutions starting from $X = X^1(s)$ do not change sign for any $t > s$. These characterizations imply the invariance of the bundles $X^i(t)$, $t \in \mathbb{R}$, $i = 1, 2$: if $u_1(t), u_2(t)$ are solutions with $u_i(s) \in X^i(s)$, then $u_i(t) \in X^i(t)$ for all $t > s$. Finally, for any such pair of solutions u_i, assuming u_1 is nontrivial, we have the estimate (which we call the exponential separation)

$$\frac{\|u_2(\cdot,t)\|_X}{\|u_1(\cdot,t)\|_X} \leqslant C\mathrm{e}^{-\gamma(t-s)} \frac{\|u_2(\cdot,s)\|_X}{\|u_1(\cdot,s)\|_X} \quad (t \geqslant s),$$

where $C, \gamma > 0$ are universal positive constants (they are determined by the quantities N, α_0, d_0, r_0, R_0, M_0 from the above assumptions). See Section 2 for the precise formulation of these results. The existence of the invariant bundles with exponential separation extends in a natural way a well known theorem on the existence of a spectral decomposition associated with the principal eigenvalue of elliptic or time-periodic parabolic operators (see [17], for example). Indeed, the principal eigenvalue of a time-independent operator L is simple and has a positive eigenfunction (which spans the first space X^1 in the spectral decomposition) and no other generalized eigenfunction is nonnegative (thus the codimension-one subspace X^2 in the spectral decomposition contains no nontrivial nonnegative function). Moreover, the principal eigenvalue is smaller than the real part of any other eigenvalue, which can be equivalently formulated as exponential separation for the corresponding parabolic equation. Similar results hold for time-periodic parabolic problems in which case one finds a Floquet decomposition $X = X^1(t) \oplus X^2(t)$ with time-periodic spaces. Again $X^1(t)$ is spanned by a positive function and the solutions in $X^2(t)$ are exponentially dominated by solutions in $X^1(t)$.

In elliptic and periodic-parabolic problems, theorems on principal eigenvalues and eigenfunctions are classical. In the non-selfadjoint case, they are usually associated with and derived from the Krein–Rutman theorem. A derivation can be found in [17] and many other monographs; for more recent results see [2–4,8].

The origins of the results on invariant bundles and exponential separation in nonautonomous parabolic equations with general time-dependence are in the papers [27,38] (an ODE predecessor is [39]). These results were motivated by certain problems in nonlinear parabolic equations where Eqs. (1.1) naturally appear as linearizations. More specifically, it is well known that if the nonlinear equation has a dissipation property, then each bounded solution approaches a set of entire solutions (that is, solutions defined for all times in $\mathbb{R}$). For further understanding of the behavior of the solution, the linearization of the equation at the entire solutions is often very useful. Now, if the nonlinear equation has a gradient structure, then the entire solutions are steady states and the linearization just gives elliptic operators. However, if no such a priori knowledge of the limit entire solutions is available (this is typically the case in time-periodic equations), to employ the linearization one needs to study Eqs. (1.1) with general time dependence. The principal Floquet bundle and exponential separation then play a similar role as the principal eigenfunction and related spectral properties of elliptic operators do in the local analysis of solutions near steady states. Perhaps the most striking application of such a linearization technique appears in the results on the large time behavior of typical solutions of periodic parabolic problems, as given in [37] and later, with improvements, in [18,41]. For other applications we refer the reader to [18,20,22,28–32,36,37,40,41]. We also remark that in one space dimension, related results on invariant bundles characterized by nodal properties of solutions, as in the classical Sturm–Liouville theory, can be found in [5,42]. See also the survey [35] for a discussion of these results, for both $N = 1$ and $N > 1$, and some perspective.

Available results on Floquet bundles and exponential separation cover a much broader class of problems than (1.1); in particular, [38,41] deal with abstract parabolic equations admitting a strong comparison principle. However, in the context of the specific problem (1.1), they are restricted by various regularity conditions on the domain and coefficients. The results we present here do not rely on any regularity assumptions other than those mentioned above (continuity of a_{ij} in the nondivergence case and Lipschitz continuity of the domain). Another major contribution of our paper is a new approach which leads to more specific results. An example is the fact that the constants C and γ in the exponential separation are determined only by explicit quantities from conditions (1.4), (1.2), and (1.7). We derive the exponential separation result from a new elliptic-type Harnack estimate for quotients of positive solutions of (1.1), which itself has many other interesting consequences regarding solutions of (1.1) (the uniqueness of positive entire solutions among them). A significant portion of this paper is devoted to perturbation results in the divergence case. We prove the continuous dependence of the bundles $X^1(t)$, $X^2(t)$, $t \in \mathbb{R}$, on the coefficients and the domain and establish a robustness property of the exponential separation. We also prove the continuous dependence of the principal spectrum (see Subsection 2.3 for the definition). These results generalize continuous dependence on the domain and coefficients of the principal eigenvalue and eigenfunctions of uniformly elliptic operators.

In this paper we only consider the Dirichlet problem which has many specific features. Indeed, the behavior of the solutions near the boundary $\partial\Omega$, where they vanish, is a major concern in this paper. For the oblique derivative problem, an exponential separation theorem is proved in a similar generality in [19].

The paper is organized as follows. Section 2 contains the statements of our main results. We have grouped them in three subsections. Subsection 2.1 contains the Harnack-type estimate for the quotients of positive solutions and a result on exponential separation between any positive solution and any solution that changes sign for all times. Among corollaries of these theorems, we state the uniqueness of positive entire solutions and a theorem on a universal spectral gap for elliptic operators. Subsection 2.2 contains a theorem on invariant bundles and exponential separation, as discussed above, and in Subsection 2.3 we formulate our perturbation results for the divergence case. We have included two preliminary sections; Section 3 contains basic estimates of positive solutions, relying on various Harnack inequalities. In these estimates we make no reference to existence theorems on initial-boundary value problems, thus we do not need the continuity assumption on the coefficients a_{ij} in the nondivergence case. In Section 5 we list several properties of the evolution operator associated with Eq. (1.1). The remaining sections consist of the proofs of our theorems. The appendix contains the proof of an existence result from Section 5.

2. Statement of the main results

Recall the standing hypothesis that Ω is a bounded Lipschitz domain satisfying (1.7). In the whole section, L is as in (D) or (ND) with coefficients satisfying (1.2), (1.4). We use the notation $C_0(\overline{\Omega})$ for the space of continuous functions on $\overline{\Omega}$ vanishing on $\partial\Omega$. Whenever needed, we assume it is equipped with the supremum norm.

We say that a function changes sign if it assumes both positive and negative values. We use the standard notions of solutions of the equation $u_t + Lu = 0$ as well as for the boundary value problem (1.1) and for the initial value problem

$$\begin{aligned} &u_t + Lu = 0 \quad \text{in } \Omega \times (s, T), \\ &u = 0 \quad \text{on } \partial\Omega \times (s, T), \\ &u = u_0 \quad \text{in } \Omega \times \{s\}, \end{aligned} \tag{2.1}$$

where $s, T \in \mathbb{R}$, $s < T$. In the divergence case, assuming $u_0 \in L^2(\Omega)$, we consider the usual weak solutions (see [1,24, 26]). Under our standing assumptions, (2.1) has a unique weak solution and this solution is continuous on $\overline{\Omega} \times (s, T)$ (more properties of weak solutions are recalled in Sections 3 and 5). In the nondivergence case, we take $u_0 \in C_0(\overline{\Omega})$ and the solution always refers to a strong solution [26], that is, a function $u \in W^{2,1,N+1}_{\mathrm{loc}}(\Omega \times (s, T)) \cap C(\overline{\Omega} \times [s, T])$ which satisfies the initial and boundary conditions everywhere and the equation almost everywhere. The assumed regularity guarantees that the solution is unique. If the coefficients a_{ij}, $i, j = 1, \dots, N$, are continuous on $\Omega \times \mathbb{R}$ then the solution exists for each $u_0 \in C_0(\overline{\Omega})$ (see Section 5).

2.1. Harnack inequality for quotients and exponential separation

Our first theorem contains a new Harnack-type estimate for quotients of positive solutions. This is our basic technical tool in the proofs of exponential separation theorems, but it is a result of independent interest.

Theorem 2.1. *Let $\delta_0 > 0$ and let u_1, u_2 ($u_2 \not\equiv 0$) be two nonnegative solutions of* (1.1) *on $\Omega \times (0, \infty)$. Then for all $t \geqslant \delta_0$ the following estimate holds*

$$\sup_{x\in\Omega} \frac{u_2(x,t)}{u_1(x,t)} \leqslant C \inf_{x\in\Omega} \frac{u_2(x,t)}{u_1(x,t)}, \tag{2.2}$$

with a constant $C > 1$ depending only on $\delta_0, N, \alpha_0, d_0, r_0, R_0, M_0$.

Theorem 2.1 is an extension of [13, Theorem 4.3] and [12, Theorem 5], where the authors prove boundedness of quotients of positive solutions of (1.1).

The next theorem states that sign-changing solutions are exponentially dominated by positive solutions.

Theorem 2.2. *In case L is as in* (ND) *assume that $a_{ij} \in C(\Omega \times \mathbb{R})$, $i, j = 1, \dots, N$. Let $u_0 \in C_0(\overline{\Omega})$, $T = \infty$, and assume that* (2.1) *has a solution $u(\cdot, t; s, u_0)$ which changes sign for all $t > s$. Let moreover v be a positive solution (vanishing on $\partial\Omega$) of the same equation on $\Omega \times (s - 1, \infty)$. Then there are constants $C, \gamma > 0$, depending only on $N, \alpha_0, d_0, r_0, R_0, M_0$, such that*

$$\frac{\|u(\cdot, t; s, u_0)\|_{L^\infty(\Omega)}}{\|v(\cdot, t)\|_{L^\infty(\Omega)}} \leqslant C\mathrm{e}^{-\gamma(t-s)} \frac{\|u_0\|_{L^\infty(\Omega)}}{\|v(\cdot, s)\|_{L^\infty(\Omega)}} \quad (t \geqslant s). \tag{2.3}$$

If L is in the divergence form (D), the statement remains valid if instead of $C_0(\overline{\Omega})$ the initial conditions are taken in $L^p(\Omega)$, for some $2 \leqslant p \leqslant \infty$, and the L^∞-norms in (2.3) are replaced by the L^p-norms. We refer to estimate (2.3) as an *exponential separation* between sign-changing and positive solutions of (1.1). We emphasize that the constants C and γ in (2.3) depend only on the specific bounds in conditions (1.4), (1.2) and (1.7), and not on the solutions, or directly on L and Ω.

We next state several interesting and useful consequences of the previous results. The first one deals with the oscillation in the space domain Ω of the quotient of two solutions of (1.1). Below we shall often abuse the notation slightly and omit the arguments x of the functions considered, for example, we write $u(t)$ for a solution of (1.1). This should cause no confusion. For any continuous function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, we define

$$\operatorname*{osc}_{\Omega} f(t) := \sup_{x\in\Omega} f(x,t) - \inf_{x\in\Omega} f(x,t),$$

whenever either the supremum or infimum is finite.

Corollary 2.3. *Let u and v be solutions of* (1.1) *on $\Omega \times (0,\infty)$, and let $v > 0$ on $\Omega \times (0,\infty)$. Then, assuming the quantities below are defined and finite, we have*

$$\omega(t) := \operatorname*{osc}_{\Omega} \frac{u(t)}{v(t)} \leqslant \omega(s) \quad \textit{for } t \geqslant s > 0, \tag{2.4}$$

and

$$\omega(t) \leqslant \mu \cdot \omega(s) \quad (t \geqslant s+1) \tag{2.5}$$

where $\mu := 1 - C^{-1} \in (0,1)$, $C > 1$ being the constant from Theorem 2.1 *with δ_0 set equal to* 1.

The proof is given in Section 4. It can be verified (see the proof of Lemma 6.1) that the finiteness assumption in Corollary 2.3 is always satisfied in the divergence case and assuming $a_{ij} \in C(\Omega \times \mathbb{R})$, $i, j = 1,\dots,N$, also in the nondivergence case.

In the next corollary we state the uniqueness (up to scalar multiples) of positive solutions of (1.1) defined on $\Omega \times \mathbb{R}$; we refer to such solutions as positive *entire* solutions.

Corollary 2.4. *If u_1 and u_2 are positive entire solutions of* (1.1)*, then there is a constant q such that $u_2 \equiv q u_1$.*

For nonautonomous parabolic equations in the divergence form, under additional conditions on the coefficients and the domain, the uniqueness of entire solutions was proved in earlier papers; the first one seems to be that of Nishio [34], later results can be found in [29,30,36]. We gave a simple proof of Corollary 2.4 for equations in divergence form in [20]. The approach we used there originated in [36]. Our present proof, see Section 4, is different, still rather simple, and, being based on Theorem 2.1, it works for both types of equations.

The fact that the constants appearing in the estimates of Theorems 2.1, 2.2 depend only on the bounds in (1.4), (1.2), and (1.7), and not directly on L, has an interesting consequence for elliptic operators. Namely, it allows us to establish a universal gap between the first (principal) eigenvalue and the rest of the spectrum. This result is nontrivial for non-divergence form operators and to our knowledge it has not been noted so far (a proof for operators without lower order terms is given in [21]). With L independent of t, consider the eigenvalue problem

$$\begin{aligned} Lu &= \lambda u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2.6}$$

The principal eigenvalue λ_1 of this problem is the eigenvalue which is real and has a positive eigenfunction. It is well known (see for example [3,10]) that λ_1 is well defined by these requirements.

Corollary 2.5. *Assume the coefficients of L are independent of t. In case L is as in* (ND) *assume that $a_{ij} \in C(\Omega)$, $i, j = 1,\dots,N$. Let λ_1 be the principal eigenvalue and let λ be any other eigenvalue of* (2.6)*. Then*

$$\operatorname{Re}(\lambda) - \lambda_1 \geqslant \gamma > 0, \tag{2.7}$$

where $\gamma = \gamma(N, \alpha_0, d_0, r_0, R_0, M_0)$ is as in (2.3).

Proof. Let $\varphi_1 > 0$ be an eigenfunction associated with λ_1. The function $v_1(x,t) := \mathrm{e}^{-\lambda_1 t}\varphi_1(x)$ is a positive entire solution of (1.1). Let $\lambda \neq \lambda_1$, $\lambda = a + \mathrm{i}b$ $(a, b \in \mathbb{R})$, be an eigenvalue of (2.6) and let φ_λ, $\varphi_\lambda(x) = f(x) + \mathrm{i}g(x)$, $x \in \Omega$, denote an eigenfunction corresponding to λ. For $(x,t) \in \Omega \times \mathbb{R}$ define

$$u_1(x,t) := \mathrm{Re}\big(\mathrm{e}^{-\lambda t}\varphi_\lambda(x)\big) = \mathrm{e}^{-at}\big(\cos(bt)f(x) + \sin(bt)g(x)\big).$$

This function is also an entire solution of (1.1). Since $\mathrm{e}^{at}u_1(\cdot,t)$ is periodic in t, the maximum principle implies that $u_1(\cdot,t)$ is either of one sign for $t \in \mathbb{R}$ or it changes sign for all $t \in \mathbb{R}$. Assume $u_1(\cdot,t)$ is, say, positive for $t \in \mathbb{R}$. In view of $\lambda \neq \lambda_1$, the functions u_1, v_1 are then two linearly independent positive entire solutions of (1.1), violating Corollary 2.4. Thus $u_1(\cdot,t)$ changes sign for all $t \in \mathbb{R}$ and applying Theorem 2.2, we conclude that $a - \lambda_1 \geqslant \gamma$, with γ as in (2.3). □

2.2. *Exponential separation and Floquet bundles*

In the following result, we state the conclusion of Theorem 2.2 in a different form and complement it by additional information, introducing the principal Floquet bundles. We denote by $u(\cdot,t;s,u_0)$ the solution of (1.1) with the initial condition $u(\cdot,s) = u_0 \in X$. Here X stands for $L^2(\Omega)$ with the standard norm in case L is as in (D) and $X = C_0(\overline{\Omega})$ if L is as in (ND). The existence and uniqueness theorems for the initial-boundary value problems are recalled in Section 5.

For any continuous $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ define

$$\underline{\lambda}(f) = \liminf_{t-s\to\infty} \frac{\log\|f(\cdot,t)\|_X - \log\|f(\cdot,s)\|_X}{t-s}, \tag{2.8}$$

and

$$\bar{\lambda}(f) = \limsup_{t-s\to\infty} \frac{\log\|f(\cdot,t)\|_X - \log\|f(\cdot,s)\|_X}{t-s}. \tag{2.9}$$

Theorem 2.6. *In case L is as in* (ND) *assume that $a_{ij} \in C(\Omega \times \mathbb{R})$, $i, j = 1, \ldots, N$. The following statements hold true.*

(i) *There exists a unique positive entire solution φ_L of* (1.1) *satisfying $\|\varphi_L(\cdot,0)\|_{L^2(\Omega)} = 1$. This solution satisfies $-C \leqslant \underline{\lambda}(\varphi_L) \leqslant \bar{\lambda}(\varphi_L) \leqslant C$ for some positive constant $C = C(N, \alpha_0, d_0, r_0, R_0, M_0)$.*

(ii) *Set*

$$X_L^1(t) := \mathrm{span}\big\{\varphi_L(\cdot,t)\big\},$$
$$X_L^2(t) := \big\{u_0 \in X\colon\ u(\cdot,\tilde{t},t,u_0) \text{ has a zero in } \Omega \text{ for all } \tilde{t} > t\big\}.$$

These sets are closed subspaces of X. They are invariant under (1.1) *in the following sense: if $i \in \{1,2\}$, $u_0 \in X_L^i(s)$, then $u(\cdot,t;s,u_0) \in X_L^i(t)$ $(t \geqslant s)$. Moreover, $X_L^1(t)$, $X_L^2(t)$ are complementary subspaces of X:*

$$X = X_L^1(t) \oplus X_L^2(t) \quad (t \in \mathbb{R}). \tag{2.10}$$

(iii) *There are constants $C, \gamma > 0$, depending only on N, α_0, d_0, r_0, R_0, M_0, such that for any $u_0 \in X_L^2(s)$ one has*

$$\frac{\|u(\cdot,t;s,u_0)\|_X}{\|\varphi_L(\cdot,t)\|_X} \leqslant C\mathrm{e}^{-\gamma(t-s)}\frac{\|u_0\|_X}{\|\varphi_L(\cdot,s)\|_X} \quad (t \geqslant s). \tag{2.11}$$

We refer to the collection of the one-dimensional spaces $X_L^1(t)$, $t \in \mathbb{R}$, as the *principal Floquet bundle* of (1.1) and to $X_L^2(t)$, $t \in \mathbb{R}$, as its *complementary Floquet bundle*. Property (iii) as stated is an *exponential separation* between these two bundles. Following [31,32], we call the interval $[\underline{\lambda}(\varphi_L), \bar{\lambda}(\varphi_L)]$ *the principal spectrum* of (1.1). As discussed in the introduction, the existence of the Floquet bundles with exponential separation extends in a natural way results on spectral decomposition associated with the principal eigenvalue of time-independent or time-periodic parabolic problems. The principal spectrum $[\underline{\lambda}(\varphi_L), \bar{\lambda}(\varphi_L)]$ and the positive entire solution φ_L serve as analogues of the principal eigenvalue and eigenfunction.

2.3. Perturbation results in the divergence case

In this subsection we assume that L is in the divergence form (D). In this case, the invariant bundle $X_L^2(t)$, $t \in \mathbb{R}$, as introduced in the previous section, can be characterized using the principal Floquet bundle for the adjoint equation to (1.1). This way we are also able to study continuity properties of both bundles $X_L^1(t)$, $X_L^2(t)$ under perturbations of the coefficients and the domain. We also examine the robustness properties of the associated exponential separation and continuity properties of the principal spectrum.

Let us first introduce the adjoint problem to (1.1):

$$\begin{aligned} & -v_t + L^* v = 0 \quad \text{in } \Omega \times J, \\ & v = 0 \quad \text{on } \partial\Omega \times J, \end{aligned} \tag{2.12}$$

with L^* defined by

$$L^* v = -\partial_j\big(a_{ij}(x,t)\partial_i v + b_j(x,t)v\big) + a_j(x,t)\partial_j v + c_0(x,t)v. \tag{2.13}$$

The next theorem characterizes the invariant bundle $X_L^2(t)$, $t \in \mathbb{R}$, using an entire solution of (2.12).

Theorem 2.7. *Assume L is as in* (D). *There exists a unique positive entire solution ψ_L of* (2.12)*, satisfying $\|\psi_L(0)\|_{L^2(\Omega)} = 1$. It satisfies $-C \leqslant \underline{\lambda}(\psi_L) \leqslant \bar{\lambda}(\psi_L) \leqslant C$ for some positive constant $C = C(N, \alpha_0, d_0, r_0, R_0, M_0)$. The space $X_L^2(t)$ defined in Theorem* 2.6 *can be characterized as*

$$X_L^2(t) := \left\{ v \in L^2(\Omega)\colon \int_\Omega \psi_L(x,t)v(x)\,\mathrm{d}x = 0 \right\} \quad (t \in \mathbb{R}).$$

In our first perturbation result, we assume that the domain Ω in (1.1) is fixed. Given an operator L, we say that *L admits exponential separation with bound C and exponent γ* if for the entire solution φ_L and the invariant bundle $X_L^2(t)$, $t \in \mathbb{R}$, inequality (2.11) holds for each $u_0 \in X_L^2(s)$. Theorem 2.6 guarantees the existence of a bound and exponent which are common to all operators satisfying (1.2) and (1.4), but now we are interested in robustness of the exponential separation with a possibly larger specific exponent for L.

Theorem 2.8. *Assume L is as in* (D) *and let $\tilde{L}$ be another operator of the form* (D) *with coefficients $(\tilde{a}_{ij})_{i,j=1}^N$, $\tilde{a}_i, \tilde{b}_i, \tilde{c}_0$, $i = 1, \ldots, N$, satisfying* (1.2) *and* (1.4). *Assume that for some $\delta > 0$*

$$\big\{\|a_{ij} - \tilde{a}_{ij}\|, \|a_i - \tilde{a}_i\|, \|b_i - \tilde{b}_i\|, \|c_0 - \tilde{c}_0\|\big\} \subseteq [0, \delta] \quad (i, j = 1, \ldots, N), \tag{2.14}$$

where $\|\cdot\|$ is the norm of $L^\infty(\Omega \times \mathbb{R})$. Then the following statements hold true.

(i) *For each $\varepsilon > 0$ there exist a constant $\delta_1 > 0$, depending only on ε and N, α_0, d_0, r_0, R_0, M_0, such that if $\delta \leqslant \delta_1$ then*

$$\max\big\{\big|\underline{\lambda}(\varphi_L) - \underline{\lambda}(\varphi_{\tilde{L}})\big|, \big|\bar{\lambda}(\varphi_L) - \bar{\lambda}(\varphi_{\tilde{L}})\big|\big\} \leqslant \varepsilon \tag{2.15}$$

and

$$\left\| \frac{\varphi_L(t)}{\|\varphi_L(t)\|_{L^\infty(\Omega)}} - \frac{\varphi_{\tilde{L}}(t)}{\|\varphi_{\tilde{L}}(t)\|_{L^\infty(\Omega)}} \right\|_{L^\infty(\Omega)} \leqslant \varepsilon \quad (t \in \mathbb{R}). \tag{2.16}$$

The statement is also valid with φ_L and $\varphi_{\tilde{L}}$ replaced by ψ_L and $\psi_{\tilde{L}}$, respectively.

(ii) *Suppose that L admits exponential separation with bound C_L and exponent γ_L. For each $\varepsilon > 0$ there exists δ_2 depending only on ε, C_L, γ_L, N, α_0, d_0, r_0, R_0, M_0, such that if $\delta \leqslant \delta_2$ then $\tilde{L}$ admits exponential separation with some bound $C(\varepsilon, C_L, \gamma_L) > 0$ and exponent $\gamma_{\tilde{L}} \geqslant \gamma_L - \varepsilon$.*

Remark 2.9. If the L^∞-norms in (2.16) are replaced by the L^2-norms, the following more precise result can be proved (see Section 8). There exist positive constants δ_0 and C depending only on N, α_0, d_0, r_0, R_0, M_0, such that if (2.14) holds with $\delta \leqslant \delta_0$, then

$$\left\| \frac{\varphi_L(t)}{\|\varphi_L(t)\|_{L^2(\Omega)}} - \frac{\varphi_{\tilde{L}}(t)}{\|\varphi_{\tilde{L}}(t)\|_{L^2(\Omega)}} \right\|_{L^2(\Omega)} \leqslant C\delta \quad (t \in \mathbb{R}). \tag{2.17}$$

In our last theorem we assume that L is fixed and we vary the domain Ω. We use the phrase that Ω *admits exponential separation with bound C and exponent γ*, in an analogous way as when varying L above. For each bounded Lipschitz domain Ω denote by φ_Ω, ψ_Ω the unique positive entire solutions of (1.1), (2.12), respectively satisfying $\|\varphi_\Omega(0)\|_{L^2(\Omega)} = \|\psi_\Omega(0)\|_{L^2(\Omega)} = 1$. By $d(\partial\Omega, \partial\widetilde{\Omega})$ we denote the Hausdorff distance of $\partial\Omega$ and $\partial\widetilde{\Omega}$:

$$d(\partial\Omega, \partial\widetilde{\Omega}) = \max\{\operatorname{dist}(\partial\Omega, \partial\widetilde{\Omega}), \operatorname{dist}(\partial\widetilde{\Omega}, \partial\Omega)\},$$
$$\operatorname{dist}(\partial\Omega, \partial\widetilde{\Omega}) = \sup_{x\in\partial\Omega} \operatorname{dist}(x, \partial\widetilde{\Omega}).$$

Theorem 2.10. *Assume L is a fixed operator as in* (D) *with coefficients defined on $\mathbb{R}^{N+1}$ and satisfying the ellipticity and boundedness conditions* (1.2), (1.4) *on $\mathbb{R}^{N+1}$. Let Ω, $\widetilde{\Omega}$ be two Lipschitz domains such that their Lipschitz constants satisfy* (1.7). *Then the following statements hold.*

(i) *For each $\varepsilon > 0$ there exists $\delta_1 > 0$ depending only on ε, N, α_0, d_0, r_0, R_0, M_0, such that if $d(\partial\Omega, \partial\widetilde{\Omega}) \leqslant \delta_1$, then*

$$\max\{|\underline{\lambda}(\varphi_\Omega) - \underline{\lambda}(\varphi_{\widetilde{\Omega}})|, |\bar{\lambda}(\varphi_\Omega) - \bar{\lambda}(\varphi_{\widetilde{\Omega}})|\} \leqslant \varepsilon \tag{2.18}$$

and

$$\left\| \frac{\varphi_\Omega(t)}{\|\varphi_\Omega(t)\|_{L^\infty(\Omega)}} - \frac{\varphi_{\widetilde{\Omega}}(t)}{\|\varphi_{\widetilde{\Omega}}(t)\|_{L^\infty(\widetilde{\Omega})}} \right\|_{L^\infty(\mathbb{R}^N)} \leqslant \varepsilon \quad (t \in \mathbb{R}). \tag{2.19}$$

Estimates (2.18) *and* (2.19) *are also valid with φ_Ω and $\varphi_{\widetilde{\Omega}}$ replaced by ψ_Ω and $\psi_{\widetilde{\Omega}}$, respectively.*

(ii) *Suppose that Ω admits exponential separation with bound C_Ω and exponent γ_Ω. Then for each $\varepsilon > 0$ there exists $\delta_2 > 0$ depending only on $\varepsilon, C_\Omega, \gamma_\Omega, N, \alpha_0, d_0, r_0, R_0, M_0$, such that if $d(\partial\Omega, \partial\widetilde{\Omega}) \leqslant \delta_2$, then $\widetilde{\Omega}$ admits exponential separation with some bound $C(\varepsilon, C_\Omega, \gamma_\Omega) > 0$ and exponent $\gamma_{\widetilde{\Omega}} \geqslant \gamma_\Omega - \varepsilon$.*

Remark 2.11. In estimate (2.19) the functions $\varphi_\Omega(\cdot, t)$ and $\varphi_{\widetilde{\Omega}}(\cdot, t)$, $t \in \mathbb{R}$, are thought of as extended by zero outside Ω and $\widetilde{\Omega}$, respectively.

Theorems 2.8 and 2.10 give robustness of the exponential separation and continuity of the principal spectrum and principal Floquet bundle under perturbations of the operator L and the domain Ω. More precisely, (2.15) and (2.18) imply that the principal spectrum depends continuously on such perturbations. Similarly, (2.17), (2.19) give the continuous dependence of the principal Floquet bundles. The interesting and nontrivial part here is that in our general time-dependent case the estimates are uniform with respect to $t \in \mathbb{R}$ (cp. [20, Theorem 1.1(i)]). These results extend the well-known continuity properties of the principal eigenvalue and eigenfunction of elliptic and time-periodic parabolic operators (see [2,3,6–8], for example). It is also well-known that in the elliptic or time-periodic case, the gap between the principal eigenvalue and the rest of the spectrum of the corresponding operator is a lower semicontinuous function of the domain and the coefficients. Our lower estimate on $\gamma_{\tilde{L}}$ ($\gamma_{\widetilde{\Omega}}$) extends this result to general time-dependent equations.

The robustness results do not seem to hold in the non-divergence case in the same generality. The main obstacle is the lack of similar continuity properties of the evolution operator as in the divergence case, see Sections 8, 9.

3. Preliminaries I: Harnack inequalities and estimates of positive solutions

For the results in this section we do not need any existence theorem for (1.1). In particular, no continuity assumption on the coefficients of L are needed. For simplicity of notation we state the results on the interval $(0, \infty)$, the results being true on any interval.

First we state the comparison principle (see [26]).

Theorem 3.1. *Let u_1, u_2 be two solutions of* (1.1) *on $\Omega \times (0, \infty)$, and let $u_1 \geqslant u_2$ on $\Omega \times \{0\}$. Then $u_1 \geqslant u_2$ on $\Omega \times (0, \infty)$.*

Corollary 3.2. *Let u_1, u_2 be two solutions of* (1.1) *on $\Omega \times (0,\infty)$, and let $u_1 > 0$ on $\Omega \times (0,\infty)$. Then we have for any $s > 0$*

$$\sup_{\Omega\times(s,\infty)} \frac{u_2}{u_1} = M(s) := \sup_{\Omega\times\{s\}} \frac{u_2}{u_1}, \qquad \inf_{\Omega\times(s,\infty)} \frac{u_2}{u_1} = m(s) := \inf_{\Omega\times\{s\}} \frac{u_2}{u_1}. \tag{3.1}$$

In particular,

$$M(t) \leqslant M(s), \quad m(t) \geqslant m(s) \quad \textit{for } t \geqslant s > 0. \tag{3.2}$$

Proof. The inequality for $M(t)$ is trivial if $M(s) = \infty$, and it follows directly from Theorem 3.1 if $M(s) = 0$ or $M(s) = 1$. In case $M(s) \notin \{0, \infty\}$, dividing u_2 by $M(s)$, we can reduce the proof of the first equality in (3.1) to the case $M(s) = 1$. The equality for $m(t)$ follows by changing the sign of u_2. □

In case L is as in (ND), the following theorem is a simple consequence of the standard maximum principle (see e.g. [26, Theorem 7.1]). In case L is as in (D), it is proved in [24, Theorem III.7.1].

Theorem 3.3. *Let u be a solution of* (1.1) *on $\Omega \times (0,\infty)$ such that $u \in C(\overline{\Omega} \times [0,\infty))$. Then there is a constant C depending on N, α_0, and d_0, such that for any $t \geqslant 0$ one has*

$$\big\|u(t+\tau)\big\|_{L^\infty(\Omega)} \leqslant C\big\|u(t)\big\|_{L^\infty(\Omega)} \quad (\tau \in [0,1]). \tag{3.3}$$

Next we state the interior Harnack inequality [23,26,33]. For any $\delta > 0$, define

$$\Omega^\delta := \big\{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \delta\big\}.$$

Theorem 3.4. *Let v be a nonnegative solution of $v_t + Lv = 0$ on $\Omega \times (0,\infty)$ and let $T > 0$ be arbitrary. Suppose $\delta \in (0,T)$ is such that $\delta \leqslant r_0/2$ (r_0 as in* (1.7)*) and let $X = (x,t)$, $Y = (y,s)$ be such that $x, y \in \Omega^\delta$, $s \geqslant \delta^2$, and $T \geqslant t - s \geqslant \delta^2$. Then there is a positive constant C depending only on δ, T, N, α_0, d_0, R_0 such that one has*

$$v(Y) \leqslant Cv(X). \tag{3.4}$$

To state the next results, we introduce some notation. For $X = (x,t) \in \mathbb{R}^{N+1}$, we define a parabolic cylinder to be

$$C_r(X) = C_r(x,t) \equiv B_r(x) \times \big(t - r^2, t + r^2\big).$$

Further, let us denote

$$Q_r(X) = \big(\Omega \times (0,\infty)\big) \cap C_r(X).$$

According to the Lipschitz properties of Ω, for $y \in \partial\Omega$ there is an orthonormal system with y as the origin $(0,0)$ and $(0,r) \in \Omega$ for all $r \in (0, r_0]$. The new coordinates for $Y = (y,s) \in \mathbb{R}^{N+1}$ are $(0,0,s)$. In this coordinate system write

$$\overline{Y}_r = \big(0, r, s + 2r^2\big), \qquad \underline{Y}_r = \big(0, r, s - 2r^2\big).$$

We now recall two results from [12,13]. The first one is often referred to as a *boundary Harnack inequality*. We say that a function defined on an open set $Q \subset \mathbb{R}^{N+1}$ continuously vanishes on $\Gamma \subset \partial Q$ if it has a continuous extension to $Q \cup \Gamma$, which vanishes on Γ.

Theorem 3.5. *Let $Y = (y,s) \in \partial\Omega \times (0,\infty)$ and $0 < r \leqslant \frac{1}{2}\min(r_0, \sqrt{s})$. Then for any nonnegative solution v of $v_t + Lv = 0$ on $\Omega \times (0,\infty)$ which continuously vanishes on $(\partial\Omega \times (0,\infty)) \cap C_{2r}(Y)$, we have*

$$\sup_{Q_r(Y)} v \leqslant Cv(\overline{Y}_r).$$

The constant C depends only on N, d_0, α_0, M_0.

The next estimate states that the quotient of two positive solutions is bounded near the portion of the lateral boundary where each solution vanishes.

Theorem 3.6. *Let $X_0=(x_0,t_0)\in\partial\Omega\times(0,\infty)$. Assume that u and v are two positive solutions of $w_t+Lw=0$ on $\Omega\times(0,\infty)$ which continuously vanish on $(\partial\Omega\times(0,\infty))\cap C_{2r}(X_0)$ with $0<r\leqslant\frac{1}{2}\min(r_0,\sqrt{t_0})$. Then*

$$\sup_{Q_r(X_0)}\frac{u}{v}\leqslant C\frac{u(\overline{X_0}_r)}{v(\underline{X_0}_r)}.$$

The constant C depends only on N, d_0, α_0, M_0.

Theorems 3.5, 3.6 are proved in [12] for equations of divergence form and in [13] for equations of nondivergence form. Although only equations without lower order terms are considered in these papers, the results hold also in the general case. Indeed, since these results are local (solutions are required to vanish on a portion of the boundary only), they carry over to the present situation by considering the method of additional variable. This is done in two steps. First, one assumes that the coefficients of the equation in question are smooth. Applying the method of additional variable (see, e.g., [14] or [20] for details), one derives the results for equations containing lower order terms. Finally, since all the estimates are independent of smoothness, one can take limits (as in [14]) to obtain the results in our more general setting.

Combining the above results, we will be able to prove the following elliptic-type Harnack inequality.

Theorem 3.7. *Suppose v is a nonnegative solution of* (1.1) *on $\Omega\times(0,\infty)$ and let $T\in(0,\infty)$, $0<r\leqslant\frac{1}{2}\min(r_0,\sqrt{T})$. Then*

$$\sup_{\Omega^r\times(r^2,T)} v\leqslant C\inf_{\Omega^r\times(r^2,T)} v. \tag{3.5}$$

The constant C depends only on r, T, N, α_0, d_0, M_0, R_0.

Let us mention that this inequality is known for equations without lower order terms (see [11,15]). The general case follows from the above results as in the proof of Theorem 1.3 in [11]. Since the proof is short we include it here for the reader's convenience (note that the method of additional variable does not apply here as the result is not local).

Proof of Theorem 3.7. Let $(x_0,t_0),(x_1,t_1)\in\Omega^r\times(r^2,T)$. Theorems 3.5 and 3.4 imply that for all $x_2\in\Omega$ such that $\operatorname{dist}(x_2,\partial\Omega)\geqslant r$ one has $\sup_{x\in\Omega}v(x,r^2/8)\leqslant Cv(x_2,r^2/4)$. Applying (3.4) to the right-hand side of the last inequality and using Theorem 3.3, we obtain for $i=0,1$

$$\sup_{x\in\Omega}v\big(x,r^2/8\big)\leqslant C_1v(x_i,t_i)\leqslant C_1\sup_{x\in\Omega}v(x,t_i)\leqslant C_2\sup_{x\in\Omega}v\big(x,r^2/8\big), \tag{3.6}$$

with a constant C_2 depending only on r, T, N, α_0, d_0, M_0, R_0. Inequality (3.6) with $i=0,1$ implies (3.5). □

The next proposition will be used in the proof of Lemma 3.9 below. It is proved in [12,13].

Proposition 3.8. *Let v be a nonnegative solution of $v_t+Lv=0$ on $\Omega\times(0,\infty)$. Take $Y=(y,s)\in\partial\Omega\times(0,\infty)$ and $0<r\leqslant\frac{1}{2}\min(r_0,\sqrt{s})$. Then*

$$v(\underline{Y}_r)\leqslant Cr^\theta\inf_{Q_r(Y)}\big(d^{-\theta}v\big),$$

where $d=d(x)\equiv\operatorname{dist}(x,\partial\Omega)$, and C,θ are positive constants depending only on N, d_0, α_0, r_0, R_0, M_0.

The following lemma contains an important pointwise estimate, which is also of independent interest. In a slightly weaker form the lemma has been proved in [20].

Lemma 3.9. *For any $\delta>0$ and any positive solution v of* (1.1) *on $\Omega\times(0,\infty)$ one has*

$$\frac{v(x,t)}{\|v(t)\|_{L^\infty(\Omega)}}\geqslant C\big(d(x)\big)^\theta\quad\big((x,t)\in\Omega\times[\delta,\infty)\big), \tag{3.7}$$

where $d(x)=\operatorname{dist}(x,\partial\Omega)$ and θ is as in Proposition 3.8. *The constant C depends only on δ, N, d_0, α_0, r_0, R_0, M_0.*

Proof. Fix $t \geqslant \delta$, $r_1 = \frac{1}{2}\min(r_0, \sqrt{\delta})$. Note that if $x \in \Omega$ is such that $d(x) = \operatorname{dist}(x, \partial\Omega) \geqslant r_1/\sqrt{M_0^2+1}$, then (3.6) implies (3.7) with a constant C depending only on δ, N, d_0, α_0, r_0, R_0, M_0. We now consider $x \in \Omega$ such that $d(x) \leqslant r_1/\sqrt{M_0^2+1}$. Using the notation introduced just before Theorem 3.5, we let $y = (0,0) \in \partial\Omega$. An easy geometric argument shows that for $r \in (0, r_0)$ we have

$$\operatorname{dist}\big((0,r), \partial\Omega\big) \in \left[\frac{r}{\sqrt{M_0^2+1}}, r\right]. \tag{3.8}$$

Our assumption on $d(x)$ implies that there exists $Y = (y,t) \in \partial\Omega \times \{t\}$ such that $(x,t) \in Q_{r_1}(Y)$. Moreover, we can find $\tilde{x} \in \Omega$ such that $\underline{Y}_{r_1} = (\tilde{x}, t - 2r_1^2)$. Using Proposition 3.8, we see that

$$v(x,t) \geqslant C^{-1} r_1^{-\theta} d(x)^\theta v\big(\tilde{x}, t - 2r_1^2\big).$$

Notice that by (3.8) we have $d(\tilde{x}) \in [r_1/\sqrt{M_0^2+1}, r_1]$. Thus, by what we said at the beginning of this proof, we have $C_1 v(\tilde{x}, t - 2r_1^2) \geqslant \|v(t - 2r_1^2)\|_{L^\infty(\Omega)}$ for some positive constant C_1 depending only on r_1, N, d_0, α_0, r_0, R_0, M_0. Finally, by Theorem 3.3, $C_2\|v(t-2r_1^2)\|_{L^\infty(\Omega)} \geqslant \|v(t)\|_{L^\infty(\Omega)}$. Combining the above estimates we obtain the desired inequality with $C = C^{-1}C_1C_2$. □

As an immediate consequence we get

Corollary 3.10. *Let $\delta > 0$. Then there exists a positive constant C depending only on δ, N, d_0, α_0, r_0, R_0, M_0, such that if v is a positive solution of* (1.1) *on $\Omega \times (0,\infty)$ one has for any $\tau \in [0,1]$*

$$\frac{\|v(t+\tau)\|_{L^\infty(\Omega)}}{\|v(t)\|_{L^\infty(\Omega)}} \in \left[\frac{1}{C}, C\right] \quad (t > \delta). \tag{3.9}$$

Proof. The upper bound is a trivial consequence of Theorem 3.3. The lower bound is proved as follows. Choose a point $x_0 \in \Omega$ such that $\operatorname{dist}(x_0, \partial\Omega) \geqslant r_0/(1+M_0)$. The Harnack inequality (3.4) implies $v(x_0,t) \leqslant Cv(x_0,t+1)$ with some positive constant C. Our special choice of x_0 and Lemma 3.9 imply that $v(x_0, t+i)$ is comparable to $\|v(t+i)\|_{L^\infty(\Omega)}$, $i = 0, 1$, and we thus get the desired lower bound for $\tau = 1$. Combining it with Theorem 3.3 we get the lower bound for any $\tau \in [0,1]$. □

4. Proofs of Theorem 2.1 and its corollaries

Proof of Theorem 2.1. The desired estimate will be obtained by combining the results from the previous section. We start by a preliminary estimate. Fix an arbitrary $X_0 = (x_0, t_0)$ with $x_0 \in \partial\Omega$ and $t_0 \geqslant \delta_0$. Set

$$r_1 := \min\left\{\frac{r_0}{2}, \frac{\sqrt{\delta_0}}{2}\right\}, \qquad \rho := \frac{r_1}{1+M_0},$$
$$Q_{r_1} := \big(\Omega \cap B_{r_1}(x_0)\big) \times \big(t_0 - r_1^2, t_0 + r_1^2\big).$$

By Theorem 3.6,

$$\sup_{Q_{r_1}} \frac{u_2}{u_1} \leqslant C_1 \frac{u_2(\overline{X_0}_{r_1})}{u_1(\underline{X_0}_{r_1})} \tag{4.1}$$

with a constant $C_1 \geqslant 1$ depending only on N, d_0, α_0, M_0. As in the proof of Lemma 3.9 we can write $\underline{X_0}_{r_1} = (y_0, t_0 - 2r_1^2)$ for some $y_0 \in \Omega \cap B_{r_1}(x_0)$, such that $d(y_0) := \operatorname{dist}(y_0, \partial\Omega) > \rho$. Note that the points $(y_0, t_0 \pm 2r_1^2)$ belong to the cylinder

$$Q^\rho := \Omega^\rho \times \left(t_0 - \frac{\delta_0}{4}, t_0 + \frac{\delta_0}{4}\right), \quad \text{where } \Omega^\rho := \big\{x \in \Omega\colon d(x) > \rho\big\}. \tag{4.2}$$

Therefore, the previous estimate implies

$$\sup_{Q_{r_1}} \frac{u_2}{u_1} \leqslant C_1 M_2 m_1^{-1}, \quad \text{where } M_j := \sup_{Q^\rho} u_j,\ m_j := \inf_{Q^\rho} u_j. \tag{4.3}$$

We now derive inequality (2.2) for any $t \geqslant \delta_0$. Take $t_0 = t$. For each $x \in \Omega$ consider separately the two possible cases: (i) $d(x) := \operatorname{dist}(x, \partial\Omega) \leqslant \rho$, and (ii) $d(x) > \rho$. In the case (i), we take $x_0 \in \partial\Omega$ such that $|x - x_0| = d(x) \leqslant \rho < r_1$, so that $(x, t_0) \in Q_{r_1}$. Then, by (4.3), $u_2(x, t_0)/u_1(x, t_0)$ does not exceed $C_1 M_2 m_1^{-1}$. In the other case (ii), $(x, t_0) \in Q^\rho$, so that $u_2(x, t_0)/u_1(x, t_0) \leqslant M_2 m_1^{-1} \leqslant C_1 M_2 m_1^{-1}$ as well. We have thus shown that

$$M(t) := \sup_{\Omega\times\{t\}} \frac{u_2}{u_1} \leqslant \frac{C_1 M_2}{m_1}. \tag{4.4}$$

Interchanging u_1 and u_2, we also get

$$m(t) := \inf_{\Omega\times\{t\}} \frac{u_2}{u_1} = \left(\sup_{\Omega\times\{t\}} \frac{u_1}{u_2}\right)^{-1} \geqslant \left(\frac{C_1 M_1}{m_2}\right)^{-1} = \frac{m_2}{C_1 M_1}. \tag{4.5}$$

Now it remains to apply (3.5) which guarantees that $M_j \leqslant C_2 m_j$ for $j = 1, 2$, with another constant $C_2 \geqslant 1$ depending only on $\delta_0, N, \alpha_0, d_0, r_0, R_0, M_0$. This gives the desired estimate (2.2) with $C := C_1^2 C_2^2$. □

Proof of Corollary 2.3. Using Corollary 3.2 with $u_1 := v$ and $u_2 := u$, we immediately get (2.4):

$$\omega(t) = M(t) - m(t) \leqslant M(s) - m(s) = \omega(s) \quad (t \geqslant s).$$

For the proof of (2.5), we use Corollary 3.2 with $u_1 := v$ and

$$u_2(x, t) := u(x, t) - m_0(s) \cdot u_1(x, t), \quad \text{where } m_0(s) := \inf_{\Omega\times\{s\}} \frac{u}{u_1}.$$

(Note that we are assuming that $m_0(s)$ is finite.)

Since $m(s) = 0$, the corollary gives $u_2 \geqslant 0$ in $\Omega \times (s, \infty)$, and (2.5) then follows from Theorem 2.1:

$$\omega(t) = M(t) - m(t) \leqslant \left(1 - C^{-1}\right) M(t) = \mu M(t) \leqslant \mu M(s) = \mu\bigl(M(s) - m(s)\bigr) = \mu\omega(s) \quad (t \geqslant s + 1).$$

Thus the corollary is proved. □

Proof of Corollary 2.4. For any $t \in \mathbb{R}$ define

$$\varrho_{\max}(t) := \sup_\Omega \left(\frac{u_2(t)}{u_1(t)}\right), \qquad \varrho_{\min}(t) := \inf_\Omega \left(\frac{u_2(t)}{u_1(t)}\right).$$

Thus $\operatorname{osc}_\Omega(u_2(t)/u_1(t)) = \varrho_{\max}(t) - \varrho_{\min}(t)$. Theorem 2.1 implies that $\operatorname{osc}_\Omega(u_2(t)/u_1(t)) < \infty$ for all $t \in \mathbb{R}$. By Corollary 3.1, $\varrho_{\max}(t)$ is nonincreasing and $\varrho_{\min}(t)$ is nondecreasing. By Corollary 2.3, we have

$$\lim_{t\to\infty} \operatorname{osc}_\Omega \left(\frac{u_2(t)}{u_1(t)}\right) = 0.$$

Thus for some $q \in (0, \infty)$

$$\lim_{t\to\infty} \varrho_{\max}(t) = \lim_{t\to\infty} \varrho_{\min}(t) = q.$$

It follows that the function $u = u_2 - qu_1$ is a solution of (1.1) on $\Omega \times \mathbb{R}$, which vanishes somewhere in Ω for all $t \in \mathbb{R}$ and consequently $\inf_\Omega(u_2(t)/qu_1(t)) \leqslant 1$ for all t. Using this fact and Theorem 2.1 we discover that

$$\sup_\Omega \frac{u_2(t)}{qu_1(t)} \leqslant C \inf_\Omega \frac{u_2(t)}{qu_1(t)} \leqslant C,$$

with a constant C independent of $t \in \mathbb{R}$. This implies that $\operatorname{osc}_\Omega(u_2(t)/qu_1(t))$ is bounded on all of $\mathbb{R}$. Moreover, by Corollary 2.3, this function is exponentially decreasing on $\mathbb{R}$. This is only possible if $\operatorname{osc}_\Omega(u_2(t)/qu_1(t)) \equiv 0$ for all $t \in \mathbb{R}$ and that happens only if $u_2 \equiv qu_1$. The proof is complete. □

5. Preliminaries II: The evolution operator

Consider the following initial value problem

$$\begin{aligned} &u_t + Lu = 0 \quad \text{in } \Omega \times (s, T), \\ &u = 0 \quad \text{on } \partial\Omega \times (s, T), \\ &u = u_0 \quad \text{in } \Omega \times \{s\}, \end{aligned} \tag{5.1}$$

where $s, T \in \mathbb{R}$, $s < T$.

In this section we give the basic existence result for this problem and list several properties of the corresponding evolution operator. Since the assumptions differ slightly in the divergence and non-divergence cases, we treat these cases separately.

5.1. Divergence case

In this subsection we assume that L is as in (D) with coefficients satisfying (1.4) and (1.2).

Recall that our notion of the solution of (5.1) coincides with the notion of the weak solution from [24,26]. For any $u_0 \in L^2(\Omega)$, and $s < T$, the weak solution of (5.1) exists, it is unique and can be (uniquely) extended to a solution on (s, ∞). Denote the solution by $U(t, s)u_0$, $t \geqslant s$. Let $\|\cdot\|_{p,q}$ stand for the operator norm of the space $\mathcal{L}(L^p(\Omega), L^q(\Omega))$ of bounded linear operators from $L^p(\Omega)$ to $L^q(\Omega)$. It is well known that the evolution operator $U(t, s)$, $t \geqslant s$, satisfies the following $L^p - L^q$ estimates (see [9], for example).

Proposition 5.1. *For all $1 \leqslant p \leqslant q \leqslant \infty$, $t, s \in \mathbb{R}$, $t > s$, one has $U(t, s) \in \mathcal{L}(L^p(\Omega), L^q(\Omega))$ and*

$$\big\|U(t, s)\big\|_{\mathcal{L}(L^p(\Omega),L^q(\Omega))} \leqslant C(t-s)^{-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})} \mathrm{e}^{\omega(t-s)},$$

where $C \geqslant 1$ and $\omega \in \mathbb{R}$ are constants depending only on N, d_0, α_0. Moreover, for any $u_0 \in L^2(\Omega)$ and $T \geqslant s$ one has $U(\cdot, s)u_0 \in C([s, T]; L^2(\Omega))$.

Another property of the evolution operator $U(t, s)$ we will use is positivity. For any $p \in (1, \infty)$ and $u_0 \in L^p(\Omega)$, $u_0 \geqslant 0$, we have $U(t, s)u_0 \geqslant 0$ for all $t \geqslant s$ (see [9]). This can be improved on: nonnegative nontrivial solutions are strictly positive. It is a direct consequence of the Harnack inequality (3.4).

Besides positivity, the evolution operator has a smoothing property. In this regard, we mention the following standard regularity result [24, Chapter III, Theorem 10.1]. We use the usual notation for the parabolic Hölder spaces.

Theorem 5.2. *Let u be the (weak) solution of (5.1) with $u_0 \in L^2(\Omega)$. Then for any $T > s$ we have $u \in C^{\alpha,\frac{\alpha}{2}}_{\mathrm{loc}}(\overline{\Omega} \times (s, T])$, and for any $\delta > 0$ ($\delta < T - s$) the norm $\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[s+\delta,T])}$ is estimated from above by a constant depending only on N, d_0 in (1.2), $\sup_{\Omega\times(s,T)}|u|$, α_0 in (1.4), constants r_0, R_0, M_0 in (1.7) and δ. The exponent $\alpha > 0$ is determined only by N, d_0, α_0.*

If, in addition, it is known that $u_0 \in C^\beta(\overline{\Omega}) \cap C_0(\overline{\Omega})$ for some $\beta > 0$, then the norm $\|u\|_{C^{\alpha,\frac{\alpha}{2}}(\overline{\Omega}\times[s,T])}$ is estimated from above by a constant depending only on N, d_0, $\sup_{\Omega\times(s,T)}|u|$, α_0, r_0, R_0, M_0, β and the norm $\|u_0\|_{C^\beta(\overline{\Omega})}$. The exponent α belongs to $(0, \beta]$ and is determined by N, d_0, α_0.

Combining L^p–L^q estimates, Corollary 3.10, and Lemma 3.9, one proves the following result.

Corollary 5.3. *There exists a constant C depending only on N, d_0, α_0, r_0, R_0, M_0, such that for any $1 \leqslant p, q \leqslant \infty$ and any positive entire solution u of (1.1) the following statements hold:*

(i) $$\frac{u(x,t)}{\|u(t)\|_{L^p(\Omega)}} \geqslant C\big(d(x)\big)^\theta \quad \big((x,t) \in \Omega \times (-\infty, \infty)\big), \tag{5.2}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$ and θ is as in Lemma 3.9 (it is a constant depending only on N, d_0, α_0, r_0, R_0 and M_0).

(ii) $$\sup_{\substack{|t-s|\leqslant 1\\ s,t\in\mathbb{R}}} \frac{\|u(s)\|_{L^p(\Omega)}}{\|u(t)\|_{L^q(\Omega)}} \leqslant C. \tag{5.3}$$

Let us now turn our attention to (2.12). One can prove (see [9]) that there is a well defined evolution operator, henceforth denoted by $U^*(t,s)$, $t \leqslant s$, for the adjoint problem (2.12). Reversing time, we obtain

$$U^*(t,s) = \widetilde{U}(-t,-s) \quad (t \leqslant s),$$

where $\widetilde{U}(t,s)$, $t \geqslant s$, is the ("forward") evolution operator for the problem

$$w_t + \tilde{L}^* w = 0, \quad x \in \Omega,$$
$$w = 0 \quad x \in \partial\Omega,$$

where $\tilde{L}^*$ is obtained from L^* by replacing t with $-t$. This problem is of the same form as (5.1) and thus $U^*(t,s)$ has the same smoothing and positivity properties as $U(t,s)$. We will use this fact frequently without notice.

The following proposition summarizes some properties of weak Green's functions we will need later. The first three statements below are proved in [1] and for the last one we refer the reader to [9].

Proposition 5.4. *There exists a unique weak Green's function $k(x,t;\xi,s)$ associated with* (5.1) *with the following properties.*

(i) *For any $u_0 \in L^2(\Omega)$ the solution $u = U(t,s)u_0$ of* (5.1) *is given by*

$$u(x,t) = \int_\Omega k(x,t;\xi,s)u_0(\xi)\,\mathrm{d}\xi \quad ((x,t) \in \Omega \times (s,\infty)),$$

and the solution $v = U^*(s,t)u_0$ of the adjoint problem (2.12) *is given by*

$$v(\xi,s) = \int_\Omega k(x,t;\xi,s)u_0(x)\,\mathrm{d}x \quad ((\xi,s) \in \Omega \times (-\infty,t)).$$

(ii) *The function $k(\cdot,\cdot;\xi,s)$: $\Omega \times (s,\infty) \to \mathbb{R}$ is a positive solution of* (1.1) *on $\Omega \times (s,\infty)$.*
(iii) *The function $k(\cdot,\cdot;\xi,s)$ is bounded on $\Omega \times (s+\tau, s+1)$ by a constant $C(\tau)$ depending only on $\tau \in (0,1)$, N, α_0 and d_0.*
(iv) *Let $\Omega_1 \subseteq \Omega_2$ be two bounded domains in $\mathbb{R}^N$ and let $k_i(x,t;\xi,s)$, $i = 1,2$, be the corresponding weak Green's functions. Then, extending $k_1(\cdot,\cdot;\xi,s)$ by zero outside Ω_1, we have*

$$k_1(x,t;\xi,s) \leqslant k_2(x,t;\xi,s) \quad ((x,t) \in \Omega_2 \times (s,\infty)).$$

The integral representation of solutions, as given in statement (i), and the Fubini theorem readily imply the following identity

$$\langle U(t,s)u, v\rangle = \langle u, U^*(s,t)v\rangle \quad (u,v \in L^2(\Omega), t \geqslant s). \tag{5.4}$$

5.2. Non-divergence case

Let us now handle the case when L is as in (ND). In this case we assume, in addition to (1.4) and (1.2), that $a_{ij} \in C(\Omega \times \mathbb{R})$, $i,j = 1,\dots,N$. Consider again the initial value problem (5.1). Recall that this time we consider strong solutions as defined in the beginning of Section 2. We have the following result regarding the well-posedness.

Proposition 5.5. *Assume $u_0 \in C_0(\overline{\Omega})$. Then the initial value problem* (5.1) *has a unique solution u; the solution is contained in $W^{2,1,p}_{\mathrm{loc}}(\Omega \times (s,T)) \cap C(\overline{\Omega} \times [s,T])$ for all $p > 1$ and it satisfies the following estimate*

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant \mathrm{e}^{m(t-s)}\|u_0\|_{L^\infty(\Omega)} \quad (t \in [s,T]), \tag{5.5}$$

where $m = \sup_{\Omega\times(s,T)}(-c_0) \leqslant d_0$ (d_0 is as in (1.2)).

This result can be derived from uniform estimates of the moduli of continuity of solutions vanishing on the lateral side of a Lipschitz cylinder. In a more general setting, these estimates are contained in Theorem 6.32 (in the case (D)) and Corollary 7.30 (in the case (ND)) of the monograph [26]. For completeness of presentation, we give an alternative proof in Appendix A.

6. Proof of Theorem 2.2

As a preparatory step we prove the following technical lemma.

Lemma 6.1. *Let u and v be as in the statement of Theorem 2.2. Suppose that for some $k \geqslant s$ and some constant $\eta > 0$ the following inequality holds*

$$\|u(k)\|_{L^\infty(\Omega)} \leqslant \eta \|u(k+1)\|_{L^\infty(\Omega)}. \tag{6.1}$$

Then one has

$$\sup_\Omega \frac{|u(k+1)|}{v(k+1)} \leqslant C\eta \frac{\|u(k+1)\|_{L^\infty(\Omega)}}{\|v(k+1)\|_{L^\infty(\Omega)}}, \tag{6.2}$$

where C is a constant depending only on $N, d_0, \alpha_0, r_0, R_0, M_0$.

For future use we note that for all positive bounded functions a, b on Ω we have the following elementary inequalities

$$\inf_\Omega \frac{a}{b} \leqslant \frac{\|a\|_{L^\infty(\Omega)}}{\|b\|_{L^\infty(\Omega)}} \leqslant \sup_\Omega \frac{a}{b}. \tag{6.3}$$

Proof of Lemma 6.1. For any real valued function f, denote by f_+ (f_-) the positive (negative) part of f. As above let $u(\cdot, t; s, u_0)$ denote the solution of (1.1) on $\Omega \times (s, \infty)$ with the initial condition $u(\cdot, s) = u_0$. By uniqueness of solutions of initial value problems, we see that for all $(x, t) \in \overline{\Omega} \times [k, \infty)$

$$u(x,t) = u_1(x,t) - u_2(x,t),$$

where $u_1(x,t) = u(x,t;k,u_+(k))$, $u_2(x,t) = u(x,t;k,u_-(k))$. Our assumption on u implies that for each $t > k$ the functions $u_1(\cdot, t)$ and $u_2(\cdot, t)$ are equal at some point $x(t) \in \Omega$. Since they are nonnegative, by Lemma 3.4 either both of them are identically zero or they are positive in $\Omega \times (k, \infty)$. The first case is trivial so we shall assume that u_1 (and hence also u_2) is positive. Taking this into account and applying Theorem 2.1, we get

$$\sup_\Omega \frac{u_2(k+1)}{u_1(k+1)} \leqslant C_1 \inf_\Omega \frac{u_2(k+1)}{u_1(k+1)} \leqslant C_1, \tag{6.4}$$

where $C_1 = C_1(N, \alpha_0, d_0, r_0, R_0, M_0)$. Using Theorem 3.3 and our assumption (6.1), we derive

$$\|u_1(k+1)\|_{L^\infty(\Omega)} \leqslant C\|u_1(k)\|_{L^\infty(\Omega)} = C\|u_+(k)\|_{L^\infty(\Omega)} \leqslant C\|u(k)\|_{L^\infty(\Omega)} \leqslant C\eta\|u(k+1)\|_{L^\infty(\Omega)}. \tag{6.5}$$

Estimates in (6.3) and Theorem 2.1 imply the following inequalities

$$\inf_\Omega \frac{u_1(k+1)}{v(k+1)} \leqslant \frac{\|u_1(k+1)\|_{L^\infty(\Omega)}}{\|v(k+1)\|_{L^\infty(\Omega)}} \leqslant \sup_\Omega \frac{u_1(k+1)}{v(k+1)} \leqslant C_1 \inf_\Omega \frac{u_1(k+1)}{v(k+1)}. \tag{6.6}$$

Utilizing (6.6), (6.4), and (6.5), we finally get

$$\begin{aligned} \sup_\Omega \frac{|u(k+1)|}{v(k+1)} &= \sup_\Omega \frac{u_1(k+1)}{v(k+1)} \frac{|u(k+1)|}{u_1(k+1)} \leqslant \sup_\Omega \frac{u_1(k+1)}{v(k+1)} \sup_\Omega \frac{|u(k+1)|}{u_1(k+1)} \\ &\leqslant C_1 \inf_\Omega \frac{u_1(k+1)}{v(k+1)} \sup_\Omega \frac{u_1(k+1)+u_2(k+1)}{u_1(k+1)} \leqslant C_1(1+C_1)\frac{\|u_1(k+1)\|_{L^\infty(\Omega)}}{\|v(k+1)\|_{L^\infty(\Omega)}} \\ &\leqslant C_1 C(1+C_1)\eta \frac{\|u(k+1)\|_{L^\infty(\Omega)}}{\|v(k+1)\|_{L^\infty(\Omega)}}, \end{aligned}$$

completing the proof. □

We are ready to give the proof of Theorem 2.2.

Proof of Theorem 2.2. Let u and v be as in the statement of the theorem. Theorem 3.3 gives

$$\|u(t)\|_{L^\infty(\Omega)} \leqslant C\|u(s+n)\|_{L^\infty(\Omega)} \quad \text{for } t\in[s+n,s+n+1],\ n\geqslant 0$$

with a constant C independent of u, s, n. This estimate and Corollary 3.10 imply

$$\frac{\|u(t)\|_{L^\infty(\Omega)}}{\|v(t)\|_{L^\infty(\Omega)}} \leqslant C_1\frac{\|u(s+n)\|_{L^\infty(\Omega)}}{\|v(s+n)\|_{L^\infty(\Omega)}} \quad (t\in[s+n,s+n+1], n\geqslant 0) \tag{6.7}$$

with a possibly bigger constant C_1, which is again independent of u, v, s, n.

Let $\mu\in(0,1)$ be as in (2.5) and denote by C_2 the constant in Corollary 3.10 (with δ set equal to 1 in it). We have the following two (mutually exclusive) possibilities:

(a) For all $k\geqslant s$

$$\frac{\|u(k+1)\|_{L^\infty(\Omega)}}{\|u(k)\|_{L^\infty(\Omega)}} < \frac{\mu}{C_2} \quad \left(\leqslant \mu\frac{\|v(k+1)\|_{L^\infty(\Omega)}}{\|v(k)\|_{L^\infty(\Omega)}}\right)$$

or else

(b) the assumptions of Lemma 6.1 are satisfied for some k with $\eta = C_2/\mu$.

Assume (a) holds. Using first (6.7) and afterward the assumed inequality repeatedly, we find that

$$\begin{aligned}\frac{\|u(t)\|_{L^\infty(\Omega)}}{\|v(t)\|_{L^\infty(\Omega)}} &\leqslant C_1\frac{\|u(s+[t-s])\|_{L^\infty(\Omega)}}{\|v(s+[t-s]\|_{L^\infty(\Omega)}} \leqslant C_1\mu^{[t-s]}\frac{\|u_0\|_{L^\infty(\Omega)}}{\|v(s)\|_{L^\infty(\Omega)}} \quad (t\geqslant s)\\ &\leqslant \frac{C_1}{\mu}\,\mathrm{e}^{-(-\log\mu)(t-s)}\frac{\|u_0\|_{L^\infty(\Omega)}}{\|v(s)\|_{L^\infty(\Omega)}},\end{aligned} \tag{6.8}$$

where $[t]$ stands for the integer part of t. We thus get the estimate of Theorem 2.2 with $C=\frac{C_1}{\mu}$ and $\gamma=-\log\mu$.

Assume now that (b) occurs, i.e., for some $k\geqslant s$

$$\frac{\|u(k+1)\|_{L^\infty(\Omega)}}{\|u(k)\|_{L^\infty(\Omega)}} \geqslant \frac{\mu}{C_2}.$$

Let us call k_0 the smallest $k\geqslant s$ such that the above inequality holds. Just as in (a), we get estimate (6.8) for all $t\in[s,k_0]$. Note that, in view of (6.7), the same estimate holds for $t\in[s,k_0+1]$ possibly after C_1 is made larger. Suppose now $t\geqslant k_0+1$. We claim $\sup_\Omega(|u(\tilde t)|/v(\tilde t))<\infty$ for all $\tilde t > s$. Indeed, we can write

$$u(x,\tilde t) = u_1(x,\tilde t) - u_2(x,\tilde t),$$

where $u_1(x,\tilde t)=u(x,\tilde t;s,u_+(s))$, $u_2(x,\tilde t)=u(x,\tilde t;s,u_-(s))$ are nonnegative solutions of (1.1). By Theorem 2.1 one has $\sup_\Omega(u_i(\tilde t)/v(\tilde t))<\infty$, $i=1,2$, which proves the claim. This observation enables us to use Corollary 2.3. Using successively estimates (6.7), (6.3), (2.5), estimate (6.2) with $k=k_0$ and $\eta=\frac{C_2}{\mu}$, and (6.8) for $t\in[s,k_0+1]$, we deduce

$$\begin{aligned}\frac{\|u(t)\|_{L^\infty(\Omega)}}{\|v(t)\|_{L^\infty(\Omega)}} &\leqslant C_1\frac{\|u(k_0+[t-k_0])\|_{L^\infty(\Omega)}}{\|v(k_0+[t-k_0])\|_{L^\infty(\Omega)}} \leqslant C_1\sup_\Omega\frac{|u(k_0+[t-k_0])|}{v(k_0+[t-k_0])} \leqslant C_1\operatorname*{osc}_\Omega\frac{u(k_0+[t-k_0])}{v(k_0+[t-k_0])}\\ &\leqslant C_1\mu^{[t-k_0]-1}\operatorname{osc}_\Omega\frac{u(k_0+1)}{v(k_0+1)} \leqslant 2C_1\mu^{[t-k_0]-1}\sup_\Omega\frac{|u(k_0+1)|}{v(k_0+1)}\\ &\leqslant 2C_1\mu^{[t-k_0]-1}\frac{CC_2}{\mu}\frac{\|u(k_0+1)\|_{L^\infty(\Omega)}}{\|v(k_0+1)\|_{L^\infty(\Omega)}} \leqslant 2C_1^2CC_2\mu^{[t-k_0]-2}\frac{\|u(s+[k_0+1-s])\|_{L^\infty(\Omega)}}{\|v(s+[k_0+1-s])\|_{L^\infty(\Omega)}}\\ &\leqslant 2C_3\mu^{[t-k_0]-2}\mu^{[k_0+1-s]}\frac{\|u_0\|_{L^\infty(\Omega)}}{\|v(s)\|_{L^\infty(\Omega)}} \leqslant \frac{2C_3}{\mu^3}\mu^{(t-s)}\frac{\|u_0\|_{L^\infty(\Omega)}}{\|v(s)\|_{L^\infty(\Omega)}}.\end{aligned}$$

Thus in this case we have established estimate (2.3) from Theorem 2.2 with $C=\frac{2C_3}{\mu^3}$ and $\gamma=-\log\mu$, as desired. Since all possibilities were covered, the proof is complete. □

Remark 6.2. We remark that in our general setting, the unique continuation theorem may not hold. Therefore it may happen, in principle, that for some $u_0 \in C_0(\overline{\Omega})$ the solution $u(\cdot, t; s, u_0)$ changes sign for all t in some interval (s, t_1) and it is identical to 0 for $t = t_1$ (hence, by Theorem 3.3, also for all $t \geqslant t_1$). The statement of Theorem 2.2 remains valid for such initial data as well. Indeed, as in the above proof, one shows that the estimate is valid for $t \in (s, t_1)$. For $t \geqslant t_1$, the estimate is trivial.

7. Proofs of Theorem 2.6 and Theorem 2.7

Proof of Theorem 2.6. For the proof of existence of the positive entire solution φ_L, we refer the reader to [20]. The uniform bound on $\underline{\lambda}(\varphi_L)$, $\bar{\lambda}(\varphi_L)$ follows immediately from Corollaries 3.10 and 5.3. This proves statement (i).

We continue with the proof of (iii). Let $u_0 \in X_L^2(s)$ (with $X_L^2(s)$ as defined in statement (ii)). Then, either $u(\cdot, t; s, u_0)$ changes sign for all t, or it becomes (and remains) nonnegative or nonpositive. In the latter case it must be identically zero for all sufficiently large t, for the Harnack inequality rules out the possibility of it being nontrivial and having a zero in Ω. Thus Theorem 2.2 applies to $u(\cdot, t; s, u_0)$ (see also Remark 6.2). Estimate (2.11) now follows from (2.3) and an easy combination of Corollary 3.10 and the L^2–L^∞ estimates from Proposition 5.1.

Finally, we prove (ii). The invariance properties of $X_L^i(t)$, $t \in \mathbb{R}$, $i = 1, 2$, are obvious from the definitions. To prove that $X_L^2(t)$ is a closed subspace of X (for $X_L^1(t)$ this is trivial), we first give an equivalent characterization of the set $X_L^2(s)$, for any $s \in \mathbb{R}$. Fix γ as in (iii). Then

$$X_L^2(s) = \left\{ u_0 \in X \colon \frac{\|u(\cdot, t; s, u_0)\|_X}{\|\varphi_L(\cdot, t)\|_X} \mathrm{e}^{\gamma(t-s)} \text{ is bounded for } t \geqslant s \right\}. \tag{7.1}$$

Indeed, by statement (iii), the expression in (7.1) is bounded for each $u_0 \in X_L^2(s)$. On the other hand, if $u_0 \notin X_L^2(s)$, then $u(\cdot, t; s, u_0)$ is of one sign (positive or negative) for all large t. Then Theorem 2.1 with $u_1 = \varphi_L$, $u_2 = |u(\cdot, t; s, u_0)|$ readily implies that the expression in (7.1) is unbounded.

From (7.1) and the linear dependence of $u(\cdot, t; s, u_0)$ on u_0 it clearly follows that $X_L^2(s)$ is a subspace of X. To prove that it is closed, consider a sequence $u_n \in X_L^2(s)$ approaching some $u_0 \in X$. It follows from statement (iii) that the expression in (7.1) with u_0 replaced by u_n is bounded by a constant independent of n. Taking the limit we obtain that the expression with u_0 is bounded, hence $u_0 \in X_L^2(s)$.

It remains to prove (2.10). Obviously, $X_L^1(t) \cap X_L^2(t) = \{0\}$. Let $u_0 \in X$. The arguments in the proof of Theorem 2.3 show that $\sup_\Omega |u(\tilde{t}, t, u_0)|/\varphi_L(\tilde{t}) < \infty$ for all $\tilde{t} > t$. As in the proof of Corollary 2.4, for some $q \in \mathbb{R}$ we have

$$\sup_\Omega \frac{u(\tilde{t}, t, u_0)}{\varphi_L(\tilde{t})} \nearrow q, \qquad \inf_\Omega \frac{u(\tilde{t}, t, u_0)}{\varphi_L(\tilde{t})} \searrow q$$

as $\tilde{t} \to \infty$. Then $w(\tilde{t}) := u(\tilde{t}, t, u_0) - q\varphi_L(\tilde{t})$ is a solution of (1.1) on $\Omega \times (t, \infty)$ which has a zero in Ω for all $\tilde{t} \geqslant t$. In particular, $w(t) \in X_L^2(t)$. Since $u_0 = w(t) + q\varphi_L(t)$, (2.10) is proved. □

Proof of Theorem 2.7. The proof of the existence of ψ_L with the required properties is the same as the proof of (i) of Theorem 2.6. To prove the characterization for $X_{\tilde{L}}^2(t)$, $t \in \mathbb{R}$, we refer to the following general fact. If u is a solution of (1.1) on $(-\infty, t_0)$ and v is the solution of (2.12) with $v(\cdot, t_0) = v_0 \in L^2(\Omega)$ then, using (5.4), one easily verifies that we have

$$\langle u(\cdot, t), v(\cdot, t) \rangle := \int_\Omega u(x, t) v(x, t)\, \mathrm{d}x \equiv \text{const}. \tag{7.2}$$

This relation implies that the orthogonal complement (in $L^2(\Omega)$) of $\psi_L(t)$ is a codimension one subspace of $L^2(\Omega)$ contained in $X_{\tilde{L}}^2(t)$. Therefore it must be equal to $X_{\tilde{L}}^2(t)$. □

8. Proof of Theorem 2.8

We shall use the same notation as in the statement of the theorem. Let $U_L(\cdot, \cdot)$, $U_{\tilde{L}}(\cdot, \cdot)$ be the evolution operators associated with L, $\tilde{L}$, respectively, see Subsection 5.1. They have the following continuity property with respect to the coefficients.

Lemma 8.1. *Let L, $\tilde L$ be as in Theorem* 2.8 *and assume that* (2.14) *holds for some* $\delta > 0$. *Then there is a constant* $C = C(N, \alpha_0, d_0, R_0)$ *such that for any* $t, s \in \mathbb{R}$ *with* $0 \leqslant t - s \leqslant 1$ *we have*

$$\left\| U_L(t,s) - U_{\tilde L}(t,s) \right\|_{\mathcal{L}(L^2(\Omega))} \leqslant C\delta. \tag{8.1}$$

Proof. Let $u_0 \in L^2(\Omega)$ and let u be the solution of (5.1). Similarly, for the same u_0, let $\tilde u$ be the solution of (5.1), where L is replaced by $\tilde L$. We will use some well known facts from [24] (see Chapter III, in particular) and adhere to the notation used in that book. Let $w = u - \tilde u$. Then it is the solution of the following initial value problem

$$\begin{aligned} &v_t + Lv = \partial_i f_i - f \quad \text{in } \Omega \times (s, \infty),\\ &v = 0 \quad \text{on } \partial\Omega \times (s, \infty),\\ &v = 0 \quad \text{in } \Omega \times \{s\}, \end{aligned} \tag{8.2}$$

where

$$\begin{aligned} f_i &= (a_{ij} - \tilde a_{ij})\partial_j \tilde u + (a_i - \tilde a_i)\tilde u,\\ f &= (b_i - \tilde b_i)\partial_i \tilde u + (c_0 - \tilde c_0)\tilde u \end{aligned}$$

(we use the summation convention as above).

The energy inequality [24, Theorem III.2.1] implies

$$\begin{aligned} \max_{s \leqslant t \leqslant s+1} \left\| w(t) \right\|_{L^2(\Omega)} + \|w\|_{L^2((s,s+1),W_0^{1,2}(\Omega))} &\leqslant C\left(\sum_{i=1}^N \|f_i\|_{L^2(\Omega\times(s,s+1))} + \|f\|_{L^2(\Omega\times(s,s+1))} \right)\\ &\leqslant C\delta \|\tilde u\|_{L^2((s,s+1),W_0^{1,2}(\Omega))}, \end{aligned} \tag{8.3}$$

where C depends only on N, α_0, d_0, R_0. Now, the same energy inequality applied to the solution $\tilde u$ of (5.1) gives that the right-hand side of (8.3) is bounded above by $C\delta\|u_0\|_{L^2(\Omega)}$ with a possibly larger C. Putting these estimates together, we obtain

$$\max_{s \leqslant t \leqslant s+1} \left\| w(t) \right\|_{L^2(\Omega)} \leqslant C\delta \|u_0\|_{L^2(\Omega)}.$$

Since $w(t) = U_L(t,s)u_0 - U_{\tilde L}(t,s)u_0$ and $u_0 \in L^2(\Omega)$ was arbitrary, the assertion of Lemma 8.1 is proved. □

Remark 8.2. Note that by Proposition 5.1 we have $\|U_L(t,s)\|_{\mathcal{L}(L^2(\Omega))} \leqslant \overline{C}$ for a suitable constant $\overline{C}$, whenever $0 \leqslant t - s \leqslant 1$. Combining this with estimate (8.1) and the fact that $U_L(\cdot,\cdot)$ satisfies the usual composition property of evolution operators, we get $\|U_L(t,s) - U_{\tilde L}(t,s)\|_{\mathcal{L}(L^2(\Omega))} \leqslant n\overline{C}C^{n-1}\delta$, where C is as in Lemma 8.1, $n \geqslant 1$ is an integer and $t, s \in \mathbb{R}$ are such that $0 \leqslant t - s \leqslant n$. Obviously, $n\overline{C}C^{n-1}$ is estimated from above by $(2\max\{\overline{C}, C\})^n$, which, in turn, can be written as C^n for another constant C depending only on the constants listed in Lemma 8.1. Thus, replacing the assumption $0 \leqslant t - s \leqslant 1$ in Lemma 8.1 by $0 \leqslant t - s \leqslant n$, n being an integer with $n \geqslant 1$, we get estimate (8.1) with C replaced by C^n. We will use this observation in the sequel.

We next prove the continuity properties of $\bar\lambda(\varphi_L)$, $\underline{\lambda}(\varphi_L)$.

Proposition 8.3. *Let L, $\tilde L$ be as in Theorem* 2.8. *For each* $\varepsilon > 0$ *there exist numbers* $C(\varepsilon) > 0$, $\delta(\varepsilon) > 0$, *depending only on* ε *and* N, α_0, d_0, r_0, R_0, M_0, *such that if* (2.14) *holds with* $\delta \leqslant \delta(\varepsilon)$ *then for all* $t \geqslant s$

$$\frac{e^{-\varepsilon(t-s)}}{C(\varepsilon)} \frac{\|\varphi_{\tilde L}(t)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde L}(s)\|_{L^\infty(\Omega)}} \leqslant \frac{\|\varphi_L(t)\|_{L^\infty(\Omega)}}{\|\varphi_L(s)\|_{L^\infty(\Omega)}} \leqslant C(\varepsilon) e^{\varepsilon(t-s)} \frac{\|\varphi_{\tilde L}(t)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde L}(s)\|_{L^\infty(\Omega)}}. \tag{8.4}$$

Remark 8.4. By Corollary 5.3, the L^∞-norms in Proposition 8.3 can be replaced by the L^p-norms for any $p \in [1, \infty)$.

Proof of Proposition 8.3. We start with several estimates of quotients appearing in (8.4). Fix $s \in \mathbb{R}$ and let $u_L(t) = U_L(t,s)1$, that is, u_L is the solution of (5.1) with the initial condition $u_L(\cdot, s) \equiv 1$. For any $t > s$ define $\varrho_{\max}(t) := \sup_\Omega(u_L(t)/\varphi_L(t))$, $\varrho_{\min}(t) := \inf_\Omega(u_L(t)/\varphi_L(t))$. Thus $\operatorname{osc}_\Omega(u_L(t)/\varphi_L(t)) = \varrho_{\max}(t) - \varrho_{\min}(t)$. Then Corollary 2.3

implies that $\lim_{t\to\infty} \mathrm{osc}_\Omega(u_L(t)/\varphi_L(t)) = 0$ and by Corollary 3.2 we also have that $\varrho_{\max}(t)$ ($\varrho_{\min}(t)$) is a positive nonincreasing (nondecreasing) function of t. Thus for some $q(s) > 0$ the following holds

$$\lim_{t\to\infty} \varrho_{\max}(t) = \lim_{t\to\infty} \varrho_{\min}(t) = q(s). \tag{8.5}$$

Utilizing (7.2), one easily shows that

$$\int_\Omega 1\psi_L(s)\,\mathrm{d}x = \lim_{t\to\infty}\int_\Omega u_L(t)\psi_L(t)\,\mathrm{d}x = \lim_{t\to\infty}\int_\Omega \frac{u_L(t)}{\varphi_L(t)}\varphi_L(t)\psi_L(t)\,\mathrm{d}x = q(s)\big\langle\varphi_L(s),\psi_L(s)\big\rangle.$$

Thus $q(s) = \int_\Omega \psi_L(s)dx/k_L$, where $k_L := \langle\varphi_L(s),\psi_L(s)\rangle$ is independent of s by (7.2).

By Proposition 5.1, $\|u_L(s+1)\|_{L^\infty(\Omega)} \leqslant C_1$ for some constant C_1. Equality (8.5) combined with inequalities in (6.3) (applied to $a = u_L$ and $b = \varphi_L$), a repeated application of estimate (2.5), Theorem 2.1, the upper bound on $\|u_L(s+1)\|_{L^\infty(\Omega)}$ and Corollary 3.10 imply

$$\begin{aligned}\left|\frac{\|u_L(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_L(s+n)\|_{L^\infty(\Omega)}} - q(s)\right| &\leqslant \mathrm{osc}_\Omega \frac{u_L(s+n)}{\varphi_L(s+n)} \leqslant \mu^{n-1}\,\mathrm{osc}_\Omega \frac{u_L(s+1)}{\varphi_L(s+1)} \leqslant \mu^{n-1}\sup_\Omega \frac{u_L(s+1)}{\varphi_L(s+1)}\\ &\leqslant C\mu^{n-1}\inf_\Omega \frac{u_L(s+1)}{\varphi_L(s+1)} \leqslant C\mu^{n-1}\frac{\|u_L(s+1)\|_{L^\infty(\Omega)}}{\|\varphi_L(s+1)\|_{L^\infty(\Omega)}}\\ &\leqslant \mu^{n-1}\frac{CC_1}{\|\varphi_L(s+1)\|_{L^\infty(\Omega)}} \leqslant \mu^{n-1}\frac{C^2C_1}{\|\varphi_L(s)\|_{L^\infty(\Omega)}},\end{aligned} \tag{8.6}$$

where $n \geqslant 1$ is an arbitrary integer and μ is as in (2.5). Define

$$c_L(s) := \big\|\varphi_L(s)\big\|_{L^\infty(\Omega)}\frac{\int_\Omega \psi_L(s)\,\mathrm{d}x}{\langle\varphi_L(s),\psi_L(s)\rangle} = \big\|\varphi_L(s)\big\|_{L^\infty(\Omega)}q(s). \tag{8.7}$$

Obviously $c_L(s) \geqslant 1$. Applying Lemma 3.9 to both φ_L and ψ_L, one finds a constant $C_2 > 1$ such that $c_L(s) \in [1, C_2]$ for any $s \in \mathbb{R}$. Using (8.6) and (8.7), we obtain

$$\left|\big\|u_L(s+n)\big\|_{L^\infty(\Omega)} - c_L(s)\frac{\|\varphi_L(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_L(s)\|_{L^\infty(\Omega)}}\right| \leqslant C_3\mu^{n-1}\frac{\|\varphi_L(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_L(s)\|_{L^\infty(\Omega)}}, \tag{8.8}$$

where $C_3 = C^2C_1$. Carrying out again the same procedure with L replaced in all places by $\tilde{L}$, we derive

$$\left|\big\|u_{\tilde{L}}(s+n)\big\|_{L^\infty(\Omega)} - c_{\tilde{L}}(s)\frac{\|\varphi_{\tilde{L}}(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde{L}}(s)\|_{L^\infty(\Omega)}}\right| \leqslant C_3\mu^{n-1}\frac{\|\varphi_{\tilde{L}}(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde{L}}(s)\|_{L^\infty(\Omega)}}, \tag{8.9}$$

with obvious meanings of the new notation.

Assume now that (2.14) holds for some $\delta > 0$. Using the definitions of u_L, $u_{\tilde{L}}$ and Lemma 8.1 (see also Remark 8.2), we get the following bound

$$\Big|\big\|u_L(s+n)\big\|_{L^\infty(\Omega)} - \big\|u_{\tilde{L}}(s+n)\big\|_{L^\infty(\Omega)}\Big| \leqslant \big\|u_{\tilde{L}}(s+n) - u_L(s+n)\big\|_{L^\infty(\Omega)} \leqslant C^n\delta. \tag{8.10}$$

Obviously, (8.8), (8.9) and (8.10) imply

$$\begin{aligned}&\left|c_L(s)\frac{\|\varphi_L(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_L(s)\|_{L^\infty(\Omega)}} - c_{\tilde{L}}(s)\frac{\|\varphi_{\tilde{L}}(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde{L}}(s)\|_{L^\infty(\Omega)}}\right|\\ &\quad\leqslant C_3\mu^{n-1}\frac{\|\varphi_L(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_L(s)\|_{L^\infty(\Omega)}} + C_3\mu^{n-1}\frac{\|\varphi_{\tilde{L}}(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde{L}}(s)\|_{L^\infty(\Omega)}} + C^n\delta,\end{aligned} \tag{8.11}$$

where, as before, $n \geqslant 1$ is an arbitrary integer and $\mu \in (0,1)$ is as in (2.5). Note that we have proved above that for a suitable constant C_2 one has $1 \leqslant c_L(s) \leqslant C_2$ and the same is true for $c_{\tilde{L}}(s)$, $s \in \mathbb{R}$. Choose now an integer $n_0 \geqslant 1$ such that in (8.11) we have $C_3\mu^{n-1} \leqslant 1/2$ for all $n \geqslant n_0$ (n_0 is determined only by $N, \alpha_0, d_0, r_0, R_0, M_0$). The choice of n_0, the two sided bounds on $c_L(s)$, $c_{\tilde{L}}(s)$, and (8.11) imply that for all $n \geqslant n_0$

$$\frac{\|\varphi_L(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_L(s)\|_{L^\infty(\Omega)}} \leqslant C_4\frac{\|\varphi_{\tilde{L}}(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde{L}}(s)\|_{L^\infty(\Omega)}} + C^n\delta, \tag{8.12}$$

where C_4 is some positive constant. Although we fixed $s \in \mathbb{R}$ at the beginning of this proof, all estimates that we derived are independent of s, hence (8.12) is valid for any $s \in \mathbb{R}$.

To complete the proof of the proposition, let $\varepsilon > 0$ be arbitrary. Let C_5 be the maximum of the constants appearing in (8.12), (3.9), and (5.3), and fix $n \geqslant n_0$ so large that $\log(C_5+1)/n < \varepsilon$. Set $\delta = C_5^{-2n}$, and assume that (2.14) holds.

With this choice of δ, (8.12) implies

$$\frac{\|\varphi_L(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_L(s)\|_{L^\infty(\Omega)}} \leqslant (C_5+1)\frac{\|\varphi_{\tilde{L}}(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde{L}}(s)\|_{L^\infty(\Omega)}}. \tag{8.13}$$

Now, given any $t \geqslant s$, we write $t - s = kn + r$, where $k \geqslant 0$ is an integer and $0 \leqslant r < n$. Then, repeatedly using (3.9), (8.13), and (5.3) (with $p = q = \infty$), we obtain

$$\begin{aligned}\frac{\|\varphi_L(t)\|_{L^\infty(\Omega)}}{\|\varphi_L(s)\|_{L^\infty(\Omega)}} &\leqslant C_5^n\frac{\|\varphi_L(s+kn)\|_{L^\infty(\Omega)}}{\|\varphi_L(s)\|_{L^\infty(\Omega)}} \leqslant C_5^n(C_5+1)^k\frac{\|\varphi_{\tilde{L}}(s+kn)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde{L}}(s)\|_{L^\infty(\Omega)}}\\ &\leqslant C_5^{2n}(C_5+1)^k\frac{\|\varphi_{\tilde{L}}(t)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde{L}}(s)\|_{L^\infty(\Omega)}}.\end{aligned} \tag{8.14}$$

We can rewrite (8.14) as follows

$$\frac{\|\varphi_L(t)\|_{L^\infty(\Omega)}}{\|\varphi_L(s)\|_{L^\infty(\Omega)}} \leqslant C_5^{2n}\mathrm{e}^{C_5(t-s)/n}\frac{\|\varphi_{\tilde{L}}(t)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde{L}}(s)\|_{L^\infty(\Omega)}} \leqslant C\mathrm{e}^{\varepsilon(t-s)}\frac{\|\varphi_{\tilde{L}}(t)\|_{L^\infty(\Omega)}}{\|\varphi_{\tilde{L}}(s)\|_{L^\infty(\Omega)}} \quad (t \geqslant s), \tag{8.15}$$

with $C = C_5^{2n}$. Interchanging the role of L and $\tilde{L}$ in the above arguments, starting from (8.11), we obtain (8.15) with L replaced with $\tilde{L}$. These estimates give (8.4). We also see that $\delta(\varepsilon)$, $C(\varepsilon)$ in Proposition 8.3 can be expressed as $\delta(\varepsilon) = C^{-1/\varepsilon}$, $C(\varepsilon) = C^{1/\varepsilon}$, respectively, where C depends only on N, α_0, d_0, r_0, R_0, M_0. Thus the proof of Proposition 8.3 is complete. □

Proposition 8.3 and Remark 8.4 imply the statement of Theorem 2.8(i) regarding $\underline{\lambda}(\varphi_L)$ and $\bar{\lambda}(\varphi_L)$.

We derive the remaining statements of Theorem 2.8 from well-known robustness properties of exponential dichotomies for abstract evolution operators. We first introduce the concept of exponential dichotomy.

By an *evolution operator* T on a Banach space X we mean a family $\{T(t,s); t,s \in \mathbb{R},\ t \geqslant s\} \subset \mathcal{L}(X)$ with the following two properties (I denotes the identity on X):

$$T(t,s)T(s,r) = T(t,r), \quad T(t,t) = I \quad (t,s,r \in \mathbb{R}, t \geqslant s \geqslant r),$$

$$t \mapsto T(t,s)x\colon [s,\infty) \to X \text{ is continuous for all } x \in X,\ t,s, \in \mathbb{R}.$$

Definition 8.5. We say that an evolution operator T admits exponential dichotomy (on $\mathbb{R}$) if there are positive constants ϱ, κ and projections $P(t) \in \mathcal{L}(X)$, $t \in \mathbb{R}$, such that

(i) $T(t,s)P(s) = P(t)T(t,s)$ $(t,s \in \mathbb{R},\ t \geqslant s)$;
(ii) For each $t \geqslant s$, the restriction $T(t,s)|_{R(P(s))}$ of $T(t,s)$ to the range of $P(s)$ is an isomorphism onto $R(P(t))$; we define $T(s,t)$ as the inverse of this isomorphism;
(iii) $\|T(t,s)(I-P(s))\|_{\mathcal{L}(X)} \leqslant \kappa \mathrm{e}^{-\varrho(t-s)}$ $(t,s \in \mathbb{R},\ t \geqslant s)$;
(iv) $\|T(t,s)P(s)\|_{\mathcal{L}(X)} \leqslant \kappa \mathrm{e}^{-\varrho(s-t)}$ $(t,s \in \mathbb{R},\ t < s)$.

The constants ϱ and κ (which are of course not unique) are referred to as an *exponent* and *bound* of the dichotomy. It is not difficult to prove (cf. Exercise 4 in [16, Section 7.6]) that if T admits exponential dichotomy, then the projections $P(t)$ are uniquely determined (interestingly, a similar uniqueness property for exponential separations is not valid). Given the projections $P(t)$, one defines the associated Green's function $G(t,s)$ by

$$G(t,s) = \begin{cases} T(t,s)(I-P(s)) & \text{if } t > s,\\ T(t,s)P(s) & \text{if } t < s.\end{cases}$$

Observe that ϱ and κ being an exponent and bound for the dichotomy is equivalent to the Green's function satisfying the estimate

$$\|G(t,s)\|_{\mathcal{L}(X)} \leqslant \kappa \mathrm{e}^{-\varrho|s-t|} \quad (t \neq s).$$

Also note that $\lim_{t\to s+} G(t,s) = I - P(s)$ in the strong (pointwise) operator topology. In the following proposition we summarize the standard robustness properties of exponential dichotomies.

Proposition 8.6. *Let T, $\widetilde{T}$ be evolution operators on a Banach space X satisfying*

$$K := \sup_{\substack{t,s\in\mathbb{R}\\ 0\leqslant t-s\leqslant 1}} \|T(t,s)\|_{\mathcal{L}(X)} < \infty, \qquad \sup_{\substack{t,s\in\mathbb{R}\\ t-s=1}} \|T(t,s)-\widetilde{T}(t,s)\|_{\mathcal{L}(X)} < \varepsilon.$$

Assume T admits exponential dichotomy with exponent ϱ, bound κ, and projections $P(t)$. Given any $\tilde{\varrho} < \varrho$, $\tilde{\kappa} > \kappa$, there exists ε_1 depending only on κ, $\tilde{\kappa}$, ϱ, $\tilde{\varrho}$, and K, with the following property. If $\varepsilon < \varepsilon_1$, then $\widetilde{T}$ admits exponential dichotomy with exponent $\tilde{\varrho}$, bound $\tilde{\kappa}$, and projections $\widetilde{P}(t)$ satisfying

$$\|P(t)-\widetilde{P}(t)\|_{\mathcal{L}(X)} < C\varepsilon \quad (t\in\mathbb{R}), \tag{8.16}$$

where C is a constant depending only on κ, ϱ, and K.

Proof. With the exception of (8.16), the statement is the same as that of Theorem 7.6.10 in [16]. Estimate (8.16) follows from the estimate on $\|\widetilde{G}(t,s)-G(t,s)\|_{\mathcal{L}(X)}$, for the associated Green's functions, as given in the proof of [16, Theorem 7.6.10]. Note that the proofs of these results in [16] are based on the relation between discrete and continuous dichotomies and apply to abstract evolution operators, independently of any underlying parabolic equation. □

To apply the above abstract result, we define the following evolution operators on $X = L^2(\Omega)$

$$U_{\varrho,L}(t,s) := \mathrm{e}^{\varrho(t-s)}\frac{\|\varphi_L(s)\|_{L^2(\Omega)}}{\|\varphi_L(t)\|_{L^2(\Omega)}}U_L(t,s), \tag{8.17}$$

$$V_{\varrho,\tilde{L}}(t,s) := \mathrm{e}^{\varrho(t-s)}\frac{\|\varphi_L(s)\|_{L^2(\Omega)}}{\|\varphi_L(t)\|_{L^2(\Omega)}}U_{\tilde{L}}(t,s) \quad (t\geqslant s). \tag{8.18}$$

Here U_L and $U_{\tilde{L}}$ are the evolution operators of (1.1) and (2.12), respectively, and ϱ is a suitable positive number. A key observation employed below is that if L admits exponential separation with bound C and exponent γ, and $\varrho \in (0,\gamma/2]$, then $U_{\varrho,L}$ admits exponential dichotomy with exponent ϱ and some bound κ. Indeed, define projections $P_L(t)$ by

$$P_L(t)u = \frac{\langle u,\psi_L(t)\rangle}{\langle\varphi_L(t),\psi_L(t)\rangle}\varphi_L(t) \quad (u\in X). \tag{8.19}$$

Note that $P_L(t)$ has the range and kernel given by the invariant subspaces $X_L^1(t)$, $X_L^2(t)$ of Theorem 2.6. Also note that $P_L(t)$ is bounded: by Corollary 5.3, for some $C_0 > 0$

$$C_0 \leqslant \left\langle \frac{\varphi_L(t)}{\|\varphi_L(t)\|_{L^2(\Omega)}}, \frac{\psi_L(t)}{\|\psi_L(t)\|_{L^2(\Omega)}}\right\rangle \leqslant 1, \tag{8.20}$$

which gives the bound

$$\|P_L(t)\|_{\mathcal{L}(X)} \leqslant C_0^{-1}. \tag{8.21}$$

Using this bound, properties of $X_L^1(t)$, $X_L^2(t)$, and (2.11), it is straightforward to verify that $U_{\varrho,L}$ admits exponential dichotomy with exponent ϱ, bound $C(1+C_0^{-1})$ and projections $P_L(t)$.

Building on this observation, our first aim is to show that if L is close to $\tilde{L}$, then the projection $P_{\tilde{L}}(t)$ corresponding to $\tilde{L}$ is close to $P_L(t)$ and, consequently, statement (i) of Theorem 2.8 holds.

Lemma 8.7. *There exist constants δ_0 and C depending only on N, α_0, d_0, r_0, R_0, M_0, such that if* (2.14) *holds with $\delta \leqslant \delta_1$, then*

$$\|P_L(t)-P_{\tilde{L}}(t)\|_{\mathcal{L}(L^2(\Omega))} \leqslant C\delta \quad (t\in\mathbb{R}) \tag{8.22}$$

and

$$\left\| \frac{\varphi_L(t)}{\|\varphi_L(t)\|_{L^2(\Omega)}} - \frac{\varphi_{\tilde{L}}(t)}{\|\varphi_{\tilde{L}}(t)\|_{L^2(\Omega)}} \right\|_{L^2(\Omega)} \leqslant C\delta \quad (t \in \mathbb{R}). \tag{8.23}$$

Proof. Let $\gamma = \gamma(N, \alpha_0, d_0, r_0, R_0, M_0)$ be as in Theorem 2.6 and set $\varrho = \gamma/2$. As remarked above, $U_{\varrho,L}$ admits exponential dichotomy with exponent ϱ and projections $P_L(t)$. Suppose (2.14) holds with $\delta > 0$ sufficiently small (as specified below). Using Corollary 5.3 and Lemma 8.1, one gets

$$\sup_{\substack{t-s=1\\ s,t\in\mathbb{R}}} \left\| U_{\varrho,L}(t,s) - V_{\varrho,\tilde{L}}(t,s) \right\|_{\mathcal{L}(L^2(\Omega))} \leqslant e^{\gamma/2} C\delta = C_1\delta \tag{8.24}$$

and also, by Proposition 5.1 and Corollary 5.3,

$$\sup_{\substack{t,s\in\mathbb{R}\\ 0\leqslant t-s\leqslant 1}} \left\| U_{\varrho,L}(t,s) \right\|_{\mathcal{L}(X)} = K < \infty, \tag{8.25}$$

where C_1 and K depend only on N, α_0, d_0, r_0, R_0, M_0. By Proposition 8.6, if δ is small enough, say $\delta < \tilde{\delta} = \tilde{\delta}(N, \alpha_0, d_0, r_0, R_0, M_0)$, then $V_{\varrho,\tilde{L}}$ admits exponential dichotomy with exponent $\gamma/4$ and some projections $\widetilde{P}_{\tilde{L}}(t)$ satisfying

$$\left\| P_L(t) - \widetilde{P}_{\tilde{L}}(t) \right\|_{\mathcal{L}(L^2(\Omega))} < C_2\delta \quad (t \in \mathbb{R}),$$

with $C_2 = C_2(N, \alpha_0, d_0, r_0, R_0, M_0)$. Now assume also that $\delta \leqslant \delta(\gamma/8)$, where $\delta(\gamma/8)$ is given by Proposition 8.3 (with $\varepsilon = \gamma/8$). Then, as one easily verifies, $V_{\varrho,\tilde{L}}$ at the same time admits exponential dichotomy with exponent $\gamma/4$ and projections $P_{\tilde{L}}(t)$. As remarked above, the projections of exponential dichotomies are uniquely determined. Thus $P_{\tilde{L}}(t) = \widetilde{P}_{\tilde{L}}(t)$, which proves (8.22).

In the remaining part of this proof $\|\cdot\|$ means $\|\cdot\|_{L^2(\Omega)}$. By (8.22),

$$\begin{aligned} C\delta &\geqslant \left\| P_L(t)\frac{\varphi_L(t)}{\|\varphi_L(t)\|} - P_{\tilde{L}}(t)\frac{\varphi_L(t)}{\|\varphi_L(t)\|} \right\| = \left\| \frac{\varphi_L(t)}{\|\varphi_L(t)\|} - P_{\tilde{L}}(t)\frac{\varphi_L(t)}{\|\varphi_L(t)\|} \right\| \\ &\geqslant \left\| \frac{\varphi_L(t)}{\|\varphi_L(t)\|} - \frac{\varphi_{\tilde{L}}(t)}{\|\varphi_{\tilde{L}}(t)\|} \right\| - \left\| \frac{\varphi_{\tilde{L}}(t)}{\|\varphi_{\tilde{L}}(t)\|} - P_{\tilde{L}}(t)\frac{\varphi_L(t)}{\|\varphi_L(t)\|} \right\|. \end{aligned} \tag{8.26}$$

We obviously have

$$\frac{\varphi_{\tilde{L}}(t)}{\|\varphi_{\tilde{L}}(t)\|} = P_{\tilde{L}}(t)\frac{\varphi_L(t)}{\|P_{\tilde{L}}(t)\varphi_L(t)\|}.$$

Thus

$$\left\| \frac{\varphi_{\tilde{L}}(t)}{\|\varphi_{\tilde{L}}(t)\|} - P_{\tilde{L}}(t)\frac{\varphi_L(t)}{\|\varphi_L(t)\|} \right\| = \left| 1 - \frac{\|P_{\tilde{L}}(t)\varphi_L(t)\|}{\|\varphi_L(t)\|} \right| \leqslant \left\| \frac{\varphi_L(t)}{\|\varphi_L(t)\|} - P_{\tilde{L}}(t)\frac{\varphi_L(t)}{\|\varphi_L(t)\|} \right\| \leqslant C\delta. \tag{8.27}$$

Comparing (8.26) and (8.27), we conclude

$$\left\| \frac{\varphi_L(t)}{\|\varphi_L(t)\|_{L^2(\Omega)}} - \frac{\varphi_{\tilde{L}}(t)}{\|\varphi_{\tilde{L}}(t)\|_{L^2(\Omega)}} \right\|_{L^2(\Omega)} \leqslant 2C\delta \quad (t \in \mathbb{R}). \qquad \square$$

We can now complete the proof of Theorem 2.8(i). By Corollary 5.3 and Proposition 5.2, the function

$$\frac{\varphi_L(t)}{\|\varphi_L(t)\|_{L^2(\Omega)}} - \frac{\varphi_{\tilde{L}}(t)}{\|\varphi_{\tilde{L}}(t)\|_{L^2(\Omega)}}$$

is bounded in $L^\infty(\Omega)$ and consequently in a Hölder space $C^\alpha(\overline{\Omega})$, by a constant depending on $N, \alpha_0, d_0, r_0, R_0, M_0$. This and (8.23) imply that the L^∞-norm of this function tends to 0 as $\delta \to 0$ (uniformly in t). Finally, we note that the normalizations in $L^2(\Omega)$ can be replaced by the normalizations in $L^\infty(\Omega)$. Indeed, the previous L^∞-estimate and Corollary 5.3 imply that

$$\frac{\|\varphi_L(t)\|_{L^2(\Omega)}}{\|\varphi_L(t)\|_{L^\infty(\Omega)}} - \frac{\|\varphi_{\tilde{L}}(t)\|_{L^2(\Omega)}}{\|\varphi_{\tilde{L}}(t)\|_{L^\infty(\Omega)}}$$

tends to 0 as $\delta \to 0$. Estimating the function

$$\frac{\varphi_L(t)}{\|\varphi_L(t)\|_{L^\infty(\Omega)}} - \frac{\varphi_{\tilde L}(t)}{\|\varphi_{\tilde L}(t)\|_{L^\infty(\Omega)}} = \frac{\varphi_L(t)}{\|\varphi_L(t)\|_{L^2(\Omega)}} \frac{\|\varphi_L(t)\|_{L^2(\Omega)}}{\|\varphi_L(t)\|_{L^\infty(\Omega)}} - \frac{\varphi_{\tilde L}(t)}{\|\varphi_{\tilde L}(t)\|_{L^2(\Omega)}} \frac{\|\varphi_{\tilde L}(t)\|_{L^2(\Omega)}}{\|\varphi_{\tilde L}(t)\|_{L^\infty(\Omega)}}$$

using the triangle inequality we obtain the desired conclusion.

We are finished with the proof of statement (i) of Theorem 2.8 regarding φ_L and $\varphi_{\tilde L}$. The conclusion regarding ψ_L, $\psi_{\tilde L}$ is obtained by applying the proved statement to the operators L^* and $\tilde L^*$ (see the discussion following Theorem 5.2).

We next prove Theorem 2.8(ii).

Lemma 8.8. *Suppose that L admits exponential separation bound C_L and exponent γ_L. For each $\varepsilon > 0$ there is $\delta_3 = \delta_3(\varepsilon, C_L, \gamma_L, N, \alpha_0, d_0, r_0, R_0, M_0) > 0$ such that if* (2.14) *holds with $\delta \leqslant \delta_3$, then $\tilde L$ admits exponential separation with some bound $C(\varepsilon, C_L, \gamma_L) > 0$ and exponent $\gamma_{\tilde L} \geqslant \gamma_L - \varepsilon$.*

Proof. Initially, we proceed as in the proof of Lemma 8.7. Suppose that (2.14) holds with $\delta > 0$ sufficiently small (as specified below). Setting $\varrho = \gamma_L/2$, we observe that $U_{\varrho,L}$ admits exponential dichotomy with exponent $\gamma_L/2$, bound $C_L(1 + C_0^{-1})$ (cp. (8.21)), and projections $P_L(t)$, and that (8.24), (8.25) hold, where the constants C_1 and K now also depend on γ_L. Using Proposition 8.6, we find $\delta_3 = \delta_3(\varepsilon, C_L, \gamma_L, N, \alpha_0, d_0, r_0, R_0, M_0) > 0$ such that if $\delta < \delta_3$, then $V_{\varrho,\tilde L}$ admits exponential dichotomy with exponent $(\gamma_L - \varepsilon)/2$ and some projections $\widetilde P_{\tilde L}(t)$. These projections are close to $P_L(t)$, in particular, their range is one-dimensional. We make δ_3 smaller, if necessary, so that

$$\delta_3 < \min\{\delta(\varepsilon/2), \delta(\gamma/4)\}, \tag{8.28}$$

where $\delta(\cdot)$ is given by Proposition 8.3 and γ is as in Theorem 2.6. We claim that

$$\widetilde P_{\tilde L}(t) = P_{\tilde L}(t). \tag{8.29}$$

Although this time we do not know a priori that $V_{\varrho,\tilde L}$ admits exponential dichotomy with the latter projections (hence we cannot refer to the uniqueness), we do know, as in the proof of Lemma 8.7, that

$$V_{\gamma/2,\tilde L}(t,s) = e^{\frac{\gamma-\gamma_L}{2}(t-s)} V_{\varrho,\tilde L}(t,s)$$

admits such a dichotomy. This is sufficient for (8.29). For example, by definition of exponential dichotomy, the range of $I - \widetilde P_{\tilde L}(s)$ consists of all $v \in X$ such that $e^{(\gamma_L/2-\varepsilon/2)(t-s)}\|V_{\varrho,\tilde L}(t,s)v\|_X$ is bounded as $t \to \infty$. If $\varepsilon < 2\gamma_L - \gamma$, which we may assume without loss of generality (note that, by Theorem 2.6, Lemma 8.8 is trivial if $\gamma_L \leqslant \gamma$), then for any such v also $\|V_{\gamma/2,\tilde L}(t,s)v\|_X$ is bounded. Hence, again by definition of exponential dichotomy, v is in the range of $I - P_{\tilde L}(s)$. Similarly one proves that the range of $P_{\tilde L}(s)$ is contained in the range of $\widetilde P_{\tilde L}(s)$. These two inclusions give (8.29).

Using (8.29), the exponential dichotomy for $V_{\varrho,\tilde L}$, and Proposition 8.3 (recall (8.28)), a straightforward estimate implies the conclusion of the lemma. □

Lemma 8.8 implies statement (ii) of Theorem 2.8. The proof of the theorem is now complete.

9. Proof of Theorem 2.10

This proof will be carried out in several steps analogous to those in Section 5. We work with a fixed operator L as in (D), whose coefficients satisfy (1.4) and (1.2) on $\mathbb{R}^{N+1}$. Since in this part we consider varying domains, we will henceforth denote by $U_\Omega(\cdot,\cdot)$ the evolution operator associated with problem (1.1). It has the following continuity property with respect to the domain.

Lemma 9.1. *Let Ω_1, Ω_2, with $\Omega_1 \subset \Omega_2$, be any two Lipschitz domains, whose Lipschitz constants satisfy* (1.7)*. Then for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ depending only on ε and N, α_0, d_0, r_0, R_0, M_0, such that if $d(\partial\Omega_1, \partial\Omega_2) \leqslant \delta(\varepsilon)$, then*

$$\|U_{\Omega_1}(s+1,s) - U_{\Omega_2}(s+1,s)\|_{\mathcal{L}(L^\infty(\mathbb{R}^N))} \leqslant \varepsilon \quad (s \in \mathbb{R}). \tag{9.1}$$

Remark 9.2. As a matter of definition, the evolution operator $U_\Omega(t,s)$ acts on $L^\infty(\Omega)$, but it has a natural extension to $L^\infty(\mathbb{R}^N)$. Specifically, given any $u_0 \in L^\infty(\mathbb{R}^N)$, we first compute $U_\Omega(t,s)(u_0|_\Omega)$, where $u_0|_\Omega$ is the restriction of u_0 to Ω and then extend the resulting function by zero outside Ω. This way, $U_\Omega(t,s)$ can be viewed as a continuous operator on $L^\infty(\mathbb{R}^N)$. We use this convention throughout the section and, abusing the notation slightly, we use the same symbol $U_\Omega(t,s)$ for the extended operator.

Proof of Lemma 9.1. Given any constant $\delta > 0$ and a domain Ω, we set

$$\Omega^\delta = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) > \delta\}.$$

For each $\sigma > 0$ (small enough) let $\zeta_\sigma \in C_0^\infty(\mathbb{R}^N)$, $0 \leqslant \zeta_\sigma \leqslant 1$, be a function with $\zeta_\sigma \equiv 1$ on $\Omega_1^{2\sigma}$, $\zeta_\sigma \equiv 0$ on $\mathbb{R}^N \setminus \Omega_1^\sigma$ and $\|\nabla\zeta_\sigma\|_{L^\infty(\Omega_1)} \leqslant C/\sigma$ for some constant C. We can always choose ζ_σ so that the constant C depends only on N, r_0, R_0, and M_0. Let u_i, $i = 1, 2$, be the solutions of (1.1) on $\Omega_i \times (s, \infty)$, respectively, such that $u_i(x, s) = \zeta_\sigma(x)$ in Ω_i, $i = 1, 2$, where $s \in \mathbb{R}$ is fixed from now on. Note that u_1, u_2 are positive. If we define $w = u_1 - u_2$, then w satisfies (weakly) $w_t + Lw = 0$ in $\Omega_1 \times (s, \infty)$ with $w(\cdot, s) \equiv 0$ on Ω_1 and $w(x,t) = u_2(x,t)$ for $(x,t) \in \partial\Omega_1 \times (s, \infty)$. Using a standard L^∞ estimate (see [24, Chapter III, Theorem 7.1]), we get

$$\sup_{\Omega_1\times[s,s+1]} |u_1 - u_2| \leqslant C \sup_{\partial\Omega_1\times[s,s+1]} u_2, \tag{9.2}$$

where C depends only on N, α_0, d_0, R_0. Theorem 3.3 guarantees that u_2 is bounded on $\Omega_2 \times [s, s+1]$ by a constant depending only on N, α_0, d_0. This fact, our choice of ζ_σ and Theorem 5.2 imply that

$$\sup_{\partial\Omega_1\times[s,s+1]} u_2 \leqslant C(\sigma)\big(d(\partial\Omega_1, \partial\Omega_2)\big)^\alpha \tag{9.3}$$

for some $\alpha > 0$, which depends only on N, d_0, α_0. The constant $C(\sigma)$ depends only on σ, α, N, α_0, d_0, r_0, R_0, M_0. Combining the last two inequalities, we obtain

$$\sup_{\Omega_1\times[s,s+1]} |u_1 - u_2| \leqslant C(\sigma)\big(d(\partial\Omega_1, \partial\Omega_2)\big)^\alpha. \tag{9.4}$$

By Proposition 5.4, we can write

$$u_i(x,t) = \int_{\Omega_i} k_i(x,t;\xi,s)\zeta_\sigma(\xi)\,\mathrm{d}\xi \quad \big((x,t) \in \Omega_i \times (s,\infty)\big), \tag{9.5}$$

where $k_i(x,t;\xi,s)$ is the weak Green's function associated with problem (1.1) with Ω replaced by Ω_i, $i = 1, 2$. Statements (ii), (iii) in Proposition 5.4 in combination with Theorem 5.2 imply that we have the following inequality

$$k_i(x, s+1; \xi, s) \leqslant C\big(d(\partial\Omega_1, \partial\Omega_2) + 2\sigma\big)^\alpha \quad \big(x \in \Omega_i \setminus \Omega_1^{2\sigma}, \xi \in \Omega_i, i = 1, 2\big), \tag{9.6}$$

with $\alpha > 0$ the same as in (9.4) and C depends only on α, N, α_0, d_0, r_0, R_0, M_0. Using (iv) in Proposition 5.4, the choice of ζ_σ, (9.5), (9.4) and (9.6), we derive

$$\begin{aligned}
\big\|U_{\Omega_1}(s+1,s) - U_{\Omega_2}(s+1,s)\big\|_{\mathcal{L}(L^\infty(\mathbb{R}^N))} &= \sup_{x\in\Omega_2} \int_{\Omega_2} \big(k_2(x,s+1;\xi,s) - k_1(x,s+1;\xi,s)\big)\,\mathrm{d}\xi \\
&\leqslant \sup_{x\in\Omega_2} \int_{\Omega_2\setminus\Omega_1^{2\sigma}} \big(k_2(x,s+1;\xi,s) - k_1(x,s+1;\xi,s)\big)\,\mathrm{d}\xi \\
&\quad + \sup_{x\in\Omega_2\setminus\Omega_1^{2\sigma}} \int_{\Omega_1^{2\sigma}} \big(k_2(x,s+1;\xi,s) - k_1(x,s+1;\xi,s)\big)\,\mathrm{d}\xi \\
&\quad + \sup_{x\in\Omega_1^{2\sigma}} \int_{\Omega_1^{2\sigma}} \big(k_2(x,s+1;\xi,s) - k_1(x,s+1;\xi,s)\big)\,\mathrm{d}\xi \\
&\leqslant C\big|\Omega_2 \setminus \Omega_1^{2\sigma}\big| + C\big|\Omega_1^{2\sigma}\big|\big(d(\partial\Omega_1, \partial\Omega_2) + 2\sigma\big)^\alpha + \sup_{\Omega_1^{2\sigma}}\big|u_1(s+1) - u_2(s+1)\big| \\
&\leqslant C\big|\Omega_2 \setminus \Omega_1^{2\sigma}\big| + C_1\big(d(\partial\Omega_1, \partial\Omega_2) + 2\sigma\big)^\alpha + C(\sigma)\big(d(\partial\Omega_1, \partial\Omega_2)\big)^\alpha,
\end{aligned} \tag{9.7}$$

where $|\Omega|$ denotes the measure of Ω, the constants C, C_1, $C(\sigma)$ depend on α, N, α_0, d_0, r_0, R_0, M_0 and $C(\sigma)$ in addition depends on σ. Notice that, under our assumptions on Ω_1, Ω_2, the measure $|\Omega_2 \setminus \Omega_1^{2\sigma}|$ can be estimated from above by $C(d(\partial\Omega_1, \partial\Omega_2) + 2\sigma)$, where C depends only on N, r_0, R_0, M_0. Hence,

$$\left\| U_{\Omega_1}(s+1,s) - U_{\Omega_2}(s+1,s) \right\|_{\mathcal{L}(L^\infty(\mathbb{R}^N))} \leqslant C\big(d(\partial\Omega_1, \partial\Omega_2) + 2\sigma\big)^\alpha + C(\sigma)\big(d(\partial\Omega_1, \partial\Omega_2)\big)^\alpha. \tag{9.8}$$

Here, we assumed without loss of generality that σ, $d(\partial\Omega_1, \partial\Omega_2)$ are sufficiently small and $\alpha \in (0,1)$. It is now apparent that, given any $\varepsilon > 0$, it is sufficient to choose σ of order $\varepsilon^{1/\alpha}$ and then $d(\partial\Omega_1, \partial\Omega_2)$ of order $(\varepsilon/C(\sigma))^{1/\alpha}$ to get the assertion of Lemma 9.1. □

In the next corollary we remove the restriction $\Omega_2 \subset \Omega_1$.

Corollary 9.3. *Let Ω_1 be a Lipschitz domain, whose Lipschitz constants satisfy* (1.7) *and let Ω_2 be any bounded Lipschitz domain. Then for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, depending only on ε and N, α_0, d_0, r_0, R_0, M_0, such that if $d(\partial\Omega_1, \partial\Omega_2) \leqslant \delta(\varepsilon)$, then*

$$\left\| U_{\Omega_1}(s+1,s) - U_{\Omega_2}(s+1,s) \right\|_{\mathcal{L}(L^\infty(\mathbb{R}^N))} \leqslant \varepsilon \quad (s \in \mathbb{R}). \tag{9.9}$$

More precisely, there exists $\delta_1 = \delta_1(N, \alpha_0, d_0, r_0, R_0, M_0)$ such that (9.8) *holds, provided $d(\partial\Omega_1, \partial\Omega_2) \leqslant \delta_1$ and $\sigma \leqslant \delta_1$.*

Proof. Assume $d(\partial\Omega_1, \partial\Omega_2)$ is small compared to r_0 in (1.7). We can then find two domains $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$ such that $d(\partial\widetilde{\Omega}_1, \partial\widetilde{\Omega}_2) \leqslant 2d(\partial\Omega_1, \partial\Omega_2)$, $\widetilde{\Omega}_1 \subset \Omega_i \subset \widetilde{\Omega}_2$, $i = 1, 2$, and $\widetilde{\Omega}_1$, $\widetilde{\Omega}_2$ are Lipschitz domains satisfying (1.5), (1.6), replacing the constants r_0, M_0, R_0 by $r_0/2$, $2M_0$, $2R_0$, if necessary. The result now follows from the monotonicity of the weak Green's functions with respect to the domain (see Proposition 5.4) and the estimates of Lemma 9.1. □

Remark 9.4. A similar extension as in Remark 8.2 applies here. Using the boundedness of the evolution operator (Theorem 3.3) one shows that the statements of Corollary 9.3 remain valid if one replaces $s+1$ by $s+n$, $n \geqslant 1$ being any integer; in this case, the constants C, $C(\sigma)$ in (9.8) have to be replaced by C^n, $C(\sigma)^n$, where C, $C(\sigma)$ may have to be made larger, but they do not depend on $n \in \mathbb{N}$.

With the above estimates on the evolution operators, the proof of Theorem 2.10 can be carried out along the lines of the proof of Theorem 2.8. We limit the further exposition to indicating the necessary modifications.

The next proposition is analogous to Proposition 8.3.

Proposition 9.5. *Let Ω, $\widetilde{\Omega}$ be as in Theorem* 2.10. *Then for each $\varepsilon > 0$ there exist numbers $C(\varepsilon)$, $\delta(\varepsilon) > 0$, depending only on ε and N, α_0, d_0, r_0, R_0, M_0, such that if $d(\partial\Omega, \partial\widetilde{\Omega}) < \delta(\varepsilon)$, then for all $t \geqslant s$*

$$\frac{e^{-\varepsilon(t-s)}}{C(\varepsilon)} \frac{\|\varphi_{\widetilde{\Omega}}(t)\|_{L^\infty(\widetilde{\Omega})}}{\|\varphi_{\widetilde{\Omega}}(s)\|_{L^\infty(\widetilde{\Omega})}} \leqslant \frac{\|\varphi_{\Omega}(t)\|_{L^\infty(\Omega)}}{\|\varphi_{\Omega}(s)\|_{L^\infty(\Omega)}} \leqslant C(\varepsilon)e^{\varepsilon(t-s)} \frac{\|\varphi_{\widetilde{\Omega}}(t)\|_{L^\infty(\widetilde{\Omega})}}{\|\varphi_{\widetilde{\Omega}}(s)\|_{L^\infty(\widetilde{\Omega})}}. \tag{9.10}$$

Proof. The only noteworthy difference from the proof of Proposition 8.3. is that we use (9.8) (and Remark 9.4) in place of (8.1) (and Remark 8.2). Thus the estimate corresponding to (8.12) reads as follows

$$\frac{\|\varphi_{\Omega}(s+n)\|_{L^\infty(\Omega)}}{\|\varphi_{\Omega}(s)\|_{L^\infty(\Omega)}} \leqslant C_4 \frac{\|\varphi_{\widetilde{\Omega}}(s+n)\|_{L^\infty(\widetilde{\Omega})}}{\|\varphi_{\widetilde{\Omega}}(s)\|_{L^\infty(\widetilde{\Omega})}} + C^n\big(d(\partial\Omega, \partial\widetilde{\Omega}) + 2\sigma\big)^\alpha + \big(C(\sigma)\big)^n\big(d(\partial\Omega, \partial\widetilde{\Omega})\big)^\alpha. \tag{9.11}$$

One now easily derives the estimate corresponding to (8.13) by first choosing σ sufficiently small followed by a suitable choice of $d(\partial\Omega, \partial\widetilde{\Omega})$. The rest of the arguments used in the proof of Proposition 8.3 are straightforward to modify. We omit the details. □

By Corollary 5.3, we can replace in (9.10) the L^∞-norms on the respective domains by the L^2-norms. This implies the statement of Theorem 2.10(i) regarding $\underline{\lambda}(\varphi_\Omega)$ and $\bar{\lambda}(\varphi_\Omega)$.

In the following it is convenient to view the functions $\varphi_\Omega(t)$, $\psi_\Omega(t)$ as elements of $Y := L^\infty(\mathbb{R}^N)$ (extending them by 0 outside Ω). Understanding that $\langle\cdot,\cdot\rangle$ now stands for the standard inner product in $L^2(\mathbb{R}^N)$, we define a continuous projection $P_\Omega(t)$ on Y by

$$P_\Omega(t)u = \frac{\langle u, \psi_\Omega(t)\rangle}{\langle \varphi_\Omega(t), \psi_\Omega(t)\rangle}\varphi_\Omega(t). \tag{9.12}$$

With these conventions and notation, arguing similarly as in the proof of Lemma 8.7, one proves the following proposition.

Proposition 9.6. *Let Ω, $\widetilde{\Omega}$ be as in Theorem 2.10. Then for each $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$, depending only on ε and N, α_0, d_0, r_0, R_0, M_0, such that if $d(\partial\Omega, \partial\widetilde{\Omega}) \leqslant \delta(\varepsilon)$, then*

$$\left\| P_\Omega(t) - P_{\widetilde{\Omega}}(t)\right\|_{\mathcal{L}(L^\infty(\mathbb{R}^N))} \leqslant \varepsilon \quad (t \in \mathbb{R}), \tag{9.13}$$

and

$$\left\| \frac{\varphi_\Omega(t)}{\|\varphi_\Omega(t)\|_{L^\infty(\Omega)}} - \frac{\varphi_{\widetilde{\Omega}}(t)}{\|\varphi_{\widetilde{\Omega}}(t)\|_{L^\infty(\widetilde{\Omega})}} \right\|_{L^\infty(\mathbb{R}^N)} \leqslant \varepsilon \quad (t \in \mathbb{R}). \tag{9.14}$$

With this result we have completed the proof of the statement of Theorem 2.10(i) regarding $\varphi_\Omega, \varphi_{\widetilde{\Omega}}$. The conclusion regarding ψ_Ω, $\psi_{\widetilde{\Omega}}$ is again taken care of by the discussion following Theorem 5.2. The proof of statement (ii) of Theorem 2.10 is completely analogous to the proof of Lemma 8.8 and is omitted.

Appendix A

Proof of Proposition 5.5. By a standard modification (multiplication of u by $\mathrm{e}^{-m(t-s)}$), one reduces the proof to the case with $m = \sup_{\Omega\times(s,T)}(-c_0) = 0$. Henceforth we assume this extra condition.

Let $\mathcal{D}(\Omega)$ denote the space of smooth functions with compact support in Ω. The proof consist of a two step approximation procedure. First, assuming the initial condition is in $\mathcal{D}(\Omega)$, we find the solution as the limit of solutions of approximating problems on smooth subdomains of Ω. Then we approximate a general initial condition by functions in $\mathcal{D}(\Omega)$ and take the limit of the solutions of these approximate problems.

We start with preliminary estimates of solutions on approximating subdomains. Fix any $\varrho_0 \in (0, r_0/4]$, where r_0 is as in (1.7). Choose a family of smooth domains Ω_ε, $\varepsilon \in (0, \varrho_0)$, which is decreasing in ε (with respect to inclusion) and such that for each ε one has $\Omega_\varepsilon \subset \overline{\Omega}_\varepsilon \subset \Omega$, $d(\partial\Omega_\varepsilon, \partial\Omega) \leqslant \varepsilon$, and (1.7) is satisfied with Ω replaced by Ω_ε, and r_0, M_0 replaced by $r_0/2$, $2M_0$, respectively.

Let $f_\varepsilon \in \mathcal{D}(\Omega)$ be any real-valued function with

$$\operatorname{dist}(x, \partial\Omega) \geqslant 2\varrho_0 \quad (x \in \operatorname{supp} f_\varepsilon)$$

and consider the following problem

$$\begin{aligned} &u_t + Lu = 0 \quad \text{in } \Omega_\varepsilon \times (s, T),\\ &u = 0 \quad \text{on } \partial\Omega_\varepsilon \times (s, T),\\ &u = f_\varepsilon \quad \text{in } \Omega_\varepsilon \times \{s\}. \end{aligned} \tag{A.1}$$

It has a unique solution u_ε and the solution is contained in $W^{2,1,p}(\Omega_\varepsilon \times (s, T)) \cap C(\overline{\Omega}_\varepsilon \times [s, T])$ for all $p > 1$ (see [26, Theorem 7.17]). We claim that for all $\varrho \in (0, \varrho_0]$ one has

$$\omega^\varepsilon(\varrho) := \sup_{\operatorname{dist}(x,\partial\Omega_\varepsilon)\leqslant\varrho} \left|u_\varepsilon(x, t)\right| \leqslant \sup_{x\in\Omega_\varepsilon} \left|f_\varepsilon(x)\right| \left(\frac{2\varrho}{\varrho_0}\right)^{\theta_0} \quad (t \in [s, T]), \tag{A.2}$$

where $\theta_0 > 0$ depends only on N, d_0, α_0, r_0, R_0, M_0. Note that, by the maximum principle,

$$\sup_{x\in\Omega_\varepsilon} \left|u_\varepsilon(x, t)\right| \leqslant \sup_{x\in\Omega_\varepsilon} \left|f_\varepsilon(x)\right| \quad (t \in [s, T]). \tag{A.3}$$

This implies that in order to verify (A.2) it suffices to show that $\omega^\varepsilon(\varrho) \leqslant 2^{-\theta_0}\omega^\varepsilon(2\varrho)$ for any fixed $\varrho \in [0, \varrho_0/2]$. For this purpose, let (x_0, t_0) be any point in $\Omega_\varepsilon \times [s, T]$ with $\operatorname{dist}(x_0, \partial\Omega_\varepsilon) \leqslant \varrho$. The assumptions on Ω_ε imply that there are two positive numbers K_1, K_2, depending only on N, r_0, R_0, M_0, and a point $y_0 \in \mathbb{R}^N \setminus \Omega_\varepsilon$ such that the following holds. If we set $r = \operatorname{dist}(x_0, \partial\Omega_\varepsilon)/K_1$, then $|x_0 - y_0| = K_2 r$ and $B_r(y_0) \cap \Omega_\varepsilon = \emptyset$. Following ideas of [25], we define the slant cylinder V by

$$V := \left\{(x,t) \in \mathbb{R}^N \times \mathbb{R}\colon \left|x - y_0 - \frac{(t-(t_0-r^2))}{r^2}(x_0 - y_0)\right| < r,\ t_0 - r^2 < t < t_0\right\}. \tag{A.4}$$

We will show at the end of this proof that there is a function $v \in C^\infty(\mathbb{R}^{N+1})$ such that v vanishes on the lateral side of ∂V, $0 \leqslant v \leqslant 1$, $v_t + Lv \leqslant 0$ in V, and $v(x_0, t_0) \geqslant \kappa_0 > 0$, where $\kappa_0 \in (0,1)$ is a constant depending only on N, d_0, α_0, r_0, R_0, M_0. Assume for now such a function v exists. In $W := (\Omega_\varepsilon \times (s,T)) \cap V$ define the function $w := u_\varepsilon + \omega^\varepsilon(2\varrho)v$. Clearly, $w_t + Lw \leqslant 0$ in W. Moreover, since the base of V is outside $\Omega \times (s,T)$, properties of v imply that $w \leqslant \omega^\varepsilon(2\varrho)$ on the parabolic boundary $\partial_p W$ of W. Thus, by the maximum principle, $w \leqslant \omega^\varepsilon(2\varrho)$ in W. In particular, $w(x_0,t_0) \leqslant \omega^\varepsilon(2\varrho)$ and hence

$$u_\varepsilon(x_0,t_0) \leqslant \big(1 - v(x_0,t_0)\big)\omega^\varepsilon(2\varrho) \leqslant (1-\kappa_0)\omega^\varepsilon(2\varrho). \tag{A.5}$$

This inequality holds for $-u$ as well and since (x_0,t_0) was arbitrary we conclude that $\omega^\varepsilon(\varrho) \leqslant 2^{-\theta_0}\omega^\varepsilon(2\varrho)$ with $\theta_0 = -\log_2(1-\kappa_0)$. Thus estimate (A.2) is established.

Consider now an arbitrary $\tilde f \in \mathcal{D}(\Omega)$ and set

$$\varrho_0 := \inf\big(\big\{\operatorname{dist}(x, \partial\Omega)\colon x \in \operatorname{supp} \tilde f\big\} \cup \{r_0/4\}\big).$$

For $\varepsilon \in (0, \varrho_0/2)$ take $f_\varepsilon := \tilde f|_{\Omega_\varepsilon}$ in (A.1). Extending the corresponding solution u_ε by zero outside Ω_ε, we view it as a function on Ω. Let $0 < \varepsilon_1 < \varepsilon_2 < \varrho_0/2$ be arbitrary. Recall that $\Omega_{\varepsilon_2} \subset \Omega_{\varepsilon_1}$, by the monotonicity of the family. The maximum principle and (A.2) imply

$$\begin{aligned}\sup_{\substack{x\in\Omega\\ s\leqslant t\leqslant T}} \big|u_{\varepsilon_1}(x,t) - u_{\varepsilon_2}(x,t)\big| &\leqslant \sup_{\substack{x\in\Omega_{\varepsilon_1}\setminus\Omega_{\varepsilon_2}\\ s\leqslant t\leqslant T}} \big|u_{\varepsilon_1}(x,t)\big| + \sup_{\substack{x\in\partial\Omega_{\varepsilon_2}\\ s\leqslant t\leqslant T}} \big|u_{\varepsilon_1}(x,t)\big| \\ &\leqslant 2\Big(\sup_{x\in\Omega_{\varepsilon_1}} \big|f_{\varepsilon_1}(x)\big|\Big)\left(\frac{2\varepsilon_2}{\varrho_0}\right)^{\theta_0} = 2\Big(\sup_{x\in\Omega}\big|\tilde f(x)\big|\Big)\left(\frac{2\varepsilon_2}{\varrho_0}\right)^{\theta_0}.\end{aligned}$$

This estimate implies that u_ε, $\varepsilon \in (0, \varrho_0/2)$, is a Cauchy family in $C(\overline{\Omega} \times [s,T])$ and, by the interior L^p-estimates [26, Theorem 7.13], also in (the Fréchet space) $W^{2,1,p}_{\mathrm{loc}}(\Omega \times (s,T))$ for each $p \in (1,\infty)$. It follows that, as $\varepsilon \to 0$, u_ε converges to a solution $\tilde u$ of (5.1) with $u_0 = \tilde f$ and the solution is in $W^{2,1,p}_{\mathrm{loc}}(\Omega \times (s,T))$ for each $p \in (1,\infty)$. By the maximum principle, the solution (as a function in $W^{2,1,N+1}_{\mathrm{loc}}(\Omega \times (s,T)) \cap C(\overline{\Omega} \times [s,T])$) is uniquely determined by $\tilde f$.

We have thus proved Proposition 5.5 under the extra assumption $u_0 = \tilde f \in \mathcal{D}(\Omega)$. To remove this assumption, take an arbitrary $f \in C_0(\overline{\Omega})$ and choose a sequence $f_n \in \mathcal{D}(\Omega)$ such that $f_n \to f$ in $C_0(\overline{\Omega})$. For the solution u_n of (5.1) with $u_0 = f_n$ we have, by the maximum principle,

$$\sup_{\substack{x\in\Omega\\ s\leqslant t\leqslant T}} \big|u_m(x,t) - u_n(x,t)\big| \leqslant \sup_{x\in\Omega}\big|f_m(x) - f_n(x)\big| \quad (m,n = 1,2,\ldots).$$

This and the interior L^p-estimates imply that u_n is a Cauchy sequence in $C(\overline{\Omega} \times [s,T])$ and in $W^{2,1,p}_{\mathrm{loc}}(\Omega \times (s,T))$ for each $p \in (1,\infty)$. Arguing as above, we conclude that u_n converges to a unique solution of (5.1) with $u_0 = f$ and the solution has the regularity as stated in Proposition 5.5.

To finish this proof, we still need to construct the function v with the properties stated above. Define

$$v(x,t) := \frac{1}{r^4} \mathrm{e}^{-\beta\frac{(t-(t_0-r^2))}{r^2}} \left(r^2 - \left|x - y_0 - \frac{(t-(t_0-r^2))}{r^2}(x_0-y_0)\right|^2\right)^2, \tag{A.6}$$

where $(x,t) \in V$ and $\beta > 0$ is to be determined. It is obvious that $0 \leqslant v \leqslant 1$ in V and v vanishes on the lateral side of ∂V. Denoting

$$A := \left|x - y_0 - \frac{(t-(t_0-r^2))}{r^2}(x_0-y_0)\right| \quad \text{and} \quad B := r^2 - A^2,$$

by a simple computation, using (ND), (1.4) and (1.2), one shows

$$-v_t - Lv \geqslant \frac{1}{r^4}\mathrm{e}^{-\beta\frac{(t-(t_0-r^2))}{r^2}}\left(8\alpha_0 A^2 - 4Nd_0B - 4Nd_0AB - d_0B^2 - \frac{4N}{r^2}|x_0-y_0|AB + \frac{\beta}{r^2}B^2\right). \tag{A.7}$$

From now on, we assume $r \leqslant 1$ without loss of generality. Let us first consider the case when $A \geqslant (1-\varepsilon)r$ and $\varepsilon > 0$ is small. Using this assumption, the fact that $A \leqslant r$, $|x_0 - y_0| = K_2 r$, $r \leqslant 1$ and (A.7), we compute

$$-v_t - Lv \geqslant \frac{1}{r^4}\mathrm{e}^{-\beta\frac{(t-(t_0-r^2))}{r^2}}(r^2)\big(8\alpha_0(1-\varepsilon)^2 - 16Nd_0\varepsilon - 4d_0\varepsilon^2 - 8NK_2\varepsilon\big).$$

The right-hand side of this inequality is nonnegative, independently of $\beta > 0$, provided ε is chosen sufficiently small. The choice of ε depends only on N, d_0, α_0 and K_2. Fix such a number and call it ε_0. If, on the other hand, $A \leqslant (1-\varepsilon_0)r$, an elementary inequality implies that $\varepsilon_0 r^2 \leqslant B \leqslant r^2 \leqslant 1$. Using this fact and (A.7), we obtain

$$-v_t - Lv \geqslant \frac{1}{r^4}\mathrm{e}^{-\beta\frac{(t-(t_0-r^2))}{r^2}}B(-8Nd_0 - d_0 - 4NK_2 + \beta\varepsilon_0).$$

If β is big enough, as determined by N, d_0, K_2, ε_0, the right hand side of this inequality becomes positive. In view of the dependences of K_2 and ε_0, β can be chosen depending only on N, d_0, α_0, r_0, R_0, M_0. With this β we have $v_t + Lv \leqslant 0$ in V and $v(x_0, t_0) = \mathrm{e}^{-\beta}$. Defining $\kappa_0 = \mathrm{e}^{-\beta}$, the function v has all the properties claimed above. The proof is now complete. □

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