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The two-species Vlasov–Maxwell–Landau system in $\mathbb{R}^3$ ☆

Yanjin Wang

School of Mathematical Sciences, Xiamen University, Xiamen, Fujian 361005, China

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Abstract

We consider the global classical solutions near the Maxwellians to the two-species Vlasov–Maxwell–Landau system in the whole space. It is shown that the cancelation properties between two species coupled with the electric effect yield the faster time decay of the electric field, which leads to our construction of global solutions.

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Keywords: Vlasov–Maxwell–Landau system; Global solution; Time decay; Energy method

1. Introduction

The dynamics of charged dilute particles (e.g., electrons and ions) is described by the Vlasov–Maxwell–Landau system:

$$\begin{aligned}
&\partial_t F_+ + v\cdot\nabla_x F_+ + (E+v\times B)\cdot\nabla_v F_+ = Q(F_+,F_+) + Q(F_-,F_+),\\
&\partial_t F_- + v\cdot\nabla_x F_- - (E+v\times B)\cdot\nabla_v F_- = Q(F_+,F_-) + Q(F_-,F_-),\\
&F_\pm(0,x,v) = F_{0,\pm}(x,v).
\end{aligned} \tag{1.1}$$

Here $F_\pm(t,x,v)\ge 0$ are the number density functions for the ions $(+)$ and electrons $(-)$ respectively, at time $t\ge 0$, position $x=(x_1,x_2,x_3)\in\mathbb{R}^3$ and velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3$. The collision between charged particles is given by the Landau (Fokker–Planck) operator:

$$Q(G_1,G_2)(v) = \nabla_v\cdot\int_{\mathbb{R}^3}\Phi\big(v-v'\big)\big(G_1\big(v'\big)\nabla_v G_2(v) - G_2(v)\nabla_{v'}G_1\big(v'\big)\big)dv', \tag{1.2}$$

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E-mail address: yanjin_wang@xmu.edu.cn.

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where

$$\Phi(v)=\frac{1}{|v|}\left(I-\frac{v\otimes v}{|v|^2}\right). \tag{1.3}$$

The self-consistent electromagnetic field $(E(t,x),B(t,x))$ in (1.1) is coupled with $F_\pm(t,x,v)$ through the Maxwell system

$$\begin{aligned}
&\partial_t E-\nabla_x\times B=-\int_{\mathbb{R}^3} v(F_+-F_-)\,dv,\quad \nabla_x\cdot E=\int_{\mathbb{R}^3}(F_+-F_-)\,dv,\\
&\partial_t B+\nabla_x\times E=0,\quad \nabla_x\cdot B=0,\\
&E(0,x)=E_0(x),\qquad B(0,x)=B_0(x).
\end{aligned} \tag{1.4}$$

It turns out that all the physical constants will not create essential mathematical difficulties along our analysis, for notational simplicity, we have normalized all constants in the Vlasov–Maxwell–Landau system to be one. Accordingly, we normalize the global Maxwellian as

$$\mu(v)\equiv\mu_+(v)=\mu_-(v)=\mathrm{e}^{-|v|^2}. \tag{1.5}$$

We define the standard perturbation $f_\pm(t,x,v)$ to μ as

$$F_\pm=\mu+\sqrt{\mu}f_\pm. \tag{1.6}$$

Letting $f(t,x,v)=\binom{f_+(t,x,v)}{f_-(t,x,v)}$, the Vlasov–Maxwell–Landau system for the perturbation now takes the form

$$\begin{aligned}
&\{\partial_t+v\cdot\nabla_x+q_0(E+v\times B)\cdot\nabla_v\}f-2E\cdot v\sqrt{\mu}q_1+Lf=\Gamma(f,f)+q_0E\cdot vf,\\
&\partial_t E-\nabla_x\times B=-\int_{\mathbb{R}^3} v\sqrt{\mu}(f_+-f_-)\,dv,\quad \nabla_x\cdot E=\int_{\mathbb{R}^3}\sqrt{\mu}(f_+-f_-)\,dv,\\
&\partial_t B+\nabla_x\times E=0,\quad \nabla_x\cdot B=0,
\end{aligned} \tag{1.7}$$

for the matrix $q_0=\mathrm{diag}(1,-1)$ and the vector $q_1=\binom{1}{-1}$. For $g=\binom{g_1}{g_2}$, the linearized collision operator Lg in (1.7) is given by the vector

$$Lg\equiv\begin{pmatrix}L_+g\\ L_-g\end{pmatrix}\equiv-\begin{pmatrix}\frac{2}{\sqrt{\mu}}Q(\mu,\sqrt{\mu}g_1)+\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}(g_1+g_2),\mu)\\ \frac{2}{\sqrt{\mu}}Q(\mu,\sqrt{\mu}g_2)+\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}(g_1+g_2),\mu)\end{pmatrix}. \tag{1.8}$$

For $g=\binom{g_1}{g_2}$ and $h=\binom{h_1}{h_2}$, the nonlinear collision operator $\Gamma(g,h)$ in (1.7) is given by the vector

$$\Gamma(g,h)\equiv\begin{pmatrix}\Gamma_+(g,h)\\ \Gamma_-(g,h)\end{pmatrix}\equiv\begin{pmatrix}\frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}(g_1+g_2),\sqrt{\mu}h_1)\\ \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}(g_1+g_2),\sqrt{\mu}h_2)\end{pmatrix}. \tag{1.9}$$

For the Landau operator (1.2), we define

$$\sigma^{ij}(v)=\Phi^{ij}*\mu=\int_{\mathbb{R}^3}\Phi^{ij}(v-v')\mu(v')\,dv'. \tag{1.10}$$

We denote $|\cdot|_2$ to be the $L^2(\mathbb{R}^3_v)$ norm, and we define $L^2_\sigma(\mathbb{R}^3_v)$ to be the space with norm

$$|f|_\sigma^2=\int_{\mathbb{R}^3}\left[\sigma^{ij}\partial_i f\partial_j f+\sigma^{ij}v_iv_jf^2\right]dv. \tag{1.11}$$

From Lemma 3 in [9], we have

$$C^{-1}|f|_\sigma\le\left|\langle v\rangle^{-\frac12}f\right|_2+\left|\langle v\rangle^{-\frac32}\nabla_v f\cdot\frac{v}{|v|}\right|_2+\left|\langle v\rangle^{-\frac12}\nabla_v f\times\frac{v}{|v|}\right|_2\le C|f|_\sigma \tag{1.12}$$

with $\langle v\rangle=\sqrt{1+|v|^2}$. It is well known [1,9] that the linear collision operator $L\ge 0$ and

$$\langle Lf,f\rangle \gtrsim \big|\{\mathbf{I}-\mathbf{P}\}f\big|_\sigma^2, \tag{1.13}$$

where $\langle\cdot,\cdot\rangle$ is the L^2 inner product in $\mathbb{R}^3_v$ and $\mathbf{P}$ denotes the L^2_v orthogonal projection on the null space of L:

$$N(L)\equiv \operatorname{span}\left\{\sqrt{\mu}\begin{pmatrix}1\\0\end{pmatrix},\sqrt{\mu}\begin{pmatrix}0\\1\end{pmatrix},v\sqrt{\mu}\begin{pmatrix}1\\1\end{pmatrix},|v|^2\sqrt{\mu}\begin{pmatrix}1\\1\end{pmatrix}\right\}. \tag{1.14}$$

There are three major mathematical difficulties in the study of the Vlasov–Maxwell–Landau system (1.7): the intrinsic difficulty associated with the Landau kernel (cf. (1.12)); the velocity-growth in the nonlinear term $E\cdot vf$ (and $v\times B\cdot\nabla_v f$); the regularity-loss of the electromagnetic field. We may refer to [11] and [2] for the more detailed explanation. For the Landau equation, Guo [9] proved the first result of the global unique solution near the Maxwellians. The Vlasov–Maxwell–Boltzmann system with hard-sphere interaction was solved by Guo [10] due to that the hard-sphere kernel has the stronger dissipation $|\langle v\rangle^{1/2}f|_2^2$. Also, the relativistic Vlasov–Maxwell–Landau system was solved by Strain and Guo [16] since the relativistic velocity is bounded and $|f|_2^2$ is controlled by the relativistic Landau dissipation. Recently, for the Vlasov–Poisson–Landau system (in which $B=0$ and so $E=-\nabla_x\phi$ with the electric potential ϕ satisfying the Poisson equation), Guo [11] made the important progress of proving the first result of the global unique solution. The key point in [11] is to introduce the exponential weight factor $e^{\pm\phi}$. The observation is that upon multiplying by $e^{\pm\phi}$ one can rewrite

$$e^{\pm\phi}\{v\cdot\nabla_x f_\pm \pm \nabla_x\phi\cdot vf_\pm\}=v\cdot\nabla_x\big\{e^{\pm\phi}f_\pm\big\}, \tag{1.15}$$

and such a perfect derivative leads to no contribution in the integration. The pay of this trick is that one needs to drive a strong enough decay rate of ϕ in order to control the nonlinear terms resulting from when $e^{\pm\phi}$ hits $\partial_t f\mp\nabla_x\phi\cdot\nabla_v f$. As remarked by Guo, the strong decay rate of ϕ in [11] is a consequence of the periodic box. To get the sufficient decay rate of ϕ in the whole space, Strain and Zhu [19] assumed additionally that the $L^2_vL^1_x$ norm of the initial data is small so as to use the linear decay analysis. Later in [20], the author showed that the cancelation property between two species coupled with the Poisson equation yield the electric potential decaying at the same rates as the periodic box case, which then allows one to remove the $L^2_vL^1_x$ assumption in [19]. Guo's observation works for all forces given by a potential, but it does not work for the Vlasov–Maxwell–Landau system. On the other hand, Duan, Yang and Zhao [4] used a different approach, which was previously developed in Duan, Yang and Zhao [5,6] for the Vlasov–Poisson–Boltzmann system with general angular cutoff potentials, to construct the global solutions to the one-species Vlasov–Poisson–Landau system in the whole space. The key point in [4] is to introduce a time-velocity exponential weight factor $e^{q\langle v\rangle^2/(1+t)^\vartheta}$. The observation is that upon multiplying by $e^{q\langle v\rangle^2/(1+t)^\vartheta}$ one can rewrite

$$e^{\frac{q\langle v\rangle^2}{(1+t)^\vartheta}}\partial_t f=\partial_t\Big\{e^{\frac{q\langle v\rangle^2}{(1+t)^\vartheta}}f\Big\}+\frac{\vartheta q}{(1+t)^{1+\vartheta}}\langle v\rangle^2 e^{\frac{q\langle v\rangle^2}{(1+t)^\vartheta}}f, \tag{1.16}$$

and this induces an artificial dissipation term with the good factor $\langle v\rangle^2$ if $q,\vartheta>0$, which then can be used to control the nonlinear velocity growth if $\nabla_x\phi$ decays faster than $(1+t)^{-(1+\vartheta)}$. This observation works for all forces. Subsequently, Duan [2] used this observation to prove the first result of the global unique solution to the Vlasov–Maxwell–Landau system, and a time-weighted energy method motivated by [13] was employed to overcome the difficulty caused by the regularity-loss feature. The argument in [2] was further adapted by Duan, Liu, Yang and Zhao [7] to construct the global unique solution to the Vlasov–Maxwell–Boltzmann system without angular cutoff. As [19], in order to get the sufficient decay of the solution for closing the energy estimates, [2] also assumed that the $L^2_vL^1_x$ norm of the initial data is small. Motivated by our previous work [20] for the Vlasov–Poisson–Landau system, it is our purpose in this paper to construct global solutions to the Vlasov–Maxwell–Landau system without any low frequency assumption.

For notational simplicity, we use $\|\cdot\|_p$ to denote L^p norms in $\mathbb{R}^3_x\times\mathbb{R}^3_v$ or $\mathbb{R}^3_x$ and we use $|\cdot|_p$ for the L^p norms in $\mathbb{R}^3_v$. Letting $w(v)\ge 1$ be a weight function, we denote $|\cdot|_{2,w}$ for the weighted $L^2(\mathbb{R}^3_v)$ norm and $|\cdot|_{\sigma,w}$ for the weighted norm of (1.12). We will write $\|\cdot\|_{2,w}=\||\cdot|_{2,w}\|_2$, $\|\cdot\|_{\sigma,w}=\||\cdot|_{\sigma,w}\|_2$ and $\|\cdot\|_\sigma=\|\cdot\|_{\sigma,1}$. We use ∇^ℓ with $\ell\in\mathbb{R}$ for the usual spatial derivatives. Letting the multi-indices α and β be $\alpha=[\alpha_1,\alpha_2,\alpha_3]$, $\beta=[\beta_1,\beta_2,\beta_3]$, we define $\partial^\alpha_\beta\equiv\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}$. If each component of θ is not greater than that of $\bar\theta$'s, we denote by $\theta\le\bar\theta$; $\theta<\bar\theta$

means $\theta \le \bar{\theta}$, and $|\theta| < |\bar{\theta}|$. Throughout the paper we let C denote positive universal constants. We will use $A \lesssim B$ ($A \gtrsim B$ and $A \sim B$) if $A \le CB$, and we will use $\partial_t A + B \le D$ if $\partial_t A + C^{-1}B \le D$, etc.

Motivated by [11] and [2], we define the following time-velocity weight

$$w(\alpha,\beta)(v) \equiv e^{\frac{q\langle v\rangle^2}{2(1+t)^{\vartheta}}}\langle v\rangle^{2(l-|\alpha|-|\beta|)}, \quad l \ge |\alpha|+|\beta|,\ 0 \le q \ll 1 \tag{1.17}$$

for a fixed constant $\vartheta \ge 0$. Letting $l \ge m \ge 2$, we define the instant energy by

$$\mathcal{E}_{m;l,q}(t) \sim \sum_{|\alpha|+|\beta|\le m} \left\|\partial^\alpha_\beta f\right\|^2_{2,w(\alpha,\beta)} + \left\|(E,B)\right\|^2_{H^m}, \tag{1.18}$$

and the corresponding dissipation rate by

$$\begin{aligned}\mathcal{D}_{m;l,q}(t) \equiv &\sum_{|\alpha|+|\beta|\le m} \left\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{\sigma,w(\alpha,\beta)} + \sum_{1\le|\alpha|\le m}\left\|\partial^\alpha \mathbf{P}f\right\|^2_2 \\ &+ \|E\|^2_{H^{m-1}} + \|\nabla B\|^2_{H^{m-2}} + \left\|(f_+ - f_-)\right\|^2_\sigma,\end{aligned} \tag{1.19}$$

and

$$\mathcal{F}_{m;l,q}(t) \equiv \sum_{|\alpha|+|\beta|\le m}\left\|\langle v\rangle\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{\sigma,w(\alpha,\beta)}. \tag{1.20}$$

Note that there is a cascade of velocity weights in (1.18)–(1.20) so that fewer derivatives of f demand stronger velocity weights. We also define the instant energy and dissipation rate for the pure spatial derivatives of the solution without the velocity weight:

$$\mathcal{E}_m(t) \sim \sum_{k\le m}\left\|\nabla^k f\right\|^2_2 + \left\|(E,B)\right\|^2_{H^m} \tag{1.21}$$

and

$$\begin{aligned}\mathcal{D}_m(t) \equiv &\sum_{k\le m}\left\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right\|^2_\sigma + \sum_{1\le k\le m}\left\|\nabla^k\mathbf{P}f\right\|^2_2 \\ &+ \|E\|^2_{H^{m-1}} + \|\nabla B\|^2_{H^{m-2}} + \left\|(f_+ - f_-)\right\|^2_\sigma.\end{aligned} \tag{1.22}$$

Our main result is as follows.

Theorem 1.1. *Let $m \ge 9$, $l \ge m + 1/4$ and $0 < q \ll 1$. Fix the constants $0 < \vartheta \le 1/5$ and $0 < \epsilon_0 \le 7/5$. Assume that f_0 satisfies $F_{0,\pm}(x,v) = \mu + \sqrt{\mu} f_{0,\pm}(x,v) \ge 0$, and that (f_0, E_0, B_0) satisfy the compatibility conditions $\nabla_x \cdot E_0 = \int_{\mathbb{R}^3}\sqrt{\mu}(f_{0,+} - f_{0,-})dv$ and $\nabla_x \cdot B_0 = 0$. There exists a sufficiently small $M > 0$ such that if $\mathcal{E}_{m;l,q}(0) \le M$, then there exists a unique global solution $(f(t,x,v), E(t,x), B(t,x))$ to the Vlasov–Maxwell–Landau system (1.7) with $F_\pm(t,x,v) = \mu + \sqrt{\mu} f_\pm(t,x,v) \ge 0$ and*

$$\begin{aligned}&\mathcal{E}_m(t) + \mathcal{E}_{m-1;l,q}(t) + (1+t)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{m;l,q}(t) \\ &+ \int_0^t \left\{\mathcal{D}_m(\tau) + \mathcal{D}_{m-1;l,q}(\tau) + \frac{\vartheta q}{(1+\tau)^{1+\vartheta}}\mathcal{F}_{m-1;l,q}(\tau)\right\}d\tau \\ &+ \int_0^t (1+\tau)^{-\frac{1+\epsilon_0}{2}}\left\{\mathcal{D}_{m;l,q}(\tau) + \frac{\vartheta q}{(1+\tau)^{1+\vartheta}}\mathcal{F}_{m;l,q}(\tau)\right\}d\tau \le C\mathcal{E}_{m;l,q}(0).\end{aligned} \tag{1.23}$$

Furthermore, for $k = 0, 1, 2, 3$,

$$\left\|\nabla^k f(t)\right\|_2 + \left\|\nabla^k(E,B)(t)\right\|_2 \le C\sqrt{\mathcal{E}_{m;l,q}(0)}(1+t)^{-\frac{k}{2}}, \tag{1.24}$$

and for $k = 0, 1, 2$,

$$\left\|\nabla^k(f_+ - f_-)(t)\right\|_2 + \left\|\nabla^k E(t)\right\|_2 \le C\sqrt{\mathcal{E}_{m;l,q}(0)}(1+t)^{-\frac{k+1-\varepsilon}{2}}, \tag{1.25}$$

where $\varepsilon = 3/(4l-7)$.

The followings are several remarks for our theorem.

Remark 1.2. $k = 3$ in the decay estimates (1.24) and $k = 2$ in (1.25) are the largest derivative indexes we can derive. But fortunately, it is just enough for our use. $\epsilon_0 > 0$ in the energy estimates (1.23) is introduced to overcome the difficulty caused by the regularity-loss of E. The extra $1/4$ in the velocity moment l is crucial for guaranteeing the derivation of (1.23); otherwise, the restriction (5.12) would change to be $(1+\epsilon_0)/2 + 1 + \vartheta \le 5/4 - \varepsilon/2$. Notice that we have no choice of $\epsilon_0 > 0$ for this new restriction, and hence we may fail to close our energy estimates.

Remark 1.3. $m = 9$ is the lowest spatial regularity we require in our analysis. We remark that if we impose some kind of low frequency assumption on the initial data, then we can lower the regularity. For example, if the initial data is small in $L^2_v \dot{H}^{-s}_x$ with $s > 0$, then we need $m \ge 8$; if $s > 1$, then we need $m \ge 6$. Note that $L^p \subset \dot{H}^{-s}$ with $s = 3(1/p - 1/2) \in [0, 3/2)$ for $p \in (1, 2]$.

Remark 1.4. We can consider the generalized Landau operator with, see [12,1,9],

$$\Phi(v) = \frac{1}{|v|^{\gamma+2}}\left(I - \frac{v \otimes v}{|v|^2}\right), \quad \gamma \ge -3. \tag{1.26}$$

One may conclude from our proof that the global unique solution to the Vlasov–Maxwell–Landau system exists for all $\gamma \ge -3$. If $\gamma \ge -2$, then we can take $\varepsilon = 0$ in (1.25). We also believe that our observation can be used to remove the $L^2_v L^1_x$ assumption of the initial data in [7]. Furthermore, as in [20] if we assume further that the initial data belongs to $L^2_v \dot{H}^{-s}_x$ with $s \in [0, 3/2)$, then the decay rates in (1.24)–(1.25) can be enhanced with the index $s/2$. These improve the decay result of the Vlasov–Maxwell–Boltzmann system (cf. $\gamma \ge -1$) in [3] since we do not require the $L^2_v \dot{H}^{-s}_x$ norm of the initial data to be small and the L^∞ norm of E decays faster than that of [3].

Theorem 1.1 will be proved in Section 5 by combining the energy estimates of Section 3 and the decay estimates of Section 4. We will prove in Section 3 by the nonlinear weighted energy method that

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}_{m;l,q} + \mathcal{D}_{m;l,q} + \frac{\vartheta q}{(1+t)^{1+\vartheta}}\mathcal{F}_{m;l,q} \\ &\quad \lesssim \left(\|E\|_\infty + \|\nabla B\|_\infty^2\right)\mathcal{F}_{m;l,q} + \left\|\nabla^m E\right\|_2\sqrt{\mathcal{D}_m}. \end{aligned} \tag{1.27}$$

The derivation of the energy estimates (1.27) is very delicate, and the great advantage of the weight (1.17) is intensively exploited as in [10]. Note that (1.27) is a refined one of [2], especially for the factor $\|E\|_\infty + \|\nabla B\|_\infty^2$ which decays faster than $\|\nabla(E, B)\|_{H^{N_0}}$ with some integer N_0 obtained in [2] (this forces the author in [2] to impose the $L^2_v L^1_x$ assumption). The reason for this refinement is as follows. From the linear $L^2 - L^2$ decay estimates [3,2] one may expect the L^2 decay rate of the whole solution is as stated in (1.24). By the Sobolev interpolation, $\|\nabla B\|_\infty^2$ then decays at the rate of $(1+t)^{-5/2}$ which is more than sufficient for closing the B term. But $\|E\|_\infty$ only decays at the rate of $(1+t)^{-3/4}$ which is definitely not fast enough. Our intuition is that since E is included in the dissipation (cf. [15,3,2]), it would suggest that E may decay at a faster rate. In fact, one can see from (1.25) that the decay of E can be improved with the index $(1-\varepsilon)/2$. This thus can close the E term since $\varepsilon > 0$ can be made small.

The decay estimates (1.24)–(1.25) will be established in Section 4 by the energy method. As noted in [20], the real thing we need to close the first term in (1.27) is a strong decay rate of (E, B) rather than the whole solution! Look back at the system (1.7), and we note that (E, B) in the Maxwell system depends only on $f_+ - f_-$ and also that there are some cancelations between the "+" and "−" equations. This motivates us to consider the sum and difference of f_+ and f_-:

$$f_1 = f_+ + f_- \quad \text{and} \quad f_2 = f_+ - f_-. \tag{1.28}$$

The Vlasov–Maxwell–Landau system (1.7) can be equivalently rewritten as

$$\partial_t f_1 + v\cdot\nabla_x f_1 + \mathcal{L}_1 f_1 = \Gamma_*(f_1,f_1) + E\cdot(v-\nabla_v)f_2 - v\times B\cdot\nabla_v f_2,$$
$$\partial_t f_2 + v\cdot\nabla_x f_2 - 4E\cdot v\sqrt{\mu} + \mathcal{L}_2 f_2 = \Gamma_*(f_1,f_2) + E\cdot(v-\nabla_v)f_1 - v\times B\cdot\nabla_v f_1,$$
$$\partial_t E - \nabla_x\times B = -\int_{\mathbb{R}^3} v\sqrt{\mu} f_2\,dv,\quad \nabla_x\cdot E = \int_{\mathbb{R}^3}\sqrt{\mu} f_2\,dv,$$
$$\partial_t B + \nabla_x\times E = 0,\quad \nabla_x\cdot B = 0, \tag{1.29}$$

where the linearized collision operators are given by

$$\mathcal{L}_1 g \equiv -\frac{2}{\sqrt{\mu}}\big\{Q(\mu,\sqrt{\mu}g) + Q(\sqrt{\mu}g,\mu)\big\} \tag{1.30}$$

and

$$\mathcal{L}_2 g \equiv -\frac{2}{\sqrt{\mu}}Q(\mu,\sqrt{\mu}g), \tag{1.31}$$

and the nonlinear collision operator is given by

$$\Gamma_*(g,h) \equiv \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}g,\sqrt{\mu}h). \tag{1.32}$$

Notice that $[\mathcal{L}_1 f_1, \mathcal{L}_2 f_2]$ is equivalent to Lf, and their null spaces are

$$N(\mathcal{L}_1) \equiv \operatorname{span}\big\{\sqrt{\mu}, v\sqrt{\mu}, |v|^2\sqrt{\mu}\big\},\quad \text{but } N(\mathcal{L}_2) \equiv \operatorname{span}\{\sqrt{\mu}\}. \tag{1.33}$$

Let $\mathbf{P}_i$ be the L^2_v orthogonal projection on the null space of $\mathcal{L}_i$ respectively, then

$$\langle \mathcal{L}_i g, g\rangle \ge \delta_0 \big|\{\mathbf{I}-\mathbf{P}_i\}g\big|_\sigma^2,\quad i=1,2. \tag{1.34}$$

Then our key observation in [20] is that we can include the full term $\|f_2\|_\sigma^2$ in the dissipation: $\mathcal{L}_2$ controls the microscopic part $\{\mathbf{I}-\mathbf{P_2}\}f_2$ by (1.34); while the Poisson equation can control the hydrodynamic part $\mathbf{P_2}f_2$!

To get the sufficiently fast decay of (E,B) from the system (1.29) by the energy method is much more subtle than that of [20] for the Vlasov–Poisson–Landau system. In [20], we can plug out the subsystem for f_2 and ϕ to derive a differential inequality similar as [11] which allow us to extract a fast enough decay rate of ϕ. But as noted in [3] the magnetic field B decays at the lowest rate, so at the first stage we could not expect to use the subsystem for f_2 and (E,B) to derive the faster decay. However, as far as our strategy of showing the decay by the energy method the system (1.29) is still more effective than the original system (1.7): when doing the energy estimates for (1.7) we will encounter the typical difficult term $B\times\mathbf{P}f\times\mathbf{P}f$ since both B and $\mathbf{P}f$ are not included in the dissipation; but when doing the energy estimates for (1.29), the situation is a bit better since now the term changes to be $B\times\mathbf{P}_1 f_1\times\mathbf{P}_2 f_2$ but $\mathbf{P}_2 f_2$ is included in the dissipation! This observation makes us be able to derive the decay (1.24) in Proposition 4.2. With (1.24) in hand, we then further explore the structure of the subsystem for f_2 and E (c.f. (4.32)) to derive the faster decay (1.25) in Proposition 4.4.

Note that the second term in (1.27) is due to the incoordination of the regularity-loss of E with the weighted energy method. In order to get around this difficulty, as in [2] a time-weighted energy estimates with the time rate of negative power will be employed in Section 5. The pay of this trick is that $\mathcal{E}_{m;l,q}(t)$ may increase in time as stated in (1.23).

The rest of our paper is organized as follows. In Section 2, we recall some useful estimates for the collision operators and collect some analytic tools. In Section 3, we establish the nonlinear energy estimates. In Section 4, we establish the decay estimates. Finally, we complete the proof of Theorem 1.1 in Section 5.

2. Preliminary

In this section, we use $\widetilde{L}$ to uniformly denote the linear collision operators L, $\mathcal{L}_1$ and $\mathcal{L}_2$, and we use $\widetilde{\Gamma}$ to denote the nonlinear collision operators Γ and Γ_*. We first recall the basic property of the linear collision operator $\widetilde{L}$.

Lemma 2.1. *We have* $\langle \widetilde{L}g, h\rangle = \langle g, \widetilde{L}h\rangle$, $\langle \widetilde{L}g, g\rangle \geq 0$, *and* $\widetilde{L}g = 0$ *if and only if* $g = \widetilde{P}g$, *where* $\widetilde{P}$ *is the* L^2_v *orthogonal projection onto the null space of* $\widetilde{L}$, *correspondingly. Moreover,*

$$\langle \widetilde{L}g, g\rangle \gtrsim \big|\{I - \widetilde{P}\}g\big|_\sigma^2. \tag{2.1}$$

Proof. We refer to Lemma 5 in [9] and Lemma 2.1 in [20]. □

Next we recall the weighted estimates for $\widetilde{L}$.

Lemma 2.2. *Let* $w = w(\alpha, \beta)$ *in* (1.17). *For any small* $\eta > 0$, *there exists* $C_\eta > 0$ *such that*

$$\big\langle \partial^\alpha \widetilde{L}g, w^2(\alpha, 0)\partial^\alpha g\big\rangle \gtrsim \big(1 - q^2 - \eta\big)\big|\partial^\alpha g\big|^2_{\sigma, w(\alpha,0)} - C_\eta\big|\partial^\alpha g\big|^2_\sigma, \tag{2.2}$$

and for $\beta \neq 0$

$$\big\langle \partial^\alpha_\beta \widetilde{L}g, w^2(\alpha, \beta)\partial^\alpha_\beta g\big\rangle \gtrsim \big|\partial^\alpha_\beta g\big|^2_{\sigma, w(\alpha,\beta)} - \eta \sum_{|\beta_1| = |\beta|} \big|\partial^\alpha_{\beta_1} g\big|^2_{\sigma, w(\alpha,\beta)} - C_\eta \sum_{|\beta_1| < |\beta|} \big|\partial^\alpha_{\beta_1} g\big|^2_{\sigma, w(\alpha,\beta_1)} \tag{2.3}$$

Proof. We refer to Lemmas 8 and 9 in [18]. □

We then recall the following refined estimates for the nonlinear collision operator $\widetilde{\Gamma}$.

Lemma 2.3. *Let* $w = w(\alpha, \beta)$ *in* (1.17). *Then we have*

$$\big\langle \partial^\alpha_\beta \widetilde{\Gamma}[g_1, g_2], w^2 \partial^\alpha_\beta g_3\big\rangle \lesssim \sum_{\substack{\alpha_1 \leq \alpha \\ \bar{\beta} \leq \beta_1 \leq \beta}} \big|\mu^\delta \partial^{\alpha_1}_{\bar{\beta}} g_1\big|_2 \big|\partial^{\alpha - \alpha_1}_{\beta - \beta_1} g_2\big|_{\sigma, w} \big|\partial^\alpha_\beta g_3\big|_{\sigma, w}. \tag{2.4}$$

Hereafter $\delta > 0$ *is a sufficiently small universal number.*

Proof. We refer to Lemma 2.3 in [20]. □

In what follows, we will collect the analytic tools which will be used in this paper. The first one is the Sobolev interpolation among the spatial regularity:

Lemma 2.4. *Let* $2 \leq p \leq \infty$ *and* $k, \ell, m \in \mathbb{R}$, *then we have*

$$\big\|\nabla^k g\big\|_p \lesssim \big\|\nabla^\ell g\big\|_2^\theta \big\|\nabla^m g\big\|_2^{1-\theta}. \tag{2.5}$$

Here $0 \leq \theta \leq 1$ *(if* $p = +\infty$, *then we require that* $0 < \theta < 1$*) and* k *satisfies*

$$k + 3\left(\frac{1}{2} - \frac{1}{p}\right) = \ell\theta + m(1 - \theta). \tag{2.6}$$

Proof. For the case $2 \leq p < \infty$, we refer to Lemma 2.4 in [20]; for the case $p = \infty$, we refer to Exercise 6.1.2 in [8]. □

In many places, we will use the Minkowski's integral inequality to interchange the orders of integration over x and v without mentioning.

Lemma 2.5. *For* $1 \leq p \leq q \leq \infty$, *we have*

$$\|g\|_{L^q_z L^p_y} \leq \|g\|_{L^p_y L^q_z}. \tag{2.7}$$

Proof. It is standard, see [8] for instance. □

The following product estimates of ∇^k will simply our some calculations.

Lemma 2.6. *Let $k \in \mathbb{Z}_+$, then we have*

$$\left\|\nabla^k(gh)\right\|_{p_0} \lesssim \|g\|_{p_1}\left\|\nabla^k h\right\|_{p_2} + \left\|\nabla^k g\right\|_{p_3}\|h\|_{p_4} \tag{2.8}$$

with $p_0, p_2, p_3 \in (1, +\infty)$ and $1/p_0 = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$.

Proof. We refer to Lemma 3.1 in [14]. □

The last one we need is the basic time decay estimates of certain integrals.

Lemma 2.7. *Suppose that $0 \le \varepsilon < 1$, $\lambda > 0$ and $\theta \ge 0$, then*

$$\int_0^t e^{-\lambda((1+t)^{1-\varepsilon}-(1+\tau)^{1-\varepsilon})}(1+\tau)^{-\theta}d\tau \le C_{\lambda,\theta,\varepsilon}(1+t)^{-\theta+\varepsilon}. \tag{2.9}$$

Proof. We refer to Lemma 18 in [19]. □

3. Nonlinear energy estimates

In this section, we will do a refined energy estimates for the solution to the Vlasov–Maxwell–Landau system (1.7). We begin with the estimates for the spatial derivatives of the solution without the velocity weight. We use $\int g$ to denote the integration of g over $\mathbb{R}^3_x \times \mathbb{R}^3_v$ or $\mathbb{R}^3_x$.

Proposition 3.1. *Let $m \ge 4$. Assume that $\mathcal{E}_m \le M$ is small, then we have*

$$\frac{d}{dt}\mathcal{E}_m + \mathcal{D}_m \lesssim \sqrt{M}\mathfrak{D}_{m-1;m,0} + \|E\|_\infty^2\mathcal{F}_{m;m+1/4,0} + \|\nabla B\|_\infty^2\mathcal{F}_{m-1;m-1/4,0}. \tag{3.1}$$

Proof. Applying ∇^k with $k \le m$ to the first equation in (1.7) and then taking the $L^2_xL^2_v$ inner product with $\nabla^k f$, using the Maxwell system and the collision invariant property, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\left\|\nabla^k f\right\|_2^2 + 2\left\|\nabla^k(E,B)\right\|_2^2\right\} + \int\left\langle L\nabla^k f, \nabla^k f\right\rangle \\
&\quad = \int \nabla^k(f,f)\nabla^k\{\mathbf{I}-\mathbf{P}\}f + \int q_0 E\cdot v\nabla^k f\nabla^k f \\
&\qquad + \sum_{0\ne j\le k} C_k^j\int q_0\nabla^j E\cdot(v-\nabla_v)\nabla^{k-j}f\nabla^k f \\
&\qquad - \sum_{0\ne j\le k} C_k^j\int q_0 v\times\nabla^j B\cdot\nabla_v\nabla^{k-j}f\nabla^k f := \sum_{i=1}^4 I_i.
\end{aligned} \tag{3.2}$$

We now estimate $I_1 - I_4$. For the term I_1, we apply Lemma 2.3 to obtain

$$I_1 \lesssim \sum_{j\le k}\int\left|\nabla^j f\right|_2\left|\nabla^{k-j}f\right|_\sigma\left|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\right|_\sigma. \tag{3.3}$$

When $k=0$, we take $L^2 - L^\infty - L^2$ in (3.3) to have an upper bound of

$$\|f\|_2\sum_{\ell=1,2}\left\|\nabla^\ell f\right\|_\sigma\left\|\{\mathbf{I}-\mathbf{P}\}f\right\|_\sigma \lesssim \sqrt{\mathcal{E}_2}\mathcal{D}_2. \tag{3.4}$$

When $k \ge 1$, if $j=0$ we take $L^\infty - L^2 - L^2$; if $j=k$ we take $L^2 - L^\infty - L^2$; if $1 \le j \le k-1$ (it only occurs when $k \ge 2$) we take $L^3 - L^6 - L^2$ in (3.3) respectively to have an upper bound of

$$\sum_{\ell=1,2}\|\nabla^\ell f\|_2\|\nabla^k f\|_\sigma\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\|_\sigma+\|\nabla^k f\|_2\sum_{\ell=1,2}\|\nabla^\ell f\|_\sigma\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\|_\sigma$$
$$+\sum_{\ell=0,1}\|\nabla^\ell\nabla^j f\|_2\|\nabla\nabla^{k-j} f\|_\sigma\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\|_\sigma\lesssim\sqrt{\mathcal{E}_m}\mathcal{D}_m. \tag{3.5}$$

For the term I_2, when $k=0$, we take $L^2-L^3-L^6$ to have

$$I_2\lesssim\|E\|_2\sum_{\ell=0,1}\|\nabla^\ell f\|_2\|\langle v\rangle^{3/2}\nabla f\|_\sigma$$
$$\lesssim\sqrt{\mathcal{D}_2}\sqrt{\mathcal{E}_2}\sqrt{\mathcal{D}_{2;2,0}}; \tag{3.6}$$

When $k\ge 1$, we take $L^\infty-L^2-L^2$ to have

$$I_2\lesssim\|E\|_\infty\|\langle v\rangle^{3/2}\nabla^k f\|_2\|\nabla^k f\|_\sigma$$
$$\lesssim\|E\|_\infty\sqrt{\mathcal{F}_{m;m+1/4,0}}\sqrt{\mathcal{D}_m}. \tag{3.7}$$

Notice that the term I_4 only occurs when $k\ge 1$, and we perform an integration by parts in v to have

$$I_4=\sum_{0\ne j\le k}C_k^j\int\nabla^{k-j}f q_0 v\times\nabla^j B\cdot\nabla_v\nabla^k f$$
$$\lesssim\sum_{0\ne j\le k}\int|\nabla^j B||\langle v\rangle^{5/2}\nabla^{k-j}f|_2|\nabla^k f|_\sigma. \tag{3.8}$$

When $k=1$, then so $j=1$, we take $L^3-L^6-L^2$ in (3.8) to have an upper bound of

$$\sum_{\ell=0,1}\|\nabla^\ell\nabla B\|_2\|\langle v\rangle^3\nabla f\|_\sigma\|\nabla f\|_\sigma\lesssim\sqrt{\mathcal{E}_2}\sqrt{\mathcal{D}_{2;5/2,0}}\sqrt{\mathcal{D}_2}. \tag{3.9}$$

For $k\ge 2$, if $j=1$, we take $L^\infty-L^2-L^2$ in (3.8) to have an upper bound of

$$\|\nabla B\|_\infty\|\langle v\rangle^{5/2}\nabla^{k-1}f\|_2\|\nabla^k f\|_\sigma\lesssim\|\nabla B\|_\infty\sqrt{\mathcal{F}_{m-1;m-1/4,0}}\sqrt{\mathcal{D}_m}. \tag{3.10}$$

When $k=2$, the remaining case is $j=2$, we take $L^3-L^6-L^2$ to have an upper bound of

$$\sum_{\ell=0,1}\|\nabla^\ell\nabla^2 B\|_2\|\langle v\rangle^3\nabla f\|_\sigma\|\nabla^2 f\|_\sigma\lesssim\sqrt{\mathcal{E}_3}\sqrt{\mathcal{D}_{2;5/2,0}}\sqrt{\mathcal{D}_2}. \tag{3.11}$$

For $k\ge 3$, if $j=2$, we take $L^\infty-L^2-L^2$ in (3.8) to have an upper bound of

$$\sum_{\ell=1,2}\|\nabla^\ell\nabla^2 B\|_2\|\langle v\rangle^3\nabla^{k-2}f\|_\sigma\|\nabla^k f\|_\sigma\lesssim\sqrt{\mathcal{E}_4}\sqrt{\mathcal{D}_{m-2;m-1/2,0}}\sqrt{\mathcal{D}_m}. \tag{3.12}$$

When $k=3$, the remaining case is $j=3$, we take $L^3-L^6-L^2$ in (3.8) to have an upper bound of

$$\sum_{\ell=0,1}\|\nabla^\ell\nabla^3 B\|_2\|\langle v\rangle^3\nabla f\|_\sigma\|\nabla^3 f\|_\sigma\lesssim\sqrt{\mathcal{E}_4}\sqrt{\mathcal{D}_{2;5/2,0}}\sqrt{\mathcal{D}_3}. \tag{3.13}$$

Now when $k\ge 4$, if $j=k$, we take $L^2-L^\infty-L^2$; the remaining cases are of $3\le j\le k-1$, and we take $L^3-L^6-L^2$ in (3.8), respectively, to have an upper bound of

$$\|\nabla^k B\|_2\sum_{\ell=1,2}\|\langle v\rangle^3\nabla^\ell f\|_\sigma\|\nabla^k f\|_\sigma+\sum_{\ell=0,1}\|\nabla^\ell\nabla^j B\|_2\|\langle v\rangle^3\nabla\nabla^{k-j}f\|_\sigma\|\nabla^k f\|_\sigma$$
$$\lesssim\sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{2;7/2,0}}\sqrt{\mathcal{D}_m}+\sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{m-2;m-1/2,0}}\sqrt{\mathcal{D}_m}. \tag{3.14}$$

Hence, we conclude that

$$I_4 \lesssim \sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{m-2;m-1/2,0}}\sqrt{\mathcal{D}_m} + \|\nabla B\|_\infty \sqrt{\mathcal{F}_{m-1;m-1/4,0}}\sqrt{\mathcal{D}_m}. \tag{3.15}$$

Similarly, for the rest term I_3, we again perform an integration by parts in v to have

$$\begin{aligned} I_3 &= \sum_{0\neq j\le k} C_k^j \int \nabla^{k-j} f q_0 \nabla^j E\cdot (v+\nabla_v)\nabla^k f \\ &\lesssim \sum_{0\neq j\le k} \int |\nabla^j E| \big|\langle v\rangle^{3/2}\nabla^{k-j} f\big|_2 \big|\nabla^k f\big|_\sigma. \end{aligned} \tag{3.16}$$

For $k\ge 2$, if $j=1$, we take $L^\infty - L^2 - L^2$ in (3.16) to have an upper bound of

$$\sum_{\ell=1,2} \|\nabla^\ell \nabla E\|_2 \|\langle v\rangle^2 \nabla^{k-1} f\|_\sigma \|\nabla^k f\|_\sigma \lesssim \sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{m-1;m,0}}\sqrt{\mathcal{D}_m}. \tag{3.17}$$

For the remaining cases, we may argue in the same as that for B, together with the bound (3.17), to conclude that

$$I_3 \lesssim \sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{m-1;m,0}}\sqrt{\mathcal{D}_m}. \tag{3.18}$$

Collecting the estimates for $I_1 - I_4$, together with Lemma 2.1 and by Cauchy's inequality, we deduce from (3.2) that

$$\begin{aligned} &\frac{d}{dt}\sum_{k\le m}\big\{\|\nabla^k f\|_2^2 + 2\|\nabla^k (E,B)\|_2^2\big\} + \sum_{k\le m}\|\nabla^k\{\mathbf{I}-\mathbf{P}\} f\|_\sigma^2 \\ &\quad\lesssim (\sqrt{\mathcal{E}_m}+\eta)\mathcal{D}_m + \sqrt{\mathcal{E}_m}\mathcal{D}_{m-1;m,0} + C_\eta\big(\|E\|_\infty^2 \mathcal{F}_{m;m+1/4,0} + \|\nabla B\|_\infty^2 \mathcal{F}_{m-1;m-1/4,0}\big). \end{aligned} \tag{3.19}$$

Notice that the dissipation estimates in (3.19) only controls the microscopic part $\{\mathbf{I}-\mathbf{P}\}f$, but it follows from [10,15, 3,2] that there exists a function $\mathcal{G}_m(t)$ with $|\mathcal{G}_m(t)| \lesssim \mathcal{E}_m(t)$ such that

$$\frac{d}{dt}\mathcal{G}_m + \sum_{1\le k\le m}\|\nabla^k \mathbf{P} f\|_2^2 + \|E\|_{H^{m-1}}^2 + \|\nabla B\|_{H^{m-2}}^2 \lesssim \sum_{k\le m}\|\nabla^k\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2 + \mathcal{E}_m\mathcal{D}_m. \tag{3.20}$$

A suitable linear combination of (3.19) and (3.20), taking η sufficiently small and since $\mathcal{E}_m \le M$, implies that there is an instant energy $\mathcal{E}_m(t)$ satisfying (1.21) such that (3.1) holds. We may refer to Lemma 4.3 later for the reason that we can include $\|(f_+ - f_-)\|_\sigma^2$ in the dissipation. □

Next, we turn to the energy estimates with the velocity weight, and we first deal with the pure spatial derivatives of the solution.

Lemma 3.2. *Let $l\ge m\ge 4$. Assume that $\mathcal{E}_m \le M$ is small. Let $w = w(\alpha, 0)$ in (1.17). Then for $1\le|\alpha|\le m$, we have that for any $\eta>0$*

$$\begin{aligned} &\frac{d}{dt}\|\partial^\alpha f\|_{2,w}^2 + \|\partial^\alpha f\|_{\sigma,w}^2 + \frac{\vartheta q}{(1+t)^{1+\vartheta}}\|\partial^\alpha f\|_{2,w}^2 \\ &\quad\lesssim (\sqrt{M}+\eta)\mathcal{D}_{m;l,q} + \|E\|_\infty \mathcal{F}_{m;l,q} + \|\nabla B\|_\infty^2 \mathcal{F}_{m-1;l,q} + \|\partial^\alpha E\|\|\partial^\alpha f\|_\sigma + \|\partial^\alpha f\|_\sigma^2. \end{aligned} \tag{3.21}$$

Proof. Applying ∂^α with $1\le|\alpha|\le m$ to the first equation in (1.7) and then taking the $L_x^2 L_v^2$ inner product with $w^2\partial^\alpha f$, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\partial^\alpha f\|_{2,w}^2 + \frac{1}{2}\frac{\vartheta q}{(1+t)^{1+\vartheta}}\|\partial^\alpha f\|_{2,w}^2 + \int\langle L\partial^\alpha f, w^2\partial^\alpha f\rangle \\ &\quad= 2\int \partial^\alpha E\cdot v\sqrt{\mu} q_1 w^2 \partial^\alpha f + \int\left[1+\frac{q}{(1+t)^\vartheta} + \frac{2(l-|\alpha|)}{1+|v|^2}\right] q_0 E\cdot v\partial^\alpha f w^2 \partial^\alpha f \\ &\qquad + \int \partial^\alpha \Gamma(f,f) w^2\partial^\alpha f + \sum_{0\neq\alpha_1\le\alpha} C_\alpha^{\alpha_1}\int q_0 \partial^{\alpha_1} E\cdot(v-\nabla_v)\partial^{\alpha-\alpha_1} f w^2 \partial^\alpha f \end{aligned}$$

$$-\sum_{0\neq\alpha_1\leq\alpha} C_\alpha^{\alpha_1}\int q_0 v\times\partial^{\alpha_1}B\cdot\nabla_v\partial^{\alpha-\alpha_1}f w^2\partial^\alpha f := \sum_{i=1}^5 I_i. \tag{3.22}$$

First, by Lemma 2.2, we have

$$\int\langle L\partial^\alpha f, w^2\partial^\alpha f\rangle dx \gtrsim \|\partial^\alpha f\|_{\sigma,w}^2 - C\|\partial^\alpha f\|_\sigma^2. \tag{3.23}$$

We now estimate $I_1 - I_5$. Clearly,

$$I_1 \lesssim \|\partial^\alpha E\|\|\partial^\alpha f\|_\sigma, \tag{3.24}$$

and

$$I_2 \lesssim \int |E|\left|\langle v\rangle^{1/2}\partial^\alpha f\right|_{2,w} \lesssim \|E\|_\infty\mathcal{F}_{m;l,q}. \tag{3.25}$$

Next, we apply Lemma 2.3 to obtain

$$I_3 \lesssim \sum_{\alpha_1\leq\alpha}\int\left|\partial^{\alpha_1}f\right|_2\left|\partial^{\alpha-\alpha_1}f\right|_{\sigma,w}\left|\partial^\alpha f\right|_{\sigma,w}. \tag{3.26}$$

If $\alpha_1 = 0$, we take $L^\infty - L^2 - L^2$ in (3.26) to have an upper bound of

$$\sum_{1\leq|\gamma|\leq 2}\|\partial^\gamma f\|_2\|\partial^\alpha f\|_{\sigma,w}^2 \lesssim \sqrt{\mathcal{E}_2}\mathcal{D}_{m;l,q}. \tag{3.27}$$

When $|\alpha| = 1$, the remaining case is $\alpha_1 = \alpha$, and we take $L^3 - L^6 - L^2$ in (3.26) to have an upper bound of

$$\sum_{|\gamma|\leq 1}\|\partial^\gamma\partial^\alpha f\|_2\sum_{|\gamma|=1}\|\partial^\gamma f\|_{\sigma,w}\|\partial^\alpha f\|_{\sigma,w} \lesssim \sqrt{\mathcal{E}_2}\mathcal{D}_{2;l,q}. \tag{3.28}$$

Here we have used the fact $w(\alpha,0) = w(\gamma,0)$ for $|\gamma| = 1$. This concludes the case $|\alpha| = 1$.

Now when $|\alpha| \geq 2$, if $\alpha_1 = \alpha$, we take $L^2 - L^\infty - L^2$; the remaining cases are of $1 \leq |\alpha_1| \leq |\alpha| - 1$, and we take $L^3 - L^6 - L^2$ in (3.26), respectively, to have an upper bound of

$$\begin{aligned}&\|\partial^\alpha f\|_2\sum_{1\leq|\gamma|\leq 2}\|\partial^\gamma f\|_{\sigma,w}\|\partial^\alpha f\|_{\sigma,w} + \sum_{|\gamma|\leq 1}\|\partial^\gamma\partial^{\alpha_1}f\|_2\sum_{|\gamma|=1}\|\partial^\gamma\partial^{\alpha-\alpha_1}f\|_{\sigma,w}\|\partial^\alpha f\|_{\sigma,w}\\ &\quad\lesssim \sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{2;l,q}}\sqrt{\mathcal{D}_{m;l,q}} + \sqrt{\mathcal{E}_m}\mathcal{D}_{m;l,q}.\end{aligned} \tag{3.29}$$

Here we have used the facts $w(\alpha,0) \leq w(\gamma,0)$ for $1 \leq |\gamma| \leq 2$ and $w(\alpha,0) \leq w(\alpha-\alpha_1+\gamma,0)$ for $|\alpha_1| \geq 1$ and $|\gamma| = 1$. This concludes the case $|\alpha| \geq 2$. Hence we conclude that

$$I_3 \lesssim \sqrt{\mathcal{E}_m}\mathcal{D}_{m;l,q}. \tag{3.30}$$

For the term I_5, we perform an integration by parts in v to have

$$\begin{aligned}I_5 &= \sum_{0\neq\alpha_1\leq\alpha} C_\alpha^{\alpha_1}\int w^2\partial^{\alpha-\alpha_1}f q_0 v\times\partial^{\alpha_1}B\cdot\nabla_v\partial^\alpha f\\ &\lesssim \sum_{0\neq\alpha_1\leq\alpha}\int\left|\partial^{\alpha_1}B\right|\left|\langle v\rangle^{5/2}\partial^{\alpha-\alpha_1}f\right|_{2,w}\left|\partial^\alpha f\right|_{\sigma,w}.\end{aligned} \tag{3.31}$$

When $|\alpha| = 1$, then so $\alpha_1 = \alpha$, we use the split $f = \mathbf{P}f + \{\mathbf{I}-\mathbf{P}\}f$ in the factor $|\langle v\rangle^{5/2}f|_{2,w}$ in (3.31) and then take $L^2 - L^\infty - L^2$ and $L^\infty - L^2 - L^2$, respectively, to have an upper bound of

$$\begin{aligned}&\|\partial^\alpha B\|_2\sum_{1\leq|\gamma|\leq 2}\|\partial^\gamma\mathbf{P}f\|_2\|\partial^\alpha f\|_{\sigma,w} + \|\partial^\alpha B\|_\infty\|\langle v\rangle^{5/2}\{\mathbf{I}-\mathbf{P}\}f\|_{2,w}\|\partial^\alpha f\|_{\sigma,w}\\ &\quad\lesssim \sqrt{\mathcal{E}_2}\sqrt{\mathcal{D}_2}\sqrt{\mathcal{D}_{2;l,q}} + \|\nabla B\|_\infty\sqrt{\mathcal{F}_{2;l,q}}\sqrt{\mathcal{D}_{2;l,q}}.\end{aligned} \tag{3.32}$$

Here we have used the fact $\langle v\rangle^2 w(\alpha,0)=w(0,0)$.

For $|\alpha|\ge 2$, if $|\alpha_1|=1$, we take $L^\infty-L^2-L^2$ in (3.31) to have an upper bound of

$$\left\|\partial^{\alpha_1}B\right\|_\infty\left\|\langle v\rangle^{5/2}\partial^{\alpha-\alpha_1}f\right\|_{2,w}\left\|\partial^\alpha f\right\|_{\sigma,w}\lesssim\|\nabla B\|_\infty\sqrt{\mathcal{F}_{m-1;l,q}}\sqrt{\mathcal{D}_{m;l,q}}. \tag{3.33}$$

Here we have used the fact $\langle v\rangle^2 w(\alpha,0)=w(\alpha-\alpha_1,0)$ for $|\alpha_1|=1$.

When $|\alpha|=2$, the remaining case is $\alpha_1=\alpha$, and we use the split $f=\mathbf{P}f+\{\mathbf{I}-\mathbf{P}\}f$ in the factor $|\langle v\rangle^{5/2}f|_{2,w}$ in (3.31) and then take $L^2-L^\infty-L^2$ and $L^\infty-L^2-L^2$, respectively, to have an upper bound of

$$\begin{aligned}&\left\|\partial^\alpha B\right\|_2\sum_{1\le|\gamma|\le 2}\left\|\partial^\gamma\mathbf{P}f\right\|_\sigma\left\|\partial^\alpha f\right\|_{\sigma,w}+\sum_{1\le|\gamma|\le 2}\left\|\partial^\gamma\partial^\alpha B\right\|_2\left\|\langle v\rangle^3\{\mathbf{I}-\mathbf{P}\}f\right\|_{\sigma,w}\left\|\partial^\alpha f\right\|_{\sigma,w}\\&\quad\lesssim\sqrt{\mathcal{E}_2}\sqrt{\mathcal{D}_2}\sqrt{\mathcal{D}_{2;l,q}}+\sqrt{\mathcal{E}_4}\mathcal{D}_{2;l,q}.\end{aligned} \tag{3.34}$$

Here we have used the fact $\langle v\rangle^3 w(\alpha,0)<w(0,0)$. This concludes the case $|\alpha|=2$.

For $|\alpha|\ge 3$, if $|\alpha_1|=2$, we take $L^\infty-L^2-L^2$ in (3.31) to have an upper bound of

$$\sum_{1\le|\gamma|\le 2}\left\|\partial^\gamma\partial^{\alpha_1}B\right\|_2\left\|\langle v\rangle^3\partial^{\alpha-\alpha_1}f\right\|_{\sigma,w}\left\|\partial^\alpha f\right\|_{\sigma,w}\lesssim\sqrt{\mathcal{E}_4}\sqrt{\mathcal{D}_{m-2;l,q}}\sqrt{\mathcal{D}_{m;l,q}}. \tag{3.35}$$

Here we have used the fact $\langle v\rangle^3 w(\alpha,0)<w(\alpha-\alpha_1,0)$ for $|\alpha_1|=2$.

When $|\alpha|=3$, the remaining case is $\alpha_1=\alpha$, and we take $L^3-L^6-L^2$ in (3.31) to have an upper bound of

$$\sum_{|\gamma|\le 1}\left\|\partial^\gamma\partial^\alpha B\right\|_2\sum_{|\gamma|=1}\left\|\langle v\rangle^3\partial^\gamma f\right\|_{\sigma,w}\left\|\partial^\alpha f\right\|_{\sigma,w}\lesssim\sqrt{\mathcal{E}_4}\sqrt{\mathcal{D}_{2;l,q}}\sqrt{\mathcal{D}_{3;l,q}}. \tag{3.36}$$

Here we have used the fact $\langle v\rangle^3 w(\alpha,0)<w(\gamma,0)$ for $|\gamma|=1$. This concludes the case $|\alpha|=3$.

Now when $|\alpha|\ge 4$, if $\alpha_1=\alpha$, we take $L^2-L^\infty-L^2$; the remaining cases are of $3\le|\alpha_1|\le|\alpha|-1$, and we take $L^3-L^6-L^2$ in (3.31), respectively, to have an upper bound of

$$\begin{aligned}&\left\|\partial^\alpha B\right\|_2\sum_{1\le|\gamma|\le 2}\left\|\langle v\rangle^3\partial^\gamma f\right\|_{\sigma,w}\left\|\partial^\alpha f\right\|_{\sigma,w}\\&\qquad+\sum_{|\gamma|\le 1}\left\|\partial^\gamma\partial^{\alpha_1}B\right\|_2\sum_{|\gamma|=1}\left\|\langle v\rangle^3\partial^\gamma\partial^{\alpha-\alpha_1}f\right\|_{\sigma,w}\left\|\partial^\alpha f\right\|_{\sigma,w}\\&\quad\lesssim\sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{2;l,q}}\sqrt{\mathcal{D}_{m;l,q}}+\sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{m-2;l,q}}\sqrt{\mathcal{D}_{m;l,q}}.\end{aligned} \tag{3.37}$$

Here we have used the facts $\langle v\rangle^3 w(\alpha,0)<w(\gamma,0)$ for $1\le|\gamma|\le 2$ and $\langle v\rangle^3 w(\alpha,0)<w(\alpha-\alpha_1+\gamma,0)$ for $|\alpha_1|\ge 3$ and $|\gamma|=1$. This concludes the case $|\alpha|\ge 4$. Hence we conclude that

$$I_5\lesssim\sqrt{\mathcal{E}_m}\mathcal{D}_{m;l,q}+\|\nabla B\|_\infty\sqrt{\mathcal{F}_{m-1;l,q}}\sqrt{\mathcal{D}_{m;l,q}}. \tag{3.38}$$

Similarly, for the rest term I_4, we again perform an integration by parts in v to have

$$\begin{aligned}I_4&=\sum_{0\ne\alpha_1\le\alpha}C_\alpha^{\alpha_1}\int w^2\partial^{\alpha-\alpha_1}fq_0\partial^{\alpha_1}E\cdot\left(\left[1+\frac{q}{(1+t)^\vartheta}+\frac{2(l-|\alpha|)}{1+|v|^2}\right]v+\nabla_v\right)\partial^\alpha f\\&\lesssim\sum_{0\ne\alpha_1\le\alpha}\int\left|\partial^{\alpha_1}E\right|\left|\langle v\rangle^{3/2}\partial^{\alpha-\alpha_1}f\right|_{2,w}\left|\partial^\alpha f\right|_{\sigma,w}.\end{aligned} \tag{3.39}$$

When $|\alpha|=1$, then so $\alpha_1=\alpha$, we use the split $f=\mathbf{P}f+\{\mathbf{I}-\mathbf{P}\}f$ in the factor $|\langle v\rangle^{3/2}f|_{2,w}$ in (3.31) and then take $L^2-L^\infty-L^2$ and $L^\infty-L^2-L^2$, respectively, to have an upper bound of

$$\begin{aligned}&\left\|\partial^\alpha E\right\|_2\sum_{1\le|\gamma|\le 2}\left\|\partial^\gamma\mathbf{P}f\right\|_2\left\|\partial^\alpha f\right\|_{\sigma,w}+\sum_{1\le|\gamma|\le 2}\left\|\partial^\gamma\partial^\alpha E\right\|_2\left\|\langle v\rangle^2\{\mathbf{I}-\mathbf{P}\}f\right\|_{\sigma,w}\left\|\partial^\alpha f\right\|_{\sigma,w}\\&\quad\lesssim\sqrt{\mathcal{E}_2}\sqrt{\mathcal{D}_2}\sqrt{\mathcal{D}_{2;l,q}}+\sqrt{\mathcal{E}_3}\mathcal{D}_{2;l,q}.\end{aligned} \tag{3.40}$$

Here we have used the fact $\langle v\rangle^2 w(\alpha,0) = w(0,0)$.

For $|\alpha| \ge 2$, if $|\alpha_1| = 1$, we take $L^\infty - L^2 - L^2$ in (3.39) to have an upper bound of

$$\sum_{1\le|\gamma|\le 2} \left\|\partial^\gamma \partial^{\alpha_1} E\right\|_2 \left\|\langle v\rangle^2 \partial^{\alpha-\alpha_1} f\right\|_{\sigma,w} \left\|\partial^\alpha f\right\|_{\sigma,w} \lesssim \sqrt{\mathcal{E}_3}\sqrt{\mathcal{D}_{m-1;l,q}}\sqrt{\mathcal{D}_{m;l,q}}. \tag{3.41}$$

Here we have used the fact $\langle v\rangle^2 w(\alpha,0) = w(\alpha-\alpha_1,0)$ for $|\alpha_1| = 1$.

For the remaining cases, we may argue in the same as that for B, together with the bounds (3.40) and (3.41), to conclude that

$$I_4 \lesssim \sqrt{\mathcal{E}_m}\mathcal{D}_{m;l,q}. \tag{3.42}$$

Plugging these estimates for $I_1 - I_5$ and (3.23) into (3.22), by Cauchy's inequality we thus conclude the estimate (3.21). □

We now turn to the mixed spatial-velocity derivatives of the solution. Notice that in view of Proposition 3.1 and Lemma 3.2, it suffices to estimate the remaining microscopic part $\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f$ for $|\alpha|+|\beta|\le m$ and $|\alpha|\le m-1$.

Lemma 3.3. *Let $l \ge m \ge 4$. Assume that $\mathcal{E}_m \le M$ is small. Let $w = w(\alpha,\beta)$ in (1.17). Then for $|\alpha|+|\beta|\le m$ with $|\alpha|\le m-1$, we have that for any $\eta>0$*

$$\begin{aligned}
&\frac{d}{dt}\left\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{2,w} + \left\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{\sigma,w} + \frac{\vartheta q}{(1+t)^{1+\vartheta}}\left\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{2,w}\\
&\quad\lesssim (\sqrt{\mathcal{E}_m}+\eta)\mathcal{D}_{m;l,q} + \|E\|_\infty \mathcal{F}_{m;l,q} + \|\nabla B\|^2_\infty \mathcal{F}_{m-1;l,q}\\
&\qquad + C_\eta \sum_{|\beta'|<|\beta|} \left\|\partial^\alpha_{\beta'}\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{\sigma,w(\alpha,\beta')} + \mathcal{D}_m.
\end{aligned} \tag{3.43}$$

Proof. We will use the following macro-micro decomposition from (1.7):

$$\begin{aligned}
&\left\{\partial_t + v\cdot\nabla_x + q_0(E+v\times B)\cdot\nabla_v\right\}\{\mathbf{I}-\mathbf{P}\}f - 2E\cdot v\sqrt{\mu}q_1 + Lf\\
&\quad = \Gamma(f,f) + q_0 E\cdot v\{\mathbf{I}-\mathbf{P}\}f + S_{E,B} + \mathbf{P}(v\cdot\nabla_x f) - v\cdot\nabla_x \mathbf{P}f,
\end{aligned} \tag{3.44}$$

where we have denoted by

$$S_{E,B} \equiv q_0 E\cdot v\mathbf{P}f - \mathbf{P}(q_0 E\cdot vf) - q_0(E+v\times B)\cdot\nabla_v\mathbf{P}f + \mathbf{P}\big(q_0(E+v\times B)\cdot\nabla_v f\big). \tag{3.45}$$

Applying ∂^α_β with $|\alpha|+|\beta|\le m$ and $|\alpha|\le m-1$ to (3.44) and then taking the $L^2_x L^2_v$ inner product with $w^2\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f$, we obtain

$$\begin{aligned}
&\frac12\frac{d}{dt}\left\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{2,w} + \frac12\frac{\vartheta q}{(1+t)^{1+\vartheta}}\left\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\|^2_{2,w} + \int\left\langle\partial^\alpha_\beta L\{\mathbf{I}-\mathbf{P}\}f, w^2\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\right\rangle\\
&\quad = 2\int \partial^\alpha E\cdot\partial_\beta(v\sqrt{\mu})q_1 w^2\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f - \int \delta^{\mathbf{e}_i}_\beta \partial^{\alpha+\mathbf{e}_i}_{\beta-\mathbf{e}_i}\{\mathbf{I}-\mathbf{P}\}f w^2\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\\
&\qquad + \int\left[\frac{q}{(1+t)^\vartheta} + \frac{2(l-|\alpha|-|\beta|)}{1+|v|^2}\right] q_0 E\cdot v\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f w^2\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\\
&\qquad + \int \partial^\alpha_\beta\Gamma(f,f)w^2\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f + \int q_0 E\cdot\partial_\beta\left[v\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\right]w^2\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\\
&\qquad + \sum_{0\ne\alpha_1\le\alpha} C^{\alpha_1}_\alpha \int q_0\partial^{\alpha_1}E\cdot\partial_\beta\left[(v-\nabla_v)\partial^{\alpha-\alpha_1}\{\mathbf{I}-\mathbf{P}\}f\right]w^2\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\\
&\qquad - \sum_{0\ne(\alpha_1,\beta_1)\le(\alpha,\beta)} C^{\alpha_1}_\alpha \int q_0\partial_{\beta_1}v\times\partial^{\alpha_1}B\cdot\nabla_v\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f w^2\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\\
&\qquad + \sum_{\alpha_1\le\alpha} C^{\alpha_1}_\alpha \int \partial^\alpha_\beta S_{E,B} w^2\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f
\end{aligned}$$

$$+\int \partial_\beta^\alpha\big(\mathbf{P}(v\cdot\nabla_x f)-v\cdot\nabla_x\mathbf{P}f\big)w^2\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f:=\sum_{i=1}^{9} I_i. \tag{3.46}$$

Here $\delta_\beta^{\mathbf{e}_i}=1$ if $\mathbf{e}_i\le\beta$; otherwise, $\delta_\beta^{\mathbf{e}_i}=0$.

First, by Lemma 2.2, we have that if $\beta=0$,

$$\int\big\langle\partial^\alpha L\{\mathbf{I}-\mathbf{P}\}f,w^2\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\rangle\gtrsim\big\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}^2-C\big\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_\sigma^2; \tag{3.47}$$

if $\beta\ne0$, then for any $\eta>0$,

$$\begin{aligned}&\int\big\langle\partial_\beta^\alpha L\{\mathbf{I}-\mathbf{P}\}f,w^2\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\rangle\\&\quad\ge\big\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}^2-\eta\sum_{|\beta'|=|\beta|}\big\|\partial_{\beta'}^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w(\alpha,\beta')}^2-C_\eta\sum_{|\beta'|<|\beta|}\big\|\partial_{\beta'}^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w(\alpha,\beta')}^2.\end{aligned} \tag{3.48}$$

We now estimate I_1-I_9. Clearly,

$$I_1\lesssim\big\|\partial^\alpha E\big\|^2+\big\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_\sigma^2\lesssim\mathcal{D}_m, \tag{3.49}$$

and

$$\begin{aligned}I_3+I_5&\lesssim\|E\|_\infty\sum_{\beta'\le\beta}\big\|\langle v\rangle^{1/2}\partial_{\beta'}^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\big\|\langle v\rangle^{1/2}\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\\&\lesssim\|E\|_\infty\mathcal{F}_{m;l,q}.\end{aligned} \tag{3.50}$$

For any $\eta>0$ and $\beta\ge\mathbf{e}_i$, by Lemma 6 in [11] we have

$$\begin{aligned}I_2&\lesssim\big\|\delta_\beta^{\mathbf{e}_i}\partial_{\beta-\mathbf{e}_i}^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w(\alpha,\beta-\mathbf{e}_i)}\big\|\partial_{\beta-\mathbf{e}_i}^{\alpha+\mathbf{e}_i}\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w(\alpha+\mathbf{e}_i,\beta-\mathbf{e}_i)}\\&\le\eta\mathcal{D}_{m;l,q}(f)+C_\eta\big\|\delta_\beta^{\mathbf{e}_i}\partial_{\beta-\mathbf{e}_i}^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w(\alpha,\beta-\mathbf{e}_i)}^2.\end{aligned} \tag{3.51}$$

Next, we apply Lemma 2.3 to obtain

$$I_4\lesssim\sum_{\substack{\alpha_1\le\alpha\\ \bar\beta\le\beta_1\le\beta}}\int\big|\mu^\delta\partial_{\bar\beta}^{\alpha_1}f\big|_2\big|\partial_{\beta-\beta_1}^{\alpha-\alpha_1}f\big|_{\sigma,w}\big|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\big|_{\sigma,w}. \tag{3.52}$$

If $(\alpha_1,\beta_1)=0$, we take $L^\infty-L^2-L^2$ in (3.52) and we use the split $f=\mathbf{P}f+\{\mathbf{I}-\mathbf{P}\}f$ in the factor $|\partial_\beta^\alpha f|_{\sigma,w}$ to have an upper bound of

$$\begin{aligned}&\sum_{1\le|\gamma|\le2}\big\|\mu^\delta\partial^\gamma f\big\|_2\big\|\partial^\alpha\mathbf{P}f\big\|_2\big\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\\&\qquad+\sum_{1\le|\gamma|\le2}\big\|\mu^\delta\partial^\gamma f\big\|_2\big\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\big\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\\&\quad\lesssim\sqrt{\mathcal{D}_2}\sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{m;l,q}}+\sqrt{\mathcal{E}_2}\mathcal{D}_{m;l,q}.\end{aligned} \tag{3.53}$$

Note that we have concluded the case $|\alpha|+|\beta|=0$.

When $|\alpha|+|\beta|=1$, the remaining case is $(\alpha_1,\beta_1)=(\alpha,\beta)$, and we take $L^3-L^6-L^2$ in (3.52) to have an upper bound of

$$\sum_{|\gamma|\le1}\big\|\mu^\delta\partial^\gamma\partial_\beta^\alpha f\big\|_2\sum_{|\gamma|=1}\big\|\partial^\gamma f\big\|_{\sigma,w}\big\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\lesssim\sqrt{\mathcal{E}_2}\mathcal{D}_{2;l,q}. \tag{3.54}$$

Here we have used the fact $w(\alpha,\beta)=w(\gamma,0)$ for $|\gamma|=1$. This concludes the case $|\alpha|+|\beta|=1$.

Now when $|\alpha|+|\beta|\ge2$, if $(\alpha_1,\beta_1)=(\alpha,\beta)$ we take $L^2-L^\infty-L^2$; the remaining cases are of $1\le|\alpha_1|+|\beta_1|\le|\alpha|+|\beta|-1$, and we take $L^3-L^6-L^2$ in (3.52), respectively, to have an upper bound of

$$
\begin{aligned}
&\big\|\mu^\delta\partial^\alpha_{\bar\beta}f\big\|_2\sum_{1\le|\gamma|\le 2}\big\|\partial^\gamma f\big\|_{\sigma,w}\big\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\\
&\quad+\sum_{|\gamma|\le 1}\big\|\mu^\delta\partial^\gamma\partial^{\alpha_1}_{\bar\beta}f\big\|_2\sum_{|\gamma|=1}\big\|\partial^\gamma\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f\big\|_{\sigma,w}\big\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\\
&\lesssim\sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{2;l,q}}\sqrt{\mathcal{D}_{m;l,q}}+\sqrt{\mathcal{E}_m}\mathcal{D}_{m;l,q}.
\end{aligned}\tag{3.55}
$$

Here we have used the facts $w(\alpha,\beta)\le w(\gamma,0)$ for $1\le|\gamma|\le 2$ and $w(\alpha,\beta)\le w(\alpha-\alpha_1+\gamma,\beta-\beta_1)$ for $|\alpha_1|+|\beta_1|\ge 1$ and $|\gamma|=1$. This concludes the case $|\alpha|+|\beta|\ge 2$. Hence we conclude that

$$
I_4\lesssim\sqrt{\mathcal{E}_m}\mathcal{D}_{m;l,q}.\tag{3.56}
$$

Notice that the term I_7 only occurs when $|\alpha|+|\beta|\ge 1$, and we perform an integration by parts in v to have

$$
\begin{aligned}
I_7&=\sum_{0\ne(\alpha_1,\beta_1)\le(\alpha,\beta)}C^{\alpha_1}_\alpha\int w^2\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f q_0\partial_{\beta_1}v\times\partial^{\alpha_1}B\cdot\nabla_v\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\\
&\lesssim\sum_{0\ne(\alpha_1,\beta_1)\le(\alpha,\beta)}\int\big|\partial^{\alpha_1}B\big|\big|\langle v\rangle^{3/2}\partial_{\beta_1}v\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f\big|_{2,w}\big|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big|_{\sigma,w}
\end{aligned}\tag{3.57}
$$

If $|\alpha_1|+|\beta_1|=1$, then either $|\alpha_1|=1,|\beta_1|=0$ or $|\alpha_1|=0,|\beta_1|=1$, and we both take $L^\infty-L^2-L^2$ in (3.57) to have an upper bound of

$$
\begin{aligned}
&\big\|\partial^{\alpha_1}B\big\|_\infty\big\|\langle v\rangle^{5/2}\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f\big\|_{2,w}\big\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\\
&\quad+\sum_{1\le|\gamma|\le 2}\big\|\partial^\gamma B\big\|_2\big\|\langle v\rangle^2\partial^\alpha_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\big\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\\
&\lesssim\|\nabla B\|_\infty\sqrt{\mathcal{F}_{m-1;l,q}}\sqrt{\mathcal{D}_{m;l,q}}+\sqrt{\mathcal{E}_2}\sqrt{\mathcal{D}_{m-1;l,q}}\sqrt{\mathcal{D}_{m;l,q}}.
\end{aligned}\tag{3.58}
$$

Here we have used the facts $\langle v\rangle^2w(\alpha,\beta)=w(\alpha-\alpha_1,\beta)$ for $|\alpha_1|=1$ and $\langle v\rangle^2w(\alpha,\beta)=w(\alpha,\beta-\beta_1)$ for $|\beta_1|=1$. Note that we have concluded the case $|\alpha_1|+|\beta_1|=1$.

When $|\alpha|+|\beta|=2$, the remaining case is $(\alpha_1,\beta_1)=(\alpha,\beta)$, and we take $L^\infty-L^2-L^2$ in (3.57) to have an upper bound of

$$
\sum_{1\le|\gamma|\le 2}\big\|\partial^\gamma\partial^\alpha B\big\|_2\big\|\langle v\rangle^3\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\big\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\lesssim\sqrt{\mathcal{E}_4}\mathcal{D}_{2;l,q}.\tag{3.59}
$$

Here we have used the fact $\langle v\rangle^3w(\alpha,\beta)<w(0,0)$. This concludes the case $|\alpha|+|\beta|=2$.

For $|\alpha|+|\beta|\ge 3$, if $|\alpha_1|+|\beta_1|=2$, we take $L^\infty-L^2-L^2$ in (3.57) to have an upper bound of

$$
\begin{aligned}
&\sum_{1\le|\gamma|\le 2}\big\|\partial^\gamma\partial^{\alpha_1}B\big\|_2\big\|\langle v\rangle^3\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\big\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\\
&\lesssim\sqrt{\mathcal{E}_4}\sqrt{\mathcal{D}_{m-2;l,q}}\sqrt{\mathcal{D}_{m;l,q}}.
\end{aligned}\tag{3.60}
$$

Here we have used the fact $\langle v\rangle^3w(\alpha,\beta)<w(\alpha-\alpha_1,\beta-\beta_1)$ for $|\alpha_1|+|\beta_1|=2$.

When $|\alpha|+|\beta|=3$, the remaining case is $(\alpha_1,\beta_1)=(\alpha,\beta)$, and we take $L^3-L^6-L^2$ in (3.57) to have an upper bound of

$$
\begin{aligned}
&\sum_{|\gamma|\le 1}\big\|\partial^\gamma\partial^\alpha B\big\|_2\sum_{|\gamma|=1}\big\|\langle v\rangle^3\partial^\gamma\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\big\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\big\|_{\sigma,w}\\
&\lesssim\sqrt{\mathcal{E}_4}\sqrt{\mathcal{D}_{2;l,q}}\sqrt{\mathcal{D}_{3;l,q}}.
\end{aligned}\tag{3.61}
$$

Here we have used the fact $\langle v\rangle^3w(\alpha,\beta)<w(\gamma,0)$ for $|\gamma|=1$. This concludes the case $|\alpha|+|\beta|=3$.

Now when $|\alpha|+|\beta|\ge 4$, if $(\alpha_1,\beta_1)=(\alpha,\beta)$, we take $L^2-L^\infty-L^2$; the remaining cases are of $3\le|\alpha_1|+|\beta_1|\le|\alpha|+|\beta|-1$, and we take $L^3-L^6-L^2$ in (3.57), respectively, to have an upper bound of

$$\|\partial^\alpha B\|_2 \sum_{1\le|\gamma|\le 2} \|\langle v\rangle^3\partial^\gamma\{\mathbf{I}-\mathbf{P}\}f\|_{\sigma,w}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|_{\sigma,w}$$
$$+\sum_{|\gamma|\le 1}\|\partial^\gamma\partial^{\alpha_1}B\|_2\sum_{|\gamma|=1}\|\langle v\rangle^3\partial^\gamma\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f\|_{\sigma,w}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|_{\sigma,w}$$
$$\lesssim\sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{2;l,q}}\sqrt{\mathcal{D}_{m;l,q}}+\sqrt{\mathcal{E}_m}\sqrt{\mathcal{D}_{m-2;l,q}}\sqrt{\mathcal{D}_{m;l,q}}. \tag{3.62}$$

Here we have used the facts $\langle v\rangle^3 w(\alpha,\beta)<w(\gamma,0)$ for $1\le|\gamma|\le 2$ and $\langle v\rangle^3 w(\alpha,\beta)<w(\alpha-\alpha_1+\gamma,\beta-\beta_1)$ for $|\alpha_1|+|\beta_1|\ge 3$ and $|\gamma|=1$. This concludes the case $|\alpha|+|\beta|\ge 4$. Hence we conclude that

$$I_7\lesssim\sqrt{\mathcal{E}_m}\mathcal{D}_{m;l,q}+\|\nabla B\|_\infty\sqrt{\mathcal{F}_{m-1;l,q}}\sqrt{\mathcal{D}_{m,l,q}}. \tag{3.63}$$

Similarly, for the term I_6, we again perform an integration by parts in v to have

$$I_6\lesssim\sum_{0\ne(\alpha_1,\beta_1)\le(\alpha,\beta)}\int|\partial^{\alpha_1}E||\langle v\rangle\partial_{\beta_1}v\partial^{\alpha-\alpha_1}_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f|_{\sigma,w}|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f|_{\sigma,w}$$
$$+\sum_{0\ne\alpha_1\le\alpha}\int|\partial^{\alpha_1}E||\langle v\rangle^2\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f|_{\sigma,w}|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f|_{\sigma,w}. \tag{3.64}$$

For the first term, if $|\alpha_1|+|\beta_1|=1$, then either $|\alpha_1|=1$, $|\beta_1|=0$ or $|\alpha_1|=0$, $|\beta_1|=1$; for the second term, if $|\alpha_1|=1$, we all take $L^\infty-L^2-L^2$ to have an upper bound of

$$\sum_{1\le|\gamma|\le 2}\|\partial^\gamma\partial^{\alpha_1}E\|_2\|\langle v\rangle^2\partial^{\alpha-\alpha_1}_\beta\{\mathbf{I}-\mathbf{P}\}f\|_{\sigma,w}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|_{\sigma,w}$$
$$+\sum_{1\le|\gamma|\le 2}\|\partial^\gamma E\|_2\|\langle v\rangle\partial^\alpha_{\beta-\beta_1}\{\mathbf{I}-\mathbf{P}\}f\|_{\sigma,w}\|\partial^\alpha_\beta\{\mathbf{I}-\mathbf{P}\}f\|_{\sigma,w}$$
$$\lesssim\sqrt{\mathcal{E}_3}\sqrt{\mathcal{D}_{m-1;l,q}}\sqrt{\mathcal{D}_{m;l,q}}. \tag{3.65}$$

Here we have used the facts $\langle v\rangle^2 w(\alpha,\beta)=w(\alpha-\alpha_1,\beta)$ for $|\alpha_1|=1$ and $\langle v\rangle w(\alpha,\beta)<w(\alpha,\beta-\beta_1)$ for $|\beta_1|=1$. For the remaining cases, we may argue in the same as that for B, together with the bound (3.65), to conclude that

$$I_6\lesssim\sqrt{\mathcal{E}_m}\mathcal{D}_{m;l,q}. \tag{3.66}$$

Now for the term I_8, we integrate by parts in v and make use of the exponential decay in v of the hydrodynamic part to get

$$I_8\lesssim\sum_{\alpha_1\le\alpha}\int\left(|\partial^{\alpha_1}E|+|\partial^{\alpha_1}B|\right)|\mu^\delta\partial^{\alpha-\alpha_1}f|_2|\mu^\delta\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f|_2. \tag{3.67}$$

We may then easily have the an upper bound as

$$I_8\lesssim\sqrt{\mathcal{E}_m}\mathcal{D}_m. \tag{3.68}$$

Similarly, we easily have

$$I_9\lesssim\|\partial^\alpha\{\mathbf{I}-\mathbf{P}\}f\|_\sigma^2+\|\nabla_x^{|\alpha|+1}f\|_\sigma^2\lesssim\mathcal{D}_m. \tag{3.69}$$

Plugging these estimates for I_1-I_9 and (3.47)–(3.48) into (3.46), by Cauchy's inequality we thus conclude the estimate (3.43). □

We now summarize the energy estimates in the following proposition.

Proposition 3.4. *Let $m\ge 4$ and $l\ge m+1/4$. Assume that $\mathcal{E}_m\le M$ is small, then we have*

$$\frac{d}{dt}\mathcal{E}_{m;l,q}+\mathcal{D}_{m;l,q}+\frac{\vartheta q}{(1+t)^{1+\vartheta}}\mathcal{F}_{m;l,q}$$
$$\lesssim\|E\|_\infty\mathcal{F}_{m;l,q}+\|\nabla B\|_\infty^2\mathcal{F}_{m-1;l,q}+\|\nabla^m E\|_2\sqrt{\mathcal{D}_m}. \tag{3.70}$$

Proof. We define

$$\mathcal{E}_{m;l,q}=\mathcal{E}_m+\sum_{1\le|\alpha|\le m}\eta_\alpha\left\|\partial^\alpha f\right\|_{2,w(\alpha,0)}^2+\sum_{\substack{|\alpha|+|\beta|\le m\\ |\alpha|\le m-1}}\eta_{\alpha,\beta}\left\|\partial_\beta^\alpha\{\mathbf{I}-\mathbf{P}\}f\right\|_{2,w(\alpha,\beta)}^2. \tag{3.71}$$

Then $\mathcal{E}_{m;l,q}$ satisfies (1.18). By further taking $\eta_\alpha,\eta_{\alpha,\beta},\eta$ sufficiently small orderly and choosing M sufficiently small, we deduce the proposition from Proposition 3.1 and Lemmas 3.2–3.3. □

4. Time decay rates

In light of the energy estimates (3.70) in Proposition 3.4, we now turn to derive a strong decay of the electromagnetic field (E,B). We recall the notations $f_1=f_++f_-$ and $f_2=f_+-f_-$ for the solution f to the Vlasov–Maxwell–Landau system (1.7), and we recall the system (1.29).

We first derive an energy estimates which allows us to prove the basic decay rate of (E,B).

Lemma 4.1. *There exist an instant energy $\mathfrak{E}(t)$ with*

$$\mathfrak{E}(t)\sim\sum_{k=3,4,5}\left\{\left\|\nabla^k(f_1,f_2)\right\|_2^2+\left\|\nabla^k(E,B)\right\|_2^2\right\} \tag{4.1}$$

and the corresponding dissipation rate

$$\mathfrak{D}(t)\sim\left\|\nabla^3\{\mathbf{I}-\mathbf{P_1}\}f_1\right\|_\sigma+\sum_{k=4,5}\left\|\nabla^k f_1\right\|_\sigma^2+\sum_{k=3,4,5}\left\|\nabla^k f_2\right\|_\sigma^2+\sum_{k=3,4}\left\|\nabla^k E\right\|_2^2+\left\|\nabla^4 B\right\|_2^2 \tag{4.2}$$

such that

$$\frac{d}{dt}\mathfrak{E}+\mathfrak{D}\lesssim\left(\sqrt{\mathcal{E}_7}+\sqrt{\mathcal{E}_{5;35/4,0}}\right)\mathfrak{D}. \tag{4.3}$$

Proof. The ∇^k $(k=3,5)$ energy estimates on (1.29) yield

$$\begin{aligned}
&\frac{d}{dt}\left\{\left\|\nabla^k(f_1,f_2)\right\|_2^2+2\left\|\nabla^k(E,B)\right\|_2^2\right\}+\left\|\nabla^k\big(\{\mathbf{I}-\mathbf{P_1}\}f_1,\{\mathbf{I}-\mathbf{P_2}\}f_2\big)\right\|_\sigma^2\\
&\quad\le\int\nabla^k\Gamma_*(f_1,f_1)\nabla^k f_1+\int\nabla^k\Gamma_*(f_1,f_2)\nabla^k f_2\\
&\qquad+\int\nabla^k\big(E\cdot(v-\nabla_v)f_2\big)\nabla^k f_1+\int\nabla^k\big(E\cdot(v-\nabla_v)f_1\big)\nabla^k f_2\\
&\qquad-\int\nabla^k(v\times B\cdot\nabla_v f_2)\nabla^k f_1-\int\nabla^k(v\times B\cdot\nabla_v f_1)\nabla^k f_2:=\sum_{i=1}^6 I_i.
\end{aligned} \tag{4.4}$$

We first estimate I_1-I_6 when $k=3$. For the term I_1, by the collision invariant property and Lemmas 2.3, 2.6, 2.5, we obtain

$$\begin{aligned}
I_1&=\int\nabla^3\Gamma_*(f_1,f_1)\nabla^3\{\mathbf{I}-\mathbf{P_1}\}f_1\\
&\lesssim\left(\left\|\mu^\delta f_1\right\|_{L_v^2L_x^3}\left\|\nabla^3 f_1\right\|_{L_\sigma^2L_x^6}+\left\|\mu^\delta\nabla^3 f_1\right\|_{L_v^2L_x^6}\|f_1\|_{L_\sigma^2L_x^3}\right)\left\|\nabla^3\{\mathbf{I}-\mathbf{P_1}\}f_1\right\|_\sigma\\
&\lesssim\sqrt{\mathcal{E}_2}\left\|\nabla^4 f_1\right\|_\sigma\left\|\nabla^3\{\mathbf{I}-\mathbf{P_1}\}f_1\right\|_\sigma\lesssim\sqrt{\mathcal{E}_2}\mathfrak{D}.
\end{aligned} \tag{4.5}$$

Note that I_2 can be estimated in the same way, at least. For the term I_6, we integrate by parts in v and use the split $f_1=\mathbf{P}_1f_1+\{\mathbf{I}-\mathbf{P}_1\}f_1$ to have

$$\begin{aligned}
I_6&=\int\nabla^3(v\times Bf_1)\cdot\nabla_v\nabla^3 f_2\\
&\lesssim\left(\left\|\nabla^3(B\mathbf{P}_1f_1)\right\|_2+\left\|\nabla^3\big(B\langle v\rangle^{5/2}\{\mathbf{I}-\mathbf{P}_1\}f_1\big)\right\|_2\right)\left\|\nabla^3 f_2\right\|_\sigma.
\end{aligned} \tag{4.6}$$

By Lemmas 2.6, 2.5, 2.4 and Hölder's inequality, we obtain

$$\begin{aligned}\left\|\nabla^3(B\mathbf{P}_1 f_1)\right\|_2 &\lesssim \|B\|_3\left\|\nabla^3\mathbf{P}_1 f_1\right\|_{L^2_v L^6_x} + \left\|\nabla^3 B\right\|_6 \|\mathbf{P}_1 f_1\|_{L^2_v L^3_x} \\ &\lesssim \|B\|_3\left\|\nabla^4\mathbf{P}_1 f_1\right\|_2 + \left\|\nabla^4 B\right\|_2 \|\mathbf{P}_1 f_1\|_{L^2_v L^3_x} \lesssim \sqrt{\mathcal{E}_2}\sqrt{\mathfrak{D}}\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}&\left\|\nabla^3\big(B\langle v\rangle^{5/2}\{\mathbf{I}-\mathbf{P}_1\}f_1\big)\right\|_2 \\ &\lesssim \|B\|_\infty \left\|\langle v\rangle^{5/2}\nabla^3\{\mathbf{I}-\mathbf{P}_1\}f_1\right\|_{L^2_v L^2_x} + \left\|\nabla^3 B\right\|_6 \left\|\langle v\rangle^{5/2}\{\mathbf{I}-\mathbf{P}_1\}f_1\right\|_{L^2_v L^3_x} \\ &\lesssim \|B\|_2^{5/8}\left\|\nabla^4 B\right\|_2^{3/8}\left\|\langle v\rangle^{15/2}\nabla^3\{\mathbf{I}-\mathbf{P}_1\}f_1\right\|_2^{3/8}\left\|\langle v\rangle^{-1/2}\nabla^3\{\mathbf{I}-\mathbf{P}_1\}f_1\right\|_2^{5/8} \\ &\quad + \left\|\nabla^4 B\right\|_2\left\|\langle v\rangle^{5/2}\{\mathbf{I}-\mathbf{P}_1\}f_1\right\|_{L^2_v L^3_x} \\ &\lesssim \sqrt{\mathcal{E}_{3;27/4,0}}\sqrt{\mathfrak{D}} + \sqrt{\mathcal{E}_{2;9/4,0}}\sqrt{\mathfrak{D}}.\end{aligned} \tag{4.8}$$

Hence, we conclude that

$$I_6 \lesssim \sqrt{\mathcal{E}_{3;27/4,0}}\,\mathfrak{D}. \tag{4.9}$$

Note that I_4 can be estimated in the same way. For the term I_5, we integrate by parts in both v and x to have

$$I_5 = -\int \nabla^2(v\times Bf_2)\cdot\nabla_v\nabla^4 f_1 \lesssim \left\|\nabla^2\big(B\langle v\rangle^{5/2}f_2\big)\right\|_2\left\|\nabla^4 f_1\right\|_\sigma. \tag{4.10}$$

By Lemmas 2.6, 2.5, 2.4 and Hölder's inequality, we obtain

$$\begin{aligned}\left\|\nabla^2\big(B\langle v\rangle^{5/2}f_2\big)\right\|_2 &\lesssim \|B\|_\infty\left\|\langle v\rangle^{5/2}\nabla^2 f_2\right\|_2 + \left\|\nabla^2 B\right\|_2\left\|\langle v\rangle^{5/2}f_2\right\|_{L^2_v L^\infty_x} \\ &\lesssim \|B\|_2^{5/8}\left\|\nabla^4 B\right\|_2^{3/8}\left\|\langle v\rangle^{15/2}\nabla^{1/3}f_2\right\|_2^{3/8}\left\|\langle v\rangle^{-1/2}\nabla^3 f_2\right\|_2^{5/8} \\ &\quad + \|B\|_2^{1/2}\left\|\nabla^4 B\right\|_2^{1/2}\left\|\langle v\rangle^{11/2}f_2\right\|_2^{1/2}\left\|\langle v\rangle^{-1/2}\nabla^3 f_2\right\|_2^{1/2} \\ &\lesssim \sqrt{\mathcal{E}_{2;19/4,0}}\sqrt{\mathfrak{D}} + \sqrt{\mathcal{E}_{2;11/4,0}}\sqrt{\mathfrak{D}}.\end{aligned} \tag{4.11}$$

Hence, we conclude that

$$I_4 \lesssim \sqrt{\mathcal{E}_{3;27/4,0}}\,\mathfrak{D}. \tag{4.12}$$

Note that I_3 can be estimated in the same way. This completes the estimates for $k=3$.

We now estimate $I_1 - I_6$ when $k=5$. For the term I_1, by Lemmas 2.3, 2.6 and 2.5, we obtain

$$\begin{aligned}I_1 &\lesssim \big(\left\|\mu^\delta f_1\right\|_{L^2_v L^\infty_x}\left\|\nabla^5 f_1\right\|_\sigma + \left\|\mu^\delta\nabla^5 f_1\right\|_2\|f_1\|_{L^2_\sigma L^\infty_x}\big)\left\|\nabla^5 f_1\right\|_\sigma \\ &\lesssim \sqrt{\mathcal{E}_{3;3,0}}\left\|\nabla^5 f_1\right\|_\sigma^2 \le \sqrt{\mathcal{E}_{3;3,0}}\,\mathfrak{D}.\end{aligned} \tag{4.13}$$

Note that I_2 can be estimated in the same way. For the term I_6, we integrate by parts in v to have

$$I_6 = \int \nabla^5(v\times Bf_1)\cdot\nabla_v\nabla^5 f_2 \lesssim \left\|\nabla^5\big(B\langle v\rangle^{5/2}f_1\big)\right\|_2\left\|\nabla^5 f_2\right\|_\sigma. \tag{4.14}$$

By Lemmas 2.6, 2.5, 2.4 and Hölder's and Young's inequalities, we obtain

$$\begin{aligned}\left\|\nabla^5\big(B\langle v\rangle^{5/2}f_1\big)\right\|_2 &\lesssim \|B\|_{L^\infty_x}\left\|\langle v\rangle^{5/2}\nabla^5 f_1\right\|_2 + \left\|\nabla^5 B\right\|_2\left\|\langle v\rangle^{5/2}f_1\right\|_{L^2_v L^\infty_x} \\ &\lesssim \|B\|_2^{5/8}\left\|\nabla^4 B\right\|_2^{3/8}\left\|\langle v\rangle^{15/2}\nabla^5 f_1\right\|_2^{3/8}\left\|\langle v\rangle^{-1/2}\nabla^5 f_1\right\|_2^{5/8} \\ &\quad + \left\|\nabla^4 B\right\|_2^{5/8}\left\|\nabla^{20/3} B\right\|_2^{3/8}\left\|\langle v\rangle^{43/10}f_1\right\|_2^{5/8}\left\|\langle v\rangle^{-1/2}\nabla^4 f_1\right\|_2^{3/8} \\ &\lesssim \big(\sqrt{\mathcal{E}_{5;35/4,0}} + \sqrt{\mathcal{E}_7} + \sqrt{\mathcal{E}_{2;43/20,0}}\big)\sqrt{\mathfrak{D}}.\end{aligned} \tag{4.15}$$

Hence, we conclude that

$$I_6 \lesssim (\sqrt{\mathcal{E}_7} + \sqrt{\mathcal{E}_{5;35/4,0}})\mathfrak{D}. \tag{4.16}$$

Note that $I_3 - I_5$ can be estimated in the same way. This completes the estimates for $k = 5$.

Summing over $k = 3, 5$ and summing up all these estimates above, we deduce from (4.4) that

$$\begin{aligned}&\frac{d}{dt}\sum_{k=3,5}\big\{\|\nabla^k(f_1, f_2)\|_2^2 + 2\|\nabla^k(E, B)\|_2^2\big\}\\&\quad+\sum_{k=3,5}\big\|\nabla^k\big(\{\mathbf{I}-\mathbf{P_1}\}f_1, \{\mathbf{I}-\mathbf{P_2}\}f_2\big)\big\|_\sigma^2 \lesssim (\sqrt{\mathcal{E}_7} + \sqrt{\mathcal{E}_{5;35/4,0}})\mathfrak{D}.\end{aligned} \tag{4.17}$$

Notice that the dissipation estimates in (4.17) only controls the microscopic parts, but it follows from [10,15,3,2,20] that for $k = 3, 4$ there exists a function $\mathfrak{G}^k_{f_1}(t)$ with

$$\big|\mathfrak{G}^k_{f_1}(t)\big| \lesssim \|\nabla^k f_1\|_2^2 + \|\nabla^{k+1} f_1\|_2^2 \tag{4.18}$$

such that

$$\frac{d}{dt}\mathfrak{G}^k_{f_1} + \|\nabla^{k+1}\mathbf{P_1} f_1\|_2^2 \lesssim \|\nabla^k\{\mathbf{I}-\mathbf{P_1}\}f_1\|_\sigma^2 + \|\nabla^{k+1}\{\mathbf{I}-\mathbf{P_1}\}f_1\|_\sigma^2 + \mathcal{E}_3\mathfrak{D}, \tag{4.19}$$

and that there exists a function $\mathfrak{G}_{f_2}(t)$ with

$$\mathfrak{G}_{f_2}(t) \lesssim \sum_{k=3,4,5}\big\{\|\nabla^k f_2\|_2^2 + \|\nabla^k(E, B)\|_2^2\big\} \tag{4.20}$$

such that

$$\begin{aligned}&\frac{d}{dt}\mathfrak{G}_{f_2} + \sum_{k=3,4,5}\|\nabla^k\mathbf{P_2} f_2\|_2^2 + \sum_{k=3,4}\|\nabla^k E\|_2^2 + \|\nabla^4 B\|_2^2\\&\quad\lesssim \sum_{k=3,4,5}\|\nabla^k\{\mathbf{I}-\mathbf{P_2}\}f_2\|_\sigma^2 + \mathcal{E}_3\mathfrak{D}.\end{aligned} \tag{4.21}$$

Note that the point in (4.21) is that we can include $\|\nabla^3\mathbf{P_2} f_2\|_2^2$ in the dissipation. One can check this in details later in Lemma 4.3. Hence, if we define

$$\mathfrak{E}(t) := \sum_{k=3,5}\big\{\|\nabla^k(f_1, f_2)\|_2^2 + \|\nabla^k(E, B)\|_2^2\big\} + \eta\big(\mathfrak{G}^3_{f_1} + \mathfrak{G}^4_{f_1} + \mathfrak{G}^3_{f_2}\big), \tag{4.22}$$

then for η sufficiently small, by the Sobolev interpolation, we deduce that $\mathfrak{E}(t)$ satisfies (4.2) and that (4.3) follows from (4.17), (4.19) and (4.21). □

We then record the decay estimates resulting from Lemma 4.1.

Proposition 4.2. *Let $m \geq 8$ and $l \geq \max\{m, 35/4\}$. Assume that $\mathcal{E}_m + \mathcal{E}_{5;l,0} \leq M$ is small, then for $t \in [0, T]$ and for $k = 0, 1, 2, 3$,*

$$\|\nabla^k f(t)\|_2 + \|\nabla^k(E, B)(t)\|_2 \leq C(1+t)^{-\frac{k}{2}} \sup_{0\leq\tau\leq T}\sqrt{\mathcal{E}_m(\tau) + \mathcal{E}_{5;l,0}(\tau)}. \tag{4.23}$$

Proof. Under the assumption of the proposition, we obtain from Lemma 4.1 that

$$\frac{d}{dt}\mathfrak{E} + \mathfrak{D} \leq 0. \tag{4.24}$$

In order to derive the decay from the time differential inequality (4.24), the point is to estimate $\mathfrak{E}$ in terms of $\mathfrak{D}$. We will use the interpolation as in [17,11,20]. First, by Hölder's inequality, we interpolate among velocity moments to have that for $k = 3, 4, 5$,

$$\begin{aligned}\|\nabla^k f\|_2 &= \|\langle v\rangle^{-1/2}\nabla^k f\|_2^{\frac{4(l-5)}{4(l-5)+1}}\|\langle v\rangle^{4(l-5)}\nabla^k f\|_2^{\frac{1}{4(l-5)+1}}\\ &\le \|\nabla^k f\|_\sigma^{\frac{4(l-5)}{4(l-5)+1}}\left\{\sqrt{\mathcal{E}_{5;l,0}(t)}\right\}^{\frac{1}{4(l-5)+1}}. \end{aligned}\tag{4.25}$$

This implies

$$\begin{aligned}&\|\nabla^3\{\mathbf{I}-\mathbf{P_1}\}f_1\|_2+\sum_{k=4,5}\|\nabla^k f_1\|_2+\sum_{k=3,4,5}\|\nabla^k f_2\|_2 \\ &\quad\le\left\{\sqrt{\mathcal{E}_{5;l,0}(t)}\right\}^{\frac{1}{4(l-5)+1}}\left\{\|\nabla^3\{\mathbf{I}-\mathbf{P_1}\}f_1\|_\sigma+\sum_{k=4,5}\|\nabla^k f_1\|_\sigma+\sum_{k=3,4,5}\|\nabla^k f_2\|_\sigma\right\}^{\frac{4(l-5)}{4(l-5)+1}}.\end{aligned}\tag{4.26}$$

Next, we interpolate among spatial regularity to have

$$\|\nabla^3\mathbf{P_1}f_1\|_2\le\|\nabla^4\mathbf{P_1}f_1\|_2^{\frac34}\|f_1\|_2^{\frac14}\le\mathcal{E}_2^{\frac18}\|\nabla^4 f_1\|_\sigma^{\frac34},\tag{4.27}$$

$$\|\nabla^3 B\|_2\le\|\nabla^4 B\|_2^{\frac34}\|B\|_2^{\frac14}\le\mathcal{E}_2^{\frac18}\|\nabla^4 B\|_2^{\frac34},\tag{4.28}$$

and

$$\|\nabla^5(E,B)\|_2\le\|\nabla^4(E,B)\|_2^{\frac{m-5}{m-4}}\|\nabla^m(E,B)\|_2^{\frac{1}{m-4}}\le\mathcal{E}_m^{\frac{1}{2(m-4)}}\|\nabla^4(E,B)\|_{L^2}^{\frac{m-5}{m-4}}.\tag{4.29}$$

In light of (4.26)–(4.29), we deduce from (4.24) that for $t\in[0,T]$

$$\frac{d}{dt}\mathfrak{E}+\left\{\sup_{0\le\tau\le T}\left\{\mathcal{E}_{5;l,0}(\tau)+\mathcal{E}_m(\tau)\right\}\right\}^{-\theta}\mathfrak{E}^{1+\theta}\le 0,\tag{4.30}$$

where $\theta=\max\{\frac13,\frac{1}{4(l-5)},\frac{1}{m-5}\}$. Solving the inequality (4.30) directly, we obtain

$$\begin{aligned}\mathfrak{E}(t)&\le\left(\mathfrak{E}(0)^{-\theta}+\theta\left\{\sup_{0\le\tau\le T}\left\{\mathcal{E}_{5;l,0}(\tau)+\mathcal{E}_m(\tau)\right\}\right\}^{-\theta}t\right)^{-1/\theta}\\ &\lesssim(1+t)^{-1/\theta}\sup_{0\le\tau\le T}\left\{\mathcal{E}_{5;l,0}(\tau)+\mathcal{E}_m(\tau)\right\}.\end{aligned}\tag{4.31}$$

Notice that if we have required that $m\ge 8$ (and $l\ge\frac{23}{4}$), then $\theta=1/3$. This in particular proves the decay estimates (4.23) for $k=3$. Note that (4.23) for $k=0$ is trivial, and the cases $k=1,2$ follow by the Sobolev interpolation. □

We now derive an energy estimates which allows us to prove the faster decay rate of E. The key point is to consider the following evolution of f_2 and E separating from (1.29):

$$\begin{aligned}&\partial_t f_2+v\cdot\nabla_x f_2-4E\cdot v\sqrt{\mu}+\mathcal{L}_2 f_2=\Gamma_*(f_1,f_2)+E\cdot(v-\nabla_v)f_1-v\times B\cdot\nabla_v f_1,\\ &\partial_t E-\nabla_x\times B=-\int_{\mathbb{R}^3}v\sqrt{\mu}f_2\,dv,\quad \nabla_x\cdot E=\int_{\mathbb{R}^3}\sqrt{\mu}f_2\,dv.\end{aligned}\tag{4.32}$$

Lemma 4.3. *For $k=0,1,2$, there exist an energy $\mathfrak{E}^k(t)$ with*

$$\mathfrak{E}^k(t)\sim\|\nabla^k f_2\|_2^2+\|\nabla^k E\|_2^2\tag{4.33}$$

and the corresponding dissipation rate

$$\mathfrak{D}^k(t)=\|\nabla^k f_2\|_\sigma^2+\|\nabla^k E\|_2^2\tag{4.34}$$

such that

$$\frac{d}{dt}\mathfrak{E}^k+\mathfrak{D}^k\lesssim\sqrt{\mathcal{E}_{3;21/4,0}}\mathfrak{D}^k+\|\nabla^{k+1}f\|_2^2+\|\nabla^{k+1}B\|_2^2.\tag{4.35}$$

Proof. The ∇^k ($k=0,1,2$) energy estimates on (4.32) yield

$$\begin{aligned}&\frac{d}{dt}\big\{\|\nabla^k f_2\|_2^2+4\|\nabla^k E\|_2^2\big\}+\|\nabla^k\{\mathbf{I}-\mathbf{P_2}\}f_2\|_\sigma^2\\&\quad\le\int\nabla^k\Gamma_*(f_1,f_2)\nabla^k f_2+\int\nabla^k\big(E\cdot(v-\nabla_v)f_1\big)\nabla^k f_2\\&\qquad-\int\nabla^k(v\times B\cdot\nabla_v f_1)\nabla^k f_2+4\int\nabla^k E\cdot\nabla_x\times\nabla^k B:=\sum_{i=1}^4 I_i.\end{aligned}\tag{4.36}$$

We now estimate I_1-I_4. For the first term, by Lemmas 2.3, 2.6 and 2.5, we obtain

$$\begin{aligned}I_1&\lesssim\big(\|\mu^\delta f_1\|_{L^2_vL^\infty_x}\|\nabla^k f_2\|_\sigma+\|\mu^\delta\nabla^k f_1\|_{L^2_vL^6_x}\|f_2\|_{L^2_\sigma L^3_x}\big)\|\nabla^k f_2\|_\sigma\\&\lesssim\sqrt{\mathcal{E}_{2;2,0}}\big(\|\nabla^k f_2\|_\sigma^2+\|\nabla^{k+1}f_1\|_2^2\big).\end{aligned}\tag{4.37}$$

For the term I_3, we integrate by parts in v to have

$$I_3=\int\nabla^k(v\times Bf_1)\cdot\nabla_v\nabla^k f_2\lesssim\big\|\nabla^k\big(B\langle v\rangle^{5/2}f_1\big)\big\|_2\|\nabla^k f_2\|_\sigma.\tag{4.38}$$

By Lemmas 2.6, 2.5, 2.4 and Hölder's and Young's inequalities, we obtain

$$\begin{aligned}\big\|\nabla^k\big(B\langle v\rangle^{5/2}f_1\big)\big\|_2&\lesssim\|B\|_6\|\langle v\rangle^{5/2}\nabla^k f_1\|_{L^2_vL^3_x}+\|\nabla^k B\|_6\|\langle v\rangle^{5/2}f_1\|_{L^2_vL^3_x}\\&\lesssim\|B\|_2^{k/(k+1)}\|\nabla^{k+1}B\|_2^{1/(k+1)}\|\langle v\rangle^{5(k+1)/2}\nabla^{(k+1)/2}f_1\|_2^{1/(k+1)}\|\nabla^{k+1}f_1\|_2^{k/(k+1)}\\&\quad+\|\nabla^{k+1}B\|_2\|\langle v\rangle^{5/2}f_1\|_{L^2_vL^3_x}\\&\lesssim\big(\sqrt{\mathcal{E}_{3;21/4,0}}+\sqrt{\mathcal{E}_{2;9/4,0}}\big)\big(\|\nabla^{k+1}f_1\|_2+\|\nabla^{k+1}B\|_2\big).\end{aligned}\tag{4.39}$$

Hence, we conclude that

$$I_3\lesssim\sqrt{\mathcal{E}_{3;21/4,0}}\big(\|\nabla^k f_2\|_\sigma^2+\|\nabla^{k+1}f_1\|_2^2+\|\nabla^{k+1}B\|_2^2\big).\tag{4.40}$$

Note that I_2 can be estimated in the same way. By Cauchy's inequality, we easily bound

$$I_4\le\eta\|\nabla^k E\|_2^2+C_\eta\|\nabla^{k+1}B\|_2^2.\tag{4.41}$$

Summing up the estimates above, we deduce

$$\begin{aligned}&\frac{d}{dt}\big\{\|\nabla^k f_2\|_2^2+4\|\nabla^k E\|_2^2\big\}+\|\nabla^k\{\mathbf{I}-\mathbf{P_2}\}f_2\|_\sigma^2\\&\quad\lesssim\big(\sqrt{\mathcal{E}_{3;21/4,0}}+\eta\big)\mathfrak{D}^k+C_\eta\big(\|\nabla^{k+1}f\|_2^2+\|\nabla^{k+1}B\|_2^2\big).\end{aligned}\tag{4.42}$$

Notice that the dissipation estimate in (4.42) only controls the microscopic part $\{\mathbf{I}-\mathbf{P}_2\}f_2$, we thus want to include the hydrodynamic part $\mathbf{P}_2 f_2$ and the electric field E in the dissipation. We now explain this in details, and this should confirm the statements of Lemma 4.1 and Proposition 3.1. We will use the local conservation laws and the macroscopic equations derived from the macro-micro decomposition. Taking the L^2_v inner product of the first equation in (4.32) with $\sqrt{\mu}$, plugging in $f_2=\mathbf{P_2}f_2+\{\mathbf{I}-\mathbf{P_2}\}f_2$ and denoting $\mathbf{P_2}f_2=d(t,x)\sqrt{\mu}$, we obtain the *local conservation laws*:

$$\partial_t d+\nabla_x\cdot\mathcal{A}\big(\{\mathbf{I}-\mathbf{P_2}\}f_2\big)=0,\tag{4.43}$$

where the moment function $\mathcal{A}(g)=\langle g,v\sqrt{\mu}\rangle$. We write the second equation in (4.32) as

$$\partial_t E-\nabla_x\times B=-\mathcal{A}\big(\{\mathbf{I}-\mathbf{P_2}\}f_2\big),\qquad\nabla_x\cdot E=d.\tag{4.44}$$

Notice that by plugging $f_2=\mathbf{P_2}f_2+\{\mathbf{I}-\mathbf{P_2}\}f_2$ into the first equation (4.32),

$$\partial_t\mathbf{P_2}f_2+v\cdot\nabla_x\mathbf{P_2}f_2-4E\cdot v\sqrt{\mu}=-\partial_t\{\mathbf{I}-\mathbf{P_2}\}f_2+\mathfrak{L}_2+\mathfrak{N}_2\tag{4.45}$$

where the linear term $\mathfrak{L}_2$ is denoted by

$$\mathfrak{L}_2 = -v \cdot \nabla_x \{\mathbf{I} - \mathbf{P_2}\} f_2 - \mathcal{L}_2 \{\mathbf{I} - \mathbf{P_2}\} f_2 \tag{4.46}$$

and $\mathfrak{N}_2$ is the nonlinear term in the right hand side of the first equation in (4.32). Taking the L^2_v inner product of (4.45) with $v\sqrt{\mu}$, we obtain the *macroscopic equations*:

$$\partial_t \mathcal{A}\big(\{\mathbf{I} - \mathbf{P_2}\} f_2\big) + \nabla_x d - 4E = \mathcal{A}(\mathfrak{L}_2 + \mathfrak{N}_2). \tag{4.47}$$

Applying ∇^k to (4.47), taking the L^2_x inner product of the resulting equations with $-\nabla^k E$, and then integrating by parts, since $\nabla_x \cdot E = d$, we get

$$\|\nabla^k d\|_2^2 + 4\|\nabla^k E\|_2^2 = -\int \partial_t \nabla^k \mathcal{A}\big(\{\mathbf{I} - \mathbf{P_2}\} f_2\big) \cdot \nabla^k E + \int \nabla^k \mathcal{A}(\mathfrak{L}_2 + \mathfrak{N}_2) \cdot \nabla^k E. \tag{4.48}$$

For the first term, we integrate by part in time to have, using the equation (4.44),

$$\begin{aligned} &-\int \partial_t \nabla^k \mathcal{A}\big(\{\mathbf{I} - \mathbf{P_2}\} f_2\big) \cdot \nabla^k E \\ &\quad = -\frac{d}{dt}\int \nabla^k \mathcal{A}\big(\{\mathbf{I} - \mathbf{P_2}\} f_2\big) \cdot \nabla^k E + \int \nabla^k \mathcal{A}\big(\{\mathbf{I} - \mathbf{P_2}\} f_2\big) \cdot \partial_t \nabla^k E \\ &\quad \le -\frac{d}{dt}\int \nabla^k \mathcal{A}\big(\{\mathbf{I} - \mathbf{P_2}\} f_2\big) \cdot \nabla^k E + \frac{1}{4}\|\nabla^{k+1} B\|_2^2. \end{aligned} \tag{4.49}$$

We may bound the second term as usual by

$$\begin{aligned} \int \nabla^k \mathcal{A}(\mathfrak{L}_2 + \mathfrak{N}_2) \cdot \nabla^k E &\lesssim \eta \|\nabla^k E\| + C_\eta \|\nabla^k \{\mathbf{I} - \mathbf{P_2}\} f_2\|_\sigma^2 \\ &\quad + \sqrt{\mathcal{E}_2}\big(\|\nabla^k f_2\|_\sigma^2 + \|\nabla^k E\|_2^2 + \|\nabla^{k+1} f_1\|_2^2 + \|\nabla^{k+1} B\|_2^2\big). \end{aligned} \tag{4.50}$$

Plugging (4.49)–(4.50) into (4.48) implies that there exists a functional $\mathfrak{G}^k(t)$ with

$$\big|\mathfrak{G}^k(t)\big| \lesssim \|\nabla^k f_2\|_2^2 + \|\nabla^k E\|_2^2 \tag{4.51}$$

such that

$$\begin{aligned} &\frac{d}{dt}\mathfrak{G}^k + \|\nabla^k \mathbf{P}_2 f_2\|_2^2 + \|\nabla^k E\|_2^2 \\ &\quad \lesssim \|\nabla^k \{\mathbf{I} - \mathbf{P_2}\} f_2\|_\sigma^2 + \mathcal{E}_2 \mathfrak{D}^k + \|\nabla^{k+1} f_1\|_2^2 + \|\nabla^{k+1} B\|_2^2. \end{aligned} \tag{4.52}$$

Hence, if we define

$$\mathfrak{E}^k(t) := \|\nabla^k f_2\|_2^2 + \|\nabla^k E\|_2^2 + \eta \mathfrak{G}^k, \tag{4.53}$$

then for η sufficiently small, we deduce that $\mathfrak{E}^k(t)$ satisfies (4.33) and that (4.35) follows from (4.42) and (4.52). □

We then record the decay estimates resulting from Lemma 4.3.

Proposition 4.4. *Let $m \ge 8$ and $l \ge \max\{m, 35/4\}$. Assume that $\mathcal{E}_m + \mathcal{E}_{5;l,0} \le M$ is small, then for $t \in [0, T]$ and for $k = 0, 1, 2$,*

$$\|\nabla^k f_2(t)\|_2 + \|\nabla^k E(t)\|_2 \lesssim (1+t)^{-\frac{k+1-\varepsilon}{2}} \sup_{0 \le \tau \le T} \sqrt{\mathcal{E}_m(\tau) + \mathcal{E}_{5;l,0}(\tau)}, \tag{4.54}$$

where $\varepsilon = 3/(4l - 7)$.

Proof. Under the assumption of the proposition, we obtain from Lemma 4.3 that

$$\frac{d}{dt}\mathfrak{E}^k + \mathfrak{D}^k \lesssim \|\nabla^{k+1} f_1\|_2^2 + \|\nabla^{k+1} B\|_2^2. \tag{4.55}$$

We now use a splitting method (velocity-time) as in [18,11,20]. For any $\varepsilon > 0$ and $k = 0, 1, 2$, we have

$$\begin{aligned}\|\nabla^k f_2\|_\sigma^2 &\gtrsim \int \langle v\rangle^{-1}|\nabla^k f_2|^2 = \int\limits_{|v|\le(1+t)^\varepsilon} + \int\limits_{|v|\ge(1+t)^\varepsilon} \\ &\ge (1+t)^{-\varepsilon}\int\limits_{|v|\le(1+t)^\varepsilon} |\nabla^k f_2|^2 \\ &= (1+t)^{-\varepsilon}\|\nabla^k f_2\|_2^2 - (1+t)^{-\varepsilon}\int\limits_{|v|\ge(1+t)^\varepsilon}|\nabla^k f_2|^2,\end{aligned} \tag{4.56}$$

and we bound by

$$\begin{aligned}(1+t)^{-\varepsilon}\int\limits_{|v|\ge(1+t)^\varepsilon}|\nabla^k f_2|^2 &\le (1+t)^{-\varepsilon}(1+t)^{-4(l-k)\varepsilon}\int |v|^{4(l-k)}|\nabla^k f_2|^2 \\ &\le (1+t)^{-(4l-4k+1)\varepsilon}\mathcal{E}_{2;l,0}(t).\end{aligned} \tag{4.57}$$

Plugging the estimates (4.23) of Lemma 4.1 and (4.56)–(4.57) into (4.55), we obtain

$$\begin{aligned}\frac{d}{dt}\mathfrak{E}^k + (1+t)^{-\varepsilon}\mathfrak{E}^k &\lesssim (1+t)^{-(4l-4k+1)\varepsilon}\mathcal{E}_{2;l,0} + (1+t)^{-(k+1)}\sup_{0\le\tau\le T}\{\mathcal{E}_m(\tau)+\mathcal{E}_{5;l,0}(\tau)\} \\ &\lesssim (1+t)^{-\theta}\sup_{0\le\tau\le T}\{\mathcal{E}_m(\tau)+\mathcal{E}_{5;l,0}(\tau)\},\end{aligned} \tag{4.58}$$

where $\theta = \min\{(4l-4k+1)\varepsilon, k+1\}$. By Lemma 2.7, we deduce from (4.58) that for $0 < \varepsilon < 1$,

$$\begin{aligned}\mathfrak{E}^k(t) &\lesssim e^{-\frac{\lambda(1+t)^{1-\varepsilon}}{1-\varepsilon}}\left(\mathfrak{E}^k(0) + \sup_{0\le\tau\le T}\{\mathcal{E}_m(\tau)+\mathcal{E}_{5;l,0}(\tau)\}\int_0^t e^{\frac{\lambda(1+\tau)^{1-\varepsilon}}{1-\varepsilon}}(1+\tau)^{-\theta}\,d\tau\right) \\ &\lesssim \sup_{0\le\tau\le T}\{\mathcal{E}_m(\tau)+\mathcal{E}_{5;l,0}(\tau)\}(1+t)^{-\theta+\varepsilon}.\end{aligned} \tag{4.59}$$

Notice that if we have chosen ε so that $(4l-7)\varepsilon \ge 3$, then $\theta = k+1$. Taking $\varepsilon = 3/(4l-7)$ yields the decay (4.54). □

5. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. Let $m \ge 9$ and $l \ge m + 1/4$. We denote

$$T_* = \sup\{t \ge 0 \mid \mathcal{E}_m(t) + \mathcal{E}_{m-1;l,q}(t) \le M\}. \tag{5.1}$$

Clearly $T_* > 0$ follows from the local-in-time existence theory if $\mathcal{E}_{m;l,q}(0)$ is sufficiently small (it can follow by combining the arguments in [10,11] and our energy estimates, and the detail is omitted for brevity). Our goal is to show $T_* = \infty$ if we further choose $\mathcal{E}_{m;l,q}(0)$ small.

By Proposition 4.2, Proposition 4.4 and the Sobolev interpolation of Lemma 2.4, we deduce

$$\|\nabla B(t)\|_\infty \lesssim \sqrt{M}(1+t)^{-\frac{5}{4}} \tag{5.2}$$

and

$$\|E(t)\|_\infty \lesssim \sqrt{M}(1+t)^{-\frac{5}{4}+\frac{\varepsilon}{2}}, \tag{5.3}$$

where $\varepsilon = 3/(4l-7) \le 1/10$ since $l \ge 9 + 1/4$. Substituting (5.2)–(5.3) into the estimates (3.70) of Proposition 3.4, we obtain

$$\begin{aligned}&\frac{d}{dt}\mathcal{E}_{m;l,q} + \mathcal{D}_{m;l,q} + \frac{\vartheta q}{(1+t)^{1+\vartheta}}\mathcal{F}_{m;l,q} \\ &\quad\lesssim (\|E\|_\infty + \|\nabla B\|_\infty^2)\mathcal{F}_{m;l,q} + \|\nabla^m E\|_2\sqrt{\mathcal{D}_m} \\ &\quad\lesssim \sqrt{M}(1+t)^{-\frac{5}{4}+\frac{\varepsilon}{2}}\mathcal{F}_{m;l,q} + \|\nabla^m E\|_2\sqrt{\mathcal{D}_m}.\end{aligned} \tag{5.4}$$

Note that if we have fixed $0 < \vartheta \le 1/5$, then $1 + \vartheta \le 5/4 - \varepsilon/2$. Then (5.4) implies

$$\frac{d}{dt}\mathcal{E}_{m;l,q} + \mathcal{D}_{m;l,q} + \frac{\vartheta q}{(1+t)^{1+\vartheta}}\mathcal{F}_{m;l,q} \lesssim \left\|\nabla^m E\right\|_2 \sqrt{\mathcal{D}_m}. \tag{5.5}$$

Similarly, substituting (5.2)–(5.3) into the estimates (3.1) of Proposition 3.1, we deduce

$$\begin{aligned}\frac{d}{dt}\mathcal{E}_m + \mathcal{D}_m &\lesssim \sqrt{M}\mathcal{D}_{m-1;l,q} + \left(\|E\|_\infty^2 + \|\nabla B\|_\infty^2\right)\mathcal{F}_{m;l,q} \\ &\lesssim \sqrt{M}\mathcal{D}_{m-1;l,q} + M(1+t)^{-\frac{5}{2}+\varepsilon}\mathcal{F}_{m;l,q}.\end{aligned} \tag{5.6}$$

Note that we could not simply close the estimates (5.5)–(5.6) since $\|\nabla^m E\|_2$ is not included in $\mathcal{D}_m$ but rather in $\mathcal{E}_m$ (or $\mathcal{D}_{m+1}$). This is caused by the regularity-loss property of the Maxwell system. We shall overcome this difficulty by using the time-weighted energy method with the time rate of negative power as [2]. Multiplying (5.6) by $(1+t)^{-\epsilon_0}$ with $\epsilon_0 > 0$ gives

$$\begin{aligned}&\frac{d}{dt}\left\{(1+t)^{-\epsilon_0}\mathcal{E}_m\right\} + (1+t)^{-(1+\epsilon_0)}\mathcal{E}_m + (1+t)^{-\epsilon_0}\mathcal{D}_m \\ &\quad\lesssim \sqrt{M}(1+t)^{-\epsilon_0}\mathcal{D}_{m-1;l,q} + M(1+t)^{-\epsilon_0}(1+t)^{-\frac{5}{2}+\varepsilon}\mathcal{F}_{m;l,q}.\end{aligned} \tag{5.7}$$

Multiplying (5.5) by $(1+t)^{-(1+\epsilon_0)/2}$ yields

$$\begin{aligned}&\frac{d}{dt}\left\{(1+t)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{m;l,q}\right\} + (1+t)^{-\frac{3+\epsilon_0}{2}}\mathcal{E}_{m;l,q} + (1+t)^{-\frac{1+\epsilon_0}{2}}\left\{\mathcal{D}_{m;l,q} + \frac{\vartheta q}{(1+t)^{1+\vartheta}}\mathcal{F}_{m;l,q}\right\} \\ &\quad\lesssim (1+t)^{-\frac{1+\epsilon_0}{2}}\left\|\nabla^m E\right\|_2\sqrt{\mathcal{D}_m} \lesssim (1+t)^{-(1+\epsilon_0)}\mathcal{E}_m + \mathcal{D}_m.\end{aligned} \tag{5.8}$$

We then combine (5.7) and (5.8) to obtain

$$\begin{aligned}&\frac{d}{dt}\left\{(1+t)^{-\epsilon_0}\mathcal{E}_m + (1+t)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{m;l,q}\right\} + (1+t)^{-(1+\epsilon_0)}\mathcal{E}_m + (1+t)^{-\epsilon_0}\mathcal{D}_m \\ &\qquad + (1+t)^{-\frac{3+\epsilon_0}{2}}\mathcal{E}_{m;l,q} + (1+t)^{-\frac{1+\epsilon_0}{2}}\left\{\mathcal{D}_{m;l,q} + \frac{\vartheta q}{(1+t)^{1+\vartheta}}\mathcal{F}_{m;l,q}\right\} \\ &\quad\lesssim \sqrt{M}(1+t)^{-\epsilon_0}\mathcal{D}_{m-1;l,q} + M(1+t)^{-\epsilon_0}(1+t)^{-\frac{5}{2}+\varepsilon}\mathcal{F}_{m;l,q} + \mathcal{D}_m.\end{aligned} \tag{5.9}$$

It is easy to check that $(1+\epsilon_0)/2 + 1 + \vartheta \le \epsilon_0 + 5/2 - \varepsilon$. Hence (5.9) implies

$$\begin{aligned}&\frac{d}{dt}\left\{(1+t)^{-\epsilon_0}\mathcal{E}_m + (1+t)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{m;l,q}\right\} + (1+t)^{-(1+\epsilon_0)}\mathcal{E}_m + (1+t)^{-\epsilon_0}\mathcal{D}_m \\ &\qquad + (1+t)^{-\frac{3+\epsilon_0}{2}}\mathcal{E}_{m;l,q} + (1+t)^{-\frac{1+\epsilon_0}{2}}\left\{\mathcal{D}_{m;l,q} + \frac{\vartheta q}{(1+t)^{1+\vartheta}}\mathcal{F}_{m;l,q}\right\} \\ &\quad\lesssim \sqrt{M}(1+t)^{-\epsilon_0}\mathcal{D}_{m-1;l,q} + \mathcal{D}_m.\end{aligned} \tag{5.10}$$

We now want to absorb the right hand side of (5.10). Notice that (5.5) holds for $m \ge 8$, hence we may replace m with $m-1$ (now $m \ge 9$) to have

$$\frac{d}{dt}\mathcal{E}_{m-1;l,q} + \mathcal{D}_{m-1;l,q} + \frac{\vartheta q}{(1+t)^{1+\vartheta}}\mathcal{F}_{m-1;l,q} \lesssim \left\|\nabla^{m-1} E\right\|_2\sqrt{\mathcal{D}_{m-1}} \lesssim \mathcal{D}_m. \tag{5.11}$$

Note that if we have fixed $0 < \epsilon_0 \le 7/10$, then

$$\frac{1+\epsilon_0}{2} + 1 + \vartheta \le \frac{5}{2} - \varepsilon. \tag{5.12}$$

Hence we may combine (5.10), (5.11) and (5.6) to deduce

$$\frac{d}{dt}\left\{\mathcal{E}_m+\mathcal{E}_{m-1;l,q}+(1+t)^{-\epsilon_0}\mathcal{E}_m+(1+t)^{-\frac{1+\epsilon_0}{2}}\mathcal{E}_{m;l,q}\right\}$$
$$+\mathcal{D}_m+\mathcal{D}_{m-1;l,q}+\frac{\vartheta q}{(1+t)^{1+\vartheta}}\mathcal{F}_{m-1;l,q}+(1+t)^{-(1+\epsilon_0)}\mathcal{E}_m$$
$$+(1+t)^{-\frac{3+\epsilon_0}{2}}\mathcal{E}_{m;l,q}+(1+t)^{-\frac{1+\epsilon_0}{2}}\left\{\mathcal{D}_{m;l,q}+\frac{\vartheta q}{(1+t)^{1+\vartheta}}\mathcal{F}_{m;l,q}\right\}\le 0. \tag{5.13}$$

The direct integration in time of the inequality (5.13) in particular yields (1.23) for $0\le t\le T_*$. Upon choosing the initial condition $\mathcal{E}_{m;l,q}(0)$ further smaller, we deduce that for $0\le t\le T_*$

$$\mathcal{E}_m(t)+\mathcal{E}_{m-1;l,q}(t)\le\frac{M}{2}<M. \tag{5.14}$$

This implies that $T_*=\infty$, which then implies (1.23) for all t. Thus the solution is indeed global. Also, the decay estimates (1.24)–(1.25) follows from Propositions 4.2, 4.4 and (1.23). This concludes Theorem 1.1. □

Conflict of interest statement

The author declares that there is no conflict of interest.

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