

The Korteweg–de Vries equation at H^{-1} regularity [☆]

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Abstract

In this paper we will prove the existence of weak solutions to the Korteweg–de Vries initial value problem on the real line with H^{-1} initial data; moreover, we will study the problem of orbital and asymptotic H^s stability of solitons for integers $s \geq -1$; finally, we will also prove new a priori H^{-1} bound for solutions to the Korteweg–de Vries equation. The paper will utilise the Miura transformation to link the Korteweg–de Vries equation to the modified Korteweg–de Vries equation.

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1. Introduction and statement of result

Consider the initial value problem (IVP) of the *Korteweg–de Vries* (KdV) equation:

$$\begin{cases} u_t + u_{xxx} - 6uu_x = 0 \\ u(0, x) = u_0(x), \end{cases} \quad (1)$$

for $x \in \mathbb{R}$ and rough initial data u_0 in the Sobolev space H^s .

It is well known that the KdV equation exhibits special travelling wave solutions, known as solitons – indeed such solutions provided much of the historical impetus to study the equation. Explicitly, up to a spatial translation, these solutions may be written in the form

$$u := R_c(x - ct), \quad (2)$$

where $c > 0$ and

$$R_c := -\frac{c}{2} \operatorname{sech}^2\left(\frac{\sqrt{c}x}{2}\right). \quad (3)$$

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Let us summarise the well-posedness theory and stability results. The initial value for KdV is known to be globally well-posed¹ for $s \geq -\frac{3}{4}$, (see [4,2,22,10]). The problem is known to be ill-posed for $s < -\frac{3}{4}$ in the sense that the flow map cannot be uniformly continuous [9]. One may hope for Hadamard well-posedness for $s \geq -1$, (cf. [11,14,5]). Using the inverse scattering transform, Kappeler and Topalov proved that the flow map extends continuously to H^{-1} in the periodic case, which provides motivation to address the supposedly simpler question of well-posedness in H^{-1} on the real line. In a similar situation for the Nonlinear Schrödinger equation, a-priori bounds have been obtained (cf. [1] and [11]) using heavily recent techniques in dispersive equations, in contrast to the more elementary approach of the paper at hand. The techniques have since been adapted to KdV by Liu [14] in order to obtain a-priori bounds for $s \geq -\frac{4}{5}$.

On the other hand Molinet [19] has shown that no well-posedness can possibly hold below $s = -1$: the solution map $u_0 \rightarrow u(t)$ does not extend to a continuous map from H^s , for $s < -1$, to distributions.

Orbital stability of the soliton in the energy space H^1 follows from Weinstein’s convexity argument [24], this argument even holds for other sub-critical gKdV equations. Weinstein’s argument is at the basis of a considerable amount of work since then, with one direction culminating in the seminal work of Martel and Merle to some version of asymptotic stability, again in the energy space [15]. Merle and Vega proved orbital stability and asymptotic stability of the soliton manifold in L^2 using the Miura map [16] in a similar fashion to our approach. Their approach to the stability of the kink however is closer to the arguments of Martel and Merle for generalised KdV.

We now present our principal results.

Theorem 1 (New H^{-1} a priori estimate for KdV). *Suppose $s \geq -\frac{3}{4}$ and $u \in C([0, \infty); H^s(\mathbb{R}))$ is a solution to (1), then*

$$\|u(t, \cdot)\|_{H^{-1}} \lesssim^2 \|u_0\|_{H^{-1}} + \|u_0\|_{H^{-1}}^3 \quad \text{for } t \in [0, \infty). \tag{4}$$

Remark 1. Applying scaling, the dependence on the H^{-1} norm of the initial data in (4) can be made more explicit, i.e. if λ is such that

$$0 < \lambda \leq \|u_0\|_{H^{-1}}^{-2},$$

then we have

$$\|u(t, \lambda \cdot)\|_{H^{-1}} \lesssim \|u_0(\lambda \cdot)\|_{H^{-1}} \quad \text{for } t \in [0, \infty).$$

Theorem 2 (Orbital stability of KdV solitons). *There exists an $\varepsilon > 0$ such that the following statement holds: Given any integer $s \geq -1$, let u be a solution to (1) satisfying $u \in C([0, \infty); H^{-3/4}(\mathbb{R}))$ if $s = -1$ and $u \in C([0, \infty); H^s(\mathbb{R}))$ for $s \geq 0$. Furthermore, assume u satisfies the bound $\|R_c - u_0\|_{H^{-1}} < \varepsilon c^{1/4}$ for some $c > 0$. Then there exists a continuous function $y : [0, \infty) \rightarrow \mathbb{R}$ such that*

$$\|u - R_c(\cdot - y(t))\|_{H^s} \leq \gamma_s(c, \|R_c - u_0\|_{H^s})$$

for any $t \geq 0$, where $\gamma_s : (0, \infty) \times [0, \infty)$ is a continuous function, polynomial in the second variable, which satisfies $\gamma(\cdot, 0) = 0$.

Remark 2. If we rescale c to 4, we obtain a more precise result. The smallness assumption becomes

$$\|4c^{-1}u_0(2c^{-1/2}x) - R_4\|_{H^{-1}} \leq \varepsilon,$$

which is weaker and more natural than the assumption of the theorem.

Theorem 3 (Asymptotic stability of KdV solitons). *Given real $\gamma > 0$ and integer $s \geq -1$, there exists an $\varepsilon_\gamma > 0$ such that if $u \in C([0, \infty); H^s(\mathbb{R}) \cap H^{-3/4}(\mathbb{R}))$, is a solution to (1), satisfying*

¹ See [22] for a discussion on the subtleties in the definition of well-posedness.

² Throughout this article we will adopt the notation $a \lesssim b$ to signify $a \leq Cb$, where C is an insignificant constant.

$$\|R_c - u_0\|_{H^{-1}} < \varepsilon_\gamma c^{1/4}$$

for $c > 0$, then there is a continuous function $y : [0, \infty) \rightarrow \mathbb{R}$ and $\tilde{c} > 0$ such that

$$\lim_{t \rightarrow \infty} \|u - R_{\tilde{c}}(x - y(t))\|_{H^s(\cdot, \infty)} = 0$$

for any $t \geq 0$. Moreover we have the bound $|c - \tilde{c}| \lesssim c^{3/4} \|R_c - u_0\|_{H^{-1}}$.

The decay follows from an explicit quantitative estimate in Proposition 17 for H^{-1} , and similar estimates for higher norms in Corollary 19. The estimates we obtain are sufficiently strong to obtain existence of weak solutions by a standard approximation and compactness argument.

Theorem 4 (Existence of global H^{-1} weak solutions to KdV IVP). For any $u_0 \in H^{-1}$, there exists a weak solution u to (1) satisfying

$$u \in C_\omega([0, \infty); H^{-1}(\mathbb{R})),^3 \tag{5}$$

$$u \in L^2([0, T] \times [-R, R]) \text{ for } R, T < \infty, \tag{6}$$

$$u(t, \cdot) \rightarrow u_0 \text{ in } H^{-1} \text{ as } t \downarrow 0. \tag{7}$$

Furthermore u satisfies the bounds given in Theorem 1.

A closely related problem to the initial value problem of the Korteweg–de Vries equation is that of the *modified* Korteweg–de Vries (mKdV) equation:

$$\begin{cases} u_t + u_{xxx} - 6u^2u_x = 0 \\ u(0, x) = u_0(x), \end{cases} \tag{8}$$

for $x \in \mathbb{R}$ and initial data u_0 .

An explicit family of solutions of the mKdV equation, called *kink* solutions, can be written up to translations as

$$Q_\lambda(t, x) := \lambda \tanh(\lambda x + 2\lambda^3 t),$$

for any $\lambda > 0$.

The mKdV problem and the KdV may be connected via a differential transformation known as the *Miura map*:

$$u \mapsto u_x + u^2; \tag{9}$$

which sends solutions of (8) to solutions of (1). To see this property formally, set

$$\begin{aligned} \text{KdV}(u) &= u_t + u_{xxx} - 6uu_x, \\ \text{mKdV}(u) &= u_t + u_{xxx} - 6u^2u_x, \quad \text{and} \\ M(u) &= u_x + u^2. \end{aligned}$$

One can then easily check that

$$\text{KdV}(M(u)) = (\text{mKdV}(u))_x + 2u \cdot \text{mKdV}(u), \tag{10}$$

from which it follows that $\text{KdV}(M(u)) = 0$ whenever $\text{mKdV}(u) = 0$. Additionally, note the mKdV equation satisfies the reflection symmetry: if u is a solution to (8), then $-u$ is also a solution. Hence if we define

$$M^*(u) := M(-u) = -u_x + u^2,$$

then $M^*(u)$ maps solutions to the mKdV equation to solutions to the KdV equation.

³ Here $C_\omega([0, \infty); H^{-1}(\mathbb{R}))$ denotes the space of weakly continuous functions from \mathbb{R} to H^{-1} .

The Korteweg–de Vries equation is invariant under the *Galilean transformation*:

$$u(t, x) \mapsto u(t, x - ht) - \frac{h}{6}, \quad (11)$$

for $h \in \mathbb{R}$, i.e. if u is a solution to the KdV equation, then its image under the above transformation is also a solution, which is easily verified.

The Korteweg–de Vries equation also satisfies the following scaling symmetry:

$$u(t, x) \mapsto \frac{1}{\lambda^2} u\left(\frac{t}{\lambda^3}, \frac{x}{\lambda}\right), \quad (12)$$

for $\lambda > 0$ and $\dot{H}^{-\frac{3}{2}}$ is the critical space.

In Section 2 we will show how to use the Miura map combined with the Galilean symmetry to relate mKdV solutions near a kink solutions to either KdV solutions near 0, or to KdV solutions near a soliton. This will afford us the freedom to choose the most convenient setting in order to prove the stated results. The H^{-1} a priori estimate of Theorem 1 will then follow as a consequence of the L^2 stability of the kink (Theorem 14). The orbital (Theorem 2) and asymptotic stability (Theorem 3) of the soliton in the H^{-1} norm will follow from the corresponding statement for the mKdV kink in L^2 (Theorem 14 and Theorem 18). Higher conserved energies imply stability of the trivial solution in H^s for nonnegative integers s , and Kato's local smoothing argument along a moving frame implies asymptotic stability of the trivial solution to the right. We use the Miura map to derive orbital and asymptotic stability of the soliton for KdV, as well as orbital and asymptotic stability of the kink for mKdV in higher norms, requiring smallness of the deviation only in H^{-1} , see Corollary 16 and the proof of Corollary 19.

The Miura map has been used in a simpler setting by Kappeler et al. [6]. Their results are limited by the fact that the Miura map is not invertible. Our additional ingredient is the shift of the initial data using the Galilean invariance. To the best of our knowledge the corresponding results on the Miura map are new, and we believe them to be appealing and of independent interest.

Of course this is intimately related to the integrable structure of KdV and mKdV, and also the Lax-Pair is clearly in the background. Nevertheless we do not explicitly use the integrable machinery, and the use of elementary key elements of the theory of integrable systems in combination with a PDE oriented approach seems to be new and promising.

It is worthwhile to point out that unlike the corresponding asymptotic stability results for generalised KdV, the scale \tilde{c} is independent of time. This holds since the scale of the kink is related to its size at infinity, and this does not change by adding L^2 perturbations.

2. Inverting the Miura map

Kappeler et al. showed in the paper [6] that if the initial data $u_0 \in H^{-1}$ is contained in the image of L^2 under the Miura map restricted to L^2 , then there exists a global weak solution to the IVP (1). The proof consists of constructing a weak solution to mKdV corresponding to initial data in the preimage under the Miura map of the original initial data, and then transforming the solution back, under the Miura map, to a solution to KdV. The following proposition is one of the key tools used by Kappeler et al. to characterise the range of the Miura map.

Proposition 5. (See [6].) *Let $u_0 \in H_{\text{loc}}^{-1}$. The following three statements are equivalent.*

- (1) *The Schrödinger operator $H_{u_0} := -\partial_{xx} + u_0$ is positive semi-definite.*
- (2) *There exists a strictly positive function ϕ with $-\phi_{xx} + u_0\phi = 0$.*
- (3) *$u_0 \in H_{\text{loc}}^{-1}$ is in the range of the Miura map on L_{loc}^2 .*

In order to remove the restrictions on the initial data imposed in [6], we will employ the use of the Galilean transformation in order to transform KdV into the range of the Miura map. This allows us to link rough H^{-1} KdV initial data to mKdV initial data. The corresponding mKdV initial data will be in the form of a sum of an L^2 function and a tanh kink. The authors would like to note that the original idea to use such an argument was somewhat motivated

by the papers [16] and [18] – related to the L^2 stability of soliton solutions to the KdV equation and KP-II equation respectively.

Appendix A contains results for Schrödinger operators with H^{-1} potentials, which we will use below.

Our aim now is to construct an “inverse” of the Miura map for Galilean transformed initial data. The Galilean transformation essentially adds a constant to the potential of the Schrödinger operator corresponding to the initial data. We easily achieve positive definiteness by adding a large enough constant; the caveat being that the initial data will no longer remain in H^{-1} .

Given initial data $u_0 \in H^{-1}$, applying the Galilean symmetry to u_0 , with h set to $-6\lambda^2$, for some $\lambda > 0$ and $t = 0$, yields the function $u_0 + \lambda^2$. Now consider the problem of finding a function $r \in L^2$ that is in the preimage of $u_0 + \lambda^2$ under the Miura transformation. Observe that $M(\lambda \tanh(\lambda \cdot)) = \lambda^2$; it then seems natural to consider the problem

$$(r + \lambda \tanh \lambda x)_x + (r + \lambda \tanh \lambda x)^2 = u_0 + \lambda^2. \tag{13}$$

For the problem of stability of solitons, we assume we are given some initial data $u_0 \in H^s$, where $s \geq -\frac{3}{4}$. Applying the Galilean transform with h as above, and noting that $M^*(\lambda \tanh(\lambda \cdot)) = \lambda^2 - 2\lambda^2 \operatorname{sech}^2(\lambda \cdot)$, we are led to consider the problem

$$-(r + \lambda \tanh(\lambda \cdot))_x + (r + \lambda \tanh(\lambda \cdot))^2 = u_0 + \lambda^2. \tag{14}$$

We now state sufficient and necessary conditions for the problems (13) and (14) to have a solution.

Proposition 6. *Let $\lambda > 0$. The ground state energy of H_{u_0} , $u_0 \in H^{-1}$ is $-\lambda^2$ if and only if there exists $r \in L^2 - \lambda \tanh(\lambda \cdot)$ such that*

$$M(r) = u_0 + \lambda^2.$$

The spectrum of H_{u_0} is contained in $(-\lambda^2, \infty)$ if and only if there exists $r \in L^2 + \lambda \tanh(\lambda \cdot)$ with

$$M(r) = u_0 + \lambda^2.$$

Proof. Let ϕ be the ground state. Observe then $r = \frac{d}{dx} \ln \phi$ satisfies the Riccati equation (see Appendix A)

$$r_x + r^2 = u_0 + \lambda^2.$$

Then as a consequence of Lemma 25 we have either

$$r - \lambda \in L^2(0, \infty) \quad \text{or} \quad r + \lambda \in L^2(0, \infty).$$

Note however the property that

$$e^{\int_0^x r} \in L^2$$

enforces $r + \lambda \in L^2(0, \infty)$. Similarly, we obtain $r - \lambda \in L^2(-\infty, 0)$ and thus

$$r + \lambda \tanh(\lambda x) \in L^2.$$

Hence $u_0 + \lambda^2$ is in the range of the Miura map on $-\lambda \tanh(\lambda x) + L^2$ if the ground state energy is $-\lambda^2$.

Now assume that

$$\lambda^2 + u_0 = r_x + r^2 \quad \text{and} \quad r + \lambda \tanh(\lambda x) \in L^2.$$

Then $\phi = e^{\int_0^x r}$ is a strictly positive function in H^1 satisfying

$$(H_{u_0} + \lambda^2)\phi = 0,$$

i.e. ϕ is the ground state with ground state energy $-\lambda^2$.

Now we turn to the case when the spectrum is contained in $(-\lambda^2, \infty)$. Since $H_{u_0 + \lambda^2}$ is positive semi-definite, by Proposition 5, there exists strictly positive $\phi \in L^2_{\text{loc}}$ satisfying

$$(H_{u_0} + \lambda^2)\phi = 0. \tag{15}$$

Note that $\phi \notin L^2$, otherwise ϕ would be the ground state. Since $\frac{d}{dx} \ln \phi$ solves the Riccati equation, it follows by Lemma 25 that either ϕ grows exponentially as $x \rightarrow \infty$ or as $x \rightarrow -\infty$.

We aim to construct a solution $\tilde{\phi}$ to (15) satisfying

$$\tilde{\phi}(x) \rightarrow \infty \quad \text{as } x \rightarrow \pm\infty. \tag{16}$$

Suppose ϕ is not such a solution; then without loss of generality we can assume

$$\phi(x) \rightarrow \infty \quad \text{as } x \rightarrow \infty,$$

and

$$\phi(x) \rightarrow 0 \quad \text{as } x \rightarrow -\infty.$$

We obtain using Lemma 25 that $\frac{d}{dx} \ln \phi - \lambda \in L^2$. It is then not difficult to show that

$$\tilde{\phi}(x) = C\phi(x) + \phi(x) \int_0^x \phi^{-2}(s) ds \tag{17}$$

for large $C > 0$, is a solution to (15), satisfying the growth conditions (16).

We now define $r = \frac{d}{dx} \ln \tilde{\phi}$. It satisfies the Riccati equation; moreover, by Lemma 25,

$$r - \lambda \tanh(\lambda x) \in L^2.$$

Thus $u_0 + \lambda^2$ is in the range of the Miura map restricted to $L^2 + \lambda \tanh(\lambda x)$ if the spectrum of u_0 is contained in $(-\lambda^2, \infty)$.

Now suppose that $u_0 + \lambda^2 = r_x + r^2$ for $r - \lambda \tanh(\lambda x) \in L^2$; hence $\phi = e^{\int_0^x r}$ satisfies the equation

$$-\phi'' + u_0\phi = -\lambda^2\phi,$$

and

$$\phi(x) \rightarrow \infty \quad \text{as } x \rightarrow \pm\infty.$$

Observe that Proposition 5 implies that the Schrödinger operator H_{u_0} has spectrum contained in $[-\lambda^2, \infty)$. We want to show that $-\lambda^2$ is not an eigenvalue. If it were, then there would be a non-negative, strictly positive L^2 ground state ψ . This is not possible since ϕ/ψ cannot attain a minimum (see Lemma 24). Therefore the spectrum is contained in $(-\lambda^2, \infty)$. \square

We now turn to the problem of relating the two sides of (13) and (14) by analytic diffeomorphisms. We begin with a technical statement.

Lemma 7. *The multiplication map $(u, v) \rightarrow uv$ can be extended from the bilinear map $C_0^\infty \times C_0^\infty \rightarrow C_0^\infty$ to continuous bilinear maps*

$$\begin{aligned} L^2 \times L^2 &\rightarrow L^1 \subset H^s, && \text{for any } s < -\frac{1}{2} \\ L^2 \times H^{s'} &\rightarrow H^s, && -\frac{1}{2} < s \leq 0, s' > \frac{1}{2} + s \\ H^{s_1} \times H^{s_2} &\rightarrow H^{s_1}, && \text{for any } s_2 > \frac{1}{2}, 0 \leq s_1 \leq s_2. \end{aligned}$$

Proof. The first two statements may be proved using Sobolev embedding inequalities and their corresponding dual inequalities. The last case is a particular case of Theorem 1, of Section 4.6.1 of [21], alternatively it may be proved by interpolating the second statement with the well known algebraic property of Sobolev spaces H^s for $s > \frac{1}{2}$. \square

For $s \geq -1$, let $F_\lambda : H^{s+1} \rightarrow H^s \times \mathbb{R}$ and $F^* : H^{s+1} \times (0, \infty) \rightarrow H^s$ to be the maps:

$$F_\lambda(r) = \left(r^2 + 2r\lambda \tanh(\lambda x) + r_x, \int r \operatorname{sech}^2(\lambda x) dx \right),$$

$$F^*(r, \lambda) = r^2 + 2r\lambda \tanh(\lambda x) - r_x - 2\lambda^2 \operatorname{sech}^2(\lambda \cdot).$$

It then follows from Lemma 7, that the above maps define quadratic (neglecting λ for F^* here), and hence analytic maps from $H^{s+1} \rightarrow H^s \times \mathbb{R}$ and $H^{s+1} \times (0, \infty) \rightarrow H^s$, respectively. Analyticity in λ (and joined analyticity) follows from the obvious holomorphic extension of λ into the complex plane.

The equations

$$r^2 + 2r \tanh x + r_x = u_0,$$

and

$$r^2 + 2r\lambda \tanh(\lambda \cdot) - r_x - 2\lambda^2 \operatorname{sech}^2(\lambda \cdot) = u_0;$$

relating functions in range and image come from the expansion of the left hand sides of (13) and (14) respectively.

Now let $L_{\lambda,r}$ denote the first component of the Fréchet derivative at r , and similarly let $L_{\lambda,r}^*$ denote the Fréchet derivative of F^* with respect to the first component at (r, λ) , i.e.

$$L_{\lambda,r} v := 2(\lambda \tanh(\lambda \cdot) + r)v + v_x,$$

and $L_{\lambda,r}^*$ is its formal adjoint:

$$L_{\lambda,r}^* v := 2(\lambda \tanh(\lambda \cdot) + r)v - v_x.$$

Lemma 8. For any $s \in \mathbb{R}$ and $r \in H^s \cap L^2$, the abstract operator $L_{\lambda,r}$ and its formal adjoint operator $L_{\lambda,r}^*$ define bounded operators from $H^{s+1}(\mathbb{R})$ to $H^s(\mathbb{R})$, which we denote by $L_{\lambda,r}$ and $L_{\lambda,r}^*$, respectively, suppressing s from the notation.

Both $L_{\lambda,r}$ and $L_{\lambda,r}^*$ are Fredholm operators of index 1 and -1 respectively. Moreover, setting $\phi_r = \operatorname{sech}^2(\lambda \cdot) \times e^{-2 \int_0^r dy}$, the operator $L_{\lambda,r}$ is surjective, with null space spanned by ϕ_r ; and the formal adjoint $L_{\lambda,r}^*$ is injective with closed range and cokernel spanned by ϕ_r .

Proof. It follows from Lemma 7 that the operators $L_{\lambda,r}$ and $L_{\lambda,r}^*$ define continuous linear operators from H^{s+1} to H^s . A simple calculation shows that

$$L_{\lambda,r} \operatorname{sech}^2(\lambda x) \exp\left(-2 \int_0^x r dy\right) = 0.$$

Since $L_{\lambda,r} \phi = 0$ is a scalar ordinary differential equation, every solution is a multiple of $\operatorname{sech}^2(\lambda x) \exp(-2 \int_0^x r dy)$ and the null space is one dimensional. Similarly, one can easily check that $L_{\lambda,r}^*$ is injective (since solutions to the homogeneous equation are multiples of $\cosh^2(\lambda x) \exp(2 \int_0^x r dy)$) and $\operatorname{sech}^2(\lambda x) \exp(-2 \int_0^x r dy)$ spans the cokernel of $L_{\lambda,r}^*$.

To complete the proof we need to show $L_{\lambda,r}$ is surjective, and $L_{\lambda,r}^*$ is injective with closed range. By scaling, it suffices to show the case when $\lambda = 1$. For reasons of brevity we will use the shorthand $L_r := L_{1,r}$, and $L := L_{1,0}$.

We start by defining an integral operator from L^2 to H^1 which is a right inverse of L_r . We begin with the simpler case when $r = 0$.

Let $\eta \in C_0^\infty([-2, 2])$ be a non-negative function such that $\eta \equiv 1$ on $[-1, 1]$ and consider the operator T defined by

$$T(g) = e^{-2 \int_0^x \tanh s ds} \int_{-\infty}^x e^{2 \int_0^y \tanh s' ds'} \eta(y) g(y) dy + e^{-2 \int_0^x \tanh s ds} \int_0^x e^{2 \int_0^y \tanh s' ds'} (1 - \eta(y)) g(y) dy.$$

Note that T is a well defined operator for functions in C_0^∞ . It can then be easily checked that Tg satisfies $LTg = g$. Now let $K(x, y)$ be the kernel of T :

$$K(x, y) \equiv \begin{cases} \eta(y) \cosh^2 y \operatorname{sech}^2 x & y < x \leq 0 \\ \eta(y) \cosh^2 y \operatorname{sech}^2 x & y < 0 < x \\ -(1 - \eta(y)) \cosh^2 y \operatorname{sech}^2 x & x \leq y \leq 0 \\ \cosh^2 y \operatorname{sech}^2 x & 0 \leq y < x \\ 0 & \text{otherwise.} \end{cases} \tag{18}$$

Now consider the case for general r : formally we have

$$T_r g := \exp\left(-2 \int_0^\cdot r\right) T \exp\left(2 \int_0^\cdot r\right) g,$$

satisfies $L_r T_r g = g$; furthermore the kernel of T_r is given by

$$K_r(x, y) = K(x, y) \exp\left(-2 \int_0^x r + 2 \int_0^y r\right).$$

We now claim that

$$\|T_r g\|_{H^1} \lesssim \|g\|_{L^2}.$$

Observe that

$$K_r(x, y) \lesssim e^{-2|y-x| + \sqrt{|y-x|} \|r\|_{L^2}} \lesssim e^{-|y-x| + \|r\|_{L^2}^2}, \tag{19}$$

and hence

$$\|T_r g\|_{L^2} \lesssim \|e^{-|\cdot|} * g\|_{L^2} \lesssim \|g\|_{L^2}. \tag{20}$$

The equality

$$\partial_x T_r g + 2(\tanh(x) + r) T_r g = g, \tag{21}$$

implies

$$\begin{aligned} \|\partial_x T_r g\|_{L^2} &\leq 2\|T_r g\|_{L^2} + 2\|r T_r g\|_{L^2} + \|g\|_{L^2} \\ &\lesssim \|g\|_{L^2} + \|r\|_{L^2} \|T_r g\|_{L^\infty} \\ &\lesssim \|g\|_{L^2} + \|r\|_{L^2} \|\partial_x T_r g\|_{L^2}^{1/2} \|T_r g\|_{L^2}^{1/2}, \end{aligned}$$

where we used the L^2 estimate (20), Hölder’s inequality and Gagliardo–Nirenberg’s inequality. Finally applying Young’s inequality and (20) again, we obtain

$$\|\partial_x T_r g\|_{L^2} \lesssim (1 + \|r\|_{L^2}^2) \|g\|_{L^2}. \tag{22}$$

Hence if $r \in L^2$, then T_r extends to a bounded operator from L^2 to H^1 , satisfying $L_r T_r g = g$, thus $L_r : H^1 \rightarrow L^2$ is surjective.

By duality, for every $r \in L^2$, the adjoint operator $L_r^* : L^2 \rightarrow H^1$ is injective with closed range, or equivalently

$$\|f\|_{L^2} \lesssim \|L_r^* f\|_{H^{-1}} \tag{23}$$

for all $f \in L^2$.

We will now show that given any $f \in H^{s+1}$ and $h \in H^s \cap L^2$, if

$$g := 2(\tanh(\cdot) + h) f \pm f_x,$$

then we have the following inequality

$$\|f\|_{H^{s+1}} \leq C(\|f\|_{L^2} + \|g\|_{H^s}), \tag{24}$$

for some constant C depending on $\|h\|_{H^s} + \|h\|_{L^2}$.

First note the trivial estimate

$$\begin{aligned} \|f\|_{H^{s+1}} &\lesssim \|f\|_{L^2} + \|\partial_x f\|_{H^s} \\ &\lesssim \|f\|_{L^2} + \|g\|_{H^s} + \|f\|_{H^s} + \|hf\|_{H^s}. \end{aligned} \tag{25}$$

Consider the case for $-1 \leq s < -\frac{1}{2}$: it follows from Lemma 7 and (25) that

$$\|f\|_{H^{s+1}} \lesssim \|f\|_{L^2} + \|g\|_{H^s} + (1 + \|h\|_{L^2})\|f\|_{L^2}. \tag{26}$$

Now consider the case when $s \geq -\frac{1}{2}$: again from Lemma 7 and (25) we have

$$\|f\|_{H^{s+1}} \lesssim \|f\|_{H^s} + \|g\|_{H^s} + (1 + \|h\|_{H^s} + \|h\|_{L^2})\|f\|_{H^{s+3/4}}.$$

Using (26) and applying the above estimate recursively we obtain (24).

Combining (23) with (24) ($h = r$), it follows that for all $s \geq -1$, $f \in H^{s+1}$ and $r \in L^2 \cap H^s$

$$\|f\|_{H^{s+1}} \lesssim \|L_r^* f\|_{H^s},$$

i.e. $L_r^* : H^{s+1} \rightarrow H^s$ is injective with closed range.

By duality it follows that if $r \in L^2$ then $L_r : L^2 \rightarrow H^{-1}$ is surjective; moreover, as a consequence of (24) ($h = r$) we obtain $L_r : H^{s+1} \rightarrow H^s$ is surjective for all $s \in \mathbb{R}$ and $r \in L^2 \cap H^s$, and again the full statement for L_r^* follows. \square

Let $U_{>\kappa}^s \subset H^s$ denote the set of all functions in H^s whose associated Schrödinger operator has spectrum contained in (κ, ∞) . Similarly, define $U_{<\kappa}^s$ to be the subset of H^s of all functions f whose associated Schrödinger operator has spectrum $\omega(T_f)$ such that $\omega \setminus (\kappa, \infty) \neq \emptyset$.

Theorem 9. *For any $s \geq -1$ the map $F_\lambda : H^{s+1} \rightarrow H^s \times \mathbb{R}$ is an analytic diffeomorphism onto its range. Moreover, for any $f \in U_{>-\lambda^2}^s$, there exists $\rho \in \mathbb{R}$ such that (f, ρ) is contained in the range of $F_\lambda : H^{s+1} \rightarrow H^s \times \mathbb{R}$.*

For any $s \geq -1$ the map $F^ : H^{s+1} \times (0, \infty) \rightarrow H^s$ is an analytic diffeomorphisms onto $U_{<0}^s$.*

Proof. First we show the two maps are local analytic diffeomorphisms.

Note the second component of $DF_\lambda|_r$ is simply the map $f \mapsto \langle f, \text{sech}^2(\lambda \cdot) \rangle$; hence from Lemma 8 we obtain $DF_\lambda|_r$ is invertible – here we use the fact

$$\langle \phi_r, \text{sech}^2(\lambda \cdot) \rangle > 0,$$

where ϕ_r is defined as in Lemma 8. Hence by the inverse function theorem, F_λ is a local analytic diffeomorphism.

Let $G : H^{-1} \rightarrow \mathbb{R}$ be the map from potentials to the ground state energy of their corresponding Schrödinger operator. By Proposition 6 we have that $G(F^*(f, \lambda)) = -\lambda^2$, from which it follows that the derivative of F^* with respect to the second component is not contained in the range of the derivative with respect to the first component. Then from Lemma 8 and the implicit function theorem, it follows that F^* is a local analytic diffeomorphism.

We now prove the injectivity of the two maps. Suppose $r_i, i \in \{1, 2\}$ satisfy $F_\lambda(r_1) = F_\lambda(r_2)$. Then, with $w = r_2 - r_1$ we have

$$w_x + (2\lambda \tanh(\lambda x) + r_2 + r_1)w = 0, \quad \int \text{sech}^2(\lambda x)w(x)dx = 0.$$

The same argument as for invertibility implies that $w = 0$, hence $r_1 = r_2$ and F_λ is injective.

We now show F^* is injective. First of all, if $F^*(r_1, \lambda_1) = F^*(r_2, \lambda_2)$, then by Lemma 6 we necessarily have $\lambda_1 = \lambda_2$. Next, with $w = r_2 - r_1$

$$w_x - (2\lambda \tanh(\lambda x) + r_1 + r_2)w = 0.$$

The only solution in L^2 to this equation is $w = 0$. This implies injectivity.

Now we make the following observation: if $F_\lambda(r) = (g, \rho)$ for $r \in L^2$ and $g \in H^s, s > -1$ then we also have $r \in H^{s+1}$; similarly, if $F^*(r, \lambda) = g$ for $r \in L^2$ and $g \in H^s, s > -1$ then we also have $r \in H^{s+1}$. This follows by iteratively applying (24), with $h = r/2$.

Thus as a consequence of Proposition 6, together with the above observation, we obtain that the range of the projection of $F_\lambda(r) : H^{s+1} \rightarrow H^s \times \mathbb{R}$ onto its first component is precisely $U_{>-\lambda^2}^s$, and the range of $F^* : H^{s+1} \times (0, \infty) \rightarrow H^s$ is $U_{<0}^s$. \square

Remark 3. Note that working out the details of the H^{s+1} estimates in the proof above, one may show that if

$$v(t, x) := w(t, x - y_0)^2 + 2w(t, x - y_0) \tanh(x - y_1) + w_x(t, x - y_0),$$

for some $y_0, y_1 \in \mathbb{R}$ then for every integer $s \geq -1$ there exists an $N > 0$ such that

$$\|w\|_{H^{s+1}} \lesssim (1 + \|w\|_{L^2} + \|v\|_{H^s})^N (\|v\|_{H^s} + \|w\|_{L^2}).$$

This estimate will be used later in Section 3.

3. The modified Korteweg–de Vries equation close to a kink

In Section 2, we mapped H^s KdV initial data to mKdV initial data – the mKdV initial data being in the form of a sum of an H^{s+1} function, and a kink of the form $\lambda \tanh(\lambda \cdot)$. In this section we will study the corresponding mKdV problem.

First note by scaling, we may restrict to the case $\lambda = 1$. Recall that $Q_1(t, x) \equiv Q(t, x) \equiv \tanh(x + 2t)$ is an explicit kink solution to (8). We now consider solutions to mKdV equation:

$$\begin{cases} u_t + u_{xxx} - 6u^2 u_x = 0 \\ u(0, x) = v_0(x) + \tanh(x), \end{cases} \tag{27}$$

for initial data $v_0 \in H^s, s \geq 0$, such that $u - Q \in H^s$. Equivalently, writing $u = v + Q$, we have:

$$\begin{cases} v_t + v_{xxx} - 2\partial_x(3Q^2 v + 3Qv^2 + v^3) = 0 \\ v(0, x) = v_0(x). \end{cases} \tag{28}$$

In order to construct global solutions to (27), we will need to prove a number of energy estimates. In our discussions below, we will use a number of formal calculations, which are not difficult to justify rigorously.

Lemma 10. *Let p be any $C^\infty(\mathbb{R}^2)$ function with uniformly bounded derivatives and assume $v \in C([0, \infty); H^1(\mathbb{R}))$ to be a solution to (28). Then*

$$\begin{aligned} \frac{d}{dt} \left[\int p v^2 dx \right] &= \int [p_t v^2 - 3p_x v_x^2 + p_{xxx} v^2 - 6p_x Q^2 v^2 - 8p_x Q v^3 - 3p_x v^4 \\ &\quad + 12p Q Q_x v^2 + 4p Q_x v^3] dx. \end{aligned} \tag{29}$$

Proof. Note that by (28), v also has some time regularity. Thus, Eq. (29) follows by employing (28) and applying a series of integrations by parts. \square

We are now in the position to prove global bounds on the L^2 norm of smooth solutions to (28), as well as a “Kato smoothing” type estimate.

Lemma 11. *Suppose $v \in C([0, \infty); H^1(\mathbb{R}))$ is a solution to (28); then for any $t \in [0, \infty)$, we have*

$$\|v(t, \cdot)\|_{L^2} \lesssim \|v_0\|_{L^2} + t^{1/2}. \tag{30}$$

Moreover for any $T > 0$

$$\int_0^T \int_{-\infty}^\infty Q_x(t, x) v_x(t, x)^2 dx dt \leq C(T, \|v(0, \cdot)\|_{L^2}). \tag{31}$$

Proof. From (29), with $p \equiv 1$, we have

$$\frac{d}{dt} \left[\int v^2 dx \right] = \int 12Q Q_x v^2 + 4Q_x v^3 dx. \tag{32}$$

Using (29) again, but with $p \equiv Q$, yields

$$\begin{aligned} \frac{d}{dt} \left[\int Q v^2 dx \right] = \int & [Q_t v^2 - 3Q_x v_x^2 + Q_{xxx} v^2 - 6Q^2 Q_x v^2 - 8Q Q_x v^3 - 3Q_x v^4 \\ & + 12Q^2 Q_x v^2 + 4Q Q_x v^3] dx. \end{aligned} \tag{33}$$

Observe that the terms $Q^2 Q_x v^2$, $Q Q_x v^2$, $Q_t v^2$ and $Q_{xxx} v^2$ are all bounded above by a multiple of $Q_x^{1/2} v^2$. Furthermore, we have

$$\begin{aligned} \int Q_x v^2 dx &\leq \|Q_x^{1/2}\|_{L^2} \|Q_x^{1/2} v^2\|_{L^2} \\ &\leq \frac{1}{\tau} \int Q_x v^4 dx + \tau \|Q_x^{1/2}\|_{L^2}^2, \end{aligned} \tag{34}$$

for $\tau > 0$ by Young’s inequality. Note also $|Q Q_x v^3| \leq |Q_x v^3|$ and

$$\begin{aligned} \int |Q_x v^3| dx &\leq \|Q_x^{1/2} v\|_{L^2} \|Q_x^{1/2} v^2\|_{L^2} \\ &\leq \frac{1}{\kappa} \int Q_x v^4 dx + \kappa \int Q_x v^2 dx, \end{aligned}$$

for any $\kappa > 0$. Applying (34), we find that for any $\kappa > 0$, there exists a constant $C_\kappa > 0$, such that

$$\int |Q_x v^3| dx \leq \frac{1}{\kappa} \int Q_x v^4 dx + C_\kappa. \tag{35}$$

Combining Eqs. (32)–(35) we obtain

$$\frac{d}{dt} \left[\int \left[v^2 + \frac{1}{10} Q v^2 \right] dx \right] \lesssim 1. \tag{36}$$

Since $v^2 \lesssim v^2 + \frac{1}{10} Q v^2$, we can conclude that for any $t \geq 0$

$$\|v(t, \cdot)\|_{L^2} \lesssim \|v(0, \cdot)\|_{L^2} + t^{1/2}.$$

The proof of (31) follows from (30) and the estimate (33). \square

As a consequence of the above estimates, we obtain the following well-posedness result for initial mKdV data near a kink.

Theorem 12. *Let $s \in \mathbb{N}$ satisfy $s \geq 1$. Then there exists a unique, global, strong solution to (28), for any initial data $v_0 \in H^s$. Moreover, for any $T > 0$, the solution map from H^s to $C_t([0, T]; H^s(\mathbb{R}))$ is continuous.*

The proof of this theorem follows essentially from the same arguments as those given in [8] and [16] – since the L^2 estimate (30) is available.

We now establish global a priori bounds on the deviation of a solution u to (27) from a translated kink, i.e. we aim to establish bounds on $w := u - \tanh(x - y(t))$ for some continuous function $y : \mathbb{R}^+ \rightarrow \mathbb{R}$ yet to be determined. We start by providing some motivation for the full argument given later.

From (27) we obtain

$$w_t + w_{xxx} - 2\partial_x (3 \tanh^2(x - y(t))^2 w + 3 \tanh(x - y(t)) w^2 + w^3) = (\dot{y} + 2) \operatorname{sech}^2(x - y(t)) \tag{37}$$

To define the position $y(t)$, we impose an orthogonality condition

$$\langle w, \psi(x - y) \rangle = 0, \tag{38}$$

where for the moment we choose $\psi(x) = e^x \operatorname{sech}^2(x)$ for reason which will become clear below – later we will actually choose $\psi(x) = \eta(x) \operatorname{sech}^2(x)$, where η is defined by (43). If w is sufficiently close to the kink then y exists and is unique by an application of the implicit function theorem – see Lemma 13 below.⁴ It is not hard to work out an equation for $\dot{y} + 2$ by formally differentiating the condition (38) with respect to t .

It is instructive to first consider the linearised problem at the $Q(x, t) = \tanh(x + 2t)$ kink, in a frame moving with the kink:

$$\tilde{w}_t(t, x) - 4\tilde{w}_x(t, x) + \tilde{w}_{xxx} + 6\partial_x(\operatorname{sech}^2(x)\tilde{w}(t, x)) = \alpha(t) \operatorname{sech}^2(x), \tag{39}$$

where $\alpha(t)$ is chosen as indicated above so that orthogonality condition

$$\langle \tilde{w}, e^x \operatorname{sech}^2(x) \rangle = 0 \tag{40}$$

is preserved over time. To obtain a formula for α , we differentiate the above orthogonality condition with respect to t . Thus we obtain

$$\langle \tilde{w}, (-4\partial_x + \partial_{xxx} + 6 \operatorname{sech}^2 \partial_x) e^x \operatorname{sech}^2(x) \rangle + \alpha(t) \langle \operatorname{sech}^2(x), e^x \operatorname{sech}^2(x) \rangle = 0,$$

which expresses α as a linear function of \tilde{w} .

From (39) we obtain

$$\frac{d}{dt} \int e^x \tilde{w}^2 dx = -3B(e^{x/2} \tilde{w}) + 2\alpha(t) \int \tilde{w}(t, x) e^x \operatorname{sech}^2(x) dx, \tag{41}$$

where here B is the quadratic form defined by

$$B(f) := \int f_x^2 + \left(\frac{5}{4} - 2 \operatorname{sech}^2(x) - 4 \operatorname{sech}^2(x) \tanh(x) \right) f(x)^2 dx. \tag{42}$$

Note that the second term on the right hand side of (41) vanishes due to (40).

According to the bound (97) of Proposition 26, taking into account the choice of ψ , we have

$$B(e^{x/2} w) \geq \frac{1}{10} \|e^{x/2} w\|_{H^1}^2.$$

As a consequence $\int e^x \tilde{w}^2 dx$ decays monotonically, and the time derivative controls $B(e^{x/2} \tilde{w})$.

We will pursue a non-linear variant of this simple strategy. The weight e^x will be replaced by the following bounded and monotone weight function

$$\eta_{R,\delta}(x) = \eta(x) = \tanh\left(\frac{x - R}{2}\right) + 1 + \delta. \tag{43}$$

We will also define y in terms of an orthogonality condition, similar to (40).

Lemma 13. *There exists an $\epsilon > 0$ and a unique analytic function y on $B^{L^2}(\tanh(\cdot), \epsilon)$ such that*

$$\langle f(\cdot) - \tanh(\cdot - y(f)), \eta(\cdot - y(f)) \operatorname{sech}^2(\cdot - y(f)) \rangle = 0,$$

and $y(\tanh(\cdot)) = 0$.

Proof. Consider the mapping

$$F : B^{L^2}(\tanh(\cdot), \epsilon) \times \mathbb{R} \rightarrow \mathbb{R}$$

– here $B^{L^2}(\tanh(\cdot), \epsilon)$ denotes by an abuse of notation the set of sums of L^2 functions of norm $< \epsilon$ and \tanh – defined by

$$F(f, y) = \langle f - \tanh(\cdot - y), \eta(\cdot - y) \operatorname{sech}^2(\cdot - y) \rangle.$$

⁴ In Lemma 13 the weight $\psi(x) = \eta(x) \operatorname{sech}^2(x)$ is actually used; however, the proof may be easily adapted to the case $\psi(x) = e^x \operatorname{sech}^2(x)$.

Clearly $F(\tanh(\cdot), 0) = 0$. Differentiating with respect to y at $f := \tanh(\cdot)$ we obtain

$$\frac{d}{dy} F(\tanh(\cdot), y)|_{y:=0} = \langle \operatorname{sech}^2(\cdot - y), \eta(\cdot - y) \operatorname{sech}^2(\cdot - y) \rangle > 0.$$

The implicit function theorem then yields the assertion. \square

Theorem 14. *There exists a $\delta > 0$ such that if u the solution to (27) of Theorem 12 with initial data satisfying $u(0, \cdot) - \tanh(\cdot) \in H^1(\mathbb{R})$ and $\|u(0, \cdot) - \tanh(\cdot)\|_{L^2} < \delta$, then there is a continuous function $y : [0, \infty) \rightarrow \mathbb{R}$ such that*

$$\|u(t, \cdot) - \tanh(x - y(t))\|_{L^2} \lesssim \|u(0, \cdot) - \tanh(\cdot)\|_{L^2}. \tag{44}$$

Moreover, writing $w := u - \tanh(\cdot - y(t))$ we have the estimates

$$\int_0^\infty \|\eta_x(\cdot - y(t))^{1/2} w\|_{H^1}^2 dt \lesssim \|u(0, \cdot) - \tanh(\cdot)\|_{L^2}^2, \tag{45}$$

and

$$|\dot{y} + 2| \lesssim \|\eta_x(\cdot - y(t))^{1/2} w\|_{L^2} + \|\eta_x(\cdot - y(t))^{1/2} w\|_{L^2} \|\eta_x(\cdot - y(t))^{1/2} w\|_{L^\infty}^2. \tag{46}$$

Proof. First define

$$\psi(x) = \eta(x) \operatorname{sech}^2(x). \tag{47}$$

Our aim is to find a function y satisfying the orthogonality condition:

$$\langle u(t, \cdot) - \tanh(\cdot - y(t)), \psi(x - y(t)) \rangle = 0 \tag{48}$$

for all $t \geq 0$ such that $y(0) = y_0$, where y_0 is given by Lemma 13. The existence of such a function, at least initially, for $t \in [0, T]$, for some $T > 0$, is a consequence of Lemma 13 and the fact $u - \tanh(x + 2t) \in C(\mathbb{R}; H^1(\mathbb{R}))$.

Now define $w(t, x)$ by

$$w(x, t) = u(t, x) - \tanh(x - y(t)),$$

from which we obtain

$$w_t + w_{xxx} - 2\partial_x(3 \tanh^2(x - y)w + 3 \tanh(x - y)w^2 + w^3) = (2 + \dot{y}) \operatorname{sech}^2(x - y). \tag{49}$$

Again by perhaps taking a smaller T if necessary, we can assume for $t \in [0, T]$

$$\|w\|_{L^2} \leq 2\epsilon.$$

By (49) and (48) we obtain:

$$\begin{aligned} & \frac{d}{dt} \int \eta(x - y) w^2 dx \\ &= \int \left[-3\eta_x(x - y)w_x^2 + (-\dot{y}\eta_x(x - y) + \eta_{xxx}(x - y)) \right. \\ & \quad \left. - 6 \tanh^2(x - y)\eta_x(x - y) + 12\eta(x - y) \tanh(x - y) \operatorname{sech}^2(x - y)w^2 \right. \\ & \quad \left. + (4\eta(x - y) \operatorname{sech}^2(x - y) - 8\eta_x(x - y) \tanh(x - y))w^3 - 3\eta_x(x - y)w^4 \right] dx. \end{aligned} \tag{50}$$

Rewriting the quadratic part of the above equation by using the identity $\operatorname{sech}^2(x) + \tanh^2(x) = 1$ numerous times, together with the observation

$$\eta_{xxx} = -3\eta_x^2 + \eta_x, \tag{51}$$

along with the trivial identity $\dot{y} = -2 + (\dot{y} + 2)$ we obtain

$$\begin{aligned}
 & \frac{d}{dt} \int \eta(x-y)w^2 dx \\
 &= \int \left[-3\eta_x(x-y)w_x^2 + (-3 + 6 \operatorname{sech}^2(x-y)) \right. \\
 &\quad + 24\eta(x-y) \tanh(x-y) \operatorname{sech}^2(x-y) (\eta_x(x-y))^{-1} \eta_x(x-y)w^2 \\
 &\quad - (2 + \dot{y}) \int \eta_x(x-y)w^2 dx - \int 3(\eta_x(x-y))^2 w^2 dx \\
 &\quad \left. + (4\eta(x-y) \operatorname{sech}^2(x-y) - 8\eta_x(x-y) \tanh(x-y))w^3 - 3\eta_x(x-y)w^4 \right] dx.
 \end{aligned} \tag{52}$$

Now observe

$$\begin{aligned}
 \int (\eta_x^{1/2} w)_x^2 dx &= \int \left[\eta_x w_x^2 + \frac{\eta_{xx}^2}{4\eta_x} w^2 + \eta_{xx} w w_x \right] dx \\
 &= \int \left[\eta_x w_x^2 + \left(\frac{\eta_{xx}^2}{4\eta_x} - \frac{1}{2} \eta_{xxx} \right) w^2 \right] dx \\
 &= \int \left[\eta_x w_x^2 + \left(\eta_x^2 - \frac{1}{4} \eta_x \right) w^2 \right] dx,
 \end{aligned}$$

where in the last line we used (51) in addition with the identity

$$\frac{\eta_{xx}^2}{\eta_x} = \eta_x - 2\eta_x^2.$$

We define the quadratic form:

$$\begin{aligned}
 B_{\varepsilon,R}(f) &:= \int f_x^2 + \left(\frac{5}{4} - 2 \operatorname{sech}^2(x) \right. \\
 &\quad \left. - 8 \operatorname{sech}^2(x) \tanh(x) \cosh^2\left(\frac{x-R}{2}\right) \left(1 + \varepsilon + \tanh\left(\frac{x-R}{2}\right) \right) \right) f(x)^2 dx
 \end{aligned}$$

and rewrite Eq. (52) as

$$\begin{aligned}
 \frac{d}{dt} \int \eta(x-y)w^2 dx &= -3B_{\varepsilon,R}(\eta_x(x-y)^{1/2}w) - (2 + \dot{y}) \int \eta_x(x-y)w^2 dx \\
 &\quad + \int (4\eta(x-y) \operatorname{sech}^2(x-y) - 8\eta_x(x-y) \tanh(x-y))w^3 dx \\
 &\quad - \int 3\eta_x(x-y)w^4 dx.
 \end{aligned}$$

We observe that η_x is positive and hence the last line is non-positive. We will now estimate the cubic term:

$$\begin{aligned}
 \left| \int (4\eta(x-y) \operatorname{sech}^2(x-y) - 8\eta_x(x-y) \tanh(x-y))w^3 dx \right| &\lesssim C_R \int \eta_x |w|^3 dx \\
 &\lesssim C_R \|w\|_{L^2} \|\eta_x^{1/2} w\|_{L^4}^2 \\
 &\lesssim C_R \|w\|_{L^2} \|\eta_x^{1/2} w\|_{H^1}^2.
 \end{aligned}$$

We turn to bounding $|\dot{y} + 2|$. Note we have from (49) and (47):

$$\begin{aligned}
 0 &= \frac{d}{dt} \langle w, \psi(x-y) \rangle \\
 &= \int \left[w \psi_{xxx}(x-y) - 2(3 \tanh^2(x-y)w + 3 \tanh(x-y)w^2 + w^3) \psi_x(x-y) \right. \\
 &\quad \left. + (2 + \dot{y}) \operatorname{sech}^2(x-y) \psi(x-y) - \dot{y} w(x) \psi_x(x-y) \right] dx.
 \end{aligned} \tag{53}$$

Thus we obtain

$$\begin{aligned} |\dot{y} + 2| &\lesssim C_R(1 + |\dot{y} + 2|)\|w\|_{L^2} + \|\eta_x^{1/2}w\|_{L^2}^2 + \|\eta_x^{1/2}w\|_{L^2}\|\eta_x^{1/2}w\|_{L^\infty}^2 \\ &\lesssim C_R\|w\|_{L^2} + \|w\|_{L^2}\|\eta_x^{1/2}w\|_{L^\infty}^2, \end{aligned} \tag{54}$$

where in the last line we use the fact that $\|w\|_{L^2} \ll 1$.

Collecting the above estimates together, we obtain:

$$\frac{d}{dt} \int \eta(x - y)w^2 dx \leq -3B_{\varepsilon,R}(\eta_x(x - y)^{1/2}w) + \Lambda\|w\|_{L^2}\|\eta_x^{1/2}w\|_{H^1}^2, \tag{55}$$

for some constant Λ depending on R , which will be a positive number.

We now compare the quadratic form $B_{\varepsilon,R}$ with the quadratic form B from [Appendix B](#):

$$B(f) = \int f_x^2 + \left(\frac{5}{4} - 2 \operatorname{sech}^2(x) - 4 \operatorname{sech}^2(x) \tanh(x)\right) f(x)^2 dx.$$

The difference $V(x)$ of the potentials in the quadratic forms is

$$4 \operatorname{sech}^2(x) \tanh(x) \left(2 \cosh^2\left(\frac{x - R}{2}\right) \left(1 + \varepsilon + \tanh\left(\frac{x - R}{2}\right)\right) - 1\right).$$

Observe that

$$\cosh^2\left(\frac{x - R}{2}\right) \left(1 + \tanh\left(\frac{x - R}{2}\right)\right) = \frac{1}{2}(e^{x-R} + 1)$$

hence the difference V can be bounded by

$$|V| \leq 8\varepsilon \operatorname{sech}^2(x) \cosh^2\left(\frac{x - R}{2}\right) + 4 \operatorname{sech}^2(x) e^{x-R} \leq 16\varepsilon e^R + 8e^{-R}.$$

Thus we obtain

$$|B(f) - B_{\varepsilon,R}(f)| \leq (16\varepsilon e^R + 4e^{-R})\|f\|_{L^2}^2. \tag{56}$$

Now define the modified quadratic form

$$\hat{B}_{\varepsilon,R}(f) := B_{\varepsilon,R}(f) + 2e^R \langle \eta_x^{-1/2} \eta \operatorname{sech}^2, f \rangle^2.$$

Observe that

$$\eta_{R,\varepsilon} - \eta_{R,0} = \varepsilon,$$

and

$$\cosh\left(\frac{x - R}{2}\right) \left(1 + \tanh\left(\frac{x - R}{2}\right)\right) = e^{\frac{x-R}{2}},$$

from which it follows that

$$(e^{R/2} \eta_x^{-1/2} \eta(x) - e^{x/2}) \operatorname{sech}^2(x) = \sqrt{2\varepsilon} e^{R/2} \cosh\left(\frac{x - R}{2}\right) \operatorname{sech}^2(x) \leq 2\varepsilon e^R,$$

which yields the estimate

$$|e^{R/2} \langle \eta_x^{-1/2} \eta \operatorname{sech}^2, f \rangle - \langle e^{x/2} \operatorname{sech}^2(x), f \rangle| \leq 2\varepsilon e^R \|f\|_{L^2}.$$

By estimate (56) together with the estimate (97) we obtain:

Lemma 15. *With $R = 10$ we have for all $f \in H^1$*

$$\hat{B}_{e^{-2R},R}(f) \geq \frac{1}{20} \|f\|_{H^1}^2.$$

We now fix $R = 10$ and set $\varepsilon := e^{-2R} = e^{-20}$ – noting that we only require the existence of R such that the conclusion of the lemma holds, with its size being neither optimal nor important.

If we for the moment assume that

$$\sup_{0 \leq t \leq T} \|w(t, \cdot)\|_{L^2} \leq \frac{1}{80A},$$

then it follows from the above lemma, and the orthogonality condition (48) that we can control the second term in Eq. (55) with the first term, which implies

$$\int \eta(x - y(t))w(t, x)^2 dx \leq \int \eta(x - y(0))w(0, x)^2 dx.$$

Hence we obtain

$$\begin{aligned} \varepsilon \|w(t, \cdot)\|_{L^2}^2 &\leq \int \eta(x - y)w(t, x)^2 dx \\ &\leq \int \eta(x - y(0))w(0, x)^2 dx \\ &\leq (1 + \varepsilon) \|w(0, \cdot)\|_{L^2}^2 \end{aligned}$$

and thus

$$\|w(t, \cdot)\|_{L^2} \leq 2e^R \|w(0, \cdot)\|_{L^2}.$$

A continuity argument gives the desired global bound provided (recall $\varepsilon = e^{-2R}$)

$$\|w(0, \cdot)\|_{L^2} \leq \frac{1}{120A} e^{-R}. \quad \square$$

Corollary 16. *Suppose u satisfies the conditions in the above theorem, furthermore assume $u(0, \cdot) - \tanh(\cdot) \in H^s$, where s is a positive integer, then there exist $C > 0$ and $N > 0$ depending only on s such that*

$$\|u(t, \cdot) - \tanh(x - y(t))\|_{H^s} \leq C \|u(0, \cdot) - \tanh(\cdot)\|_{H^s} (1 + \|u(0, \cdot) - \tanh(\cdot)\|_{H^s})^N. \tag{57}$$

Proof. Let w be as in Theorem 14 and define:

$$v(t, x) := w(t, x - 6t)^2 + 2w(t, x - 6t) \tanh(x - y(t) - 6t) + w_x(t, x - 6t), \tag{58}$$

i.e. v is a solution to (1).

Note as a consequence of infinite conservation laws associated with the KdV equation (see Appendix C), we have

$$\begin{aligned} \|v(t, \cdot)\|_{H^s} &\lesssim \|v(0, \cdot)\|_{H^{s-1}} (1 + \|v(0, \cdot)\|_{H^{s-1}})^{N'} \\ &\lesssim \|w(0, \cdot)\|_{H^s} (1 + \|w(0, \cdot)\|_{H^s})^{N''} \end{aligned} \tag{59}$$

for some positive integers N' and N'' .

Then from Remark 3 at the end of Section 2, Theorem 14 and (59) we obtain (57). \square

We now consider the problem of asymptotic stability of the mKdV equation near a kink. We will require an additional weight function:

$$\phi_{x_0, A}(t, x) = \phi(t, x) = 1 + \tanh\left(\frac{x - x_0 + \gamma t}{A}\right).$$

Proposition 17. *Let $\gamma < 6$, then there exists $\delta, A > 0$ such that if u is the solution to (27) with initial data satisfying $u(0, \cdot) - \tanh(\cdot) \in H^1(\mathbb{R})$ and $\|u(0, \cdot) - \tanh(\cdot)\|_{L^2} < \delta$, and $x_0 \in \mathbb{R}$ we have the bounds*

$$\int \eta(x - y(t))\phi_{A, x_0}(t, x)w(t, x)^2 dx \leq \int \eta(x - y(0))\phi_{A, x_0}(0, x)w(0, x)^2 dx, \tag{60}$$

where $t > 0$, $w := u - \tanh(\cdot - y)$ and y references to the continuous function constructed in [Theorem 14](#). Moreover, we have the following smoothing estimate:

$$\int_0^\infty \|(\eta(x - y)\phi_{A,x_0})_x^{1/2} w\|_{H^1}^2 dt \lesssim \int \eta(x - y(0))\phi_{A,x_0}(0, x)w(0, x)^2 dx. \tag{61}$$

Proof. Using the shorthand $\eta = \eta(x - y(t))$, $\eta_x = \eta_x(x - y(t))$, \dots , $\text{sech}^2 = \text{sech}^2(x - y(t))$, \dots we obtain:

$$\begin{aligned} \frac{d}{dt} \int \phi \eta w^2 dx &= -B_{\varepsilon,R}((\phi \eta_x)^{1/2} w) + \int [-(2 + \dot{y})\phi \eta_x w^2 + \phi(4\eta \text{sech}^2 - 8\eta_x \tanh)w^3 - 3\phi \eta_x w^4 \\ &\quad + \eta(-3\phi_x w_x^2 + (\gamma \phi_x + \phi_{xxx} - 6 \tanh^2 \phi_x)w^2 - 8\phi_x \tanh w^3 - 3\phi_x w^4) \\ &\quad + 4(\phi_{xx} \eta_x + \phi_x \eta_{xx})w^2 + (2 + \dot{y})\phi \eta \text{sech}^2 w] dx. \end{aligned}$$

We first note that

$$\begin{aligned} (\gamma - 6 \tanh^2(x - y))\phi_x(t, x) &= ((\gamma - 6) + 6 \text{sech}^2(x - y))\phi_x(t, x) \\ &\leq (\gamma - 6)\phi_x(t, x) + CA^{-1}\phi(t, x)\eta_x(x - y), \end{aligned}$$

where we used the fact that $\phi_x \lesssim A^{-1}\phi$. We also have the estimate

$$\int \eta(x - y)\phi_x(t, x) \tanh(x - y)w^3 \lesssim \|w\|_{L^2} \|(\eta(x - y)\phi_x(t, \cdot))^{1/2} w\|_{H^1}^2.$$

Thus if we assume $\|w\|_{L^2}$ to be suitably small and A to be large, then by the above estimates and the arguments in [Theorem 14](#) we obtain:

$$\begin{aligned} \frac{d}{dt} \int \phi \eta w^2 dx &\leq -\kappa \|(\phi \eta)_x^{1/2} w\|_{H^1}^2 + \int [4(\phi_{xx} \eta_x + \phi_x \eta_{xx})w^2 + \phi(2 + \dot{y})(\eta \text{sech}^2(x - y)w - \eta_x w^2)] dx \\ &\quad + C\langle \phi^{1/2} w, \eta(x - y) \text{sech}^2(x - y) \rangle^2. \end{aligned} \tag{62}$$

The plan is bound the last two terms on the right hand side in terms of the first term.

First note for large A we have the following simple estimates

$$\begin{aligned} |\phi(t, x) - \phi(t, y(t))| \text{sech}(x - y) &\lesssim A^{-1}e^{-2|(y-x_0+\gamma t)/A|}, \\ \phi(t, x)^{-1/2} &\lesssim e^{-(x-x_0+\gamma t)/A} + 1, \\ \phi(t, x)^{1/2} &\lesssim \frac{e^{(x-x_0+\gamma t)/A}}{e^{(x-x_0+\gamma t)/A} + 1} \quad \text{and} \\ \text{sech}(x - y) &\lesssim \eta_x(x - y). \end{aligned}$$

Applying the above estimates we obtain

$$\begin{aligned} \left| \int \phi \eta \text{sech}^2(x - y)w dx \right| &= \left| \int (\phi(t, x) - \phi(t, y))\eta \text{sech}^2(x - y)w dx \right| \\ &\lesssim A^{-1}e^{-2|(y-x_0+\gamma t)/A|} \|(\phi \eta_x)^{1/2} w\|_{L^2} \|\eta_x^{1/2} \phi^{-1/2}\|_{L^2} \\ &\lesssim A^{-1}e^{-|(y-x_0+\gamma t)/A|} \|(\phi \eta_x)^{1/2} w\|_{L^2}. \end{aligned} \tag{63}$$

Similarly we obtain

$$\begin{aligned} \left| \int \phi \eta_x w^2 dx \right| &\lesssim \|(\phi \eta_x)^{1/2}\|_{L^\infty} \|(\phi \eta_x)^{1/2} w\|_{L^2} \|w\|_{L^2} \\ &\lesssim \frac{e^{(y-x_0+\gamma t)/A}}{e^{(y-x_0+\gamma t)/A} + 1} \|(\phi \eta_x)^{1/2} w\|_{L^2} \|w\|_{L^2}. \end{aligned} \tag{64}$$

By (53) we have

$$\begin{aligned}
 |\dot{y} + 2| &\lesssim \int \operatorname{sech}^2(x - y)(|w| + |w|^3) \\
 &\lesssim \|(\phi\eta_x)^{1/2}w\|_{L^2} \|\eta_x^{1/2}\phi^{-1/2}\|_{L^2} + \|(\phi\eta_x)^{1/2}w\|_{L^\infty} \|\eta_x^{1/2}\phi^{-1/2}\|_{L^\infty} \|w\|_{L^2}^2 \\
 &\lesssim (e^{-(y-x_0+\gamma t)/A} + 1)(\|(\phi\eta_x)^{1/2}w\|_{L^2} + \|(\phi\eta_x)^{1/2}w\|_{H^1}) \\
 &\lesssim (e^{-(y-x_0+\gamma t)/A} + 1)\|(\phi\eta_x)^{1/2}w\|_{H^1}.
 \end{aligned} \tag{65}$$

Combining (63), (64) and (65) we get

$$\left| \int \phi(2 + \dot{y})(\eta \operatorname{sech}^2(x - y)w - \eta_x w^2) dx \right| \lesssim (A^{-1} + \delta)\|(\phi\eta_x)^{1/2}w\|_{H^1}^2,$$

and similarly for the last term we get

$$\begin{aligned}
 \langle \phi^{1/2}w, \eta(x - y) \operatorname{sech}^2(x - y) \rangle^2 &\lesssim \|(\phi(t, \cdot)^{1/2} - \phi(t, y(t))^{1/2}) \operatorname{sech}^{1/2}(\cdot - y)\eta\|_{L^\infty}^2 \\
 &\quad \times \|\phi^{1/2} \operatorname{sech}^{1/2}(\cdot - y)w\|_{L^2}^2 \|\operatorname{sech}(\cdot - y)\phi^{-1/2}\|_{L^2}^2 \\
 &\lesssim A^{-1}\|(\phi\eta_x)^{1/2}w\|_{H^1}^2.
 \end{aligned}$$

Then from the above estimates, if we assume A to be suitably large and δ to be suitably small we obtain

$$\frac{d}{dt} \int \phi\eta w^2 dx \leq 0,$$

from which we obtain our claim. \square

As a consequence of Proposition 17 and Theorem 14 we obtain the following theorem.

Theorem 18. *Let $\gamma < 6$. Then there exists $\delta_\gamma > 0$ such that if u is a solution to (8) with initial data u_0 , satisfying $u_0 - \tanh(x) \in H^1(\mathbb{R})$ and $\|u(0, \cdot) - \tanh(x)\|_{L^2} < \delta_\gamma$,*

$$\lim_{t \rightarrow \infty} \|u(t, \cdot) - \tanh(x - y(t))\|_{L^2((-\gamma t, \infty))} = 0 \tag{66}$$

where $y : [0, \infty) \rightarrow \mathbb{R}$ refers to the continuous function constructed in Theorem 14.

Making use of the Miura transformation to relate mKdV near the kink with KdV near zero, we will replace L^2 in the statement of the above theorem with H^s for any non-negative integer s . Specifically we have, denoting again $w = u - \tanh(\cdot - y(t))$:

Corollary 19. *Let $\gamma < 6$ and s any positive integer. Then there exists $\delta_\gamma > 0$ such that if u is a solution to (27) with initial data u_0 , satisfying $u_0 - \tanh(\cdot) \in H^s(\mathbb{R})$ and $\|u(0, \cdot) - \tanh(x)\|_{L^2} < \delta_\gamma$,*

$$\lim_{t \rightarrow \infty} \|u(t, \cdot) - \tanh(x - y(t))\|_{H^s((-\gamma t, \infty))} = 0 \tag{67}$$

where $y : [0, \infty) \rightarrow \mathbb{R}$ refers to the continuous function constructed in Theorem 14. Moreover we have the smoothing estimate

$$\int_0^\infty \|\rho_x(t, \cdot + 6t)^{1/2}w(t, \cdot)\|_{H^{s+1}}^2 dt \leq C\|\rho(0, \cdot)^{1/2}w(0, \cdot)\|_{H^s}^2, \tag{68}$$

where C depends on γ and $\|u(0, \cdot) - \tanh(\cdot)\|_{H^s}$, and ρ is defined as

$$\rho(x, t) = 1 + \tanh\left(\frac{x - x_0 + (\gamma - 6)t}{A}\right),$$

for some large constant $A > 0$.

Proof. Note that the absolute values of the derivatives of ρ are bounded above by a constant multiple of ρ . The same property is also true for the function ρ_x . This property of ρ and ρ_x will be used extensively below without further comment.

Define v as in (58), hence v is a solution to (1). Fixing $t \geq 0$, observe from (58), Lemma 7 we have for $f := \rho^{1/2}$ or $f := \rho_x^{1/2}$ the following estimate

$$\begin{aligned} \|f(t, \cdot)v(t, \cdot)\|_{H^{s-1}} &\lesssim \|f(t, \cdot + 6t)w(t, \cdot)^2\|_{H^{s-1}} + \|f(t, \cdot + 6t)w(t, \cdot) \tanh(\cdot - y)\|_{H^{s-1}} \\ &\quad + \|f(t, \cdot + 6t)w_x(t, \cdot)\|_{H^{s-1}} \\ &\lesssim (1 + \|w(t, \cdot)\|_{H^{s-1}}) \|f(t, \cdot + 6t)w(t, \cdot)\|_{H^s} \\ &\leq C \|f(t, \cdot + 6t)w(t, \cdot)\|_{H^s}, \end{aligned} \tag{69}$$

for all integers $s \geq 1$, where C depends on $\|w(t, \cdot)\|_{H^{s-1}}$.

Similarly we also have the estimate

$$\begin{aligned} \|f(t, \cdot + 6t)w(t, \cdot)\|_{H^s} &\lesssim \|f(t, \cdot + 6t)w(t, \cdot)\|_{L^2} + \|f(t, \cdot + 6t)w_x(t, \cdot)\|_{H^{s-1}} \\ &\lesssim \|f(t, \cdot + 6t)w(t, \cdot)\|_{L^2} + \|f(t, \cdot)v(t, \cdot)\|_{H^{s-1}} + \|f(t, \cdot + 6t)w(t, \cdot)^2\|_{H^{s-1}} \\ &\quad + \|f(t, \cdot + 6t)w(t, \cdot) \tanh(\cdot - y)\|_{H^{s-1}} \\ &\lesssim (1 + \|w(t, \cdot)\|_{H^{s-1}} + \|w(t, \cdot)\|_{H^1}) \|f(t, \cdot + 6t)w(t, \cdot)\|_{H^{s-1}} + \|f(t, \cdot)v(t, \cdot)\|_{H^{s-1}} \\ &\leq C(\|f(t, \cdot + 6t)w(t, \cdot)\|_{H^{s-1}} + \|f(t, \cdot)v(t, \cdot)\|_{H^{s-1}}), \end{aligned} \tag{70}$$

for all integers $s \geq 1$, where C depends on $\|w(t, \cdot)\|_{H^{s-1}} + \|w(t, \cdot)\|_{H^1}$.

The inequalities (69) and (70) will essentially allow to shift our focus from a study of mKdV near a kink to that of KdV in a neighbourhood of zero. In particular, note that by Theorem 14 and Corollary 16, the constants in (69) and (70) depend only on the initial data $\|u(0, \cdot) - \tanh(\cdot)\|_{H^{s-1}}$ and $\|u(0, \cdot) - \tanh(\cdot)\|_{H^{s-1}} + \|u(0, \cdot) - \tanh(\cdot)\|_{H^1}$ respectively.

Now consider the case $s = 1$. Below C will denote a positive constant depending on $\|u(0, \cdot) - \tanh(\cdot)\|_{H^1}$ and γ , which may change from line to line.

A simple energy estimate yields

$$\frac{d}{dt} \int \rho v^2 dx = \int \rho_t v^2 + \rho_{xxx} v^2 - 3\rho_x v_x^2 - 4\rho_x v^3 dx.$$

Note that replacing ρ with 1 we recover the L^2 conservation law for KdV.

Also, we have the simple estimate

$$\begin{aligned} \int \rho_x v^3 dx &\leq \|v\|_{L^2} \|\rho_x^{1/2} v\|_{L^2} \|\rho_x^{1/2} v\|_{L^\infty} \\ &\lesssim \|v(0, \cdot)\|_{L^2} \|\rho_x^{1/2} v\|_{L^2} \|\rho_x^{1/2} v\|_{H^1} \\ &\leq C(\varepsilon^{-1} \|\rho_x^{1/2} v\|_{L^2}^2 + \varepsilon \|\rho_x^{1/2} v\|_{H^1}^2), \end{aligned}$$

for any $\varepsilon > 0$.

Hence from the above estimates

$$\frac{d}{dt} \int \rho(t, x)v^2(t, x) dx \leq -2\|\rho_x(t, \cdot)^{1/2}v_x(t, \cdot)\|_{L^2}^2 + C\|\rho_x(t, \cdot + 6t)^{1/2}w(t, \cdot)\|_{H^1}^2.$$

Therefore from (61) we obtain

$$\int \rho(t, x)v^2(t, x) dx + 2 \int_0^t \|\rho_x(t', \cdot)^{1/2}v_x(t', \cdot)\|_{L^2}^2 dt' \lesssim \int \rho(0, x)v(0, x)^2 dx + C \int \rho(0, x)w(0, x)^2 dx.$$

Then from the above inequality, (70) (with $f = \rho^{1/2}$), and (66), we obtain (67) for $s = 1$. Similarly from the above inequality, (70) (with $f = \rho_x^{1/2}$), and (61), we obtain (67) for $s = 1$.

We now will provide a sketch of the proof for $s > 1$. We proceed by induction, assuming as our inductive hypothesis that (67) and (68) holds for a given positive integer s . Below C will denote a positive constant depending on $\|u(0, \cdot) - \tanh(\cdot)\|_{H^{s+1}}$ and γ , which may change from line to line.

From (103) we have

$$\frac{d}{dt} \int \rho T^{(s)} dx = \int \rho_t T^{(s)} + \rho_x X^{(s)} dx. \tag{71}$$

Observe that from the two monomials $2\partial_x^s u \partial_x^{s+2} u$ and $-(\partial_x^s u)^2$ in $X^{(s)}$ we recover (after a couple of integration by parts) the terms

$$-3\rho_x (\partial_x^{s+1} u)^2 + \rho_{xxx} (\partial_x^s u)^2, \tag{72}$$

in the integrand on the right hand side of (71).

We now proceed in a similar manner to the case of $s = 1$, using extensively the properties of $X^{(s)}$ and $T^{(s)}$ as stated in Appendix C. In this way, one can show that

$$\begin{aligned} \int \rho_t T^{(s)} + \rho_x X^{(s)} + 3\rho_x (\partial_x^{s+1} u)^2 dx &\lesssim (1 + \|v\|_{H^s}^{s+1}) \|\rho_x^{1/2} v\|_{H^s}^2 \\ &\leq C \|\rho_x^{1/2} v\|_{H^s}^2. \end{aligned}$$

Integrating (71) with respect to t , and using our induction hypothesis together with (69) and (70) leads to

$$\begin{aligned} \int \rho(t, x) T^{(s)}(t, x) dx + 3 \int_0^\infty \|\rho_x(t', \cdot)^{1/2} (\partial_x^{s+1} v)(t', \cdot)\|_{L^2}^2 dt' \\ \leq C \|\rho(0, \cdot)^{1/2} w(0, \cdot)\|_{H^s}^2 + \int \rho(0, x) T^{(s)}(0, x) dx. \end{aligned}$$

We observe that the terms on the left hand side of the above equation resulting from lower order terms in $T^{(s)}$ can be bounded by a constant multiple of

$$(1 + \|v\|_{H^s}^s) \|\rho^{1/2} v\|_{H^{s-1}}^2 \leq C \|\rho^{1/2} v\|_{H^{s-1}}^2.$$

Similarly, the terms on the right hand side resulting from lower order terms in $T^{(s)}$ can be bounded by $C \|\rho(0, \cdot)^{1/2} v(0, \cdot)\|_{H^{s-1}}^2$. Thus we obtain

$$\begin{aligned} \int \rho(t, x) (\partial_x^s v)^2(t, x) dx + 3 \int_0^\infty \|\rho_x(t', \cdot)^{1/2} (\partial_x^{s+1} v)(t', \cdot)\|_{L^2}^2 dt' \\ \leq \int \rho(0, x) (\partial_x^s v)^2(0, x) dx + C (\|\rho(0, \cdot)^{1/2} w(0, \cdot)\|_{H^s}^2 + \|\rho(t, \cdot)^{1/2} v(t, \cdot)\|_{H^{s-1}}^2 + \|\rho(0, \cdot)^{1/2} v(0, \cdot)\|_{H^{s-1}}^2). \end{aligned}$$

Then applying (69) and (70), together with our induction hypothesis, we obtain (67) and (68) for $s + 1$. \square

4. Existence of weak solutions to the Korteweg–de Vries equation with initial data in H^{-1}

With the help of Theorem 12, Lemma 11 and Lemma 31, we will now prove the existence of weak L^2 solutions to the IVP (28).

Proposition 20. *For any $v_0 \in L^2$, there exists a weak solution $u = v + Q$ to (28) satisfying*

$$v \in C_\omega([0, \infty); L^2), \tag{73}$$

$$v_x \in L^2([0, T] \times [-R, R]) \text{ for } R, T < \infty, \tag{74}$$

$$\|v(t, \cdot)\|_{L^2} \lesssim \|v(0, \cdot)\|_{L^2} + t^{1/2} \text{ for } t \in [0, \infty), \tag{75}$$

$$v(t, \cdot) \rightarrow v_0 \text{ in } L^2 \text{ as } t \downarrow 0. \tag{76}$$

Furthermore there exists a $\delta > 0$ such that if $\|v_0\|_{L^2} < \delta$ then there exists a continuous function $y : \mathbb{R} \rightarrow \mathbb{R}$ such that if we write $u = w + \tanh(\cdot + y(t))$, we have

$$\|w\|_{L^2} \lesssim \|v_0\|_{L^2}. \tag{77}$$

Proof. Let $v_0^{(j)} \in H^1$ be a sequence such that $v_0^{(j)} \rightarrow v_0$ in L^2 , and $\|v_0^{(j)}\|_{L^2} = \|v_0\|_{L^2}$. Define $v^{(j)} \in C([0, \infty); H^1)$ to be the solution to (28) with $v^{(j)}(0, \cdot) = r_{0,j}$, corresponding to Theorem 12.

If in addition we have $\|v_0\|_{L^2} < \delta$, and we write

$$u^{(j)}(t, x) = v^{(j)}(t, x) + Q(t, x) = w^{(j)}(t, x) + \tanh(x + y^{(j)}(t)),$$

where $y^{(j)}$ is defined as in Theorem 14, then using (46), (44), and (45) we obtain a uniform bound of $y^{(j)}$ in $H^1([0, T])$, and thus by Morrey’s inequality we have a uniform bound of $y^{(j)}$ in $C^{0,1/2}([0, T])$, for any fixed $T > 0$. By the Azelà–Ascoli theorem, and a suitable diagonal argument we can construct a subsequence $(v^{(N_j)})$ such that for all $T > 0$, $y^{(N_j)}$ converges uniformly to some continuous function $y : \mathbb{R}^+ \rightarrow \mathbb{R}$. Moreover from (44), we have for any $t \geq 0$, there exists a k such that if $j > k$

$$\|u^{(N_j)}(t, \cdot) - \tanh(\cdot - y(t))\|_{L^2} \lesssim \|v_0\|_{L^2}. \tag{78}$$

Now applying an almost identical argument to the one given in [7] to construct weak L^2 KdV solutions – here the smoothing estimate is replaced by (31), and L^2 conservation replaced by (30) – we obtain a subsequence $(v^{(N'_j)})$ such that for any $R, T > 0$ the sequence converges weakly in $L^2([0, T]; H^1([-R, R]))$, strongly in $L^2([0, T] \times [-R, R])$ and weak-* in $L^\infty([0, \infty); L^2)$ to a limit v satisfying (73)–(75), and solves (28) in the distributional sense.

In order to prove (76) we set $\tilde{v} = v\sqrt{1 + \frac{1}{10}Q}$, and observe that \tilde{v} is continuous at $t = 0$ if and only (76) is satisfied. Note that weak continuity of \tilde{v} in t follows from weak continuity of v . Estimating we obtain

$$\begin{aligned} \|\tilde{v}(t, \cdot) - \tilde{v}(0, \cdot)\|_{L^2}^2 &= \|\tilde{v}(t, \cdot)\|_{L^2}^2 + \|\tilde{v}(0, \cdot)\|_{L^2}^2 - 2\langle \tilde{v}(t, \cdot), \tilde{v}(0, \cdot) \rangle \\ &\leq (\|\tilde{v}(t, \cdot)\|_{L^2}^2 - \|\tilde{v}(0, \cdot)\|_{L^2}^2) + 2\|\tilde{v}(0, \cdot)\|_{L^2}^2 - 2\langle \tilde{v}(t, \cdot), v_0 \rangle \\ &= (\|\tilde{v}(t, \cdot)\|_{L^2}^2 - \|\tilde{v}(0, \cdot)\|_{L^2}^2) + 2\langle \tilde{v}(0, \cdot) - \tilde{v}(t, \cdot), \tilde{v}(0, \cdot) \rangle. \end{aligned}$$

Then from (36) and the weak continuity of \tilde{v} we obtain (76).

Finally, note (77) is a simple consequence of (78). \square

We will now construct weak H^{-1} solutions to the Korteweg–de Vries equation. Using the scaling symmetry, we may restrict to small initial data in H^{-1} .

Proposition 21. For any $u_0 \in H^{-1}$ satisfying $\|u_0\|_{H^{-1}} \leq \epsilon$, for $\epsilon > 0$ chosen suitably small, there exists a weak solution u to (28), and a continuous function $y : [0, \infty) \rightarrow \mathbb{R}$ satisfying

$$u \in C_\omega([0, \infty); H^{-1}), \tag{79}$$

$$u \in L^2([0, T] \times [-R, R]) \text{ for } R, T < \infty, \tag{80}$$

$$\|u\|_{H^{-1}} \lesssim \|f\|_{H^{-1}}, \tag{81}$$

$$u(t, \cdot) \rightarrow u_0 \text{ in } H^{-1} \text{ as } t \downarrow 0. \tag{82}$$

Proof. Define v_0 such that $F(v_0) = (u_0, 0)$, where F is defined as in Theorem 9. Then by Proposition 20, there exists a weak solution \tilde{u} to the mKdV equation corresponding to initial data $v_0 + \tanh(\cdot)$. Let u be map obtained by applying the Galilean transformation ($h = 6$) to $M(\tilde{u})$. It is then easy to check that u satisfies (79)–(82). What remains to be shown is that u satisfies (1) in a distributional sense, which is equivalent to $M(\tilde{u})$ satisfying (1) in a distributional sense – this is the subject of Lemma 22 below. \square

Lemma 22. Let $v_0 \in L^2$, and suppose \tilde{u} is a weak solution to (8), satisfying the properties (73)–(76), then $u := \partial_x(\tilde{u}) + (\tilde{u})^2$ satisfies (1), in a distributional sense, i.e.

$$\int_{\mathbb{R}^2} [-u\varphi_t - u\varphi_{xxx} + 3u^2\varphi_x] dt dx = 0,$$

for all $\varphi \in C_0^\infty$.

For a proof of the above lemma we refer the reader to the papers [23] and [6].

By utilising the scaling symmetry of the KdV equation and Proposition 21, one easily obtains existence of weak solutions of Theorem 4.

5. A priori bounds and soliton stability

Theorem 1. Suppose $u \in C([0, \infty); H^s(\mathbb{R}))$ is a solution to (1), for some $s \geq -\frac{3}{4}$, then

$$\|u(t, \cdot)\|_{H^{-1}} \lesssim \|u_0\|_{H^{-1}} + \|u_0\|_{H^{-1}}^3 \quad \text{for } t \in [0, \infty). \tag{83}$$

Proof. First consider the case when $s = 0$. By scaling, the problem reduces to showing that for all solutions $u \in C([0, \infty); L^2(\mathbb{R}))$ to (1) satisfying $\|u\|_{H^{-1}} \leq \epsilon$ for some suitably chosen $\epsilon > 0$, we have

$$\|u(t, \cdot)\|_{H^{-1}} \lesssim 1 \quad \text{for } t \in [0, \infty). \tag{84}$$

From Theorem 9, Theorem 12 and the well-posedness theory of the KdV equation, it follows that there exists a solution $\tilde{u} \in C([0, \infty); H^1(\mathbb{R}))$ to (8), such that the Galilean transformation ($h = 6$) of $M(\tilde{u})$ is u . Assuming we chose ϵ sufficiently small, then as a consequence of Lemma 11 and Theorem 14, we obtain (84).

The general case when $s \geq -\frac{3}{4}$ can be proven via approximation. \square

Theorem 2. There exists an $\epsilon > 0$ such that if $u \in C([0, \infty); H^s(\mathbb{R}) \cap H^{-3/4}(\mathbb{R}))$ is a solution to (1), for some integer $s \geq -1$, satisfying $\|R_c - u_0\|_{H^{-1}} < \epsilon c^{1/4}$ for some $c > 0$, then there is a continuous function $y : [0, \infty) \rightarrow \mathbb{R}$ such that

$$\|u - R_c(x - y(t))\|_{H^s} \leq \gamma_s(c, \|R_c - u_0\|_{H^s})$$

for any $t \geq 0$, where $\gamma_s : (0, \infty) \times [0, \infty)$ is a continuous function, polynomial in the second variable, which satisfies $\gamma(\cdot, 0) = 0$.

Proof. The proof follows in a similar manner to that of Theorem 1. Again, without loss of generality we may assume $u_0 \in H^1$. By scaling we may also assume that $c = 4$. Then assuming $\|R_c - u(0, \cdot)\|_{H^{-1}}$ to be suitably small, and making use of the arguments in Section 2, we may link the KdV IVP with initial data $u(0, \cdot)$ to the mKdV IVP with initial data $\tilde{u}_0 := \lambda \tanh(\lambda \cdot) + v_0$, for some $\lambda \approx 1$, such that $v_0 \in H^{s+1}$ and $\|v_0\|_{L^2} \lesssim \|R_c - u_0\|_{H^{-1}}$. By scaling on the mKdV side, we can assume $\lambda = 1$. The conclusion then follows from Theorem 12, the well posedness theory of the KdV equation, Theorem 14, and Corollary 16. \square

Making use of Theorem 18, Corollary 19, and following a similar argument to that given above we obtain:

Theorem 3. Given real $\gamma > 0$ and integer $s \geq -1$, there exists an $\epsilon_\gamma > 0$ such that if $u \in C([0, \infty); H^s(\mathbb{R}) \cap H^{-3/4}(\mathbb{R}))$, is a solution to (1), satisfying

$$\|R_c - u_0\|_{H^{-1}} < \epsilon_\gamma c^{1/4}$$

for $c > 0$, then there is a continuous function $y : [0, \infty) \rightarrow \mathbb{R}$ and $\tilde{c} > 0$ such that

$$\lim_{t \rightarrow \infty} \|u - R_{\tilde{c}}(x - y(t))\|_{H^s((\gamma t, \infty))} = 0$$

for any $t \geq 0$. Moreover we have the bound $|c - \tilde{c}| \lesssim c^{\frac{3}{4}} \|R_c - u_0\|_{H^{-1}}$.

Conflict of interest statement

None declared.

Appendix A. Schrödinger operators with rough potentials

In this section we collect a couple of useful results concerning Schrödinger operators with distributional H^{-1} potentials. This subject was partially studied by Kappeler et al. [5] in a direction similar to ours as discussed above. The Miura map is a central part of the integrable structure of KdV and mKdV, and hence it provides a link to Schrödinger operators and inverse scattering. Typically the inverse scattering methods requires integrability of the potentials and even some decay. Nevertheless trace identities allow to express the L^2 norm (as well as higher norms) in terms of the scattering data. This relation has been used by Deift and Killip [3] to study the spectral density for L^2 potentials. The available results indicate that the spectrum of L^2 potentials is a highly non-trivial and difficult object. The failure of surjectivity of the Miura map in the work of Kappeler et al. can be seen as a shadow of this complexity.

Here we aim for something considerably simpler: our main spectral object is the ground state energy, which is much more robust. We start by noting that there is a factorisation of the Schrödinger operator

$$H_q := -\partial_{xx}^2 + q = -(\partial_x + r)(\partial_x - r),$$

if q satisfies the Riccati equation $q = r_x + r^2$. Moreover, with $\phi = e^{\int_0^x r dx}$

$$\partial_{xx}\phi = \phi(r_x + r^2) = \phi q$$

and ϕ is a non-negative solution to the Schrödinger equation:

$$H_q\phi = 0.$$

Conversely, if ϕ is non-negative and satisfies

$$\phi_{xx} + q\phi = 0$$

then, with

$$r = -\partial_x \ln \phi,$$

we have

$$r_x + r^2 = -\frac{\phi_{xx}}{\phi} = q.$$

Lemma 23. *Let $q \in H^{-1}$. Then the Schrödinger operator*

$$\phi \rightarrow H_q\phi = -\phi_{xx} + q\phi$$

has a unique self-adjoint, semi-bounded below extension.

Proof. Note that it suffices to show H_q is semi-bounded below: the unique self-adjoint, semi-bounded below extension follows by Friedrichs’ construction [20]. We now turn to the bound from below.

Using a combination of duality, a product estimate, Gagliardo–Nirenberg inequality and Young’s inequality we obtain

$$\begin{aligned} \int qf^2 &\leq \|q\|_{H^{-1}} \|f^2\|_{H^1} \\ &\lesssim \|q\|_{H^{-1}} \|f\|_{H^1} \|f\|_{L^\infty} \\ &\lesssim \|q\|_{H^{-1}} \|f\|_{H^1}^{3/2} \|f\|_{L^2}^{1/2} \\ &\lesssim C^{-4/3} \|f\|_{H^1}^2 + C^4 \|q\|_{H^{-1}}^4 \|f\|_{L^2}^2. \end{aligned}$$

Thus taking C large we obtain

$$\begin{aligned} \langle Tf, f \rangle &= \int f_x^2 + qf^2 \\ &\gtrsim -(1 + \|q\|_{H^{-1}}^4) \|f\|_{L^2}^2. \quad \square \end{aligned}$$

Lemma 24. Let $q_i \in H^{-1}(a, b)$ and suppose that $\phi, \psi \in H^1(a, b)$ are strictly positive functions satisfying

$$-\phi'' + q_1\phi = 0, \quad -\psi'' + q_2\psi = 0,$$

and $q_2 \leq q_1$. Then ϕ/ψ has no interior minimum unless it is constant.

Proof. We will proceed formally, however we note that it is not difficult to make the calculations rigorous, then

$$-\frac{d^2}{dx^2} \frac{\phi}{\psi} + (q_1 - q_2) \frac{\phi}{\psi} - 2 \frac{\psi'}{\psi} \frac{d}{dx} \frac{\phi}{\psi} = 0$$

and since $q_2 \leq q_1$ we obtain

$$-\frac{d^2}{dx^2} \frac{\phi}{\psi} - 2 \frac{\psi'}{\psi} \frac{d}{dx} \frac{\phi}{\psi} \leq 0.$$

We claim that $u(x) = \phi/\psi$ cannot have an interior positive minimum. We search for a contradiction, and assume that $u(x_0) = M = \inf_{x \in (a,b)} u(x)$ and $u(a), u(b) > M$. We test with $u_\varepsilon = ((M + \varepsilon) - u)_+$; setting $U_\varepsilon = \{x : M < u < M + \varepsilon\}$ yields

$$\int_{U_\varepsilon} (u_\varepsilon)_x^2 - 2u_\varepsilon \frac{\psi'}{\psi} (u_\varepsilon)_x dx \leq 0.$$

By assumption, for ε sufficiently small, the quotient ψ'/ψ is uniformly bounded by some constant $c > 0$. Thus

$$\begin{aligned} \int_{U_\varepsilon} u_x^2 &\leq \frac{1}{2} \int_{U_\varepsilon} (u_\varepsilon)_x^2 dx + c^2 \int_{U_\varepsilon} u_\varepsilon^2 dx \\ &\leq \left(\frac{1}{2} + c^2 |U_\varepsilon| \right) \int_{U_\varepsilon} u_x^2 dx \end{aligned}$$

and hence, since the left hand side is nonzero,

$$|U_\varepsilon| \geq \frac{1}{2c}.$$

Letting ε tend to 0 we obtain a contradiction. \square

Lemma 25. Suppose $\lambda > 0$, $r \in L^2_{loc}$, $q \in H^{-1}$ and

$$r_x + r^2 = \lambda^2 + q$$

Then either

$$r - \lambda \in L^2(0, \infty) \quad \text{or} \quad r + \lambda \in L^2(0, \infty) \tag{85}$$

and either

$$r - \lambda \in L^2(-\infty, 0) \quad \text{or} \quad r + \lambda \in L^2(-\infty, 0). \tag{86}$$

Proof. By the symmetry of the problem, it suffices to restrict our attention to (85).

Since $q \in H^{-1}$, there exists functions $f, g \in L^2$ such that $q = f + g'$. Define $y = r - g$; hence f satisfies

$$y_x + y^2 + 2gy = \lambda^2 + f - g^2 \tag{87}$$

in the distribution sense.

Now for a given large x_0 , we will now investigate the behaviour of y on the interval $[x_0, x_0 + 1]$. Define $\eta := e^{2 \int_{x_0}^x g}$, $H = \int_{x_0}^x \eta(f - g^2)$ and

$$\tilde{y} = y\eta - H. \tag{88}$$

Thus, \tilde{y} satisfies

$$\tilde{y}_x + y^2\eta = \eta\lambda^2. \tag{89}$$

Taking x_0 to be sufficiently large we may assume η to be arbitrarily close to 1 and H arbitrarily small on the interval $[x_0, x_0 + 1]$. More precisely, we can show for a given $\delta > 0$, there exists $z \in \mathbb{R}$ such that if $x_0 > z$ then on the interval $[x_0, x_0 + 1]$

$$\tilde{y} - y = e_1, \tag{90}$$

and

$$\tilde{y}_x = \lambda^2 - \tilde{y}^2 + e_2 \tag{91}$$

where the functions e_1 and e_2 satisfy the bound

$$|e_{\{1,2\}}| \leq \delta|\tilde{y}| + \delta. \tag{92}$$

That is, \tilde{y} behaves like the non-linear ODE $h' = \lambda^2 - h^2$, which has a stable fixed point at λ and an unstable fixed point at $-\lambda$. Since $\tilde{y} \in L^2_{\text{loc}}$, it is then not difficult to show from (91) and (92) that $|y| \rightarrow \lambda$.

Now consider the case when $y \rightarrow \lambda$. Pick $z \in \mathbb{R}$ such that $\|y - \lambda\|_{L^\infty[z, \infty)} < \min\{1, \lambda\}$; hence from (87) we obtain

$$\|y - \lambda\|_{L^2[z, \infty)} \lesssim \frac{1}{\lambda} (1 + \|g\|_{L^2[z, \infty)}^2 + \lambda\|g\|_{L^2[z, \infty)} + \|f\|_{L^2[z, \infty)}). \tag{93}$$

Similarly for the case when $y \rightarrow -\lambda$, if we pick $z \in \mathbb{R}$ such that $\|y + \lambda\|_{L^\infty[z, \infty)} < \min\{1, \lambda\}$ we obtain

$$\|y + \lambda\|_{L^2[z, \infty)} \lesssim \frac{1}{\lambda} (1 + \|g\|_{L^2[z, \infty)}^2 + \lambda\|g\|_{L^2[z, \infty)} + \|f\|_{L^2[z, \infty)}). \quad \square \tag{94}$$

Appendix B. Quadratic form estimates

We consider the quadratic form defined by

$$B(f) := \int f_x^2 + \left(\frac{5}{4} - 2 \operatorname{sech}^2(x) - 4 \operatorname{sech}^2(x) \tanh(x) \right) f(x)^2 dx. \tag{95}$$

Proposition 26. *The quadratic form B satisfies the following inequality*

$$B(f) + 2\langle f, e^{i/2} \operatorname{sech}^2(\cdot) \rangle^2 \geq \frac{1}{3} \|f\|_{L^2}^2, \tag{96}$$

holds for all $f \in H^1$; moreover we also have the estimate

$$B(f) + 2\langle f, e^{i/2} \operatorname{sech}^2(\cdot) \rangle^2 \geq \frac{1}{20} \|f\|_{H^1}^2. \tag{97}$$

Remark 4. The inequality (97) is actually a simple consequence of (96). A straightforward calculation yields

$$\frac{5}{4} - 2 \operatorname{sech}^2(x) - 4 \operatorname{sech}^2(x) \tanh(x) \geq -5$$

and hence, using (96)

$$\|f\|_{H^1}^2 \leq B(f) + 6\|f\|_{L^2}^2 \leq 20(B(f) + 2\langle f, e^{i/2} \operatorname{sech}^2(\cdot) \rangle^2), \tag{98}$$

we obtain (97). Note also that the constant 1/20 is neither optimal nor of any particular importance in the context of the paper, as we will simply require the existence of a non-negative constant.

Proof. First consider the Schrödinger $H = -\partial_{xx} + V(x)$ operator with potential $V(x) := -2 \operatorname{sech}^2(x) - 4 \operatorname{sech}^2(x) \tanh(x)$. A celebrated theorem by Lieb and Thirring [13] gives us a bound on the moments of the bound states energies (negative eigenvalues) e_j of H :

$$\sum_j |e_j|^\gamma \leq L_{\gamma,1} \int |V(x)|_-^{\gamma+n/2}$$

for $\gamma \geq \frac{3}{2}$, where $|V(x)|_- = (|V(x)| - V(x))/2$ and

$$L_{\gamma,1} = \frac{1}{2\sqrt{\pi}} \Gamma(\gamma + 1) / \Gamma\left(\gamma + \frac{3}{2}\right).$$

In particular for $\gamma := \frac{3}{2}$ we have

$$\sum_j |e_j|^{3/2} \leq \frac{3}{16} \int |V(x)|_-^2 = 567/320, \tag{99}$$

where the second equality involves determining the support of $|V(x)|_-$, and an evaluation of the integral. This was done with the help of Mathematica, but could easily be done by hand.

It follows immediately that the ground state satisfies the bound

$$e_0 \geq -(567/320)^{2/3} > -\frac{3}{2}. \tag{100}$$

Now, let $u = \sqrt{2/\pi} e^{x/2} \operatorname{sech}^2(x)$ – this is normalised so that the L^2 norm of u is 1. Then an explicit calculation yields

$$\langle H(u), u \rangle = -5/4,$$

and thus

$$-5/4 \geq e_0 \geq -(567/320)^{2/3}.$$

Furthermore from (99) if we denote the ground state as v_0 we have

$$-5/4 = H(u) \geq e_0 |\langle u, v_0 \rangle|^2 - (567/320 - |e_0|^{3/2})^{2/3} (1 - |\langle u, v_0 \rangle|^2),$$

hence

$$|\langle u, v_0 \rangle|^2 \geq \frac{-5/4 + (567/320 - |e_0|^{3/2})^{2/3}}{e_0 + (567/320 - |e_0|^{3/2})^{2/3}}$$

Denoting the right hand side by $h(s)$ evaluated at e_0 ; then one can check – either with the help of a software package such as Mathematica, or by hand, with patience – that for s satisfying the bounds (100), h has a minimum $\frac{1701 + \sqrt{1435533}}{3402}$ at $s = -\frac{721489 + 567\sqrt{1435533}}{960000}$. Hence we obtain

$$|\langle u, v_0 \rangle|^2 \geq \frac{1701 + \sqrt{1435533}}{3402} > \frac{5}{6}. \tag{101}$$

Also as a consequence of (99) and (100), we have that for any $v \in H^2$ in the orthogonal complement of v_0

$$\langle H(v), v \rangle \geq -\left(567/320 - (5/4)^{3/2}\right)^{2/3} \|v\|_{L^2}^2 \geq -\frac{5}{9} \|v\|_{L^2}^2. \tag{102}$$

Now pick $f \in H^2$ and let $f(x) = av_0(x) + g(x)$ be an L^2 orthonormal decomposition. Then applying Young’s inequality in the first inequality and orthogonality of v_0 and g for the second inequality we have

$$\begin{aligned} \langle f, u \rangle^2 &= a^2 \langle v_0, u \rangle^2 + 2a \langle v_0, u \rangle \langle g, u \rangle + \langle g, u \rangle^2 \\ &\geq \frac{a^2}{2} \langle v_0, u \rangle - \langle g, u \rangle^2 \\ &\geq \frac{a^2}{2} \langle v_0, u \rangle^2 - \|g\|_{L^2}^2 (1 - \langle v_0, u \rangle^2), \end{aligned}$$

hence

$$\langle H(f), f \rangle + 2\langle f, u \rangle^2 \geq a^2(e_0 + \langle v_0, u \rangle^2) + \|g\|_{L^2}^2 \left(-\frac{5}{9} - 2(1 - \langle v_0, u \rangle^2) \right).$$

The claim (97) follows from the observations

$$e_0 + \langle v_0, u \rangle^2 \geq -\frac{3}{2} + \frac{5}{6} = -\frac{2}{3}, \quad \text{and} \quad -\frac{5}{9} - 2(1 - \langle v_0, u \rangle^2) \geq -\frac{5}{9} - \frac{2}{6} = -\frac{8}{9},$$

since $\frac{5}{4} - \frac{8}{9} > \frac{1}{3}$. \square

Appendix C. Higher energies

In order to study higher regularity we need to make use of higher order *polynomial conservation laws* (see [17] and [12]) associated with KdV. Specifically, if u is a smooth solution to (1), then for every integer $k \geq 0$, there exists polynomials $T^{(k)}$ and $X^{(k)}$ in u and its derivatives such that

$$\partial_t T^{(k)} + \partial_x X^{(k)} := 0, \tag{103}$$

and the following additional properties are satisfied:

- The polynomial $T^{(k)}$ is irreducible.
- The *rank*⁵ of all monomials contained in $T^{(k)}$ is $2 + k$.
- The rank of all monomials contained in $X^{(k)}$ is $3 + k$.
- The *dominant*⁶ term of $T^{(k)}$ is $(\partial_x^k u)^2$.
- The polynomial $X^{(k)}$ has two terms with maximal *derivative index*,⁷ namely $2\partial_x^k u \partial_x^{k+2} u$ and $-(\partial_x^{k+1} u)^2$.

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⁵ Here the rank of a monomial is $m + \frac{n}{2}$, where m and n are respectively the degree and total number of differentiations of the monomial. Only terms whose rank is an integer occur.

⁶ Writing a monomial in the form $cu^{a_0}u_x^{a_1} \dots (\partial_x^l u)^{a_l}$, then the dominant term is the term with the larger l , or the same l but larger a_l , or with the same l and a_l , but with larger a_{l-1} , etc.

⁷ The derivative index of a monomial refers to the number of differentiations in the monomial.

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