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Nonlinear evolution PDEs in $\mathbb{R}^+ \times \mathbb{C}^d$: existence and uniqueness of solutions, asymptotic and Borel summability properties

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Abstract

We consider a system of n-th order nonlinear quasilinear partial differential equations of the form

$$\mathbf{u}_t + \mathcal{P}(\partial_{\mathbf{x}}^{\mathbf{j}})\mathbf{u} + \mathbf{g}(\mathbf{x}, t, \{\partial_{\mathbf{x}}^{\mathbf{j}}\mathbf{u}\}) = 0; \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u_I}(\mathbf{x})$$

with $\mathbf{u} \in \mathbb{C}^r$, for $t \in (0, T)$ and large $|\mathbf{x}|$ in a poly-sector S in $\mathbb{C}^d$ ($\partial_{\mathbf{x}}^{\mathbf{j}} \equiv \partial_{x_1}^{j_1} \partial_{x_2}^{j_2} \cdots \partial_{x_d}^{j_d}$ and $j_1 + \cdots + j_d \leqslant n$). The principal part of the constant coefficient n-th order differential operator $\mathcal{P}$ is subject to a cone condition. The nonlinearity $\mathbf{g}$ and the functions $\mathbf{u_I}$ and $\boldsymbol{u}$ satisfy analyticity and decay assumptions in S.

The paper shows existence and uniqueness of the solution of this problem and finds its asymptotic behavior for large $|\mathbf{x}|$.

Under further regularity conditions on $\mathbf{g}$ and $\mathbf{u_I}$ which ensure the existence of a formal asymptotic series solution for large $|\mathbf{x}|$ to the problem, we prove its Borel summability to the actual solution $\mathbf{u}$.

The structure of the nonlinearity and the complex plane setting preclude standard methods. We use a new approach, based on Borel–Laplace regularization and Écalle acceleration techniques to control the equation.

These results are instrumental in constructive analysis of singularity formation in nonlinear PDEs with prescribed initial data, an application referred to in the paper.

In special cases motivated by applications we show how the method can be adapted to obtain short-time existence, uniqueness and asymptotic behavior for small t, of sectorially analytic solutions, without size restriction on the space variable.

Keywords: Sectorial existence for nonlinear PDEs; Asymptotic behavior; Borel summability

1. Introduction

1.1. General considerations

There are relatively few general results on existence, uniqueness and regularity of solutions of partial differential equations in the complex domain when the conditions of the classical Cauchy–Kowalewski (C–K) theorem are

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not met. The C–K theorem holds for first-order analytic systems (or those equivalent to them) with analytic non-characteristic data, and for these it guarantees local existence and uniqueness of analytic solutions. As is well known, its proof requires convergence of local power series expansions. Evolution equations with higher spatial derivatives do not satisfy the C–K assumptions and even when formal power series solutions exist their radius of convergence is zero. One of the goals of this paper is to provide a theory for existence, uniqueness and regularity of solutions in such cases, in a relatively general setting. The theory also applies to classes of equations of higher order in time and sufficiently high order in space after reduction (by well known transformations, see e.g. [16]) to evolution systems.

The present paper generalizes [7] to d dimensions and arbitrary order in the spatial variable, to r-dimensional dependent variable, proves additional results about short term existence and shows Borel summability of formal solutions. A fortiori we obtain results on the asymptotic character of these solutions. (In Appendix A.2, we briefly discuss the definition and properties of Borel summation.)

Under assumptions to allow for formal expansions for large $\mathbf{x}$, we show that series solutions are Borel summable to actual solutions of the PDE. For this purpose we make use of Écalle acceleration techniques. In special cases we obtain existence and uniqueness results for t in a compact set and large enough $\mathbf{x}$, and separately for small t and fewer restrictions on $\mathbf{x}$.

Properties of solutions of PDEs in the complex plane, apart from their intrinsic interest, are relevant for properties in the real domain, as initial singularities in $\mathbb{C}$ may give rise to blow-up at later times in the physical domain. Representation of solutions as Borel sums is instrumental in extending techniques originally developed for ODEs [6] to find the location and type of singularities of solutions to nonlinear PDEs [9].

It is certainly difficult to give justice to the existing theory of nonlinear PDEs, and we mention a number of results in the literature relevant to the current paper. For certain classes of PDEs in the complex domain Sammartino and Caflisch [13,14] proved the existence of nonlinear Prandtl boundary layer solutions for analytic initial data in a half-plane. This work involves inversion of the heat operator $\partial_t - \partial_{YY}$ and uses the abstract Cauchy–Kowalewski theorem for the resulting integral equation. While their method is likely to be generalizable to certain higher-order partial differential equations, it appears unsuitable for problems where the highest derivative terms appear in a nonlinear manner. Such terms cannot be controlled by inversion of a linear operator and estimates of the kernel, as used in [13,14].

The complex plane setting, as well as the type of nonlinearity allowed in our paper, do not allow for an adaptation of classical, Sobolev space based, techniques. This can be also seen in simple examples which show that existence fails outside the domain of validity of the results we obtain.

Certainly, many evolution equations are amenable to our setting; to illustrate canonical form transformations and the general results we chose a third order equation with quartic nonlinearity arising in fluid dynamics. Detailed singularity study [9] of solutions of this equation relies on the present analysis.

Our approach extends Borel transform regularization to a general class of nonlinear partial differential equations. A vast literature has emerged recently in Borel summability theory, starting with the fundamental contributions of Écalle (see e.g. [10]) whose consequences are far from being fully explored and it is impossible to give a quick account of the breadth of this field. See for example [6] for more references. Yet, in the context of relatively general PDEs, very little is known. For small variables, Borel summability has been recently shown for the heat equation [12,3], and generalized to linear PDEs with constant coefficients by Balser [2]. One large space variable was considered by us in [7], in special classes of higher order nonlinear PDEs. The methods in the present paper are different and apply, for large $|\mathbf{x}|$, to a wide class of equations.

1.2. Notation

We use the following conventions. For vectors in $\mathbb{C}^d$ or multiindices we write

$$|\mathbf{u}| = \sum_{j=1}^{d} |u_i|$$

and for multiindices we define

$$\mathbf{k} \succ \mathbf{m} \quad \text{if } k_i > m_i \text{ for all } i.$$

If a is a scalar we write $\mathbf{x}^a = (x_1^a, x_2^a, \ldots, x_d^a)$.

With $\mathbf{p}$, $\mathbf{x}$ and $\mathbf{j}$ vectors of same dimension d, we define

$$\mathbf{p}^{\mathbf{j}} = \prod_{i=1}^{d} p_i^{j_i}$$

and

$$\partial_{\mathbf{x}}^{\mathbf{j}} = \partial_{x_1}^{j_1} \partial_{x_2}^{j_2} \cdots \partial_{x_d}^{j_d}.$$

We write $\mathbf{1} = (1, 1, \ldots, 1)$ and more generally, if α is a scalar, we write $\boldsymbol{\alpha} = \alpha\mathbf{1}$; thus $\mathbf{x}^{\mathbf{1}} = \prod_{i=1}^{d} x_i$. For d-dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ we write

$$\int_{\mathbf{a}}^{\mathbf{b}} \cdot \,\mathrm{d}\mathbf{p} = \int_{a_1}^{b_1}\int_{a_2}^{b_2} \cdots \int_{a_d}^{b_d} \cdot \,\mathrm{d}p_1\,\mathrm{d}p_2 \cdots \mathrm{d}p_d.$$

The *directional Laplace transform* along the ray $\arg p_i = \varphi_i$, $i = 1, \ldots, d$, of F is given by

$$\{\mathcal{L}_{\boldsymbol{\varphi}} F\}(\mathbf{x}) \equiv \int_{\mathbf{0}}^{\infty \mathrm{e}^{\mathrm{i}\boldsymbol{\varphi}}} F(\mathbf{p}) \mathrm{e}^{-\mathbf{p}\cdot\mathbf{x}}\,\mathrm{d}\mathbf{p} \tag{1}$$

where $\mathbf{x}\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}}$ will denote the vector with components $x_i \mathrm{e}^{\mathrm{i}\theta_i}$. *Convolution* is defined as

$$(f * g)(\mathbf{p}) := \int_{\mathbf{0}}^{\mathbf{p}} f(\mathbf{s}) g(\mathbf{p} - \mathbf{s})\,\mathrm{d}\mathbf{s} \tag{2}$$

and ${}^{*}\prod$ denotes convolution product (see also [5]). Whenever used as *sum or product indices*, l takes all integer values between 1 and m, i is between 1 and d, As a sum or product multiindex, $|\mathbf{j}|$ indicates all $\mathbf{j}$ with positive integer components subject to the constraint $1 \leqslant |\mathbf{j}| \leqslant n$.

2. Problem statement and main results

2.1. Setting and assumptions

Consider the initial value problem for a quasilinear system

$$\mathbf{u}_t + \mathcal{P}\big(\partial_{\mathbf{x}}^{\mathbf{j}}\big)\mathbf{u} + \mathbf{g}\big(\mathbf{x}, t, \big\{\partial_{\mathbf{x}}^{\mathbf{j}}\mathbf{u}\big\}_{|\mathbf{j}| \leqslant n}\big) = 0; \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_I(\mathbf{x}). \tag{3}$$

In (3), $\mathcal{P}(\partial_{\mathbf{x}})\mathbf{u}$ collects the constant coefficient linear terms of the partial differential equation.
Emphasizing quasilinearity, we rewrite the equation as

$$\partial_t \mathbf{u} + \mathcal{P}(\partial_{\mathbf{x}})\mathbf{u} + \sum_{|\mathbf{J}|=n} \mathbf{g}_{2,\mathbf{J}}\big(\mathbf{x}, t, \big\{\partial_{\mathbf{x}}^{\mathbf{j}}\mathbf{u}\big\}_{|\mathbf{j}|<n}\big)\partial_{\mathbf{x}}^{\mathbf{J}}\mathbf{u} = \mathbf{g}_1\big(\mathbf{x}, t, \big\{\partial_{\mathbf{x}}^{\mathbf{j}}\mathbf{u}\big\}_{|\mathbf{j}|<n}\big); \quad \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_{\mathbf{I}}(\mathbf{x}). \tag{4}$$

The restrictions on $\mathbf{g}_1$, $\mathbf{g}_2$, and $\mathbf{u}_{\mathbf{I}}$ are simpler in a normalized form, more suitable for our analysis. By applying $\partial_{\mathbf{x}}^{\mathbf{j}}$ to (4) for all $\mathbf{j}$ with $1 \leqslant |\mathbf{j}| \leqslant n - 1$, we get an extended system of equations for $\mathbf{f} \in \mathbb{C}^m$, consisting in $\mathbf{u}$ and its spatial derivatives of order less than n, of the type (see Appendix A for further details):

$$\partial_t \mathbf{f} + \mathcal{P}(\partial_{\mathbf{x}})\mathbf{f} = {\sum_{\mathbf{q}\succeq 0}}' \mathbf{b}_{\mathbf{q}}(\mathbf{x}, t, \mathbf{f}) \prod_{l,|\mathbf{j}|} \big(\partial_{\mathbf{x}}^{\mathbf{j}} f_l\big)^{q_{l,\mathbf{j}}} + \mathbf{r}(\mathbf{x}, t) \quad \text{with } \mathbf{f}(\mathbf{x}, 0) = \mathbf{f}_I(\mathbf{x}) \tag{5}$$

where $\sum'$ means the sum over the multiindices $\mathbf{q}$ with

$$\sum_{l=1}^{m} \sum_{1 \leqslant |\mathbf{j}| \leqslant n} |\mathbf{j}| q_{l,\mathbf{j}} \leqslant n. \tag{6}$$

The matrix $\mathcal{P}$ is assumed to be diagonalizable, and modulo simple changes of variables we assume it is presented in diagonal form, $\mathcal{P} = \operatorname{diag} \mathcal{P}_j$, $j = 1, \ldots, m$. In (5), $\mathbf{q} = (q_{l,\mathbf{j}})$, $1 \leqslant |\mathbf{j}| \leqslant n$, $1 \leqslant l \leqslant m$, is a vector of integers and $\mathcal{P}_j$ is an n-th order polynomial. We let $\mathcal{P}_{n;j}$ be the principal part of $\mathcal{P}_j$, i.e. the part that contains all monomials of (total) degree n. The inequality (6) implies in particular that none of the $q_{l,\mathbf{j}}$ can exceed n and that the summation in (5) involves *only finitely many terms*. The fact that (6) can always be ensured leads to important simplifications in the proofs. Let $\rho > \rho_0 > 0$, $\phi < \frac{\pi}{2n}$, $\epsilon > 0$ and

$$\mathcal{D}_{\phi,\rho;\mathbf{x}} = \left\{\mathbf{x}\colon\ |\arg x_i| < \frac{\pi}{2} + \phi;\ |x_i| > \rho;\ i \leqslant d\right\}, \tag{7}$$

$$\mathcal{D}_{\phi,\rho} = \mathcal{D}_{\phi,\rho;\mathbf{x}} \times [0, T]. \tag{8}$$

Assumptions 1.

(1) There is a $\phi \in (0, \frac{\pi}{2n})$ such that for all $\mathbf{p} \neq 0$ with $\max_i |\arg p_i| \leqslant \phi$ we have

$$\Re \mathcal{P}_{n;j}(-\mathbf{p}) > 0. \tag{9}$$

(2) The functions $\mathbf{b_q}(\cdot, t, \cdot)$ are analytic in $\mathcal{D}_{\frac{\pi}{2n},\rho_0} \times \{\mathbf{f}\colon\ |\mathbf{f}| < \epsilon\}$. We write

$$\mathbf{b_q}(\mathbf{x}, t; \mathbf{f}) = \sum_{\mathbf{k} \succeq 0} \mathbf{b_{q,k}}(\mathbf{x}, t)\mathbf{f}^{\mathbf{k}}. \tag{10}$$

(3) For some constants $\alpha_r \geqslant 1$ independent of T (see also Appendix A.1), $A_r(T) > 0$, $\alpha_{\mathbf{q}} > 0$[1]

$$\sup_{\mathbf{x} \in \mathcal{D}_{\frac{\pi}{2n},\rho_0;\mathbf{x}}} \left|\mathbf{x}^{\boldsymbol{\alpha}_r}\mathbf{r}(\mathbf{x}, t)\right| = A_r(T) < \infty, \tag{11}$$

$$\sup_{\mathbf{x} \in \mathcal{D}_{\frac{\pi}{2n},\rho_0;\mathbf{x}}} \left|\mathbf{x}^{\boldsymbol{\alpha}_r}\mathbf{f}_I(\mathbf{x}, t)\right| = A_f(T) < \infty, \tag{12}$$

$$\sup_{\mathbf{k},\mathbf{q};\mathbf{x} \in \mathcal{D}_{\frac{\pi}{2n},\rho_0;\mathbf{x}}} \left|\mathbf{x}^{\boldsymbol{\alpha}_{\mathbf{q}}}\mathbf{b_{q,k}}\right| = A_b(T) < \infty. \tag{13}$$

(4) The analysis is interesting for $n > 1$, which we assume is the case.

2.2. *Existence and uniqueness for large* $|\mathbf{x}|$

Theorem 1. *Under the Assumptions* 1*, there is a unique solution* $\mathbf{f}$ *of* (5) *satisfying the following properties in* $\mathcal{D}_{\phi,\rho_0;\mathbf{x}}$: (a) $\mathbf{f}$ *analytic and* (b) $|\mathbf{x}^{\mathbf{1}}||\mathbf{f}|$ *bounded. Furthermore, this solution satisfies* $\mathbf{f} = \mathrm{O}(\mathbf{x}^{-\boldsymbol{\alpha}_r})$ *as* $\mathbf{x} \to \infty$ *in* $\mathcal{D}_{\phi,\tilde{\rho};\mathbf{x}}$*, for large* $\tilde{\rho}$*.*

The proof of Theorem 1 is given in Section 4.

Notes.

1. As shown in [7,9] for special examples, $\mathbf{f}$, in a larger sector is expected to have singularities with an accumulation point at infinity.
2. In Section 6, we also show that in some special cases, there is a duality between *small* t and *large* $\mathbf{x}$.
3. Relatively simple examples in which the assumptions apply after suitable transformations are the modified Harry–Dym equation $H_t + H_x = H^3 H_{xxx} - H^3/2$, Kuramoto–Sivashinsky $u_t + uu_x + u_{xx} + u_{xxxx} = 0$ and thin-film equation $h_t + \nabla \cdot (h^3 \nabla \Delta h) = 0$ (the latter with initial conditions such as $h(\mathbf{x}, 0) = 1 + (1 + ax_1^2 + bx_2^2)^{-1}$ in $d = 2$). The former equation is discussed in detail in [7] and the normalizing process, adapted to short time analysis, is described in Section 6.

[1] A restriction of the form $|\mathbf{x}|^{\tilde{\alpha}}|\mathbf{r}(\mathbf{x}, t)| < A_r(T)(*)$ may appear more natural. However, since every component of $\mathbf{x}$ is bounded below in $\mathcal{D}_{\phi,\rho_0,\mathbf{x}}$, it is clear that $(*)$ implies (11) with $\alpha_r = \tilde{\alpha}/d$. The same comment applies for condition (13). This form is more convenient in the present analysis. See also Note 4 following Theorem 1.

4. The condition $\alpha_r \geqslant 1$ is not particularly restrictive in problems with algebraically decaying coefficients. For these, as discussed in [7], one can redefine $\mathbf{f}$ by subtracting out from it the first few terms of its formal asymptotic expansion for large $\mathbf{x}$. The new $\mathbf{f}$ decays faster at ∞ and the condition to $\alpha_r \geqslant 1$ can be ensured.

2.3. Borel summability of power series solutions and their asymptotic character

Determining asymptotic properties of solutions of PDEs is substantially more difficult than the corresponding question for ODEs. Borel–Laplace techniques however provide a well suited modality to overcome this difficulty. The paper shows that formal series solutions are Borel summable to actual solutions (a fortiori are asymptotic to them). A few notes on Borel summability are found in Appendix A.2.

In addition to hypothesis of Theorem 1 we need, first of all, to impose restrictions to ensure that there exist series solutions, to which end the coefficients of the equation should be expandable for large $\mathbf{x}$. In many practical applications these coefficients turn out to be finite combinations of ramified inverse powers of x_i.

Condition 2. For large $|\mathbf{x}|$ and some $\mathbf{N} \in \mathbb{N}^d$, the functions $\mathbf{b}_{\mathbf{q},\mathbf{k}}(\mathbf{x}, t)$ and $\mathbf{r}(\mathbf{x}, t)$ are analytic in $(x_1^{-1/N_1}, \dots, x_d^{-1/N_d})$.

Theorem 2. *If Condition* 2 *and the assumptions of Theorem* 1 *are satisfied, then the unique solution* $\mathbf{f}$ *found there is the Borel sum of its own asymptotic series. More precisely,* $\mathbf{f}$ *can be written as*

$$\mathbf{f}(\mathbf{x}, t) = \int_{\mathbb{R}^{+d}} \mathrm{e}^{-\mathbf{p}\cdot\mathbf{x}^{\frac{n}{n-1}}} \mathbf{F}_1(\mathbf{p}, t)\, \mathrm{d}\mathbf{p} \tag{14}$$

where $\mathbf{F}_1$ *is* (a) *analytic at zero in* $(p_1^{1/(nN_1)}, \dots, p_d^{1/(nN_d)})$; (b) *analytic in* $\mathbf{p} \neq 0$ *in the poly-sector* $|\arg p_i| < \frac{n}{n-1}\phi + \frac{\pi}{2(n-1)}$, $i \leqslant d$; *and* (c) *exponentially bounded in the latter poly-sector.*

Comment. For PDEs it is known that it difficult to show, by classical methods, the existence of actual solutions given formal ones, when the formal solutions diverge. Borel summability of a formal asymptotic series solution shows in particular, using Watson's lemma [4], that there always indeed exist *actual* solutions of the PDE asymptotic to it. Borel summability also entails uniqueness of the actual solution if a sufficiently large sector of asymptoticity is prescribed (see, e.g. [1]). The Borel summability parameters proven in the present paper are optimal, as explained in the following remarks, and the sharp Gevrey class of the formal solutions follows too.

Remark 3.

(i) It follows from the same proof that $\mathbf{x}^{n/(n-1)}$ can be replaced with $\mathbf{x}^{\beta}$ for any $\beta \in [1, \frac{n}{n-1}]$. The canonical variable in Borel summation is that in which the generic Gevrey class of the formal series solution is one (i.e., the series diverge factorially, with factorial power one; [1]). This variable, in our case, is $\mathbf{x}^{n/(n-1)}$.
(ii) At least in simple examples, the sector of summability is optimal. See also Note 43.
(iii) In many problems of interest the conditions of Theorem 2 are met by the equation in more than one sector (after suitable rotation of coordinates). Then the functions $\mathbf{F}_1$ obtained in (2) are analytic continuations of each-other, as it follows from their construction.
(iv) If we had made the change of variable $\mathbf{x} \to \mathbf{x}^{n/(n-1)}$ first, (yielding the normalized Borel variable), the transformed PDE would have been more difficult to handle. Borel transforming directly from the $\mathbf{x}$ to $\mathbf{p}$ instead requires us to perform, in the proof of Theorem 2, an acceleration in the sense of Écalle to establish Borel summability, but is technically simpler.

The proof of Theorem 2 is given in Section 5.
See also Appendix A.1.

2.4. Spontaneous formation of singularities in nonlinear PDEs

Borel summability of formal solutions associated to solutions with prescribed initial data is a key ingredient in the detailed analysis of spontaneous singularities of solutions and in the study of their global properties. Applications of the present techniques in these directions, partly relying on extensions to PDEs of the methods in [6], are discussed in the paper [9].

3. Inverse Laplace transform and associated integral equation

The inverse Laplace transform (ILT) $\mathbf{G}(\mathbf{p},t)$ of a function $\mathbf{g}(\mathbf{x},t)$ analytic in $\mathbf{x}$ in $\mathcal{D}_{\phi,\rho;\mathbf{x}}$ and vanishing algebraically as $\mathbf{x}\to\infty$ (cf. Lemma 4 below and Note following it) is given by:

$$\mathbf{G}(\mathbf{p},t)=\left[\mathcal{L}^{-1}\{\mathbf{g}\}\right](\mathbf{p},t)\equiv\frac{1}{(2\pi\mathrm{i})^d}\int_{\mathcal{C}_D^d}\mathrm{e}^{\mathbf{p}\cdot\mathbf{x}}\mathbf{g}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\tag{15}$$

with a contour $\mathcal{C}_D$ as in Fig. 1 (modulo homotopies), $\mathcal{C}_D^d\subset\mathcal{D}_{\phi,\rho;\mathbf{x}}$, and $\mathbf{p}$ restricted to the dual (polar) domain $\mathcal{S}_\phi$ defined by

$$\mathcal{S}_\phi\equiv\left\{\mathbf{p}\colon\ |p_i|>0;\ \arg p_i\in(-\phi,\phi),\ i=1,\ldots,d\right\}\tag{16}$$

to ensure convergence of the integral.

The following lemma connects the $\mathbf{p}$ behavior of the ILT of functions of the type considered in this paper to their assumed behavior in $\mathbf{x}$.

Lemma 4. *If $\mathbf{g}(\mathbf{x},t)$ is analytic for $\mathbf{x}$ in $\mathcal{D}_{\phi,\rho;\mathbf{x}}$, and satisfies*

$$\left|\mathbf{x}^{\boldsymbol{\alpha}}\right|\left|\mathbf{g}(\mathbf{x},t)\right|\leqslant A(T)\tag{17}$$

for $\alpha\geqslant\alpha_0>0$, then for any $\delta\in(0,\phi)$, the ILT $\mathbf{G}=\mathcal{L}^{-1}\mathbf{g}$ exists in $\mathcal{S}_{\phi-\delta}$ and satisfies

$$\left|\mathbf{G}(\mathbf{p},t)\right|\leqslant C\frac{A(T)}{[\Gamma(\alpha)]^d}\left|\mathbf{p}^{\boldsymbol{\alpha}-\mathbf{1}}\right|\mathrm{e}^{2|\mathbf{p}|\rho}\tag{18}$$

for some $C=C(\delta,\alpha_0)$.

Proof. The proof is a higher-dimensional version of that of Lemma 3.1 in [7]. We first consider the case when $2\geqslant\alpha\geqslant\alpha_0$. Let C_{ρ_1} be a contour so that the integration path in each $\mathbf{x}$ component is as shown in Fig. 1: it passes through

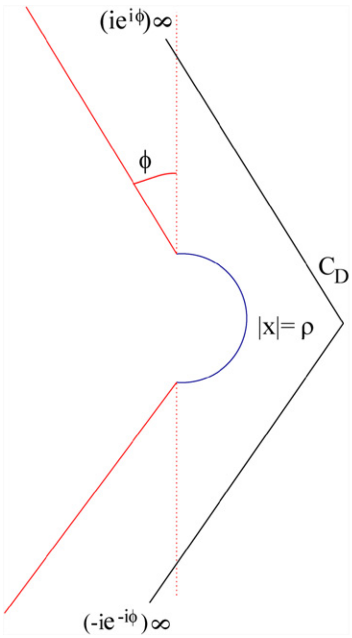

Fig. 1. Contour C_D in the $(\mathbf{p})_i$-plane.

point $\rho_1 + |p_i|^{-1}$, and $s = \rho_1 + |p_i|^{-1} + \mathrm{i}r\exp(\mathrm{i}\phi\,\mathrm{signum}(r))$ with $r \in (-\infty, \infty)$. Choosing $2\rho \geqslant \rho_1 \geqslant (2/\sqrt{3})\rho$, we have $|s| > \rho$ along the contour and therefore, with $\arg(p_i) = \theta \in (-\phi + \delta, \phi - \delta)$,

$$\left|\mathbf{g}(\mathbf{s},t)\right| \leqslant A(T)\left|\mathbf{s}^{-\boldsymbol{\alpha}}\right| \quad \text{and} \quad \left|\mathrm{e}^{\mathbf{s}\cdot\mathbf{p}}\right| \leqslant \mathrm{e}^{\rho_1|\mathbf{p}|+d}\mathrm{e}^{-r|\mathbf{p}|\sin|\phi+\theta|}.$$

Thus,

$$\left|\int_{C_{\rho_1}} \mathrm{e}^{\mathbf{s}\cdot\mathbf{p}}\mathbf{g}(\mathbf{s},t)\,\mathrm{d}\mathbf{s}\right| \leqslant 2A(T)\mathrm{e}^{\rho_1|\mathbf{p}|+d}\prod_i \int_0^\infty \left|\rho_1 + |p_i|^{-1} + \mathrm{i}r\mathrm{e}^{\mathrm{i}\phi}\right|^{-\alpha}\mathrm{e}^{-|p_i|r\sin\delta}\,\mathrm{d}r$$

$$\leqslant \widetilde{K}A(T)\mathrm{e}^{\rho_1|\mathbf{p}|}\prod_i\left\{\left|\rho_1 + |p_i|^{-1}\right|^{-\alpha}\int_0^\infty \mathrm{e}^{-|p_i|r\sin\delta}\,\mathrm{d}r\right\} \leqslant K\delta^{-d}\left|\mathbf{p}^{\boldsymbol{\alpha}-\mathbf{1}}\right|\mathrm{e}^{2\rho|\mathbf{p}|} \tag{19}$$

where $\widetilde{K}$ and K are constants independent of any parameter. Thus, the lemma follows for $2 \geqslant \alpha \geqslant \alpha_0$, if we note that $\Gamma(\alpha)$ is bounded in this range of α, the bound only depending on α_0.

For $\alpha > 2$, there exists an integer $k > 0$ so that $\alpha - k \in (1, 2]$. Taking

$$\left[(k-1)!\right]^d\mathbf{h}(\mathbf{x},t) = \int_\infty^{\mathbf{x}} \mathbf{g}(\mathbf{z},t)(\mathbf{x}-\mathbf{z})^{\mathbf{k}-\mathbf{1}}\,\mathrm{d}\mathbf{z}$$

(clearly $\mathbf{h}$ is analytic in $\mathbf{x}$, in $\mathcal{D}_{\phi,\rho}$ and $\partial_{\mathbf{x}}^{\mathbf{k}}\mathbf{h}(\mathbf{x},t) = \mathbf{g}(\mathbf{x},t)$), we get

$$\mathbf{h}(\mathbf{x},t) = \frac{(-1)^{dk}\mathbf{x}^{k\mathbf{1}}}{[(k-1)!]^d}\int_{\mathbf{1}}^\infty \mathbf{g}(\mathbf{x}\cdot\mathbf{y},t)(\mathbf{y}-\mathbf{1})^{(k-1)\mathbf{1}}\,\mathrm{d}\mathbf{y}$$

$$= \frac{(-1)^{dk}\mathbf{x}^{(k-\alpha)\mathbf{1}}}{[(k-1)!]^d}\int_{\mathbf{1}}^\infty \mathbf{A}(\mathbf{x}\cdot\mathbf{y},t)\mathbf{y}^{-\boldsymbol{\alpha}}(\mathbf{y}-\mathbf{1})^{(k-1)\mathbf{1}}\,\mathrm{d}\mathbf{y}$$

with $|\mathbf{A}(\mathbf{x}\cdot\mathbf{p},t)| \leqslant A(T)$, whence

$$\left|\mathbf{h}(\mathbf{x},t)\right| \leqslant \frac{A(T)[\Gamma(\alpha-k)]^d}{|\mathbf{x}^1|^{\alpha-k}[\Gamma(\alpha)]^d}.$$

From the arguments above with $\alpha - k$ playing the role of α, we get

$$\left|\mathcal{L}^{-1}\{\mathbf{h}\}(\mathbf{p},t)\right| \leqslant C(\delta)\frac{A(T)}{[\Gamma(\alpha)]^d}|\mathbf{p}^1|^{\alpha-k-1}\mathrm{e}^{2|\mathbf{p}|\rho}.$$

Since $\mathbf{G}(\mathbf{p},t) = (-1)^{kd}\mathbf{p}^{\mathbf{1}k}\mathcal{L}^{-1}\{\mathbf{h}\}(\mathbf{p},t)$, by multiplying the above equation by $|\mathbf{p}^1|^k$, the lemma follows for $\alpha > 2$ as well. □

Remark 5. The constant 2ρ in the exponential bound can be lowered to $\rho + 0$, but (18) suffices for our purposes. Note also that the statement also holds for $\rho = 0$, a fact that will be used in Section 6.

Remark 6. Corollary 9 below implies that for any $\mathbf{p} \in \mathcal{S}_\phi$, the ILT exists for the functions $\mathbf{r}(\mathbf{x},t)$, $\mathbf{b}_{\mathbf{q},\mathbf{k}}(\mathbf{x},t)$, as well as for the solution $\mathbf{f}(\mathbf{x},t)$, whose existence is shown in the sequel.

Remark 7. Conversely, if $\mathbf{G}(\mathbf{p},t)$ is any integrable function satisfying the exponential bound in (18), it is clear that the Laplace Transform along a ray (1) exists and defines an analytic function of $\mathbf{x}$ in the half-plane for each component defined by $\Re[\mathrm{e}^{\mathrm{i}\theta_i}x_i] > 2\rho$ for $\theta_i \in (-\phi,\phi)$. Due to the width of the sector it is easy to see, by Fubini, that $\mathcal{L}\mathbf{G} = \mathbf{g}$.

Remark 8. The next corollary finds bounds for $\mathbf{B}_{\mathbf{q},\mathbf{k}} = \mathcal{L}^{-1}\{\mathbf{b}_{\mathbf{q},\mathbf{k}}\}$ and $\mathbf{R} = \mathcal{L}^{-1}\{\mathbf{r}\}$ independent of $\arg p_i$ for $\mathbf{p} \in \mathcal{S}_\phi$, following from the properties of $\mathbf{b}_{\mathbf{q},\mathbf{k}}$ and $\mathbf{r}$ in $\mathcal{D}_{\frac{\pi}{2n},\rho_0} \supset \mathcal{D}_{\phi,\rho}$.

Corollary 9. *The ILT of the coefficients* $\mathbf{b}_{\mathbf{q},\mathbf{k}}$ *(cf.* (10)) *and of the inhomogeneous term* $\mathbf{r}(\mathbf{x},t)$ *satisfy the following upper bounds for any* $\mathbf{p}\in\mathcal{S}_\phi$

$$\left|\mathbf{B}_{\mathbf{q},\mathbf{k}}(\mathbf{p},t)\right| \leqslant \frac{C_1(\phi,\alpha_{\mathbf{q}})}{[\Gamma(\alpha_{\mathbf{q}})]^d} A_b(T)\left|\mathbf{p}^{\boldsymbol{\alpha}_{\mathbf{q}}-\mathbf{1}}\right| \mathrm{e}^{2\rho_0|\mathbf{p}|}, \tag{20}$$

$$\left|\mathbf{R}(p,t)\right| \leqslant \frac{C_2(\phi)}{[\Gamma(\alpha_r)]^d} A_r(T)\left|\mathbf{p}^{\boldsymbol{\alpha}_r-\mathbf{1}}\right| \mathrm{e}^{2\rho_0|\mathbf{p}|}. \tag{21}$$

Proof. The proof is similar to that of Corollary 3.2 in [7]. From the conditions assumed we see that $\mathbf{b}_{\mathbf{q},\mathbf{k}}$ is analytic in $\mathbf{x}\in\mathcal{D}_{\phi_1,\rho_0;\mathbf{x}}$ for any ϕ_1 satisfying $(2n)^{-1}\pi>\phi_1>\phi>0$. So Lemma 4 can be applied, with $\mathbf{g}(\mathbf{x},t)=\mathbf{b}_{\mathbf{q},\mathbf{k}}$, with $\phi_1=\phi+((2n)^{-1}\pi-\phi)/2$ replacing ϕ, and with δ replaced by $\phi_1-\phi=((2n)^{-1}\pi-\phi)/2$. The same applies to $\mathbf{R}(\mathbf{p},t)$, leading to (20) and (21). In the latter case, since $\alpha_r\geqslant 1$, α_0 in Lemma 4 can be chosen to be 1. Thus, one can choose C_2 to be independent of α_r. □

Lemma 10. *For some* $R\in\mathbb{R}^+$ *and all* $\mathbf{p}$ *with* $|\mathbf{p}|>R$ *and* $\max_{i\leqslant d}|\arg p_i|\leqslant\phi$ *we have for some* $C>0$

$$\Re\mathcal{P}_j(-\mathbf{p})>C|\mathbf{p}|^n. \tag{22}$$

Proof. For the proof, we take $B=\{\mathbf{p}\colon\ |\mathbf{p}|=1,\ \max_{j\leqslant d}|\arg p_j|\leqslant\phi\}$ and note that

$$C_0=\inf_{\substack{\mathbf{p}\in B\\ 1\leqslant j\leqslant m}}\Re\mathcal{P}_{n;j}(-\mathbf{p})>0 \tag{23}$$

(cf. definitions following (6)). Indeed, if $C_0=0$, then by continuity $\Re\mathcal{P}_{n;j}(-\mathbf{p})$ would have a root in B which is ruled out by (9). The conclusion now follows, since on a sphere of large radius R, $\mathcal{P}_j$ is given by $R^n\mathcal{P}_{n;j}(-\mathbf{p}/R)+o(R^n)$. □

The formal inverse Laplace transform (Borel transform) of (5) with respect to $\mathbf{x}$ (see also (10)) for $\mathbf{p}\in\mathcal{S}_\phi$ is

$$\partial_t\mathbf{F}+\mathcal{P}(-\mathbf{p})\mathbf{F}=\sum_{\mathbf{q}\succeq 0}{}'\sum_{\mathbf{k}\succeq 0}\mathbf{B}_{\mathbf{q},\mathbf{k}}*\mathbf{F}^{*\mathbf{k}}*{}^*\prod\nolimits_{l,|\mathbf{j}|}\big((-\mathbf{p})^{\mathbf{j}}F_l\big)^{*q_{l,\mathbf{j}}}+\mathbf{R}(\mathbf{p},t) \tag{24}$$

where $\mathbf{F}=\mathcal{L}^{-1}\mathbf{f}$. After inverting the differential operator on the left side of (24) with respect to t, we obtain the integral equation

$$\mathbf{F}(\mathbf{p},t)=\mathcal{N}(\mathbf{F})\equiv\mathbf{F}_0(\mathbf{p},t) + \int_0^t \mathrm{e}^{-\mathcal{P}(-\mathbf{p})(t-\tau)}\sum_{\mathbf{q}\succeq 0}{}'\sum_{\mathbf{k}\succeq 0}\mathbf{B}_{\mathbf{q},\mathbf{k}}(\mathbf{p},\tau)*\mathbf{F}^{*\mathbf{k}}(\mathbf{p},\tau)*{}^*\prod\nolimits_{l,|\mathbf{j}|}\big((-\mathbf{p})^{\mathbf{j}}F_l(\mathbf{p},\tau)\big)^{*q_{l,\mathbf{j}}}\,\mathrm{d}\tau \tag{25}$$

where

$$\mathbf{F}_0(p,t)=\mathrm{e}^{-\mathcal{P}(-\mathbf{p})t}\mathbf{F}_I(\mathbf{p})+\int_0^t\mathrm{e}^{-\mathcal{P}(-\mathbf{p})(t-\tau)}\mathbf{R}(p,\tau)\,\mathrm{d}\tau\quad\text{and}\quad\mathbf{F}_I=\mathcal{L}^{-1}\{\mathbf{f}_I\}. \tag{26}$$

Our strategy is to reduce the problem of existence and uniqueness of a solution of (5) to the problem of existence and uniqueness of a solution of (25), under appropriate conditions.

4. Solution to the associated integral equation

To establish the existence and uniqueness in (25) we first introduce suitable function spaces.

Definition 11. Denoting by $\overline{\mathcal{S}_\phi}$ the closure of $\mathcal{S}_\phi$ defined in (16), $\partial\mathcal{S}_\phi=\overline{\mathcal{S}_\phi}\setminus\mathcal{S}_\phi$ and $\mathcal{K}=\overline{\mathcal{S}_\phi}\times[0,T]$, we define for $\nu>0$ (later to be taken appropriately large) the norm $\|\cdot\|_\nu$ as

$$\|\mathbf{G}\|_\nu=M_0^d\sup_{(\mathbf{p},t)\in\mathcal{K}}\left(\prod_i\big(1+|p_i|^2\big)\right)\mathrm{e}^{-\nu|\mathbf{p}|}\left|\mathbf{G}(\mathbf{p},t)\right| \tag{27}$$

where the constant M_0 (about 3.76) is defined as

$$M_0 = \sup_{s\geqslant 0}\left\{\frac{2(1+s^2)(\ln(1+s^2)+s\arctan s)}{s(s^2+4)}\right\} \tag{28}$$

Note. For fixed $\mathbf{F}$, $\|\mathbf{F}\|_\nu$ is nonincreasing in ν.

Definition 12. Consider the following Banach space.

$$\mathcal{A}_\phi = \left\{\mathbf{F}\colon \mathbf{F}(\cdot,t) \text{ analytic in } \mathcal{S}_\phi \text{ and continuous in } \overline{\mathcal{S}_\phi} \text{ for } t\in[0,T] \text{ s.t. } \|\mathbf{F}\|_\nu<\infty\right\}. \tag{29}$$

Remark 13. If $\mathbf{G}\in\mathcal{A}_\phi$, then $\mathbf{g}(\mathbf{x},t) =: \mathcal{L}_{\boldsymbol{\theta}}\{\mathbf{G}\}$ exists for suitable $\boldsymbol{\theta}$ if $\rho\cos(\theta_i+\arg x_i)>\nu$. Furthermore, $\mathbf{g}(\mathbf{x},t)$ is analytic in $\mathbf{x}$, and $|\mathbf{x}^{\mathbf{1}}\mathbf{g}(\mathbf{x},t)|$ is bounded in $\mathcal{D}_{\phi,\rho;\mathbf{x}}$.

Lemma 14. *For $\nu>4\rho_0+\alpha_r$, $\mathbf{F}_I$ in* (26) *satisfies*

$$\|\mathbf{F}_I\|_\nu \leqslant C(\phi)A_{f_I}(\nu/2)^{-d\alpha_r+d}$$

while $\mathbf{R}$ *satisfies the inequality*

$$\|\mathbf{R}\|_\nu \leqslant C(\phi)A_r(T)(\nu/2)^{-d\alpha_r+d}$$

and therefore

$$\|\mathbf{F}_0\|_\nu \leqslant C(\phi)A_0(T)(\nu/2)^{-d\alpha_r+d}. \tag{30}$$

Proof. This proof is similar to that of Lemma 4.4 in [7]. We use (21), note that $\alpha_r\geqslant 1$ and also that for $\nu>4\rho_0+\alpha_r$ we have

$$\sup_{|p_1|>0}\frac{|p_1|^{\alpha_r\pm 1}}{\Gamma(\alpha_r)}\,\mathrm{e}^{-(\nu-2\rho_0)|p_1|} \leqslant \frac{(\alpha_r\pm 1)^{\alpha_r\pm 1}}{\Gamma(\alpha_r)}\,\mathrm{e}^{-\alpha_r\mp 1}(\nu-2\rho_0)^{-\alpha_r\mp 1} \leqslant K\alpha_r^{1/2\pm 1}(\nu/2)^{-\alpha_r\mp 1} \tag{31}$$

where K is independent of ν and α_r. The latter inequality follows from Stirling's formula for $\Gamma(\alpha_r)$ for large α_r.

Using the definition of the ν-norm and the two equations above, the inequality for $\|\mathbf{R}\|_\nu$ follows. Since $\mathbf{f}_I(x)$ is required to satisfy the same bounds as $\mathbf{r}(x,t)$, a similar inequality holds for $\|\mathbf{F}_I\|_\nu$. Now, from the relation (26) and the fact that $\Re\mathcal{P}_j(-\mathbf{p})$ is, by Lemma 10, bounded below for $\mathbf{p}\in\mathcal{S}_\phi$, we get the following inequality, implying (30)

$$\left|\mathbf{F}_0(\mathbf{p},t)\right| \leqslant \left|\mathbf{F}_I(\mathbf{p})\right| + T\hat{A}_0(T)\sup_{0\leqslant t\leqslant T}\left|R(\mathbf{p},t)\right|. \qquad \square$$

It is convenient to introduce a space of sectorially analytic functions possibly unbounded at the origin but integrable.

Definition 15. Let

$$\mathcal{H} := \left\{\mathbf{H}\colon \mathbf{H}(\mathbf{p},t) \text{ analytic in } \mathcal{S}_\phi,\ \left|\mathbf{H}(\mathbf{p},t)\right| \leqslant C\left|\mathbf{p}^{\boldsymbol{\alpha}-\mathbf{1}}\right|\mathrm{e}^{\rho|\mathbf{p}|}\right\}$$

(C, α and ρ may depend on $\mathbf{H}$).

Lemma 16. *If $\mathbf{H}\in\mathcal{H}$ and $\mathbf{F}\in\mathcal{A}_\phi$, then for $\nu>\rho+4$, for any j, $\mathbf{H}*F_j\in\mathcal{A}_\phi$, and:*[2]

$$\|\mathbf{H}*F_j\|_\nu \leqslant \left\||\mathbf{H}|*|F_j|\right\|_\nu \leqslant C\left[\Gamma(\alpha)\right]^d 2^{d\alpha}(\nu-\rho)^{-d\alpha}\|\mathbf{F}\|_\nu \tag{32}$$

where C is independent of α.

Proof. The proof is a vector adaptation of that of Lemma 4.6 in [7]. From the elementary properties of convolution, it is clear that $\mathbf{H}*F_j$ is analytic in $\mathcal{S}_\phi$ and continuous in $\overline{\mathcal{S}_\phi}$. Let $\theta_i=\arg p_i$. We have

$$\left|\mathbf{H}*F_j(\mathbf{p})\right| \leqslant \left||\mathbf{H}|*|F_j|(\mathbf{p})\right| \leqslant \int\limits_{\prod_i[0,|p_i|]}\left|\mathbf{H}\left(\mathbf{s}\mathrm{e}^{\mathrm{i}\theta}\right)\right|\left|F_j\left(\mathbf{p}-\mathbf{s}\mathrm{e}^{\mathrm{i}\theta}\right)\right|\mathrm{d}\mathbf{s}.$$

[2] In the following equation, $\|\cdot\|_\nu$ is extended naturally to functions which are only continuous in $\mathcal{K}$.

Now

$$\left|\mathbf{H}\left(\mathrm{s}\mathrm{e}^{\mathrm{i}\theta}\right)\right| \leqslant C\left|\mathbf{s}^{\boldsymbol{\alpha}-\mathbf{1}}\right|\mathrm{e}^{|\mathbf{s}|\rho} \tag{33}$$

and

$$\int\limits_{\prod_i [0,|p_i|]} \mathbf{s}^{\boldsymbol{\alpha}-\mathbf{1}}\mathrm{e}^{|\mathbf{s}|\rho}\left|F_j\left(\mathbf{p}-\mathbf{s}\mathrm{e}^{i\boldsymbol{\theta}}\right)\right| \mathrm{d}\mathbf{s} \leqslant \|F_j\|_\nu \mathrm{e}^{\nu|\mathbf{p}|}|\mathbf{p}^{\boldsymbol{\alpha}}|\prod_i\left[\int_0^1 \frac{s_i^{\alpha-1}\mathrm{e}^{-(\nu-\rho)|p_i|s_i}}{M_0(1+|p_i|^2(1-s_i)^2)}\,\mathrm{d}s_i\right]. \tag{34}$$

Since $\nu-\rho \geqslant 4$, we can readily use (122) in Appendix A with $\mu=|p_i|$, ν replaced by $\nu-\rho$, $\sigma=1$ and $m=1$ to conclude

$$|p_i|^\alpha \int_0^1 \frac{s_i^{\alpha-1}\mathrm{e}^{-(\nu-\rho)|p_i|s_i}}{M_0(1+|p_i|^2(1-s_i)^2)}\,\mathrm{d}s_i \leqslant \frac{K\Gamma(\alpha)2^\alpha(\nu-\rho)^{-\alpha}}{M_0(1+|p_i|^2)}. \tag{35}$$

Therefore, from (34), we obtain

$$\int\limits_{\prod_i [0,|p_i|]} \mathbf{s}^{\boldsymbol{\alpha}-\mathbf{1}}\mathrm{e}^{|\mathbf{s}|\rho}\left|F_j\left(\mathbf{p}-\mathbf{s}\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}}\right)\right| \mathrm{d}\mathbf{s} \leqslant K\left[\Gamma(\alpha)\right]^d \frac{\|F_j\|_\nu \mathrm{e}^{\nu|\mathbf{p}|}2^{d\alpha}|\nu-\rho|^{-d\alpha}}{M_0^d\prod_i(1+|p_i|^2)}. \tag{36}$$

From this relation, (32) follows by applying the definition of $\|\cdot\|_\nu$. □

Remark 17. Lemma 16 holds for $\rho=0$ as well, when $\nu>4$.

Corollary 18. *For $\mathbf{F}\in\mathcal{A}_\phi$, and $\nu>4\rho_0+4$ we have $\mathbf{B}_{\mathbf{q},\mathbf{k}} * F_l \in \mathcal{A}_\phi$ and*

$$\|\mathbf{B}_{\mathbf{q},\mathbf{k}} * F_l\|_\nu \leqslant \big\||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{F}|\big\|_\nu \leqslant KC_1(\phi,\alpha_{\mathbf{q}})(\nu/4)^{-d\alpha_{\mathbf{q}}}A_b(T)\|\mathbf{F}\|_\nu.$$

Proof. The proof follows simply by using Lemma 16, with $\mathbf{H}$ replaced by $\mathbf{B}_{\mathbf{q},\mathbf{k}}$ and using the relations in Corollary 9. □

Lemma 19. *For $\mathbf{F}\in\mathcal{A}_\phi$, with $\nu>4\rho_0+4$, for any $\mathbf{j}$, l,*

$$\left|\mathbf{B}_{\mathbf{q},\mathbf{k}} * \left(\mathbf{p}^{\mathbf{j}}F_l\right)\right| \leqslant \frac{KC_1|\mathbf{p}^{\mathbf{j}}|\mathrm{e}^{\nu|\mathbf{p}|}A_b(T)}{M_0^d\prod_i(1+|p_i|^2)}\|\mathbf{F}\|_\nu\left(\frac{\nu}{4}\right)^{-d\alpha_{\mathbf{q}}}.$$

Proof. From the definition (2), it readily follows that

$$\left|\mathbf{B}_{\mathbf{q},\mathbf{k}} * \left(\mathbf{p}^{\mathbf{j}}F_l\right)\right| \leqslant \left|\mathbf{p}^{\mathbf{j}}\right||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |F_l|.$$

The rest follows from Corollary 18, and the definition of $\|\cdot\|_\nu$. □

Lemma 20. *For $\mathbf{F}$, $\mathbf{G}\in\mathcal{A}_\phi$ and $j\geqslant 0$*

$$\left|\left(\mathbf{p}^{\mathbf{j}}F_{l_1}\right) * G_{l_2}\right| \leqslant \left|\mathbf{p}^{\mathbf{j}}\right|\left||\mathbf{F}| * |\mathbf{G}|\right|. \tag{37}$$

Proof. Let $\mathbf{p}=(p_1\mathrm{e}^{\mathrm{i}\theta_1}, p_2\mathrm{e}^{\mathrm{i}\theta_2},\ldots,p_d\mathrm{e}^{\mathrm{i}\theta_d})$. Then the result follows from the inequality

$$\left|\mathbf{p}^{\mathbf{j}}F_{l_1} * G_{l_2}\right| = \left|\int_{\mathbf{0}}^{\mathbf{p}} \tilde{\mathbf{s}}^{\mathbf{j}}F_{l_1}(\tilde{\mathbf{s}})G_{l_2}(\mathbf{p}-\tilde{\mathbf{s}})\,\mathrm{d}\tilde{\mathbf{s}}\right| \leqslant \left|\mathbf{p}^{\mathbf{j}}\right| \int\limits_{\prod_i [0,|p|_i]} \left|\mathbf{F}\left(\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}}\mathbf{s}\right)\right|\left|\mathbf{G}\left(\mathbf{p}-\mathrm{e}^{\mathrm{i}\boldsymbol{\theta}}\mathbf{s}\right)\right|\mathrm{d}\mathbf{s}. \qquad \square \tag{38}$$

Corollary 21. *If $\mathbf{F}\in\mathcal{A}_\phi$, then*

$$\left|{}^*\prod\nolimits_{l,|\mathbf{j}|}\left(\mathbf{p}^{\mathbf{j}}F_l\right)^{*q_{l,\mathbf{j}}}\right| \leqslant \prod_i |p_i|^{\sum_{l,|\mathbf{j}|} j_i q_{l,\mathbf{j}}}\left|{}^*\prod\nolimits_{l,|\mathbf{j}|}|\mathbf{F}|^{*q_{l,\mathbf{j}}}\right|. \tag{39}$$

Proof. This follows simply from repeated application of Lemma 20. □

Lemma 22. *For* $\mathbf{F}$, $\mathbf{G} \in \mathcal{A}_\phi$,

$$\big||\mathbf{F}| * |\mathbf{G}|\big| \leqslant \frac{e^{\nu|\mathbf{p}|}}{M_0^d \prod_i (1+|p_i|^2)} \|\mathbf{F}\|_\nu \|\mathbf{G}\|_\nu.$$

Proof.

$$\big||\mathbf{F}| * |\mathbf{G}|\big| = \left| \int_{\mathbf{0}}^{\mathbf{p}} \big|\mathbf{F}(\tilde{\mathbf{s}})\big|\big|\mathbf{G}(\mathbf{p}-\tilde{\mathbf{s}})\big| \, d\tilde{\mathbf{s}} \right| \leqslant \int_{\prod_i [0,|p|_i]} \big|\mathbf{F}\big(e^{i\boldsymbol{\theta}}\mathbf{s}\big)\big|\big|\mathbf{G}\big(\mathbf{p}-e^{i\boldsymbol{\theta}}\mathbf{s}\big)\big| \, d\mathbf{s}. \tag{40}$$

Using the definition of $\|\cdot\|_\nu$, the above expression is bounded by

$$\frac{e^{\nu|\mathbf{p}|}}{M_0^{2d}} \|\mathbf{F}\|_\nu \|\mathbf{G}\|_\nu \prod_i \int_0^{|p_i|} \frac{ds_i}{(1+s_i^2)[1+(|p_i|-s_i)^2]} \leqslant \frac{|\mathbf{p}^{\mathbf{j}}| e^{\nu|\mathbf{p}|}}{M_0^d \prod_i (1+|p_i|^2)} \|\mathbf{F}\|_\nu \|\mathbf{G}\|_\nu.$$

The last inequality follows from the definition (28) of M_0 since

$$\int_0^{|p_i|} \frac{ds_i}{(1+s_i^2)[1+(|p_i|-s_i)^2]} = 2\frac{\ln(|p_i|^2+1)+|p_i|\tan^{-1}|p_i|}{|p_i|(|p_i|^2+4)}. \qquad \square$$

Corollary 23. *For* $\mathbf{F}$, $\mathbf{G} \in \mathcal{A}_\phi$, *then*

$$\big\||\mathbf{F}| * |\mathbf{G}|\big\|_\nu \leqslant \|\mathbf{F}\|_\nu \|\mathbf{G}\|_\nu.$$

Proof. This is an application of Lemma 22 and the definition of $\|\cdot\|_\nu$. □

Lemma 24. *For* $\nu > 4\rho_0 + 4$,

$$\left| \mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} *^* \prod_{l,|\mathbf{j}|} \big(\mathbf{p}^{\mathbf{j}} F_l\big)^{*q_{l,\mathbf{j}}} \right| \leqslant \frac{e^{\nu|\mathbf{p}|} \prod_i |p_i|^{\sum j_i q_{l,\mathbf{j}}}}{M_0^d \prod_i (1+|p_i|^2)} \|\mathbf{F}\|_\nu^{|\mathbf{q}|+|\mathbf{k}|-1} \big\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{F}| \big\|_\nu \tag{41}$$

if $(\mathbf{q},\mathbf{k}) \neq (\mathbf{0},\mathbf{0})$ *and is zero if* $(\mathbf{q},\mathbf{k}) = (\mathbf{0},\mathbf{0})$.

Proof. For $(\mathbf{q},\mathbf{k}) = (\mathbf{0},\mathbf{0})$ we have $\mathbf{B}_{\mathbf{q},\mathbf{k}} = 0$ (see remarks after Eq. (10)). If $\mathbf{k} \neq \mathbf{0}$, Corollary 21 shows that the left-hand side of (41) is bounded by

$$\prod_i |p_i|^{\sum j_i q_{l,\mathbf{j}}} \left| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{F}| * |\mathbf{F}|^{*(|\mathbf{k}|-1)} *^* \prod_{l,|\mathbf{j}|} |\mathbf{F}|^{*q_{l,\mathbf{j}}} \right|.$$

Using Corollaries 21 and 23 and Lemma 22, the proof follows for $\mathbf{k} \neq 0$. Similar steps work for the case $\mathbf{k} = \mathbf{0}$ and $\mathbf{q} \neq \mathbf{0}$, except that $\mathbf{B}_{\mathbf{q},\mathbf{k}}$ is convolved with $\mathbf{p}^{\mathbf{j}'} F_{l_1}$ for some $(\mathbf{j}', l_1)$, for which the corresponding $q_{l_1,\mathbf{j}'} \neq 0$, and we now use Lemma 20 and the definition of $\|\cdot\|_\nu$. □

Corollary 25. *For* $\nu > 4\rho_0 + 4$,

$$\left| \mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} *^* \prod_{l,|\mathbf{j}|} \big(\mathbf{p}^{\mathbf{j}} F_l\big)^{*q_{l,\mathbf{j}}} \right| \leqslant \frac{K C_1 A_b(T) e^{\nu|\mathbf{p}|} \prod_i |p_i|^{\sum j_i q_{l,\mathbf{j}}}}{M_0^d \prod_i (1+|p_i|^2)} \left(\frac{\nu}{4}\right)^{-d\alpha_{\mathbf{q}}} \|\mathbf{F}\|_\nu^{|q|+|\mathbf{k}|}. \tag{42}$$

The proof follows immediately from Corollary 18 and Lemma 24.

Lemma 26. *For $\nu > 4\rho_0 + 4$, we have*

$$\left| \int_0^t e^{-\mathcal{P}(-\mathbf{p})(t-\tau)} \mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} * {}^*\prod_{l,|\mathbf{j}|} \big(\mathbf{p}^{\mathbf{j}} F_l\big)^{*q_{l,\mathbf{j}}} \, d\tau \right| \leqslant \frac{C \tilde{A}_b(T) e^{\nu|\mathbf{p}|}}{M_0^d \prod_i (1+|p_i|^2)} \left(\frac{\nu}{4}\right)^{-d\alpha_{\mathbf{q}}} \|\mathbf{F}\|_\nu^{|\mathbf{q}|+|\mathbf{k}|} \tag{43}$$

for some $\tilde{A}_b(T) \geqslant A_b(T)$ (evaluated in the proof) and where the constant C is independent of T, but depends on ϕ and $\alpha_{\mathbf{q}}$.

Proof. This is a consequence of Lemmas 19 and 24 and the fact that for $0 \leqslant |\mathbf{l}'| \leqslant n$ we have, for $|\mathbf{p}| \leqslant R$ (with R as in Lemma 10),

$$J := \big|\mathbf{p}^{\mathbf{l}'}\big| \int_0^t e^{-\Re\mathcal{P}(-\mathbf{p})(t-\tau)} \, d\tau \leqslant C_2(T). \tag{44}$$

For $|\mathbf{p}| > R$ we have, by Lemma 10, $\mathcal{P}(-\mathbf{p}) > C|\mathbf{p}|^n$, and J is majorized by

$$m \max_{j\leqslant m} \frac{|\mathbf{p}^{\mathbf{l}'}|}{\Re\mathcal{P}_j(-\mathbf{p})} \big[1 - e^{-\Re\mathcal{P}_j(-\mathbf{p})t}\big] \leqslant \max_{j\leqslant m} \frac{T^{1-|\mathbf{l}'|/n} |\mathbf{p}|^{|\mathbf{l}'|}}{|\Re\mathcal{P}_j(-\mathbf{p})|^{|\mathbf{l}'|/n}} \sup_{\gamma>0} \frac{1-e^{-\gamma}}{\gamma^{1-|\mathbf{l}'|/n}} \leqslant C T^{1-|\mathbf{l}'|/n} \tag{45}$$

where $\mathbf{l}' = \sum_{\mathbf{j},l} \mathbf{j} q_{l,\mathbf{j}}$. □

Definition 27. For $\mathbf{F}$ and $\mathbf{h}$ in $\mathcal{A}_\phi$, and $\mathbf{B}_{\mathbf{q},\mathbf{k}} \in \mathcal{H}$, as above, define $\mathbf{h}_0 = \mathbf{0}$ and for $k \geqslant 1$,

$$\mathbf{h}_{\mathbf{k}} \equiv \mathbf{B}_{\mathbf{q},\mathbf{k}} * \big[(\mathbf{F}+\mathbf{h})^{*\mathbf{k}} - \mathbf{F}^{*\mathbf{k}}\big]. \tag{46}$$

Lemma 28. *For $\nu > 4\rho_0 + 4$, and for $\mathbf{k} \neq 0$,*

$$\|\mathbf{h}_{\mathbf{k}}\|_\nu \leqslant |\mathbf{k}| \big(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\big)^{|\mathbf{k}|-1} \big\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}| \big\|_\nu \tag{47}$$

and is zero for $\mathbf{k} = 0$.

Proof. The cases $|\mathbf{k}| = 0, 1$ follow from the definition of $\mathbf{h}_0$ and (46) respectively. Assume formula (47) holds for all $|\mathbf{k}| \leqslant l$. Then all multiindices of length $l+1$ can be expressed as $\mathbf{k} + \hat{\mathbf{e}}_i$, where $\hat{\mathbf{e}}_i \in \mathbb{R}^m$ is the m-dimensional unit vector in the i-th direction, and $|\mathbf{k}| = l$.

$$\|\mathbf{h}_{\mathbf{k}+\hat{\mathbf{e}}_i}\|_\nu = \big\| \mathbf{B}_{\mathbf{q},\mathbf{k}} * (F_i + h_i) * (\mathbf{F}+\mathbf{h})^{*\mathbf{k}} - \mathbf{B}_{\mathbf{q},\mathbf{k}} * F_i * \mathbf{F}^{*\mathbf{k}} \big\|_\nu = \big\| \mathbf{B}_{\mathbf{q},\mathbf{k}} * h_i * (\mathbf{F}+\mathbf{h})^{*\mathbf{k}} + F_i * \mathbf{h}_{\mathbf{k}} \big\|_\nu.$$

Using (47) for $|\mathbf{k}| = l$, we get

$$\leqslant \big\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}| \big\|_\nu \big(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\big)^l + l \|\mathbf{F}\|_\nu \big(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\big)^{l-1} \big\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}| \big\|_\nu$$
$$\leqslant (l+1) \big(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\big)^l \big\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}| \big\|_\nu.$$

Thus (47) holds for $|\mathbf{k}| = l+1$. □

Definition 29. For $\mathbf{F} \in \mathcal{A}_\phi$ and $\mathbf{h} \in \mathcal{A}_\phi$, and $\mathbf{B}_{\mathbf{q},\mathbf{k}}$ as above define $\mathbf{g}_{\mathbf{0}} = \mathbf{0}$, and for $|\mathbf{q}| \geqslant 1$,

$$\mathbf{g}_{\mathbf{q}} \equiv \mathbf{B}_{\mathbf{q},\mathbf{k}} * {}^*\prod_{l,|\mathbf{j}|} \big(\mathbf{p}^{\mathbf{j}} [F_l + h_l]\big)^{*q_{l,\mathbf{j}}} - \mathbf{B}_{\mathbf{q},\mathbf{k}} * {}^*\prod_{l,|\mathbf{j}|} \big(\mathbf{p}^{\mathbf{j}} F_l\big)^{*q_{l,\mathbf{j}}}. \tag{48}$$

Lemma 30. *For $\nu > 4\rho_0 + 4$, $\mathbf{g}_{\mathbf{0}} = 0$ and for $|\mathbf{q}| \geqslant 1$*

$$|\mathbf{g}_{\mathbf{q}}| \leqslant \big|\mathbf{p}^{\sum \mathbf{j} q_{l,\mathbf{j}}}\big| \frac{e^{\nu|\mathbf{p}|} |\mathbf{q}|}{M_0^d \prod_i (1+|p_i|^2)} \big(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\big)^{|\mathbf{q}|-1} \big\| |\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}| \big\|_\nu \tag{49}$$

and is zero for $\mathbf{q} = 0$.

Proof. The cases $|\mathbf{q}| = 0, 1$ follow from the definition of $\mathbf{g_0}$ and (48) respectively (since only terms linear in $\mathbf{F}$ are involved in (48)). Assuming (49) holds if $|\mathbf{q}| \leqslant l$ we show that it holds for $\mathbf{q} + \hat{\mathbf{e}}$, where $\hat{\mathbf{e}}$ is a unit vector, say in the $(l_1, j_1', j_2', \dots, j_d')$ direction. We have

$$\begin{aligned}|\mathbf{g}_{\mathbf{q}+\hat{\mathbf{e}}}| &\leqslant \left|\mathbf{B}_{\mathbf{q},\mathbf{k}} * \left[\mathbf{p}^{\mathbf{j}'}(F_{l_1} + h_{l_1})\right] * {}^*\prod_{l,|\mathbf{j}|}\left[\mathbf{p}^{\mathbf{j}}(F_l + h_l)\right]^{*q_{l,\mathbf{j}}} - \mathbf{B}_{\mathbf{q},\mathbf{k}} * \left[\mathbf{p}^{\mathbf{j}'}F_{l_1}\right] * {}^*\prod_{l,|\mathbf{j}|}\left[\mathbf{p}^{\mathbf{j}}F_l\right]^{*q_{l,\mathbf{j}}}\right| \\ &\leqslant \left|\mathbf{B}_{\mathbf{q},\mathbf{k}} * \left(\mathbf{p}^{\mathbf{j}'}h_{l_1}\right)\right| * \left|{}^*\prod_{l,|\mathbf{j}|}\left[\mathbf{p}^{\mathbf{j}}(F_l + h_l)\right]^{*q_{l,\mathbf{j}}}\right| + \left|\left(\mathbf{p}^{\mathbf{j}'}F_{l_1}\right) * \mathbf{g_q}\right|. \end{aligned} \tag{50}$$

Using Lemma 24 and Eq. (49), we get the following upper bound implying the induction step

$$\begin{aligned}|\mathbf{g}_{\mathbf{q}+\hat{\mathbf{e}}}| &\leqslant \frac{|\mathbf{p}^{\mathbf{j}'+\sum \mathbf{j}q_{l,\mathbf{j}}}|\mathrm{e}^{\nu|\mathbf{p}|}}{M_0^d\prod_i(1+|p_i|^2)}\left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\right)^{\sum q_{l,\mathbf{j}}}\left\||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}|\right\|_\nu \\ &\quad + \frac{|\mathbf{p}^{\mathbf{j}'+\sum \mathbf{j}q_{l,\mathbf{j}}}||\mathbf{q}|\mathrm{e}^{\nu|\mathbf{p}|}}{M_0^d\prod_i(1+|p_i|^2)}\left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\right)^{|\mathbf{q}|-1}\|\mathbf{F}\|_\nu\left\||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}|\right\|_\nu \\ &\leqslant \frac{|\mathbf{p}^{\sum \mathbf{j}(q_{l,\mathbf{j}}+e_{l,\mathbf{j}})}||\mathbf{q}+\hat{\mathbf{e}}|\mathrm{e}^{\nu|\mathbf{p}|}}{\prod_i M_0^d\prod_i(1+|p_i|^2)}\left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\right)^{|\mathbf{q}|}\left\||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}|\right\|_\nu. \qquad \square\end{aligned}$$

Lemma 31. *For* $\mathbf{F}$ *and* $\mathbf{h}$ *in* $\mathcal{A}_\phi$, $\nu > 4\rho_0 + 4$,

$$\begin{aligned}&\left|\mathbf{B}_{\mathbf{q},\mathbf{k}} * (\mathbf{F}+\mathbf{h})^{*\mathbf{k}} * {}^*\prod_{l,|\mathbf{j}|}\left(\mathbf{p}^{\mathbf{j}}(F_l + h_l)\right)^{*q_{l,\mathbf{j}}} - \mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} * {}^*\prod_{l,|\mathbf{j}|}\left(\mathbf{p}^{\mathbf{j}}F_l\right)^{*q_{l,\mathbf{j}}}\right| \\ &\quad\leqslant \frac{|\mathbf{p}^{\sum \mathbf{j}q_{l,\mathbf{j}}}|(|\mathbf{q}|+|\mathbf{k}|)\mathrm{e}^{\nu|\mathbf{p}|}}{M_0^d\prod_i(1+|p_i|^2)}\left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\right)^{|\mathbf{k}|+|\mathbf{q}|-1}\left\||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}|\right\|_\nu\end{aligned} \tag{51}$$

if $(\mathbf{q},\mathbf{k}) \neq (\mathbf{0},\mathbf{0})$ *and is zero otherwise.*

Proof. It is clear from (46) that the left side of (51) is simply

$$\left|\mathbf{h_k} * {}^*\prod_{l,|\mathbf{j}|}\left(\mathbf{p}^{\mathbf{j}}(F_l + h_l)\right)^{*q_{l,\mathbf{j}}} + \mathbf{F}^{*\mathbf{k}} * \mathbf{g_q}\right|.$$

However, from Corollary 21, Lemmas 22 and 28,

$$\left|\mathbf{h_k} * {}^*\prod_{l,|\mathbf{j}|}\left(\mathbf{p}^{\mathbf{j}}(F_l + h_l)\right)^{*q_{l,\mathbf{j}}}\right| \leqslant \frac{|\mathbf{p}^{\sum \mathbf{j}q_{l,\mathbf{j}}}||\mathbf{k}|\mathrm{e}^{\nu|\mathbf{p}|}}{M_0^d\prod_i(1+|p_i|^2)}\left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\right)^{|\mathbf{k}|+|\mathbf{q}|-1}\left\||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}|\right\|_\nu$$

and from Corollary 21, Lemmas 22 and 30,

$$\left|\mathbf{F}^{*\mathbf{k}} * \mathbf{g_q}\right| \leqslant \frac{|\mathbf{p}^{\sum \mathbf{j}q_{l,\mathbf{j}}}||\mathbf{q}|\mathrm{e}^{\nu|\mathbf{p}|}}{M_0^d\prod_i(1+|p_i|^2)}\left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\right)^{|\mathbf{k}|+|\mathbf{q}|-1}\left\||\mathbf{B}_{\mathbf{q},\mathbf{k}}| * |\mathbf{h}|\right\|_\nu.$$

Combining these two inequalities, the proof of the lemma follows. $\square$

Lemma 32. *For* $\nu > 4\rho_0 + 4$ *we have*

$$\begin{aligned}&\left\|\int_0^t \mathrm{e}^{-\mathcal{P}(-\mathbf{p})(t-\tau)}\left[\mathbf{B}_{\mathbf{q},\mathbf{k}} * (\mathbf{F}+\mathbf{h})^{*\mathbf{k}} * {}^*\prod_{l,|\mathbf{j}|}\left(\mathbf{p}^{\mathbf{j}}(F_l + h_l)\right)^{*q_{l,\mathbf{j}}} - \mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} * {}^*\prod_{l,|\mathbf{j}|}\left(\mathbf{p}^{\mathbf{j}}F_l\right)^{*q_{l,\mathbf{j}}}\right]\mathrm{d}\tau\right\|_\nu \\ &\quad\leqslant \tilde{A}_b(T)C(\phi)\left(|\mathbf{q}|+|\mathbf{k}|\right)\left(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\right)^{|\mathbf{k}|+|\mathbf{q}|-1}\left(\frac{\nu}{4}\right)^{-d\alpha_{\mathbf{q}}}\|\mathbf{h}\|_\nu.\end{aligned} \tag{52}$$

Proof. This follows from Corollary 18 and Lemma 31 and the definition of $\|\cdot\|_\nu$ together with the bounds (44) and (45). $\square$

Lemma 33. *For $\mathbf{F} \in \mathcal{A}_\phi$, and $\nu > 4\rho_0 + \alpha_r + 3$ large enough (see Note after Definition 11), $\mathcal{N}(\mathbf{F})$ defined in (25) satisfies the following bounds*

$$\left\|\mathcal{N}(\mathbf{F})\right\|_\nu \leqslant \|\mathbf{F}_0\|_\nu + C(\phi)\tilde{A}_b(T)\sum_{\mathbf{q}\succeq 0}{}' \sum_{\mathbf{k}\succeq 0}\left(\frac{\nu}{4}\right)^{-d\alpha_{\mathbf{q}}}\|\mathbf{F}\|_\nu^{|\mathbf{q}|+|\mathbf{k}|}, \tag{53}$$

$$\left\|\mathcal{N}(\mathbf{F}+\mathbf{h}) - \mathcal{N}(\mathbf{F})\right\|_\nu \leqslant C(\phi)\tilde{A}_b(T)\|\mathbf{h}\|_\nu \sum_{\mathbf{q}\succeq 0}{}' \sum_{\mathbf{k}\succeq 0}\left(\frac{\nu}{4}\right)^{-d\alpha_{\mathbf{q}}}\big(|\mathbf{q}|+|\mathbf{k}|\big)\big(\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu\big)^{|\mathbf{q}|+|\mathbf{k}|-1}. \tag{54}$$

Proof. The proofs are immediate from the expression (25) of $\mathcal{N}(\mathbf{F})$ and Lemmas 26, 28 and 32. Note also that the sum with respect to $\mathbf{q}$ only involves finitely many terms, see (6). □

Remark 34. Lemma 33 is the key to showing the existence and uniqueness of a solution in $\mathcal{A}_\phi$ to (25), since it provides the conditions for the nonlinear operator $\mathcal{N}$ to map a ball into itself as well the necessary contractivity condition.

Lemma 35. *If there exists some $b > 1$ so that*

$$b\|\mathbf{F}_0\|_\nu < 1 \tag{55}$$

and

$$C(\phi)\tilde{A}_b(T)\sum_{\mathbf{q}\succeq 0}{}' \sum_{\mathbf{k}\succeq 0}\left(\frac{\nu}{4}\right)^{-d\alpha_{\mathbf{q}}}\|b\mathbf{F}_0\|_\nu^{|\mathbf{k}|+|\mathbf{q}|} < 1 - \frac{1}{b} \tag{56}$$

then the nonlinear mapping $\mathcal{N}$, as defined in (25), maps a ball of radius $b\|\mathbf{F}_0\|_\nu$ into itself. Furthermore, if

$$C(\phi)\tilde{A}_b(T)\sum_{\mathbf{q}\succeq 0}{}' \sum_{\mathbf{k}\succeq 0}\big(|\mathbf{q}|+|\mathbf{k}|\big)\left(\frac{\nu}{4}\right)^{-d\alpha_{\mathbf{q}}}(3b)^{|\mathbf{k}|+|\mathbf{q}|-1}\|\mathbf{F}_0\|_\nu^{|\mathbf{k}|+|\mathbf{q}|-1} < 1 \tag{57}$$

then $\mathcal{N}$ is a contraction there.

Proof. This is a simple application of Lemma 33, if we note that in the ball of radius $b\|\mathbf{F}_0\|$, $\|\mathbf{F}\|_\nu^k < b^k\|\mathbf{F}_0\|_\nu^k$ and using in (54) the fact that $\|\mathbf{F}\|_\nu + \|\mathbf{h}\|_\nu \leqslant 3b\|\mathbf{F}_0\|_\nu$ if $\max\{\|\mathbf{F}\|_\nu, \|\mathbf{F}+\mathbf{h}\|_\nu\} < b\|\mathbf{F}_0\|$. □

Lemma 36. *Consider $T > 0$ and $\phi \in (0, (2n)^{-1}\pi)$ so that (9) is satisfied. Then, for all sufficiently large ν, there exists a unique $\mathbf{F} \in \mathcal{A}_\phi$ that satisfies the integral equation (25).*

Proof. We choose $b = 2$ for definiteness. It is clear from the bounds on $\|\mathbf{F}_0\|_\nu$ in Lemma 14 that for given T, since $\alpha_r \geqslant 1$, we have $b\|\mathbf{F}_0\|_\nu < 1$ for all ν large. Further, it is clear by inspection that all conditions (55), (56) and (57) are satisfied for all sufficiently large ν. The lemma now follows from the contractive mapping theorem. □

4.1. Behavior of ${}^s\mathbf{F}$ near $\mathbf{p} = 0$

In the following proposition, we denote by ${}^s\mathbf{F}$ the solution $\mathbf{F}$ of Lemma 36.

Proposition 37. *For some $K_1 > 0$ and small $\mathbf{p}$ we have $|{}^s\mathbf{F}| \leqslant K_1|\mathbf{p}^1|^{\alpha_r-1}$ and thus $|{}^s\mathbf{f}| \leqslant K_2|\mathbf{x}^1|^{-\alpha_r}$ for some $K_2 > 0$ in $\mathcal{D}_{\phi,\rho}$ as $|\mathbf{x}| \to \infty$.*

Proof. The idea of the proof is to note that, once we have found ${}^s\mathbf{F}$, this function also satisfies in a neighborhood of the origin $\overline{\mathcal{S}}_a = \overline{\mathcal{S}} \cap \{\mathbf{p}\colon |p_i| \leqslant a_i\}$ a linear equation of the form

$${}^s\mathbf{F} = \mathcal{G}\big({}^s\mathbf{F}\big) + \mathbf{F}_0 \quad \text{or} \quad {}^s\mathbf{F} = (1-\mathcal{G})^{-1}\mathbf{F}_0 \tag{58}$$

where, of course, $\mathcal{G}$ depends on the previously found ${}^s\mathbf{F}$; there are many choices of $\mathcal{G}$ that work. Every term in the sum in (25) is a convolution product; in each of them we replace all but one component of $\mathbf{F}$ by the corresponding component of ${}^{s}\mathbf{F}$; $\mathcal{G}\mathbf{F}$ is defined as the sum of the terms thus constructed. Estimates of the form used for Lemma 33 show uniform convergence of the sum for large enough ν (or small $\mathbf{a}$). The result is a $\mathcal{G}$ as below, where the sum over $\boldsymbol{\mu}$ contains only finitely many terms and which has manifestly small norm if $\mathbf{a}$ is small (or ν is large)

$$\mathcal{G}\mathbf{F} = \int_0^t \mathrm{e}^{-\mathcal{P}(-\mathbf{p})(t-\tau)} \left[\sum_l \mathbf{G}_l * F_l + \sum_{\boldsymbol{\mu}} \widehat{\mathbf{G}}_{\boldsymbol{\mu}} * \left((-\mathbf{p})^{\boldsymbol{\mu}} F_{l_{\boldsymbol{\mu}}} \right) \right] \mathrm{d}\tau. \tag{59}$$

By (11), (12), (26) and Lemma 4, we see that $\|\mathbf{F}_0\|_\infty \leqslant K_3 |\mathbf{a}^{\alpha_r - 1}|$ in $\overline{\mathcal{S}}_{\mathbf{a}}$ for some $K_3 > 0$ independent of $\mathbf{a}$. Then, from (58) for small enough $|\mathbf{a}|$, we have

$$\max_{\overline{\mathcal{S}}_a} \left| {}^{s}\mathbf{F}(p,t) \right| = \left\| {}^{s}\mathbf{F} \right\| \leqslant \left(1 - \|\mathcal{G}\| \right)^{-1} \max_{\overline{\mathcal{S}}_a} \|\mathbf{F}_0\| \leqslant 2K_3 \left| \mathbf{a}^{\alpha_r - 1} \right|$$

and thus for small $|\mathbf{p}|$, we have $|\mathbf{F}(\mathbf{p},t)| \leqslant 2K_3 |\mathbf{p}^{\alpha_r - 1}|$ and the proposition follows. Indeed, the arguments also show that the same estimates hold when any component $p_i \to 0$, if the others are bounded. □

4.2. *End of proof of Theorem 1*

Lemma 4 shows that if $\mathbf{f}$ is a solution of (5) satisfying $|\mathbf{x}^{\mathbf{1}}||\mathbf{f}| \leqslant A(T)$ for $\mathbf{x} \in \mathcal{D}_{\phi,\rho,\mathbf{x}}$, then $\mathcal{L}^{-1}\{\mathbf{f}\} \in \mathcal{A}_{\phi-\delta}$ for $0 < \delta < \phi$ for ν sufficiently large. For large enough ρ, the series (10) converges uniformly for $\mathbf{x} \in \mathcal{D}_{\phi,\rho,\mathbf{x}}$ and thus $\mathbf{F} = \mathcal{L}^{-1}\{\mathbf{f}\}$ satisfies (25), which by Lemma 36 has a unique solution in $\mathcal{A}_\phi$ for any $\phi \in (0,\ (2n)^{-1}\pi)$ for which (9) holds. Conversely, if ${}^{s}\mathbf{F} \in \mathcal{A}_{\tilde{\phi}}$ is the solution of (25) for $\nu > \nu_1$, then, for sufficiently large ρ, ${}^{s}\mathbf{f} = \mathcal{L}\,{}^{s}\mathbf{F}$ is analytic in $\mathbf{x}$ in $\mathcal{D}_{\phi,\rho}$ for $0 < \phi < \tilde{\phi} < (2n)^{-1}\pi$ (cf. Remark 13). Proposition 37 shows that ${}^{s}\mathbf{f} = \mathrm{O}(\mathbf{x}^{-\alpha_{\mathbf{r}}})$ and entails uniform convergence of the series in (5). By the properties of Laplace transforms, ${}^{s}\mathbf{f}$ solves the problem (5).

5. Borel summability of formal solutions to the PDE

We now assume Condition 1 in addition to Assumption 1. In our approach it was technically convenient to use oversummation, in that the inverse Laplace transform was performed with respect to $\mathbf{x}$. Showing Borel summability in the appropriate variable ($\mathbf{x}^{\frac{n}{n-1}}$, as explained) requires further arguments.

5.1. *Behavior of* $\mathbf{F}$ *for large* $|\mathbf{p}|$ *outside* $\mathcal{S}_\phi$

For the purpose of showing Borel summability of formal series solutions we need to control $\mathbf{F}$ for large $|\mathbf{p}|$ uniformly in $\mathbb{C}^d$. For this purpose we introduce two other Banach spaces, relevant to the properties we are aiming to show. Firstly, let $\mathfrak{B}(\nu,n,\mathcal{S})$ be the Banach space of functions analytic in the sector $\mathcal{S} = \{\mathbf{p}\colon\ |p_i| > 0,\ \arg(p_i) \in (a_i, b_i)\}$ and continuous in its closure, where $b_i - a_i$ will be chosen larger than $2\pi N_i$ (cf. Condition 2) The Banach space is equipped with the norm

$$\|\boldsymbol{\Psi}\|_{\nu n} = \sup_{\mathbf{p}\in\mathcal{S};\ t\in[0,T]} \left| \boldsymbol{\Psi}(\mathbf{p},t) \mathrm{e}^{-\nu(t+1)\sum_j (|p_j| + |p_j|^n)} \right|. \tag{60}$$

Lemma 38. *For any intervals* (a_i, b_i), $i = 1, \ldots, d$, *the solution* $\mathbf{F}$ *of* (25) *given in Lemma* 36 *is in* $\mathfrak{B}(\nu,n,\mathcal{S})$.

Proof. Because of the obvious embeddings, it suffices to show that for any $\mathcal{S}$, (25) has a unique solution in $\mathfrak{B}(\nu,n,\mathcal{S})$. The proof of this property is very close to that of Lemma 36, after adaptations of the inequalities to the new norms, which are explained in Appendix A.4. □

5.2. *Ramification of* $\mathbf{F}$ *at* $\mathbf{p} = 0$ *and global properties*

We define $\mathfrak{B}(\nu,n,\epsilon_1)$ to be the Banach space of functions defined on $S_{\epsilon_1}^d = \{\mathbf{p}\colon\ \max_i |p_i| \leqslant \epsilon_1\}$ in the norm (60) with $\mathcal{S}$ replaced by $S_{\epsilon_1}^d$.

Lemma 39. *Let*

$$G(\mathbf{p}) = \sum_{0 \preceq \mathbf{j} \prec \mathbf{N}} p_1^{j_1/N_1} \cdots p_d^{j_d/N_d} A_{j_1,\ldots,j_d}(\mathbf{p}) \tag{61}$$

where $A_{j_1,\ldots,j_d}$ *are analytic at* $\mathbf{p} = 0$. *Then the functions* $A_{j_1,\ldots,j_d}$ *are unique and for some constants* C_1 *and* C_2 *and large* $\mathbf{p}$ *we have*

$$\left|A_{j_1,\ldots,j_d}(\mathbf{p})\right| \leqslant C_1|\mathbf{p}|^{C_2} \max_{\mathbf{0} \preceq \mathbf{j} \prec \mathbf{N}} \left|G\left(p_1 e^{2j_1\pi i}, \ldots, p_d e^{2j_d\pi i}\right)\right|. \tag{62}$$

In particular, in S_1^d *we have, for some constants* C_3 *and* C_4,

$$\begin{aligned} C_3 \max_{\mathbf{0} \preceq \mathbf{j} \prec \mathbf{N}} \sup_{|\mathbf{p}| \in S_1^d} \left|G\left(p_1 e^{2j_1\pi i}, \ldots, p_d e^{2j_d\pi i}\right)\right| &\leqslant \sup_{|\mathbf{p}| \in S_1^d} \left|A_{j_1,\ldots,j_d}(\mathbf{p})\right| \\ &\leqslant 2C_4 \max_{\mathbf{0} \preceq \mathbf{j} \prec \mathbf{N}} \sup_{|\mathbf{p}| \in S_1^d} \left|G\left(p_1 e^{2j_1\pi i}, \ldots, p_d e^{2j_d\pi i}\right)\right|. \end{aligned} \tag{63}$$

Remark 40. We note that in (62) the order of analytic continuations is immaterial.

Proof. The proof is by induction on d. We take $d \geqslant 1$, assume (39) with $A_{\mathbf{j}}$ analytic and write $\mathbf{p} = (p_1, \mathbf{p}^\perp)$. We have

$$G(\mathbf{p}) = \sum_{0 \leqslant j_1 < N_1} p_1^{j_1/N_1} \left(\sum_{\{j_m < N_m; m=2,\ldots,d\}} p_2^{j_2/N_2} \cdots p_d^{j_d/N_d} A_{j_1,\ldots,j_d}(\mathbf{p}) \right) =: \sum_{0 \leqslant j_1 < N_1} p_1^{j_1/N_1} G_{j_1}\left(p_1, \mathbf{p}^\perp\right) \tag{64}$$

(with the convention that $G_{j_1} = A_{j_1}$ if $d = 1$). We write the system

$$G\left(p_1 e^{2k\pi i}, \mathbf{p}^\perp\right) = \sum_{0 \leqslant j_1 < N_1} e^{2kj_1\pi i/N_1} p_1^{j_1/N_1} G_{j_1}\left(p_1, \mathbf{p}^\perp\right); \quad k = 0, 1, \ldots, N_1 - 1 \tag{65}$$

which has nonzero Vandermonde determinant, from which $G_{j_1}(p_1, \mathbf{p}^\perp)$ are uniquely determined, which in turn, by the induction hypothesis determine $A_{j_1,\ldots,j_d}$, with the required estimates. □

Lemma 41. *Under Assumption* 1 *and Condition* 1, *the solution in Lemma* 36 *can be decomposed as follows*:

$$\mathbf{F}(\mathbf{p}, t) = \sum_{\mathbf{0} \preceq \mathbf{j} \prec \mathbf{N}} p_1^{j_1/N_1} \cdots p_d^{j_d/N_d} \mathbf{A}_{\mathbf{j}}(\mathbf{p}, t) \tag{66}$$

where $\mathbf{A}_{\mathbf{j}}(\mathbf{p}, t) \in \mathfrak{B}(\nu, n, \mathcal{S})$ *are analytic at* $\mathbf{p} = 0$. *Furthermore, in analyzing the continuations in restricted sectors* $\mathbf{p}e^{2\pi i \mathbf{j}} \in \mathcal{S}_\phi$ *we have for some* ν, *in the norm defined in* (27) (*cf. also Remark* 40)

$$\max\left\{\left\|\mathbf{F}\left(\cdot e^{2\pi i \mathbf{j}}, \cdot\right)\right\|_\nu, \left\{\left\|\mathbf{A}_{\mathbf{j}}(\cdot, \cdot)\right\|_\nu\right\}; \mathbf{0} \preceq \mathbf{j} \prec \mathbf{N}\right\} = K < \infty. \tag{67}$$

Proof. We consider Eq. (25) on $\mathfrak{B}(\nu, n, \mathcal{S})^{\widetilde{N}}$ where $\widetilde{N}$ counts the $\mathbf{A}_{\mathbf{j}}(\cdot, t)$ via the decomposition (66). Noting that

$$p^\alpha * p^\beta = \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)} p^{\alpha+\beta+1} \tag{68}$$

it is straightforward to show that the space of functions of the form (61) is stable under convolution. Since $R(\mathbf{p}, t)$ and therefore $\mathbf{F}_0(\mathbf{p}, t)$ are of the form (66) it follows that $\mathcal{N}$ leaves the space of $\mathbf{F}$ of the form (66) invariant. Using the estimates (63) we see that $\mathcal{N}$ is well defined in a small ball of radius ϵ_2 in $\mathfrak{B}(\nu, n, \mathcal{S})$ and that it is a contraction there. Therefore the solution to (25) is of the form (66). For $\mathbf{p}e^{2\pi i \mathbf{j}} \in \mathcal{S}_\phi$, $\|\mathbf{F}(\mathbf{p}e^{2\pi i \mathbf{j}})\|_\nu$ are well defined. Using again Lemma 39 the first statement follows. To show finiteness of $\|\mathbf{A}_{\mathbf{j}}(\cdot, t)\|_\nu$ it suffices to prove finiteness of $\|\mathbf{F}(\mathbf{p}e^{2\pi i \mathbf{j}})\|_\nu$. To this end, we note that all these continuations satisfy equations of the type (2) with coefficients satisfying the requirements in Section 3 and thus the result follows from Lemma 36. □

Lemma 42. *Assume* $\mathbf{G}$ *is an entire function of exponential order n, more precisely satisfying the inequality* $|\mathbf{G}(\mathbf{p})| \leqslant C\mathrm{e}^{\nu|\mathbf{p}|^n}$ *for some constants* C, ν *and that in a sector* $\mathcal{S}_\phi = \{\mathbf{p}\colon\ |\mathbf{p}| > 0,\ \max_i |\arg(p_i)| < \phi\}$, *it grows at most exponentially,* $|\mathbf{G}(\mathbf{p})| \leqslant C\mathrm{e}^{\nu_1|\mathbf{p}|}$. *Then there exists a function* $\mathbf{G}_1$ *increasing at most exponentially* $|\mathbf{G}_1(\mathbf{p})| \leqslant C\mathrm{e}^{\nu_2|\mathbf{p}|}$ *in any proper subsector of* $\mathcal{S}_{\phi_1}$ *where* $\phi_1 = \frac{\pi}{2(n-1)} + \frac{n\phi}{n-1}$ *and such that* $\mathbf{G}(\mathbf{z}^n)$ *is analytic at* $\mathbf{z} = 0$, *such that*

$$\mathbf{g}(\mathbf{x}) := \int_0^\infty \mathrm{e}^{-\mathbf{p}\cdot\mathbf{x}}\mathbf{G}(\mathbf{p})\,\mathrm{d}\mathbf{p} = \int_0^\infty \mathrm{e}^{-\mathbf{p}\cdot\mathbf{x}^{n/(n-1)}}\mathbf{G}_1(\mathbf{p})\,\mathrm{d}\mathbf{p}. \tag{69}$$

Proof. We start with the case when $\mathbf{G}$, $\mathbf{x}$ and $\mathbf{p}$ are scalar, the general case following in a quite straightforward way as outlined at the end.

The assumptions on G ensure that the first integral in (69) exists and $g(x)$ has an asymptotic power series in powers of x^{-1} in a sector of opening $\pi + 2\phi$ centered on $\mathbb{R}^+$. The function $g_1(x) = g(x^{(n-1)/n})$ has a (noninteger) power series asymptotics in a sector of opening $\frac{n}{n-1}(\pi + 2\phi)$ and by the general theory of Laplace transforms, $G_1 := \mathcal{L}^{-1}g_1$ is analytic in a sector of opening $\frac{n}{n-1}(\pi + 2\phi) - \pi$ centered on $\mathbb{R}^+$, Laplace transformable, with Laplace transform g_1. It follows that

$$G_1(p) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{pu}\int_0^\infty \mathrm{e}^{-qu^{(n-1)/n}}G(q)\,\mathrm{d}q\,\mathrm{d}u =: \int_0^\infty K_{\frac{n-1}{n}}(p,q)G(q)\,\mathrm{d}q. \tag{70}$$

We show that G_1 has a convergent expansion in powers of $p^{1/n}$ at zero. The function

$$K_{\frac{n-1}{n}}(p,q) = \left(\frac{q}{p}\right)^n C_{\frac{n-1}{n}}\left(\frac{q^n}{p^{n-1}}\right). \tag{71}$$

is Écalle's acceleration kernel [1,11]. For $\alpha \in (0,1)$, with $\beta = 1-\alpha$, $c = \beta\alpha^{\alpha/\beta}$, the function C_α is an entire function and has the following asymptotic behavior [1,11]:

$$C_\alpha(x) \sim \frac{\alpha^{1/(2\beta)}}{\sqrt{2\pi\beta}}x^{1/2}\mathrm{e}^{-cx}; \quad |x| \to \infty,\ |\arg x| < \frac{\pi}{2}. \tag{72}$$

Using (71) we see that

$$\int_0^\infty K_{\frac{n-1}{n}}(p,q)q^k\,\mathrm{d}q = p^{(nk-k-1)/n}\int_0^\infty s^{k+n}C_{\frac{n-1}{n}}\big(s^n\big)\,\mathrm{d}s. \tag{73}$$

We expand the entire function G in series about the origin, $G(q) = \sum_{k=1}^{N-1} g_k q^k + R_N(q)$ and note that

$$\big|R_N(q)\big| \leqslant \sum_{k=N}^\infty \big|G^{(k)}(0)\big|\frac{|q|^k}{k!} \leqslant \sum_{k=0}^\infty \big|G^{(k)}(0)\big|\frac{|q|^k}{k!} \leqslant C\mathrm{e}^{\nu_5|q|^n} = E(q) \tag{74}$$

uniformly in $\mathbb{C}$. By (72) and (74) $E(q)C_\alpha(q^n/p^{n-1})$ is, for small enough p, in $L_1[0,\infty]$ in q. By dominated convergence, we have

$$\int_0^\infty K_{\frac{n-1}{n}}(p,q)G(q)\,\mathrm{d}q = \lim_{N\to\infty}\int_0^\infty K_{\frac{n-1}{n}}(p,q)\sum_{k=1}^{N-1} g_k q^k\,\mathrm{d}q$$

and, using (73) it follows that for small p, G_1 is the sum of a convergent series in powers of $p^{1/n}$, as stated.[3]

The argument for d variables and vectorial $\mathbf{G}$ is nearly the same: a vectorial $\mathbf{G}$ is treated componentwise, while the assumptions ensure that the multidimensional integrals involved can be taken iteratively, the estimates being preserved in the process. □

[3] To estimate the radius of convergence of this series it is convenient to start from the duality (69) and apply Watson's lemma, using Cauchy's formula on a circle of radius $k^{1/n}/(n\nu)^{1/n}$ to bound $|\mathbf{G}^{(k)}(0)|$.

Collecting the results of Lemmas 41 and 42 applied to each of the $\mathbf{A_j}$, the proof of Theorem 2 follows.

Note 43. In the example $\partial_t u + (-\partial_x)^n u = 0$ we have $\phi = \frac{\pi}{2n}$. Formal exponential solutions have the behavior, to leading order, $\exp(c_n(-x)^{n/(n-1)} t^{-1/n})$ with $c_n = (n-1)/4/n^{n/(n-1)}$ (for all determinations of $(-x)^{n/(n-1)}$). This also points to $x^{n/(n-1)}$ as natural variable and indicates that the sector of summability cannot be improved since it is bordered by (anti)stokes lines.

6. Short time existence and asymptotics in special cases

In some cases, the Borel summation approach can be adapted to study short time existence of sectorial solutions and study small time asymptotics. One important application is in the analysis of singularity formation in PDEs [9]. For simplicity, and since some assumptions are less general than in the rest of the paper, we restrict to $d = 1$ (scalar case) in this section.

We motivate the assumptions made by looking at a particular example arising in Hele–Shaw flow with surface tension

$$H_t = -\frac{H^3}{2} + H^3 H_{zzz}, \qquad H(z,0) = z^{-1/2} \tag{75}$$

the modified Harry–Dym equation (see [15]), where it arises with $\xi = z + t$ (as a local approximation near an initial zero of the derivative of a conformal mapping).

6.1. *Formal series, preparation of normal form*

Note. To simplify notation, in the following we let $\mathfrak{p}$ stand for generic *polynomials*, $\mathfrak{p}^+$ for polynomials with *nonnegative coefficients*, and $\mathfrak{p}_{(n)}$ for polynomials of *degree n*. Similar conventions are followed for $\mathfrak{h}$ which represents *homogeneous* polynomials.

Substituting in (75) a power-series of the form $\sum_{n=0}^{\infty} t^n H_n(z)$ where $H_0 = z^{-1/2}$ yields the recurrence

$$(n+1)H_n = -\frac{1}{2} \sum_{n_j \geqslant 0,\ \sum_{j=1}^{3} n_j = n} H_{n_1} H_{n_2} H_{n_3} + \sum_{n_j \geqslant 0,\ \sum_{j=1}^{4} n_j = n} H_{n_1} H_{n_2} H_{n_3} H'''_{n_4} \tag{76}$$

which inductively shows that $H_n = z^{-1/2} \mathfrak{h}_{(n)}(z^{-9/2}, z^{-1})$. We let

$$g_N(x,t) := \sum_{k=0}^{N} t^n H_n(z) = x^{-1/3} \sum_{n=0}^{N} \mathfrak{h}_{(n)}\big(tx^{-3}, tx^{-2/3}\big); \quad \text{where } x = \frac{2}{3} z^{3/2}. \tag{77}$$

In terms of x, (75) becomes,

$$\mathcal{N}(H) := H_t + \frac{1}{2}H^3 - \frac{3x}{2} H^3 H_{xxx} - \frac{3}{2} H^3 H_{xx} + \frac{1}{6x} H^3 H_x = 0. \tag{78}$$

It is straightforwardly shown that

$$\mathcal{N} g_N(x,t) = t^{-1} x^{-1/3} \mathfrak{p}_{(4N+1)}\big(tx^{-3}, tx^{-2/3}\big) \tag{79}$$

where for small x_1, x_2 we have moreover

$$\mathfrak{p}_{(4N+1)}(x_1,x_2) = \mathfrak{h}_{(N+1)}(x_1,x_2)\big[1 + \mathrm{O}(x_1,x_2)\big]. \tag{80}$$

It is then natural to substitute:

$$H\big(z(x),t\big) = g_N(x,t) + x^{-2} f(x,t) \tag{81}$$

into (75); we choose without loss of generality $N \geqslant 3$.

It will follow from the analysis that $|f(x,t)| = \mathrm{o}(x^{5/3} \mathfrak{h}_{(N)}(tx^{-3}, tx^{-2/3}))$ for small $t^{1/3} x^{-1}$ with $\arg x \in (-\frac{\pi}{2} - \phi, \frac{\pi}{2} + \phi)$ and $\phi \in (0, \frac{\pi}{6})$, thus $H \sim \sum_{n=0}^{\infty} t^n H_n(z)$ for small $t^{1/3} x^{-1}$ (see Corollary 44).

Substitution shows that $f(x,t)$ satisfies an equation of the form (5), with $n=3$ (third order, $m=1$ (scalar case)), with (cf. also (10), and (114) below)

$$r(x,t)=t^{-1}x^{5/3}\mathfrak{p}_{(4N+1)}\big(tx^{-3},tx^{-2/3}\big);\qquad b_{\mathbf{q},k}=x^{-\beta k}\sum_{j=1}^{J_{\mathbf{q}}}x^{-\alpha_{\mathbf{q},k}}\mathfrak{p}_{\mathbf{q},k;j}\big(tx^{-3},tx^{-2/3}\big).\tag{82}$$

Note. By (80), $r(x,t)$ is small for small t or large x, in spite of the prefactor $t^{-1}x^{5/3}$.

6.2. More general setting

Setting 1. We take $\rho_0=0$, suitable for algebraic initial conditions in the domain, and consider the domain $\mathcal{D}_{\phi,0,x}$, with $\phi<\frac{\pi}{2n}$ small enough to ensure (9). Taking $\mathbf{f}(x,t)-\mathbf{f}_I(x)$ as the unknown function we may assume

$$\mathbf{f}_I(x)=\mathbf{0}$$

(see Note 3 after Theorem 3 below) and require that

$$\big|\mathbf{r}(x,t)\big|\leqslant t^{-1}\sum_{j=1}^{J_r}|x|^{\omega_j}\mathfrak{h}^+_{(n'_j)}\big(t^{\gamma_1}|x|^{-\beta_1},\dots,t^{\gamma_K}|x|^{-\beta_K}\big)\tag{83}$$

where the degrees n'_j satisfy

$$n'_j\beta_l-\omega_j\geqslant 1,\quad \text{for } 1\leqslant l\leqslant K,\ 1\leqslant j\leqslant J_r\tag{84}$$

(as before, (84) implies that $r(x,t)$ is small for large x or small t). The positive constants $\omega_1,\omega_2,\dots,\omega_{J_r}$, $\beta_1,\beta_2,\dots,\beta_K$ and $\gamma_1,\gamma_2,\dots,\gamma_K$, are restricted by the condition

$$\hat n:=\frac{\beta_1}{\gamma_1}\geqslant n.\tag{85}$$

The labeling is chosen so that

$$\hat n=\frac{\beta_1}{\gamma_1}\geqslant\frac{\beta_2}{\gamma_2}\geqslant\cdots\geqslant\frac{\beta_K}{\gamma_K}.\tag{86}$$

Also, if for some $1\leqslant j\leqslant K-1$, $\frac{\beta_j}{\gamma_j}=\frac{\beta_{j+1}}{\gamma_{j+1}}$, we arrange $\beta_j>\beta_{j+1}$. The ω_j are arranged increasingly:

$$\omega_1<\omega_2<\cdots<\omega_{J_r}.\tag{87}$$

Furthermore, for any $x\in\mathcal{D}_{\phi,0,x}$, we require

$$\big|\mathbf{b}_{\mathbf{q},\mathbf{k}}(x,t)\big|\leqslant|x|^{-\beta|\mathbf{k}|}\sum_{j=1}^{J_{\mathbf{q}}}|x|^{-\alpha_{\mathbf{q},j}}\mathfrak{p}^+_{\mathbf{q},\mathbf{k},j}\big(t^{\gamma_1}|x|^{-\beta_1},\dots,t^{\gamma_K}|x|^{-\beta_K}\big),\tag{88}$$

$$\beta>0,\quad \alpha_{\mathbf{q},1}>\alpha_{\mathbf{q},2}>\cdots>\alpha_{\mathbf{q},J_{\mathbf{q}}};\quad \mathbf{b}_{\mathbf{q},\mathbf{k}}\neq 0\Rightarrow\alpha_{\mathbf{q},\mathbf{j}}+\beta|\mathbf{k}|\geqslant 0.\tag{89}$$

If only finitely many $\mathbf{b}_{\mathbf{q},\mathbf{k}}$ are nonzero we allow

$$\beta\geqslant 0.\tag{90}$$

We also require that for all $\mathbf{q}$, $\mathbf{k}$ for which $\mathbf{b}_{\mathbf{q},\mathbf{k}}\neq 0$ we have

$$m_{\mathbf{q},\mathbf{k}}:=\hat n+\omega_1\big(|\mathbf{q}|-1\big)-\alpha_{\mathbf{q},1}+(\omega_1-\beta)|\mathbf{k}|-\frac{\hat n}{n}\sum_{j,l}jq_{l,j}\geqslant 0.\tag{91}$$

Note. Assumption (91) is satisfied by modified Harry–Dym and by certain classes of nonlinear PDEs and initial conditions – for instance, the thin-film equation $h_t+(h^3h_{xxx})_x=0$, with singular initial condition $h(x,0)=x^{-\alpha}$ for $\alpha>0$, but is generally quite restrictive. Weakening it requires more substantial modifications of the framework and will not be discussed here.

Setting 2. Better properties are obtained under the assumptions described below.

$$\hat n = n,$$
$$\mathcal{P}(-s) = s^n,$$
$$\mathbf{r}(x,t) = \frac{1}{t}\sum_{j=1}^{J_r} x^{\omega_j}\mathfrak{a}_j\left(t^{\gamma_1}x^{-\beta_1},\ldots,t^{\gamma_K}x^{-\beta_K}\right),$$
$$\mathbf{b}_{\mathbf{q},\mathbf{k}}(x,t) = x^{-\beta|\mathbf{k}|}\sum_{j=1}^{J_\mathbf{q}} x^{-\alpha_{\mathbf{q},j}}\mathfrak{a}_{\mathbf{q},\mathbf{k},j}\left(t^{\gamma_1}x^{-\beta_1},\ldots,t^{\gamma_K}x^{-\beta_K}\right) \tag{92}$$

where $\mathfrak{a}_j$, $\mathfrak{a}_{\mathbf{q},\mathbf{k},j}$ are *analytic* near the origin and for small $|\mathbf{z}|$ we require, with the same restriction (84) on n'_j,

$$\left|\mathfrak{a}_j(\mathbf{z})\right| \leqslant \mathfrak{h}^+_{(n'_j)}\left(|z_1|,\ldots,|z_n|\right). \tag{93}$$

The restrictions on the numbers $\beta_1,\beta_2,\ldots,\beta_K,\gamma_1,\gamma_2,\ldots,\gamma_K,\alpha_{\mathbf{q},j}$, etc. are as in Setting 1. Furthermore, we assume that there is an $\omega\in\mathbb{R}^+$ so that the nonnegative numbers

$$m_{\mathbf{q},k},\omega_2-\omega_1,\ldots,\omega_{J_r}-\omega_1,\alpha_{\mathbf{q},1}-\alpha_{\mathbf{q},2},\ldots,\alpha_{\mathbf{q},1}-\alpha_{\mathbf{q},J_\mathbf{q}},n\gamma_2-\beta_2,\ldots,n\gamma_K-\beta_K \tag{94}$$

are *integer multiples of* $n\omega$. This condition, satisfied for the problem (75), comes out naturally in a number of examples and ensures the existence of a ramified variable in which the solutions are analytic. We choose $\omega>0$ to be the largest with the property above. Define

$$\zeta = yt^{-1/n}, \qquad \hat{\mathbf{f}}(\zeta,t) = \mathbf{f}\left(t^{1/n}\zeta,t\right) \tag{95}$$

and

$$\widehat{D}_{\phi,\rho} = \left\{\zeta\colon\ |\zeta|>\rho;\ |\arg\zeta|<\phi\right\}. \tag{96}$$

Theorem 3.

(i) *In Setting* 1*, under Assumption* 1*, there exists for large enough* ρ *a unique solution* $\hat{\mathbf{f}}(xt^{-1/\hat n},t)$ *to* (5)*, for* $\zeta = xt^{-1/\hat n}\in\widehat{\mathcal{D}}_{\phi,\rho}$ *and, with* n'_j *as in* (84),

$$\left|\hat{\mathbf{f}}(\zeta,t)\right| \leqslant \sum_{j=1}^{J_r}|\zeta|^{\omega_j}t^{\omega_j/\hat n}\mathfrak{h}_{(n'_j)}\left(|\zeta|^{-\beta_1},t^{\gamma_2-\beta_2/\hat n}|\zeta|^{-\beta_2},\ldots,t^{\gamma_K-\beta_K/\hat n}|\zeta|^{-\beta_K}\right). \tag{97}$$

(ii) *In Setting* 2*, under Assumption* 1*, for any* $T>0$ *there is a* $\rho=\rho(T)>0$ *so that the mapping*

$$(\zeta,\theta)\to\theta^{-\frac{\omega_1}{n\omega}}\hat{\mathbf{f}}\left(\zeta,\theta^{1/\omega}\right)$$

is analytic in $\widehat{\mathcal{D}}_{\phi,\rho}\times\{\theta\colon\ |\theta|<T\}$.

Notes.

1. The function ρ will, generally, increase with T.
2. The restriction $d=1$ is not essential, but made for the sake of simplicity.
3. In these settings, there is a duality between large x and small t in the asymptotics: ζ can be large either due to largeness of x or smallness of t. For t in a fixed interval, there exists some ρ so that the asymptotic bounds are satisfied for $\zeta\in\widehat{D}_{\phi,\rho}$.
4. The following example shows that the requirement $\hat n\geqslant n$ is natural. In the equation $g_t+(-\partial_x)^n g=0$ with $g(x,0)=x^{-\alpha}$, substituting the expansion $g(x,t)=x^{-\alpha}+\sum_{n\in\mathbb{N}}t^n g_n(x)$, we get $g_n(x)=\mathrm{O}(x^{-\alpha-n})$. Thus one of the scales that emerge in the formal expansion is t/x^n. On the other hand, in view of (83) and (88) the most singular term as $x\to 0$ is of the order $t/x^{\hat n}$ since $\hat n=\beta_1/\gamma_1$. Combining with the above discussion we see that $\hat n\geqslant n$.

5. The leading order term in the Taylor expansion of $\theta^{-\frac{\omega_1}{n\omega}}\hat{\mathbf{f}}$, $\hat{\mathbf{f}}_{\mathbf{0}}$, satisfies an easily obtained ODE. The convergence of the series in part (ii) implies that singularities of $\hat{\mathbf{f}}_{\mathbf{0}}$ can be related to actual singularities of the PDE for small time and this is the subject of another paper [9].

Corollary 44. *For the initial value problem* (75), *for any $T > 0$ there is a $\rho = \rho(T)$ such that*

$$H(z,t) = \sum_{k=0}^{\infty} t^{\frac{7k+1}{9}} G_k\big(zt^{-2/9}\big) \tag{98}$$

where the series converges in the region $\{(z,t)\colon |t| < T, |z| > \rho, |\arg z| < \frac{4}{9}\pi\}$ and $G_k(\zeta)$ are analytic in the sector $\{\zeta\colon |\zeta| > \rho, |\arg\zeta| < \frac{4}{9}\pi\}$.

6.3. Proof of Theorem 3(i)

It is convenient to make rescalings of variables in Borel space as well. We note that

$$\hat{\mathbf{f}}(\zeta,t) = t^{-1/\hat{n}} \int_0^{\infty} \mathrm{e}^{-s\zeta}\hat{\mathbf{F}}(s,1;t)\,\mathrm{d}s \tag{99}$$

where

$$s = pt^{1/\hat{n}}, \qquad \widehat{\mathbf{F}}(s,\lambda;t) = \mathbf{F}\big(t^{-1/\hat{n}}s, t\lambda\big). \tag{100}$$

We use similar rescaling to define $\widehat{\mathbf{R}}(s,\lambda;t)$, $\widehat{\mathbf{B}}_{\mathbf{q},\mathbf{k}}(s,\lambda;t)$ and $\widehat{\mathbf{F}}_{\mathbf{0}}(s,\lambda;t)$ where now

$$\widehat{\mathbf{F}}_{\mathbf{0}}(s,\lambda;t) = t\lambda \int_0^1 \mathrm{e}^{-t\lambda\mathcal{P}(-st^{-1/\hat{n}})(1-\tau)}\widehat{\mathbf{R}}(s,\lambda\tau;t)\,\mathrm{d}\tau. \tag{101}$$

We let

$$\mu_{\mathbf{q},k} = 1 - \hat{n}^{-1}\left(|\mathbf{q}| + |\mathbf{k}| + \sum_{j=1}^{n}\sum_{l=1}^{m} j q_{l,j}\right).$$

Using (25), straightforward calculations show that

$$\begin{aligned}\widehat{\mathbf{F}}(s,\lambda;t) = \widehat{\mathcal{N}}(\widehat{\mathbf{F}})(s,\lambda;t) &\equiv \widehat{\mathbf{F}}_0(s,\lambda;t) + \sum_{\mathbf{q}\succeq 0}{}' \sum_{\mathbf{k}\succeq 0} \lambda t^{\mu_{\mathbf{q},k}} \\ &\times \int_0^1 \mathrm{e}^{-t\lambda\mathcal{P}(-st^{-1/\hat{n}})(1-\tau)}\left\{\widehat{\mathbf{B}}_{\mathbf{q},\mathbf{k}} * \widehat{\mathbf{F}}^{*\mathbf{k}} * {}^*\prod_{l=1}^{m} {}^*\prod_{j=1}^{n}\big((-s)^j \widehat{F}_l\big)^{*q_{l,j}}\right\}(s,\lambda\tau,t)\,\mathrm{d}\tau.\end{aligned} \tag{102}$$

With slight abuse of notation we drop the hats from the newly defined functions. Let now

$$\mathcal{S}_\phi \equiv \left\{s\colon \arg s \in (-\phi,\phi),\ 0 < |s| < \infty,\ 0 < \phi < \frac{\pi}{2n}\right\} \tag{103}$$

and consider the Banach space $\mathcal{A}_\phi$ of analytic functions in $\mathcal{S}_\phi$, continuous in $\overline{S}_\phi$ in the norm

$$\big\|\mathbf{F}(\cdot,\cdot;t)\big\|_\nu = \sup_{0\leqslant\lambda\leqslant 1, s\in\mathcal{S}_\phi} \big(1+|s|^2\big)\mathrm{e}^{-\nu|s|}\big|\mathbf{F}(s,\lambda;t)\big|. \tag{104}$$

Lemma 45. *With $\mathbf{r}(x,t)$ satisfying* (83) *we have*

$$\big\|\mathbf{F}_0(\cdot,\cdot;t)\big\|_\nu \leqslant \mathrm{e}^{at}\sum_{j=1}^{J_r} \nu^{\omega_j+1} t^{(\omega_j+1)/\hat{n}} \mathfrak{h}^{+}_{n'_j}\big(\nu^{-\beta_1}, t^{\gamma_2-\beta_2/\hat{n}}\nu^{-\beta_2}, \ldots, t^{\gamma_K-\beta_K/\hat{n}}\nu^{-\beta_K}\big)$$

for ν large (independent of t for small t), where $-a$ is the lower bound of $\Re\mathcal{P}(p)$.

Proof. From (83), (84) and applying Lemma 4 (with $\rho = 0$; see Remark 5) we have

$$\left|\mathbf{R}(s,\lambda;t)\right| \leqslant \frac{1}{t\lambda}\sum_{j=1}^{J_r}|s|^{-\omega_j-1}t^{(\omega_j+1)/\hat{n}}\mathfrak{h}^+_{n'_j}\left(\lambda^{\gamma_1}|s|^{\beta_1},\lambda^{\gamma_2}t^{\gamma_2-\beta_2/\hat{n}}|s|^{\beta_2},\ldots,\lambda^{\gamma_K}t^{\gamma_K-\beta_K/\hat{n}}|s|^{\beta_K}\right).$$

For $\lambda \in (0,1)$ we have $|\mathrm{e}^{-t\mathcal{P}(-st^{-1/\hat{n}})\lambda(1-\tau)}| \leqslant \mathrm{e}^{at}$ and thus (cf. (101))

$$\left|\mathbf{F}_0(s,\lambda;t)\right| \leqslant \mathrm{e}^{at}\sum_{j=1}^{J_r}|s|^{-\omega_j-1}t^{(\omega_j+1)/\hat{n}}\mathfrak{h}^+_{n'_j}\left(\lambda^{\gamma_1}|s|^{\beta_1},\lambda^{\gamma_2}t^{\gamma_2-\beta_2/\hat{n}}|s|^{\beta_2},\ldots,\lambda^{\gamma_K}t^{\gamma_K-\beta_K/\hat{n}}|s|^{\beta_K}\right). \tag{105}$$

Bounding each term of the polynomial $\mathfrak{h}^+_{n'_j}$ in $\|\cdot\|_\nu$ we obtain

$$\left\|\widehat{\mathbf{F}}_0(\cdot,\cdot;t)\right\|_\nu \leqslant \mathrm{e}^{at}\sum_{j=1}^{J_r}\nu^{\omega_j+1}t^{(1+\omega_j)/\hat{n}}\mathfrak{h}^+_{n'_j}\left(\nu^{-\beta_1},t^{\gamma_2-\beta_2/\hat{n}}\nu^{-\beta_2},\ldots,t^{\gamma_K-\beta_K/\hat{n}}\nu^{-\beta_K}\right).$$

The proof now follows, choosing ν sufficiently large and using (84) and (86), (87). □

Lemma 46. *For large ν, we have*

$$\|\mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}\|_\nu \leqslant c_{\mathbf{q},\mathbf{k}}(\nu,t)\|\mathbf{F}\|_\nu, \quad \textit{where}$$

$$c_{\mathbf{0},\mathbf{0}} = \mathbf{0}; \quad c_{\mathbf{q},\mathbf{k}}(\nu,t) = \nu^{-\beta|\mathbf{k}|}t^{(1-\beta|\mathbf{k}|)/\hat{n}}\sum_{j=1}^{J_\mathbf{q}}K_j\nu^{-\alpha_{\mathbf{q},j}}t^{-\alpha_{\mathbf{q},j}/\hat{n}} \ \left((\mathbf{q},\mathbf{k}) \neq \mathbf{0}\right) \tag{106}$$

with K_j constants independent of $\mathbf{q}$, $\mathbf{k}$, *ν and t.*

Proof. Note first that $\mathbf{b}_{\mathbf{0},\mathbf{0}} = \mathbf{0}$ hence $c_{\mathbf{0},\mathbf{0}} = \mathbf{0}$. From (88) and Lemma 4 (with $\rho = 0$),

$$\left|\mathbf{B}_{\mathbf{q},\mathbf{k}}(p,t)\right| \leqslant |p|^{\beta|\mathbf{k}|-1}\sum_{j=1}^{J_\mathbf{q}}|p|^{\alpha_{\mathbf{q},j}}\mathfrak{p}^+_{\mathbf{q},\mathbf{k},j}\left(t^{\gamma_1}|p|^{\beta_1},t^{\gamma_2}|p|^{\beta_2},\ldots,t^{\gamma_K}|p|^{\beta_K}\right).$$

Switching from (p,t) to $(s,\lambda;t)$,

$$\left|\mathbf{B}_{\mathbf{q},\mathbf{k}}(s,\lambda;t)\right| \leqslant t^{(1-\beta|\mathbf{k}|)/\hat{n}}|s|^{\beta|\mathbf{k}|-1}\sum_{j=1}^{J_\mathbf{q}}|s|^{\alpha_{\mathbf{q},j}}t^{-\alpha_{\mathbf{q},j}/\hat{n}}\mathfrak{p}^+_{\mathbf{q},\mathbf{k},j}\left(\lambda^{\gamma_1}|s|^{\beta_1},\lambda^{\gamma_2}t^{\gamma_2-\beta_2/\hat{n}}|s|^{\beta_2},\ldots,\lambda^{\gamma_K}t^{\gamma_K-\beta_K/\hat{n}}|s|^{\beta_K}\right).$$

For large ν, using Lemma 16 (with $\rho = 0$) to bound in norm the terms of $\mathfrak{p}^+_{\mathbf{q},\mathbf{k},j}$

$$\begin{aligned}\|\mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}| &\leqslant \|\mathbf{F}\|_\nu\ t^{(1-\beta|\mathbf{k}|)/\hat{n}}|\nu|^{-\beta|\mathbf{k}|}\\ &\quad\times\sum_{j=1}^{J_\mathbf{q}}|\nu|^{-\alpha_{\mathbf{q},j}}t^{-\alpha_{\mathbf{q},j}/\hat{n}}\mathfrak{p}^+_{\mathbf{q},\mathbf{k},j}\left(\lambda^{\gamma_1}\nu^{-\beta_1},\lambda^{\gamma_2}t^{\gamma_2-\beta_2/\hat{n}}\nu^{-\beta_2},\ldots,\lambda^{\gamma_K}t^{\gamma_K-\beta_K/\hat{n}}\nu^{-\beta_K}\right).\end{aligned} \tag{107}$$

Clearly, for large ν, $\mathfrak{p}^+_{\mathbf{q},\mathbf{k}}$ can be replaced in (107) by a constant K_j. Using (86) and (89) the conclusion follows. □

Let now

$$C(\phi,T) = \max\left\{\sup_{p\in\mathcal{S}_\phi,\ |p|>R,\ 0\leqslant l'\leqslant n,\ \gamma>0}\left(\frac{|p|^n}{\Re\mathcal{P}(-p)}\right)^{l'/n}\frac{1-\mathrm{e}^{-\gamma}}{\gamma^{1-l'/n}},\ \sup_{p\in\mathcal{S}_\phi,\ |p|\leqslant R,\ 0\leqslant l'\leqslant n}t^{l'/n}|p|^{l'}\mathrm{e}^{-t\Re\mathcal{P}(-p)}\right\}$$

where R is the same as in the proof of Lemma 10.

Lemma 47. *For ν large enough, $\mathcal{N}$ is contractive, and thus there exists unique solution* $\mathbf{F}$ *of* (102).

Proof. For ν large enough, (91), Lemmas 45 and 46 imply

$$C(\phi,T){\sum_{\mathbf{q}\succeq 0}}'\sum_{\mathbf{k}\succeq 0} t^{\mu_{\mathbf{q},\mathbf{k}}} c_{\mathbf{q},\mathbf{k}}(\nu,t)\|2\mathbf{F_0}\|^{|\mathbf{k}|+|\mathbf{q}|} \leqslant \|\mathbf{F_0}\|_\nu \tag{108}$$

and

$$C(\phi,T){\sum_{\mathbf{q}\succeq 0}}'\sum_{\mathbf{k}\succeq 0} t^{\mu_{\mathbf{q},\mathbf{k}}} c_{\mathbf{q},\mathbf{k}}(\nu,t)\big(|\mathbf{q}|+|\mathbf{k}|\big)\|6\mathbf{F_0}\|^{|\mathbf{k}|+|\mathbf{q}|-1} \leqslant 1. \tag{109}$$

Now, Lemma 24 (with $\rho_0=0$, $d=1$ and s replacing p), and Lemma 46 imply

$$\left|\left\{\mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} * {}^*\prod_{l=1}^m {}^*\prod_{j=1}^n \big(s^j F_l\big)^{*q_{l,j}}\right\}(s,\lambda\tau;t)\right| \leqslant \frac{\mathrm{e}^{\nu|s|}|s|^{\sum jq_{l,j}}}{M_0(1+|s|^2)} c_{\mathbf{q},\mathbf{k}}(\nu,t)\|\mathbf{F}\|_\nu^{|\mathbf{q}|+|\mathbf{k}|}.$$

Also, note that if $l'\geqslant 0$, $s\in S_\phi$ with $|st^{-1/\hat{n}}|>R$

$$\left|\int_0^1 s^{l'}\lambda \mathrm{e}^{-t\mathcal{P}(-st^{-1/\hat{n}})\lambda(1-\tau)}\,\mathrm{d}\tau\right| \leqslant \lambda\left\{\frac{1-\mathrm{e}^{-t\lambda\Re\mathcal{P}(-st^{-1/\hat{n}})}}{t\lambda\Re P(-st^{-1/\hat{n}})}\right\}s^{l'} \leqslant C(\phi,T)t^{l'/\hat{n}-l'/n}. \tag{110}$$

The definition of $C(\phi,T)$ implies that for $l'\geqslant 0$, $s\in S_\phi$ with $|st^{-1/\hat{n}}|\leqslant R$ we have

$$\left|\int_0^1 s^{l'}\lambda \mathrm{e}^{-t\mathcal{P}(-st^{-1/\hat{n}})\lambda(1-\tau)}\,\mathrm{d}\tau\right| \leqslant C(\phi,T)t^{l'/\hat{n}-l'/n}. \tag{111}$$

Setting $l'=\sum jq_{l,j}$, using (110) and (111), we find after time integration

$$\left\|\int_0^1 \lambda \mathrm{e}^{-t\mathcal{P}(-st^{-1/\hat{n}})\lambda(1-\tau)}\mathbf{B}_{\mathbf{q},\mathbf{k}} * \mathbf{F}^{*\mathbf{k}} * {}^*\prod_{l=1}^m {}^*\prod_{j=1}^n \big(s^j F_l\big)^{*q_{l,j}}(s,\lambda\tau;t)\,\mathrm{d}\tau\right\|_\nu$$
$$\leqslant t^{l'/\hat{n}-l'/n} C(\phi,T) c_{\mathbf{q},\mathbf{k}}(\nu,t)\|\mathbf{F}\|_\nu^{|\mathbf{q}|+|\mathbf{k}|}. \tag{112}$$

Using (91), (102), (108) and (112) , it follows that $\mathcal{N}$ maps a ball of radius $2\|\mathbf{F_0}\|_0$ into itself. Using Lemma 31, (110) and (111), we obtain

$$\Bigg\|\int_0^1 \lambda\mathbf{B}_{\mathbf{q},\mathbf{k}} * \Bigg\{(\mathbf{F}+\mathbf{h})^{*\mathbf{k}} * {}^*\prod_{l=1}^m {}^*\prod_{j=1}^n \big(s^j[F_l+h_l]\big)^{*q_{l,j}}$$
$$- \mathbf{F}^{*\mathbf{k}} * {}^*\prod_{l=1}^m {}^*\prod_{j=1}^n \big(s^j F_l\big)^{*q_{l,j}}\Bigg\}(s,\lambda\tau;t)\mathrm{e}^{-t\mathcal{P}(-st^{1/\hat{n}})\lambda(1-\tau)}\,\mathrm{d}\tau\Bigg\|_\nu$$
$$\leqslant t^{l'/\hat{n}-l'/n}C(\phi,T)\big(|\mathbf{q}|+|\mathbf{k}|\big)c_{\mathbf{q},\mathbf{k}}(\nu,t)\big(\|\mathbf{h}\|_\nu+\|\mathbf{F}\|_\nu\big)^{|\mathbf{q}|+|\mathbf{k}|-1}\|\mathbf{h}\|_\nu$$

where $l'=\sum jq_{l,j}$ from which the conclusion using (106) and (91). □

Behavior of ${}^s\mathbf{F}$ *near* $s=0$.

In the following proposition, we denote by ${}^s\mathbf{F}$ the solution $\mathbf{F}$ of Lemma 47.

Proposition 48. *For small s we have*

$$\big|{}^s\mathbf{F}\big| \leqslant \sum_{j=1}^{J_r} |s|^{-\omega_j-1}t^{(1+\omega_j)/\hat{n}}\mathfrak{h}^+_{n'_j}\big(|s|^{\beta_1}, t^{\gamma_2-\beta_2/\hat{n}}|s|^{\beta_2},\ldots,t^{\gamma_K-\beta_K/\hat{n}}|s|^{\beta_K}\big).$$

Proof. The proof is similar to that of Proposition 37, using (105), (83) and (84). ${}^s\mathbf{F}$ to (102) solves a linear equation

$$ {}^s\mathbf{F} = \mathcal{G}\big({}^s\mathbf{F}\big) + \mathbf{F}_0 \quad \text{or} \quad {}^s\mathbf{F} = (1-\mathcal{G})^{-1}\mathbf{F}_0 \tag{113}$$

with $\mathcal{G}$ very similar to that given in Section 4. □

End of proof of Theorem 3(i). The proof is a direct application of Lemma 47 and Proposition 48. Using (99) and properties of Laplace transform, (97) follows for large $|\zeta|$, in the sector $\arg\zeta \in (-\frac{\pi}{2}-\phi, \frac{\pi}{2}+\phi)$.

6.4. Proof of Theorem 3(ii)

An important difference is that infinite sums appear in some estimates. Analyticity of the functions $\mathfrak{a}$ and the estimate

$$\big\|\mathcal{L}^{-1}y^{-\alpha}\big\|_\nu = \left\|\frac{p^{\alpha-1}}{\Gamma(\alpha)}\right\|_\nu \leqslant C\big(1+\alpha^2\big)\nu^{-\alpha+1},$$

for $\nu > 1$ with C is independent of α and ν, show convergence of the corresponding series. Also, the proof of Lemma 47 holds if the following norm was used instead:

$$\|F\|_\nu^u = \sup_{0\leqslant\lambda\leqslant 1,\ |t|\leqslant T,\ s\in\mathcal{S}_\phi} \big(1+|s|^2\big)\mathrm{e}^{-\nu|s|}\big|F(s,\lambda;t)\big|$$

since for $\hat{n} = n$, $\Re t\mathcal{P}(-st^{-1/n}) = \Re s^n$, is independent of t in the exponent in (102). To show analyticity, we let $\widehat{G}(s,\lambda;\theta) = \theta^{-(1+\omega_1)/(n\omega)}\widehat{F}(s,\lambda;\theta^{1/\omega})$; then $\widehat{G}$ satisfies an equation of the form

$$\widehat{G} = \mathcal{N}_1(\widehat{G})$$

where the conditions in Setting 2 and the choice of ω are such that $\mathcal{N}_1$, as it is seen after straightforward algebra, manifestly preserves analyticity in θ. Using (99), analyticity of $t^{-\omega_1/n}f(\zeta,t)$ in t^ω follows provided $|\zeta|$ is large enough (depending on T).

6.5. Proof of Corollary 44

Substitution gives for $f(x,t)$, defined by (81), an equation of the form (5), with $m=1$, $d=1$. Then in (10), $\mathbf{k}$ is scalar. The vector $\mathbf{q}$ is 3-dimensional, indexed by (l,j), $l=1$, $j=1,2,3$. The nonlinearity is quartic and the equation is linear in the derivatives of f, thus the only nonzero values of $b_{\mathbf{q},k}$ are when $\mathbf{q}$ is $\mathbf{0}$ (and $k=1,\ldots,4$) or a unit vector $\hat{\mathbf{e}}_i \in \mathbb{R}^3$ (and $k=0,\ldots,3$). Further, it is found that

$$J_r = 1,\quad K=2,\quad \omega_1 = \frac{5}{3} = \beta,\quad \gamma_1=\gamma_2=1,\quad \beta_1 = 3,\quad \beta_2=\frac{2}{3},\quad \hat{n}=3$$

and in (82) we have

$$\mathfrak{a}_{\mathbf{0},1} = \frac{4}{3},\quad \mathfrak{a}_{\mathbf{0},2} = -1,\quad \mathfrak{a}_{\hat{\mathbf{e}}_1,1} = 2,\quad \mathfrak{a}_{\hat{\mathbf{e}}_2,1} = 1,\quad \mathfrak{a}_{\hat{\mathbf{e}}_3,1} = 0. \tag{114}$$

This is sufficient to check that Theorem 3 applies.

Since $|z|t^{-2/9}$ large corresponds to $|\zeta| = |x|t^{-1/3}$ large, and $\arg z \in (-\frac{4}{9}\pi, \frac{4}{9}\pi)$ corresponds to $\arg\zeta \in (-\frac{2}{3}\pi, \frac{2}{3}\pi)$, Theorem 3 implies that for any $\phi \in (0, \frac{\pi}{6})$ for large $x \in \mathcal{D}_\phi$ and large $\zeta = x/t^{1/3}$ we have

$$\big|f(x,t)\big| = \mathrm{O}\big(|x|^{5/3}\mathfrak{h}_{(N+1)}\big(t|x|^{-3}, t|x|^{-2/3}\big)\big) = \mathrm{O}\big(|x|^{5/3}t^{N+1}\mathfrak{h}_{(N+1)}\big(|x|^{-3}, |x|^{-2/3}\big)\big).$$

Changing variables, this implies

$$x(z)^{-2}f\big(x(z,t),t\big) = \mathrm{O}\big(t^{N+1}|z|^{-1/2}\mathfrak{h}_{(N+1)}\big(|z|^{-9/2},|z|^{-1}\big)\big) = \mathrm{o}\big(t^N|z|^{-1/2}\mathfrak{h}_{(N)}\big(|z|^{-9/2},|z|^{-1}\big)\big)$$

as needed for asymptoticity. The convergence in the series representation in $t^{7/9}$ follows from Theorem 3(ii). It is seen from (94) that all the exponents of t are integer multiples of $\frac{7}{9}$. □

Note 49. Large ζ includes part of the region where Theorems 1 and 2 imply Borel summability of the expansion in inverse powers of z. Together, the results provide uniform control of the solution.

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Appendix A

A.1. Asymptotic behavior: further comments

In the assumptions of Theorem 2, by the remark following it, formal series solutions to the initial value problem are asymptotic to the actual unique solution. The discussion below addresses the issue of deriving this series, or, when less regularity is provided and only the first few terms of the expansion exist, how to show their asymptoticity.

Heuristic calculation. Assuming algebraic behavior of $\mathbf{f}$ in our assumptions on the nonlinearity, it is seen that the most important terms for large $\mathbf{x}$ (giving the "dominant balance") are $\mathbf{f}_t$, $\mathcal{P}_0\mathbf{f}$, coming from the constant part of $\mathcal{P}$, and $\mathbf{r}(\mathbf{x},t)$. This suggests that, to leading order,

$$\mathbf{f}(\mathbf{x},t) \sim \mathbf{f}_I(\mathbf{x}) + \int_0^t \mathrm{e}^{-\mathcal{P}_0(t-\tau)}\mathbf{r}(\mathbf{x},\tau)\,\mathrm{d}\tau.$$

If we substitute

$$\mathbf{f}(\mathbf{x},t) = \mathbf{A}_1(t)\mathbf{x}^{-\alpha_r\mathbf{1}} + \tilde{\mathbf{f}} \tag{115}$$

into (5), $\tilde{\mathbf{f}}$ will generally satisfy an equation of the form (5), for an *increased* value of α_r; if the process can be iterated, as is the case in the examples in [7], it generates a formal series solution.

To obtain rigorous estimates, one writes the equation for $\tilde{\mathbf{f}}$ defined in (115) and applies Theorem 1 to show $\tilde{\mathbf{f}} = \mathrm{o}(\mathbf{x}^{-\alpha_r\mathbf{1}})$. If the coefficients of the equation allow it, this procedure can be repeated to obtain more asymptotic terms for $\mathbf{f}$. This is the case for instance in the assumptions of Theorem 2, where a complete series is obtained, which is furthermore Borel summable to $\mathbf{f}$.

The discussion also shows that the assumption $\alpha_r \geqslant 1$ can be often be circumvented by subtracting the higher powers of $\mathbf{x}$ from $\mathbf{f}$.

A.2. Simple examples of Borel regularization

In this section we discuss informally and using rather trivial examples, the regularizing features of Borel summation. An excellent account of Écalle's modern theory of generalized summability is found in [10]; see [8] as well. Many interesting results, using more classical tools can be found in [1].

Singular perturbations give rise to nonanalytic behavior and divergent series. Infinity is an irregular singular point of the ODE $f' - f = 1/x$, and the formal power series solution $\tilde{f} = \sum_{k=0}^{\infty}(-1)^k k! x^{-k-1}$ diverges. In the context of PDEs, the solution h of the heat equation $h_t - h_{xx} = 0$ with $h(0,x)$ real-analytic but not entire, has a factorially divergent expansion in *small t*, the recurrence relation for the terms of which is $kH_k = H''_{k-1}$.

The *Borel transform* of a series, is by definition its term-wise inverse Laplace transform, which improves convergence since $\mathcal{L}^{-1}x^{-k-1} = p^k/k!$. If the Borel transformed of a series converges to a function which can be continued analytically along $\mathbb{R}^+$ *and* is exponentially bounded, then its Laplace transform is by definition the *Borel sum* of the series. Since on a formal level Borel summation is $\mathcal{L}\mathcal{L}^{-1}$, the identity, it can be shown to be an extended isomorphism between series and functions; in particular, the Borel sum of $\tilde{f}$ above, $\mathcal{L}(1+p)^{-1}$ is an actual solution of the equation. Another way to view this situation is that Borel transform maps singular problems into more regular ones. The Borel transform of the ODE discussed is $(p+1)\mathcal{L}^{-1}f + 1 = 0$. The inverse Laplace transform of $h_t = h_{xx}$ in $1/t$ is $\hat{h}_{xx} - p\hat{h}_{pp} - \frac{3}{2}\hat{h}_p = 0$ which becomes regular, $u_{xx} - u_{zz} = 0$ by taking $\hat{h}(p,x) = p^{-1/2}u(2p^{1/2},x)$, $z = 2p^{1/2}$.

It is in its latter role, of a regularizing tool, that we use Borel summation in PDEs.

A.3. Derivation of Eq. (5) from (4)

We define an m-dimensional vector $\mathbf{f}$ by ordering the set $\{\partial_{\mathbf{x}}^{\mathbf{j}}\mathbf{u}\colon\ 0\leqslant|\mathbf{j}|<n\}$. It is convenient to introduce $\hat{\mathbf{g}}_2(\mathbf{x},t,\mathbf{f})$ so that

$$\sum_{|\mathbf{J}|=n}\mathbf{g}_{2,\mathbf{J}}\big(\mathbf{x},t,\big\{\partial_{\mathbf{x}}^{\mathbf{j}}\mathbf{u}\big\}_{|\mathbf{j}|\leqslant n-1}\big)\partial_{\mathbf{x}}^{\mathbf{J}}\mathbf{u}=-\sum_i\hat{\mathbf{g}}_{2,i}(\mathbf{x},t,\mathbf{f})\partial_{x_i}\mathbf{f}.$$

So, for showing that (4) implies (5) it is enough to show that for $1\leqslant n'\leqslant n$, for $|\mathbf{J}'|=n'-1$,

$$\partial_{\mathbf{x}}^{\mathbf{J}'}\left[\mathbf{g}_1(\mathbf{x},t,\mathbf{f})+\sum_i\hat{\mathbf{g}}_{2,i}(\mathbf{x},t,\mathbf{f})\partial_{x_i}\mathbf{f}\right]$$

is of the form on the right-hand side of (5). We do so in three steps.

Lemma 50. *Consider for $k\geqslant 1$,*

$$\mathbf{E}(\mathbf{x},t)=\sum_{\mathbf{q}\succeq 0}^{\ddagger}\mathbf{b}_{\mathbf{q}}(\mathbf{x},t,\mathbf{f})\prod_{\{m;k\}}\big(\partial_{\mathbf{x}}^{\mathbf{j}}f_l\big)^{q_{l,\mathbf{j}}}\tag{116}$$

where $\{m;k\}$ denotes the set $\{(l,\mathbf{j})\colon\ 1\leqslant l\leqslant m;\ 1\leqslant|\mathbf{j}|\leqslant k\}$, and $\ddagger$ means summation over $\mathbf{q}$ with the restriction

$$\sum_{\{m;k\}}|\mathbf{j}|q_{l,\mathbf{j}}\leqslant k.\tag{117}$$

Then, for $i=1,2,\ldots,d$, $\partial_{x_i}\mathbf{E}(\mathbf{x},t)$ has the same form as (116) *with restriction* (117)*, provided k is replaced by $k+1$.*

Proof. The proof is straightforward, keeping track of the number of derivatives and the powers involved: note that

$$\partial_{x_i}\mathbf{E}(\mathbf{x},t,\mathbf{f})=\sum_{\mathbf{q}\succeq 0}\left(\sum_{l=1}^{m}\frac{\partial}{\partial f_l}\mathbf{b}_{\mathbf{q}}(\mathbf{x},t,\mathbf{f})\partial_{x_i}f_l+\partial_{x_i}\mathbf{b}_{\mathbf{q}}(\mathbf{x},t,\mathbf{f})\right)\prod_{\{m;k\}}\big(\partial_{\mathbf{x}}^{\mathbf{j}}f_l\big)^{q_{l,\mathbf{j}}}$$
$$+\sum_{\mathbf{q}\succeq 0}\mathbf{b}_{\mathbf{q}}(\mathbf{x},t,\mathbf{f})\sum_{l'=1}^{m}\sum_{|\mathbf{j}'|=1}^{k}q_{l',\mathbf{j}'}\big(\partial_{\mathbf{x}}^{\mathbf{j}'}f_{l'}\big)^{q_{l',\mathbf{j}'}-1}\partial_{x_i}\big(\partial_{\mathbf{x}}^{\mathbf{j}'}f_{l'}\big)\prod_{\{m;k\}}^{\dagger}\big(\partial_x^{j}f_l\big)^{q_{l,\mathbf{j}}}$$

where $\prod^{\dagger}$ indicates that the term $l=l'$, $\mathbf{j}=\mathbf{j}'$ is missing from the product. Manifestly, this is of the form (116) with a suitable redefinition of $\mathbf{b}_{\mathbf{q}}$ and with the product of the number of derivatives times the power totaling at most

$$|\mathbf{j}'|+1+|\mathbf{j}'|(q_{l',\mathbf{j}'}-1)+\sum_{\{m;k\}}^{\dagger}|\mathbf{j}|q_{l,\mathbf{j}}=1+\sum_{\{m;k\}}|\mathbf{j}|q_{l,\mathbf{j}}\leqslant k+1.$$

Hence restriction (117) holds, now with $k+1$ instead of k. $\square$

Lemma 51. *For any $n'\geqslant 1$, and any $\mathbf{J}'$ with $|\mathbf{J}'|=n'-1$,*

$$\partial_{\mathbf{x}}^{\mathbf{J}'}\mathbf{g}_1\big(y,t,\mathbf{f}(y,t)\big)=\sum_{\mathbf{q}\succeq 0}^{\ddagger}\mathbf{b}_{\mathbf{q}}(\mathbf{x},t,\mathbf{f})\prod_{\{m;n'-1\}}\big(\partial_{\mathbf{x}}^{\mathbf{j}}f_l\big)^{q_{l,\mathbf{j}}}\tag{118}$$

for some $\mathbf{b}_{\mathbf{q}}$, depending on n', $\mathbf{g}_1$, and its first $n'-1$ derivatives, and where $\sum^{\ddagger}$ means the sum over $\mathbf{q}$ with the further restriction

$$\sum_{\{m;n'-1\}}|\mathbf{j}|q_{l,\mathbf{j}}\leqslant n'-1.$$

Proof. The proof is by induction. We have, with obvious notation,

$$\partial_{x_i}\mathbf{g}_1\big(\mathbf{x},t,\mathbf{f}(\mathbf{x},t)\big)=\mathbf{g}_{1,x_i}+\mathbf{g}_{1,\mathbf{f}}\cdot\partial_{x_i}\mathbf{f}$$

which is of the form (118). Assume (118) holds for $n' = k \geqslant 1$, i.e. for all $\mathbf{J}'$ satisfying $|\mathbf{J}'| = k-1$,

$$\partial_{\mathbf{x}}^{\mathbf{J}'} \mathbf{g_1}(\mathbf{x},t,\mathbf{f}) = \sum_{\mathbf{q}\succeq 0}^{\ddagger} \mathbf{b_q}(\mathbf{x},t,\mathbf{f}) \prod_{\{m;k-1\}} \left(\partial_{\mathbf{x}}^{\mathbf{j}} f_l\right)^{q_{l,\mathbf{j}}}.$$

Taking a x_i derivative, and applying Lemma 50, $\partial_{\mathbf{x}}^{\mathbf{J}} \mathbf{g_1}(y,t,\mathbf{f})$ for $|\mathbf{J}| = k$ will have the form above, with $k-1$ replaced by k and with restriction

$$\sum_{\{m;k\}} |\mathbf{j}| q_{l,\mathbf{j}} \leqslant k.$$

Thus, (118) holds for $n' = k+1$, with a different $\mathbf{b}$. The induction step is proved. □

Lemma 52. *For $n' = 1, 2, \ldots, n$, and any $\mathbf{J}$ with $|\mathbf{J}| = n' - 1$ we have*

$$\partial_{\mathbf{x}}^{\mathbf{J}} \left[\hat{\mathbf{g}}_{\mathbf{2},\mathbf{i}'}(\mathbf{x},t,\mathbf{f}) \partial_{\mathbf{x}_{\mathbf{i}'}} \mathbf{f}\right] = \sum_{\mathbf{q}\succeq 0}^{\ddagger} \mathbf{b_q}(\mathbf{x},t,\mathbf{f}) \prod_{\{m;n'\}} \left(\partial_{\mathbf{x}}^{\mathbf{j}} f_l\right)^{q_{l,\mathbf{j}}} \tag{119}$$

for some $\mathbf{b_q}$, depending on n', $\mathbf{g}_2$ and its first $n'-1$ derivatives, where $\sum_{\mathbf{q}\succeq 0}^{\ddagger}$ denotes summation with the restriction

$$\sum_{\{m;n'\}} |\mathbf{j}| q_{l,\mathbf{j}} \leqslant n'. \tag{120}$$

Proof. Clearly (119) with restriction (120) holds for $n' = 1$. Suppose it holds for $n' = k$. Then we note that if $|\mathbf{J}| = k+1$, then there exists some index $1 \leqslant i \leqslant d$ and some $\mathbf{J}'$, with $|\mathbf{J}'| = k$ so that $\partial_{\mathbf{x}}^{\mathbf{J}} = \partial_{x_i}[\partial_{\mathbf{x}}^{\mathbf{J}'}]$; hence applying Lemma 50, we obtain (119) and (120) for $n' = (k+1)$. □

A.4. Some useful inequalities

(1) We start with a simple inequality for $\alpha > 1$ and $\mu > 0$:

$$\left(1+\mu^{\alpha}\right) \int_0^1 s^{\alpha-1} \mathrm{e}^{-\mu s} \,\mathrm{d}s \leqslant 2\Gamma(\alpha). \tag{121}$$

This is clear for $\mu \leqslant 1$, while for $\mu > 1$ we write $(1+\mu^{\alpha}) \leqslant 2\mu^{\alpha}$ and note that $\int_0^{\infty} s^{\alpha-1} \mathrm{e}^{-\mu s} \,\mathrm{d}s = \mu^{-\alpha}\Gamma(\alpha)$.

(2) For $\alpha > 0$, $\mu > 0$, $\sigma = 0, 1$, $\nu > 2$ and $m \in \mathbb{N}$,

$$\mu^{\alpha}\nu^{\alpha} \int_0^1 \frac{\mathrm{e}^{-\nu\mu[1-(1-s)^m]}}{[1+\mu^2(1-s)^2]^{\sigma}} s^{\alpha-1} \,\mathrm{d}s \leqslant 8\left(2^{\alpha}+1\right)\Gamma(\alpha)\left[1+\mu^2\right]^{-\sigma} \tag{122}$$

where $C(m)$ is independent of μ, α and ν. Indeed, the integral is bounded by

$$\begin{aligned}
\left(\int_0^{1/2} \mathrm{d}u + \int_{1/2}^1 \mathrm{d}u\right) \frac{\mathrm{e}^{-\mu\nu s} s^{\alpha-1} \,\mathrm{d}s}{[1+\mu^2(1-s)^2]^{\sigma}} &\leqslant \frac{1}{(1+\mu^2/4)^{\sigma}} \int_0^1 \mathrm{e}^{-\mu\nu s} s^{\alpha-1} \,\mathrm{d}s + \max_{s\in[1/2,1]} \frac{\mathrm{e}^{-\mu\nu s}}{[1+\mu^2(1-s)^2]^{\sigma}} \int_0^1 s^{\alpha-1} \,\mathrm{d}s \\
&\leqslant \frac{2\Gamma(\alpha)(\mu\nu)^{-\alpha}}{(1+\mu^2/4)^{\sigma}} + \frac{\mathrm{e}^{-\mu\nu/2}}{\alpha(1+\mu^2/4)^{\sigma}} \\
&\leqslant \frac{2\Gamma(\alpha)(\mu\nu)^{-\alpha}}{(1+\mu^2/4)^{\sigma}} + \frac{2^{\alpha+1}\Gamma(\alpha)(\mu\nu)^{-\alpha}}{(1+\mu^2/4)^{\sigma}} \sup_{\alpha\in\mathbb{R}^+} \sup_{\mu\nu\in\mathbb{R}^+} \frac{(\mu\nu)^{\alpha}\mathrm{e}^{-\mu\nu/2}}{2^{\alpha+1}\alpha\Gamma(\alpha)} \\
&\leqslant \frac{2\Gamma(\alpha)(\mu\nu)^{-\alpha}}{(1+\mu^2/4)^{\sigma}} + \frac{2^{\alpha+1}\Gamma(\alpha)(\mu\nu)^{-\alpha}}{(1+\mu^2/4)^{\sigma}}.
\end{aligned}$$

(3) For $n>1$ the function

$$(1+\mu)\mathrm{e}^{-\mu}\int_0^1 \mathrm{e}^{\mu[u^n+(1-u)^n]}\,\mathrm{d}u$$

is bounded in $\mathbb{R}^+$, as it can be checked applying Watson's lemma for large μ and noting its continuity on $[0,\infty)$. Thus, for some constant C and $\nu>1$ we have

$$\int_0^{|p|} \mathrm{e}^{\nu|s|^n+\nu|p-s|^n}\,\mathrm{d}s \leqslant \frac{C|p|}{1+|p|^n}\mathrm{e}^{\nu|p|^n}. \tag{123}$$

(4) We have $|\mathbf{p}^{\mathbf{k}}| \leqslant \max_{i\leqslant d}|p|_i^{|\mathbf{k}|} \leqslant \sum_{i\leqslant d}|p_i|^{|\mathbf{k}|}$ and thus for some constant C and all $j\leqslant m$ we have

$$\left|\mathcal{P}_j(-\mathbf{p})\right| \leqslant C\sum_i\left(1+|p_i|^n\right). \tag{124}$$

Also, for some $C_2>0$, $|\mathcal{P}_j(-\mathbf{p})| \leqslant C_2\sum_i(1+|p_i|+|p_i^n|) =: C_2(d+q)$ and thus, for $\nu>C_2+1$ we have, for $0\leqslant l'\leqslant n$,

$$\begin{aligned}|\mathbf{p}|^{l'}\int_0^t \mathrm{e}^{|\mathcal{P}_j(-\mathbf{p})|(t-\tau)}\mathrm{e}^{\nu(\tau+1)q}\,\mathrm{d}\tau &\leqslant |\mathbf{p}|^{l'}\mathrm{e}^{q\nu+C_2td}\int_0^t \mathrm{e}^{(\nu-C_2)q\tau}\,\mathrm{d}\tau\\ &\leqslant T^{1-l'/n}\mathrm{e}^{\nu q(t+1)+C_2td}\frac{|\mathbf{p}|^{l'}}{[(\nu-C_2)q]^{l'/n}}\sup_{\gamma>0}\frac{1-\mathrm{e}^{-\gamma}}{\gamma^{1-l'/n}}\\ &\leqslant \frac{C_3(T)}{(\nu-C_2)^{l'/n}}\mathrm{e}^{\nu q(t+1)+C_2td}.\end{aligned} \tag{125}$$

A.5. Modified estimates for Lemma 38

From (123) it follows that for a constant C independent of $\boldsymbol{\Psi}$, $\boldsymbol{\Phi}$ we have

$$|\boldsymbol{\Psi}*\boldsymbol{\Phi}| \leqslant C\mathrm{e}^{\nu(t+1)\sum_i(|p_l|+|p_l|^n)}\|\boldsymbol{\Psi}\|_{\nu n}\|\boldsymbol{\Phi}\|_{\nu n}. \tag{126}$$

In particular $\mathfrak{B}(\nu,n,\mathcal{S})$ is a Banach algebra. For the equivalent of Lemma 16, we use the following bounds.

$$I=\int_0^{|p_1|} s^{\alpha-1}\mathrm{e}^{-\nu(t+1)[|p_1|^n-(|p_1|-s)^n]}\mathrm{e}^{-\nu(t+1)s}\,\mathrm{d}s \leqslant \int_0^{|p_1|} s^{\alpha-1}\mathrm{e}^{-\nu(t+1)s}\,\mathrm{d}s \leqslant \frac{\nu^{-\alpha}}{\Gamma(\alpha)(t+1)^\alpha} \tag{127}$$

and

$$I\leqslant |p_1|^\alpha\int_0^1 s^{\alpha-1}\mathrm{e}^{-\nu(t+1)|p_1|^n[1-(1-s)^n]}\,\mathrm{d}s \leqslant C\frac{2^\alpha\Gamma(\alpha)|p_1|^\alpha}{[\nu(t+1)|p_1|^n]^\alpha}$$

where we used (122) for $\sigma=0$. From (127) it is clear that

$$\|\mathbf{H}*F_j\|_{\nu n} \leqslant \big\||\mathbf{H}|*|F_j|\big\|_{\nu n} \leqslant C\left[\Gamma(\alpha)\right]^d c^\alpha\left(\nu(t+1)\right)^{-d\alpha}\|\mathbf{F}\|_{\nu n}. \tag{128}$$

In Lemma 22, we get instead

$$\big||\mathbf{F}|*|\mathbf{G}|\big| \leqslant \mathrm{e}^{\nu(t+1)\sum_i(|p_i|+|p_i|^n)}\|\mathbf{F}\|_{\nu n}\|\mathbf{G}\|_{\nu n}.$$

Very similar changes are made in Lemma 24, Corollary 25, and in Lemma 26 where in the proof we use (125) instead of (45). Definition 27, Lemma 28 and Definition 29 do not change. Lemmas 30 and 31 change in the same way as above. In Lemma 32 we use again (125) instead of (45) to make corresponding changes. Finally, in Lemma 33, $\nu/4$ changes to $\nu/4/c$.

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