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An extension theorem to rough paths

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Abstract

We show that any continuous path of finite p-variation can be lifted to a geometric q-rough path, where $q > p$.

Résumé

Nous montrons que tout chemin continu de p-variation finie peut être relevé en un « geometric q-rough path », pour $q > p$.

Keywords: Differential equations; Rough paths

1. Introduction

Let

$$x : [0,1] \to \mathbb{R}^n,$$
$$t \to \big(x_1(t), \dots, x_n(t)\big)$$

be a continuous function of bounded variation, and $V_1, \dots, V_n$ some smooth functions from $\mathbb{R}^d$ into itself. Then there exists a (unique) solution to the control differential equation

$$\begin{cases} \mathrm{d}y(t) = \sum_{i=1}^n V_i(y(t))\,\mathrm{d}x_i(t), \\ y(0) = y_0. \end{cases} \tag{1}$$

But without the smoothness assumption on x (which is for example almost surely not satisfied by Brownian motion), classical theory fails to give a meaning to the above equation. Rough paths theory [12,13,11] gives a meaning to Eq. (1), whenever x is a continuous path of finite p-variation lifted to a "geometric p-rough path".

To understand what a geometric p-rough path is, consider a smooth path $x : [0,1] \to \mathbb{R}^n$, and define

$$S(x)_{0,t} = 1 + \int\limits_{0<s_1<t} \mathrm{d}x_{s_1} + \int\limits_{s<s_{i_1}<\cdots<s_{i_{[p]}}<t} \mathrm{d}x_{s_{i_1}} \otimes \cdots \otimes \mathrm{d}x_{s_{i_{[p]}}}.$$

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$t \to S(x)_{0,t}$ takes its values in $G^{[p]}(\mathbb{R}^n)$, the free nilpotent group of step $[p]$ over $\mathbb{R}^n$, thought of a manifold immersed in the tensor algebra $\bigoplus_{k=0}^{[p]}(\mathbb{R}^n)^{\otimes k}$ [12,14]. For a given $p \geqslant 1$, the set of geometric rough paths is the closure under a given p-variation metric of the set $\{S(x), x \text{ smooth}\}$ (see definition (12)). A weak geometric p-rough path is a $G^{[p]}(\mathbb{R}^n)$-valued path of finite p-variation, where the p-variation is computed using a homogeneous metric associated to the group. The distinction between these two spaces was glossed over in the paper [12] by the first name author. It is obvious that a geometric p-rough path is a weak geometric p-rough path and that a weak geometric p-rough path is a geometric q-rough path for any $q > p$. There are examples of weak geometric p-rough paths that are not geometric p-rough paths [6]. The difference between weak geometric rough paths and geometric rough paths is a bit like the difference between Lipschitz functions and C^1 functions.

If $\mathbf{x}$ is a (weak) geometric p-rough path, $\mathbf{x}$ projects onto a path x with values in $\mathbb{R}^n$. The solution to Eq. (1) is uniquely defined for any $\mathbf{x}$, and the solution is also a (weak) geometric p rough path $\mathbf{y}$. Moreover, the map $\mathbf{x} \to \mathbf{y}$ is continuous in an appropriate topology. In the classical setting where x is smooth, $p = 1$. There is a functional relationship between x and the solution of the differential equation (1). For $p \geqslant 2$, there will be infinitely many choices for $\mathbf{x}$ projecting onto x. The corresponding solution $\mathbf{y}$ and its projection y will in general depend on this choice.

There is often a "canonical" choice for the lift $\mathbf{x}$ of x (for example, if x is smooth, $S(x)$ is a canonical lift of x to a geometric p-rough path, for any $p \geqslant 1$). "Canonical" lifts have been constructed for Brownian motion [13,10], fractional Brownian motion with Hurst parameter greater than 1/4 [3,13], free Brownian motion [2,18], and a large class of random paths on fractals [1,8].

1.1. Our goal

Consider the following natural question: can every continuous path of finite p-variation in V (a Banach space) be lifted to a weak geometric p-rough path (a $G^{[p]}(V)$-valued path of finite p-variation)? We will see, that provided that p is not an integer number greater than or equal to 2, the answer is affirmative. This is optimal, as a counter example for $p = 2$ was provided in [18]. In particular, any path of finite p-variation in V can be lifted to a geometric q-rough path, for any $q > p$. The theorem we prove is actually stronger.

Theorem 1. *We fix $p \in [1, +\infty)$. Let V be a Banach space and K a closed normal subgroup of $G^{([p])}(V)$. If x is a $(G^{([p])}(V)/K, \|\cdot\|_{G^{([p])}(V)/K})$ continuous path of finite p-variation, with $p \notin \mathbb{N}\backslash\{0,1\}$, then one can lift x to a weak geometric p-rough path.*

Consider a path x of finite p-variation with values in $(G^{([p])}(V)/K, \|\cdot\|_{G^{([p])}(V)/K})$, where K is as above. x projects to a $\mathbb{R}^n$-valued path, and so one can consider again the differential equation (1). This one only makes sense once we lift x to a geometric q-rough path $\mathbf{x}$, for $q > p$. The solution depends in general on the choice of the lift $\mathbf{x}$. We will identify conditions on the Lie algebra generated by the vector fields $(V_i)_{1\leqslant i\leqslant n}$ in (1) so that the projection y of the rough path solution $\mathbf{y}$ of Eq. (1) does not depend on the lift of x. In general, $\mathbf{y}$ and y will depend on the lift $\mathbf{x}$ of x.

To help the comprehension of the paper, we start by presenting the main theorem for $p \in (2,3)$ and $V = \mathbb{R}^2$, where no algebra is necessary and result are quite intuitive.

2. A simple case

We start with a non-surprising technical lemma, whose proof is inspired from the Kolmogorov–Centsov criteria [9].

Lemma 2. *Let y be a map from $\bigcup_{n\geqslant 0}\bigcup_{k=0}^{2^n}\{k2^{-n}\}$ into (E,d), a metric space, such that for all $n, k \in \{0,\dots,2^n\}$,*

$$d(y_{\frac{k}{2^n}}, y_{\frac{k+1}{2^n}}) \leqslant C2^{-n/p}. \tag{2}$$

Then, there exists a unique continuous path $\tilde{y}:[0,1] \to (E,d)$ which coincides with y on $\bigcup_{n\geqslant 0}\bigcup_{k=0}^{2^n}\{k2^{-n}\}$. Moreover, $\tilde{y}$ is $1/p$-Hölder.

Proof. We fix $r \in \mathbb{N}$, and show by induction on m that for all $s, t \in D_m$ such that $0 < t - s < 2^{-r}$,

$$d(y_s, y_t) \leqslant 2C \sum_{k=r+1}^{m} 2^{-k/p}. \tag{3}$$

When $m = r + 1$, necessarily, (s, t) is of the form $(\frac{k}{2^m}, \frac{k+1}{2^m})$, $k \in \{0, \ldots, 2^m - 1\}$, and so (3) is exactly formula (2). Suppose now that formula (3) is valid for $m = r + 1, \ldots, M - 1$. Take $s, t \in D_M$ such that $0 < t - s < 2^{-r}$, and consider $t_1 = \max\{u \in D_{M-1}; u \leqslant t\}$ and $s_1 = \max\{u \in D_{M-1}; u \geqslant s\}$. Notice that $d(y_s, y_{s_1})$ and $d(y_{t_1}, y_t)$ are both bounded by $C2^{-M/p}$, and, by the induction assumption, that

$$d(y_{s_1}, y_{t_1}) \leqslant 2C \sum_{k=r+1}^{M-1} 2^{-k/p}.$$

Therefore,

$$\begin{aligned} d(y_s, y_t) &\leqslant 2C2^{-M/p} + 2C \sum_{k=r+1}^{M-1} 2^{-k/p} \\ &= 2C \sum_{k=r+1}^{M} 2^{-k/p}, \end{aligned}$$

which concludes the induction.

Now let us consider $(s, t) \in \bigcup_{m \geqslant 0} D_m$, and let r be the natural number such that $2^{-(r+1)} < t - s < 2^{-r}$. From the induction, we obtain

$$\begin{aligned} d(y_s, y_t) &\leqslant 2C \sum_{k=r+1}^{\infty} 2^{-k/p} \leqslant \widetilde{C}_p 2^{-(r+1)/p} \\ &\leqslant \widetilde{C}_p |t - s|^{1/p}. \end{aligned} \tag{4}$$

We finally define $\tilde{y}_t$ for $0 \leqslant t \leqslant 1$ by

$$\tilde{y}_t = \lim_{r \to \infty} y_{\frac{[2^r t]}{2^r}}.$$

From (4), the limit exists and $\tilde{y}$ satisfies $d(\tilde{y}_s, \tilde{y}_t) \leqslant \widetilde{C}|t - s|z^{1/p}$. □

Let x be a $\mathbb{R}^2$-valued path, which is $1/p$-Hölder with $p \in (2, 3)$. We want to prove that we can lift x to a $1/p$-Hölder path with values in the Heisenberg group H^1 equipped with its Carnot–Caratheodory metric. Let us first recall a few fact about this group and its metric. The Heisenberg group H^1 is equal to $\mathbb{R}^3$ equipped with the product

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = \left(x_1 + x_2, y_1 + y_2, z_1 + z_2 + \frac{1}{2}(y_1 x_2 - y_2 x_1)\right).$$

The Carnot–Caratheodory distance will be introduced later, the only property we need for this preliminary chapter is that there exists positive constants c, C such that

$$c \max\left\{|x_1 - x_2|, |y_1 - y_2|, \left|z_1 - z_2 + \frac{1}{2}(y_1 z_2 - y_2 z_1)\right|^{1/2}\right\} \leqslant d\big((x_1, y_1, z_1), (x_2, y_2, z_2)\big),$$

and

$$d\big((x_1, y_1, z_1), (x_2, y_2, z_2)\big) \leqslant C \max\left\{|x_1 - x_2|, |y_1 - y_2|, \left|z_1 - z_2 + \frac{1}{2}(y_1 z_2 - y_2 z_1)\right|^{1/2}\right\}.$$

It is easy to see that to lift x to a $1/p$-Hölder H^1-valued path, we need to construct the Levy area of x, i.e. a map $A : \{0 \leqslant s < t \leqslant 1\} \to \mathbb{R}$ such that

- for all $s < t < u \in [0, 1]$,

$$A_{s,u} = A_{s,t} + A_{t,u} + \frac{1}{2}\left(x_{s,t}^1 x_{t,u}^2 - x_{s,t}^2 x_{t,u}^1\right); \tag{5}$$

- for some constant C, for all $s, t \in [0, 1]$,

$$|A_{s,t}| \leqslant C|t - s|^{2/p}. \tag{6}$$

Of course, if x is of bounded variation, $A_{s,t} = \frac{1}{2}\iint_{s<v_1<v_2<t}(\mathrm{d}x_{v1}^1\,\mathrm{d}x_{v_2}^2 - \mathrm{d}x_{v2}^1\,\mathrm{d}x_{v_1}^2)$ satisfies the above condition. As we do not assume x smooth, we cannot use this area.

Proposition 3. *Let x be a $\mathbb{R}^2$-valued path, which is $1/p$-Hölder with $p \neq 2$. Then, one can lift x to a $1/p$-Hölder path with values in the Heisenberg group H^1 equipped with its Carnot–Caratheodory metric.*

In rough paths language, using the fact that path of finite p-variation are $1/p$-Hölder after a time change, this means that we can lift any $\mathbb{R}^2$-valued path of finite p-variation to a geometric p-rough path, whenever $p \in (2, 3)$.

Proof. If $p < 2$, the result is just a easy consequence of Theorem 1 in [12], or just properties of Young integrals. We therefore assume $p > 2$.

Let C_x be the Hölder constant of x. We construct inductively the area of x between dyadic times, $A_{\frac{k}{2^n},\frac{k+1}{2^n}}$ for $k = 0, \ldots, 2^n$. We also define inductively

$$a_n = 2^{2n/p} \max_{0\leqslant k\leqslant 2^n} |A_{\frac{k}{2^n},\frac{k+1}{2^n}}|.$$

First, we set $A_{0,1} = 0$, and therefore we have $a_0 = 0$. Assume then, for a fixed n, that we have constructed $A_{\frac{k}{2^n},\frac{k+1}{2^n}}$ for $k = 0, \ldots, 2^n$. We define $A_{\frac{2k}{2^{n+1}},\frac{2k+1}{2^{n+1}}}$ and $A_{\frac{2k+1}{2^{n+1}},\frac{2k+2}{2^{n+1}}}$ so that they are both equal. Eq. (5) therefore forces them to be equal to

$$\frac{1}{2}A_{\frac{k}{2^n},\frac{k+1}{2^n}} - \frac{1}{4}\left(x^1_{\frac{k}{2^n},\frac{2k+1}{2^{n+1}}} x^2_{\frac{2k+1}{2^{n+1}},\frac{k+1}{2^n}} - x^2_{\frac{k}{2^n},\frac{2k+1}{2^{n+1}}} x^1_{\frac{2k+1}{2^{n+1}},\frac{k+1}{2^n}}\right).$$

In particular

$$a_{n+1}2^{-2(n+1)/p} \leqslant 2^{-2n/p}\frac{a_n}{2} + \frac{C_x^2}{2}2^{-2(n+1)/p},$$

i.e.

$$a_{n+1} \leqslant 2^{2/p-1}a_n + \frac{1}{2}C_x^2.$$

It is easy to see by induction that, if $p > 2$, the sequence a_n is bounded. Transferring this information in terms of the path $\mathbf{x} = (x, A)$, we see that we have constructed elements $\mathbf{x}_{k2^{-n}}$ of the metric space (H^1, d) such that for all $n, k \in \{0, \ldots, 2^n\}$,

$$d(\mathbf{x}_{\frac{k}{2^n}}, \mathbf{x}_{\frac{k+1}{2^n}}) \leqslant M_p 2^{-n/p}. \tag{7}$$

We conclude the proof with Lemma 2. □

The above construction and idea will be the main argument of the proof of the main theorem. To be able to explain it, we need to introduce a few algebraic and geometric notions.

3. Algebraic preliminaries

3.1. Carnot groups

If G is a simply connected nilpotent Lie group with Lie algebra $\mathcal{G}$, then the Lie group exponential map $\exp : \mathcal{G} \to G$ is a diffeomorphism [15,17]. In this case we let $\ln : G \to \mathcal{G}$ denote the inverse of the exponential function. We start with a couple definitions.

Definition 4. A Carnot group[1] is a connected nilpotent Lie group G, such that its Lie algebra $\mathcal{G}$ can be written as

$$\mathcal{G} = W_1 \oplus \cdots \oplus W_n,$$

where for all i, $W_{i+1} = [W_1, W_i]$. For an element $g = \exp(w_1 + \cdots + w_n) \in G$, with $w_i \in W_i$, we let, for $t \in \mathbb{R}$,

$$\delta_t g = \exp\bigl(t w_1 + \cdots + t^n w_n\bigr).$$

δ is called the dilation operator.

Definition 5. A (symmetric sub-additive) homogeneous norm [5] on a Carnot group G is a function $\|\cdot\|_G : G \to \mathbb{R}^+$ such that

(i) $\|g\|_G = 0$ if and only if g is the neutral element of the group, $1 = \exp(0)$,
(ii) $\|\delta_t g\|_G = |t| \|g\|_G$,
(iii) for all $g, h \in G$, $\|g \otimes h\|_G \leqslant \|g\|_G + \|h\|_G$,
(iv) for all g, $\|g\|_G = \|g^{-1}\|_G$.

Such a norm define a left invariant distance on the group by $d_G(g,h) = \|h^{-1} \otimes g\|_G$. We will say that $(G, \|\cdot\|_G)$ is a normed Carnot group.

If G is a fixed Carnot group with finite dimensional Lie algebra, all homogeneous norms on G are equivalent. The Carnot–Caratheodory norm is an example of a homogeneous norm on a Carnot group [7]. Any homogeneous norms $\|\cdot\|_G$ on G leads to a left invariant distance $d_G(x,y) = \|y^{-1}x\|$ (in particular, the Carnot–Caratheodory norm leads to the Carnot–Caratheodory distance). Let G be a normed Carnot group with Lie algebra $\mathcal{G}$, K a Lie subgroup of G, with Lie algebra $\mathcal{K}$. If K is a closed normal Lie subgroup of G, or equivalently if $\mathcal{K}$ is closed ideal of $\mathcal{G}$, then G/K is then a Carnot group with Lie algebra $\mathcal{G}/\mathcal{K}$ [15]. If G is equipped with a homogeneous norm $\|\cdot\|_G$, then we equip G/K with the quotient homogeneous norm on G/K

$$\|\cdot\|_{G/K} : G/K \to \mathbb{R},$$
$$gK \to \inf_{k \in K} \|g \otimes k\|_G.$$

We will denote by $\pi_{G,G/K}$ the canonical homomorphism from G onto G/K. Sometimes, it will be more convenient to write gK for $\pi_{G,G/K}(g)$.

Proposition 6. *Let $(G, \|\cdot\|_G)$ be a normed Carnot group, K a closed normal Lie subgroup of G. There exists an injection $i_{G/K,G} : G/K \to G$ such that*

(i) *$\pi_{G,G/K} \circ i_{G/K,G}$ is the identity map of G/K,*
(ii) *for all $t \in \mathbb{R}^+$, $gK \in G/K$, $\delta_t(i_{G/K,G}(gK)) = i_{G/K,G}(\delta_t(gK))$,*
(iii) *for all $gK \in G/K$, $\|gK\|_{G/K} \leqslant \|i_{G/K,G}(gK)\|_G \leqslant 2\|gK\|_{G/K}$.*

Proof. By definition of the homogeneous norm on G/K, for all $g \in G$ such that $\|gK\|_{G/K} = 1$, the set

$$M_g = \bigl\{g \otimes k \text{ such that } k \in K \text{ and } 1 \leqslant \|g \otimes k\|_G \leqslant 2\bigr\}$$

is non-empty. We define $i_{G/K,G}$ on the set of elements

$$\bigl\{gK, \text{ such that } \|gK\|_{G/K} = 1\bigr\}$$

to be any function which at gK associates an element of $\bigcup_{m \in \pi_{G,G/K}^{-1}(gK)} M_m$; such function exists by the axiom of choice. We then extend $i_{G/K,G}$ to G/K with the help of the formula $i_{G/K,G}(\delta_t gK) = \delta_t i_{G/K,G}(gK)$. □

[1] In most definitions of a Carnot group, $\mathcal{G}$ is assumed to be finite dimensional. We do not make such an assumption here.

3.2. Free nilpotent groups

We know introduce a fundamental example of a Carnot group.

We fix (for the rest of the paper) a normed vector space $(V, \|\cdot\|_1)$. We let $T(V)=\bigoplus_{n=0}^{\infty} V^{\otimes n}$ be the tensor algebra over V. $T(V)$ equipped with standard addition $+$, tensor multiplication $\otimes$ and scalar product is an associative algebra. $T^{(n)}(V)$, the quotient algebra of $T(V)$ by the ideal $\bigoplus_{m=n+1}^{\infty} V^{\otimes m}$, inherits this algebraic structure. One can define on $T^{(n)}(V)$ a Lie bracket by the formula

$$[a,b]=a\otimes b-b\otimes a,$$

which makes $T^{(n)}(V)$ into a Lie algebra. We let $\mathcal{G}^{(n)}(V)$ be the Lie subalgebra of $T^{(n)}(V)$ generated by elements in V. Note that

$$\mathcal{G}^{(n)}(V)\simeq\bigoplus_{i=1}^{n} V_i,$$

where

$$V_1=V \quad \text{and} \quad V_{i+1}=[V,V_i]. \tag{8}$$

$\mathcal{G}^{(n)}(V)$ is the free nilpotent Lie algebra of step n [12–14]. The exponential, logarithm and inverse function are defined on $T^{(n)}(V)$ by mean of their power series. We denote by $G^{(n)}(V)=\exp(\mathcal{G}^{(n)}(V))$. By the Baker–Campbell–Hausdorff formula, $(G^{(n)}(V),\otimes)$ is a connected nilpotent Lie group, called the free nilpotent Lie group of step n over V. By construction, $(G^{(n)}(V),\otimes)$ is a Carnot group, with Lie algebra $\mathcal{G}^{(n)}(V)$.

We are now going to equip $G^{(n)}(V)$ with a homogeneous norm. We first let $\|\cdot\|_i$ be some norms on $V^{\otimes i}$ such that for all $(a_i,a_j)\in V^{\otimes i}\times V^{\otimes j}$, $\|a_i\otimes a_j\|_{i+j}\leqslant\|a_i\|_i+\|a_j\|_j$. To simplify notations, we will write $\|\cdot\|$ for all these norms. Now define

$$\|g\|_{G^{(n)}(V)}=\max_{i=1,\ldots,n}\big(i!\|g_i\|\big)^{1/i}+\max\big(i!\big\|\big(g^{-1}\big)_i\big\|\big)^{1/i},$$

where $g=1+g_1+\cdots+g_n$, $g_i\in V^{\otimes i}$ is an element of the group $G^{(n)}(V)$, and g^{-1} is its inverse. The binomial equality quickly shows that $g\to\max_{i=1,\ldots,n}(i!\|g_i\|)^{1/i}$ is a sub-additive homogeneous norm (but a priori not symmetric). That implies that $\|\cdot\|_{G^{(n)}(V)}$ defines a homogeneous norm on $G^{(n)}(V)$. We also let

$$d_{G^{(n)}(V)}(g,h)=\big\|h^{-1}\otimes g\big\|_{G^{(n)}(V)}.$$

Proposition 7. *Let $g=\exp(l_1+\cdots+l_n)$, with $l_i\in V_i$. Then,*

$$c_n\max_{i=1,\ldots,n}\|l_i\|^{1/i}\leqslant\|g\|_{G^{(n)}(V)}\leqslant C_n\max_{i=1,\ldots,n}\|l_i\|^{1/i},$$

for some constants c_n and C_n which depends only on n.

Proof. Let us fix $i\in\{1,\ldots,n\}$ and write $g=1+g_1+\cdots+g_n$, with $g_i\in V^{\otimes i}$. By definition of the exponential function,

$$g_k=\sum_{i=1}^{k}\frac{1}{i!}\sum_{\substack{j_1,\ldots,j_i\\ j_1+\cdots+j_i=k}} l_{j_1}\otimes\cdots\otimes l_{j_i}.$$

Hence,

$$\begin{aligned}\big(k!\|g_k\|_k\big)^{1/k}&\leqslant\left(\sum_{i=1}^{k}\frac{k!}{i!}\sum_{\substack{j_1,\ldots,j_i\\ j_1+\cdots+j_i=k}}\|l_{j_1}\|\cdots\|l_{j_i}\|\right)^{1/k}\\&\leqslant\big(k!(\exp k-1)\big)^{1/k}\max_{i=1,\ldots,n}\|l_i\|^{1/i}.\end{aligned}$$

Applying this to g^{-1}, we obtain

$$\big(k!\big\|\big(g^{-1}\big)_k\big\|_k\big)^{1/k} \leqslant \big(k!(\exp k - 1)\big)^{1/k} \max_{i=1,\dots,n} \|-l_i\|^{1/i} = \big(k!(\exp k - 1)\big)^{1/k} \max_{i=1,\dots,n} \|-l_i\|^{1/i}.$$

That gives us the upper bound. For the lower bound, observe that by definition of the logarithm function,

$$l_k = \sum_{i=1}^{k} \frac{(-1)^i}{i} \sum_{\substack{j_1,\dots,j_i \\ j_1+\dots+j_i=k}} g_{j_1} \otimes \dots \otimes g_{j_i},$$

which, when applied to both g and its inverse, gives that for all $1 \leqslant k \leqslant n$,

$$\|l_k\|^{1/k} \leqslant c_n^{-1} \|g\|_{G^{(n)}(V)}$$

for a constant $c_n > 0$. □

Corollary 8. *Let $K = \exp(\mathcal{K})$ be a closed normal subgroup of $G^{(n)}(V)$. Then, if $g = \exp(l_1 + \dots + l_n)$ with $l_i \in V_i$,*

$$c_n \leqslant \frac{\|gK\|_{G^{(n)}(V)/K}}{\max_{i=1,\dots,n}(\inf_{k_i \in \mathcal{K} \cap V_i} \|l_i + k_i\|)^{1/i}} \leqslant C_n.$$

Corollary 9. *Let $C(G^{(n)}(V))$ be the centre of $G^{(n)}(V)$ and θ the canonical isomorphism between $G^{(n-1)}(V)$ and $G^{(n)}(V)/C(G^{(n)}(V))$. Then the homogeneous norm $\|\cdot\|_{G^{(n-1)}(V)}$ and $\|\theta(\cdot)\|_{G^{(n)}(V)/C(G^{(n)}(V))}$ are equivalent. We will therefore not distinguish between them.*

4. Rough paths

In this paper, by E-valued path, we mean a function from [0, 1] into E.

4.1. On p-variation

Definition 10. Let (E, d) be a metric space. A (E, d)-valued path x is said to have finite p-variation if

$$\sup_D \sum_{i=1}^{\#D-1} d(x_{t_i}, x_{t_{i+1}})^p < \infty,$$

where the supremum runs over all subdivisions $D = (0 \leqslant t_1 \leqslant \dots \leqslant t_{\#D} \leqslant 1)$ of the interval [0, 1].

Note that x is continuous and of finite regular p-variation if and only if for all $s \leqslant t$, $d(x_s, x_t) \leqslant \omega(s, t)$, where

(i) $\omega : \{(s,t), 0 \leqslant s \leqslant t \leqslant 1\} \to \mathbb{R}^+$ is continuous.
(ii) ω is super-additive, i.e. $\forall\, s < t < u$, $\omega(s,t) + \omega(t,u) \leqslant \omega(t,u)$. (9)
(iii) $\omega(t,t) = 0$ for all $t \in [0,1]$.

We will say in such case that x has finite p-variation controlled by ω.

We are going to show that a continuous (E, d)-valued path of finite p-variation is, up to reparametrisation of time, $1/p$-Hölder continuous. If ω satisfies (9), then

$$(s,t) \to \omega(0,1)\left(\frac{\omega(0,t)}{\omega(0,1)} - \frac{\omega(0,s)}{\omega(0,1)}\right)$$

is a continuous additive map, equal to zero on the diagonal, and $\omega(0,t) - \omega(0,s) \geqslant \omega(s,t)$ (by the super-additivity of ω). Therefore, a path is of finite p-variation if and only if there exists an non-decreasing and continuous surjection γ from [0, 1] onto [0, 1] and a positive constant C such that

$$\text{for all } s \leqslant t, \quad d(x_s, x_t)^p \leqslant C\big|\gamma(t) - \gamma(s)\big|.$$

For such a γ, we define

$$\gamma^{-1} : [0,1] \to [0,1], \qquad t \to \inf\{u, \gamma(u) = t\}.$$

The following is straightforward to check.

Lemma 11. *Let x be a continuous (E,d)-valued path of finite p-variation controlled by $(s,t) \to C|\gamma(t) - \gamma(s)|$, where γ is a continuous increasing surjection from $[0,1]$ onto $[0,1]$. Define*

$$y : [0,1] \to E, \qquad t \to x_{\gamma^{-1}(t)}.$$

Then, y is a $1/p$-Hölder (E,d)-valued path.

Reciprocally, if y is a $1/p$-Hölder (E,d)-valued path then, it is a continuous path of finite p-variation controlled by $(s,t) \to C|t-s|$.

4.2. Geometric p-rough paths

Definition 12. A weak geometric p-rough path is a $(G^{([p])}(V), \|\cdot\|_{G^{([p])}(V)})$-valued path which has finite p-variation.[2]

When x is a path with values in a group $(G, \otimes)$, we will write $x_{s,t} = x_s^{-1} \otimes x_t$.

5. The extension theorem

We first need an important lemma.

Lemma 13. *Let $(G, \|\cdot\|_G)$ be a normed Carnot group with graded Lie algebra*

$$\mathcal{G} = W_1 \oplus W_2 \oplus \cdots \oplus W_n.$$

Define K to be a closed subgroup of $\exp(W_n)$, which gives us a normed Carnot group $(G/K, \|\cdot\|_{G/K})$. Let x be a $1/p$-Hölder $(G/K, \|\cdot\|_{G/K})$-valued path. Then, if $p > n$, there exists a $1/p$-Hölder $(G, \|\cdot\|_G)$-valued path $\tilde{x}$ such that $\pi_{G,G/K}(\tilde{x}) = x$.

Proof. As $\exp(W_n)$ is in the center of G, K is a subgroup of the center of G. In particular, K is a closed normal subgroup of G.

To construct our path $\tilde{x}$, we are first going to construct its increments $\tilde{x}_{s,t}$ when $s,t \in D_m = \{\frac{k}{2^m}, k \in \{0, \ldots, 2^{m-1}\}\}$ with $t - s = 2^{-m}$, doing this for all m. $\tilde{x}_{s,t}$ will be constructed in such a way that $\|\tilde{x}_{s,t}\|_G \leqslant C|t-s|^{1/p}$ for a given $C < \infty$. Multiplying the increments, we will then have defined x on all dyadics, and the proof will be finished thanks to Lemma 2.

So we define recursively on m some elements $y_{\frac{k}{2^m}, \frac{k+1}{2^m}} \in K$, $k \in \{0, \ldots, 2^m - 1\}$, $m \in \mathbb{N}$, in the aim of defining the elements $\tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}}$ with the formula

$$\tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}} = i_{G/K,G}\left(x_{\frac{k}{2^m}, \frac{k+1}{2^m}}\right) \otimes y_{\frac{k}{2^m}, \frac{k+1}{2^m}}$$

where $i_{G/K,G}$ is the injection of Proposition 6. This will ensure that $\pi_{G,G/K}(\tilde{x}) = x$. First, we let, $y_{0,1} = \exp(0)$. Then, we assume that $y_{\frac{k}{2^m}, \frac{k+1}{2^m}}$ (and hence $\tilde{x}_{\frac{k}{2^m}, \frac{k+1}{2^m}}$) has been constructed for all $0 \leqslant k \leqslant 2^m - 1$ and a fixed m, and we define the two elements $y_{\frac{2k}{2^{m+1}}, \frac{2k+1}{2^{m+1}}}$ and $y_{\frac{2k+1}{2^{m+1}}, \frac{2k+2}{2^{m+1}}}$ to be both equal, and equal to the inverse of

$$\delta_{2^{-1/n}}\left(i_{G/K,G}\left(x_{\frac{2k}{2^{m+1}}, \frac{2k+1}{2^{m+1}}}\right) \otimes i_{G/K,G}\left(x_{\frac{2k+1}{2^{m+1}}, \frac{2k+2}{2^{m+1}}}\right) \otimes \tilde{x}^{-1}_{\frac{k}{2^m}, \frac{k+1}{2^m}}\right).$$

[2] The definition of a geometric p-rough path is presented quite differently in [12], as the notion of a homogeneous norm was not mentioned there. Nonetheless, the difference is easily seen to be only notational.

We easily check that $\pi_{G,G/K}(y_{\frac{2k}{2^{m+1}},\frac{2k+1}{2^{m+1}}}) = \exp(0)$, i.e. that $y_{\frac{2k}{2^{m+1}},\frac{2k+1}{2^{m+1}}} = y_{\frac{2k+1}{2^{m+1}},\frac{2k+2}{2^{m+1}}} \in K$. As elements of K commute with elements of G, and with the help of the formula $\delta_{2^{1/n}}(y) = y^{\otimes 2}$ for $y \in K$, we check that this choice for $y_{\frac{2k}{2^{m+1}},\frac{2k+1}{2^{m+1}}}$ and $y_{\frac{2k+1}{2^{m+1}},\frac{2k+2}{2^{m+1}}}$ gives

$$\tilde{x}_{\frac{k}{2^m},\frac{k+1}{2^m}} = \tilde{x}_{\frac{2k}{2^{m+1}},\frac{2k+1}{2^{m+1}}} \otimes \tilde{x}_{\frac{2k+1}{2^{m+1}},\frac{2k+2}{2^{m+1}}}.$$

We then define $a_m = 2^{m/p} \sup_{k\in\{0,\dots,2^m-1\}} \|y_{\frac{k}{2^m},\frac{k+1}{2^m}}\|_G$. By the assumption that x is $1/p$-Hölder and by the definition of $i_{G/K,G}$,

$$\left\|i_{G/K,G}(x_{\frac{k}{2^m},\frac{k+1}{2^m}})\right\|_G \leqslant 2\|x_{\frac{k}{2^m},\frac{k+1}{2^m}}\|_{G/K} \leqslant 2C2^{-m/p}.$$

Hence, from the previous inequality, we obtain that

$$2^{1/n}2^{-(m+1)/p}a_{m+1} \leqslant a_m 2^{-m/p} + 2^{-(m+1)/p}2^{2+1/p}C,$$

i.e.

$$a_{m+1} \leqslant 2^{1/p-1/n}a_m + 2^{2+1/p-1/n}C.$$

As $n < p$, we have $2^{1/p-1/n} < 1$ which forces the sequence a_m to be bounded. So we have constructed for every $m \geqslant 0$, $k \in \{0,\dots,2^m-1\}$ some elements $\tilde{x}_{\frac{k}{2^m},\frac{k+1}{2^m}} = i_{G/K,G}(x_{\frac{k}{2^m},\frac{k+1}{2^m}}) \otimes y_{\frac{k}{2^m},\frac{k+1}{2^m}} \in G$, such that

$$\begin{aligned}\|\tilde{x}_{\frac{k}{2^m},\frac{k+1}{2^m}}\|_G &\leqslant 2\|x_{\frac{k}{2^m},\frac{k+1}{2^m}}\|_{G/K} + \|y_{\frac{k}{2^m},\frac{k+1}{2^m}}\|_G \\ &\leqslant \Big(2C + \sup_m a_m\Big)2^{-m/p} = C_{p,n}2^{-m/p}.\end{aligned}$$

Remember also that for all dyadic $\frac{k}{2^m}$,

$$\tilde{x}_{\frac{k}{2^m},\frac{k+1}{2^m}} = \tilde{x}_{\frac{2k}{2^{m+1}},\frac{2k+1}{2^{m+1}}} \otimes \tilde{x}_{\frac{2k+1}{2^{m+1}},\frac{2k+2}{2^{m+1}}}.$$

That allows us to define

$$\tilde{x}_{\frac{k}{2^m},\frac{l}{2^m}} = \bigotimes_{j=k}^{l-1} \tilde{x}_{\frac{j}{2^m},\frac{j+1}{2^m}}.$$

We have proved that for all $n, k \in \{0,\dots,2^n\}$,

$$\|\tilde{x}_{\frac{k}{2^m},\frac{k+1}{2^m}}\|_G \leqslant C'2^{-m/p}. \tag{10}$$

The proof is therefore finished using Lemma 2. □

We are now ready for our main theorem.

Theorem 14. *We fix $p \in [1,+\infty)$. Let K be a closed normal subgroup of $G^{([p])}(V)$. If x is a $(G^{([p])}(V)/K, \|\cdot\|_{G^{([p])}(V)/K})$ continuous path of finite p-variation, with $p \notin \mathbb{N}\backslash\{0,1\}$, then there exists a continuous $(G^{([p])}(V), \|\cdot\|_{G^{([p])}(V)})$-valued path $\tilde{x}$ of finite p-variation such that*

$$\pi_{G^{([p])}(V),G^{([p])}(V)/K}(\tilde{x}) = x.$$

Proof. As noticed in Section 4.1, we assume without loss of generalities that x is $1/p$-Hölder. We denote by $\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_{[p]}$ the Lie algebra of K, with $\mathcal{K}_i \subset V_i$. We define for $k = 1,\dots,n$,

$$H^{(k)} = G^{(k)}(V)/\exp(\mathcal{K}_k)$$

and

$$\begin{aligned}M^{(k)} &= G^{(k)}(V)/\exp(\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_k) \\ &\simeq H^{(k)}/\exp(\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_{k-1}).\end{aligned}$$

We are going to construct recursively some $H^{(k)}$-valued paths $y^{(k)}$ and $G^{(k)}(V)$-valued paths $x^{(k)}$ which are $1/p$-Hölder and such that the projections of x, $x^{(k)}$, and $y^{(k)}$ onto $M^{(k)}$ are equal, i.e. such that

$$\pi_{H^{(k)},M^{(k)}}\big(y^{(k)}\big) = \pi_{M([p]),M^{(k)}}(x), \tag{11}$$

$$\pi_{G^{(k)}(V),M^{(k)}}\big(x^{(k)}\big) = \pi_{M([p]),M^{(k)}}(x). \tag{12}$$

Using Lemma 13, $x^{(k)}$ is easily constructed from $y^{(k)}$, so we only need to construct the paths $y^{(k)}$. Constructing those paths are not very difficult intuitively, as to construct $y^{(k)}$, we just need to "paste" together $x^{(k-1)}$ and x.

For $k = 1$ $y^{(1)} = \pi_{M([p]),M^{(1)}}(x)$ is a $H^{(1)} = G^{(1)}(V)/\exp(\mathcal{K}_1)$ valued path, which is $1/p$-Hölder.

We now assume that we have a $G^{(k)}(V)$-valued path $x^{(k)}$ which is $1/p$-Hölder and which satisfies equality (12), and we aim to construct a $1/p$-Hölder $H^{(k+1)}$-valued path $y^{(k+1)}$.

The set $Z^{(k+1)}$ defined by

$$\big\{(g,m) \in G^{(k)}(V) \times M^{(k+1)} \text{ such that } \pi_{G^{(k)}(V),M^{(k)}}(g) = \pi_{M^{(k+1)},M^{(k)}}(m)\big\}$$

equipped with the product

$$(g_1, m_1) \otimes (g_2, m_2) = (g_1 \otimes g_2, m_1 \otimes m_2)$$

is a group, and the application $\Psi : Z^{(k+1)} \to H^{(k+1)}$ defined by the formula: if $\ell_k \in V_1 \oplus \cdots \oplus V_k$ and $l^{k+1} \in V_{k+1}$,

$$\Psi\big(\exp(\ell_k), \exp\big(\ell_k + l^{k+1}\big)\exp\big(\mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_{k+1}\big)\big) = \exp(\ell_k)\exp\big(l^{k+1}\big)\exp(\mathcal{K}_{k+1}),$$

is easy seen to be an isomorphism by Baker–Campbell–Hausdorff formula.

Using Proposition 7 and Corollary 8, we see that there exists a constant C_{k+1} such that for all $(g,m) \in Z^{(k+1)}$,

$$\big\|\Psi(g,m)\big\|_{H^{(k+1)}} \leqslant C_{k+1}\big(\|g\|_{G^{(k)}(V)} + \|m\|_{M^{(k+1)}}\big).$$

For all $t \in [0,1]$, $(x_t^{(k)}, \pi_{M([p]),M^{(k+1)}}(x_t)) \in Z^{(k+1)}$, hence we can define

$$y_t^{(k+1)} = \Psi\big(x_t^{(k)}, \pi_{M([p]),M^{(k+1)}}(x_t)\big).$$

Note first that $y^{(k+1)}$ satisfies the equality (11). Because Ψ is an isomorphism, $y_{s,t}^{(k+1)} = \Psi(x_{s,t}^{(k)}, \pi_{M([p]),M^{(k+1)}}(x_{s,t}))$ and hence

$$\begin{aligned}\big\|y_{s,t}^{(k+1)}\big\|_{H^{(k+1)}} &\leqslant C_{k+1}\big(\big\|x_{s,t}^{(k)}\big\|_{G^{(k)}(V)} + \big\|\pi_{M([p]),M^{(k+1)}}(x_{s,t})\big\|_{M^{(k+1)}}\big)\\ &\leqslant C_{k+1}\big(\big\|x_{s,t}^{(k)}\big\|_{G^{(k)}(V)} + \|x_{s,t}\|_{M([p])}\big).\end{aligned}$$

By hypothesis and induction hypothesis,

$$\big\|x_{s,t}^{(k)}\big\|_{G^{(k)}(V)} + \|x_{s,t}\|_{M([p])} \leqslant (C + C_k')|t-s|^{1/p},$$

hence $\|y_{s,t}^{(k+1)}\|_{H^{(k+1)}} \leqslant C_{k+1}'|t-s|^{1/p}$.

Using the induction step until we reach the level $[p]$, we obtain a $G^{([p])}(V)$-valued path $x^{([p])}$ which is $1/p$-Hölder and such that

$$\pi_{G^{([p])}(V),M([p])}\big(x^{([p])}\big) = \pi_{M([p]),M([p])}(x) = x. \qquad \square$$

We ought to make a couple comments on our main theorem.

Remark 15. Note that we could have considered a continuous path of finite p-variation with values in a quotient space of $G^{(n)}(V)$, with $n > [p]$. If $K^{(n)}$ is a closed normal subgroup of $G^{(n)}(V)$ and x is a $1/p$-Hölder path with values in $(G^{(n)}(V)/K^{(n)}, \|\cdot\|_{G^{(n)}(V)/K^{(n)}})$, with $p \notin \mathbb{N}\backslash\{0,1\}$, then there exists a $1/p$-Hölder $G^{(n)}(V)$-valued path $\tilde{x}$ such that

$$\pi_{G^{(n)}(V),G^{(n)}(V)/K^{(n)}}(\tilde{x}) = x.$$

To prove this, first let $K^{([p])} = \pi_{G^{(n)},G^{([p])}}(K^{(n)})$. The canonical projection of x into $G^{[p]}(V)/K^{([p])}$ is a $1/p$-Hölder path. Hence, by the previous theorem, there exists a $1/p$-Hölder $G^{[p]}(V)$-valued path $x^{([p])}$ such that

$$\pi_{G^{([p])}(V),G^{[p]}(V)/K^{([p])}}\big(x^{([p])}\big) = \pi_{G^{(n)}(V)/K^{(n)},G^{[p]}(V)/K^{([p])}}(x).$$

Then, by Theorem 1 in [12], $x^{([p])}$ can be (uniquely) extended to a $1/p$-Hölder $(G^{(n)}(V), \|\cdot\|_{G^{(n)}(V)})$-valued path $x^{(n)}$. By the uniqueness statement, $x^{(n)}$ must satisfy

$$\pi_{G^{(n)}(V),G^{(n)}(V)/K^{(n)}}\big(x^{(n)}\big) = x.$$

Remark 16. As already pointed out in [12,13], if $p \geqslant 2$ and if there exists one $(G^{([p])}(V), \|\cdot\|_{G^{([p])}(V)})$-valued path $\tilde{x}$ of finite p variation such that

$$\pi_{G^{([p])}(V),G^{([p])}(V)/K}(\tilde{x}) = x,$$

then there exists infinitely many such paths.

Remark 17. The condition $p \notin \mathbb{N}\backslash\{0,1\}$ is necessary. In [18], it was proven that, for a particular choice of tensor norm, there does not exist a 2-rough path lying above the free Brownian motion (which is a path of finite 2-variation).

Remark 18. If p is a natural number greater than or equal to 2, keeping the notation of the previous theorem, we can find, for any fixed $\varepsilon > 0$, a continuous $G^{([p])}(V)$-valued path $\tilde{x}$ of finite $p+\varepsilon$ variation such that

$$\pi_{G^{(p)}(V),G^{(p)}(V)/K}(\tilde{x}) = x.$$

This is obtained just by noticing that a path of finite p-variation has finite $(p+\varepsilon)$-variation.

We end up with a corollary, which was the original motivation of this paper.

Corollary 19. *If $p \in [1,\infty)\backslash\{2,3,\ldots\}$, a continuous V-valued path of finite p-variation can be lifted to a geometric p-rough path. For any p, a continuous path of finite p-variation can be lifted to a geometric $(p+\varepsilon)$-rough path.*

Proof. Apply Theorem 14 to $K = \exp(\bigoplus_{i=2}^{[p]} V_i)$ and use the previous remark. □

That means, in particular, that one can always define a notion of solution to differential equations controlled by a continuous path of finite p-variation, whatever the p is.

6. Rough differential equations for which the extension does not matter

We fix a real $p \geqslant 1$. $\mathcal{X}^{k+\varepsilon}(\mathbb{R}^d)$ denotes the class of k-times differentiable vector fields with the kth-derivatives being ε-Hölder and with all the first k-derivatives being bounded. We consider $A_1,\ldots,A_m$ some elements of $\mathcal{X}^{\gamma}(\mathbb{R}^d)$, with $\gamma > p$. We fix a basis $e_1,\ldots,e_m$ of $\mathbb{R}^m$, and extend the linear application

$$\mathbb{R}^m \to \mathcal{X}^{\gamma}(\mathbb{R}^d),$$
$$e_i \to A_i$$

to an algebra homomorphism $\mathcal{F}^A_{[p]}$ from $T^{[p]}(\mathbb{R}^m)$ into the space of continuous differential operators, in other words, for a smooth function g, we have

$$\mathcal{F}^A_{[p]}\Big(\sum \alpha_{i_1,\ldots,i_n} e_{i_1} \otimes \cdots \otimes e_{i_n}\Big) g = \sum \alpha_{i_1,\ldots,i_n} A_{i_1}\cdots A_{i_n} g.$$

Note that $\mathcal{F}^A_{[p]}$ restricted to the free Lie algebra $\mathcal{G}^{([p])}(\mathbb{R}^m)$, i.e. $(\mathcal{F}^A_{[p]})_{|\mathcal{G}^{([p])}(\mathbb{R}^n)}$ is a Lie homomorphism into $\mathcal{X}^0(\mathbb{R}^d)$.

Recall that if $\mathbf{x}$ is a p-geometric rough path, a solution of the differential equation

$$\mathrm{d}\mathbf{y}_t = A(y_t)\,\mathrm{d}\mathbf{x}_t,$$
$$y_0 = a$$

is an extension of $\mathbf{x}$ to $\mathbf{z} \in G\Omega(\mathbb{R}^{m+d})$ that projects onto $(\mathbf{x},\mathbf{y})$, $(x_0,y_0) = (0,a)$, and such that

$$\mathbf{z}_{s,t} = \int_s^t h(z_u)\,\mathrm{d}\mathbf{z}_u,$$

with

$$h : \mathbb{R}^d \oplus \mathbb{R}^m \to \mathrm{Hom}\big(\mathbb{R}^d \oplus \mathbb{R}^m, \mathbb{R}^d \oplus \mathbb{R}^m\big),$$
$$(x, y) \to \big((\mathrm{d}X, \mathrm{d}Y) \to (\mathrm{d}X, V(y)\,\mathrm{d}X)\big).$$

The map $\mathbf{x} \to \mathbf{z}$ is called the Itô map, denoted $I_V : G\Omega(\mathbb{R}^n) \to G\Omega(\mathbb{R}^{n+d})$.

Theorem 20. *Let x be a $1/p$-Hölder path in $G^{([p])}(\mathbb{R}^m)/K$, where K is a normal subgroup of $G^{([p])}(\mathbb{R}^m)$ with Lie algebra $\mathcal{K}$ and $\tilde{\mathbf{x}}$ an extension of $\mathbf{x}$ to a $1/p$-Hölder path in $G^{([p])}(\mathbb{R}^m)$ ($1/(p+\varepsilon)$ if p is an integer). Assume that the kernel of the Lie algebra homomorphism $(\mathcal{F}^A_{[p]})_{|\mathcal{G}^{([p])}(\mathbb{R}^m)}$ contains $\mathcal{K}$. Then $I_A(\tilde{\mathbf{x}})$ is a $1/p$-Hölder path in $G^{([p])}(\mathbb{R}^{m+d})$ which, in general depends on the extension of x to $\mathbf{x}$. Nevertheless, the projection of $I_A(\mathbf{x})$ onto $\mathbb{R}^d$ depends only on x.*

Proof. To see that $I_A(\mathbf{x})$ depends on general of the extension of x to $\mathbf{x}$, just consider the Itô map which is the identity. Now let y be the projection of $I_A(\mathbf{x})$ onto $\mathbb{R}^d$. From [12], we know that

$$\big|y_t - \mathcal{F}^A_{[p]}(\mathbf{x}_{s,t})(y_s)\big| \leqslant C|t-s|^\theta, \tag{13}$$

where $\theta > 1$ and $C \geqslant 0$. Define the path y^n by the inductive formula

$$y^n_0 = a,$$
$$y^n_{\frac{k+1}{2^n}} = \mathcal{F}^A_{[p]}(\mathbf{x}_{\frac{k}{2^n},\frac{k+1}{2^n}})\big(y^n_{\frac{k+1}{2^n}}\big), \quad k = 0,\dots,2^n-1,$$
$$y^n_t = \big(k+1-2^n t\big)y^n_{\frac{k}{2^n}} + \big(2^n t - k\big)y^n_{\frac{k+1}{2^n}}, \quad t \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right].$$

By Eq. (13) and an argument similar to Euler construction of a solution to an ordinary differential equation, we see that y^n converges to y in uniform topology. Due to our assumption on the vector fields $A_1,\dots,A_m$, $\mathcal{F}^A_{[p]}(\mathbf{x}_{s,t})(y)$ only depends on $x_{s,t}$ (and not on the choice of the lift). In particular, y^n does not depend on the choice of the lift. Letting n tends to infinity, we obtain our theorem. □

A simple case of the above is the following:

Example 21. Let $A_1,\dots,A_d$ be d vector fields which commute, i.e. such that $[A_i, A_j] = 0$ for all i, j. Let $x : [0,1] \to \mathbb{R}^d$ be a continuous path of finite p-variation, lifted to a geometric $(p+\varepsilon)$-rough path $\mathbf{x}$. Then, the projection of $I_A(\mathbf{x})$ into $\mathbb{R}^d$ depends only on x, and not on the choice of the lift. This could be seen more directly from Doss–Sussman's theorem [4,16]

The following less trivial example should illustrate a bit more the interest of Theorem 20.

Example 22. Let A_1, A_2, A_3 be 3 vector fields, such that $[A_1, A_2] = [A_1, A_3] = 0$, but we do not assume that $[A_2, A_3]$ is equal to 0. Let $x = (x^1, x^2, x^3) : [0,1] \to \mathbb{R}^3$ be a $1/p$-Hölder path ($p > 2$), equipped with a Levy area $A^{2,3}$ between x^2 and x^3, such that $|A^{2,3}_{s,t}| \leqslant C|t-s|^{2/p}$. We lift (x, A) to a geometric $(p+\varepsilon)$-rough path $\mathbf{x}$. Then, the projection of $I_A(\mathbf{x})$ into $\mathbb{R}^d$ depends only on x and $A^{2,3}$, and not on the choice of the lift.

References

[1] R.F. Bass, B.M. Hambly, T.J. Lyons, Extending the Wong–Zakai theorem to reversible Markov processes, J. Eur. Math. Soc. 4 (2002) 237–269.

[2] M. Capitaine, C. Donati-Martin, The Lévy area process for the free Brownian motion, J. Funct. Anal. 179 (1) (2001) 153–169.

[3] L. Coutin, Z. Qian, Stochastic analysis, rough path analysis and fractional Brownian motions, Probab. Theory Related Fields 122 (1) (2002) 108–140.

[4] H. Doss, Liens entre équations différentielles stochastiques et ordinaires, Ann. Inst. H. Poincaré 13 (1977) 99–125.

[5] G.B. Folland, E.M. Stein, Hardy spaces on homogeneous groups, Math. Notes 28 (1982).

[6] P. Friz, N. Victoir, On the notion of geometric rough paths, preprint, 2004.

[7] M. Gromov, Carnot–Caratheodory spaces seen from within, in: A. Bellaiche, J.-J. Risler (Eds.), Sub-Riemannian Geometry, in: Progress in Mathematics, vol. 144, Birkhäuser, 1996, pp. 79–323.
[8] B.M. Hambly, T.J. Lyons, Stochastic area for Brownian motion on the Sierpinski gasket, Ann. Probab. 26 (1) (1998) 132–148.
[9] I. Karatzas, S.E. Shreve, Brownian Motion and Stochastic Calculus, second ed., Graduate Texts in Mathematics, vol. 113, Springer-Verlag, New York, 1991.
[10] M. Ledoux, T. Lyons, Z. Qian, Lévy area of Wiener processes in Banach spaces, Ann. Probab. 30 (2) (2002) 546–578.
[11] A. Lejay, Introduction to Rough Paths, Séminaire de probabilités, Lecture Notes in Mathematics, vol. XXXVII, 2003.
[12] T. Lyons, Differential equations driven by rough signals, Rev. Mat. Iberoamericana 14 (2) (1998) 215–310.
[13] T. Lyons, Z. Qian, System Control and Rough Paths, Oxford University Press, 2002.
[14] C. Reutenauer, Free Lie Algebras, London Mathematical Society Monographs (N.S.), vol. 7, Oxford Science Publications, 1993.
[15] J.P. Serre, Lie Algebras and Lie Groups, Lecture Notes in Mathematics, vol. 1500, 1992.
[16] H.J. Sussman, On the gap between deterministic and stochastic ordinary differential equations, Ann. Probab. 6 (1978) 19–41.
[17] V.S. Varadarajan, Lie Groups, Lie Algebras, and their Representations, Graduate Texts in Mathematics, vol. 102, 1984.
[18] N.B. Victoir, Levy area for the free Brownian motion: existence and non-existence, J. Funct. Anal., in press.