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Erratum

Erratum to: "Classical solvability in dimension two of the second boundary value problem associated with the Monge–Ampère operator" [Ann. Inst. H. Poincaré Analyse Non Linéaire 8 (5) (1991) 443–457]

Ph. Delanoë [1]

Univ. Nice-Sophia Antipolis, Laboratoire Dieudonné, Parc Valrose, 06108 Nice Cedex 2, France

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Recently, Simon Brendle (whom I would like to thank) pointed out to me that the assertion $u_t = tu_1 + (1-t)u_0 \in \mathrm{S}(\mathrm{D}, \mathrm{D}^*)$ made in [1, p. 449, 14 lines from top] is incorrect (unless $u_1 - u_0$ is constant). So we must fix the uniqueness proof in which it enters. Since uniqueness has been asserted without proof in several subsequent articles where the same nonlinear boundary condition is considered (see e.g. [3–7]), we will provide a fairly general proof valid for all.

We require a lemma, nowhere stated in that generality although its proof (given here for completeness) has become standard [6, pp. 870–871], [7, p. 65]:

Lemma 1 *(strict obliqueness). Let* D *(resp.* D^**) be a bounded domain of* $\mathbb{R}^n$ *(resp. of* $(\mathbb{R}^n)^*$*) with* C^2 *(resp.* C^1*) boundary. The boundary condition* $du(\mathrm{D}) = \mathrm{D}^*$*, considered on real functions* $u \in C^2(\bar{\mathrm{D}})$ *which are strictly convex (meaning they have a positive definite Hessian matrix at each point) on* $\bar{\mathrm{D}}$*, this condition, is strictly oblique.*

Proof. Fix $u \in C^2(\bar{\mathrm{D}})$ strictly convex and $x_0 \in \partial\mathrm{D}$. Set $p_0 = du(x_0) \in \partial\mathrm{D}^*$ and h^* for a C^1 real function defined in $(\mathbb{R}^n)^*$ near p_0 and satisfying on $\partial\mathrm{D}^*$: $h^* = 0$ and $dh^* \neq 0$. Consider the vector field: $x \in \partial\mathrm{D} \to \xi_u(x) := dh^*[du(x)]$ near x_0. Finally, denote by a dot (resp. by ∇) the standard euclidean scalar product (resp. the canonical flat connection) of $\mathbb{R}^n$, by N, the outward unit normal field to $\partial\mathrm{D}$ and set $H_u := h^* \circ du$.

The asserted strict obliqueness means: $\xi_u \cdot N(x_0) \neq 0$. To establish it, note that $dH_u(x) = (\nabla du)(\xi_u, \cdot)$ does not vanish, while $H_u(x) = 0$ on $\partial\mathrm{D}$ near x_0: so there, the 1-form $\pm dH_u/|dH_u|$ (setting $|\cdot|$ for the standard euclidean norm) is equal to the euclidean scalar product with N. In other words, at x_0, we have: $\xi_u \cdot N = \pm(1/|dH_u|)(\nabla du)(\xi_u, \xi_u)$ which indeed does not vanish. □

Proposition 1 *(uniqueness). Assume for* D^* *the existence of a global convex function* $h^* \in C^1(\overline{\mathrm{D}^*})$ *such that:* $h^* = 0$ *and* $dh^* \neq 0$ *on* $\partial\mathrm{D}^*$*. Let* $u \mapsto F(u)$ *be a second order (possibly nonlinear) differential operator on* D *satisfying at each strictly convex* $u \in C^2(\bar{\mathrm{D}})$ *the following conditions:*

(i) *$F(u)$ is well-defined on* $\bar{\mathrm{D}}$*;*

E-mail address: Philippe.DELANOE@unice.fr.

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(ii) $dF(u)$ *is* (*linear*) *elliptic with positive-definite symbol in* $\bar{\mathrm{D}}$;
(iii) $dF(u)(1) \leqslant 0$.

Then there exists at most one strictly convex solution $u \in C^2(\bar{\mathrm{D}})$ *of the problem*:

$$F(u) = 0 \quad \text{in } \mathrm{D}, \qquad du(\mathrm{D}) = \mathrm{D}^*, \tag{1}$$

unless $dF(u)(1) \equiv 0$*, in which case the solution is defined up to an additive constant.*

Proof. If u_0 and u_1 are two strictly convex solutions of (1) in $C^2(\bar{\mathrm{D}})$, for $t \in [0, 1]$, set $u_t = u_0 + tv$ with $v = u_1 - u_0$. Under the assumption made on F and since u_t is strictly convex, we may write as usual: $F(u_1) - F(u_0) = L(v)$, where $L := \int_0^1 dF(u_t)\,dt$ is a second order linear elliptic operator with positive definite symbol and $L(1) \leqslant 0$, throughout $\bar{\mathrm{D}}$; moreover, v satisfies $Lv = 0$ in D. To exploit the boundary condition, we fix $x \in \partial\mathrm{D}$, set for short $p_t = du_t(x)$ and observe that, by the convexity of h^*, we have:

$$dh^*(p_0)(p_1 - p_0) \leqslant h^*(p_1) - h^*(p_0) \leqslant dh^*(p_1)(p_1 - p_0),$$

hence:

$$dv(x)\big[\xi_0(x)\big] \leqslant 0 \leqslant dv(x)\big[\xi_1(x)\big], \tag{2}$$

where $\xi_i := \xi_{u_i}$ for $i \in \{0, 1\}$ (with the notation ξ_u introduced in the proof of Lemma 1). The left (resp. right) inequality of (2), used at the point $x = x_{\max}$ (resp. $x = x_{\min}$) where the function v assumes its maximum (resp. minimum) on $\partial\mathrm{D}$, and combined with the strict obliqueness of the ξ_i's (Lemma 1), implies:

$$\frac{\partial v}{\partial N}(x_{\max}) \leqslant 0, \qquad \frac{\partial v}{\partial N}(x_{\min}) \geqslant 0.$$

Now the proposition readily follows from Hopf's lemma combined with his strong maximum principle [2]. □

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