Available online at www.sciencedirect.com

SciVerse ScienceDirect

Ann. I. H. Poincaré – AN 30 (2013) 547–571

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Homogenization of convex functionals which are weakly coercive and not equi-bounded from above

Marc Briane [a,∗], Juan Casado-Díaz [b]

[a] *Institut de Recherche Mathématique de Rennes, INSA de Rennes, France*
[b] *Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Universidad de Sevilla, Spain*

Received 17 February 2012; received in revised form 19 October 2012; accepted 23 October 2012

Available online 15 November 2012

Abstract

This paper deals with the homogenization of nonlinear convex energies defined in $W_0^{1,1}(\Omega)$, for a regular bounded open set Ω of $\mathbb{R}^N$, the densities of which are not equi-bounded from above, and which satisfy the following weak coercivity condition: There exists $q > N-1$ if $N > 2$, and $q \geqslant 1$ if $N = 2$, such that any sequence of bounded energy is compact in $W_0^{1,q}(\Omega)$. Under this assumption the Γ-convergence of the functionals for the strong topology of $L^\infty(\Omega)$ is proved to agree with the Γ-convergence for the strong topology of $L^1(\Omega)$. This leads to an integral representation of the Γ-limit in $C_0^1(\Omega)$ thanks to a local convex density. An example based on a thin cylinder with very low and very large energy densities, which concentrates to a line shows that the loss of the weak coercivity condition can induce nonlocal effects.

MSC: 35B27; 35B50; 35J60

Keywords: Homogenization; Convex functionals; Nonlinear elliptic equations; Weak coercivity; Maximum principle

1. Introduction

Since the beginning of the seventies the homogenization theory has greatly developed through the G-convergence of operators [30], the H-convergence of PDE's [29] (see also [31] and the references therein), and the Γ-convergence of functionals [19,21] (see also [18] for a review and the references therein). The De Giorgi Γ-convergence has been a powerful mathematical tool for studying the asymptotic behavior of minima of functionals defined for a regular bounded open set Ω of $\mathbb{R}^N$, by

$$F_n(v) := \int_\Omega f_n(x, \nabla v)\, dx, \quad \text{for } v \in W_0^{1,1}(\Omega). \tag{1.1}$$

∗ Corresponding author.
E-mail addresses: mbriane@insa-rennes.fr (M. Briane), jcasadod@us.es (J. Casado-Díaz).

0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2012.10.005

The seminal results in this sense were obtained in [20,15,17]. Assuming that f_n is convex with respect to the second argument and satisfies the boundedness from above:

$$f_n(x,\xi) \leqslant a_n(x)\big(1+|\xi|^p\big), \quad \text{for a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N,\ \forall n \in \mathbb{N}, \tag{1.2}$$

for a fixed $p > 1$ and for a given nonnegative bounded sequence a_n in $L^1(\Omega)$, any Γ-limit F of F_n for the topology of $C_0^0(\Omega)$ was shown in [15,17] to have a similar integral representation, namely

$$F(v) = \int_\Omega f(x,\nabla v)\,d\mu, \quad \text{for } v \in C_0^1(\Omega), \tag{1.3}$$

where f is convex with respect to the second argument and μ is a Radon measure on $\bar{\Omega}$. Under the additional assumption of equi-integrability of the sequence a_n the previous representation also holds for the strong topology of $L^1(\Omega)$ as shown first in [20]. A few years later, it was proved in [22] that the loss of equi-integrability for equicoercive quadratic densities f_n may induce nonlocal effects in dimension three. A connection between this type of degeneracy and the Beurling–Deny [5] representation formula of the Dirichlet forms was established in [28] for quadratic functionals. Then, the closure set of the three-dimensional quadratic functionals with respect to the Γ-convergence for the strong topology of $L^2(\Omega)$ was obtained in [16] according to the Beurling–Deny theory. In the same spirit, the three-dimensional examples from [4] of $W^{1,p}(\Omega)$-equicoercive functionals for $p > 1$, i.e.

$$\exists C > 0,\ \forall n \in \mathbb{N},\ \forall v \in C_c^1(\Omega), \quad F_n(v) \geqslant C \int_\Omega |\nabla v|^p\,dx, \tag{1.4}$$

show no degeneracy of their Γ-limits for the strong topology of $L^p(\Omega)$ provided that $p > 2$, while nonlocal effects appear when $p \in (1,2]$, like in [22,4]. On the contrary, the case of dimension two with equicoercive functionals is quite different, since it was proved in [12–14] for quadratic functionals, and in [8] for $W^{1,p}$-equicoercive, $p > 1$, convex periodic functionals, that the Γ-limits have a representation of type (1.3) in dimension two. On the other hand, the loss of coercivity may also induce degenerate limit behaviors in terms of coupled systems as shown for example in [24,28,25,9,7]. Also note that in periodic homogenization very weak coercivity conditions including perforated domains were treated using extension operators, weak notions of connectedness or multi-scale convergence approaches (see, e.g., [1,2,10,32,33]). From a certain point of view all these works deal with the same question:

Under what conditions the Γ-limits of convex functionals of type (1.1) *remain of type* (1.3)?

The present work is an attempt to give a unified answer in any dimension $N \geqslant 2$, for sequences of convex functionals F_n the densities of which are neither equi-bounded from above nor equi-bounded from below. Our approach is based on the combination of three independent results:

- In Section 2 we recover the result of [17] (see Theorem 2.4) but replacing the Γ-convergence for the topology of $C_0^0(\Omega)$ by the Γ-convergence for the strong topology of $L^\infty(\Omega)$. We also make an assumption on the convex densities f_n, which is less restrictive than (1.2) (see conditions (2.11), (2.12), and Remark 2.6), and needs an alternative approach.
- In Section 3 we establish a general framework (see Corollary 3.5) in which the Γ-convergence for the strong topology of $L^\infty(\Omega)$ agrees with the Γ-convergence for the strong topology of $L^1(\Omega)$. This is the most original part of the paper. The strong equicoercivity condition (1.4) is now replaced by the following weaker condition: There exists a real number q with $q > N-1$ if $N > 2$, and $q = 1$ if $N = 2$, such that

$$\begin{cases} \forall n \in \mathbb{N},\ \forall c > 0, & \{F_n \leqslant c\} \text{ is sequentially compact in } W_0^{1,q}(\Omega) \text{ weak},\\ \forall u_n \in W_0^{1,1}(\Omega), & \limsup\limits_{n\to\infty} F_n(u_n) < \infty \Rightarrow \begin{array}{l} u_n \text{ converges weakly in } W_0^{1,q}(\Omega), \\ \text{up to a subsequence,}\end{array} \end{cases} \tag{1.5}$$

  which holds for a large class of functions f_n (see Proposition 3.2). Under this condition we prove (see Theorem 3.4) a uniform convergence for (roughly speaking) minimizers u_n of F_n which converge weakly in $W_0^{1,q}(\Omega)$ to a function in $C^0(\bar{\Omega})$. The key ingredient is a maximum principle type result (see Lemma 3.7) following the idea of [27] (see also [23]), which allows us to deduce a uniform estimate for u_n from the compact embedding of $W^{1,q}(\partial B)$ into $C^0(\partial B)$ for any ball $B \subset \Omega$, due to the condition on q. The Aubin compactness theorem [3]

is also used in the case $N > 2$ and $q > N - 1$, while it is replaced by the Kuratowski, Ryll-Nardzewski selection theorem [26] in the much more delicate case $N = 2$ and $q = 1$.
- Section 4 is devoted to a counter-example, separating the cases $N > 2$ (Theorem 4.2) and $N = 2$ (Theorem 4.4), which shows that the weak coercivity condition (1.5) is actually crucial to obtain the local Γ-limit representation (1.3). Indeed, the loss of condition (1.5) may induce nonlocal effects. The counter-example is based on a columnar structure like in [22,4]. But contrary to the three-dimensional periodic fiber reinforcement of [22,4], here the energy density f_n takes both very low and very large values in one cylinder if $N > 2$, and one strip if $N = 2$, which concentrates along a line as n tends to ∞. Based on the counter-example the importance of the weak coercivity condition (1.5) as well as the more precise conditions of Proposition 3.2, for deriving a local Γ-limit is discussed in Remark 4.1.

Therefore, the three previous results allow us to answer to the above question through the following:

Theorem 1.1. *Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geqslant 2$, with a Lipschitz boundary. Consider a sequence of nonnegative functions $f_n : \Omega \times \mathbb{R}^N \to [0, \infty)$, $n \in \mathbb{N}$, satisfying the properties* (2.1)–(2.4), (2.11), (2.12) *below. Also assume that the associated convex functional F_n defined by* (1.1) *satisfies the condition* (1.5).

Then, there exist a subsequence of n, still denoted by n, a Radon measure μ on $\bar{\Omega}$, and a function $f : \Omega \times \mathbb{R}^N \to [0, \infty)$ satisfying the properties (2.13)–(2.16) *below, such that the sequence F_n Γ-converges in $C_0^1(\Omega)$ for the strong topology of $L^1(\Omega)$ to the functional F defined by* (1.3).

Focus on the particular two-dimensional case with quadratic densities $f_n(x, \xi) = A_n(x)\xi \cdot \xi$, for $(x, \xi) \in \Omega \times \mathbb{R}^2$, where A_n is a sequence of positive definite symmetric matrix-valued functions defined on Ω. Then, Theorem 1.1 and Proposition 3.2 for $N = 2$ lead to a local Γ-limit of type (1.3) under the sole assumption that the inverse of the smallest eigenvalue λ_n of A_n is bounded and equi-integrable in $L^1(\Omega)$, without any prescribed bound from above. Moreover, the two-dimensional counter-example of Section 4 (see Theorem 4.4) shows that the equi-integrability of λ_n^{-1} in $L^1(\Omega)$ is actually essential. This extends the result of [14] obtained through an approach based on the Dirichlet forms, but for a sequence λ_n which is bounded from below by a positive constant.

A few recalls and notations

We recall the definition of the De Giorgi Γ-convergence and some of its properties which will be used in the sequel. We refer to [18] for an exhaustive presentation of Γ-convergence (see also [6] for an elementary approach).

Definition 1.2. Let V be a metric space, and let $F_n : V \to [0, \infty]$, $n \in \mathbb{N}$, be a sequence of functionals. For $v \in V$, F_n is said to Γ-converge to $F(v) \in [0, \infty]$ at v if

i) the Γ-liminf inequality holds

$$\forall v_n \to v \text{ in } V, \quad F(v) \leqslant \liminf_{n\to\infty} F_n(v_n), \tag{1.6}$$

ii) the Γ-limsup inequality holds

$$\exists \bar{v}_n \to v \text{ in } V, \quad F(v) = \lim_{n\to\infty} F_n(\bar{v}_n). \tag{1.7}$$

Any sequence satisfying (1.7) is called a recovery sequence for F_n of limit v.

Let W be a subset of V. The sequence F_n is said to Γ-converge in W to $F : W \to [0, \infty]$ if for any $v \in W$, F_n Γ-converges to $F(v)$ at v.

Notations

- S_{N-1} denotes the unit sphere of $\mathbb{R}^N$ for any integer $N \geqslant 2$.
- $|E|$ denotes the Lebesgue measure of any measurable set $E \subset \mathbb{R}^N$.
- $\fint_E = \frac{1}{|E|} \int_E$ denotes the average-value over a measurable set $E \subset \mathbb{R}^N$.

- For any bounded open set Ω of $\mathbb{R}^N$, $C^1(\bar{\Omega})$ denotes the space of the restrictions to $\bar{\Omega}$ of the functions in $C_c^1(\mathbb{R}^N)$. Note that $C^1(\bar{\Omega})$ is not generally a Banach space if Ω is not regular. But this property will not be used.
- $\mathscr{M}(X)$ denotes the set of the Radon measures on a locally compact set X.

2. Γ-convergence in L^∞

Let Ω be a bounded open set of $\mathbb{R}^N$. Let $f_n, g_n : \Omega \times \mathbb{R}^N \to [0, \infty)$, $n \in \mathbb{N}$, be two sequences of nonnegative functions satisfying:

$$f_n(\cdot, \xi),\ g_n(\cdot, \xi) \text{ are measurable for any } \xi \in \mathbb{R}^N \quad \text{and} \quad f_n(\cdot, 0) = g_n(\cdot, 0) = 0, \tag{2.1}$$

$$f_n(x, \cdot),\ g_n(x, \cdot) \text{ are convex for a.e. } x \in \Omega, \tag{2.2}$$

$$f_n(x, \xi) \leqslant g_n(x, \xi) \quad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N,\ \forall n \in \mathbb{N}, \tag{2.3}$$

there exist $K \geqslant 2$ and a nonnegative bounded sequence b_n in $L^1(\Omega)$ such that

$$g_n(x, 2\xi) \leqslant K g_n(x, \xi) + b_n \quad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N,\ \forall n \in \mathbb{N}. \tag{2.4}$$

An easy consequence of (2.4) is the following estimate:

Proposition 2.1. *There exists a constant $\rho \geqslant 1$ such that*

$$g_n(x, t\xi) \leqslant K t^\rho \big(g_n(x, \xi) + b_n\big) \quad \text{a.e. } x \in \Omega,\ \forall t \geqslant 1,\ \forall \xi \in \mathbb{R}^N,\ \forall n \in \mathbb{N}. \tag{2.5}$$

Remark 2.2. Conversely to Proposition 2.1, estimate (2.5) implies (2.4) replacing K by $2^\rho K$. Also note that by convexity we have

$$g_n(x, t\xi) \leqslant t g_n(x, \xi) \quad \text{a.e. } x \in \Omega,\ \forall t \in [0, 1],\ \forall \xi \in \mathbb{R}^N,\ \forall n \in \mathbb{N}. \tag{2.6}$$

On the other hand, taking into account the convexity of g_n and (2.4), the following inequality holds

$$g_n(x, \xi + \eta) \leqslant \frac{K}{2}\big(g_n(x, \xi) + g_n(x, \eta)\big) + b_n \quad \text{a.e. } x \in \Omega,\ \forall \xi, \eta \in \mathbb{R}^N,\ \forall n \in \mathbb{N}. \tag{2.7}$$

Remark 2.3. Despite of the convexity and the inequality $f_n \leqslant g_n$, the sequence f_n does not satisfy in general a bound of type (2.4). Indeed, consider the following example:

Let $\theta : \mathbb{R} \to [0, \infty)$ be defined by

$$\theta(t) := \begin{cases} t^2 & \text{if } t \leqslant 1, \\ (t - \sqrt{k!})(\sqrt{(k+1)!} + \sqrt{k!}) + k! & \text{if } t \in [\sqrt{k!}, \sqrt{(k+1)!}],\ k \in \mathbb{N}. \end{cases} \tag{2.8}$$

The function θ is convex, $\theta(\sqrt{k!}) = k!$ for any $k \in \mathbb{N}$, and $\theta(t) \leqslant t^4 + 1$ for any $t \in \mathbb{R}$. Now define the function $f_n : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ by

$$f_n(x, \xi) := a_n(x)\big(\theta(\xi_1) + |\xi_2|^4 + \cdots + |\xi_N|^4\big) \quad \text{for } (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N, \tag{2.9}$$

where $a_n : \mathbb{R}^N \to (0, \infty)$ is a positive function. Therefore, we have

$$f_n(x, \xi) \leqslant g_n(x, \xi) := a_n(x)\big(|\xi|^4 + 1\big) \quad \text{for any } (x, \xi) \in \mathbb{R}^N \times \mathbb{R}^N,$$

hence conditions (2.1)–(2.4) are clearly satisfied. However, we have for $\xi_n := (\sqrt{n!}, 0, \ldots, 0)$,

$$\frac{f_n(2\xi_n)}{f_n(\xi_n)} = \frac{\theta(2\sqrt{n!})}{\theta(\sqrt{n!})} \underset{n\to\infty}{\approx} (\sqrt{2} - 1)\sqrt{n},$$

which shows that f_n cannot satisfy a bound of type (2.4).

Let Ω be a bounded open set of $\mathbb{R}^N$, with a Lipschitz boundary. Consider the sequence of functionals F_n defined by

$$F_n(u) := \begin{cases} \int_\Omega f_n(x,\nabla u)\,dx & \text{if } u \in W^{1,1}(\Omega)\cap L^\infty(\Omega), \\ \infty & \text{if } u \in L^\infty(\Omega)\setminus W^{1,1}(\Omega), \end{cases}$$
$$G_n(u) := \begin{cases} \int_\Omega g_n(x,\nabla u)\,dx & \text{if } u \in W^{1,1}(\Omega)\cap L^\infty(\Omega), \\ \infty & \text{if } u \in L^\infty(\Omega)\setminus W^{1,1}(\Omega). \end{cases} \tag{2.10}$$

We make the following assumption: for any $x \in \bar\Omega$, there exist $N+1$ functions $w^i \in C^1(\bar\Omega)$, $0 \leqslant i \leqslant N$, such that

$$0 \text{ belongs to the interior of the convex envelop of } \big(\nabla w^0(x),\dots,\nabla w^N(x)\big), \tag{2.11}$$

and $N+1$ sequences $w_n^i \in W^{1,1}(\Omega)\cap L^\infty(\Omega)$ such that

$$w_n^i \to w^i \text{ strongly in } L^\infty(\Omega) \quad \text{and} \quad G_n\big(w_n^i\big) \text{ is bounded.} \tag{2.12}$$

We have the following representation result:

Theorem 2.4. *Assume that* (2.1), (2.2), (2.3), (2.4)*, and* (2.11), (2.12) *hold. Then, there exist a subsequence of n, still denoted by n, a Radon measure μ on $\bar\Omega$, and two functions $f, g : \bar\Omega \times \mathbb{R}^N \to [0,\infty)$ satisfying the following properties*:

$$f(\cdot,\xi),\ g(\cdot,\xi) \text{ are } \mu\text{-measurable for any } \xi \in \mathbb{R}^N \quad \text{and} \quad f(\cdot,0) = g(\cdot,0) = 0, \tag{2.13}$$
$$f(x,\cdot),\ g(x,\cdot) \text{ are convex for } \mu\text{-a.e. } x \in \bar\Omega, \tag{2.14}$$
$$f(x,\xi) \leqslant g(x,\xi) \quad \mu\text{-a.e. } x \in \bar\Omega,\ \forall \xi \in \mathbb{R}^N,\ \forall n \in \mathbb{N}, \tag{2.15}$$
$$g(x,2\xi) \leqslant K g(x,\xi) + b \quad \mu\text{-a.e. } x \in \bar\Omega,\ \forall \xi \in \mathbb{R}^N,\ \forall n \in \mathbb{N}, \tag{2.16}$$

where K is the constant in (2.4) *and b is given by the convergence*

$$b_n \rightharpoonup b\mu \quad \text{weakly-}* \text{ in } \mathscr{M}(\bar\Omega), \tag{2.17}$$

such that the sequences F_n, G_n defined by (2.10) *Γ-converge in $C^1(\bar\Omega)$* (*see Definition* 1.2) *for the strong topology of $L^\infty(\Omega)$ to the functionals F, G given by*

$$F(u) := \int_{\bar\Omega} f(x,\nabla u)\,d\mu, \qquad G(u) := \int_{\bar\Omega} g(x,\nabla u)\,d\mu, \quad \text{for } u \in C^1(\bar\Omega). \tag{2.18}$$

Moreover, for any open set $\omega \subset \Omega$, the sequence of functionals F_n^ω, G_n^ω defined by

$$F_n^\omega(u) := \begin{cases} \int_\omega f_n(x,\nabla u)\,dx & \text{if } u \in W_0^{1,1}(\omega)\cap L^\infty(\omega), \\ \infty & \text{if } u \in L^\infty(\omega)\setminus W_0^{1,1}(\omega), \end{cases}$$
$$G_n^\omega(u) := \begin{cases} \int_\omega g_n(x,\nabla u)\,dx & \text{if } u \in W_0^{1,1}(\omega)\cap L^\infty(\omega), \\ \infty & \text{if } u \in L^\infty(\omega)\setminus W_0^{1,1}(\omega). \end{cases} \tag{2.19}$$

Γ-converge in $C_0^1(\omega)$ to the functionals F^ω, G^ω given by

$$F^\omega(u) := \int_\omega f(x,\nabla u)\,d\mu, \quad G^\omega(u) := \int_\omega g(x,\nabla u)\,d\mu, \quad \text{for } u \in C_0^1(\omega). \tag{2.20}$$

Remark 2.5. In the proof of Theorem 2.4 below we will also prove that for any $u \in C^1(\bar\Omega)$ and any recovery sequence u_n for F_n of limit u, the weak convergence of the energy density holds

$$f_n(\cdot,\nabla u_n) \rightharpoonup f(\cdot,\nabla u)\mu \quad \text{weakly-}* \text{ in } \mathscr{M}(\bar\Omega). \tag{2.21}$$

Remark 2.6. Carbone and Sbordone [17] obtained a representation formula for the Γ-convergence in $C^0(\Omega)$ of a sequence of convex functionals F_n the density of which satisfies

$$f_n(x,\xi) \leqslant a_n(x)\big(|\xi|^p + 1\big) \quad \text{a.e. } x \in \Omega,\ \forall \xi \in \mathbb{R}^N,\ \forall n \in \mathbb{N}, \tag{2.22}$$

where $p > 1$ and a_n is a bounded sequence in $L^1(\Omega)$. The condition (2.4) is sharper than (2.22). Indeed, the function $g_n(x,\xi) := a_n(x)(|\xi|^p + 1)$ clearly satisfies the inequality (2.4). Moreover, the L^1-boundedness of a_n in [17] is here replaced by the weaker condition (2.12). Indeed, it is easy to construct a sequence a_n which is not bounded in $L^1(\Omega)$ such that the extra condition (2.12) holds and for which the representation Theorem 2.4 applies. Think for example of the sequence $f_n(x,\xi) := (1 + \beta_n 1_{B(0,n^{-1})})|\xi|^p$, where $\beta_n n^{-N} \to \infty$.

Remark 2.7. Assumptions (2.11), (2.12) are needed to ensure that the domain D_F of the Γ-limit F of the sequence F_n contains the set of regular functions $C^1(\bar{\Omega})$. More precisely, at each point $x \in \bar{\Omega}$, the gradients of $(N+1)$ functions in $C^1(\bar{\Omega}) \cap D_F$ have to span a sufficiently large convex set in order to derive any regular function as an L^∞-limit of a sequence of bounded energy G_n in the neighborhood of x. This is given by the barycenter condition (2.11) combined with the convergence condition (2.12) which are the key ingredients of Lemma 2.10 below.

Lemma 2.8. *Let $u_n \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ which converges strongly to u in $L^\infty(\Omega)$. Then, there exists a sequence $\tilde{u}_n \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ which strongly converges to u in $L^\infty(\Omega)$ such that*

$$\tilde{u}_n = 0 \quad \text{a.e. in } \{u = 0\} \quad \text{and} \quad f_n(\cdot, \nabla \tilde{u}_n) \leqslant f_n(\cdot, \nabla u_n) \quad \text{a.e. in } \Omega. \tag{2.23}$$

Proof. Set $\varepsilon_n := \|u_n - u\|_{L^\infty(\Omega)}$. Then, the sequence $\tilde{u}_n$ defined by

$$\tilde{u}_n := \begin{cases} u_n + \varepsilon_n & \text{if } u_n < -\varepsilon_n, \\ 0 & \text{if } -\varepsilon_n \leqslant u_n \leqslant \varepsilon_n, \\ u_n - \varepsilon_n & \text{if } u_n > \varepsilon_n, \end{cases}$$

clearly satisfies (2.23). □

We have the following result:

Proposition 2.9. *Assume that conditions (2.1)–(2.4), and (2.11), (2.12) hold. Then, for any $u \in C^1(\bar{\Omega})$ there exists a sequence $u_n \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ such that*

$$u_n \to u \text{ strongly in } L^\infty(\Omega) \quad \text{and} \quad G_n(u_n) \text{ is bounded.} \tag{2.24}$$

Proof. We need the following lemma which is a simple extension of Proposition 4.4 in [8] to dimension $N \geqslant 2$, extending $g_n(x,\cdot)$ by 0 for $x \in \mathbb{R}^N \setminus \Omega$. So, we omit its proof.

Lemma 2.10. *For any $x_0 \in \bar{\Omega}$, there exist three constants $\varepsilon, \delta, C > 0$ such that for any u in $C^1(\bar{B}(x_0,\delta) \cap \bar{\Omega})$, with $\|\nabla u\|_{L^\infty(B(x_0,\delta)\cap\Omega)} \leqslant \varepsilon$, there exists a sequence u_n satisfying*

$$\begin{cases} u_n \in W^{1,1}\big(B(x_0,\delta)\cap\Omega\big) \cap L^\infty\big(B(x_0,\delta)\cap\Omega\big), \\ u_n \to u \text{ strongly in } L^\infty\big(B(x_0,\delta)\cap\Omega\big), \\ \displaystyle\sup_{n\geqslant 0} \int_{B(x_0,\delta)\cap\Omega} g_n(x,\nabla u_n)\,dx \leqslant C. \end{cases} \tag{2.25}$$

Due to the compactness of $\bar{\Omega}$, Lemma 2.10 implies the existence of k balls $B(z_i,\delta_i)$, for $i = 1,\dots,k$, covering $\bar{\Omega}$, and constants $\varepsilon, C > 0$ such that (2.25) holds in $B(z_i,\delta_i)$ for any u in $C^1(\bar{B}(z_i,\delta_i)\cap\bar{\Omega})$, with $\|\nabla u\|_{L^\infty(B(z_i,\delta_i)\cap\Omega)} \leqslant \varepsilon$, and any $i = 1,\dots,k$. Consider a partition of the unity φ^i, $1 \leqslant i \leqslant N$, such that $\varphi^i \in C_c^1(B(z_i,\delta_i))$, $0 \leqslant \varphi^i \leqslant 1$, $\sum_{i=1}^k \varphi^i = 1$ in $\bar{\Omega}$. Then, there exist k sequences u_n^i in $W^{1,1}(B(z_i,\delta_i)\cap\Omega)$ such that

$$\begin{cases} u_n^i \xrightarrow[n\to\infty]{} \dfrac{\varepsilon\varphi^i u}{\|\nabla(\varphi^i u)\|_{L^\infty(\Omega)} + 1} \text{ strongly in } L^\infty\big(B(z_i,\delta_i)\cap\Omega\big), \\ \displaystyle\int_{B(z_i,\delta_i)\cap\Omega} g_n\big(\cdot,\nabla u_n^i\big)\,dx \text{ is bounded.} \end{cases} \tag{2.26}$$

By Lemma 2.8 with the open set $B(z_i,\delta_i)\cap\Omega$, we can also assume that $u_n^i=0$ a.e. in $\{\varphi^i=0\}$. Therefore, extending u_n^i by 0 in $\Omega\setminus B(z_i,\delta_i)$, the sequence u_n defined by

$$u_n:=\sum_{i=1}^k \varepsilon^{-1}\big(\big\|\nabla\big(\varphi^i u\big)\big\|_{L^\infty(\Omega)}+1\big)u_n^i$$

strongly converges to u in $L^\infty(\Omega)$. Moreover, by the convexity of g_n and (2.5) combined with estimate (2.26) we get that

$$G_n(u_n)\leqslant\frac{1}{k}\sum_{i=1}^k\int_\Omega g_n\big(x,k\varepsilon^{-1}\big(\big\|\nabla\big(\varphi^i u\big)\big\|_{L^\infty(\Omega)}+1\big)u_n^i\big)\,dx\leqslant c.\qquad\square$$

Consider for any $\varphi\in C^1(\bar\Omega)$, the sequence of functionals F_n^φ, G_n^φ defined by

$$\begin{aligned}&F_n^\varphi(v):=\begin{cases}\int_\Omega\varphi f_n(x,\nabla v)\,dx & \text{if } v\in W^{1,1}(\Omega)\cap L^\infty(\Omega),\\ \infty & \text{if } v\in L^\infty(\Omega)\setminus W^{1,1}(\Omega),\end{cases}\\ &G_n^\varphi(v):=\begin{cases}\int_\Omega\varphi g_n(x,\nabla v)\,dx & \text{if } v\in W^{1,1}(\Omega)\cap L^\infty(\Omega),\\ \infty & \text{if } v\in L^\infty(\Omega)\setminus W^{1,1}(\Omega).\end{cases}\end{aligned}\tag{2.27}$$

These sequences allow us to derive local properties for the Γ-convergence of the sequences F_n, G_n.

Proposition 2.11. *Assume that conditions (2.1)–(2.4), and (2.11), (2.12) hold. Then, there exist a constant $M>0$ and a Radon measure μ on $\bar\Omega$, such that for any $\varphi\in C^1(\bar\Omega)$ with $\varphi\geqslant 0$, and for any $u\in C^1(\bar\Omega)$, there exists a sequence u_n in $W^{1,1}(\Omega)$ strongly converging to u in $L^\infty(\Omega)$ satisfying*

$$\limsup_{n\to\infty}G_n^\varphi(u_n)\leqslant M\big(\|\nabla u\|^\rho_{L^\infty(\operatorname{supp}\varphi)^N}+\|\nabla u\|_{L^\infty(\operatorname{supp}\varphi)^N}\big)\int_{\bar\Omega}\varphi\,d\mu.\tag{2.28}$$

Proof. Define the linear functions

$$w^0(x):=-\frac{1}{2N}\sum_{i=1}^N x_i\quad\text{and}\quad w^i(x):=x_i+w^0(x),\quad\text{for } 1\leqslant i\leqslant N.$$

By Proposition 2.9 there exist sequences w_n^i in $W^{1,1}$ strongly converging to w^i in $L^\infty(\Omega)$, with $F_n(w_n^i)$ bounded, for $i=0,\dots,N$. Define the Radon measure μ on $\bar\Omega$ by

$$\mu:=\nu+\sum_{i=0}^N\mu^i\quad\text{with}\ \begin{cases}b_n\rightharpoonup\nu,\\ g_n(\cdot,\nabla w_n^i)\rightharpoonup\mu^i\end{cases}\ \text{weakly-}*\text{ in }\mathscr{M}(\bar\Omega),\tag{2.29}$$

where the $N+2$ weak-$*$ convergences hold true up to a subsequence of n still denoted by n. Let $u\in C^1(\bar\Omega)$. We can assume that ∇u is non-zero in $\operatorname{supp}\varphi$, otherwise the sequence $u_n:=u$ does the job. Define the sequences z_n and u_n by

$$\begin{cases}z_n:=\big(w_n^1-w_n^0,\dots,w_n^N-w_n^0\big)\in W^{1,1}(\Omega)^N\cap L^\infty(\Omega)^N,\\ u_n:=u(z_n)+4N\gamma\big(w_n^0-w^0(z_n)\big),\quad\text{with }\gamma:=\|\nabla u\|_{L^\infty(\operatorname{supp}\varphi)^N}.\end{cases}\tag{2.30}$$

Since z_n converges strongly to the identity function in $L^\infty(\Omega)^N$, the sequence u_n clearly converges strongly to u in $L^\infty(\Omega)$. On the other hand, we have

$$\nabla u_n=4N\gamma\Bigg[\sum_{i=1}^N\frac{1}{4N}\bigg(2+\frac{\partial_i u(z_n)}{\gamma}\bigg)\nabla w_n^i+\frac{1}{4N}\bigg(2N-\sum_{i=1}^N\frac{\partial_i u(z_n)}{\gamma}\bigg)\nabla w_n^0\Bigg].$$

Since the term in brackets is a convex combination for large enough n (due to the strong convergence of $\partial_i u(z_n)$ to $\partial_i u$), the convexity of g_n together with estimates (2.5) and (2.6) yields

$$g_n(\cdot, \nabla u_n) \leqslant \big(K(4N\gamma)^\rho + 4N\gamma\big)\left(b_n + g_n\big(\cdot, \nabla w_n^0\big) + \sum_{i=1}^N g_n\big(\cdot, \nabla w_n^i\big)\right).$$

This combined with the definition (2.29) of the measure μ implies the desired estimate (2.28). □

Proposition 2.12. *Assume that conditions* (2.1)–(2.4)*, and* (2.11), (2.12) *hold. Consider $u \in C^1(\bar{\Omega})$ and a recovery sequence u_n for F_n at u strongly converging in $L^\infty(\Omega)$. Then, for any sequence v_n strongly converging to u in $L^\infty(\Omega)$ with $F_n(v_n)$ bounded, and for any $\varphi \in C^1(\bar{\Omega})$ with $\varphi \geqslant 0$, we have*

$$\liminf_{n\to\infty} \int_\Omega \varphi\big(f_n(x, \nabla v_n) - f_n(x, \nabla u_n)\big)\, dx \geqslant 0. \tag{2.31}$$

Remark 2.13. Proposition 2.12 implies some local property of the Γ-convergence of F_n, where the strong topology of $L^\infty(\Omega)$ plays an essential role (contrary to Proposition 2.11 which only gives an energy bound).

Proof of Proposition 2.12. Let $u \in C^1(\bar{\Omega})$. Consider a recovery sequence u_n for F_n strongly converging to u in $L^\infty(\Omega)$. Clearly it is enough to prove the result for any $\varphi \in C^1(\bar{\Omega})$ with $0 \leqslant \varphi \leqslant 1/2$. By Proposition 2.9 there exist two sequences $\tilde{\varphi}_n, \tilde{\psi}_n$ strongly converging respectively to $\varphi, -\varphi$ in $L^\infty(\Omega)$ with $F_n(\tilde{\varphi}_n), F_n(\tilde{\psi}_n)$ bounded. Then, the truncations $\varphi_n := (0 \vee \tilde{\varphi}_n) \wedge 1/2$, $\psi_n := (-1/2 \vee \tilde{\psi}_n) \wedge 0$ strongly converge respectively to $\varphi, -\varphi$ in $L^\infty(\Omega)$, and satisfy

$$0 \leqslant \varphi_n, -\psi_n \leqslant 1/2 \quad \text{and} \quad g_n(\cdot, \varphi_n) \leqslant g_n(\cdot, \tilde{\varphi}_n), \quad g_n(\cdot, \psi_n) \leqslant g_n(\cdot, \tilde{\psi}_n) \text{ a.e. in } \Omega,$$

so that the energies $G_n(\varphi_n), G_n(\psi_n)$ are bounded.

Consider a sequence $v_n \in W^{1,1}(\Omega)$ strongly converging to u in $L^\infty(\Omega)$ with $F_n(v_n)$ bounded and a sequence $z_n \in W^{1,1}(\Omega)$ strongly converging to u in $L^\infty(\Omega)$ with $G_n(z_n)$ bounded, and define for $\varepsilon \in (0, 1/2)$ the sequence

$$w_n := (1-\varepsilon)u_n + \varphi_n(v_n - u_n)^+ + \psi_n(v_n - u_n)^- + \varepsilon z_n.$$

In the set $\{u_n \leqslant v_n\}$ we have

$$\begin{aligned}\nabla w_n &= (1-\varepsilon-\varphi_n)\nabla u_n + \varphi_n \nabla v_n + \varepsilon\big(\nabla z_n + \varepsilon^{-1}(v_n - u_n)\nabla\varphi_n\big)\\ &= (1-\varepsilon-\varphi_n)\nabla u_n + \varphi_n \nabla v_n + \varepsilon\big(\big(1-(v_n-u_n)\big)\nabla z_n + (v_n-u_n)\big(\nabla z_n + \varepsilon^{-1}\nabla\varphi_n\big)\big).\end{aligned}$$

Note that the last term is a linear combination for n large enough, since we work in the set $\{u_n \leqslant v_n\}$ and $v_n - u_n$ converges strongly in $L^\infty(\Omega)$. Then, by the convexity of f_n and (2.3), (2.7) we get that

$$\begin{aligned}\int_{\{u_n\leqslant v_n\}} f_n(x, \nabla w_n)\, dx \leqslant{}& \int_{\{u_n\leqslant v_n\}} \big((1-\varepsilon-\varphi_n) f_n(x, \nabla u_n) + \varphi_n f_n(x, \nabla v_n)\big)\, dx\\ &+ \varepsilon \int_{\{u_n\leqslant v_n\}} \big(\big(1-(v_n-u_n)\big) g_n(x, \nabla z_n) + (v_n-u_n) g_n\big(x, \nabla z_n + \varepsilon^{-1}\nabla\varphi_n\big)\big)\, dx.\end{aligned}$$

From the estimates (2.5), (2.6), (2.7) satisfied by g_n, the boundedness of $F_n(u_n)$, $F_n(v_n)$, $G_n(z_n)$, $G_n(\varphi_n)$, and the strong convergence of φ_n to φ and $v_n - u_n$ to 0 in $L^\infty(\Omega)$, we deduce that

$$\int_{\{u_n\leqslant v_n\}} f_n(x, \nabla w_n)\, dx \leqslant \int_{\{u_n\leqslant v_n\}} \big((1-\varepsilon-\varphi) f_n(x, \nabla u_n) + \varphi f_n(x, \nabla v_n)\big)\, dx + \varepsilon \int_{\{u_n\leqslant v_n\}} g_n(x, \nabla z_n)\, dx + o(1),$$

where $o(1)$ tends to 0 as $n \to \infty$ for a fixed $\varepsilon \in (0, 1/2)$. Similarly for the set $\{v_n < u_n\}$ with the sequence ψ_n, we obtain that

$$\int_{\{v_n<u_n\}} f_n(x, \nabla w_n)\, dx \leqslant \int_{\{v_n<u_n\}} \big((1-\varepsilon-\varphi) f_n(x, \nabla u_n) + \varphi f_n(x, \nabla v_n)\big)\, dx + \varepsilon \int_{\{v_n<u_n\}} g_n(x, \nabla z_n)\, dx + o(1).$$

Using that w_n converges strongly to u in $L^\infty(\Omega)$ and that u_n is a recovery sequence for F_n, and adding the two previous inequalities, it follows that

$$\begin{aligned}\int_\Omega f_n(x,\nabla u_n)\,dx &\leqslant \int_\Omega f_n(x,\nabla w_n)\,dx + o(1)\\ &\leqslant \int_\Omega f_n(x,\nabla u_n)\,dx + \int_\Omega \varphi\big(f_n(x,\nabla v_n)-f_n(x,\nabla u_n)\big)\,dx\\ &\quad + \varepsilon\int_\Omega \big(g_n(x,\nabla z_n)-f_n(x,\nabla u_n)\big)\,dx + o(1),\end{aligned}$$

which implies

$$\liminf_{n\to\infty}\int_\Omega \varphi\big(f_n(x,\nabla v_n)-f_n(x,\nabla u_n)\big)\,dx \geqslant -\varepsilon \limsup_{n\to\infty}\int_\Omega \big(g_n(x,\nabla z_n)-f_n(x,\nabla u_n)\big)\,dx.$$

Finally, the arbitrariness of ε yields (2.31). □

Proposition 2.14. *Assume that (2.1)–(2.4), and (2.11), (2.12) hold. Then, there exists a constant $C>0$ such that for any $u,v\in C^1(\bar\Omega)$ and any recovery sequences u_n,v_n for F_n respectively at u,v, converging respectively to u,v in $L^\infty(\Omega)$, and for any $\varphi\in C^1_c(\bar\Omega)$ with $\varphi\geqslant 0$, the following estimate holds*

$$\begin{aligned}&\limsup_{n\to\infty}\left|\int_\Omega \varphi\big(f_n(x,\nabla u_n)-f_n(x,\nabla v_n)\big)\,dx\right|\\ &\quad\leqslant C\|\nabla u-\nabla v\|_{L^\infty(\operatorname{supp}\varphi)}\big(\|\nabla u\|^\rho_{L^\infty(\operatorname{supp}\varphi)}+\|\nabla v\|^\rho_{L^\infty(\operatorname{supp}\varphi)}+1\big)\int_{\bar\Omega}\varphi\,d\mu.\end{aligned}\tag{2.32}$$

Proof. Let $u,v,\varphi\in C^1(\bar\Omega)$, and set $\gamma:=\|\nabla u-\nabla v\|_{L^\infty(\operatorname{supp}\varphi)}$. Consider two recovery sequences u_n,v_n for F_n respectively at u,v, converging respectively to u,v in $L^\infty(\Omega)$. First, note that by virtue of Proposition 2.11 and Proposition 2.12 (applied to the recovery sequences u_n,v_n) combined with the inequality $F_n^\varphi\leqslant G_n^\varphi$, estimate (2.32) holds when $\gamma\geqslant 1$. From now on, we assume that $\gamma<1$.

Take $\varepsilon\in(0,1-\gamma)$. By Proposition 2.11 there exists a sequence $\zeta_n\in W^{1,1}(\Omega)$ strongly converging to $\zeta:=v+(\gamma+\varepsilon)^{-1}(u-v)$ in $L^\infty(\Omega)$, and satisfying the bound (2.28) with the pair (ζ_n,ζ). The sequence $\tilde u_n:=(1-\gamma-\varepsilon)v_n+(\gamma+\varepsilon)\zeta_n$ converges strongly to u in $L^\infty(\Omega)$. Then, since u_n is a recovery sequence for F_n, by Proposition 2.12 and by the convexity of f_n we have

$$\begin{aligned}F_n^\varphi(u_n)&\leqslant F_n^\varphi(\tilde u_n)+o(1)\leqslant(1-\gamma-\varepsilon)F_n^\varphi(v_n)+(\gamma+\varepsilon)G_n^\varphi(\zeta_n)+o(1)\\ &\leqslant F_n^\varphi(v_n)+M(\gamma+\varepsilon)\big(\|\nabla\zeta\|^\rho_{L^\infty(\operatorname{supp}\varphi)^N}+\|\nabla\zeta\|_{L^\infty(\operatorname{supp}\varphi)^N}\big)\int_{\bar\Omega}\varphi\,d\mu+o(1)\\ &\leqslant F_n^\varphi(v_n)+2^\rho M(\gamma+\varepsilon)\big(\|\nabla v\|^\rho_{L^\infty(\operatorname{supp}\varphi)^N}+1\big)\int_{\bar\Omega}\varphi\,d\mu+o(1).\end{aligned}$$

Changing the roles of u_n and v_n we also get that

$$F_n^\varphi(v_n)\leqslant F_n^\varphi(u_n)+2^\rho M(\gamma+\varepsilon)\big(\|\nabla u\|^\rho_{L^\infty(\operatorname{supp}\varphi)^N}+1\big)\int_{\bar\Omega}\varphi\,d\mu+o(1).$$

Therefore, from the two previous inequalities we deduce that

$$\limsup_{n\to\infty}\big|F_n^\varphi(u_n)-F_n^\varphi(v_n)\big|\leqslant 2^\rho M(\gamma+\varepsilon)\big(\|\nabla u\|^\rho_{L^\infty(\operatorname{supp}\varphi)^N}+\|\nabla v\|^\rho_{L^\infty(\operatorname{supp}\varphi)^N}+2\big)\int_{\bar\Omega}\varphi\,d\mu.$$

Finally, the arbitrariness of $\varepsilon>0$ implies the desired estimate (2.32). □

Proof of Theorem 2.4. First, note that it is enough to prove the results for the sequence f_n, since the properties of f_n are clearly satisfied by g_n. Thanks to Proposition 2.9, Proposition 2.11, and using a diagonal extraction there exists a subsequence of n still denoted by n, such that for any linear function $w^\xi : x \mapsto \xi \cdot x$, with $\xi \in \mathbb{Q}^N$, there exists a recovery sequence w_n^ξ for F_n at w^ξ strongly converging in $L^\infty(\Omega)$ to w^ξ, and a function $h^\xi \in L^1_\mu(\bar{\Omega})$ such that

$$f_n\big(\cdot, \nabla w_n^\xi\big) \rightharpoonup h^\xi \mu \quad \text{weakly-}* \text{ in } \mathscr{M}(\bar{\Omega}), \tag{2.33}$$

where by estimates (2.28) and (2.32) the function h^ξ satisfies

$$\begin{cases} 0 \leqslant h^\xi \leqslant M\big(|\xi|^p + |\xi|\big) & \mu\text{-a.e. in } \bar{\Omega},\ \forall \xi \in \mathbb{Q}^N, \\ \big|h^\xi - h^\eta\big| \leqslant C|\xi - \eta|\big(|\xi|^p + |\eta|^p + 1\big) & \mu\text{-a.e. in } \bar{\Omega},\ \forall \xi, \eta \in \mathbb{Q}^N. \end{cases} \tag{2.34}$$

By the Lipschitz estimate of (2.34) there exists a unique Caratheodory function $f : \bar{\Omega} \times \mathbb{R}^N \to [0, \infty)$ defined by

$$f(x, \xi) := h^\xi(x) \quad \mu\text{-a.e. } x \in \bar{\Omega},\ \forall \xi \in \mathbb{Q}^N. \tag{2.35}$$

First step: Γ-convergence of F_n in $C^1(\bar{\Omega})$.

Let $u \in C^1(\bar{\Omega})$. There exists a subsequence n' of n such that $F_{n'}$ Γ-converges at u. Consider a recovery sequence $u_{n'}$ for $F_{n'}$, which strongly converges to u in $L^\infty(\Omega)$. Up to extract a new subsequence, thanks to Proposition 2.11 and Proposition 2.12 combined with $F_n^\varphi \leqslant G_n^\varphi$, we can also assume that there exists $h^u \in L^1_\mu(\bar{\Omega})$ such that

$$f_{n'}(\cdot, \nabla u_{n'}) \rightharpoonup h^u \mu \quad \text{weakly-}* \text{ in } \mathscr{M}(\bar{\Omega}). \tag{2.36}$$

Applying (2.32) (after a localization) with the recovery sequences $u_{n'}, w_{n'}^\xi$, for $\xi \in \mathbb{Q}^N$, we obtain that

$$\big|h^u - f(\cdot, \xi)\big| \leqslant C|\nabla u - \xi|\big(|\nabla u|^p + |\nabla u| + |\xi|^p + |\xi| + 1\big) \quad \mu\text{-a.e. in } \bar{\Omega},\ \forall \xi \in \mathbb{Q}^N,$$

which by continuity of $f(x, \cdot)$ implies that $h^u = f(\cdot, \nabla u)$ a.e. in $\bar{\Omega}$. Hence, by convergence (2.36) and a uniqueness argument the whole sequence F_n Γ-converges at u to $F(u)$ defined by (2.18). This also implies convergence (2.21) for any recovery sequence u_n for F_n.

Second step: Properties of the density f.

Let us prove the convexity of $f(x, \cdot)$. The proofs of the other properties are similar. Let $\xi, \eta \in \mathbb{R}^N$, and set $\lambda := t\xi + (1-t)\eta$ for $t \in [0, 1]$. Consider recovery sequences w_n^ξ, w_n^η, w_n^λ, for F_n of limits w^ξ, w^η, w^λ respectively. Applying inequality (2.31) with $u_n := w_n^\lambda$ and $v_n := t w_n^\xi + (1-t) w_n^\eta$ and using the convergence (2.21) of the energy density, we get that for any $\varphi \in C^1(\bar{\Omega})$ with $\varphi \geqslant 0$,

$$\begin{aligned} \int_{\bar{\Omega}} \varphi f\big(x, t\xi + (1-t)\eta\big)\, d\mu &= \lim_{n\to\infty} \int_{\Omega} \varphi f_n\big(x, \nabla w_n^\lambda\big)\, dx \\ &\leqslant \liminf_{n\to\infty} \int_{\Omega} \varphi f_n\big(x, t\nabla w_n^\xi + (1-t)\nabla w_n^\eta\big)\, dx \\ &\leqslant t \int_{\bar{\Omega}} \varphi f(x, \xi)\, d\mu + (1-t) \int_{\bar{\Omega}} \varphi f(x, \eta)\, d\mu, \end{aligned}$$

which implies the convexity of $f(x, \cdot)$ due to the arbitrariness of φ.

Third step: Γ-convergence of F_n^ω for any open set $\omega \subset \Omega$.

Let us prove the Γ-liminf and the Γ-limsup properties for the sequence F_n^ω (see Definition 1.2). Let $u \in C_0^1(\omega)$. Let u_n be a sequence of $W_0^{1,1}(\omega)$ which strongly converges to u in $L^\infty(\omega)$. Extending u and u_n by 0 in $\Omega \setminus \omega$, the Γ-liminf property for F_n implies that

$$F(u) \leqslant \liminf_{n\to\infty} F_n(u_n) = \liminf_{n\to\infty} F_n^\omega(u_n).$$

On the other hand, consider a recovery sequence $\bar{u}_n \in W^{1,1}(\Omega)$ for F_n strongly converging to u in $L^\infty(\Omega)$. Then, by Lemma 2.8 there exist a sequence $\tilde{u}_n$ strongly converging to u in $L^\infty(\Omega)$, such that $\tilde{u}_n = 0$ in $\Omega \setminus \omega$ and $F_n(\tilde{u}_n) \leqslant F_n(\bar{u}_n)$. Therefore, we obtain that

$$F(u) = \lim_{n\to\infty} F_n(\bar{u}_n) \geqslant \limsup_{n\to\infty} F_n(\tilde{u}_n) = \limsup_{n\to\infty} F_n^\omega(\tilde{u}_n),$$

which shows the Γ-limsup property. □

3. Conditions for that the Γ-limits in L^∞ and in L^1 agree

In this section we show the existence of a general class of convex functionals which have the same Γ-limits for the strong topology of $L^\infty(\Omega)$ and the strong topology of $L^1(\Omega)$.

Let Ω be a bounded open set of $\mathbb{R}^N$, $N \geqslant 2$. Consider a sequence $f_n : \Omega \times \mathbb{R}^N \to [0, \infty)$ satisfying

$$\begin{cases} f_n(\cdot, \xi) \text{ are measurable for any } \xi \in \mathbb{R}^N, \\ f_n(x, \cdot) \text{ are convex for a.e. } x \in \Omega, \end{cases} \quad \text{and} \quad f_n(\cdot, 0) = 0. \tag{3.1}$$

Also consider the associated sequence of functionals F_n defined in $L^1(\Omega)$ by

$$F_n(v) := \begin{cases} \int_\Omega f_n(x, \nabla v)\, dx & \text{if } v \in W_0^{1,1}(\Omega), \\ \infty & \text{if } v \in L^1(\Omega) \setminus W_0^{1,1}(\Omega). \end{cases} \tag{3.2}$$

In addition, we assume that F_n satisfies the following weak coercivity condition:

There exists a real number q with $q > N - 1$ if $N > 2$, and $q = 1$ if $N = 2$, such that

$$\begin{cases} \forall n \in \mathbb{N},\ \forall c > 0, \quad \{F_n \leqslant c\} \text{ is sequentially compact in } W_0^{1,q}(\Omega) \text{ weak}, \\ \forall u_n \in W_0^{1,1}(\Omega), \quad \limsup\limits_{n\to\infty} F_n(u_n) < \infty \Rightarrow \begin{array}{l} u_n \text{ converges weakly in } W_0^{1,q}(\Omega), \\ \text{up to a subsequence.} \end{array} \end{cases} \tag{3.3}$$

Remark 3.1. Note that for any $p \in (1, \infty)$, the weak compactness in $W^{1,p}(\Omega)$ is equivalent to the boundedness in $W^{1,p}(\Omega)$. Therefore, the compactness in assumption (3.3) is actually essential in the case $q = 1$.

The following result provides a large class of sequences f_n which satisfy condition (3.3):

Proposition 3.2. *Consider a sequence of functions $f_n : \Omega \times \mathbb{R}^N \to [0, \infty)$ satisfying (3.1) and the following estimate from below*:

- *If $N > 2$, there exist $p > N - 1$, $r > (N - 1)/(p - N + 1)$, $S > 0$, and a sequence of nonnegative measurable functions λ_n in Ω, with λ_n^{-r} bounded in $L^1(\Omega)$, such that*

$$f_n(x, \xi) \geqslant \lambda_n(x)|\xi|^p - S, \quad \forall \xi \in \mathbb{R}^N,\ a.e.\ x \in \Omega.$$

- *If $N = 2$, there exist $p > 1$, $S > 0$, and a sequence of nonnegative measurable functions λ_n in Ω, with $\lambda_n^{-\frac{1}{p-1}}$ weakly compact in $L^1(\Omega)$, such that*

$$f_n(x, \xi) \geqslant \lambda_n(x)|\xi|^p - S, \quad \forall \xi \in \mathbb{R}^2,\ a.e.\ x \in \Omega.$$

Define $q \geqslant 1$ by

$$q := \begin{cases} \frac{pr}{1+r} & \text{if } N > 2, \\ 1 & \text{if } N = 2. \end{cases} \tag{3.4}$$

Then, the assertions of (3.3) are fulfilled with q.

Proof. First of all, note that $q > N-1$ if $N > 2$. Let us first prove the second assertion of (3.3). Let u_n be a sequence in $W_0^{1,1}(\Omega)$ such that

$$\limsup_{n\to\infty} \int_\Omega f_n(x, \nabla u_n)\, dx < \infty.$$

- If $N > 2$, then by the Hölder inequality we have

$$\int_\Omega |\nabla u_n|^q\, dx \leqslant \left(\int_\Omega \lambda_n |\nabla u_n|^p\, dx\right)^{\frac{r}{1+r}} \left(\int_\Omega \lambda_n^{-r}\, dx\right)^{\frac{1}{1+r}} \leqslant c. \tag{3.5}$$

  Hence, u_n is bounded in $W_0^{1,q}(\Omega)$, for $q > N-1 > 1$, thus converges weakly in $W^{1,q}(\Omega)$, up to a subsequence.
- If $N = 2$, then we get that for any measurable set $E \subset \Omega$,

$$\int_E |\nabla u_n|\, dx \leqslant \left(\int_E \lambda_n |\nabla u_n|^p\, dx\right)^{\frac{1}{p}} \left(\int_E \lambda_n^{-\frac{1}{p-1}}\, dx\right)^{1-\frac{1}{p}} \leqslant c\left(\int_E \lambda_n^{-\frac{1}{p-1}}\, dx\right)^{1-\frac{1}{p}}. \tag{3.6}$$

  Hence, by virtue of the Dunford–Pettis theorem, u_n converges weakly in $W_0^{1,1}(\Omega)$, up to a subsequence.

This establishes the second assertion of (3.3).

Now, let us check the first assertion of (3.3). Fix $n \in \mathbb{N}$ and $c > 0$. Consider a sequence v_k in $W_0^{1,q}(\Omega)$ such that $F_n(v_k) \leqslant c$, for any $k \in \mathbb{N}$. Proceeding as for (3.5) and (3.6), v_k converges weakly, up to a subsequence, to some function v in $W_0^{1,q}(\Omega)$. Therefore, due to the convexity of F_n the desired inequality $F_n(v) \leqslant c$ follows from the lower semicontinuity of F_n for the strong topology of $W_0^{1,q}(\Omega)$. It thus remains to prove the strong lower semicontinuity of F_n. Since f_n is convex with respect to the second argument in $\mathbb{R}^N$, f_n is continuous with respect to the second argument. Then, for any sequence w_k converging strongly to w in $W_0^{1,q}(\Omega)$, $f_n(x, \nabla w_k)$ converges to $f_n(x, \nabla w)$ for a.e. $x \in \Omega$. Hence, by the Fatou lemma applied to the nonnegative sequence $f_n(\cdot, \nabla w_k)$, we get that

$$\int_\Omega f_n(x, \nabla w)\, dx \leqslant \liminf_{k\to\infty} \int_\Omega f_n(x, \nabla w_k)\, dx, \tag{3.7}$$

which implies the strong lower semicontinuity of F_n. □

Remark 3.3. Consider F_n defined by (3.2) satisfying the assumptions of Proposition 3.2. As a consequence of the first assertion of (3.3), the nonnegative functional F_n is lower semicontinuous for the strong topology of $W_0^{1,q}(\Omega)$. Hence by (3.5) and (3.6), for any G in $W^{-1,q'}(\Omega)$, the convex functional $v \mapsto F_n(v) - \langle G, v\rangle$ is $W_0^{1,q}(\Omega)$-coercive, and thus has a minimum u_n in $W_0^{1,q}(\Omega)$ for any $n \in \mathbb{N}$. Thanks to the second assertion of (3.3), the sequence u_n is weakly compact in $W_0^{1,q}(\Omega)$, and is thus compact in $L^1(\Omega)$. Therefore, the study of the asymptotic behavior of u_n is equivalent to study the Γ-convergence of the sequence F_n for the strong topology of $L^1(\Omega)$.

Then, the main results of this section are the following:

Theorem 3.4. *Assume that the conditions (3.1) and (3.3) hold. Consider a function u in $C^0(\bar{\Omega}) \cap W_0^{1,q}(\Omega)$, such that there exists a sequence u_n converging weakly to u in $W_0^{1,q}(\Omega)$ with*

$$\limsup_{n\to\infty} F_n(u_n) < \infty.$$

Then, there exists a sequence $\hat{u}_n$ which converges to u weakly in $W_0^{1,q}(\Omega)$ and strongly in $L^\infty(\Omega)$, such that

$$\liminf_{n\to\infty} F_n(\hat{u}_n) \leqslant \liminf_{n\to\infty} F_n(u_n). \tag{3.8}$$

Corollary 3.5. *Under the assumptions of Theorem 3.4, the Γ-limit of F_n at any point u in $C^0(\bar{\Omega})$ for the strong topology of $L^\infty(\Omega)$ agrees with its Γ-limit for the strong topology of $L^1(\Omega)$.*

Proof. If $u \in W_0^{1,q}(\Omega)$, the result is an immediate consequence of Theorem 3.4. Otherwise, applying the condition (3.3) to recovery sequences for F_n of limit u, the two Γ-limits at u are equal to ∞. □

In the sequel we will use the following:

Remark 3.6. Let $v \in W^{1,q}(\Omega)$, let $x_0 \in \Omega$ and $R > 0$ be such that $B(x_0, R) \subset \Omega$. Denoting by S_{N-1} the unit sphere of $\mathbb{R}^N$, we have

$$\int_0^R \left(\int_{S_{N-1}} \left|\nabla v(x_0 + r\zeta) \cdot \zeta\right|^q d\sigma \right) r^{N-1}\, dr \leqslant \int_0^R \left(\int_{S_{N-1}} \left|\nabla v(x_0 + r\zeta)\right|^q d\sigma \right) r^{N-1}\, dr$$
$$= \int_{B(x_0,R)} |\nabla v|^q\, dx < \infty.$$

Hence, the function $\zeta \mapsto v(x_0 + r\zeta)$ belongs to $W^{1,q}(S_{N-1})$ for a.e. $r \in (0, R)$. Therefore, by the embedding of $W^{1,q}(S_{N-1})$ into $C^0(S_{N-1})$ since $q > N - 1$, the restriction $v_{|\partial B(x_0,r)}$ has a continuous representative on $\partial B(x_0, r)$. This gives a sense to the bounds of v on $\partial B(x_0, r)$ for a.e. $r > 0$ such that $B(x_0, r) \subset \Omega$.

Now, the key ingredient of the proof of Theorem 3.4 is the following:

Lemma 3.7. *Consider a sequence u_n which converges weakly to a function u in $W^{1,q}(\Omega)$, with $q > N - 1$ if $N > 2$, and $q = 1$ if $N = 2$. Assume that u belongs to $C^0(\bar{\Omega})$, and that u_n satisfies the following maximum property:*

For any $x_0 \in \Omega$, and for a.e. $r > 0$ such that $B(x_0, r) \subset \Omega$, we have

$$\min\left\{ \min_{\partial B(x_0,r)} u_n, \min_{\bar{B}(x_0,r)} u \right\} \leqslant \inf_{B(x_0,r)} u_n \leqslant \sup_{B(x_0,r)} u_n \leqslant \max\left\{ \max_{\partial B(x_0,r)} u_n, \max_{\bar{B}(x_0,r)} u \right\}. \tag{3.9}$$

Then, the sequence u_n converges strongly to u in $L^\infty_{\mathrm{loc}}(\Omega)$.

Proof. Assume for the moment that for any $x_0 \in \Omega$, and for any $r > 0$ such that $B(x_0, 2r) \subset \Omega$, we have

$$\limsup_{n\to\infty} \Big(\sup_{B(x_0,r)} |u_n - u| \Big) \leqslant \max_{\bar{B}(x_0,2r)} u - \min_{\bar{B}(x_0,2r)} u. \tag{3.10}$$

Given a compact set $K \subset \Omega$ and $\varepsilon > 0$, there exist $r > 0$ and $x_0^1, \ldots, x_0^m \in \Omega$ such that

$$\forall i \in \{1, \ldots, m\}, \quad B\big(x_0^i, 2r\big) \subset \Omega \quad \text{and} \quad K \subset \bigcup_{i=1}^m B\big(x_0^i, r\big),$$

and due to the uniform continuity of u on $\bar{\Omega}$

$$\forall x, y \in \Omega, \quad |x - y| < 2r, \qquad \big|u(x) - u(y)\big| < \varepsilon.$$

Then, by (3.10) we get that

$$\limsup_{n\to\infty} \|u_n - u\|_{L^\infty(K)} \leqslant \max_{1\leqslant i\leqslant m} \Big[\limsup_{n\to\infty} \Big(\sup_{B(x_0^i,r)} |u_n - u| \Big) \Big] \leqslant \varepsilon,$$

for any $\varepsilon > 0$, which implies the strong convergence of u_n to u in $L^\infty(K)$.

In order to prove (3.10) we distinguish the case $N > 2$, and the more delicate case $N = 2$.

Case $N > 2$: Define the functions $v_n, v : (r, 2r) \times S_{N-1} \to \mathbb{R}$ by

$$v_n(s, \zeta) := u_n(x_0 + s\zeta), \qquad v(s, \zeta) := u(x_0 + s\zeta), \quad \text{for } (s, \zeta) \in (r, 2r) \times S_{N-1}.$$

Since the sequence u_n converges weakly to u in $W^{1,q}(B(x_0,r))$, the sequence v_n converges weakly to v in $L^q(r,2r;W^{1,q}(S_{N-1}))$ and $\partial_s v_n$ converges weakly to $\partial_s v$ in $L^q(r,2r;L^q(S_{N-1}))$. Therefore, taking into account that, due to $q>N-1$, the space $W^{1,q}(S_{N-1})$ is compactly embedded in $C^0(S_{N-1})$, and that $C^0(S_{N-1})$ is continuously embedded in $L^q(S_{N-1})$, the Aubin compactness theorem [3] implies that the sequence v_n converges strongly to v in $L^q(r,2r;C^0(S_{N-1}))$. In particular, using Remark 3.6 this yields, up to a subsequence,

$$\max_{\partial B(x_0,s)} u_n = \max_{\zeta\in S_{N-1}} v_n(s,\zeta) \xrightarrow[n\to\infty]{} \max_{\zeta\in S_{N-1}} v(s,\zeta) = \max_{\partial B(x_0,s)} u, \quad \text{a.e. } s\in(r,2r), \tag{3.11}$$

$$\min_{\partial B(x_0,s)} u_n = \min_{\zeta\in S_{N-1}} v_n(s,\zeta) \xrightarrow[n\to\infty]{} \min_{\zeta\in S_{N-1}} v(s,\zeta) = \min_{\partial B(x_0,s)} u, \quad \text{a.e. } s\in(r,2r). \tag{3.12}$$

Hence, using inequality (3.9) with r replaced by $s\in(r,2r)$, we get that

$$\min_{\bar B(x_0,2r)} u - \max_{\bar B(x_0,2r)} u \leqslant \liminf_{n\to\infty}\Big(\inf_{B(x_0,r)}(u_n-u)\Big) \leqslant \limsup_{n\to\infty}\Big(\sup_{B(x_0,r)}(u_n-u)\Big) \leqslant \max_{\bar B(x_0,2r)} u - \min_{\bar B(x_0,2r)} u,$$

which is equivalent to (3.10).

Case $N=2$: Denote by $\mathbb{T}$ the torus $\mathbb{R}/(2\pi\mathbb{Z})$. Similarly to the functions v_n, v defined in the case $N>2$, define the functions $w_n, w:(r,2r)\times\mathbb{T}\to\mathbb{R}$ by

$$w_n(s,t) := u_n\big(x_0+s(\cos t,\sin t)\big), \qquad w(s,t) := u\big(x_0+s(\cos t,\sin t)\big), \quad \text{a.e. } (s,t)\in(r,2r)\times\mathbb{T}.$$

Fix $\varepsilon>0$. Since ∇u_n is equi-integrable, there exists $\delta>0$ such that

$$\int_E |\nabla u_n|\,dx < \varepsilon r, \quad \forall n\in\mathbb{N},\ \forall E\subset\Omega,\ |E|\leqslant\delta. \tag{3.13}$$

Set $h:=2\delta/(3r^2)$, which can be chosen less than 2π. Let us prove that for any $n\in\mathbb{N}$, there exists a Lebesgue point for $w_n\in L^1(r,2r;W^{1,1}(\mathbb{T}))$, $s_n\in(r,2r)$ such that

$$\int_\tau^{\tau+h} \big|\partial_t w_n(s_n,t)\big|\,dt < \varepsilon, \quad \forall\tau\in\mathbb{T},\ \forall n\in\mathbb{N}. \tag{3.14}$$

We reason by contradiction. If this assertion is not true, then the multifunction $\Psi:(r,2r)\to\mathscr{P}(\mathbb{T})$ defined by

$$\Psi(s) := \left\{\tau\in\mathbb{T}:\ \int_\tau^{\tau+h}\big|\partial_t w_n(s,t)\big|\,dt \geqslant \varepsilon\right\}$$

takes values in nonempty closed sets of $\mathbb{T}$ for a.e. $s\in(r,2r)$. The multifunction Ψ is measurable in the sense that for any closed set $C\subset\mathbb{T}$, the set $\Psi^{-1}(C):=\{s\in(r,2r):\ \Psi(s)\cap C\neq\emptyset\}$ is measurable. Indeed, we have

$$\Psi^{-1}(C) = \Phi^{-1}\big([\varepsilon,\infty)\big) \quad \text{with } \Phi(s) := \max_{\tau\in C}\int_\tau^{\tau+h}\big|\partial_t w_n(s,t)\big|\,dt,$$

and $\Phi\in L^1(r,2r)$ by the Fubini theorem. Then, by virtue of the Kuratowski, Ryll-Nardzewski selection theorem [26], there exists a measurable function $\psi:(r,2r)\to\mathbb{T}$ such that $\psi(s)\in\Psi(s)$ for a.e. $s\in(r,2r)$. Now, define the measurable set E by

$$E := \big\{x\in\mathbb{R}^2:\ x=x_0+s(\cos t,\sin t),\ s\in(r,2r),\ \psi(s)<t<\psi(s)+h\big\}.$$

By the Fubini theorem we have

$$|E| = h\int_r^{2r} s\,ds = \delta,$$

and

$$\int_E |\nabla u_n|\,dx = \int_r^{2r} \int_{\psi(s)}^{\psi(s)+h} s\big|\nabla u_n\big(x_0 + s(\cos t, \sin t)\big)\big|\,dt\,ds \geqslant \int_r^{2r} \int_{\psi(s)}^{\psi(s)+h} |\partial_t w_n|\,dt\,ds \geqslant \varepsilon r,$$

which contradicts (3.13).

Up to extract a subsequence we can assume that s_n converges to some $\bar{s} \in [r, 2r]$. By the compact embedding of $W^{1,1}((r,2r)\times\mathbb{T})$ into $L^1((r,2r)\times\mathbb{T}) = L^1(r,2r; L^1(\mathbb{T}))$, the sequence w_n converges strongly to w in $L^1(r,2r; L^1(\mathbb{T}))$. Hence, from the estimate (3.18) below we deduce that for any $\gamma > 0$,

$$\limsup_{n\to\infty}\bigg\| w_n(s_n,.) - \fint_{I_\gamma} w(s,.)\,ds\bigg\|_{L^1(\mathbb{T})} \leqslant \limsup_{n\to\infty}\int_{I_\gamma} \|\partial_s w_n\|_{L^1(\mathbb{T})}\,ds, \tag{3.15}$$

where $I_\gamma := (r,2r)\cap(\bar{s}-\gamma, \bar{s}+\gamma)$. Using that ∇u_n is equi-integrable, we get that the right-hand of (3.15) tends to zero as γ tends to zero. The continuity of the function w also shows that

$$\fint_{I_\gamma} w(s,.)\,ds \underset{\gamma\to 0}{\longrightarrow} w(\bar{s},.) \quad \text{in } C^0(\mathbb{T}).$$

Therefore, passing to the limit in (3.15) as γ tends to zero, we obtain that

$$w_n(s_n,.) \underset{n\to\infty}{\longrightarrow} w(\bar{s},.) \quad \text{strongly in } L^1(\mathbb{T}). \tag{3.16}$$

On the other hand, since $w_n(s_n,.)$ belongs to $W^{1,1}(\mathbb{T})$ which is continuously embedded into $C^0(\mathbb{T})$, there exists $t_n \in \mathbb{T}$ such that

$$w_n(s_n,t_n) = \max_{t\in\mathbb{T}} w_n(s_n,t).$$

Due to the compactness of $\mathbb{T}$ we can assume that t_n converges to some $\bar{t}$. Applying the inequality (3.18) below to the sequence $w_n(s_n,.) \in W^{1,1}(\mathbb{T})$ with $\gamma < h/2$, we get that

$$\limsup_{n\to\infty}\bigg| w_n(s_n,t_n) - \fint_{\bar{t}-\gamma}^{\bar{t}+\gamma} w(\bar{s},t)\,dt\bigg| \leqslant \limsup_{n\to\infty}\int_{\bar{t}-\gamma}^{\bar{t}+\gamma} \big|\partial_t w_n(s_n,t)\big|\,dt \leqslant \varepsilon,$$

which combined with the continuity of w gives

$$\limsup_{n\to\infty}\big|w_n(s_n,t_n) - w(\bar{s},\bar{t})\big| \leqslant \varepsilon.$$

From this equality we deduce that

$$\limsup_{n\to\infty}\Big(\sup_{\partial B(x_0,s_n)} u_n\Big) = \limsup_{n\to\infty} w_n(s_n,t_n) \leqslant w(\bar{s},\bar{t}) + \varepsilon \leqslant \max_{\partial B(x_0,\bar{s})} u + \varepsilon.$$

Similarly we can prove that

$$\liminf_{n\to\infty}\Big(\inf_{\partial B(x_0,s_n)} u_n\Big) \geqslant \min_{\partial B(x_0,\bar{s})} u - \varepsilon.$$

Finally, the two previous inequalities combined with condition (3.9) easily yield

$$\limsup_{n\to\infty}\ \sup_{B(x_0,r)} |u_n - u| \leqslant \max_{\bar{B}(x_0,2r)} u - \min_{\bar{B}(x_0,2r)} u + \varepsilon,$$

for any $\varepsilon > 0$, which implies (3.10). □

Proof of Theorem 3.4. Let $t_n > 0$ be a sequence such that $t_n \to \infty$ and $t_n\|u_n - u\|_{L^1(\Omega)} \to 0$. Thanks to the first assertion of (3.3) combined with the compact embedding of $W_0^{1,q}(\Omega)$ into $L^q(\Omega)$, the nonnegative convex functional $v \mapsto F_n(v) + t_n\|v - u\|_{L^1(\Omega)}$ defined in $W_0^{1,q}(\Omega)$ has a mininum $\hat{u}_n \in W_0^{1,q}(\Omega)$. From the inequality

$$\liminf_{n\to\infty}\big(F_n(\hat{u}_n) + t_n\|\hat{u}_n - u\|_{L^1(\Omega)}\big) \leqslant \liminf_{n\to\infty}\big(F_n(u_n) + t_n\|u_n - u\|_{L^1(\Omega)}\big) = \liminf_{n\to\infty} F_n(u_n),$$

we deduce that (3.8) holds and that $\hat u_n$ converges strongly to u in $L^1(\Omega)$. Hence, by the boundedness of $F_n(\hat u_n)$ combined with the second assertion of (3.3), the sequence $\hat u_n$ converges weakly to u in $W_0^{1,q}(\Omega)$.

It thus remains to prove that $\hat u_n$ converges strongly to u in $L^\infty(\Omega)$. Extending $\hat u_n$ and u by zero outside of Ω, and using Lemma 3.7 applied with an open set $\tilde\Omega$ containing $\bar\Omega$, we just need to show that $\hat u_n$ satisfies the inequalities (3.9) for any ball $B \subset \mathbb{R}^N$, the radius of which belongs to a full measure subset of $(0,\infty)$ (see Remark 3.6). To this end, consider the function

$$w_n := \hat u_n - (\hat u_n - M)^+ \chi_B \quad \text{where } M = \max\Big\{\max_{\partial B} \hat u_n, \max_{\bar B} u\Big\},$$

so that $w_n \in W_0^{1,q}(\Omega)$. By the definition of $\hat u_n$ we have

$$\begin{aligned}\int_\Omega f_n(x,\nabla \hat u_n)\,dx + t_n\|\hat u_n - u\|_{L^1(\Omega)} &\leqslant \int_\Omega f_n(x,\nabla w_n)\,dx + t_n\|w_n-u\|^1_{L^1(\Omega)} \\ &= \int_{\Omega\setminus(B\cap\{\hat u_n>M\})} f_n(x,\nabla\hat u_n)\,dx + t_n\|\hat u_n - u\|_{L^1(\Omega\setminus(B\cap\{\hat u_n>M\}))} \\ &\quad + t_n\|M-u\|_{L^1(B\cap\{\hat u_n>M\})}.\end{aligned} \tag{3.17}$$

Note that in the set $B\cap\{\hat u_n > M\}$, we have $u \leqslant M \leqslant \hat u_n$ and thus $|M-u| < |\hat u_n - u|$. Hence, it follows the inequality

$$\|M-u\|_{L^1(B\cap\{\hat u_n>M\})} \leqslant \|\hat u_n - u\|_{L^1(B\cap\{\hat u_n>M\})},$$

where the equality holds only if $|B\cap\{\hat u_n>M\}| = 0$. Therefore, (3.17) implies that $\hat u_n \leqslant M$ a.e. in B. This yields the second inequality of (3.9). The first one can be shown in a similar way. □

Lemma 3.8. *Let X be a Banach space, and let $a,b\in\mathbb{R}$, with $a<b$. Consider a sequence $z_n \in W^{1,1}(a,b;X)$ which converges strongly in $L^1(a,b;X)$ to some function z. Then, for any $\bar s\in[a,b]$, any $s_n\in[a,b]$ which converges to $\bar s$, and any $\gamma>0$, we have*

$$\limsup_{n\to\infty}\Big\| z_n(s_n) - \fint_{I_\gamma} z\,ds\Big\|_X \leqslant \limsup_{n\to\infty}\int_{I_\gamma}\Big\|\frac{dz_n}{ds}\Big\|_X\,ds, \tag{3.18}$$

where $I_\gamma := [a,b]\cap(\bar s-\gamma, \bar s+\gamma)$.

Proof. For any $n\in\mathbb{N}$, consider $\zeta_n' \in X'$ such that

$$\|\zeta_n'\|_{X'} = 1, \qquad \Big\langle \zeta_n', z_n(s_n) - \fint_{I_\gamma} z\,ds\Big\rangle_{X',X} = \Big\| z_n(s_n) - \fint_{I_\gamma} z\,ds\Big\|_X.$$

Taking $s\in I_\gamma$ and n large enough, such that $s_n \in I_\gamma$, we have

$$\langle \zeta_n', z_n(s_n) - z(s)\rangle_{X',X} \leqslant \int_{I_\gamma}\Big|\Big\langle\zeta_n', \frac{dz_n}{ds}\Big\rangle_{X',X}\Big|\,ds \leqslant \int_{I_\gamma}\Big\|\frac{dz_n}{ds}\Big\|_X\,ds.$$

Integrating this inequality with respect to s in I_γ and dividing by $|I_\gamma|$, we get that

$$\Big\| z_n(s_n) - \fint_{I_\gamma} z_n\,ds\Big\|_X \leqslant \int_{I_\gamma}\Big\|\frac{dz_n}{ds}\Big\|_X\,ds.$$

Finally, taking the limsup in n in the previous inequality and using the strong convergence of z_n to z in $L^1(a,b;X)$, it follows (3.18). □

4. An example with loss of compactness

In this section we study a sequence of nonlinear conductivity equations which induces a nonlocal limit behavior due to the loss of the coercivity condition (3.3).

4.1. Statement of the problem

Let $\Omega := \omega \times (0,1)$ be the cylinder of $\mathbb{R}^N$, for $N \geqslant 2$, the basis of which ω is a regular bounded connected open set of $\mathbb{R}^{N-1}$ containing the origin. Any point $x \in \Omega$ is represented by the pair (x', x_N), where $x' \in \omega$ and $x_N \in (0,1)$. For any $0 < r < s$, denote by B'_r the open ball in $\mathbb{R}^{N-1}$ of radius r, by $A'_{r,s}$ the open annulus in $\mathbb{R}^{N-1}$ of inner radius r and outer radius s, by C_r the open cylinder in $\mathbb{R}^N$ of basis B'_r and of height 1, and by $C_{r,s}$ the open cylinder in $\mathbb{R}^N$ of basis $A'_{r,s}$ and of height 1, i.e.

$$\begin{cases} B'_r := \{x \in \mathbb{R}^{N-1}\colon |x'| < r\}, & C_r := B'_r \times (0,1), \\ A'_{r,s} := \{x \in \mathbb{R}^{N-1}\colon r < |x'| < s\}, & C_{r,s} := A'_{r,s} \times (0,1). \end{cases} \tag{4.1}$$

Let ε_n, $n \in \mathbb{N}$, be a positive sequence which converges to 0, simply denoted by ε. For a given $p > N-1$, consider the columnar conductivity function a_ε defined in Ω by

$$a_\varepsilon(x) = a_\varepsilon(x', x_N) := \begin{cases} \varepsilon^{1-N} \gg 1 & \text{if } |x'| < \varepsilon, \\ \varepsilon^{p-N+1} \ll 1 & \text{if } \varepsilon \leqslant |x'| \leqslant 2\varepsilon, \\ 1 & \text{if } |x'| > 2\varepsilon. \end{cases} \tag{4.2}$$

Our aim is to derive the Γ-limit for the strong topology of $L^1(\Omega)$ of the sequence $F_\varepsilon : L^1(\Omega) \to \mathbb{R}$ defined by

$$F_\varepsilon(v) := \begin{cases} \int_\Omega a_\varepsilon |\nabla v|^p\,dx, & \text{if } v \in W_0^{1,1}(\Omega), \\ \infty & \text{if } v \in L^1(\Omega) \setminus W_0^{1,1}(\Omega). \end{cases} \tag{4.3}$$

Remark 4.1. On the one hand, the weak coercivity condition (3.3) is not satisfied by the functionals F_ε of (4.3). Indeed, consider the function $v_\varepsilon \in W_0^{1,1}(\Omega)$ defined by

$$v_\varepsilon(x) = v_\varepsilon(x', x_N) := \begin{cases} 0 & \text{if } |x'| < \varepsilon, \\ (\frac{|x'|}{\varepsilon} - 1)\theta(x) & \text{if } \varepsilon \leqslant |x'| \leqslant 2\varepsilon, \\ \theta(x) & \text{if } |x'| > 2\varepsilon, \end{cases} \tag{4.4}$$

where $\theta \in C_0^1(\Omega)$ and $\theta \equiv 1$ in $\frac{1}{2}\omega \times (\frac{1}{4}, \frac{3}{4}) \Subset \Omega$. The sequence v_ε satisfies

$$F_\varepsilon(v_\varepsilon) \leqslant c_\theta \left(\varepsilon^{p-N+1} \int_\varepsilon^{2\varepsilon} \varepsilon^{-p} r^{N-2}\,dr + 1 \right) \leqslant c,$$

and for any $q > N-1$,

$$\int_\Omega |\nabla v_\varepsilon|^q\,dx \geqslant c\left(\int_\varepsilon^{2\varepsilon} \varepsilon^{-q} r^{N-2}\,dr - 1 \right) \geqslant c\varepsilon^{N-1-q} \xrightarrow[\varepsilon \to 0]{} \infty.$$

Therefore, the sequence v_ε has a bounded energy, but is not compact in $W^{1,q}(\Omega)$ weak for any $q > N-1$. This contradicts the second assertion of assumption (3.3).

More precisely, it is easy to check that the energy density $f_n(x, \xi) := a_\varepsilon(x)|\xi|^p$, defined with a_ε of (4.2), satisfies the assumptions (2.1)–(2.4) (with $g_n = f_n$), and (2.11), (2.12) (with $w^i_\varepsilon(x) = x_i$), of the homogenization Theorem 2.4 about the local Γ-limit for the strong topology of $L^\infty(\Omega)$ of the sequence F_ε. However, in contrast with the class of admissible convex densities defined in Proposition 3.2, a_ε^{-r} is bounded in $L^1(\Omega)$, with $r = (N-1)/(p-N+1)$, but is not equi-integrable in $L^1(\Omega)$. Moreover, due to estimates (3.5) and (3.6) any sequence of bounded energy F_ε, is bounded in $W^{1,1}(\Omega)$ and thus compact in $L^1(\Omega)$. Therefore, the nonlocal results of Theorem 4.2 and Theorem 4.4 below show that the weak coercivity condition (3.3), or the one of Proposition 3.2, is crucial for deriving the Γ-limit representation of Theorem 1.1.

On the other hand, Theorem 3.4 of [17] implies that the Γ-limit for the strong topology of $L^1(\Omega)$ of the sequence F_ε defined by (4.3), is local if a_ε is bounded and equi-integrable in $L^1(\Omega)$. However, the sequence a_ε defined by (4.2) is bounded in $L^1(\Omega)$, but is not equi-integrable in $L^1(\Omega)$. In dimension $N = 2$, the result of [17] and Theorem 1.1 with the assumption of Proposition 3.2, prove actually that the Γ-limit for the strong topology of $L^1(\Omega)$ of the sequence F_ε is local if a_ε or $a_\varepsilon^{-r} = a_\varepsilon^{1/1-p}$ is equi-integrable in $L^1(\Omega)$. This is exactly the opposite to the sequence a_ε of (4.2). Therefore, the equi-integrability in $L^1(\Omega)$ of a_ε or $a_\varepsilon^{1/1-p}$, is essential for preventing the appearance of nonlocal effects in dimension two.

We have the following result when $N > 2$:

Theorem 4.2. *Assume that $N > 2$ and $p > N - 1$. Then, defining $\gamma_{N,p}$ as the positive constant*

$$\gamma_{N,p} := \left(\frac{q}{2^q - 1}\right)^{p-1}, \quad \text{with } q := \frac{p - N + 1}{p - 1}, \tag{4.5}$$

and S_{N-2} as the unit sphere of $\mathbb{R}^{N-1}$, the sequence F_ε defined by (4.3) Γ-converges for the strong topology of $L^1(\Omega)$ to the functional F defined by

$$\begin{cases} F(v) := \displaystyle\int_\Omega |\nabla v|^p\,dx + |S_{N-2}| \min_{\hat{v} \in W_0^{1,p}(0,1)} \left\{ \int_0^1 \left(\frac{1}{N-1} \left| \frac{d\hat{v}}{dx_N} \right|^p + \gamma_{N,p} \left| \hat{v} - v(0, x_N) \right|^p \right) dx_N \right\}, \\ \quad \text{if } v \in W_0^{1,p}(\Omega), \\ F(v) := \infty \quad \text{if } v \in L^1(\Omega) \setminus W_0^{1,p}(\Omega). \end{cases} \tag{4.6}$$

Remark 4.3. Since $L^p(0, 1; W^{1,p}(\omega))$ is embedded in $L^p(0, 1; C^0(\bar{\omega}))$ for $p > N - 1$ (recall that ω is regular), so is $W^{1,p}(\Omega) \subset L^p(0, 1; W^{1,p}(\omega))$. This shows that any function $v \in W^{1,p}(\Omega)$, with $p > N - 1$, has a trace $v(0, \cdot)$ in $L^p(0, 1)$ on the line $\{x' = 0\}$ of Ω.

The case $N = 2$ is different since $\omega \setminus \{0\}$ (ω is an open interval containing 0) is not connected. Denoting $\Omega^L := \Omega \cap \{x_1 < 0\}$ and $\Omega^R := \Omega \cap \{x_1 > 0\}$, we have the following result:

Theorem 4.4. *Assume $N = 2$ and $p > 1$. Then, the sequence F_ε defined by (4.3) Γ-converges for the strong topology of $L^1(\Omega)$ to the functional F defined by*

$$\begin{cases} F(v) := \displaystyle\int_{\Omega^L \cup \Omega^R} |\nabla v|^p\,dx + \min_{\hat{v} \in W_0^{1,p}(0,1)} \left\{ \int_0^1 \left(2\left| \frac{d\hat{v}}{dx_2} \right|^p + \left| \hat{u} - u_L(0, x_2) \right|^p + \left| \hat{u} - u_R(0, x_2) \right|^p \right) dx_2 \right\}, \\ \quad \text{if } v \in W^{1,p}(\Omega^L \cup \Omega^L),\ v = 0 \text{ on } \partial\Omega,\ v = \chi_{\Omega^L} v_L + \chi_{\Omega^R} v_R, \\ F(v) := \infty \quad \text{elsewhere}. \end{cases} \tag{4.7}$$

The result of Theorem 4.4 is similar to the result of Theorem 4.2 in each connected part of $\Omega \setminus \{x_1 = 0\}$. As a consequence we will prove only Theorem 4.2.

Remark 4.5. The asymptotic behavior of F_ε induces a nonlocal Γ-limit F. A similar result was obtained in [4] for $N = 3$ and $p \leqslant 2 = N - 1$, with a unit conductivity medium reinforced by a periodic distribution of high conductivity cylinders. Here we have $p > N - 1$, and the nonlocal effect is due to the columnar arrangement of the low conductivity region separating the unit conductivity region and the high conductivity one. Moreover, our result is not obtained by a homogenization procedure as in [4], but by a concentration effect on a line.

4.2. *Proof of Theorem 4.2*

We need the following technical results (using notations (4.1) and (4.5)):

Lemma 4.6. *Let $p > N - 1 \geqslant 2$. There exists a constant $C > 0$ such that for any $0 < 2r \leqslant s$ and any $v \in W^{1,p}(C_{r,s})$, we have, with $q := (p - N + 1)/(p - 1)$,*

$$\int_0^1 \left| \fint_{\partial B'_r} v(\cdot, x_N)\, d\sigma' - \fint_{\partial B'_s} v(\cdot, x_N)\, d\sigma' \right|^p dx_N \leqslant C \left| r^q - s^q \right|^{p-1} \int_{C_{r,s}} |\nabla_{x'} v|^p\, dx, \tag{4.8}$$

$$\left\| v - \fint_{\partial B'_s} v\, d\sigma' \right\|^p_{L^p(0,1;C^0(\partial B'_r))} \leqslant C s^{p-N+1} \int_{C_{r,s}} |\nabla_{x'} v|^p\, dx. \tag{4.9}$$

Lemma 4.7. *Let $u \in W_0^{1,p}(\Omega)$ with $p > N-1$. Then, there exists a sequence $\tilde{u}_\varepsilon$ which strongly converges to u in $W_0^{1,p}(\Omega)$, and such that $\tilde{u}_\varepsilon$ only depends on the variable x_N in $C_{2\varepsilon}$.*

Lemma 4.8. *Let $p > N-1 \geqslant 2$. There exists a constant $C > 0$ such that for any $\varepsilon > 0$, the functional (4.3) satisfies the estimate from below*

$$\forall v \in W_0^{1,p}(\Omega), \quad \int_\Omega |v|^p\, dx \leqslant C F_\varepsilon(v). \tag{4.10}$$

Lemma 4.9. *Let u_ε be a sequence in $W_0^{1,p}(\Omega)$ such that $F_\varepsilon(u_\varepsilon)$ is bounded. Define the rescaled function in the cylinder C_2 by*

$$\hat{u}_\varepsilon(y) := u_\varepsilon\big(\varepsilon y', y_N\big), \quad \textit{for } \left|y'\right| < 2,\ y_N \in (0,1). \tag{4.11}$$

Then, there exist $u \in W_0^{1,p}(\Omega)$, $\hat{u}^- \in W_0^{1,p}(0,1)$, $\hat{u}^+ \in L^p(0,1; W^{1,p}(A'_{1,2}))$, such that the following convergences hold up to a subsequence:

$$\begin{cases} u_\varepsilon \rightharpoonup u & \textit{weakly in } L^p(\Omega), \\ u_\varepsilon \rightharpoonup u & \textit{weakly in } W^{1,p}(\Omega \setminus C_\delta) \textit{ for small enough } \delta > 0, \\ \hat{u}_\varepsilon \rightharpoonup \hat{u}^- & \textit{weakly in } W^{1,p}(C_1), \\ \hat{u}_\varepsilon \rightharpoonup \hat{u}^+ & \textit{weakly in } L^p\big(0,1; W^{1,p}\big(A'_{1,2}\big)\big), \\ \varepsilon\hat{u}_\varepsilon \rightharpoonup 0 & \textit{weakly in } W^{1,p}(C_{1,2}), \end{cases} \tag{4.12}$$

together with the boundary conditions

$$\hat{u}^+\big(y', y_N\big) = \begin{cases} \hat{u}^-(y_N) & \textit{if } |y'| = 1, \\ u(0, y_N) & \textit{if } |y'| = 2. \end{cases} \tag{4.13}$$

Proof of Theorem 4.2. We need to prove the Γ-liminf and the Γ-limsup inequalities (1.6) and (1.7).
Proof of the Γ-liminf inequality. Consider a sequence u_ε in $W_0^{1,p}(\Omega)$ which converges strongly to a function u in $L^1(\Omega)$, and such that $F_\varepsilon(u_\varepsilon)$ is bounded. Defining $\hat{u}_\varepsilon$ by (4.11) and applying Lemma 4.9 it follows that u belongs to $W_0^{1,p}(\Omega)$ and up to a subsequence, there exist $\hat{u}^- \in W_0^{1,p}(0,1)$ and $\hat{u}^+ \in L^p(0,1; W^{1,p}(A'_{1,2}))$ such that (4.12) and (4.13) hold. Then, the lower semicontinuity of the norm for the weak convergence in L^p implies that

$$\begin{aligned} \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) &= \liminf_{\varepsilon\to 0} \bigg(\int_{C_1} \big(\left|\varepsilon^{-1} \nabla_{y'} \hat{u}_\varepsilon\right|^2 + |\partial_{y_N} \hat{u}_\varepsilon|^2 \big)^{\frac{p}{2}}\, dy \\ &\quad + \int_{C_{1,2}} \big(|\nabla_{y'} \hat{u}_\varepsilon|^2 + |\varepsilon \partial_{y_N} \hat{u}_\varepsilon|^2 \big)^{\frac{p}{2}}\, dy + \int_{\Omega \setminus C_{2\varepsilon}} |\nabla u_\varepsilon|^p\, dx \bigg) \\ &\geqslant \frac{|S_{N-2}|}{N-1} \int_0^1 \left| \frac{d\hat{u}^-}{dy_N} \right|^p dy_N + \int_{C_{1,2}} \left|\nabla_{y'} \hat{u}^+\right|^p dy + \int_\Omega |\nabla u|^p\, dx. \end{aligned}$$

Minimizing the right-hand side of the previous inequality with respect to the functions $\hat{u}^-$ in $W_0^{1,p}(0,1)$ and $\hat{u}^+$ in $L^p(0,1;W^{1,p}(A'_{1,2}))$ satisfying (4.13), we thus obtain that

$$\liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \geqslant F(u).$$

Proof of the Γ-limsup inequality. For $u \in W_0^{1,p}(\Omega)$, consider the sequence $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ defined by (4.15) in Lemma 4.7 above. Then, from the minimizer $\hat{u} \in W_0^{1,p}(0,1)$ of the left-hand side of (4.6) with v replaced by u and $q = (p-N+1)/(p-1)$, we define u_ε in $W_0^{1,p}(\Omega)$ by

$$u_\varepsilon(x) := \begin{cases} \hat{u}(x_N) & \text{if } |x'| < \varepsilon, \\ \hat{u}(x_N) + \frac{\tilde{u}_\varepsilon(0,x_N)-\hat{u}(x_N)}{2^q-1}\left(\frac{|x'|^q}{\varepsilon^q} - 1\right) & \text{if } \varepsilon \leqslant |x'| \leqslant 2\varepsilon, \\ \tilde{u}_\varepsilon(x) & \text{if } 2\varepsilon < |x'|. \end{cases}$$

The sequence u_ε converges strongly to u in $L^1(\Omega)$. Moreover, making the change of variable $r = |x'|/\varepsilon$ in $C_{\varepsilon,2\varepsilon}$, we get that

$$F_\varepsilon(u_\varepsilon) = \int_{\Omega\setminus C_{2\varepsilon}} |\nabla\tilde{u}_\varepsilon|^p\,dx + \frac{|S_{N-2}|}{N-1}\int_0^1 \left|\frac{d\hat{u}}{dx_N}\right|^p dx_N$$
$$+ |S_{N-2}|\int_0^1\int_1^2 \left(\left|\frac{\tilde{u}_\varepsilon(0,x_N)-\hat{u}}{2^q-1}qr^{q-1}\right|^2 + \varepsilon^2\left|\frac{d\hat{u}}{dx_N} + \left(\partial_{x_N}\tilde{u}_\varepsilon - \frac{d\hat{u}}{dx_N}\right)\frac{r^q-1}{2^q-1}\right|^2\right)^{\frac{p}{2}} r^{N-2}\,dr\,dx_N.$$

In the last term of this expression we use that

$$\int_0^1\int_1^2 \varepsilon^p\left|\frac{d\hat{u}}{dx_N} + \left(\partial_{x_N}\tilde{u}_\varepsilon - \frac{d\hat{u}}{dx_N}\right)\frac{r^q-1}{2^q-1}\right|^p r^{N-2}\,dr\,dx_N$$
$$\leqslant \int_0^1\int_1^2 \varepsilon^p\left(\left|\frac{d\hat{u}}{dx_N}\right|^p + |\partial_{x_N}\tilde{u}_\varepsilon|^p\right) r^{N-2}\,dr\,dx_N$$
$$= \frac{2^{N-1}-1}{N-1}\varepsilon^p\int_0^1\left|\frac{d\hat{u}}{dx_N}\right|^p dx_N + \frac{\varepsilon^{p-N+1}}{|S_{N-1}|}\int_{C_{\varepsilon,2\varepsilon}} |\nabla\tilde{u}_\varepsilon|^p\,dx \xrightarrow[\varepsilon\to 0]{} 0,$$

which implies

$$\lim_{\varepsilon\to 0}\int_0^1\int_1^2 r^{N-2}\left(\left|\frac{\tilde{u}_\varepsilon(0,x_N)-\hat{u}}{2^q-1}qr^{q-1}\right|^2 + \varepsilon^2\left|\frac{d\hat{u}}{dx_N} - \left(\partial_{x_N}\tilde{u}_\varepsilon - \frac{d\hat{u}}{dx_N}\right)\frac{r^q-1}{2^q-1}\right|^2\right)^{\frac{p}{2}} dr\,dx_N$$
$$= \left(\frac{q}{2^q-1}\right)^p \lim_{\varepsilon\to 0}\int_0^1\int_1^2 r^{-\frac{N-2}{p-1}}\left|\tilde{u}_\varepsilon(0,x_N)-\hat{u}\right|^p dr\,dx_N = \gamma_{N,p}\int_0^1 \left|u(0,x_N)-\hat{u}\right|^p dx_N.$$

Therefore, we obtain

$$\lim_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) = \int_\Omega |\nabla u|^p\,dx + |S_{N-2}|\int_0^1\left(\frac{1}{N-1}\left|\frac{d\hat{u}}{dx_N}\right|^p + \gamma_{N,p}|\hat{u}-u(0,x_N)|^p\right)dx_N = F(u). \qquad \square$$

4.3. Proof of the technical lemmas

Proof of Lemma 4.6. Let $0 < 2r \leqslant s$ and let $v \in C_c^1(\mathbb{R}^{N-1})$. By the Hölder inequality we have

$$\left| \fint_{\partial B'_r} v \, d\sigma' - \fint_{\partial B'_s} v \, d\sigma' \right|^p = \left| \fint_{S_{N-2}} \int_r^s \nabla v(ty) \cdot y \, dt \, d\sigma' \right|^p$$
$$\leqslant \left(\int_r^s t^{\frac{2-N}{p-1}} \, dt \right)^{p-1} \fint_{S_{N-2}} \int_r^s \left| \nabla_{x'} v(ty) \right|^p t^{N-2} \, dt \, d\sigma'$$
$$\leqslant c \left(s^q - r^q \right)^{p-1} \int_{A'_{r,s}} |\nabla_{x'} v|^p \, dx'. \tag{4.14}$$

By a density argument estimate (4.14) also holds for $v \in W^{1,p}(A'_{r,s})$. Now, for $v \in W^{1,p}(C_{r,s})$ and for a.e. $x_N \in (0,1)$, the function $v(\cdot, x_N)$ belongs to $W^{1,p}(A'_{r,s})$ and satisfies (4.14). Hence, integrating with respect to $x_N \in (0,1)$ it follows (4.8).

On the other hand, by the Morrey embedding of $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^{N-1})$ into $C^0_{\mathrm{loc}}(\mathbb{R}^{N-1})$ for $p > N-1$ (see, e.g., [11]), for a.e. $x_N \in (0,1)$, the function $v(\cdot, x_N)$ is continuous in the closed annulus $\bar{A}'_{r,s}$. Then, r-rescaling the inequality associated with the Morrey embedding $W^{1,p}(A'_{1,2}) \hookrightarrow C^0(\bar{A}'_{1,2})$ we get that for any $x' \in \partial B'_r$ and for a.e. $x_N \in (0,1)$,

$$\left| v\left(x', x_N\right) - \fint_{\partial B'_r} v(\cdot, x_N) \, d\sigma' \right|^p \leqslant C r^{p-N+1} \int_{A'_{r,2r}} \left| \nabla_{y'} v(\cdot, x_N) \right|^p dy'.$$

This combined with estimate (4.14) and $2r < s$, implies that for any $x' \in \partial B'_r$ and for a.e. $x_N \in (0,1)$,

$$\left| v\left(x', x_N\right) - \fint_{\partial B'_s} v(\cdot, x_N) \, d\sigma' \right|^p \leqslant c \left| v\left(x', x_N\right) - \fint_{\partial B'_r} v(\cdot, x_N) \, d\sigma' \right|^p + c \left| \fint_{\partial B'_r} v \, d\sigma' - \fint_{\partial B'_s} v \, d\sigma' \right|^p$$
$$\leqslant c s^{p-N+1} \int_{A'_{r,s}} \left| \nabla_{y'} v(\cdot, x_N) \right|^p dy',$$

which yields estimate (4.9) by integrating over $(0,1)$. □

Proof of Lemma 4.7. Let $u \in W_0^{1,p}(\Omega)$ and let $\psi_\varepsilon \in C^1(\bar{\omega}; [0,1])$ be such that

$$\psi_\varepsilon = 1 \quad \text{in } B'_{2\varepsilon}, \qquad \psi_\varepsilon = 0 \quad \text{in } \omega \setminus B'_{3\varepsilon} \quad \text{and} \quad \|\nabla_{x'} \psi_\varepsilon\|_{C^0(\bar{\omega})} \leqslant \frac{c}{\varepsilon}.$$

Consider the function $\tilde{u}_\varepsilon \in W_0^{1,p}(\Omega)$ defined by

$$\tilde{u}_\varepsilon(x) := \psi_\varepsilon\left(x'\right) \fint_{\partial B'_{6\varepsilon}} u(\cdot, x_N) \, d\sigma' + \left(1 - \psi_\varepsilon\left(x'\right)\right) u(x). \tag{4.15}$$

It is clear that $\tilde{u}_\varepsilon$ only depends on x_N in $C_{2\varepsilon}$, and converges strongly to u in $L^p(\Omega)$. Moreover, using estimate (4.9) we get that

$$\|\nabla \tilde{u}_\varepsilon - \nabla u\|^p_{L^p(\Omega)^N} \leqslant \left\| \nabla_{x'} \psi_\varepsilon \left(u - \fint_{\partial B'_{6\varepsilon}} u \, d\sigma' \right) \right\|^p_{L^p(\Omega)^N} + \|\psi_\varepsilon \nabla u\|^p_{L^p(\Omega)^N}$$
$$\leqslant \varepsilon^{-p} \int_{2\varepsilon}^{3\varepsilon} \left\| u - \fint_{\partial B'_{6\varepsilon}} u \, d\sigma' \right\|^p_{L^p(0,1;C^0(\partial B'_r))} r^{N-2} \, dr + \|\psi_\varepsilon \nabla u\|^p_{L^p(\Omega)^N}$$
$$\leqslant c \|\nabla_{x'} u\|^p_{L^p(C_{2\varepsilon,3\varepsilon})} + \|\psi_\varepsilon \nabla u\|^p_{L^p(\Omega)^N}$$

which tends to 0 by virtue of Lebesgue's dominated convergence theorem. Therefore, the sequence $\tilde{u}_\varepsilon$ converges strongly to u in $W^{1,p}(\Omega)$. $\square$

Proof of Lemma 4.8. It is well known that there exists a constant $c > 0$ such that

$$\forall V \in W^{1,p}(C_2), \qquad \int_{C_{1,2}} |V|^p\,dy \leqslant \int_{C_2} |V|^p\,dy \leqslant c\int_{C_2} |\nabla V|^p\,dy + c\int_{C_1} |V|^p\,dy. \tag{4.16}$$

Hence, by ε-rescaling (4.16) with the function $v(x) = V(x/\varepsilon)$ and noting that $a_\varepsilon \geqslant \varepsilon^{p-N+1}$ in the cylinder $C_{2\varepsilon}$, it follows that

$$\int_{C_{\varepsilon,2\varepsilon}} |v|^p\,dx \leqslant c\varepsilon^p \int_{C_{2\varepsilon}} |\nabla v|^p\,dx + c\int_{C_\varepsilon} |v|^p\,dx \leqslant c\varepsilon^{N-1}\int_{C_{2\varepsilon}} a_\varepsilon|\nabla v|^p\,dx + c\int_{C_\varepsilon} |v|^p\,dx. \tag{4.17}$$

On the other hand, using that $v(x) = \int_0^{x_N} \partial_t v(x',t)\,dt$ in Ω, for $v \in W_0^{1,p}(\Omega)$, we have

$$\int_{C_\varepsilon} |v|^p\,dx \leqslant c\int_{C_\varepsilon} |\partial_{x_N} v|^p\,dx \leqslant c\varepsilon^{N-1}\int_{C_\varepsilon} a_\varepsilon|\nabla v|^p\,dx,$$

$$\int_{\Omega\setminus C_{2\varepsilon}} |v|^p\,dx \leqslant c\int_{\Omega\setminus C_{2\varepsilon}} |\partial_{x_N} v|^p\,dx \leqslant c\int_{\Omega\setminus C_{2\varepsilon}} a_\varepsilon|\nabla v|^p\,dx.$$

This combined with (4.17) yields

$$\int_\Omega |v|^p\,dx = \int_{C_\varepsilon} |v|^p\,dx + \int_{C_{\varepsilon,2\varepsilon}} |v|^p\,dx + \int_{\Omega\setminus C_{2\varepsilon}} |v|^p\,dx \leqslant C\int_\Omega a_\varepsilon|\nabla v|^p\,dx,$$

which implies the desired estimate (4.10). $\square$

Proof of Lemma 4.9.
Proof of the two first convergences of (4.12) *with* $u \in W_0^{1,p}(\Omega)$: Let u_ε be a function in $W_0^{1,p}(\Omega)$, with $F_\varepsilon(u_\varepsilon) \leqslant c$. By estimate (4.10) the sequence u_ε is bounded in $L^p(\Omega)$ and up to a subsequence converges weakly to some u in $L^p(\Omega)$. Moreover, for any $\delta > 0$, u_ε is clearly bounded in $W^{1,p}(\Omega \setminus C_\delta)$, which implies that $u \in W^{1,p}(\Omega \setminus C_\delta)$ and by lower semicontinuity

$$\int_{\Omega\setminus C_\delta} |\nabla u|^p\,dx \leqslant \liminf_{\varepsilon\to 0}\int_{\Omega\setminus C_\delta} |\nabla u_\varepsilon|^p\,dx \leqslant \liminf_{\varepsilon\to 0} F_\varepsilon(u_\varepsilon) \leqslant c, \tag{4.18}$$

where the bound is independent of δ. This establishes the second convergence of (4.12).

It remains to prove that $u \in W_0^{1,p}(\Omega)$. Let $\varphi \in C^1(\bar{\Omega})$ and let $\delta > 0$ be small enough. By the Hölder inequality we have for $i \in \{1,\dots,N\}$,

$$\left|\int_\Omega u\partial_i\varphi\,dx - \int_{\Omega\setminus C_\delta} u\partial_i\varphi\,dx\right| = \left|\int_{C_\delta} u\partial_i\varphi\,dx\right| \leqslant c_\varphi\|u\|_{L^p(\Omega)}|C_\delta|^{\frac{1}{p'}} \leqslant c_\varphi\delta^{\frac{N-1}{p'}}. \tag{4.19}$$

Then, integrating by parts, using that $u(x',0) = u(x',1) = 0$ for $x' \in \omega \setminus B'_\delta$ (as a consequence of the second convergence of (4.12)) and the uniform estimate (4.18), we get that

$$\left|\int_{\Omega\setminus C_\delta} u\partial_i\varphi\,dx - \int_{\partial B'_\delta\times(0,1)} u\varphi n_i\,d\sigma'\right| = \left|\int_{\Omega\setminus C_\delta} \partial_i u\varphi\,dx\right| \leqslant c\|\varphi\|_{L^{p'}(\Omega)}, \tag{4.20}$$

where the constant c is independent of δ. On the other hand, by estimates (4.8), (4.9) the boundary integral satisfies

$$\begin{aligned}\Bigg|\int_{\partial B'_\delta\times(0,1)} u\varphi n_i\,d\sigma'\Bigg| &\leqslant \Bigg|\int_{\partial B'_\delta\times(0,1)}\Big(u-⨍_{\partial B'_{2\delta}} u\Big)\varphi n_i\,d\sigma'\Bigg| + \Bigg|\int_{\partial B'_\delta\times(0,1)}\Big(⨍_{\partial B'_{2\delta}} u\Big)\varphi n_i\,d\sigma'\Bigg| \\ &\leqslant c_\varphi\int_0^1\Bigg\|u-⨍_{\partial B'_{2\delta}} u\,d\sigma'\Bigg\|_{C^0(\partial B'_\delta)} dx_N + c_\varphi|\partial B'_\delta|\int_0^1\Bigg|⨍_{\partial B'_{2\delta}} u\Bigg|\,dx_N \\ &\leqslant c_\varphi\big(\delta^{\frac{p-N+1}{p}}+\delta^{N-2}\big).\end{aligned} \tag{4.21}$$

Therefore, combining estimates (4.19), (4.20), (4.21), and passing to the limit $\delta\to 0$, we obtain that there exists a constant $c>0$ such that

$$\forall\varphi\in C^1(\bar\Omega),\qquad \Bigg|\int_\Omega u\,\partial_i\varphi\,dx\Bigg|\leqslant c\|\varphi\|_{L^{p'}(\Omega)},$$

which implies that $u\in W_0^{1,p}(\Omega)$.

Now, making the change of variables $x'=\varepsilon y'$ with the function $\hat u_\varepsilon$ defined by (4.11), we obtain the estimate

$$\int_{C_1}\big(\varepsilon^{-p}|\nabla_{y'}\hat u_\varepsilon|^p+|\partial_{y_N}\hat u_\varepsilon|^p\big)\,dy+\int_{C_{1,2}}\big(|\nabla_{y'}\hat u_\varepsilon|^p+\varepsilon^p|\partial_{y_N}\hat u_\varepsilon|^p\big)\,dy\leqslant F_\varepsilon(u_\varepsilon)\leqslant c, \tag{4.22}$$

which easily implies the fourth and the fifth convergences of (4.12) up to a subsequence.

Proof of $\hat u^+(y',y_N)=u(0,y_N)$ *for* $|y'|=2$: Taking into account the fourth convergence of (4.12), it is enough to prove that

$$\big\|\hat u_\varepsilon-u(0,\cdot)\big\|_{L^p(0,1;C^0(\partial B'_2))}\xrightarrow[\varepsilon\to 0]{}0. \tag{4.23}$$

Let be a fixed $\delta>0$. By estimate (4.9) and $F_\varepsilon(u_\varepsilon)\leqslant c$, we have for $4\varepsilon<\delta$,

$$\Bigg\|u_\varepsilon-⨍_{\partial B'_\delta} u_\varepsilon\,d\sigma'\Bigg\|^p_{L^p(0,1;C^0(\partial B'_{2\varepsilon}))}\leqslant C\delta^{p-N+1}\int_{C_{2\varepsilon,\delta}}|\nabla_{x'}u_\varepsilon|^p\,dx\leqslant c\delta^{p-N+1}.$$

This combined with the strong convergence of u_ε to u in $L^p(\partial B'_\delta\times(0,1))$ (as a consequence of the weak convergence in $W^{1,p}(\Omega\setminus C_\delta)$) gives

$$\limsup_{\varepsilon\to 0}\Bigg\|u_\varepsilon-⨍_{\partial B'_\delta} u\,d\sigma'\Bigg\|^p_{L^p(0,1;C^0(\partial B'_{2\varepsilon}))}\leqslant c\delta^{p-N+1}. \tag{4.24}$$

On the other hand, by the Morrey embedding of $W^{1,p}(B'_\delta)\hookrightarrow C^0(\bar B'_\delta)$ we have

$$\lim_{r\to 0}\Bigg(⨍_{\partial B'_r} u(\cdot,x_N)\,d\sigma'\Bigg)=u(0,x_N)\quad\text{for a.e. } x_N\in(0,1).$$

Using this limit and the Fatou lemma in (4.8) with $s=\delta$, we get that

$$\Bigg\|u(0,x_N)-⨍_{\partial B'_\delta} u(\cdot,x_N)\,d\sigma'\Bigg\|_{L^p(0,1)}\leqslant c\delta^{p-N+1}.$$

This combined with estimate (4.24) yields

$$\limsup_{\varepsilon\to 0}\big\|u_\varepsilon-u(0,x_N)\big\|^p_{L^p(0,1;C^0(\partial B'_{2\varepsilon}))}\leqslant c\delta^{p-N+1},$$

which implies

$$\big\|u_\varepsilon-u(0,x_N)\big\|^p_{L^p(0,1;C^0(\partial B'_{2\varepsilon}))}\xrightarrow[\varepsilon\to 0]{}0.$$

This limit is equivalent to (4.23).

Proof of the third convergence of (4.12): By estimate (4.22) the sequence $\hat{u}_\varepsilon$ is bounded in $W^{1,p}(C_1)$ with $\hat{u}_\varepsilon(\cdot,0)=\hat{u}_\varepsilon(\cdot,1)=0$, and the sequence $\nabla_{y'}\hat{u}_\varepsilon$ strongly converges to 0 in $L^p(C_1)^{N-1}$. Therefore, $\hat{u}_\varepsilon$ converges weakly to $\hat{u}^- \in W^{1,p}(0,1)$ in $W^{1,p}(C_1)$.

Proof of $\hat{u}(y',y_N)=\hat{u}^-(y_N)$ *for* $|y'|=1$: By the inequality associated with the Morrey embedding $W^{1,p}(B_1')\hookrightarrow C^0(\bar{B}_1')$, and by estimate (4.22), we have

$$\int_0^1 \left\| \hat{u}_\varepsilon - \fint_{B_1'} \hat{u}_\varepsilon\, dy' \right\|^p_{C^0(S_{N-2})} dy_N \leqslant c \int_0^1 \int_{B_1'} |\nabla_{y'}\hat{u}_\varepsilon|^p\, dy'\, dy_N \underset{\varepsilon\to 0}{\longrightarrow} 0,$$

and by the third convergence of (4.12) we also have

$$\fint_{B_1'} \hat{u}_\varepsilon\big(y',y_N\big)\, dy' \longrightarrow \fint_{B_1'} \hat{u}^-\big(y',y_N\big)\, dy' = \hat{u}^-(y_N) \quad \text{strongly in } L^p(0,1).$$

Hence, we deduce that

$$\int_0^1 \big\| \hat{u}_\varepsilon(\cdot,y_N) - \hat{u}^-(y_N)\big\|^p_{C^0(S_{N-2})}\, dy_N \underset{\varepsilon\to 0}{\longrightarrow} 0. \tag{4.25}$$

Moreover, by the third convergence of (4.12) and the Morrey compactness embedding of $W^{1,p}(A'_{1,2})$ into $C^0(\bar{A}'_{1,2})$, we have

$$\big\|\hat{u}_\varepsilon(\cdot,y_N) - \hat{u}^-(y_N)\big\|_{C^0(S_{N-2})} \underset{\varepsilon\to 0}{\longrightarrow} \big\|\hat{u}^+(\cdot,y_N) - \hat{u}^-(y_N)\big\|_{C^0(S_{N-2})} \quad \text{for a.e. } y_N \in (0,1).$$

This combined with the Fatou lemma and the strong convergence (4.25) implies the boundary condition $\hat{u}^+(y',y_N)=\hat{u}^-(y_N)$ for $|y'|=1$. □

Acknowledgement

The authors would like to thank the anonymous referees for their comments and suggestions, especially about Section 4, which have improved the presentation of this paper. This paper has been partially supported by the project MTM2011-24457 of the Spanish Ministry of "Econoía y Competitividad". The second author is grateful to the University of Rennes 1 for its invitation and to the INSA of Rennes for its hospitality in June 2011.

References

[1] E. Acerbi, V. Chiadò Piat, G. Dal Maso, D. Percivale, An extension theorem from connected sets, and homogenization in general periodic domains, Nonlinear Anal. 18 (5) (1992) 481–496.
[2] Allaire, Homogenization and two-scale convergence, SIAM J. Math. Anal. 23 (6) (1992) 1482–1518.
[3] J.P. Aubin, Un théorème de compacité, C. R. Math. Acad. Sci. Paris 256 (1963) 5042–5044.
[4] M. Bellieud, G. Bouchitté, Homogenization of elliptic problems in a fiber reinforced structure. Nonlocal effects, Ann. Sc. Norm. Super. Pisa Cl. Sci. 26 (4) (1998) 407–436.
[5] A. Beurling, J. Deny, Espaces de Dirichlet, Acta Matematica 99 (1958) 203–224.
[6] A. Braides, Γ-Convergence for Beginners, Oxford University Press, Oxford, 2002.
[7] A. Braides, M. Briane, Homogenization of non-linear variational problems with thin low-conducting layers, Appl. Math. Optim. 55 (1) (2007) 1–29.
[8] A. Braides, M. Briane, J. Casado-Díaz, Homogenization of non-uniformly bounded periodic diffusion energies in dimension two, Nonlinearity 22 (2009) 1459–1480.
[9] A. Braides, V. Chiadò Piat, A. Piatnitski, A variational approach to double-porosity problems, Asymptot. Anal. 39 (3–4) (2004) 281–308.
[10] A. Braides, A. Defranceschi, Homogenization of Multiple Integrals, Oxford University Press, Oxford, 1998.
[11] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, Springer, New York, 2011, p. 599.
[12] M. Briane, Nonlocal effects in two-dimensional conductivity, Arch. Ration. Mech. Anal. 182 (2) (2006) 255–267.
[13] M. Briane, J. Casado-Díaz, Two-dimensional div-curl results. Application to the lack of nonlocal effects in homogenization, Comm. Partial Differential Equations 32 (2007) 935–969.

[14] M. Briane, J. Casado-Díaz, Asymptotic behavior of equicoercive diffusion energies in two dimension, Calc. Var. Partial Differential Equations 29 (4) (2007) 455–479.
[15] G. Buttazzo, G. Dal Maso, Γ-limits of integral functionals, J. Anal. Math. 37 (1980) 145–185.
[16] M. Camar-Eddine, P. Seppecher, Closure of the set of diffusion functionals with respect to the Mosco-convergence, Math. Models Methods Appl. Sci. 12 (8) (2002) 1153–1176.
[17] L. Carbone, C. Sbordone, Some properties of Γ-limits of integral functionals, Ann. Mat. Pura Appl. 122 (1979) 1–60.
[18] G. Dal Maso, An Introduction to Γ-Convergence, Birkhäuser, Boston, 1993.
[19] E. De Giorgi, Sulla convergenza di alcune successioni di integrali del tipo dell'area, Rend. Mat. Roma 8 (1975) 277–294.
[20] E. De Giorgi, Γ-convergenza e G-convergenza, Boll. Unione Mat. Ital. A 14 (1977) 213–220.
[21] E. De Giorgi, T. Franzoni, Su un tipo di convergenza variazionale, Rend. Acc. Naz. Lincei Roma 58 (6) (1975) 842–850.
[22] V.N. Fenchenko, E.Ya. Khruslov, Asymptotic of solution of differential equations with strongly oscillating matrix of coefficients which does not satisfy the condition of uniform boundedness, Dokl. AN Ukr. SSR 4 (1981).
[23] B. Franchi, R. Serapioni, F. Serra Cassano, Irregular solutions of linear degenerate elliptic equations, Potential Anal. 9 (1998) 201–216.
[24] E.Ya. Khruslov, Homogenized models of composite media, in: G. Dal Maso, G.F. Dell'Antonio (Eds.), Composite Media and Homogenization Theory, in: Progr. Nonlinear Differential Equations Appl., Birkhäuser, 1991, pp. 159–182.
[25] E.Ya. Khruslov, V.A. Marchenko, Homogenization of Partial Differential Equations, Prog. Math. Phys., vol. 46, Birkhäuser, Boston, 2006.
[26] K. Kuratowski, C. Ryll-Nardzewski, A general theorem on selectors, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965) 397–403.
[27] J.J. Manfredi, Weakly monotone functions, J. Geom. Anal. 4 (3) (1994) 393–402.
[28] U. Mosco, Composite media and asymptotic Dirichlet forms, J. Funct. Anal. 123 (2) (1994) 368–421.
[29] F. Murat, H-convergence, in: Séminaire d'Analyse Fonctionnelle et Numérique, Université d'Alger, 1977–1978, multicopied, 34 pp.; English translation: F. Murat, L. Tartar, H-convergence, in: L. Cherkaev, R.V. Kohn (Eds.), Topics in the Mathematical Modelling of Composite Materials, in: Progr. Nonlinear Differential Equations Appl., vol. 31, Birkhäuser, Boston, 1998, pp. 21–43.
[30] S. Spagnolo, Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche, Ann. Sc. Norm. Super. Pisa Cl. Sci. 22 (3) (1968) 571–597.
[31] L. Tartar, The General Theory of Homogenization: A Personalized Introduction, Lect. Notes Unione Mat. Ital., Springer-Verlag, Berlin, Heidelberg, 2009, p. 471.
[32] V.V. Zhikov, Connectedness and averaging. Examples of fractal conductivity, Mat. Sb. 187 (8) (1996) 3–40 (in Russian); translation in: Sb. Math. 187 (8) (1996) 1109–1147.
[33] V.V. Zhikov, On an extension and an application of the two-scale convergence method, Mat. Sb. 191 (7) (2000) 31–72 (in Russian. Russian summary); translation in: Sb. Math. 191 (7–8) (2000) 973–1014.