A lower bound on the blow-up rate for the Davey–Stewartson system on the torus

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Abstract

We consider the hyperbolic–elliptic version of the Davey–Stewartson system with cubic nonlinearity posed on the two-dimensional torus. A natural setting for studying blow-up solutions for this equation takes place in H^s, $1/2 < s < 1$. In this paper, we prove a lower bound on the blow-up rate for these regularities.

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1. Introduction

We consider the Davey–Stewartson system defined on the two-dimensional torus $T^2 := \mathbb{R}^2/2\pi\mathbb{Z}^2$:

$$\begin{cases} i\partial_t u - \partial_x^2 u + \partial_y^2 u = -|u|^2 u + 2u\partial_x\phi, \\ (\partial_x^2 + \partial_y^2)\phi = \partial_x|u|^2, \end{cases} \tag{1}$$

where $u : \mathbb{R} \times T^2 \to \mathbb{C}$ and $\phi : \mathbb{R} \times T^2 \to \mathbb{R}$ are the unknowns. Rearranging the second equation, we may see this system as a dispersive equation with a hyperbolic linear part and a nonlocal nonlinearity:

$$i\partial_t u + Pu = -|u|^2 u - E(|u|^2)u, \quad (t, x) \in \mathbb{R} \times T^2, \tag{2}$$

where $P = -\partial_x^2 + \partial_y^2$ and E is the nonlocal operator such that:

$$\widehat{E(f)}(m, n) = \frac{2m^2}{m^2 + n^2}\widehat{f}(m, n), \quad (m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\},$$

$$\widehat{E(f)}(0, 0) = 0.$$

The Cauchy problem and the blow-up theory for this equation have been studied essentially in the case where the system is posed in $\mathbb{R}^2$. In this case, the system is locally well posed in the Sobolev spaces L^2, H^1 (see [5]) and more easily for higher regularities H^s, $s > 1$. In [14], T. Ozawa proved that the equation posed on $\mathbb{R}^2$ enjoys a pseudo-

conformal type symmetry and as for NLS, this allows to construct a blow-up solution by applying this transformation to an explicit stationary (periodic in time for NLS) solution:

$$u(t, x, y) = \frac{1}{1 + x^2 + y^2}.$$

This function is then transformed into

$$v(t, x, y) = \frac{1}{a + bt} \exp\left(\frac{ib}{4(a + bt)}(-x^2 + y^2)\right) \frac{1}{1 + (\frac{x}{a+bt})^2 + (\frac{y}{a+bt})^2}, \tag{3}$$

with $(a, b) \in \mathbb{R}^2$. Note that $v(t)$ is in $H^s(\mathbb{R}^2)$ (see [14]) for every $s < 1$ with $\|v(t)\|_{L^2} = \sqrt{\pi}$ but is not in $H^1(\mathbb{R}^2)$. The solution v blows up at time $T = -a/b$ in L^2 in the sense that the L^2 blow-up criteria (the $L^4_{[0,t]}L^4$ norm goes to infinity when t goes to T) is satisfied:

$$\|v\|_{L^4([0,t])L^4(\mathbb{R}^2)} \sim \frac{C}{(T - t)^{1/4}} \to \infty \quad \text{as } t \text{ goes to } T,$$

and accumulates all the mass in the origin:

$$|v(t)|^2 \to \pi \delta_{(0,0)} \quad \text{as } t \to T \text{ in } \mathcal{D}'(\mathbb{R}^2).$$

The explosion also occurs in H^s, $s < 1$ with the pseudo-conformal bound [14]:

$$\|v(t)\|_{H^s} \sim \frac{C}{(T - t)^s}.$$

Note that in $\mathbb{R}^2$, we have a scaling symmetry; if u solves (2) then for all $\lambda > 0$, $(t, x, y) \mapsto \lambda u(t, \lambda^2 t, \lambda x, \lambda y)$ also solves (2). It is a classical fact that this symmetry automatically implies a lower bound on the blow-up rate: for all blow-up solution u with maximal time $T(u) < \infty$, we have

$$\|u(t)\|_{H^s} \geqslant \frac{C}{(T(u) - t)^{s/2}}. \tag{4}$$

By analogy with NLS in $\mathbb{R}^2$, we may ask the question of existence of ground states of the type $u(t, x, y) = \exp(i\omega t)Q(x, y)$ for (1) posed on $\mathbb{R}^2$ with an exponentially decaying profile Q but the hyperbolicity of the operator $-\partial_x^2 + \partial_y^2$ forbids the existence of such solutions at least in the case where the nonlinearity is $-|u|^2u$ [6]. Moreover, numerics [11] seems to show that the L^2-norm of the solution u (or v) is the minimal mass for which we may have singularities in finite time. Thus, the function u plays the role of a ground state but is only polynomially decaying and this requires to work with low regularities H^s, $s < 1$.

The aim of this paper is twofold: first give an H^s framework for studying blow-up theory for (2) i.e. show the local well-posedness of (2) for initial data in $H^s(T^2)$, $s < 1$ and secondly show that the lower bound on the blow-up rate (4) still holds even if a scaling symmetry does not strictly make sense on the torus. The proof relies on local existence arguments on the dilated torus $\mathbb{R}^2/2\pi L\mathbb{Z}^2$, $L \to \infty$ and more precisely on bilinear Strichartz estimates. The classical method [2] giving well-posedness from bilinear Strichartz estimates does not work in our setting because of the nonlocal term; we will have to refine the bilinear approach with more general localizations. An interesting question, not solved here, is the localization of the solution v (3) i.e. construct from v a solution of (2). The nonexponential decay of v is reflected in the estimate

$$\|v(t)\|_{H^s(\varepsilon \leqslant |(x,y)| \leqslant A)} \leqslant C(T - t)^{(1-s)},$$

which make perturbation arguments around v difficult to apply even on a compact domain. In particular, for $s > 1$, blow-up is not localized and this explains our choice to treat low regularities.

Remark 1. The system we study is called the hyperbolic–elliptic version of the generalized Davey–Stewartson system:

$$\begin{cases} i\partial_t u + \varepsilon_1 \partial_x^2 u + \partial_y^2 u = -|u|^2u + 2u\partial_x\phi, \\ (\varepsilon_2 \partial_x^2 + \partial_y^2)\phi = \partial_x|u|^2, \end{cases} \tag{5}$$

where $\varepsilon_i \in \{-1, +1\}$. Depending on the values of ε_i, the local well-posedness holds [5,1,13,10,4] but blow-up theory is really well understood only in the elliptic–elliptic case $\varepsilon_1 = \varepsilon_2 = 1$ [12,17,15] where results are similar to those for NLS.

2. Statement of the result and remarks

Let us now give our result.

Theorem 1. *Let $s > 1/2$.*

1) *Eq.* (2) *is locally well posed in the space $H^s(T^2)$ in the following sense. There exists $b > 1/2$ such that the following holds true: for all $u_0 \in H^s(T^2)$, there exist a time $T > 0$ and a unique $u \in X_T^{s,b}(T^2) \subset \mathcal{C}([0, T], H^s(T^2))$ satisfying $u(0) = u_0$ and* (2). *Here, $X_T^{s,b}$ denotes the Bourgain space associated to* (2) *and defined in* (17).
2) *Let $u_0 \in H^s(T^2)$ and u be the corresponding solution. If $T(u_0)$ denotes the maximal time of existence of u, then we have the following possibilities: either $T(u_0) = +\infty$ or $T(u_0) < +\infty$ and in this case, there exists $C > 0$ such that for all $t \in [0, T(u_0))$:*

$$\|u(t)\|_{H^s(T^2)} \geqslant \frac{C}{(T(u_0) - t)^{s/2}}. \tag{6}$$

Before giving the proof of Theorem 1, let us give some comments. Consider the equation

$$i\partial_t u + Pu = -|u|^2 u, \quad t > 0, \ (x, y) \in T^2. \tag{7}$$

Strichartz type estimates hold for the operator P (see [8,16]) and this with an analysis similar to [2] gives the local well-posedness of (7) in $H^s(T^2)$ for all $s > 1/2$. Following [8], it is easy to check that the function defined by

$$u(t, x, y) = e^{it|u_0(x+y)|^2} u_0(x + y) \tag{8}$$

is a solution of (7) for all $u_0 \in \mathcal{C}^\infty(\mathbb{R}/2\pi\mathbb{Z}, \mathbb{R})$. If u_0 is not a constant function and if $s \geqslant 0$, we can check that there exists $C > 0$ such that for $t \geqslant 1$,

$$\|u(t)\|_{H^s} \geqslant Ct^s. \tag{9}$$

Thus, for $s > 1/2$, we obtain an explicit solution which blows up in infinite time; this contrasts with the usual Schrödinger equation. Using a suitable rescaling of the explicit solution (8), one may also show the local ill-posedness of (7) in H^s, $s < 1/2$ (see the appendix of [3] for a similar discussion). It would be interesting to know if we may construct solutions behaving like (8) for Eq. (2).

3. Proof of the result

Strategy of the proof. To prove Theorem 1, the idea is to rescale the torus $T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ by considering $T_L^2 = \mathbb{R}^2/2\pi L\mathbb{Z}^2$ where $L > 0$ will tend to infinity. In a first step, we perform a Banach fixed point argument in the dilated Bourgain space $X_{T,L}^{s,b}$ to obtain a local well-posedness result in $H^s(T_L^2)$ with the bound on the blow-up time:

$$T(u_0) \geqslant F(\|u_0\|_{H^s(T_L^2)}), \tag{10}$$

for some function F independent of L. This relies on a uniform bilinear Strichartz estimate. In our analysis, it is of importance that dispersive estimates are local in space and time. In particular, if we take $L = 1$, this step will give the first point of Theorem 1. In the step 2, we deduce the blow-up lower bound from a scaling argument. The bound (6) which is the same as $\mathbb{R}^2$ is in accordance with the fact that when L goes to infinity T_L^2 looks like $\mathbb{R}^2$ formally. Note that the machinery of the Bourgain spaces is natural to study such questions but we do not exclude the possibility of working on other spaces by adapting harmonic analysis results to the case of T_L^2 to treat the operator E.

Notations. We denote by $e_{m,n}(x, y) = (2\pi)^{-1}\exp(imx + iny)$ the usual orthonormal basis of $L^2(T_1^2)$. When working on T_L^2, we will keep the same notation $e_{m,n}$ for the rescaled basis: $e_{m,n}(x, y) = (2\pi L)^{-1}\exp(i(m/L)x + i(n/L)y)$. For a function u defined on T_L^2, we note

$$\Delta_Q(u) = \sum_{(m,n)\in Q} c(m,n)e_{m,n},$$

where $c(m, n)$ are the Fourier coefficients of u:

$$c(m, n) = \frac{1}{2\pi L} \int_{T_L^2} u(x, y)e^{i(\frac{m}{L}x+\frac{n}{L}y)}\, dx\, dy.$$

If $Q \subset \mathbb{Z}^2$ and R is a dyadic number, we set for a function $u(t, x)$ defined on $\mathbb{R} \times T_L^2$:

$$\Delta_{Q,R}u = \sum_{(m,n)\in Q}\left(\int_{R\leqslant\langle\tau-\frac{m^2}{L^2}+\frac{n^2}{L^2}\rangle\leqslant 2R} \widehat{c_{n,m}}(\tau)e^{2i\pi t\tau}\, d\tau\right)e_{m,n},$$

where $c_{m,n}(t)$ are the Fourier coefficients of $u(t)$. When Q is the cube $Q = \{(m, n) \in \mathbb{Z}^2,\ N \leqslant \mathrm{Max}(|m/L|, |n/L|) \leqslant 2N\}$, we will note $\Delta_N = \Delta_Q$ and $\Delta_{N,R} = \Delta_{Q,R}$.

Step 1. We prove: for all $s > 1/2$, $L \geqslant 1$ and $u_0 \in H^s(T_L^2)$, there exists a solution u of

$$i\partial_t u + Pu = -|u|^2 u - E(|u|^2)u, \quad (x, y) \in T_L^2, \tag{11}$$

and $\alpha > 0$, $C > 0$ independent of L satisfying the lower bound on the blow-up time:

$$T(u_0) \geqslant \frac{C}{\|u_0\|_{H^s(T_L^2)}^\alpha}. \tag{12}$$

The point here is that the lower bound depends only on the size of the initial data and not on L. On T_L^2, we denote (without changing notations) by P and E the natural extensions of the operators P and E defined above on T^2. Hence, symbols are respectively $(-m^2 + n^2)/L^2$ and $2m^2/(m^2 + n^2)$.

High regularity. Before looking at low regularities and to convince the reader that (12) holds, let us focus on the easier case of more regular data i.e. $s \in \mathbb{N} \setminus \{0, 1\}$. Let us prove (12) in this case. Let $L \geqslant 1$ and consider Eq. (11) and its equivalent formulation

$$u(t) = e^{itP}u_0 + i\int_0^t e^{i(t-\tau)P}\left(|u(\tau)|^2 u(\tau) + E(|u(\tau)|^2)u(\tau)\right) d\tau. \tag{13}$$

Let $u_0 \in H^s(T_L^2)$. Taking the H^s-norm in (13) and using the triangle inequality, we get for a constant $C > 0$ independent of L:

$$\|u(t)\|_{H^s} \leqslant \|u_0\|_{H^s} + C\int_0^T \left(\|u(\tau)^2 u(\tau)\|_{H^s} + \|E(|u(\tau)|^2)u(\tau)\|_{H^s}\right) d\tau. \tag{14}$$

Now we need a Sobolev type inequality with constants independent of the size of the torus.

Lemma 1. *Let s be an integer with $s \geqslant 2$. There exists a constant $C > 0$ such that for all $L > 0$ and $v, w \in H^s(T_L)$,*

$$\|vw\|_{H^s(T_L^2)} \leqslant C\|v\|_{H^s(T_L^2)}\|w\|_{H^s(T_L^2)}.$$

Proof. We first prove that for all $\sigma > 1$ and $v \in H^\sigma(T_L)$, $\|v\|_{L^\infty} \leqslant C\|v\|_{H^\sigma}$ for a constant $C > 0$ depending only on σ. Indeed, expanding v in Fourier series, we first get

$$|v| \leqslant \frac{1}{2\pi L} \sum_{(m,n)\in\mathbb{Z}^2} |v_{m,n}|.$$

We make appear the H^σ-norm of u and use Cauchy–Schwarz inequality to obtain

$$\|v\|_{L^\infty} \leqslant \frac{1}{2\pi L} \sum_{(m,n)\in\mathbb{Z}^2} |v_{m,n}| \left(1 + \frac{m^2}{L^2} + \frac{n^2}{L^2}\right)^{\sigma/2} \left(1 + \frac{m^2}{L^2} + \frac{n^2}{L^2}\right)^{-\sigma/2}$$

$$\leqslant \frac{1}{2\pi L} \|v\|_{H^\sigma} \left(\sum_{(m,n)\in\mathbb{Z}^2} \left(1 + \frac{m^2}{L^2} + \frac{n^2}{L^2}\right)^{-\sigma} \right)^{1/2}. \tag{15}$$

But we can easily compute the dependence in L of the last sum above by comparing with an integral as follows:

$$\sum_{(m,n)\in\mathbb{Z}^2} \left(1 + \frac{m^2}{L^2} + \frac{n^2}{L^2}\right)^{-\sigma} \leqslant C \sum_{(m,n)\in\mathbb{Z}^2} \left(1 + \frac{m^2}{L^2}\right)^{-\sigma/2} \left(1 + \frac{n^2}{L^2}\right)^{-\sigma/2}$$

$$\leqslant C \left(\int \frac{dx}{(1 + \frac{x^2}{L^2})^{\sigma/2}} \right)^2$$

$$\leqslant CL^2.$$

Thus, there is not more dependence on L in (15) and we obtain the claim. Now we can prove the lemma. Indeed, we first write the Leibniz rule then use the previous claim and an interpolation argument to get

$$\left\| (-\Delta)^{s/2}(vw) \right\|_{L^2} \leqslant C\|v\|_{H^s}\|w\|_{H^s}.$$

Here, the constant C contains binomial coefficients and therefore is independent of L. Moreover, again with the embedding $H^s \hookrightarrow L^\infty$, we have

$$\|vw\|_{L^2} \leqslant \|v\|_{L^\infty}\|w\|_{L^2} \leqslant C\|v\|_{H^s}\|w\|_{H^s},$$

and the last two inequalities end the proof of the lemma. $\quad\square$

Therefore, coming back to (14), using the boundedness of E in H^s and Lemma 1, we have

$$\left\|u(t)\right\|_{H^s} \leqslant \|u_0\|_{H^s} + CT\left\|u(t)\right\|_{H^s}^3,$$

with $C > 0$ independent of the period. This last estimate allows us to perform a Banach fixed point argument (the Lipschitz property is proved with similar arguments) in a ball of the space $\mathcal{C}([0,T], H^s)$ of radius $M = 2\|u_0\|_{H^s}$ and with $T = C/\|u_0\|_{H^s}^2$, and this proves (12).

Low regularity. This part is the more interesting since, as said above, the explicit blow-up solution $v(t)$ of the introduction lives only in H^s with $s < 1$. So let $s > 1/2$. We define the Bourgain spaces associated with Eq. (11) as the completion of the space of smooth compactly supported functions on $\mathbb{R} \times T_L^2$ for the norm defined by

$$\|u\|_{X_L^{s,b}} = \left\| \langle i\partial_t + P \rangle^b \langle(-\Delta)^{\frac{1}{2}}\rangle^s u \right\|_{L^2(\mathbb{R}\times T_L^2)},$$

where $\langle\alpha\rangle = (1+\alpha^2)^{1/2}$. Note that there exist more convenient equivalent definitions of this space: we may also check that the norm is equivalent to the following

$$\|u\|_{X_L^{s,b}}^2 = \sum_{(m,n)\in\mathbb{Z}^2} \left(1 + \frac{m^2}{L^2} + \frac{n^2}{L^2}\right)^s \int_{\mathbb{R}} \left\langle \tau - \frac{m^2}{L^2} + \frac{n^2}{L^2} \right\rangle^{2b} \left|\widehat{c_{m,n}}(\tau)\right|^2 d\tau,$$

where $\widehat{c_{m,n}}$ is the Fourier transform of $c_{m,n}$. A last definition is possible linking the Bourgain norm with Sobolev norm of the free dynamic:

$$\|u\|_{X_L^{s,b}} = \left\|e^{-itP}u(t)\right\|_{H^b(\mathbb{R}, H^s(T_L^2))}. \tag{16}$$

We will work on a finite time interval so that we have to define the localized version of the Bourgain spaces; for $u : [0, T] \times T_L^2 \to \mathbb{C}$:

$$\|u\|_{X_{L,T}^{s,b}} = \inf\{\|v\|_{X_L^{s,b}}, \; v \in X_L^{s,b} \text{ such that } v(t) = u(t) \text{ for all } t \in [0, T]\}. \tag{17}$$

Let us recall the integral formulation (13):

$$u(t) = e^{itP}u_0 + i \int_0^t e^{i(t-\tau)P}\left(|u(\tau)|^2 u(\tau) + E\left(|u(\tau)|^2\right)u(\tau)\right) d\tau.$$

First, the linear term is easily bounded: if $T \leqslant 1$ and $\psi(t)$ denotes a smooth real cut-off function equal to 1 on $[0, 1]$ and with compact support, we get using the definition of the Bourgain spaces (16):

$$\left\|e^{itP}u_0\right\|_{X_{L,T}^{s,b}} \leqslant \left\|e^{itP}\psi(t)u_0\right\|_{X_L^{s,b}} \leqslant \left\|\psi(t)u_0\right\|_{H^b(\mathbb{R}, H^s(T_L^2))} \leqslant C\|u_0\|_{H^s(T_L^2)}, \tag{18}$$

where $C = \|\psi\|_{H^b(\mathbb{R})}$ is independent of the period L.

Lemma 2. *There exists $C > 0$ such that for all $L \geqslant 1$, $T \leqslant 1$ and all pair (b, b') satisfying $0 < b' < 1/2 < b$, $b + b' < 1$,*

$$\left\|\int_0^t e^{i(t-\tau)P} F(\tau)\, d\tau\right\|_{X_{L,T}^{s,b}} \leqslant C T^{1-b-b'}\|F\|_{X_{L,T}^{s,-b'}}.$$

Proof. For a fixed $L > 0$, this estimate is classical in the context of the Bourgain spaces. To see that we may choose C independent of L, we remark (see [2]) that the proof of such an estimate for a fixed L relies on the one-dimensional inequality (proved in [7]):

$$\left\|\phi\left(\frac{t}{T}\right) \int_0^t g(\tau)\, d\tau\right\|_{H^b(\mathbb{R})} \leqslant C T^{1-b-b'}\|g\|_{H^{-b'}(\mathbb{R})}, \tag{19}$$

for a cut-off function ϕ. Then we apply this estimate pointwise with $g(\tau) = (F(\tau, x), e_{m,n})e_{m,n}$, take the square, integrate on T_L^2, multiply by $(-m^2 + n^2)/L^2$ and sum for $(m, n) \in \mathbb{Z}^2$. We then obtain the desired estimate with the same constant C as in (19) thus independent of L. $\quad\square$

To treat the nonlinearity in the fixed point argument, we will need the following proposition.

Proposition 1 *(Trilinear estimate). There exist a pair (b, b') satisfying $0 < b' < 1/2 < b$, $b + b' < 1$ and a constant $C > 0$ such that for every $L \geqslant 1$, $T > 0$, $u_1, u_2, u_3 \in X_{L,T}^{s,b}$,*

$$\|u_1 u_2 u_3\|_{X_{L,T}^{s,-b'}} \leqslant C\|u_1\|_{X_{L,T}^{s,b}} \|u_2\|_{X_{L,T}^{s,b}} \|u_3\|_{X_{L,T}^{s,b}},$$

$$\left\|E(u_1 u_2)u_3\right\|_{X_{L,T}^{s,-b'}} \leqslant C\|u_1\|_{X_{L,T}^{s,b}} \|u_2\|_{X_{L,T}^{s,b}} \|u_3\|_{X_{L,T}^{s,b}}.$$

Proof. Let us start with a lemma.

Lemma 3 *(Uniform periodic bilinear Strichartz estimate). There exists $C > 0$ such that for every $N_1, N_2 \geqslant 1$ dyadic numbers, $(a_1, b_1), (a_2, b_2) \in \mathbb{Z}^2$, $L \geqslant 1$ and $u_1, u_2 \in L^2(T_L^2)$ writing*

$$u_1 = \sum_{N_1 \leqslant \text{Max}(|\frac{m}{L}-a_1|,|\frac{n}{L}-b_1|) \leqslant 2N_1} c_1(m,n)e_{m,n}, \qquad u_2 = \sum_{N_2 \leqslant \text{Max}(|\frac{m}{L}-a_2|,|\frac{n}{L}-b_2|) \leqslant 2N_2} c_2(m,n)e_{m,n},$$

we have the bilinear estimate

$$\left\| e^{itP}(u_1)e^{itP}(u_2) \right\|_{L^2([0,1])L^2(T_L^2)} \leqslant C \min(N_1, N_2)^{1/2} \|u_1\|_{L^2(T_L^2)} \|u_2\|_{L^2(T_L^2)}. \tag{20}$$

Proof. Note that for $L = 1$, linear Strichartz estimates have been proved recently in [16,8]. We first prove the property in the case where $u_1 = u_2$ and $a_1 = b_1 = a_2 = b_2 = 0$. So let $u = u_1 = u_2$ and $N = N_1 = N_2$. We recall the semiclassical Strichartz estimate on the torus of size 1 (see [8]): for all $h \in (0, 1)$, for all interval J of size h and for all u_0 writing

$$v_0 = \sum_{h^{-1} \leqslant \text{Max}(|m|,|n|) \leqslant 2h^{-1}} c(m,n)e_{m,n},$$

for some coefficient $c(m, n)$, we have

$$\left\| e^{itP}v_0 \right\|_{L^4(J)L^4(T_1^2)} \leqslant C \|v_0\|_{L^2(T_1^2)}. \tag{21}$$

Similarly to the case where P is the Laplace operator (see [9]), we apply a scaling argument on this estimate to derive a linear Strichartz estimate on T_L^2 on the time interval $[0, 1]$. Let $u_0 \in L^2(T_L^2)$ localized in frequency in $[0, N]$ i.e.

$$u_0 = \sum_{N \leqslant \text{Max}(|\frac{m}{L}|,|\frac{n}{L}|) \leqslant 2N} c(m,n)e_{m,n}, \tag{22}$$

and $v_0 \in L^2(T_1^2)$ defined by $v_0(x) = u_0(Lx)$. Then computing the $L^4([0, 1])L^4(T_L^2)$ of $\exp(itP)u_0$ in term of v_0 and applying a change of variable, we get

$$\left\| e^{itP}u_0 \right\|_{L^4([0,1])L^4(T_L^2)} = L \left\| e^{itP}v_0 \right\|_{L^4([0,L^{-2}])L^4(T_1^2)}.$$

Remark that v_0 writes

$$v_0 = \sum_{LN \leqslant \text{Max}(|m|,|n|) \leqslant 2LN} c(m,n)e_{m,n},$$

so that we may apply (21) with $h \sim LN$. We need to consider two cases. If $L \geqslant N$, then $[0, L^{-2}] \subset [0, (LN)^{-1}]$ and so

$$\left\| e^{itP}u_0 \right\|_{L^4([0,1])L^4} \leqslant L \left\| e^{itP}v_0 \right\|_{L^4([0,(LN)^{-1}])L^4} \leqslant CL\|v_0\|_{L^2} \leqslant C\|u_0\|_{L^2}.$$

If $L < N$, we write $[0, L^{-2}]$ as a union of intervals $[t_k, t_{k+1}]$ with $t_{k+1} - t_k \sim (LN)^{-1}$ and $k \sim N/L$. We apply (21) on each $[t_k, t_{k+1}]$ and this gives

$$\left\| e^{itP}u_0 \right\|_{L^4([0,1])L^4} \leqslant C \left(\frac{N}{L} \right)^{1/4} L\|v_0\|_{L^2} \leqslant C \left(\frac{N}{L} \right)^{1/4} \|u_0\|_{L^2}.$$

If $L \geqslant 1$, we may in particular summarize the last two inequalities as

$$\left\| e^{itP}u_0 \right\|_{L^4([0,1])L^4} \leqslant CN^{1/4}\|u_0\|_{L^2},$$

and this proves (20) if $u_1 = u_2$ and $a_1 = b_1 = a_2 = b_2 = 0$. Now we treat the case $u = u_1 = u_2$ but without the assumption $a_1 = b_1 = a_2 = b_2 = 0$. Let $(a, b) \in \mathbb{Z}^2$ and write

$$u = \sum_{N \leqslant \text{Max}(|\frac{m}{L}-a|,|\frac{n}{L}-b|) \leqslant 2N} e^{\frac{it}{L^2}(n^2-m^2)}c(m,n)e_{m,n}$$

$$= e^{iax}e^{iby}e^{it(a^2-b^2)} \sum_{N \leqslant \text{Max}(|\frac{p}{L}|,|\frac{q}{L}|) \leqslant 2N} c(aL+p, bL+q)e^{\frac{-it}{L^2}(p^2-q^2+2aLp-2bLq)}e_{p,q}.$$

Then

$$\|u\|_{L^4 L^4}^4 = \left\| \sum_{N \leqslant \mathrm{Max}(|\frac{p}{L}|,|\frac{q}{L}|) \leqslant 2N} \frac{1}{2\pi L} c(aL+p, bL+q) e^{\frac{-it}{L^2}(p^2-q^2)} e^{\frac{ip}{L}(x-2at)} e^{\frac{iq}{L}(y+2bt)} \right\|_{L^4 L^4}^4$$

$$= \int_t \int_{\substack{-2at \leqslant \alpha \leqslant 2\pi L - 2at \\ 2bt \leqslant \beta \leqslant 2\pi L + 2bt}} \left| \sum_{N \leqslant \mathrm{Max}(|\frac{p}{L}|,|\frac{q}{L}|) \leqslant 2N} e^{\frac{-it}{L^2}(p^2-q^2)} \frac{1}{2\pi L} c(aL+p, bL+q) e^{\frac{i}{L} p\alpha} e^{\frac{i}{L} q\beta} \right|^4 d\alpha\, d\beta\, dt$$

$$= \int_t \int_{\substack{0 \leqslant \alpha \leqslant 2\pi L \\ 0 \leqslant \beta \leqslant 2\pi L}} \left| \sum_{N \leqslant \mathrm{Max}(|\frac{p}{L}|,|\frac{q}{L}|) \leqslant 2N} e^{\frac{it}{L^2}(q^2-p^2)} c(aL+p, bL+q) e_{p,q}(\alpha, \beta) \right|^4 d\alpha\, d\beta\, dt.$$

We apply the linear result proved above with $(a, b) = (0, 0)$ and this gives

$$\|u\|_{L^4 L^4}^4 \leqslant CN \left(\sum_{k,l} |c(k,l)|^2 \right)^2$$

$$\leqslant CN \|u_0\|_{L^2}^4.$$

This proves the result when $u_1 = u_2$. Note that if we assume another type of localization for u

$$u = \sum_{\mathrm{Max}(|\frac{m}{L} - a|, |\frac{n}{L} - b|) \leqslant N} c(m,n) e_{m,n},$$

the $L^4 L^4$ estimate still holds. It may be seen by remarking that estimate (21) also holds if u_0 is spectrally localized in $\{(m,n) \in \mathbb{Z}^2,\ \mathrm{Max}(|m|,|n|) \leqslant 2h^{-1}\}$ (see [8]) and using the same analysis as above. Now, we can prove the bilinear estimate in the general case. We assume for instance $N_1 \leqslant N_2$ and decompose the set $A = \{(m,n) \in \mathbb{Z}^2,\ N_2 \leqslant \mathrm{Max}(|a_2 - m/L|, |b_2 - n/L|) \leqslant 2N_2\}$ in small disjoint cubes of the form $Q_\alpha = Q_{(k,l)} = \{(m,n) \in A,\ \mathrm{Max}(|k - m/L|, |l - n/L|) \leqslant N_1\}$ for $\alpha = (k,l)$ running over a set I. Then for different $\alpha's$, the functions $e^{itP}(u_0) e^{itP}(\Delta_{Q_\alpha} v_0)$ are almost orthogonal since each function is localized in Fourier in the set $D_\alpha := \{(m,n) \in \mathbb{Z}^2,\ N_1 \leqslant \mathrm{Max}(|m/L - a_1|, |n/L - b_1|) \leqslant 2N_1\} + Q_\alpha$ and the sets D_α are almost disjoint in the sense that each point of $\mathbb{Z}^2$ belongs to a finite number of sets D_α. Indeed, if $(m,n) \in D_{\alpha_1} \cap D_{\alpha_2}$, then in particular we may write

$$(m,n) = c + d = e + f,$$

with $d \in Q_{\alpha_1}$, $f \in Q_{\alpha_2}$, and $c, e \in \{(m,n) \in \mathbb{Z}^2,\ N_1 \leqslant \mathrm{Max}(|m/L - a_1|, |n/L - b_1|) \leqslant 2N_1\}$. We deduce $|c_1 - e_1| = |f_1 - d_1| \leqslant 4N_1 L$. But each Q_α is of size less than $4N_1 L$ and there is a finite number of Q_α whose distance to Q_{α_2} is less than $4N_1 L$. So if we fix α_1, then α_2 runs in a finite number of indexes. Thus, this orthogonality property implies

$$\left\| e^{itP}(u_1) e^{itP}(u_2) \right\|_{L^2 L^2}^2 \leqslant C \sum_{\alpha \in I} \left\| e^{itP}(u_1) e^{itP}(\Delta_{Q_\alpha} u_2) \right\|_{L^2 L^2}^2$$

$$\leqslant C \left\| e^{itP}(u_1) \right\|_{L^4 L^4}^2 \sum_{\alpha \in I} \left\| e^{itP}(\Delta_{Q_\alpha} u_2) \right\|_{L^4 L^4}^2$$

$$\leqslant C N_1^{1/2} \|u_1\|_{L^2}^2 N_1^{1/2} \sum_{\alpha \in I} \|\Delta_{Q_\alpha} u_2\|_{L^2}^2$$

$$\leqslant C N_1 \|u_1\|_{L^2}^2 \|u_2\|_{L^2}^2.$$

This proves the lemma. □

Remark 2. Note that if Q_i denotes the set

$$Q_i = \left\{ (m,n) \in \mathbb{Z}^2,\ N_i \leqslant \mathrm{Max}\left(\left| \frac{m}{L} - a_i \right|, \left| \frac{n}{L} - b_i \right| \right) \leqslant 2N_i \right\},$$

then $(N_i L)^2 \leqslant |Q_i|$ and we may rewrite the Strichartz estimate (20) as

$$\left\| \Delta_{Q_1}(e^{itP}u)\Delta_{Q_2}(e^{itP}v) \right\|_{L^2 L^2} \leqslant C \left(\frac{\min(|Q_1|, |Q_2|)}{L^2} \right)^{\frac{1}{4}} \|u\|_{L^2} \|v\|_{L^2}. \tag{23}$$

Once we have proved (23), from covering arguments, we may deduce the same estimate for other shapes of Q_i typically $Q_i = \{(m, n) \in \mathbb{Z}^2, \text{ Max}(|m/L|, |n/L|) \leqslant 2N_i\}$ or translated sets of the previous one. In the sequel, we will use (23) for these kinds of Q_i. More precisely, we have the following.

Lemma 4. *For all $b > 1/2$, there exist $C(b) > 0$, $\beta(b) \in (0, 1 - b)$ and $\varepsilon(b) > 0$ such that for all dyadic square $Q_1, Q_2 \subset \mathbb{Z}^2$, R_1, R_2 dyadic number, $L \geqslant 1$ and $u_0, v_0 \in L^2(\mathbb{R}, L^2(T_L^2))$,*

$$\|\Delta_{Q_1, R_1} u_0 \Delta_{Q_2, R_2} v_0\|_{L^2 L^2} \leqslant C(b) \left(\frac{\text{Min}(|Q_1|, |Q_2|)}{L^2} \right)^{1/4 + \varepsilon(b)} (R_1 R_2)^{\beta(b)}$$
$$\times \|\Delta_{Q_1, R_1} u_0\|_{L^2 L^2} \|\Delta_{Q_2, R_2} v_0\|_{L^2 L^2}, \tag{24}$$

where $|Q_i|$ denotes the number of points in Q_i. Moreover, we may choose $\varepsilon(b)$ such that $\varepsilon(b)$ goes to 0 as b goes to 1/2.

Proof. As for the proof of (20), we first assume $u = u_0 = v_0$. Next, from (20), we get for all $b > 1/2$ and $f \in X_L^{0,b}$ localized in frequency in Q,

$$\|f\|_{L^4 L^4} \leqslant C \left(\frac{|Q|}{L^2} \right)^{1/8} \|f\|_{X_L^{0,b}}.$$

Again the constant C does not depend on L since the proof (see [2]) relies only on manipulations in time. In particular, for all u,

$$\|\Delta_{Q, R} u\|_{L^4 L^4} \leqslant C \left(\frac{|Q|}{L^2} \right)^{1/8} \|\Delta_{Q, R} u\|_{X_L^{0,b}}, \tag{25}$$

for all $b > 1/2$. And this gives using properties of the Bourgain spaces

$$\|\Delta_{Q, R} u\|_{L^4 L^4} \leqslant C \left(\frac{|Q|}{L^2} \right)^{1/8} R^b \|\Delta_{Q, R} u\|_{L^2 L^2}. \tag{26}$$

The fact that $b > 1/2$ in the above estimate will not be enough to conclude so that we need to refine this $L^4 L^4$ estimate. To do so, we compute the $L^\infty L^\infty$ norm of $\Delta_{Q, R} u$. From the definition of the projection $\Delta_{Q, R}$, we get using twice Cauchy–Schwarz inequality

$$\|\Delta_{Q, R} u\|_{L^\infty L^\infty} \leqslant \frac{1}{L} \sum_{\substack{(m,n) \in Q \\ R \leqslant \langle \tau - \frac{m^2}{L^2} + \frac{n^2}{L^2} \rangle \leqslant 2R}} \int |\widehat{c_{m,n}}(\tau)| \, d\tau$$

$$\leqslant \frac{R^{1/2}}{L} \sum_{(m,n) \in Q} \left(\int_{R \leqslant \langle \tau - \frac{m^2}{L^2} + \frac{n^2}{L^2} \rangle \leqslant 2R} |\widehat{c_{m,n}}(\tau)|^2 \, d\tau \right)^{1/2}$$

$$\leqslant R^{1/2} \left(\frac{|Q|}{L^2} \right)^{1/2} \left(\sum_{\substack{(m,n) \in Q \\ R \leqslant \langle \tau - \frac{m^2}{L^2} + \frac{n^2}{L^2} \rangle \leqslant 2R}} \int |\widehat{c_{m,n}}(\tau)|^2 \, d\tau \right)^{1/2}$$

$$\leqslant \left(\frac{|Q|}{L^2} \right)^{1/2} R^{1/2} \|\Delta_{Q, R} u\|_{L^2 L^2}. \tag{27}$$

By interpolation between the trivial inequality $\|\Delta_{Q, R} u\|_{L^2 L^2} \leqslant \|\Delta_{Q, R} u\|_{L^2 L^2}$ and (27), we have

$$\|\Delta_{Q, R} u\|_{L^4 L^4} \leqslant \left(\frac{|Q|}{L^2} \right)^{1/4} R^{1/4} \|\Delta_{Q, R} u\|_{L^2 L^2}. \tag{28}$$

Let $\varepsilon(b) > 0$ such that $\delta(b) := b(1 - 8\varepsilon(b)) + 8\varepsilon(b)\frac{1}{4} \in (0, 1 - b)$ and $\varepsilon(b) \to 0$ as $b \to 1/2$. For instance choose $\delta(b) = 3/2 - 2b$. Next, by interpolation between (26) with weight $1 - 8\varepsilon(b)$ and (28) with weight $8\varepsilon(b)$, we get the expected estimate:

$$\|\Delta_{Q,R}u_0\|_{L^4 L^4} \leqslant C\left(\frac{|Q|}{L^2}\right)^{1/8+\varepsilon(b)} R^{\delta(b)} \|\Delta_{Q,R}u_0\|_{L^2 L^2}. \tag{29}$$

To deduce (24) from (29), we proceed as in the proof of Strichartz estimate (20): if for instance $|Q_1| < |Q_2|$ then we decompose Q_2 in pieces of size $|Q_1|$ and next apply an almost orthogonality argument. We omit this argument and the proof is over. □

To prove Lemma 1, it is enough to prove the trilinear estimate for the global space $X_L^{s,b}$ i.e. $T = \infty$, then we recover the local in time estimate by taking the infimum on all extensions of $u_1, u_2, u_3 \in X_{L,T}^{s,b}$. Moreover, we only prove the second estimate; the first one is easier. By a duality argument, we have to show the quadrilinear estimate: there exists $C > 0$ such that for all $L \geqslant 1$, $u_1, u_2, u_3, u_4 \in X_L^{s,b}$:

$$\left| \int\limits_{\mathbb{R} \times T_L^2} E(u_1 u_2)u_3 u_4 \right| \leqslant C \|u_1\|_{X_L^{s,b}} \|u_2\|_{X_L^{s,b}} \|u_3\|_{X_L^{s,b}} \|u_4\|_{X_L^{-s,b'}}.$$

In the sequel, we will note $Q_i = \{(m, n) \in \mathbb{Z}^2, \; N_i \leqslant \text{Max}(|m/L|, |n/L|) < 2N_i\}$. Decomposing each u_i as

$$u_i = \sum_{N_i, R_i} \Delta_{N_i, R_i}(u_i),$$

we have that

$$G = \int\limits_{\mathbb{R} \times T_L^2} E(u_1 u_2)u_3 u_4$$

becomes

$$G = \int\limits_{\mathbb{R} \times T_L^2} \sum_{\substack{N_1, N_2, N_3, N_4 \\ R_1, R_2, R_3, R_4}} E\big(\Delta_{N_1, R_1}(u_1)\Delta_{N_2, R_2}(u_2)\big)\Delta_{N_3, R_3}(u_3)\Delta_{N_4, R_4}(u_4).$$

In the summation above, we may restrict indexes to $N_4 \leqslant 2(N_1 + N_2 + N_3)$. Indeed, the function

$$U = E\big(\Delta_{N_1, R_1}(u_1)\Delta_{N_2, R_2}(u_2)\big)\Delta_{N_3, R_3}(u_3)$$

is localized in Fourier in the set $\{(m, n) \in \mathbb{Z}^2, \; m = m_1 + m_2 + m_3, \; n = n_1 + n_2 + n_3, \; (m_1, n_1) \in Q_1, \; (m_2, n_2) \in Q_2, \; (m_3, n_3) \in Q_3\}$. Thus, if $N_4 > 2(N_1 + N_2 + N_3)$, the integral over T_L^2 of $U \Delta_{N_4, R_4}(u_4)$ is zero. Therefore

$$G = \sum_{\substack{N_4 \leqslant 2(N_1 + N_2 + N_3) \\ R_1, R_2, R_3, R_4}} \alpha(N_1, N_2, N_3, N_4, R_1, R_2, R_3, R_4), \tag{30}$$

where

$$\alpha(N_1, N_2, N_3, N_4, R_1, R_2, R_3, R_4) = \int\limits_{\mathbb{R} \times T_L^2} E\big(\Delta_{N_1, R_1}(u_1)\Delta_{N_2, R_2}(u_2)\big)\Delta_{N_3, R_3}(u_3)\Delta_{N_4, R_4}(u_4).$$

Contrary to the case of a typical cubic nonlinearity, α is not symmetric in N_1, N_2, N_3, N_4 and we need to split the analysis in several cases. The worst situation is when the two lowest frequencies appear in the nonlocal term. Let us first treat this case.

Case $N_3 = \max(N_1, N_2, N_3)$. Without loss of generality, we may assume $N_1 \leqslant N_2 \leqslant N_3$. In this situation, we decompose the set Q_3 in small pieces of size $N_2 L$. Hence, we may write Q_3 as a disjoint union of sets of the form

$Q_\alpha = Q_{(a,b)} = \{(m,n) \in Q_3, \; \text{Max}(|a - m/L|, |b - n/L|) \leqslant N_2\}$ for some well-chosen set I of pairs $\alpha = (a,b) \in Q_3$ so that the union is disjoint. Using again an orthogonality argument, α is then

$$\alpha(N_i, R_i) = \int\limits_{\mathbb{R} \times T_L^2} E\big(\Delta_{N_1,R_1}(u_1)\Delta_{N_2,R_2}(u_2)\big)\Delta_{Q_\alpha,R_3}(u_3)\Delta_{\tilde{Q}_\alpha,R_4}(u_4)$$

where

$$\tilde{Q}_\alpha = \big\{(m_4,n_4) \in Q_4, \; m = -m_1 - m_2 - m_3, \; n = -n_1 - n_2 - n_3, \; (m_i,n_i) \in Q_i, \; i = 1,2, \; (m_3,n_3) \in Q_\alpha\big\}.$$

From Cauchy–Schwarz inequality in space and time and the boundedness of E on $L^2(T_L^2)$,

$$\big|\alpha(N_i, R_i)\big| \leqslant \big\|\Delta_{N_1,R_1}(u_1)\Delta_{N_2,R_2}(u_2)\big\|_{L^2 L^2} \big\|\Delta_{Q_\alpha,R_3}(u_3)\Delta_{\tilde{Q}_\alpha,R_4}(u_4)\big\|_{L^2 L^2}.$$

Note that since $|Q_\alpha| \leqslant (N_2 L)^2$, we deduce by the triangle inequality that we also have $|\tilde{Q}_\alpha| \leqslant C(LN_2)^2$ and thus we can apply Lemma 4 to get

$$\alpha(N_i, R_i) \leqslant C N_1^{\frac{1}{2}+\varepsilon(b)} N_2^{\frac{1}{2}+\varepsilon(b)} (R_1 R_2 R_3 R_4)^{\beta(b)} \big\|\Delta_{N_1,R_1}(u_1)\big\|_{L^2 L^2}$$
$$\times \big\|\Delta_{N_2,R_2}(u_1)\big\|_{L^2 L^2} \sum_{\alpha \in I} \big\|\Delta_{Q_\alpha,R_3}(u_3)\big\|_{L^2 L^2} \big\|\Delta_{\tilde{Q}_\alpha,R_4}(u_4)\big\|_{L^2 L^2}. \tag{31}$$

Next from Cauchy–Schwarz inequality, we may write

$$\sum_{\alpha \in I} \big\|\Delta_{Q_\alpha,R_3}(u_3)\big\|_{L^2 L^2} \big\|\Delta_{\tilde{Q}_\alpha,R_0}(u_0)\big\|_{L^2 L^2} \leqslant \bigg(\sum_{\alpha \in I} \big\|\Delta_{Q_\alpha,R_3}(u_3)\big\|_{L^2 L^2}^2\bigg)^{\frac{1}{2}} \bigg(\sum_{\alpha \in I} \big\|\Delta_{\tilde{Q}_\alpha,R_4}(u_4)\big\|_{L^2 L^2}^2\bigg)^{\frac{1}{2}}.$$

First, since $(Q_\alpha)_\alpha$ is a partition of Q_3, by orthogonality, we have for the first term on the right hand side above:

$$\bigg(\sum_{\alpha \in I} \big\|\Delta_{Q_\alpha,R_3}(u_3)\big\|_{L^2 L^2}^2\bigg)^{1/2} = \big\|\Delta_{Q_3,R_3}(u_3)\big\|_{L^2 L^2}.$$

For the second term, the $\tilde{Q}_\alpha$'s recover Q_4 but since there are not disjoint, strict orthogonality is broken. However, using the same argument of almost orthogonality as for the proof of Strichartz estimate (each point of Q_4 belongs to a finite number of $\tilde{Q}_\alpha$), we deduce

$$\bigg(\sum_{\alpha \in I} \big\|\Delta_{\tilde{Q}_\alpha,R_4}(u_4)\big\|_{L^2 L^2}^2\bigg)^{1/2} \leqslant C \big\|\Delta_{Q_4,R_4}(u_0)\big\|_{L^2 L^2}.$$

Thus,

$$\alpha(N_i, R_i) \leqslant C(N_1 N_2)^{1/2+\varepsilon(b)} (R_1 R_2 R_3 R_4)^{\beta(b)} \prod_{i=0}^{3} \big\|\Delta_{Q_i,R_i}(u_i)\big\|_{L^2 L^2}. \tag{32}$$

We reorder terms to make appear Bourgain's norms of u_i. The quantity

$$H = \sum_{\substack{N_4 \leqslant 2(N_1+N_2+N_3) \\ R_1,R_2,R_3,R_4 \\ N_3 = \text{Max}(N_1,N_2,N_3)}} \alpha(N_i, R_i)$$

is bounded by

$$|H| \leqslant \sum_{N_1,R_1} N_1^{\frac{1}{2}+\varepsilon(b)-s} R_1^{\beta(b)-b} N_1^s R_1^b \big\|\Delta_{N_1,R_1}(u_1)\big\|_{L^2 L^2}$$
$$\times \sum_{N_2,R_2} N_2^{\frac{1}{2}+\varepsilon(b)-s} R_2^{\beta(b)-b} N_1^s R_1^b \big\|\Delta_{N_2,R_2}(u_2)\big\|_{L^2 L^2}$$
$$\times \sum_{N_4 \leqslant 6N_3} \sum_{R_4,R_3} R_0^{\beta(b)-b'} R_4^{b'} R_3^{\beta(b)-b} R_3^b \big\|\Delta_{N_4,R_4}(u_4)\big\|_{L^2 L^2} \big\|\Delta_{N_3,R_3}(u_3)\big\|_{L^2 L^2}.$$

For the first two sums above, we use Cauchy–Schwarz inequality to recover Bourgain's norm of u_i. For instance for the first term, we have if $s > 1/2 + \varepsilon(b)$, and since $b > \beta(b)$,

$$\sum_{N_1, R_1} N_1^{\frac{1}{2}+\varepsilon(b)-s} R_1^{\beta(b)-b} N_1^s R_1^b \|\Delta_{N_1, R_1}(u_1)\|_{L^2 L^2} \leqslant \|u_1\|_{X_L^{s,b}} \left(\sum_{N_1, R_1} N_1^{1+2\varepsilon(b)-2s} R_1^{2(\beta(b)-b)} \right)^{\frac{1}{2}}$$

$$\leqslant C \|u_1\|_{X_L^{s,b}}.$$

For the third sum, using again Cauchy–Schwarz inequality, and choosing $b' > \beta(b)$ (this condition is compatible with $1 - b - b' > 0$ since $\beta(b) < 1 - b$), we write

$$\sum_{R_4} R_4^{\beta(b)-b'} R_4^{b'} \|\Delta_{N_4, R_4}(u_4)\|_{L^2 L^2} \leqslant \left(\sum_{R_4} R_4^{2\beta(b)-2b'} \right)^{\frac{1}{2}} \left(\sum_{R_4} R_4^{2b'} \|\Delta_{N_4, R_4}(u_4)\|_{L^2 L^2}^2 \right)^{\frac{1}{2}}$$

$$\leqslant C \|\Delta_{N_4}(u_4)\|_{X_L^{0,b'}}.$$

We treat the sum over R_3 in the same way. Therefore,

$$|H| \leqslant \|u_1\|_{X_L^{s,b}} \|u_2\|_{X_L^{s,b}} \sum_{N_4 \leqslant 6N_3} \frac{N_4^s}{N_3^s} N_4^{-s} \|\Delta_{N_4}(u_4)\|_{X_L^{0,b'}} N_3^s \|\Delta_{N_3}(u_3)\|_{X_L^{0,b}}.$$

Now we need the following lemma (see [3], Lemma 4.5 for a proof) to conclude.

Lemma 5. *For every $s > 0$, there exists a constant $C > 0$ such that for all sequence $(a_{N_4})_{N_4 \in 2^{\mathbb{N}}}$, $(b_{N_3})_{N_3 \in 2^{\mathbb{N}}}$, we have*

$$\sum_{N_4 \leqslant 6N_3} \left(\frac{N_4}{N_3} \right)^s |a_{N_4} b_{N_3}| \leqslant C \left(\sum_{N_4} a_{N_4}^2 \right)^{1/2} \left(\sum_{N_3} a_{N_3}^2 \right)^{1/2}.$$

To conclude in this case, we apply the lemma with

$$a_{N_4} = N_4^{-s} \|\Delta_{N_4}(u_4)\|_{X_L^{0,b'}}, \qquad b_{N_3} = N_3^s \|\Delta_{N_3}(u_3)\|_{X_L^{0,b}},$$

and obtain

$$|H| \leqslant C \|u_1\|_{X_L^{s,b}} \|u_2\|_{X_L^{s,b}} \|u_3\|_{X_L^{s,b}} \|u_4\|_{X_L^{-s,b'}}. \tag{33}$$

Case $N_3 < \max(N_1, N_2, N_3)$. In the summation (30), we assume for instance $N_1 \leqslant N_3 \leqslant N_2$. This case is easier since we do not need to decompose high frequencies. With the definition of $\alpha(N_i, R_i)$ and from Cauchy–Schwarz inequality:

$$|\alpha(N_i, R_i)| \leqslant \|\Delta_{N_1, R_1}(u_1) \Delta_{N_2, R_2}(u_2)\|_{L^2 L^2} \|\Delta_{N_3, R_3}(u_3) \Delta_{N_4, R_4}(u_4)\|_{L^2 L^2}.$$

Coming back to Lemma 4, we have directly

$$\alpha(N_i, R_i) \leqslant (N_1 N_3)^{1/2+\varepsilon} (R_4 R_1 R_2 R_3)^{\beta(\varepsilon)} \prod_{i=1}^4 \|\Delta_{N_i, R_i}(u_i)\|_{L^2 L^2}.$$

Once we have this estimate, the end of the proof in this case is the same as the previous one and we obtain

$$\sum_{\substack{N_3 < \max(N_1, N_2, N_3) \\ R_1, R_2, R_3, R_4 \\ N_4 \leqslant 6 \max(N_1, N_2, N_3)}} \alpha(N_i, R_i) \leqslant C \|u_1\|_{X_L^{s,b}} \|u_2\|_{X_L^{s,b}} \|u_3\|_{X_L^{s,b}} \|u_4\|_{X_L^{-s,b'}}. \tag{34}$$

Estimates (33) and (34) provides Proposition 1. $\quad\square$

Writing

$$\Phi(u) = e^{itP}u_0 + i\int_0^t e^{i(t-\tau)P}\big(|u(\tau)|^2 u(\tau) + E\big(|u(\tau)|^2\big)u(\tau)\big)\,d\tau$$

and using (18), Lemma 2 and Proposition 1, we have easily

$$\big\|\Phi(u)\big\|_{X_{L,T}^{s,b}} \leqslant C\|u_0\|_{H^s} + CT^{1-b-b'}\|u\|_{X_{L,T}^{s,b}}^3,$$

and

$$\big\|\Phi(u) - \Phi(v)\big\|_{X_{L,T}^{s,b}} \leqslant CT^{1-b-b'}\big(\|u\|_{X_{L,T}^{s,b}}^2 + \|v\|_{X_{L,T}^{s,b}}^2\big)\|u - v\|_{X_{L,T}^{s,b}}.$$

Therefore, we may close the fixed point argument in the ball $B(0, R)$ of $X_{L,T}^{s,b}$ with $R = 2C\|u_0\|_{H^s}$ and $T \geqslant D/\|u_0\|_{H^s}^{2/(1-b-b')}$ with $D > 0$ independent of the period $L \geqslant 1$. This proves (12) for low regularities and also the first point (take $L = 1$) in Theorem 1.

Step 2. Let us now finish the proof of the lower bound (6). Let $u \in H^s(T^2)$ solution to (2) and consider the family for $\tau \in [0, T]$:

$$v^\tau(t, x, y) = \lambda(\tau)u\big(\lambda^2(\tau)t + \tau, \lambda(\tau)x, \lambda(\tau)y\big),$$

where $\lambda(\tau) = \|u(\tau)\|_{H^s(T^2)}^{-1/s}$. For all τ, v^τ is a function on the torus $T_{1/\lambda(\tau)}$ and satisfies Eq. (11) for $L = 1/\lambda(\tau)$. Moreover, it is easy to check that $\|v^\tau(0)\|_{L^2} = \|u(0)\|_{L^2}$ and $\|(-\Delta)^{s/2}(v^\tau(0))\|_{L^2} \leqslant 1$. If we denote by T_τ the maximal time for v^τ, from (12), we deduce the uniform bound, $T_\tau \geqslant C > 0$. But $T_\tau = (T - \tau)/\lambda^2(\tau)$ where T is the maximal time for u and this with the uniform lower bound on T_τ proves the lower bound (6).

Acknowledgements

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