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# Global existence for a nonlinear Schroedinger–Chern–Simons system on a surface

# L'existence d'une solution globale régulière pour un système non-linéaire d'équations de Schroedinger–Chern–Simons sur une surface

Sophia Demoulini

*Centre for Mathematical Sciences, Wilberforce Road, Cambridge, CB3 OWB, UK*

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**Abstract**

Global existence of regular solutions for a nonlinear Schroedinger–Chern–Simons system of equations on a two-dimensional compact Riemannian manifold is proved.


**Résumé**

L'existence d'une solution globale régulière est démontrée pour un système non-linéaire d'équations de Schroedinger–Chern–Simons sur une variété Riemannienne compacte de deux dimensions.


*Keywords:* Nonlinear Schroedinger; Chern–Simons; Global existence; Regularity

## 1. Introduction and statement of global existence

The Ginzburg–Landau energy functional on an oriented two-dimensional compact surface $\Sigma$ without boundary with a fixed Riemannian metric $g$ is given by the integral

$$\int_{\Sigma} \mathcal{V}_\lambda(A, \Phi)\, \mathrm{d}\mu_g$$

*E-mail address:* s.demoulini@dpmms.cam.ac.uk (S. Demoulini).

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where $\mathrm{d}\mu_g$ is the associated area form and

$$\mathcal{V}_\lambda(A,\Phi)=\frac{1}{2}\left[|B|^2+g^{ij}\langle(\nabla_A)_i\Phi,(\nabla_A)_j\Phi\rangle+\frac{\lambda}{4}\left(|\Phi|^2-1\right)^2\right]. \tag{1}$$

Here $\nabla_A$ is an $S^1$ connection on a complex line bundle $L\to\Sigma$ of which $\Phi$ is a section and the 2-form $B\,\mathrm{d}\mu_g$ is the curvature associated to $\nabla_A$. If $\nabla_a$ is a fixed connection with curvature $b\,\mathrm{d}\mu_g$ then there exists a real 1-form $A$ such that $\nabla_A=\nabla_a-\mathrm{i}A$ and $B\,\mathrm{d}\mu_g=b\,\mathrm{d}\mu_g+\mathrm{d}A$. Various time dependent models associated to this functional have been considered:

(i) The gradient flow, which is essentially parabolic (once gauge invariance is properly handled). This was studied in [6].
In addition to the gradient flow there are two associated conservative dynamical models:
(ii) the Abelian Higgs model which forms (modulo gauge invariance) a hyperbolic system of semi-linear wave equations. Vortex dynamics for this model were studied in [12,13];
(iii) the Schroedinger–Chern–Simons equations (*SCS*) introduced in [10], to be studied here (see also [7]). These form (modulo gauge invariance) a system of coupled nonlinear Schroedinger equations for $(A,\Phi)$ together with a constraint.

### 1.1. The (SCS) system and statement of the main result

In the time-dependent case $L$ extends to a line bundle $\mathbf{L}=\mathbb{R}\times L$ over $\mathbb{R}\times\Sigma$. Explicitly the dependent variables consist of a time-dependent 1-form $\mathbf{A}=(A_0,A)\equiv(A_0,A_1,A_2)$ on $\mathbb{R}\times\Sigma$ where $A=A_1\,\mathrm{d}x^1+A_2\,\mathrm{d}x^2$ is a 1-form on $\Sigma$, $A_0(t,x)\in\mathbb{R}$ and a time dependent section $\Phi$ of $\mathbf{L}$. Thus at each time $t$ we have a section $\Phi(t)$ of $L$ and a connection $\nabla_{A(t)}$ on $L$ as well as the real valued function $A_0(t)$. The equations are

$$\begin{aligned}&2\mu(\partial_t A-\nabla A_0)+\nabla B=-\epsilon\langle \mathrm{i}\Phi,\nabla_A\Phi\rangle,\\ &\mathrm{i}\gamma(\partial_t-\mathrm{i}A_0)\Phi=-\frac{1}{2}\Delta_A\Phi-\frac{\lambda}{4}\left(1-|\Phi|^2\right)\Phi\end{aligned} \tag{2}$$

together with a third *constraint* equation

$$2\mu B=\gamma\left(1-|\Phi|^2\right).$$

Here $\epsilon:T^*\Sigma\to T^*\Sigma$ is the complex structure (Hodge dual operation), $\langle a,b\rangle$ denotes a real inner product on $L$, $\langle\Phi,\Phi\rangle=|\Phi|^2$ and $\gamma,\mu,\lambda$ are positive constants. Note that the constraint equation is preserved by the evolution (since $\partial_t B=\frac{1}{2\mu}\langle \mathrm{i}\Phi,\Delta_A\Phi\rangle=\frac{\gamma}{2\mu}\partial_t(1-|\Phi|^2)$), however, it is slightly more general to consider the following additional equation for $h\in H^2(\Sigma)$:

$$h(x)=B(t,x)-\frac{\gamma}{2\mu}\left(1-|\Phi|^2\right)(t,x)=B(0,x)-\frac{\gamma}{2\mu}\left(1-|\Phi|^2\right)(0,x). \tag{3}$$

From $\mathbf{A}=(A_0,A)$ we form the operator $D_{\mathbf{A}}$ by

$$D_{\mathbf{A}}\Phi=(\partial_t-\mathrm{i}A_0)\Phi\,\mathrm{d}t+\nabla_A\Phi$$

which is a connection on $\mathbf{L}\to\mathbb{R}\times\Sigma$ and, writing $E_j=\partial_t A_j-\partial_j A_0$, the 2-form $-\mathrm{i}E_j\,\mathrm{d}t\wedge\mathrm{d}x^j-\mathrm{i}B\,\mathrm{d}\mu_g$ is the associated curvature. Thus by the first equation in (2) above

$$2\mu E+\nabla B=-\epsilon\langle \mathrm{i}\Phi,\nabla_A\Phi\rangle.$$

In conformal co-ordinates $g=\mathrm{e}^{2\rho}((\mathrm{d}x^1)^2+(\mathrm{d}x^2)^2)$ and the area form is then $\mathrm{d}\mu_g=\mathrm{e}^{2\rho}\mathrm{d}x^1\wedge\mathrm{d}x^2$. For the connection $\nabla_A$ defined above we let $\Delta_A=\mathrm{e}^{2\rho}\nabla_A\cdot\nabla_A$. See the first appendix for further explanation of notational conventions and interpretation.

For the case $\Sigma=\mathbb{R}^2$ this system was proposed by Manton [10], who derived it as the Euler–Lagrange equations from a Lagrangian extending the Ginzburg–Landau functional by a Chern–Simons term and the Schroedinger term $\langle \mathrm{i}\Phi,(\partial_t-\mathrm{i}A_0)\Phi\rangle$. In [2] and [3] the authors prove local existence and blow-up in $H^2$ (and global existence under conditions on the initial data in $H^1$) on $\Sigma = \mathbb{R}^2$ for a closely related system having negative energy density (an "attractive" nonlinearity, corresponding to $\lambda < 0$). In this paper we study the case of a positive energy density (a "repulsive" nonlinearity, corresponding to $\lambda > 0$) with $\Sigma$ a two-dimensional surface as described above. We prove a local existence theorem for (2), (3), and then global existence for $\Phi$ in $H^2$. In [5] the authors show a global existence for a related system for $\Sigma = \mathbb{R}^2$ in which $\Phi$ solves a wave equation. A study of vortex dynamics of (2), (3) is currently undertaken along the lines of [12,13] for the Abelian Higgs model.

Two useful ways to think of (2), (3) are:

(i) In Coulomb gauge $\operatorname{div} A = 0$ it is possible to reformulate this system as a *nonlocal* Schroedinger equation for $\Phi$, as (3) and the divergence of the first equation in (2) then determine $A_0$ and $A$ in terms of $\Phi$ as nonlocal functionals (by solving elliptic equations) in $x$ at each time $t$. This is the approach adopted in [2]. Here we make a different gauge choice, the parabolic gauge $A_0 = \operatorname{div} A$ in which $A_0$ and $A$ (through the same equations) are determined by $\Phi$ nonlocally in $(t, x)$ (by solving a heat equation).

(ii) With appropriate choice of symplectic structure the equation is a constrained Hamiltonian system with $\mathcal{V}_\lambda$ as Hamiltonian and (3) is a constraint (i.e. its time-derivative vanishes identically as a consequence of (2)). A useful consequence of this second formulation is the conservation of $\mathcal{V}_\lambda$ which will be used later.

In view of interpretation (i) it is to be expected that control of $\Phi$ in a sufficiently strong norm for all time ensures the existence of a global solution and this idea is used to prove the following in the parabolic gauge (the spaces used are explained below):

**Theorem 1.1** *(Global existence). Consider the Cauchy problem for* (2), (3) *with initial data* $\Phi(0) \in H^2(\Sigma)$ *and* $\mathbf{A}(0) = (A_0(0), A(0)) \in H^3(\Sigma)$ *satisfying* $A_0(0) = 0 = \operatorname{div} A(0)$. *In the parabolic gauge with* $A_0 = \operatorname{div} A$, *there exists a unique global solution* $(A, \Phi) \in C([0,\infty); H^1(\Sigma) \times H^2(\Sigma)) \cap C^1([0,\infty); L^2(\Sigma) \times L^2(\Sigma))$ *such that* $\Phi$ *satisfies the estimate*

$$\left|\Phi(t)\right|_{H^2(\Sigma)} \leqslant c\, \mathrm{e}^{\alpha \mathrm{e}^{\beta t}} \tag{4}$$

*for some positive constants* $c, \alpha, \beta$ *depending only on* $(\Sigma, g)$, *the constants* $\gamma, \mu, \lambda$ *and the initial data.*

Brezis and Gallouet in [4] prove an analogous result for the nonlinear Schroedinger equation

$$\mathrm{i}\partial_t u - \Delta u + |u|^2 u = 0.$$

The crucial point there was the use of the inequality (valid, e.g., for $u \in H^2(\mathbb{R}^2)$)

$$|u|_{L^\infty} \leqslant C\left[1 + \sqrt{\ln(1 + |u|_{H^2})}\right] \tag{5}$$

with $C = C(|u|_{H^1})$, in conjunction with standard $L^2$ estimates for the differentiated equation to provide control of the $H^2$ norm of the solution at each time. For this to work the cubic structure of the nonlinearity and the square root in (5) turn out to be important. The main point of the present article is that for (2) *the same balancing of nonlinear effects occurs* even in the presence of the additional nonlinearity provided by the presence of $\mathbf{A}$ in the equation: the constraint equations ensure that the overall strength of these can be estimated in the same manner as the cubic nonlinearity. This is achieved by careful use of the constraint equations to estimate the various commutator terms which appear on differentiation of the equation, together with a covariant form of the Brezis–Gallouet inequality (5).

Before stating the inequality we discuss the spaces in which we work (more details can be found in the appendix). The space of $H^k$ connections is the space of operators $\nabla_A$ of the form $\nabla_A = \nabla_a - \mathrm{i}A$ with $A \in H^k(\Omega^1(\Sigma))$, which usually will be written $H^k$ or $H^k(\Sigma)$ suppressing $\Omega^1$. For any given measurable connection 1-form $A = (A_1, A_2)$ with measurable $k$-order derivatives, we define the space of $H^k_A$ sections $\Phi$ as

$$H^k_A(\Sigma) = \left\{\Phi\colon\ \Sigma \to \mathcal{C}\colon \sum_{|\alpha| \leqslant k} |\nabla_A^\alpha \Phi| \in L^2\right\}$$

with the usual norm. For $x \in \Sigma$, $|\Phi(x)|^2 = \langle\Phi, \Phi\rangle_h(x)$ and $|\nabla_A\Phi|^2 = |\nabla_A\Phi|^2_{g\times h} = g^{ij}\langle\nabla_{A_i}\Phi, \nabla_{A_j}\Phi\rangle_h$ where $g$ is the metric on $\Sigma$ and $h$ is the inner product on $L$ (cf. Appendix A); we suppress $h$ in notation. Also when the background connection $a$ is implied we often write only $\nabla$ in place of $\nabla_a$ and the norm $|\Phi|_{H^k}$ rather than $|\Phi|_{H^k_a}$. Certain Sobolev imbedding theorems are valid, see Lemma 1.2 and Appendix A. If $A$ is time dependent then $H^1_{A(t)}$ is a time-dependent norm. Supposing (as will be the case below) that $A(t,\cdot)$ varies continuously with $t$ in $H^1(\Sigma)$ then the corresponding $H^1_{A(t)}$ norms for $\Phi \in H^1(\Sigma)$ are equivalent and continuously varying in $t$, both of which can be seen from

$$\begin{aligned}|\nabla_{A(t)}\Phi|_{L^2} &\leqslant |\nabla_{A(\tau)}\Phi|_{L^2} + \big|\big(A(t)-A(\tau)\big)\Phi\big|_{L^2}\\ &\leqslant |\nabla_{A(\tau)}\Phi|_{L^2} + c\big|A(t)-A(\tau)\big|_{H^1}|\Phi|_{H^1}\end{aligned}$$

by the Kato and Sobolev inequalities (see below).

In the remainder of this section in Lemmas 1.2, 1.3 and 1.4 we show covariant versions of known inequalities. These are derived for a complex section $\Phi$ of a line bundle $L$ with connection $A$ (the time variable is fixed and $A$, $\Phi$ are time-independent) and obtained in two stages from their corresponding statements on Euclidean space: once derived for the two-dimensional Riemannian manifold $\Sigma$, the covariant version on $L$ is then derived from that. The first stage is easily achieved in the usual way with a partition of unity, see Appendix A.

**Lemma 1.2** *(Covariant version of the Sobolev and Gagliardo–Nirenberg inequalities). For $(\Sigma, g)$ as above and for $(A,\Phi) \in (H^1 \times H^2_A)(\Sigma)$ then $\nabla_A\Phi \in L^4(\Sigma)$ and*

$$|\nabla_A\Phi|_{L^4} \leqslant c|\nabla_A\Phi|_{H^1_A} \tag{6}$$

*and also for all $1 \leqslant p < \infty$, $H^2_A \hookrightarrow W^{1,p}_A \hookrightarrow L^\infty$ continuously on $\Sigma$. Also*

$$|\nabla_A\Phi|_{L^4} \leqslant c|\nabla_A\Phi|^{1/2}_{L^2}\big(|\nabla_A\Phi|^{1/2}_{L^2} + |\nabla_A\nabla_A\Phi|^{1/2}_{L^2}\big) \tag{7}$$

*where $c$ depends only on $(\Sigma, g)$.*

**Proof.** For real valued $u \in H^1(\Sigma)$ we have the Sobolev and Gagliardo–Nirenberg inequalities, respectively,

$$|u|_{L^4} \leqslant c|u|_{H^1} \quad \text{and} \quad |u|_{L^4} \leqslant c|u|^{1/2}_{L^2}|u|^{1/2}_{H^1}. \tag{8}$$

(Both of these follow the corresponding standard forms of the Sobolev and Gagliardo–Nirenberg inequalities on $\mathbb{R}^2$ by a partition of unity cf. Appendix A.) We recall the Kato inequality,

$$\big|\nabla|\Phi|\big|_{L^p} \leqslant |\nabla_A\Phi|_{L^p}$$

and let $u = |\nabla_A\Phi| \in H^1(\Sigma)$. By (8) we have

$$|\nabla_A\Phi|^2_{L^4} \leqslant c|\nabla_A\Phi|_{L^2}\big(|\nabla_A\Phi|_{L^2} + \big|\nabla|\nabla_A\Phi|\big|_{L^2}\big)$$

and by the Kato inequality

$$\leqslant c|\nabla_A\Phi|_{L^2}\big(|\nabla_A\Phi|_{L^2} + |\nabla_A\nabla_A\Phi|_{L^2}\big)$$

which proves (7). The Sobolev inequality and imbeddings follow in the same way. □

**Lemma 1.3** *(Covariant version of the Garding inequality). For $\Psi = (A,\Phi)$ such that the norms on $\Sigma$ appearing below are finite we have*

$$|\nabla_A\nabla_A\Phi|_{L^2} \leqslant |\Delta_A\Phi|_{L^2} + c|B|^{1/2}_{L^\infty}|\nabla_A\Phi|_{L^2} + c|\Phi|^{1/2}_{L^\infty}|\nabla_A\Phi|^{1/2}_{L^2}|\nabla B|^{1/2}_{L^2} \tag{9}$$

*where $c$ is a number depending only on $(\Sigma, g)$.*

**Proof.** Recall that $\langle\cdot,\cdot\rangle$ is the inner product on $L$. Using a local co-ordinate system $\{x^j\}_{j=1}^2$ on $\Sigma$ and using the upper/lower index notation we can define two real 1-forms by

$$\alpha_j \equiv \big\langle(\nabla_A)_j\Phi, \Delta_A\Phi\big\rangle, \qquad \beta_k \equiv \big\langle(\nabla_A)^j\Phi, (\nabla_A)_k(\nabla_A)_j\Phi\big\rangle.$$

Stokes theorem implies $\int \nabla^j\alpha_j \,\mathrm{d}\mu_g = \int \nabla^k\beta_k \,\mathrm{d}\mu_g = 0$ since $\partial\Sigma = \emptyset$; but expanding out these divergences as

$$\nabla^j \alpha_j - \nabla^k \beta_k = \langle \Delta_A \Phi, \Delta_A \Phi \rangle - \big\langle (\nabla_A)^j (\nabla_A)_k \Phi, (\nabla_A)^k (\nabla_A)_j \Phi \big\rangle + g^{jl} g^{km} \big\langle (\nabla_A)_m \Phi, \big[(\nabla_A)_j, (\nabla_A)_k\big] (\nabla_A)_l \Phi \big\rangle + g^{jl} g^{km} \big\langle (\nabla_A)_j \Phi, (\nabla_A)_k \big(\big[(\nabla_A)_m, (\nabla_A)_l\big] \Phi\big) \big\rangle$$

and integrating then implies (9) by use of (A.5) and (A.6). □

**Lemma 1.4** *(Covariant version of the Brezis–Gallouet inequality). If $A \in H^1(\Sigma)$ and $\Phi \in H^2_A(\Sigma)$ then*

$$|\Phi|_{L^\infty(\Sigma)} \leqslant c\Big(1 + |\Phi|_{H^1_A} \sqrt{\ln\big(1 + |\Phi|_{H^2_A}\big)}\Big) \tag{10}$$

*where $c$ depends only on $(\Sigma, g)$.*

**Proof.** The form of this inequality for real $u \in H^2(\Sigma)$ is

$$|u|_{L^\infty(\Sigma)} \leqslant c\Big(1 + |u|_{H^1(\Sigma)} \sqrt{\ln\big(1 + |u|_{H^2(\Sigma)}\big)}\Big) \tag{11}$$

where, throughout in this proof, $c$ is a generic constant independent of $u$ depending only on $(\Sigma, g)$. This follows from the inequality for $u \in H^2 \cap H^1(\mathbb{R}^2)$ in [4] using a partition of unity (see Appendix A). Now apply (11) with $u(x) = |\Phi(x)|^2$ for $x \in \Sigma$; by the Kato inequality (and using the unitarity property of $A$, cf. Appendix A)

$$|\nabla u|_{L^2} \leqslant c|\Phi|_{L^\infty} |\nabla_A \Phi|_{L^2}$$

and

$$|\nabla\nabla u|_{L^2} \leqslant c\big(|\nabla_A \Phi|^2_{L^4} + |\Phi|_{L^\infty} |\nabla_A \nabla_A \Phi|_{L^2}\big) \leqslant c\big(|\nabla_A \Phi|^2_{H^1_A} + |\Phi|_{L^\infty} |\nabla_A \nabla_A \Phi|_{L^2}\big) \leqslant c|\Phi|^2_{H^2_A}$$

using Lemma 7. Altogether with (11) this leads to the inequality

$$|\Phi|^2_{L^\infty} \leqslant c\Big(1 + |\Phi|_{L^\infty} |\Phi|_{H^1_A} \sqrt{\ln\big(1 + |\Phi|_{H^2_A}\big)}\Big).$$

Take $c > 1$ without loss of generality, then this leads to (10). □

## 2. Statement of local existence

The system comprising (2), (3) is gauge invariant: for smooth real valued functions $g$ on $\mathbb{R} \times \Sigma$ the triple $(A_0, A, \Phi)$ is a smooth solution if and only if

$$\mathrm{e}^{\mathrm{i}g} \cdot (A_0, A, \Phi) \equiv \big(A_0 + \partial_t g, A + dg, \mathrm{e}^{\mathrm{i}g} \Phi\big) \tag{12}$$

is also a smooth solution. (Clearly this action can be extended to more general weak solutions.) To circumvent this degeneracy we consider the *parabolic gauge* in which

$$\operatorname{div} A = A_0.$$

This choice of gauge fixes the positive direction in time, so from now on we solve for $t \geqslant 0$. (The choice $\operatorname{div} A = -A_0$ fixes the opposite direction; the existence result obtained here is then similarly valid for $t \leqslant 0$.)

As mentioned in Section 1, we will prove local and global existence for the augmented system of equations, coupling the equations of (2) with a constraint equation

$$B - \frac{\gamma}{2\mu}\big(1 - |\Phi|^2\big) = h$$

for general $h \in H^2(\Sigma)$ and determined by the initial data. (The existence for the original system follows as the special case of $h = 0$.) Local existence is established by the following theorem which is proved in Section 4.

**Theorem 2.1** *(Local existence). For initial data $\Phi(0) \in H^2(\Sigma)$, $A(0) \in H^3(\Sigma)$ together with $A_0(0) = \operatorname{div} A(0) = 0$, there is a positive time $T_{\mathrm{loc}}$ which depends continuously on the above norms of the initial data, and there is a solution $(A_0, A, \Phi)$ of* (2), (3) *satisfying the gauge condition $A_0 = \operatorname{div} A$, and of regularity*

$$A \in C\big([0, T_{\mathrm{loc}}], H^1(\Sigma)\big) \cap C^1\big([0, T_{\mathrm{loc}}], L^2(\Sigma)\big),$$
$$\Phi \in C\big([0, T_{\mathrm{loc}}], H^2(\Sigma)\big) \cap C^1\big([0, T_{\mathrm{loc}}], L^2(\Sigma)\big).$$

*Furthermore, the solution is unique in these spaces and satisfies the conservation laws*

$$\mathcal{V}_\lambda\big(A(t), \Phi(t)\big) = \mathcal{V}_\lambda\big(A(0), \Phi(0)\big), \tag{13}$$
$$\big|\Phi(t)\big|_{L^2} = \big|\Phi(0)\big|_{L^2}. \tag{14}$$

**Remark.** The regularity of $\Phi$ implies by the constraint equation (3) that

$$*dA \in C\big([0, T_{\mathrm{loc}}], H^2(\Sigma)\big) \cap C^1\big([0, T_{\mathrm{loc}}], L^2(\Sigma)\big)$$

(however, $\operatorname{div} A$ is only proved (Lemma 4.1) to be continuous into $L^2$ in this gauge, thus overall $A$ is continuous into $H^1$).

## 3. Proof of the global existence theorem

Assuming Theorem 2.1, we have a local solution $(A, \Phi)$ of (2)–(3) defined on an interval $[0, T_{\mathrm{loc}}]$. In this section it is shown that this solution can be extended (in the same spaces) to a solution for infinite time. From the construction in the proof of Theorem 2.1 (in Section 4), the time $T_{\mathrm{loc}}$ depends continuously on the norms of the initial data, and we have $A(0) \in H^3$ and $\Phi(0) \in H^2$. By standard local existence theory there is a maximal time $T_{\max} \geqslant T_{\mathrm{loc}}$ such that for $t \in [0, T_{\max})$ a solution $(A, \Phi)$ of the system exists in the same gauge and spaces of Theorem 2.1 and for the same initial data. The bounds in the norm defined in Theorem 2.1 are not yet proved valid up to time $T_{\max}$; however, a priori bounds derived below from the energy and the equations do hold and will be used to show $T_{\max} = +\infty$.

Denote by $c$ a generic constant which depends on $(\Sigma, g)$, the Sobolev norms of the initial data, $h$, the constants $\gamma, \mu, \lambda$, and the energy. Unless stated otherwise the norms below are taken over $\Sigma$ at fixed time $t$ and the dependence on $t$ is omitted where no confusion is possible.

Differentiate in time equation (2) for $\Phi$ letting $V = \frac{\lambda}{4\gamma}(1 - |\Phi|^2)$

$$\begin{aligned}\left(\partial_t - \mathrm{i}A_0 - \frac{\mathrm{i}}{2\gamma}\Delta_A\right)(\partial_t - \mathrm{i}A_0)\Phi &= \left[-\frac{\mathrm{i}}{2\gamma}\Delta_A, \partial_t - \mathrm{i}A_0\right]\Phi + \mathrm{i}V(\partial_t - \mathrm{i}A_0)\Phi + \mathrm{i}(\partial_t V)\Phi \\ &= 2\frac{\mathrm{i}}{2\gamma}E \cdot \nabla_A\Phi + \frac{\mathrm{i}}{2\gamma}(\operatorname{div} E)\Phi + \mathrm{i}V(\partial_t - \mathrm{i}A_0)\Phi + \mathrm{i}(\partial_t V)\Phi^n\end{aligned} \tag{15}$$

(with $E \cdot \nabla_A\Phi = g^{ij}E_i\nabla_{A_j}\Phi$) and Lemma B.5 applies to give an estimate for $|(\partial_t - \mathrm{i}A_0)\Phi(t)|_{L^2}$ as in (B.11). Algebraically from (2) this implies an estimate for $|\Delta_A\Phi(t)|_{L^2}$ and by the Garding inequality (9) this in turn gives an estimate for $|\nabla_A\nabla_A\Phi(t)|_{L^2}$. Altogether we have for $t \in [0, T_{\max})$,

$$\begin{aligned}|\nabla_A\nabla_A\Phi|_{L^2} \leqslant c\Bigg(&1 + \big|\Delta_{A(0)}\Phi(0)\big|_{L^2} + \big|V(0)\Phi(0)\big|_{L^2} + |V\Phi|_{L^2} + |B|_{L^\infty}^{1/2}|\nabla_A\Phi|_{L^2} + |\Phi|_{L^\infty}^{1/2}|\nabla_A\Phi|_{L^2}^{1/2}|\nabla B|_{L^2}^{1/2} \\ &+ \int_0^t \big\{|E \cdot \nabla_A\Phi|_{L^2} + |\operatorname{div} E\Phi|_{L^2} + \big|(\partial_t V)\Phi\big|_{L^2}\big\}\,\mathrm{d}s\Bigg).\end{aligned} \tag{16}$$

We recall from Theorem 2.1 that the local solutions have constant energy,

$$\mathcal{V}_\lambda\big(A(t), \Phi(t)\big) = \mathcal{V}_\lambda\big(A(0), \Phi(0)\big).$$

Observing that for arbitrary $\epsilon > 0$

$$\frac{4}{\lambda}\mathcal{V} \geqslant \big(1 - |\Phi|^2\big)^2 \geqslant \left(1 - \frac{1}{2\epsilon}\right) + (1 - 2\epsilon)|\Phi|^4$$

we infer that $|\Phi|_{L^\infty(L^4)}$ and $|\nabla_A\Phi|_{L^\infty(L^2)}$ are bounded uniformly in $t$ and hence

$$\sup_{t>0}\big(|\Phi|_{H^1_A}(t) + |\Phi|_{L^p}(t)\big) \leqslant c = c\big(p, \mathcal{V}(0), \Sigma, g, \lambda\big) \tag{17}$$

for all $1 \leqslant p < \infty$. Now observe from Eqs. (2), (3) that $E, \nabla E, \nabla B$ are "schematically" given by $E = \nabla B + \langle \Phi, \nabla_A \Phi \rangle$, $\nabla B = \langle \Phi, \nabla_A \Phi \rangle$, $\nabla E = \langle \Phi, \nabla_A \nabla_A \Phi \rangle + |\nabla_A \Phi|^2$), so that the Sobolev (6) and Holder inequalities imply bounds in terms of the $H^2$ and $L^\infty$ norms of $\Phi$ (where any choice made is with the view that (10) will be eventually used on $|\Phi|_{L^\infty}$)

$$\begin{aligned}
&|V\Phi|_{L^2} \leqslant c\big(|\Phi|_{L^2} + |\Phi|_{L^6}^3\big) \leqslant c\big(|\Phi|_{L^2} + |\Phi|_{H_A^1}^3\big),\\
&|B|_{L^\infty} \leqslant c_1\big(1 + |\Phi|_{L^\infty}^2\big),\\
&|E|_{L^2} \leqslant c\big(1 + |\Phi|_{L^\infty}|\nabla_A \Phi|_{L^2}\big),\\
&|\nabla E|_{L^2} \leqslant c\big(1 + |\Phi|_{L^\infty}|\nabla_A \nabla_A \Phi|_{L^2} + |\nabla_A \Phi|_{L^2}|\nabla_A \Phi|_{H_A^1}\big).
\end{aligned} \tag{18}$$

Based on these, and using the Sobolev (6) and the interpolation (7) inequalities (where the choice between the two is essential to *avoid superlinear* terms in $|\nabla_A \nabla_A \Phi|_{L^2}$ which would cause the following argument to fail), we obtain

$$\begin{aligned}
&|E \cdot \nabla_A \Phi|_{L^2} \leqslant c|E|_{L^4}|\nabla_A \Phi|_{L^4},\\
&\text{(by (7))} \qquad \leqslant c|E|_{L^2}^{1/2}|E|_{H_A^1}^{1/2}|\nabla_A \Phi|_{L^2}^{1/2}|\nabla_A \Phi|_{H_A^1}^{1/2},\\
&\text{(by (18))} \qquad \leqslant c\big(1 + |\Phi|_{L^\infty}^{1/2}|\nabla_A \Phi|_{L^2}^{1/2}\big)\big(1 + |\Phi|_{L^\infty}^{1/2}|\nabla_A \Phi|_{L^2}^{1/2} + |\Phi|_{L^\infty}^{1/2}|\nabla_A \nabla_A \Phi|_{L^2}^{1/2}\\
&\qquad\qquad + |\nabla_A \Phi|_{L^2}^{1/2}\big(|\nabla_A \Phi|_{L^2}^{1/2} + |\nabla_A \nabla_A \Phi|_{L^2}\big)\big)|\nabla_A \Phi|_{L^2}^{1/2}\big(|\nabla_A \Phi|_{L^2}^{1/2} + |\nabla_A \nabla_A \Phi|_{L^2}^{1/2}\big)
\end{aligned}$$

which by (17) can be estimated as

$$\leqslant c\big(1 + |\Phi|_{L^\infty}|\nabla_A \nabla_A \Phi|_{L^2}\big). \tag{19}$$

(Here all norms bounded by the energy are absorbed in the constant $c$.) Similarly,

$$\begin{aligned}
|\operatorname{div} E\Phi|_{L^2} &\leqslant c|\nabla E|_{L^2}|\Phi|_{L^\infty}\\
&\leqslant c\big(1 + |\Phi|_{L^\infty}^2|\nabla_A \nabla_A \Phi|_{L^2}\big).
\end{aligned} \tag{20}$$

The final term under the integral is estimated using (15)

$$\begin{aligned}
\big|(\partial_t V)\Phi\big|_{L^2} &\leqslant c\big|\langle \Phi, (\partial_t - \mathrm{i}A_0)\Phi\rangle \Phi\big|_{L^2}\\
&\leqslant c|\Phi|_{L^\infty}^2|\Delta_A \Phi|_{L^2}\\
&\leqslant c|\Phi|_{L^\infty}^2|\nabla_A \nabla_A \Phi|_{L^2}
\end{aligned} \tag{21}$$

using that $V_t = 2\langle \Phi, \mathrm{i}\Delta_A \Phi \rangle$. Altogether we obtain from (16)–(21)

$$\begin{aligned}
|\nabla_A \nabla_A \Phi|_{L^2} &\leqslant c\left(1 + |\Phi|_{L^\infty} + \int_0^t \big(1 + |\Phi|_{L^\infty}|\nabla_A \nabla_A \Phi|_{L^2} + |\Phi|_{L^\infty}^2|\nabla_A \nabla_A \Phi|_{L^2}\big)\,\mathrm{d}s\right)\\
&\leqslant c\left(1 + |\Phi|_{L^\infty} + \int_0^t \big(1 + |\Phi|_{L^\infty}^2\big)|\nabla_A \nabla_A \Phi|_{L^2}\,\mathrm{d}s\right).
\end{aligned} \tag{22}$$

To this now apply the inequality (10):

$$\big|\Phi(t)\big|_{H_{A(t)}^2} \leqslant c\left(1 + \sqrt{\ln\big(1 + \big|\Phi(t)\big|_{H_{A(t)}^2}\big)} + \int_0^t \Big(1 + \big|\Phi(s)\big|_{H_A^2} + \big|\Phi(s)\big|_{H_A^2}\sqrt{\ln\big(1 + \big|\Phi(s)\big|_{H_A^2}\big)}\Big)\,\mathrm{d}s\right).$$

Now since $\sqrt{\ln(1+x)}/x \to 0$ as $x \to +\infty$ so there exists $L(c)$ such that for $|\Phi(t)|_{H_{A(t)}^2} > L(c)$

$$\sqrt{\ln\big(1 + |\Phi|_{H_A^2}\big)} \leqslant \frac{1}{2c}|\Phi|_{H_A^2}$$

and hence there exists (another) constant $c$ such that

$$\left|\Phi(t)\right|_{H^2_{A(t)}} \leqslant c\left(1+\int_0^t \left(1+\left|\Phi(s)\right|_{H^2_A}+\left|\Phi(s)\right|_{H^2_A}\sqrt{\ln\left(1+\left|\Phi(s)\right|_{H^2_A}\right)}\right)\mathrm{d}s\right) \equiv \mathcal{G}(t).$$

As the functions in the integrand are increasing we have

$$\mathcal{G}'(t) \leqslant c\left(1+\mathcal{G}(t)\right)\left(1+\ln\left(1+\mathcal{G}(t)\right)\right)$$

or $\ln(1+\ln(1+\mathcal{G}(t))) \leqslant ct+k$ (for $k$ a constant). Hence $|\Phi(t)|_{H^2_{A(t)}} \leqslant \mathcal{G}(t) \leqslant k'\mathrm{e}^{k\mathrm{e}^{ct}}$ as claimed in (4) for all time $t < T_{\max}$ which by a standard continuation argument implies that the solution $(A, \Phi)$ exists on $[0, \infty)$, and this proves the theorem. □

## 4. Proof of local existence

The proof of Theorem 2.1 follows a fixed point argument for an iteration as in the procedure layed out for conservation laws in [9]. Here the proof is based on the following two Lemmas 4.1, 4.2. The first lemma shows that a uniform time exists in which all the iterates $\Psi^n = (A^n, \Phi^n)$ are bounded in terms of the initial data only in the (high) norm $\Psi^n \in C([0, T_{\mathrm{loc}}], H^1 \times H^2(\Sigma))$. The second lemma shows that the iteration (24), (25) is a contraction in the (low) norm $C([0, T_{\mathrm{loc}}], L^2 \times H^1(\Sigma))$, to be precise, $\Psi^n$ is proved to be Cauchy in this space.

To start, we define and solve the iteration scheme.

*Smoothing of the initial data*: consider a smooth sequence $\Psi^n_0 = (A^n(0), \Phi^n(0)$ which approximates the initial data $\Psi(0) = (A(0), \Phi(0))$ in the sense that:

$$\left|\Psi^n_0 - \Psi(0)\right|_{H^1\times H^2(\Sigma)} \leqslant \epsilon_0 2^{-n} \tag{23}$$

(implying also that $\Psi^n_0$ are bounded, uniformly in $n$, in the same norm in terms of the initial data). Smoothing of the initial data ensures that the iterates below are smooth and well-defined.

*Definition of the iteration scheme*: given $(A^{n-1}, \Phi^{n-1})$, let $\Psi^n = (A^n, \Phi^n)$ be the solution of the approximating system

$$\partial_t A^n - \nabla \operatorname{div} A^n = \mathcal{F}\left(\Phi^{n-1}, \nabla_{A^{n-1}}\Phi^{n-1}\right) \equiv \mathcal{F}^{n-1}, \tag{24}$$

$$\left(\partial_t - \mathrm{i}A^n_0 - \frac{\mathrm{i}}{2\gamma}\Delta_{A^n}\right)\Phi^n = \frac{\lambda \mathrm{i}}{4\gamma}\left(1-\left|\Phi^{n-1}\right|^2\right)\Phi^n \tag{25}$$

with initial data $\Psi^n_0$ as above and where

$$\mathcal{F}^n \equiv -\frac{1}{2\mu}\nabla\left(h+\frac{\gamma}{2\mu}\left(1-\left|\Phi^n\right|^2\right)\right) - \frac{1}{2\mu}\epsilon\left\langle \mathrm{i}\Phi^n, \nabla_{A^n}\Phi^n\right\rangle \tag{26}$$

where as above $\epsilon$ is the antisymmetric $2\times 2$ tensor. Differentiating (24) and (26) will allow estimation of norms of higher order derivatives: first, taking $d$ of (24) (letting $B^n = b + *dA^n$) we have $\partial_t *dA^n = \frac{1}{2\mu}\langle \mathrm{i}\Phi^{n-1}, \Delta_{A^{n-1}}\Phi^{n-1}\rangle = \frac{\gamma}{2\mu}\partial_t(1-|\Phi^{n-1}|^2)$ and hence

$$B^n = \frac{\gamma}{2\mu}\left(1-\left|\Phi^{n-1}\right|^2\right) + h(x). \tag{27}$$

Taking divergence of (24) we obtain an inhomogeneous heat equation

$$\partial_t \operatorname{div} A^n - \Delta \operatorname{div} A^n = \nabla\cdot\mathcal{F}\left(\Phi^{n-1}, \nabla_{A^{n-1}}\Phi^{n-1}\right) = \nabla\cdot\mathcal{F}^{n-1} \tag{28}$$

where the right-hand side depends on $(\Phi^{n-1}, \nabla_{A^{n-1}}\Phi^{n-1}, \Delta_{A^{n-1}}\Phi^{n-1}, B^{n-1})$.

Finally we will use the following equations obtained by differentiation of (25). Let $E^n = \partial_t A^n - \nabla A^n_0$ and $V^n = \frac{\lambda}{4\gamma}(1-|\Phi^n|^2)$. Differentiation in time gives

$$\left(\partial_t - \mathrm{i}A_0^n - \frac{\mathrm{i}}{2\gamma}\Delta_{A^n}\right)\left(\partial_t - \mathrm{i}A_0^n\right)\Phi^n$$
$$= \left[-\frac{\mathrm{i}}{2\gamma}\Delta_{A^n}, \partial_t - \mathrm{i}A_0^n\right]\Phi^n + \mathrm{i}V^{n-1}\left(\partial_t - \mathrm{i}A_0^n\right)\Phi^n + \mathrm{i}\left(\partial_t V^{n-1}\right)\Phi^n$$
$$= 2\frac{\mathrm{i}}{2\gamma}E^n \cdot \nabla_{A^n}\Phi^n + \frac{\mathrm{i}}{2\gamma}\left(\operatorname{div} E^n\right)\Phi^n + \mathrm{i}V^{n-1}\left(\partial_t - \mathrm{i}A_0^n\right)\Phi^n + \mathrm{i}\left(\partial_t V^{n-1}\right)\Phi^n. \tag{29}$$

(Here $E^n \cdot \nabla_{A^n}\Phi^n = g^{ij}E_i\nabla_{A_{n_j}}\Phi^n$.) Similarly differentiation in $x$ gives

$$\left(\partial_t - \mathrm{i}A_0^n - \frac{\mathrm{i}}{2\gamma}\Delta_{A^n}\right)\left(\nabla_{A^n}\Phi^n\right) = \left[\partial_t - \mathrm{i}A_0^n, \nabla_{A^n}\right]\Phi^n - \frac{\mathrm{i}}{2\gamma}[\Delta_{A^n}, \nabla_{A^n}]\Phi^n + \mathrm{i}\left(\nabla V^{n-1}\right)\Phi^n + \mathrm{i}V^{n-1}\nabla_{A^n}\Phi^n$$
$$= \left(-\mathrm{i}E^n + \frac{\mathrm{i}}{2\gamma}[\Delta_{A^n}, \nabla_{A^n}] + \mathrm{i}\nabla V^{n-1}\right)\Phi^n + \mathrm{i}V^{n-1}\nabla_{A^n}\left(\Phi^n - \Phi^{n-1}\right). \tag{30}$$

*The unique solution of the iteration equations*: we solve (28), (24), (25) with the understanding $A_0^n = \operatorname{div} A^n$, with smooth initial data (23) to obtain by standard linear theory smooth solutions

$$\Psi^n = \left(A^n, \nabla_{A^n}\Phi^n\right) \in C^\infty\left([0,\infty) \times \Sigma\right) \quad \text{for } n = 1, 2, \ldots. \tag{31}$$

To see this, first solve the heat equation (28) which yields a $C^\infty([0,\infty) \times \Sigma)$ solution $divA^n$ (the regularity at $t = 0$ follows as $\partial\Sigma = \emptyset$); following the o.d.e. (24) implies $A^n$ is $C^\infty$ as well on $[0,\infty) \times \Sigma$. To solve the remaining equation (25) we apply Theorems 4.1 and 5.1 in [8] to the operator $-\mathrm{i}\Delta_{A^n(t)} - \mathrm{i}f(t,\cdot) : H^s(\Sigma) \to L^2(\Sigma)$ for smooth real $f$ and here $f = \mathrm{i}(A_0^n + \frac{\lambda}{4\gamma}V^{n-1})\Phi^n$; then the evolution operator at each time $0 \leqslant t < \infty$ preserves $H^s(\Sigma)$ in $L^2(\Sigma)$ for each $s \geqslant 2$). This implies that the solution is a smooth section $\Phi^n \in C^\infty([0,\infty) \times \Sigma)$ (recall that a fixed smooth background connection is implied for all derivatives of $\Phi$). The solution for each of these equations is unique and the regularity of the solutions justifies the manipulations in the following lemma.

**Lemma 4.1** *(Uniform bounds for the iterates). There exists a time $T_1 > 0$ and a constant $M > 0$, both depending only on the initial data and the function $h$ (and $\Sigma, g, \gamma, \mu, \lambda$), such that for each $n$ the solution of* (24), (25) *given in* (31) *$\Psi^n = (A^n, \Phi^n)$, with smooth data $\Psi_0^n$ defined in* (23)*, satisfies*

$$Y^n \in C\left([0, T_1]\right) \quad \textit{with } \left|Y^n\right|_{C([0,T_1])} \leqslant M \tag{32}$$

*where $Y^n(t)$ is the norm,*

$$Y^n(t) \equiv \sup_{s\in[0,t]}\left(\left|A^n\right|_{L^2} + \left|\operatorname{div} A^n\right|_{L^2} + \left|*dA^n\right|_{L^\infty} + \left|\Phi^n\right|_{L^\infty} + \left|\nabla_{A^n}\Phi^n\right|_{L^2} + \left|\nabla_{A^n}\nabla_{A^n}\Phi^n\right|_{L^2}\right)(s).$$

**Proof.** We assume inductively that $T_1$ exists for which (32) is valid for $n = 1, \ldots, N-1$; we denote by $c$ a generic constant depending only on $\Sigma, g, h, \gamma, \mu, \lambda$ and the initial values $Y^n(0)$ for $n = 0, \ldots, N-1$ in (i)–(iv) below. We show that for any $0 \leqslant t < T_1$, and

$$Y^N(t) \leqslant c + tp\left(Y^N(t), M\right) \tag{33}$$

where $p$ is a positive coefficient polynomial which depends (first argument) on the norm $Y^N$ of the $N$th iterate $\Psi^N$, and (second argument) on the same norm of the previous $1, \ldots, N-1$ iterates, inductively assumed to be less than $M$ for all time $0 \leqslant t \leqslant T_1$; $p$ is taken as the sum of the polynomials in (i)–(iv) below. We also assume that $M$ is chosen so that $M > c$ and set $T_1 = \frac{M-c}{2p(M,M)}$. If $\max_{t\in[0,T_1]} Y^N(t) \geqslant M$ then by continuity of $t \mapsto Y^N(t)$ there is a first time $0 < t_N \leqslant T_1$ for which $Y^N(t_N) = M$ and hence

$$\sup_{t\leqslant t_N} Y^N(t) \leqslant c + t_N p(M, M) \leqslant c + T_1 p(M, M) < M \tag{34}$$

which implies that $t_N$ can be taken to be $T_1$ otherwise we contradict the choice of $M, T_1$. Below we also write $p(M)$ in place of $p(M, M)$.

We now show Eq. (33) is valid to complete the induction argument. For the rest of this proof, all norms shown are implied to be *spatially taken over $\Sigma$ pointwise in time for t for $t \in [0, T_1]$ unless indicated by the notation, e.g. $C(L^2)$*

*indicating the norm for* $C([0,t], L^2(\Sigma))$. The following estimates (35)–(37) will be used in (i)–(iv) below (and applied with $n = N$). The right-hand side of (24) is given in (26) and is seen to satisfy

$$|\mathcal{F}^n|_{L^2} \leqslant c\big(|\nabla h|_{L^2} + |\Phi^n|_{L^\infty}|\nabla_{A^n}\Phi^n|_{L^2}\big) \tag{35}$$

and similarly in $L^p$. Differentiating (26) once,

$$\begin{aligned}|\nabla\mathcal{F}^n|_{L^2} &\leqslant c\big(|\nabla^2 h|_{L^2} + |\langle\Phi^n, \nabla_{A^n}\nabla_{A^n}\Phi^n\rangle|_{L^2} + |\nabla_{A^n}\Phi^n|^2_{L^4}\big)\\ &\leqslant c\big(|\nabla^2 h|_{L^2} + |\Phi^n|_{L^\infty}|\nabla_{A^n}\nabla_{A^n}\Phi^n|_{L^2} + |\Phi^n|^2_{H^2_{A^n}}\big)\end{aligned} \tag{36}$$

using the unitarity of the connection and the Sobolev inequality (6) applied to the last term. (Note here that either (6) or (7) can be used but for the corresponding calculation in the proof of global existence the quadratic rate for the $H^2$ norm is not suitable and therefore (7) must be used.)

In the parabolic gauge $E^n \equiv \partial_t A^n - \nabla A^n_0 = \mathcal{F}^{n-1}$ from (24). Thus by (35) we can estimate (at each $t$)

$$|E^n|_{L^2} \leqslant |\mathcal{F}^{n-1}|_{L^2} \leqslant c\big(|\nabla h|_{L^2} + |\Phi^{n-1}|_{L^\infty}|\nabla_{A^n}\Phi^{n-1}|_{L^2}\big) \tag{37}$$

and by (36) we have a similar bound for $|\nabla E^n|_{L^2}$. Also by the Sobolev inequality

$$\begin{aligned}|E^n|_{L^4} &\leqslant |\mathcal{F}^{n-1}|_{L^4} \leqslant c\big(|\nabla h|_{L^4} + |\langle\Phi^{n-1}, \nabla_{A^n}\Phi^{n-1}\rangle|_{L^4}\big)\\ &\leqslant c\big(|\nabla h|_{L^4} + |\Phi^{n-1}|_{L^\infty}|\Phi^{n-1}|_{H^2_{A^{n-1}}}\big).\end{aligned} \tag{38}$$

In Appendix B a priori estimates are shown for the heat, ordinary differential and Schroedinger equations which are now applied in turn to (28), (24), (25) and (29)–(30) leading to the following four estimates:

*(i) Estimate for* $\operatorname{div} A^N, A^N$. Apply the a priori estimate (B.1) for the heat equation in Appendix B to (28) for $2 \leqslant p < \infty$ to obtain

$$\begin{aligned}&\sup_{s\leqslant t}|\operatorname{div} A^N|^2_{L^2}(s) + \int_0^t |\nabla \operatorname{div} A^N|^2_{L^2}(s)\,\mathrm{d}s\\ &\quad\leqslant |\operatorname{div} A^N(0)|^2_{L^2} + ct\big(|\nabla h|^2_{L^2} + |\Phi^{N-1}|^2_{L^\infty(L^\infty)}|\nabla_{A^{N-1}}\Phi^{N-1}|^2_{L^\infty(L^2)}\big)\end{aligned} \tag{39}$$

and the right-hand side is of the form (33). Moreover, $\operatorname{div} A^n \in H^1$ and by (24) and (26) and with the o.d.e. estimate (B.2),

$$\sup_{\tau\leqslant t}|A^N|_{L^2}(\tau) \leqslant c + tp\big(|Y^N|_{C([0,t])}, M\big)$$

which is of the form (33). (Alternatively, note that by (39) and (27) we deduce from $L^p$ estimates for the div–curl system ((A.10)) that

$$\big(|A^N|^2_{L^2} + |\nabla A^N|^2_{L^2}\big)(\tau) \leqslant c + tp\big(\Psi^N, M\big) \tag{40}$$

with $c$, $p$, $M$ as defined above, which is again (33).)

*(ii) Estimate for* $\nabla_{A^N}\Phi^N$. From (25), and the Schroedinger equation estimate (B.5) in Lemma B.4 with $V^{N-1} = \frac{\lambda \mathrm{i}}{4\gamma}(1 - |\Phi^{N-1}|^2)$, and by (37),

$$\sup_{s\leqslant t}|\nabla_{A^N}\Phi^N|^2_{L^2}(s) \leqslant |\nabla_{A^N}\Phi^N|^2_{L^2}(0) + tp(M)\sup_{s\leqslant t}\big(|\Phi^N(s)|_{L^\infty}|\nabla_{A^N}\Phi^N(s)|_{L^2}\big), \tag{41}$$

which is of the general form of (33). Also note that as in (B.4) the $L^2$ norm is conserved,

$$|\Phi^N|_{L^2}(t) = |\Phi^N|_{L^2}(0). \tag{42}$$

*(iii) Estimate for* $(\partial_t - iA^N_0)\Phi^N$. We consider (29) (for $n = N$)

$$\big(\partial_t - \mathrm{i}A^N_0 - \mathrm{i}\Delta_{A^N}\big)\big((\partial_t - \mathrm{i}A^N_0)\Phi^N\big) = 2\mathrm{i}E^N\nabla_{A^N}\Phi^N + \mathrm{i}\operatorname{div} E^N\Phi^N + \mathrm{i}\big(\partial_t V^{N-1}\big)\Phi^N + \mathrm{i}V^{N-1}\big(\partial_t - \mathrm{i}A^N_0\big)\Phi^N.$$

To this we apply the estimate (B.11) (the last term does not contribute) to obtain,

$$\left|\left(\partial_t - \mathrm{i}A_0^N\right)\Phi^N\right|^2_{L^2}(t) \leqslant \left|\left(\partial_t - \mathrm{i}A_0^N\right)\Phi^N\right|^2_{L^2}(0) + c\sup_{[0,t]}\left|\left(\partial_t - \mathrm{i}A_0^N\right)\Phi^N\right|_{L^2}\int_0^t \left|\nabla_{A^N}\Phi^N\right|_{L^4}\left|E^N\right|_{L^4} + \left|\Phi^N\right|_{L^\infty}\left(\left|\partial_t V^{N-1}\right|_{L^2} + \left|\operatorname{div} E^N\right|_{L^2}\right) \tag{43}$$

where $((\partial_t - \mathrm{i}A_0^N)\Phi^N)(0)$ is determined by $\Psi_0^N$ and (25). The integrand is estimated uniformly in time, using (38) and either the Sobolev or the interpolation) inequalities, e.g. by (6) $|\nabla_{A^N}\Phi^N|_{L^4} \leqslant |\Phi^N|_{H^2_{A^N}}$; also, $|\partial_t V^{N-1}|_{L^2} \leqslant c|\Phi^{N-1}|_{L^\infty}|(\partial_t - \mathrm{i}A_0^{N-1})\Phi^{N-1})|_{L^2}$ which is bounded inductively using (25)). Without loss of generality we can assume $\sup_{[0,t]}|\partial_t - \mathrm{i}A_0^N)\Phi^N|_{L^2} > 1$ and dividing by this quantity the right-hand side is of the form (33).

*(iv) Estimate for* $\nabla_{A^N}\nabla_{A^N}\Phi^N$, and $\Phi^N$. Estimate (43) above, by Eq. (25), provides also an estimate for $|\Delta_{A^N}\Phi^N|_{L^2}(t)$, and together with the Garding inequality (9) implies,

$$\begin{aligned}\left|\nabla_{A^N}\nabla_{A^N}\Phi^N\right|_{L^2}(t) \leqslant{} & 1 + \left|\Delta_{A^N}\Phi^N\right|_{L^2}(0) + \left|V^{N-1}\Phi^N\right|_{L^2}(0) + \left|V^{N-1}\Phi^N\right|_{L^2}(t)\\ & + c\left|B^N\right|^{1/2}_{L^\infty}\left|\nabla_{A^N}\Phi^N\right|_{L^2} + c\left|\Phi^N\right|^{1/2}_{L^\infty}\left|\nabla_{A^N}\Phi^N\right|^{1/2}_{L^2}\left|\nabla B^N\right|^{1/2}_{L^2}\\ & + c\int_0^t \left(\left|E^N\right|_{L^4}\left|\nabla_{A^N}\Phi^N\right|_{L^4} + \left|\Phi^N\right|_{L^\infty}\left(\left|\partial_t V^{N-1}\right|_{L^2} + \left|\nabla E^N\right|_{L^2}\right)\right)\mathrm{d}s.\end{aligned} \tag{44}$$

The idea is to estimate this as in (iii) to obtain an estimate of the general form (33) for $\nabla_{A^N}\nabla_{A^N}\Phi^N$. Now $|\nabla_{A^N}\Phi^N|_{L^2}$ has already been estimated above but it is necessary to take care of the $|\Phi^N|^{1/2}_{L^\infty}$ on the right-hand side. To do this recall the Sobolev inequality $|\Phi^N|_{L^\infty} \leqslant c|\Phi^N|_{H^2_{A^N}}$, and use the square root to absorb it in the left-hand side since $\sqrt{A}\sqrt{B} \leqslant \epsilon A + B/(4\epsilon)$ for any $\epsilon > 0$. The induction is now complete and Lemma 4.1 proved. □

We now show that the iteration scheme defines a contraction for $\Psi^n$ and derivatives in $C([0, T_{\mathrm{loc}}], L^2(\Sigma))$ for some time $T_{\mathrm{loc}} \leqslant T_1$.

**Lemma 4.2** *(Contraction). For the solutions* $\Psi^n = (A^n, \Phi^n)$ *in Lemma* 4.1*, consider the vector*

$$\Xi^n = \left(A^n, \operatorname{div} A^n, *dA^n, \Phi^n, \nabla_{A^n}\Phi^n\right).$$

*If* $\epsilon_0$ *in* (23) *is sufficiently small there exists a time* $0 < T_{\mathrm{loc}} \leqslant T_1$ *and positive constants* $\gamma < 1$ *and* $\delta_n$ *with* $\Sigma_n\delta_n < \infty$ *such that for* $n = 1, 2, \ldots$

$$\left|\Xi^n - \Xi^{n-1}\right|_{C([0,T_{\mathrm{loc}}],L^2(\Sigma))} \leqslant \gamma\left|\Xi^{n-1} - \Xi^{n-2}\right|_{C([0,T_{\mathrm{loc}}],L^2(\Sigma))} + \delta_n. \tag{45}$$

**Proof.** We let $T_1$, $M$ be as in Lemma 4.1 and use the same convention on the generic constant $c$ and positive coefficient polynomial $p$ as in the previous lemma. For $0 \leqslant t \leqslant T_1$ use the notation $C(L^2)$ on $[0, t] \times \Sigma$ to indicate the norm uniform in time on $[0, t]$ and $L^2$ over $\Sigma$. (Recall that $\Xi^n \in C^\infty([0, \infty) \times \Sigma)$.)

We start by estimating $\mathcal{F}^n - \mathcal{F}^{n-1}$ (where $\mathcal{F}^n$ is the right-hand side of (26)), using Kato's inequality and the last lemma

$$\begin{aligned}\left|\mathcal{F}^n - \mathcal{F}^{n-1}\right|^2_{L^2(L^2)} \leqslant{} & c\left(\left|\Phi^n\right|^2_{C(L^\infty)} + \left|\Phi^{n-1}\right|^2_{C(L^\infty)}\right)\left|\nabla_{A^n}\Phi^n - \nabla_{A^n}\Phi^{n-1}\right|^2_{L^2(L^2)}\\ & + c\left|\left|\Phi^n - \Phi^{n-1}\right|\left(\left|\nabla_{A^n}\Phi^n\right| + \left|\nabla_{A^n}\Phi^{n-1}\right|\right)\right|^2_{L^2(L^2)}\end{aligned}$$

(applying Lemma 4.1 on the first term and Holder's inequality on the second)

$$\begin{aligned}&\leqslant cp(M)t\left|\nabla_{A^n}\Phi^n - \nabla_{A^n}\Phi^{n-1}\right|^2_{C(L^2)}\\ &\quad + ct\left|\Phi^n - \Phi^{n-1}\right|^2_{C(L^4)}\left(\left|\nabla_{A^n}\Phi^n\right|^2_{C(L^4)} + \left|\nabla_{A^n}\Phi^{n-1}\right|^2_{C(L^4)}\right)\\ &\leqslant cp(M)t\left(\left|\nabla_{A^n}\Phi^n - \nabla_{A^n}\Phi^{n-1}\right|^2_{C(L^2)} + \left|A^n - A^{n-1}\right|^2_{C(L^2)}\right).\end{aligned} \tag{46}$$

(To arrive at the last inequality in (46), the Sobolev and then the Kato inequalities imply that

$$\begin{aligned}\left|\Phi^n-\Phi^{n-1}\right|_{C(L^4)} &\leqslant c\left(\left|\nabla\left|\Phi^n-\Phi^{n-1}\right|\right|_{C(L^2)}+\left|\Phi^n-\Phi^{n-1}\right|_{C(L^2)}\right)\\ &\leqslant c\left(\left|\nabla_{A^n}\Phi^n-\nabla_{A^n}\Phi^{n-1}\right|_{C(L^2)}+\left|\Phi^{n-1}\right|_{C(L^\infty)}\left|A^n-A^{n-1}\right|_{C(L^2)}+\left|\Phi^n-\Phi^{n-1}\right|_{C(L^2)}\right).\end{aligned}$$

Altogether by Lemma 4.1

$$\left|\Phi^n-\Phi^{n-1}\right|_{C(L^4)}\leqslant cp(M)\left(\left|\nabla_{A^n}\Phi^n-\nabla_{A^n}\Phi^{n-1}\right|_{C(L^2)}+\left|A^n-A^{n-1}\right|_{C(L^2)}+\left|\Phi^n-\Phi^{n-1}\right|_{C(L^2)}\right).$$

Similarly, note that by the Sobolev inequality $|\nabla_{A^n}\Phi^n|_{C(L^4)}\leqslant c|\nabla_{A^n}\Phi^n|_{C(H^1_{A^n})}$ and again Lemma 4.1 applies to give (46).)

We use similar estimates to prove the contraction of the remaining terms. Subtract (28) at the $n$th and $(n-1)$st iterates

$$\begin{aligned}(\partial_t-\Delta)\left(\operatorname{div}A^n-\operatorname{div}A^{n-1}\right)&=\nabla\cdot\left(\mathcal{F}\left(\Phi^{n-1},\nabla_{A^{n-1}}\Phi^{n-1}\right)-\mathcal{F}\left(\Phi^{n-2},\nabla_{A^{n-2}}\Phi^{n-2}\right)\right)\\ &\equiv\nabla\cdot\left(\mathcal{F}^{n-1}-\mathcal{F}^{n-2}\right)\end{aligned}$$

and by (46) and (B.1),

$$\begin{aligned}&\left|\operatorname{div}A^n-\operatorname{div}A^{n-1}\right|^2_{C(L^2)}+\int_0^t\left|\nabla\operatorname{div}\left(A^n-A^{n-1}\right)\right|^2_{L^2}(s)\,\mathrm{d}s\\ &\quad\leqslant\left|\operatorname{div}A^n-\operatorname{div}A^{n-1}\right|^2_{L^2}(0)+cp(M)t\left(\left|\nabla_{A^n}\Phi^{n-1}-\nabla_{A^n}\Phi^{n-2}\right|^2_{C(L^2)}+\left|A^{n-1}-A^{n-2}\right|^2_{C(L^2)}\right).\end{aligned}\tag{47}$$

From this we obtain a contraction estimate for $A^n$: subtract Eqs. (24) for $n, n-1$,

$$\begin{aligned}\partial_t\left(A^n-A^{n-1}\right)&=\nabla\left(\operatorname{div}A^n-\operatorname{div}A^{n-1}\right)+\mathcal{F}^{n-1}-\mathcal{F}^{n-2}\\ &\equiv g_1+g_2\end{aligned}$$

with $g_1=\nabla(\operatorname{div}A^n-\operatorname{div}A^{n-1})$ and $g_2=\mathcal{F}^{n-1}-\mathcal{F}^{n-2}$. Use the o.d.e. estimate in Lemma B.3, followed by (47) on the $g_1$ term and (46) on the $g_2$ term (on $[0,t]\times\Sigma$ as above), to conclude

$$\begin{aligned}\left|A^n-A^{n-1}\right|_{L^\infty(L^2)}&\leqslant\left|A^n_0-A^{n-1}_0\right|_{L^2}+\sqrt{t}\,|g_1|_{L^2(L^2)}+t|g_2|_{L^\infty(L^2)}\\ &\leqslant\left|\operatorname{div}A^n-\operatorname{div}A^{n-1}\right|_{L^2}(0)+ctp(M)\left(\left|\nabla_{A^n}\Phi^{n-1}-\nabla_{A^n}\Phi^{n-2}\right|^2_{C(L^2)}\right.\\ &\qquad\left.+\left|A^{n-1}-A^{n-1}\right|^2_{C(L^2)}\right)\\ &\leqslant tp(M)\left|\Xi^n-\Xi^{n-1}\right|_{C(L^2)}.\end{aligned}\tag{48}$$

Next subtract (25) for $n, n-1$,

$$\begin{aligned}\left(\partial_t-\mathrm{i}A^n_0-\frac{\mathrm{i}}{2\gamma}\Delta_{A^n}\right)\left(\Phi^n-\Phi^{n-1}\right)&=\mathrm{i}\left(A^n_0-A^{n-1}_0\right)\Phi^{n-1}-\frac{1}{2\gamma}\nabla\cdot\left(A^n-A^{n-1}\right)\Phi^{n-1}\\ &\quad-\frac{1}{\gamma}\left(A^n-A^{n-1}\right)\cdot\left(\nabla\Phi^{n-1}+\frac{\mathrm{i}}{2}\left(A^n+A^{n-1}\right)\Phi^{n-1}\right)\\ &\quad+\frac{\mathrm{i}\lambda}{4\gamma}\left(\Phi^n-\Phi^{n-1}\right)\left(1-\left|\Phi^{n-1}\right|^2+\left|\Phi^{n-1}+\Phi^{n-2}\right|\Phi^{n-1}\right)\\ &\equiv g^n+\mathrm{i}V^{n-1}\left(\Phi^n-\Phi^{n-1}\right)\end{aligned}\tag{49}$$

where $V^{n-1}=\frac{\lambda}{4\gamma}(1-|\Phi^{n-1}|^2+|\Phi^{n-1}+\Phi^{n-2}|\Phi^{n-1})$ and $g^n$ is the sum of the remaining terms. (This separation of terms is not essential but slightly simplifies the following estimate as the last term's contribution vanishes. Also recall by $\nabla\Phi^n$ we imply the background connection which is independent of $n$ and thus this term is bounded just as $\nabla_{A^n}\Phi^n$ is.) The contraction is provided by $g^n$ since clearly (recall $A^n_0=\operatorname{div}A^n$)

$$\begin{aligned}\left|g^n\right|_{C(L^2)}&\leqslant c\left|\Phi^{n-1}\right|_{C(L^\infty)}\left|\operatorname{div}A^n-\operatorname{div}A^{n-1}\right|_{C(L^2)}\\ &\quad+\left|A^n-A^{n-1}\right|_{C(L^2)}\left(\left|\nabla\Phi^{n-1}\right|_{C(L^2)}+\left|A^n+A^{n-1}\right|_{C(L^2)}\left|\Phi^{n-1}\right|_{C(L^\infty)}\right)\end{aligned}$$

so that by applying (B.11) of Lemma B.5 as before we have

$$\begin{aligned}\left|\Phi^n-\Phi^{n-1}\right|_{C(L^2)} &\leqslant \left|\left(\Phi^n-\Phi^{n-1}\right)(0)\right|_{L^2}+t\left|g^n\right|_{C(L^2)}\\ &\leqslant \left|\left(\Phi^n-\Phi^{n-1}\right)(0)\right|_{L^2}+tp(M)\left|\Xi^n-\Xi^{n-1}\right|_{C(L^2)}.\end{aligned}\tag{50}$$

From (27) and (50) similarly, (since $||\Phi^{n-1}|^2-|\Phi^{n-2}|^2|_{C(L^2)}\leqslant p(M)|\Phi^{n-1}-\Phi^{n-2}|_{C(L^2)}$)

$$\left|{*dA^n}-{*dA^{n-1}}\right|_{C(L^2)}\leqslant cp(M)\left|\left(\Phi^n-\Phi^{n-1}\right)(0)\right|_{L^2}+tp(M)\left|\Xi^n-\Xi^{n-1}\right|_{C(L^2)}.\tag{51}$$

Together with (47), the last estimate implies that in fact the $A^n$ form a Cauchy sequence in $C(H^1)$, not just in $C(L^2)$ as implied by (48). This follows from solving the elliptic system of $\operatorname{div} A^n$ and $\operatorname{curl} A^n$ on $[0,t]\times\Sigma$ which is explained in Appendix A.

Finally we show the contraction of the sequence of $\nabla_{A^n}\Phi^n$; differentiate (49) which implies

$$\begin{aligned}\left(\partial_t-\mathrm{i}A_0^n-\frac{\mathrm{i}}{2\gamma}\Delta_{A^n}\right)\left(\nabla_{A^n}\left(\Phi^n-\Phi^{n-1}\right)\right)&=\left[\partial_t-\mathrm{i}A_0^n,\nabla_{A^n}\right]\left(\Phi^n-\Phi^{n-1}\right)-\frac{\mathrm{i}}{2\gamma}\left[\Delta_{A^n},\nabla_{A^n}\right]\left(\Phi^n-\Phi^{n-1}\right)\\ &\quad+\nabla_{A^n}g^n+\mathrm{i}\left(\nabla V^{n-1}\right)\left(\Phi^n-\Phi^{n-1}\right)+\mathrm{i}V^{n-1}\nabla_{A^n}\left(\Phi^n-\Phi^{n-1}\right)\\ &=\left(-\mathrm{i}E^n-\frac{\mathrm{i}}{2\gamma}\left[\Delta_{A^n},\nabla_{A^n}\right]+\mathrm{i}\nabla V^{n-1}\right)\left(\Phi^n-\Phi^{n-1}\right)+\nabla_{A^n}g^n\\ &\quad+\mathrm{i}V^{n-1}\nabla_{A^n}\left(\Phi^n-\Phi^{n-1}\right)\end{aligned}\tag{52}$$

where $g^n$, $V^{n-1}$ are the same as in (49). To this we apply again (B.11) (and as before, the last term does not contribute). For the term $[\Delta_{A^n},\nabla_{A^n}]$ we refer to (A.5), (A.6) in the appendix which imply,

$$\left|\left[\Delta_{A^n},\nabla_{A^n}\right]\left(\Phi^n-\Phi^{n-1}\right)\right|_{L^2}\leqslant ct\left|\Phi^n-\Phi^{n-1}\right|_{C(H^1)}.$$

Also,

$$\begin{aligned}\left|\left(-\mathrm{i}E^n+\mathrm{i}\nabla V^{n-1}\right)\left(\Phi^n-\Phi^{n-1}\right)\right|_{L^\infty(L^2)}&\leqslant c\left(\left|E^n\right|+\left|\nabla V^{n-1}\right|\right)_{C(L^4)}\left|\Phi^n-\Phi^{n-1}\right|_{C(L^4)}\\ &\leqslant ctp(M)\left|\nabla_{A^n}\left(\Phi^n-\Phi^{n-1}\right)\right|_{H^1_{A^n}}.\end{aligned}$$

For the terms in $\nabla A^n g^n$ it is useful (cf. (39)) to estimate as follows

$$\begin{aligned}\left|\nabla_{A^n}g^n\right|^2_{L^2(L^2)}&\leqslant cp(M)\left(\sqrt{t}\left|\nabla\operatorname{div}A^n-\nabla\operatorname{div}A^{n-1}\right|_{L^2(L^2)}\right.\\ &\quad\left.+t\left|\operatorname{div}A^n-\operatorname{div}A^{n-1}\right|_{C(L^2)}+t\left|A^n-A^{n-1}\right|_{C(L^2)}\right).\end{aligned}$$

Now apply (B.11) which together with the bounds of Lemma 4.1 implies that

$$\begin{aligned}\left|\nabla_{A^n}\left(\Phi^n-\Phi^{n-1}\right)\right|_{C(L^2)}&\leqslant\left|\nabla_{A^n(0)}\left(\Phi^n-\Phi^{n-1}\right)(0)\right|_{L^2}\\ &\quad+t\left|\left(-\mathrm{i}E^n+\mathrm{i}\nabla V^{n-1}\right)\left(\Phi^n-\Phi^{n-1}\right)\right|_{C(L^2)}+\sqrt{t}\left|\nabla g^n\right|_{L^2(L^2)}\\ &\leqslant\left|\nabla_{A^n(0)}\left(\Phi^n-\Phi^{n-1}\right)(0)\right|_{L^2}+tp(M)\left(\left|\Phi^n-\Phi^{n-1}\right|^2_{C(L^4)}\right.\\ &\quad\left.+\left|A^n-A^{n-1}\right|^2_{C(L^2)}+\left|\operatorname{div}A^n-\operatorname{div}A^{n-1}\right|_{C(L^2)}\right)\\ &\leqslant\left|\left(\nabla_{A^n}\Phi^n-\nabla_{A^{n-1}}\Phi^{n-1}\right)(0)\right|_{L^2}+tp(M)\left|\Xi^n-\Xi^{n-1}\right|_{C(L^2)}\end{aligned}\tag{53}$$

since

$$\left|\nabla_{A^n}\Phi^n-\nabla_{A^{n-1}}\Phi^{n-1}\right|\leqslant\left|\nabla_{A^n}\left(\Phi^n-\Phi^{n-1}\right)\right|+\left|A^n-A^{n-1}\right|\left|\Phi^{n-1}\right|.$$

From (48), (47), (51), (48), (50), and (53), and any positive time $T_{\mathrm{loc}}\leqslant\min\{\frac{1}{p(M)},T_1\}$, take $\gamma=T_{\mathrm{loc}}p(M)<1$. Recalling the regularity of $\Xi^n$, this completes the proof of Lemma 4.2. □

*Completion of the proof of local existence*: the sequence $\Xi^n\in C^\infty([0,T_{\mathrm{loc}}]\times\Sigma)^5$ of Lemma 4.2 is Cauchy in $C([0,T_{\mathrm{loc}}];L^2(\Sigma))^5$ and so $\Xi^n\to\Xi$ in $C([0,T_{\mathrm{loc}}];L^2(\Sigma))^5$ for $\Xi$ in the same space. So far we have obtained

$(A_0, A, \Phi)$ as the unique weak solution of (2)–(3) in the parabolic gauge (taking pointwise limits of the iterated equations using the smoothness of $(A^n, \Phi^n)$) with the same gauge condition $\operatorname{div} A = A_0$.) Further, because by Lemma 4.1 $(A^n, \Phi^n)$ are bounded in $L^\infty((0, T_{\mathrm{loc}}], (H^1 \times H^2)(\Sigma))$, thus $\Phi \in L^\infty((0, T_{\mathrm{loc}}), H^2(\Sigma))$. The energy is constant and (13) holds. By the interpolation inequality (7) $(\Phi^n)_n$ is Cauchy in $C(W^{1,4})$ and so also in $C(L^\infty)$ (since $|\Phi^n - \Phi^m|_{L^\infty} \leqslant c|\Phi^n - \Phi^m|_{W^{1,4}} \leqslant c|\Phi^n - \Phi^m|_{L^2}^{1/2} \sup_n |\Phi^n|_{H^2}^{1/2}$). Altogether we have $\Phi \in C(H^1) \cap C(L^\infty)$.

To complete the proof we show the further regularity for $\Phi \in C([0, T_{\mathrm{loc}}], H^2(\Sigma)) \cap C^1([0, T_{\mathrm{loc}}), L^2(\Sigma))$. (It follows immediately from what is already known that continuity in time of the $H^1$ norm of $\Phi$ is equivalent to continuity of $|\nabla_A \Phi|_{L^2}(t)$ and similarly for the $H^2$ norm). In fact it is enough to prove regularity in only one of these spaces, because if $\Phi \in C([0, T_{\mathrm{loc}}], H^2(\Sigma))$ then by the equation for $\Phi$ in (2) it follows that $\Phi \in C^1([0, T_{\mathrm{loc}}], L^2(\Sigma))$ (since $\partial_t \Phi$ is schematically given by $A_0\Phi + \Delta_A \Phi + V\Phi$ and these terms are all in $C(L^2)$). So it suffices to show $\Phi \in C([0, T_{\mathrm{loc}}], H^2)$. This is a consequence of the following

**Claim.** *$\Phi^n \to \Phi$ in $C([0, T_{\mathrm{loc}}],\ H^2_{\mathrm{weak}})$ so that $\Phi \in C([0, T_{\mathrm{loc}}],\ H^2_{\mathrm{weak}})$ and, furthermore, $t \mapsto |\Phi(t)|_{H^2}$ is continuous on* $[0, T_{\mathrm{loc}}]$.

For the first part it must be shown that for all $\psi \in H^{-2}(\Sigma)$, the map

$$t \mapsto \langle \psi, \Phi(t) \rangle$$

is (uniformly) continuous on $[0, T_{\mathrm{loc}}]$. (In this paragraph $\langle\, ,\, \rangle$ is the pairing between $H^s$ and $H^{-s}$, with $s$ indicated as a suffix only when necessary for emphasis.) By density, consider a sequence $(\psi^m)_m \subset H^{-1}$ with $\psi^m \to \psi$ in $H^{-2}$. Fix any $\epsilon > 0$, let $M = M(\epsilon)$ such that for all $m \geqslant M$, $|\psi^m - \psi|_{H^{-2}} \leqslant \epsilon/3$. As established in Lemma 4.2 $\Phi^n \to \Phi$ in $C(H^1)$, so that there is $N_1 = N(\epsilon, M)$ such that if $n \geqslant N_1$ then $|\langle \psi^M, \Phi^n(t) - \Phi(t)\rangle_1| \leqslant \epsilon/3$. Thus for all $n \geqslant N_1$

$$\begin{aligned} \big|\langle \psi, \Phi^n(t)\rangle_2 - \langle \psi, \Phi(t)\rangle_2\big| &\leqslant \big|\big\langle \big(\psi - \psi^M\big), \Phi^n(t)\big\rangle_2\big| + \big|\big\langle \psi^M, \big(\Phi^n(t) - \Phi(t)\big)\big\rangle_1\big| + \big|\big\langle \big(\psi - \psi^M\big), \Phi(t)\big\rangle_2\big| \\ &\leqslant \big|\psi^M - \psi\big|_{H^{-2}} \big|\Phi^n\big|_{C(H^2)} + \epsilon/3 + \big|\psi^M - \psi\big|_{H^{-2}} |\Phi|_{L^\infty(H^2)} \leqslant \epsilon \end{aligned}$$

by the bounds in Lemma 4.1. Thus $\Phi^n \to \Phi$ in $C(H^2_{\mathrm{weak}})$.

For the continuity of the norm, first notice that since $\Phi \in C(H^2_{\mathrm{weak}})$ the equation for $\Phi$ in (2) implies $\partial_t \Phi \in C(L^2_{\mathrm{weak}})$ (as it is schematically given by terms of type $C(L^2) \cdot C(L^\infty) + C(L^2_{\mathrm{weak}}) + C(L^\infty)$), and so for any $t_0 \in [0, T_{\mathrm{loc}})$,

$$|\partial_t \Phi|_{L^2}(t_0) \leqslant \liminf_{t \to t_0^+} |\partial_t \Phi|_{L^2}(t)$$

and we must show the reverse inequality for the limit superior. This is done using the estimate (43), however, because of the choice of parabolic gauge, right and left limits are shown separately, starting with the right limit.

Let $(A_0^n, A^n, \Phi^n)$ be iterates obtained for $t > t_0$ by solving (25), (24) with initial data $(A_0^n(t_0), A^n(t_0), \Phi^n(t_0))$ posed at $t = t_0$ and obtained as in (23) by smoothing $(A_0(t_0), A(t_0), \Phi(t_0))$ (the unique solution just constructed at $t = t_0$). These iterates will converge to a solution which, by uniqueness, will coincide on $[t_0, t_0 + T_{\mathrm{loc}}]$ with the solution obtained previously with data $\Psi(0)$ specified at $t = 0$. We apply the Schroedinger estimate (B.10) for the iterates $(A_0^n, A^n, \Phi^n)$ as in (43) on $[t_0, t_0 + \epsilon] \times \Sigma$. Consider first the limit $n \to \infty$ of (43): the integrand on the right is bounded uniformly in $n$ by Lemma 32 so

$$\big|(\partial_t - \mathrm{i}A_0)\Phi\big|_{L^2}(t) \leqslant \liminf_{n \to \infty} \big|\big(\partial_t - \mathrm{i}A_0^n\big)\Phi^n\big|_{L^2}(t) \leqslant \lim_{n \to \infty} \big|\big(\partial_t - \mathrm{i}A_0^n\big)\Phi^n\big|_{L^2}(t_0) + c(M)\epsilon$$

where $M$ is the uniform bounds of the iterates in $n$ given in Lemma 4.1, and where the strong convergence of the initial data at $t_0$ allows replacing *liminf* by *lim* on the right-hand side. Therefore,

$$\limsup_{t \to t_0^+} \big|(\partial_t - \mathrm{i}A_0)\Phi\big|_{L^2}(t) \leqslant \big|(\partial_t - \mathrm{i}A_0)\Phi\big|_{L^2}(t_0).$$

Therefore,

$$\lim_{t \to t_0^+} \big|(\partial_t - \mathrm{i}\operatorname{div} A)\Phi\big|_{L^2}(t) = \big|(\partial_t - \mathrm{i}\operatorname{div} A)\Phi\big|_{L^2}(t_0)$$

and together with $(\partial_t - \mathrm{i}\,\mathrm{div}\,A)\Phi \in C(L^2_{\mathrm{weak}})$ we conclude that $(\partial_t - \mathrm{i}\,\mathrm{div}\,A)\Phi \in C(L^2)$.

By the equation for $\Phi$ in (2) this implies that $\Delta_A \Phi \in C(L^2)$. Now recall the proof of the covariant Garding inequality in Lemma 1.3: integrating the last inequality in the proof we obtain that $|\nabla_A \nabla_A \Phi|_{L^2} \in C(L^2)$ and because $A \in C(H^1)$ this implies again that $t \mapsto |\nabla_a \nabla_a \Phi|_{L^2}(t)$ is a continuous function of $t$ (where $a$ is the background connection); recalling that $\Phi \in C(H^2_{\mathrm{weak}})$ implies that $\nabla_a \nabla_a \Phi(t)$ is weakly convergent in $L^2$ as $t \to t_0^+$, and as the norms converge the (uniform) right continuity in $H^2$ at any $t_0 \in [0, T_{\mathrm{loc}})$ is shown.

Finally, we show the left continuity at $t_0 \in (0, T_{\mathrm{loc}}]$. Fix $t_0$ and let $\tilde{\Psi} = (\tilde{A}_0, \tilde{A}, \tilde{\Phi})$ be the solution with initial data $\tilde{\Psi}(t_0) = \Psi(t_0)$ in the gauge $\tilde{A}_0 = -\,\mathrm{div}\,\tilde{A}$, defined for $t \leqslant t_0$ close to $t_0$. (Here, $\Psi = (A_0, A, \Phi)$ is the solution constructed as before with initial data specified at 0 defined for $t > 0$ in the gauge $A_0 = \mathrm{div}\,A$.) By the right continuity already proved and the following claim it follows that $\tilde{\Phi}$ is left continuous (in $H^2$) at $t_0$.

**Claim.** *Let $(A_0, A, \Phi)(t, x)$ solve* (2), (3) *on the line bundle $L$ with gauge condition $A_0 = \mathrm{div}\,A$ for $t > 0$. Define for $t < 0$ a section $\hat{\Phi}(t, x) = \Phi^*(-t, x)$ of the complex conjugate bundle $L^*$, define $\hat{A}_0(t, x) = A_0(-t, x)$ and introduce a conjugate connection $\hat{A}$ on $L^*$ by the formula $\mathbf{X} \cdot \nabla_{\hat{A}} \hat{\Phi}(t, x) = (\mathbf{X} \cdot \nabla_A \Phi)^*(-t, x)$ for every vector field $\mathbf{X}$ on $\Sigma$. Then $(\hat{A}_0, \hat{A}, \hat{\Phi})$ solves the same system on the line bundle $L^*$ with gauge condition $A_0 + \mathrm{div}\,A = 0$ for $t < 0$.*

The proof of this follows by direct calculation. Moreover, $\hat{\Psi}$ and $\Psi$ are related by a gauge transformation as in (12) for which we now solve: for $0 < t < t_0$ we find a gauge transformation $\mathrm{e}^{\mathrm{i}g}$ such that

$$(\tilde{A}_0, \tilde{A}, \tilde{\Phi}) = \mathrm{e}^{\mathrm{i}g}(A_0, A, \Phi)$$

locally solves (2), (3) under the gauge condition $\tilde{A}_0 = -\,\mathrm{div}\,\tilde{A}$, which implies that $g$ is the solution of

$$\partial_t g + \Delta g = -\tilde{A}_0 - \mathrm{div}\,\tilde{A} = -2\,\mathrm{div}\,\tilde{A} \equiv f \tag{54}$$

where $\mathrm{d}f \in L^2(\mathrm{d}x\,\mathrm{d}t)$ (as $\nabla A \in L^2(\mathrm{d}x\,\mathrm{d}t)$ and with initial condition $g(t_0) = 0$ (so $\mathrm{e}^{\mathrm{i}g(t_0)}$ is the identity on $\Sigma$, i.e. $\tilde{\Psi}(t_0) = \Psi(t_0)$). By the time reversal argument just given $\tilde{\Phi}$ is left continuous at $t_0$ so that $\Phi = \mathrm{e}^{-\mathrm{i}g(t)}\tilde{\Phi}$ is left continuous at $t_0$ provided $g$ is sufficiently regular: but it follows from Lemma B.2 that $g(t) \to 0$ uniformly as $t \to t_0$, and hence $\mathrm{e}^{-\mathrm{i}g(t)} \to 1$ uniformly, as $t \to t_0$. Thus

$$\mathrm{e}^{-\mathrm{i}g(t)}(\partial_t - \mathrm{i}\tilde{A}_0)\tilde{\Phi}(t) \longrightarrow \mathrm{e}^{-\mathrm{i}g(t_0)}(\partial_t - \mathrm{i}\tilde{A}_0)\tilde{\Phi}(t_0) \quad \text{as } t \to t_0^- \text{ in } C\big(L^2\big)$$

and thus left continuity is proved completing the proof of the regularity assertions in the local existence theorem. The fact that the solution thus constructed obeys the conservation laws (13) and (14) is proved in the usual way by computing the rates of change of the corresponding quantities for the iterates and taking a limit. This completes the proof of Theorem 2.1.

## Appendix A. Notation

Assume $(\Sigma, g)$ is a two-dimensional, oriented, compact Riemannian manifold without boundary ($\partial\Sigma = \emptyset$), with metric $g$ and let $\{U_i, V_i, \chi_i\}$ for $i = 1, \ldots, n$ be a finite, regular covering by spherical coordinate neighbourhoods i.e. $\chi_i(U_i) = B_3(0)$ and $V_i = \chi_i^{-1}(B_1(0))$. Let $(b_i)_i$ be a subordinate $C^\infty$ partition of unity such that $b_i > 0$, supp $b_i \subset V_1 = \chi_i^{-1}(\overline{B_1(0)})$, $\sum_i b_i(x) = 1$ for all $x \in \Sigma$ and the supports of the $b_i$ form a locally finite covering of $\Sigma$. For all $f$ continuous on $\Sigma$

$$\int\limits_\Sigma f\,\mathrm{d}\mu_g = \sum_{i=1}^n \int\limits_{U_i} b_i f\,\mathrm{d}\mu_g = \sum_{i=1}^n \int\limits_{B_3(0)} (b_i f) \circ \chi_i^{-1}\,\mathrm{d}\mu_g^i$$

where $\mathrm{d}\mu_g^i = \sqrt{\det g^i}\,\mathrm{d}x \in \Omega^2(B_3(0))$ is the local representation of the canonical volume element determined by $g$. (Here $g^i : \chi_i(U_i) \to \mathrm{Mat}^+(2, \mathbb{R})$ that is, $g^i(x)$ varies smoothly with $x \in B_3(0)$ and is a symmetric, positive-definite invertible $2 \times 2$ matrix with smoothly varying inverse $(g^i)^{-1}$, so both matrices are bounded on $B_1(0)$, for $i = 1, \ldots, n$.) The induced measure $\mathrm{d}\mu_g^i$ on $\mathbb{R}^2$ is equivalent to Lebesgue measure and, defining $f_i = f b_i$ and $\tilde{f}_i = (f b_i) \circ \chi_i^{-1}$, we have the integrals

$$|f|_{L^2(\Sigma)} \leqslant \sum_i |f_i|_{L^2(\Sigma)} = \Sigma_i \Bigg( \int\limits_{B_3(0)} |\tilde{f}_i|^2 \,\mathrm{d}\mu_i^g \Bigg)^{1/2}, \tag{A.1}$$

$$|\nabla f|^2_{L^2(\Sigma)} = \int\limits_{\Sigma} \nabla_\alpha f \nabla_\beta f g^{\alpha\beta} \,\mathrm{d}\mu_g \leqslant c \sum_{i=1}^n \int\limits_{B_3(0)} g_i^{\alpha\beta} \nabla_\alpha \tilde{f}_i \nabla_\beta \tilde{f}_i \sqrt{\det g^i} \,\mathrm{d}x, \tag{A.2}$$

$$\begin{aligned} |\nabla\nabla f|^2_{L^2(\Sigma)} &= \int\limits_{\Sigma} g^{\alpha\gamma} g^{\beta\delta} \nabla_\alpha \nabla_\beta f \nabla_\gamma \nabla_\delta f \,\mathrm{d}\mu^g \\ &\leqslant c \sum_{i=1}^n \int\limits_{B_3(0)} \big(g^i\big)^{\alpha\gamma} \big(g^i\big)^{\beta\delta} \nabla_\alpha \nabla_\beta \tilde{f}_i \nabla_\gamma \nabla_\delta \tilde{f}_i \sqrt{\det g^i} \,\mathrm{d}x. \end{aligned} \tag{A.3}$$

The above define norms for the spaces $L^2(\Sigma)$, $H^1(\Sigma)$ and $H^2(\Sigma)$ (e.g., $|f|^2_{H^2} = |f|^2_{L^2} + |\nabla f|^2_{L^2} + |\nabla\nabla f|^2_{L^2}$). Standard density and imbedding theorems are valid, see, for example, [1,11]. To derive the version (11) of the Brezis–Gallouet inequality used in the proof of Lemma 1.4 we start with the inequality

$$|u|_{L^\infty(\Omega)} \leqslant c\Big(1 + |u|_{H^1(\Omega)} \sqrt{\ln\big(1 + |u|_{H^2(\Omega)}\big)}\Big) \tag{A.4}$$

valid for smooth $u$ supported in $\Omega$; the constant $c$ does not depend on $\Omega$. (See [4] for a derivation.) Now for $f \in C^\infty(\Sigma)$ use the decomposition $f = \sum_i f_i$ to infer $|f|_{L^\infty(\Sigma)} \leqslant \sup |\tilde{f}_i|_{L^\infty}$. Now apply (A.4) to $\tilde{f}_i$ and observe that $|\tilde{f}_i|_{H^s(B_3} \leqslant c|f|_{H^s(\Sigma)}$ by the Leibniz and chain rules applied to $\tilde{f}_i = (f b_i) \circ \chi_i^{-1}$. This leads to

$$|f|_{L^\infty(\Sigma)} \leqslant c\Big(1 + |f|_{H^1(\Sigma)} \sqrt{\ln\big(1 + |f|_{H^2(\Sigma)}\big)}\Big)$$

and hence (11) follows for $u \in H^2(\Sigma)$ by approximation.

We shall always consider conformal co-ordinate systems on $\Sigma$ in which the metric is of the form $\mathrm{e}^{2\rho}((\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2)$ and the volume element is then $\mathrm{e}^{2\rho}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$. On functions the Hodge operator acts as $*f = f\,\mathrm{d}\mu_g = f\mathrm{e}^{2\rho}\,\mathrm{d}x^1 \wedge \mathrm{d}x^2$ and $*^2 = 1$. On 1-forms $*$ is just the complex structure in conformal co-ordinates $*(\omega_1\,\mathrm{d}x^1 + \omega_2\,\mathrm{d}x^2) = \epsilon\omega \equiv \omega_1\,\mathrm{d}x^2 - \omega_2\,\mathrm{d}x^1$, represented also by the anti-symmetric tensor $\epsilon_i^j$ with $\epsilon_2^1 = +1$, $\epsilon_1^2 = -1$, the other components being zero. For a real 1-form $A$ we write the co-differential $d^*A = -\operatorname{div} A$ with $\operatorname{div} A = \nabla_i(g^{ij}\sqrt{g}A_j)/\sqrt{g}$ and the Laplacian on real functions is $\Delta f = \nabla_i(g^{ij}\sqrt{g}\nabla_j f)/\sqrt{g} = \mathrm{e}^{-2\rho}\nabla_i\nabla_i f$. Finally we sometimes denote the inner product of two 1-forms with a dot i.e.

$$\omega \cdot \alpha \equiv g^{ij}\omega_i\alpha_j.$$

Now let $L \to \Sigma$ be a complex line bundle over $(\Sigma, g)$ with an inner product $h$ on the fibers; write $h_p$ for the induced inner product on each fiber $\pi^{-1}(p)$, $p \in \Sigma$. Consider a local trivialisation determined by choice of a local unitary frame over the co-ordinate neighbourhoods $U_i$; a smooth section $s$ of $L$ then corresponds to a family of smooth functions $s_i : U_i \to \mathcal{C}$ so that on $U_i \cap U_j$ we have $s_i = \mathrm{e}^{\mathrm{i}\theta_{ij}} s_j$ with $\mathrm{e}^{\mathrm{i}\theta_{ij}} : U_i \cap U_j \to S^1$ smooth. A smooth $S^1$ connection on $L$ is a covariant derivative operator $\nabla_A : s \mapsto \nabla_A s$ mapping any smooth section $s$ to $\nabla_A s$, a smooth section of $\Omega^1(L) = \Omega^1 \otimes L$ (the $L$-valued 1-forms, i.e. $\mathbf{X} \cdot \nabla_A s = (\nabla_A s)(\mathbf{X})$ is a section of $L$ for any vector field $\mathbf{X}$); it is required that $\nabla_A$ is unitary (preserves $h$) and satisfies the Leibniz rule. The commutator gives the curvature:

$$\big[(\nabla_A)_1, (\nabla_A)_2\big]\Phi\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 = -\mathrm{i}B\Phi\,\mathrm{d}\mu_g. \tag{A.5}$$

This generalises as follows for $\omega \in \Omega^1(L)$ to

$$\big[(\nabla_A)_1, (\nabla_A)_2\big]\omega_k\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 = -\mathrm{i}B\omega_k\,\mathrm{d}\mu_g + R^j_{12k}\omega_j\,\mathrm{d}x^1 \wedge \mathrm{d}x^2. \tag{A.6}$$

Fix a smooth connection $\nabla_a$ whose associated curvature $b\,\mathrm{d}\mu_g$ is constant (i.e. $b = \text{const.}$). A general connection is of the form $\nabla_A = \nabla_a - \mathrm{i}A$ for a real 1-form $A$; the choice of $\nabla_a$ thus gives an identification of $\Omega^1(\Sigma)$ with the space of $S^1$ connections on $L$. The curvature of $\nabla_A$ is seen from (A.5) to be $B\,\mathrm{d}\mu_g = b\,\mathrm{d}\mu_g + \mathrm{d}A$ and explicitly $B = b + (\nabla_1 A_2 - \nabla_2 A_1)/\mathrm{e}^{2\rho}$ in conformal co-ordinates as above. The dot product between 1-forms on $\Sigma$ extends in an obvious way to $L$-valued 1-forms. The integrals in (A.1)–(A.3) extend to similar integrals on 1-forms on $\Sigma$ and also $L$-valued sections and 1-forms, as do the corresponding $H^s_a$ spaces and associated Sobolev inequalities, see [11]. For more general nonsmooth connections $A$ the validity of these Sobolev inequalities can often be proved as in Lemmas 1.2–1.4 using the Kato inequality. The Laplacian on sections is, similar to the real case,

$$\Delta_A \Phi = \frac{1}{\sqrt{g}} (\nabla_A)_j \big(g^{ij}\sqrt{g}\,(\nabla_A)_i \Phi\big) = \mathrm{e}^{-2\rho}\big((\nabla_A)_i (\nabla_A)_i \Phi\big). \tag{A.7}$$

This can be further expanded in terms of the Laplacian of the background connection $\Delta_a$ as

$$\Delta_A \Phi = \Delta_a \Phi + \mathrm{e}^{-2\rho}\big(-2\mathrm{i}A_j (\nabla_a)_j \Phi - \mathrm{i}\nabla_j A_j - |A|^2 \Phi\big). \tag{A.8}$$

The system of equations

$$B = f, \qquad \operatorname{div} A = g \tag{A.9}$$

(where as above $\operatorname{div}: \Omega^1 \to \Omega^0$ is minus the adjoint of $d$) is a first order elliptic system which can be solved for $A$ subject to the condition on $\int f \,\mathrm{d}\mu_g$ dictated by the degree of $L$. It can be rewritten

$$\mathrm{d}A = (f - b)\,\mathrm{d}\mu_g, \qquad \operatorname{div} A = g \tag{A.10}$$

and solved via Hodge decomposition as long as the right-hand sides have zero integral. There is a solution unique up to addition of harmonic 1-forms which satisfies $\|A\|_{W^{1,p}} \leqslant c_p(1 + \|f\|_{L^p} + \|g\|_{L^p})$ for $p < \infty$.

In the time dependent setting of (2), (3) we take products with $\mathbb{R}$ to get a line bundle $\mathbb{R} \times L \to \mathbb{R} \times \Sigma$ over $\mathbb{R} \times \Sigma$. On this bundle a connection is of the form $D_{\mathbf{A}}\Phi = (\partial_t - \mathrm{i}A_0)\Phi\,\mathrm{d}t + \nabla_A \Phi$ and the commutator gives the associated curvature as

$$\sum_{\mu<\nu} \big[(D_{\mathbf{A}})_\mu, (D_{\mathbf{A}})_\nu\big]\Phi\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -\mathrm{i}E_{0j}\,\mathrm{d}t \wedge \mathrm{d}x^j - \mathrm{i}B\Phi\,\mathrm{d}\mu_g. \tag{A.11}$$

We consider measurable sections $\Phi$ with the norm $H^1_A$

$$|\Phi|^2_{H^1_A} = \int_\Sigma |\Phi|^2 + |\nabla_A \Phi|^2\,\mathrm{d}\mu_g.$$

In the above integral, the inner products $h$ and $g$ are implied, i.e. $|\Phi|^2 = |\Phi|^2_h = \langle \Phi, \Phi\rangle_h$ and $|\nabla_A\Phi|^2 = |\nabla_A\Phi|^2_{g\times h} = g^{ij}\langle \nabla_{A_i}\Phi, \nabla_{A_j}\Phi\rangle_h$ and accordingly for $|\nabla_A\nabla_A\Phi|_{L^2}$. The usual Sobolev-type inequalities in the spaces $H^1_A(\Sigma)$, $H^2_A(\Sigma)$ are all valid. This can be seen easily since for $\Phi \in H^1_A$, then $|\Phi|, \nabla|\Phi| \in L^2$ so that $|\Phi| \in H^1(\Sigma)$ which implies $|\Phi| \in L^p(\Sigma)$ for all $p < \infty$. If $\Phi \in H^2_A$ then $\Phi, \nabla_A\Phi \in H^1_A$; so $|\Phi|, |\nabla_A\Phi|$ and $\nabla|\Phi| \in L^p$ for all $p < \infty$, implying that $|\Phi| \in W^{1,p}(\Sigma)$ for all $p < \infty$ and so $\Phi \in L^\infty$. When $t$ is also an independent variable these facts are valid for each $t$ (with a time-dependent norm $H^1_{A(t)}$).

## Appendix B. A priori estimates

We base the proof of local existence on the following estimates for the *heat, ordinary differential and Schroedinger equations*. (Where functions appear as general, given inhomogeneities, the necessary regularity is assumed for all the norms involved to be meaningful.) Recall the notation $E_k = \partial_t A_k - \partial_{x^k} A_0$, for $k = 1, 2$.

**Lemma B.1.** *For a smooth solution $u : [0,\infty) \times \Sigma \to \mathbb{R}$ of*

$$u_t - \Delta u = \partial_{x^k}\mathcal{F}_k$$

*for given smooth $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$, $\mathcal{F}(t,x) \in \mathbb{R}^2$ it follows that*

$$\sup_{[0,T]} \int_\Sigma |u|^2\,\mathrm{d}x + \int_0^T \int_\Sigma |\nabla u|^2\,\mathrm{d}x\,\mathrm{d}t \leqslant c\left(\big|u(0)\big|^2_{L^2} + \int_0^T \int_\Sigma |\mathcal{F}|^2\,\mathrm{d}x\,\mathrm{d}t\right). \tag{B.1}$$

For the proof we multiply by $u$ and integrate over $[0,T] \times \Sigma$. (The smoothness assumption can be generalised.)

**Lemma B.2.** *There is a unique solution of the heat equation $u_t - \Delta u = f$ on $[0,T] \times \Sigma$ where $f \in C(L^2)$, $\nabla f \in L^2(L^2)$ with initial data $u(0) \equiv 0$ such that $u(t) \to 0$ as $t \to 0^+$ uniformly on $\Sigma$.*

**Proof.** Write the integral equation $u(t) = \int_0^t \mathrm{e}^{(t-s)\Delta} f(s)\,\mathrm{d}s$ and recall that since $\Sigma$ is two-dimensional

$$\left\|\mathrm{e}^{t\Delta}\right\|_{L^p \to L^\infty} \leqslant ct^{-1/p}$$

[14, Chapter 15] while Sobolev's lemma which implies that $f \in L^2(L^p)$. Therefore,

$$\left|u(t,x)\right| \leqslant c \int_0^t |t-s|^{-1/p} \left|f(s)\right|_{L^p} \mathrm{d}s \leqslant ct^{1-2/p} |f|_{L^2(L^p)}$$

which gives the result for $p > 2$. □

**Lemma B.3.** *From the ordinary differential equation $\partial_t A = f$ for $f$ a given function, $f(t,x) \in \mathbb{R}$, it follows for any spatial derivative of order $|s|$*

$$\left|\partial_x^s A(t)\right|_{L^2} - \left|\partial_x^s A(0)\right|_{L^2} \leqslant \min\bigl\{\theta_1(t), \theta_2(t)\bigr\} \tag{B.2}$$

*where $\theta_1(t) = t|\partial_x^s f|_{L^\infty(L^2)}$ and $\theta_2(t) = \sqrt{t}\,|\partial_x^s f|_{L^2(L^2)}$.*

The proof follows by differentiation in space and integration in time and standard Holder estimates.

**Lemma B.4.** *Let $\Phi$ be a smooth solution of the Schroedinger equation*

$$(\partial_t - \mathrm{i}A_0)\Phi - \mathrm{i}\Delta_A \Phi = \mathrm{i}V\Phi \tag{B.3}$$

*where $A_0, A_i, V$ are given smooth functions on $[0,\infty) \times \Sigma$. Then*

$$\left|\Phi(t)\right|_{L^2}^2 = \left|\Phi(0)\right|_{L^2}^2 \tag{B.4}$$

*and*

$$\left|\nabla_A \Phi(t)\right|_{L^2}^2 \leqslant \left|\nabla_A \Phi(0)\right|_{L^2}^2 + c\int_0^t |\Phi|_{L^\infty} |\nabla_A \Phi|_{L^2} \bigl(|\nabla V|_{L^2} + |\partial_t A - \nabla A_0|_{L^2}\bigr)\,\mathrm{d}s. \tag{B.5}$$

**Proof.** The identity (B.4) follows as for the usual Schroedinger equation, i.e., by multiplying by $\Phi$ and integrating over $\Sigma$. For (B.5) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}|\nabla_A \Phi|_{L^2}^2 &= 2\bigl\langle \nabla_A \Phi, (\partial_t - \mathrm{i}A_0)\nabla_A \Phi\bigr\rangle_{L^2} \\ &= 2\bigl\langle \nabla_A \Phi, \nabla_A(\partial_t - \mathrm{i}A_0)\Phi\bigr\rangle_{L^2} + 2\bigl\langle \nabla_A \Phi, -\mathrm{i}(\partial_t A_k - \partial_{x^k} A_0)\Phi\bigr\rangle_{L^2}\end{aligned}$$

where $\langle h, g\rangle_{L^2} = \int_\Sigma \langle h, g\rangle\,\mathrm{d}x$. Expanding the first term on the right,

$$\nabla_A(\partial_t - \mathrm{i}A_0)\Phi = \mathrm{i}\nabla_A \Delta_A \Phi + \mathrm{i}(\nabla V)\Phi + \mathrm{i}V\nabla_A \Phi \tag{B.6}$$

so that

$$\bigl\langle \nabla_A \Phi, \nabla_A(\partial_t - \mathrm{i}A_0)\Phi\bigr\rangle_{L^2} = \bigl\langle \nabla_A \Phi, \mathrm{i}(\nabla V)\Phi\bigr\rangle_{L^2} \tag{B.7}$$

which implies

$$\bigl|\bigl\langle \nabla_A \Phi, (\partial_t - \mathrm{i}A_0)\nabla_A \Phi\bigr\rangle_{L^2}\bigr| \leqslant |\nabla_A \Phi|_{L^2} |\Phi|_{L^\infty} |\nabla V|_{L^2} \tag{B.8}$$

and, in addition,

$$\bigl|\bigl\langle \nabla_A \Phi, -\mathrm{i}(\partial_t A_k - \partial_{x^k} A_0)\Phi\bigr\rangle\bigr| \leqslant |\Phi|_{L^\infty} |\nabla_A \Phi|_{L^2} |\partial_t A_k - \partial_k A_0|_{L^2} \tag{B.9}$$

and so (B.5) follows by integration in time. □

**Lemma B.5.** *Solutions of the equation for $u$,*

$$(\partial_t - \mathrm{i}A_0 - \mathrm{i}\Delta_A)u = g + \mathrm{i}Vu \tag{B.10}$$

*with $A_0$, $A_i$, $g$, $V$ are smooth given functions on $[0,T] \times \Sigma$, satisfy the two inequalities*

$$\begin{aligned} &\left|u(t)\right|_{L^2}^2 \leqslant \left|u(0)\right|_{L^2}^2 + |u|_{L^\infty_{(0,t)}(L^2)} \int_0^t \left|g(s)\right|_{L^2} \,\mathrm{d}s, \\ &|u|_{L^\infty_{(0,t)}(L^2)} \leqslant \left|u(0)\right|_{L^2} + \int_0^t \left|g(s)\right|_{L^2} \,\mathrm{d}s. \end{aligned} \tag{B.11}$$

**Proof.** Multiply (B.10) by $u$ (the last term in (B.10) vanishes) and integrate:

$$\left|u(t)\right|_{L^2}^2 \leqslant \left|u(0)\right|_{L^2}^2 + \int_0^t \left|g(s)\right|_{L^2} \left|u(s)\right|_{L^2}$$

from which (B.11) follows. □

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