Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 24 (2007) 251–278
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Random modulation of solitons for the stochastic Korteweg–de Vries equation

A. de Bouard [a,*], A. Debussche [b]

[a] *CNRS et Université Paris-Sud, UMR 8628, Bât. 425, Université de Paris-Sud, 91405 Orsay cedex, France*
[b] *ENS de Cachan, Antenne de Bretagne, Campus de Ker Lann, Av. R. Schuman, 35170 Bruz, France*

Received 8 September 2005; received in revised form 22 February 2006; accepted 3 March 2006

Available online 3 October 2006

Abstract

We study the asymptotic behavior of the solution of a Korteweg–de Vries equation with an additive noise whose amplitude ε tends to zero. The noise is white in time and correlated in space and the initial state of the solution is a soliton solution of the unperturbed Korteweg–de Vries equation. We prove that up to times of the order of $1/\varepsilon^2$, the solution decomposes into the sum of a randomly modulated soliton, and a small remainder, and we derive the equations for the modulation parameters. We prove in addition that the first order part of the remainder converges, as ε tends to zero, to a Gaussian process, which satisfies an additively perturbed linear equation.

Résumé

Nous étudions le comportement asymptotique de la solution d'une équation de Korteweg–de Vries avec un bruit additif dont l'amplitude ε tend vers 0. Le bruit est blanc en temps et spatialement corrélé, la donnée initiale est un soliton de l'équation non perturbée. Nous montrons que pour des temps inférieurs à $1/\varepsilon^2$, la solution se décompose en une onde solitaire aléatoirement modulée et un reste petit. Nous obtenons les équations des paramètres de modulation. Nous montrons également la convergence du terme d'ordre un dans le reste vers un processus gaussien centré vérifiant une équation linéaire bruitée.

MSC: 35Q53; 60H15; 76B25; 76B35

Keywords: Korteweg–de Vries equation; Stochastic partial differential equations; White noise; Central limit theorem; Solitary waves

1. Introduction

The influence of random perturbations on the propagation of solitons, either in the nonlinear Schrödinger equation or in the Korteweg–de Vries equation has been extensively studied in the physics literature; one of the method used is the so called collective coordinate approach, which consists in assuming a priori that the main part of the solution is given by a modulated soliton, and in finding then the modulation equations for the soliton parameters (see [3,11,17]).

* Corresponding author.
E-mail address: anne.debouard@math.u-psud.fr (A. de Bouard).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2006.03.009

The inverse scattering method has also been widely used; it gives more precise results, but requires particular perturbations [1,13,27,28]. See also [16,24,25] for numerical studies.

In [27], the special case of an additive perturbation which is a space independent white noise is considered. In this case, using the Galilean invariance of the homogeneous Korteweg–de Vries equation, the author was able to write explicitly the solution of the perturbed equation in terms of the solution of the homogeneous equation, that is the soliton solution. It appears that this solution is given by the sum of a randomly modulated soliton and a Brownian motion.

Our aim in the present article is to give a rigorous analysis of the validity of this kind of decomposition of the solution for more general additive perturbations, which are still white noise in time, but may also depend on the space variable. The analysis will be performed in the limit where the amplitude of the noise tends to zero. The equation we consider may be written in Itô form as

$$\mathrm{d}u + \left(\partial_x^3 u + \partial_x\left(u^2\right)\right)\mathrm{d}t = \varepsilon\,\mathrm{d}W, \tag{1.1}$$

where u is a random process defined on $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$, and the process $W(t,x)$ may be written as $W(t,x) = \phi\frac{\partial B}{\partial x}$, ϕ being a linear bounded operator on $L^2(\mathbb{R})$ and $B(t,x)$ a two parameters Brownian motion on $\mathbb{R}^+ \times \mathbb{R}$. Considering a complete orthonormal system $(e_i)_{i\in\mathbb{N}}$ in $L^2(\mathbb{R})$, we may alternatively write W as

$$W(t,x) = \sum_{i\in\mathbb{N}} \beta_i(t)\phi e_i(x), \tag{1.2}$$

$(\beta_i)_{i\in\mathbb{N}}$ being an independent family of real valued Brownian motions. Hence, the correlation function of the process W is

$$\mathbb{E}\big(W(t,x)W(s,y)\big) = c(x,y)(s\wedge t), \quad x, y \in \mathbb{R},\ s,t > 0,$$

where

$$c(x,y) = \int_{\mathbb{R}} \mathcal{K}(x,z)\mathcal{K}(y,z)\,\mathrm{d}z,$$

and $\mathcal{K}$ here stands for the kernel of ϕ, that is for any $u \in L^2(\mathbb{R})$,

$$(\phi u)(x) = \int_{\mathbb{R}} \mathcal{K}(x,y)u(y)\,\mathrm{d}y.$$

We will be led to assume some spatial smoothness for the correlation function of the process W. We indeed need enough smoothness on the solution of (1.1) we consider to be able to use the evolution of the Hamiltonian and higher order conserved quantities of the homogeneous KdV equation (see e.g. [26]). This may be translated in terms of the operator ϕ: if we want W to be a process (in the time variable) with values almost surely in a Hilbert space H, then we need ϕ to be a Hilbert–Schmidt operator from $L^2(\mathbb{R})$ with values into H; this is also sufficient, i.e. if this is the case, then the series in (1.2) converges in $L^2(\Omega; H)$. We recall that ϕ is a Hilbert–Schmidt operator form $L^2(\mathbb{R})$ into the Hilbert space H – denoted $\phi \in \mathcal{L}_2(L^2(\mathbb{R}); H)$ – if and only if the norm

$$\|\phi\|_{\mathcal{L}_2(L^2;H)} = \operatorname{tr}(\phi^*\phi) = \sum_{i\in\mathbb{N}} |\phi e_i|_H^2 \tag{1.3}$$

is finite, and that this norm does not depend on the complete orthonormal system under consideration. We will mainly deal with solutions living in the usual Sobolev space $H^1(\mathbb{R})$ of square integrable functions of the space variable x, having their first order derivative in $L^2(\mathbb{R})$. Because the Airy equation – the homogeneous linear equation associated with (1.1) – generates a unitary operator, we will then be led to assume that W lies in $H^1(\mathbb{R})$, i.e. $\phi \in \mathcal{L}_2(L^2(\mathbb{R}); H^1)$. In terms of the kernel $\mathcal{K}$, this amounts to require that $\mathcal{K} \in L^2(\mathbb{R}\times\mathbb{R})$ and $\partial_x\mathcal{K} \in L^2(\mathbb{R}\times\mathbb{R})$. Note that $H^1(\mathbb{R})$ is a natural space for the homogeneous KdV equation, and allows to use the Hamiltonian, which we recall is defined for $u \in H^1(\mathbb{R})$ by

$$H(u) = \frac{1}{2}\int_{\mathbb{R}} (\partial_x u)^2\,\mathrm{d}x - \frac{1}{3}\int_{\mathbb{R}} u^3\,\mathrm{d}x. \tag{1.4}$$

We recall also that if $\varepsilon = 0$, any time continuous solution of (1.1) with values in $H^1(\mathbb{R})$ satisfies $H(u(t)) = H(u(0))$ for all t.

For $\varepsilon > 0$ the existence and uniqueness of path-wise solutions for Eq. (1.1) supplemented with an initial condition $u(0) = u_0 \in H^1(\mathbb{R})$ has been studied in [5] (see also [7] and [8] for existence and uniqueness of less regular solutions in the case of rough spatial correlations). We recall hereafter the precise result (see [5], Theorem 3.1).

Theorem 1.1. *Let $u_0 \in H^1(\mathbb{R})$ and assume that $\phi \in \mathcal{L}_2(L^2(\mathbb{R}); H^1(\mathbb{R}))$; then there exists a solution u of (1.1), a.s. continuous in time with values in $H^1(\mathbb{R})$, defined for all $t > 0$ and with $u(0) = u_0$. Moreover, for any $T > 0$ such a solution is unique among those having paths a.s. in some space $X_T \subset C([0,T]; H^1(\mathbb{R}))$.*

Consider now the homogeneous Korteweg–de Vries equation

$$\partial_t u + \partial_x^3 u + \partial_x\big(u^2\big) = 0. \tag{1.5}$$

It is well known that this equation possesses solitary wave (soliton) solutions, propagating with a constant velocity $c > 0$, with the expression $u_{c,x_0}(t,x) = \varphi_c(x - ct + x_0)$, $x_0 \in \mathbb{R}$, and with

$$\varphi_c(x) = \frac{3c}{2\cosh^2(\sqrt{c}\,x/2)} \tag{1.6}$$

satisfying

$$\varphi_c'' - c\varphi_c + \varphi_c^2 = 0. \tag{1.7}$$

A large literature has been devoted to Eq. (1.5), and especially to solutions of the form (1.6). The most precise results have been obtained with the use of the inverse scattering transform (see [2] or [21] for a review). It is known for example that any sufficiently localized and smooth solution of (1.5) will resolve, as time goes to infinity, into a finite sum of soliton solutions, (1.6), with different velocities, entirely determined by the initial state. If the method gives precise results, however, it does not work for arbitrary perturbations of Eq. (1.5). If e.g. the nonlinear term $\partial_x(u^2)$ is replaced by a more general term $\partial_x f(u)$, then the inverse scattering method does not apply in general, even though solutions of the form (1.6) still exist for a wide range of functions f. Stability properties of such solutions (1.6) for those generalizations of Eq. (1.5) have also been the object of several studies, starting with the work of Benjamin [4] on orbital stability, and improving recently with the work of Pego and Weinstein [23] or Martel and Merle [18] dealing with asymptotic stability. Note that another conserved quantity for Eq. (1.5) is given by

$$m(u) = \frac{1}{2}\int_{\mathbb{R}} u^2(x)\,\mathrm{d}x, \tag{1.8}$$

i.e. we have $m(u(t)) = m(u(0))$ for any solution $u \in C(\mathbb{R}; H^1)$ of (1.5), and that Eq. (1.7) may be written as $H'(\varphi_c) + cm'(\varphi_c) = 0$. The proof of orbital stability is based on the use of the functional

$$Q_c(u) = H(u) + cm(u), \quad u \in H^1(\mathbb{R}), \tag{1.9}$$

as a Lyapunov functional. It uses the fact that the set $\{\varphi_c(\cdot - s), s \in \mathbb{R}\}$, that is the orbit of φ_c, is a set of local minima of Q_c restricted to the manifold $\{u \in H^1(\mathbb{R}), m(u) = m(\varphi_c)\}$. Indeed, the linearized operator

$$L_c = -\partial_x^2 + c - 2\varphi_c \tag{1.10}$$

is not a positive operator on $H^1(\mathbb{R})$, but it is when restricted to the subspace of H^1 of functions orthogonal in $L^2(\mathbb{R})$ to both φ_c and $\partial_x\varphi_c$ (see [4] or [14]). Now, the operator arising in the linearization of Eq. (1.5) is $\partial_x L_c$. This operator has no unstable eigenvalue – see [22] – but it has a two dimensional generalized null-space spanned by $\partial_c\varphi_c$ and $\partial_x\varphi_c$. Indeed, it is easily checked that

$$\partial_x L_c \partial_c \varphi_c = -\partial_x\varphi_c \quad \text{and} \quad \partial_x L_c \partial_x \varphi_c = 0.$$

The asymptotic stability is then obtained via the use of modulations of the solitary wave φ_c in both the phase parameter x_0, and the velocity c, in order to get rid of these two secular modes (see [18] and [23]). It is then proved e.g. in [18] that a solution of (1.5) (or some of its generalizations), initially close in $H^1(\mathbb{R})$ to a solitary wave of the form (1.6), will converge weakly in $H^1(\mathbb{R})$ as time goes to infinity to another solitary wave of the form (1.6), but where the velocity c and the phase x_0 have been shifted.

Our aim here is to investigate the influence of random perturbations of the form given in Eq. (1.1) on the propagation of solitary waves of the form (1.6). Consider indeed the solution $u^\varepsilon(t,x)$ of Eq. (1.1), given by Theorem 1.1, and with $u^\varepsilon(0,x) = \varphi_{c_0}(x)$ where $c_0 > 0$ is fixed. We may expect that, if ε is small, the main part of the solution $u^\varepsilon(t,x)$ is a solitary wave, randomly modulated in its velocity and phase. We will prove in Section 3 that this is true for time less than C/ε^2 where C is a constant. Note that this order of time C/ε^2 for the persistence of the soliton is natural and was numerically observed in [10] and [24].

Our next purpose – achieved in Sections 4 to 6 – is to investigate more precisely the behavior at order one in ε of the remaining term in the preceding decomposition, as ε goes to zero. More precisely, the preceding decomposition says that the solution $u^\varepsilon(t,x)$ is written as

$$u^\varepsilon(t,x) = \varphi_{c^\varepsilon(t)}\big(x - x^\varepsilon(t)\big) + \varepsilon\eta^\varepsilon\big(x - x^\varepsilon(t)\big),$$

where $c^\varepsilon(t)$ and $x^\varepsilon(t)$ are the modulation parameters – these are random processes, and more precisely semi-martingales. Then, in the spirit of [12], Chapter 7, we show that the process η^ε converges as ε goes to zero, in probability, to a centered Gaussian process which satisfies an additively driven linear equation, with a conservative deterministic part. Moreover, x^ε and c^ε can be developed up to order one in ε, and we get

$$\begin{cases} \mathrm{d}x^\varepsilon = c_0\,\mathrm{d}t + \varepsilon y\,\mathrm{d}t + \varepsilon B_1\,\mathrm{d}t + \varepsilon\,\mathrm{d}B_2 + \mathrm{o}(\varepsilon), \\ \mathrm{d}c^\varepsilon = \varepsilon\,\mathrm{d}B_1 + \mathrm{o}(\varepsilon), \end{cases}$$

where B_1 and B_2 are time changed Brownian motions, and y is a centered Gaussian process. Let us mention that the parameters c^ε and x^ε have been numerically computed in [10], and that our results agree with those computations. All these results are precisely stated and discussed in Section 2.

We end the introduction with a few notations. In all the paper, $(\cdot,\cdot)$ will denote the inner product in $L^2(\mathbb{R})$, or sometimes the duality product between the usual Sobolev space $H^m(\mathbb{R})$, $m \in \mathbb{N}$, of functions having m derivatives in $L^2(\mathbb{R})$, and its dual space $H^{-m}(\mathbb{R})$.

If A and B are Banach spaces, $\mathcal{L}(A;B)$ will stand for the space of linear bounded operators from A into B, and $\mathcal{L}_2^m$ will be an abbreviation for $\mathcal{L}_2(L^2(\mathbb{R}); H^m(\mathbb{R}))$, the space of Hilbert–Schmidt operators from $L^2(\mathbb{R})$ into $H^m(\mathbb{R})$, endowed with the norm defined as in (1.3), with $H = H^m(\mathbb{R})$.

In all the paper, $(e_j)_{j\in\mathbb{N}}$ is a fixed complete orthonormal system of $L^2(\mathbb{R})$.

For a sequence $\gamma = (\gamma_n)_{n\in\mathbb{N}}$ of real positive numbers, with $\lim_{n\to+\infty}\gamma_n = +\infty$, we denote by X_γ the space

$$X_\gamma = \left\{ u \in L^2(\mathbb{R}), \sum_{\ell\in\mathbb{N}} \gamma_\ell (u, e_\ell)^2 = |u|^2_{X_\gamma} < +\infty \right\}. \tag{1.11}$$

Note that X_γ is compactly embedded in $L^2(\mathbb{R})$. The definition of X_γ a priori depends on the basis $(e_j)_{j\in\mathbb{N}}$, but this will not cause any trouble, since this basis is from now on fixed.

Also, for $x_0 \in \mathbb{R}$, we denote $\mathcal{T}_{x_0}$ the translation operator defined for $\varphi \in C(\mathbb{R})$ by $(\mathcal{T}_{x_0}\varphi)(x) = \varphi(x + x_0)$.

Finally, we assume from now on that a stochastic basis $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geqslant 0}, (\widetilde{W}(t))_{t\geqslant 0})$ is given, i.e. $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $(\mathcal{F}_t)_{t\geqslant 0}$ is a filtration and $(\widetilde{W}(t))_{t\geqslant 0}$ is a cylindrical Wiener process associated with this filtration. We then consider an operator ϕ with $\phi \in \mathcal{L}_2^1$ and we define

$$W = \phi\widetilde{W}. \tag{1.12}$$

More restrictive assumptions will be required concerning ϕ in Theorem 2.6, and they are stated there.

The paper is organized as follows. Section 2 is devoted to the statement and discussion of the results. In Section 3, we prove Theorem 2.1, i.e. we explain how we can define the modulation parameters. We also estimate the exit time, i.e the time up to which the modulation procedure is available. In Section 4, we give the equations for the modulation parameters and start to estimate these parameters. Section 5 is devoted to estimates on the remainder term. These estimates will allow us to pass to the limit and conclude the proof of Theorem 2.6 in Section 6. Finally, in Section 7, we collect the proofs of a few technical estimates which will be used in Section 6.

2. Statement of the results

Let $c_0 > 0$ be fixed, and consider for $\varepsilon > 0$ the solution $u^\varepsilon(t,x)$ of Eq. (1.1), with $u^\varepsilon(0,x) = \varphi_{c_0}(x)$, given by Theorem 1.1. The next theorem says that u^ε may be decomposed as the sum of a modulated solitary wave, and a remaining part with small H^1 norm, for t less than some stopping time τ^ε which goes to infinity in probability as ε goes to zero. We shall then show that this remaining part is of order one with respect to ε.

More precisely, we will write

$$u^\varepsilon(t,x) = \varphi_{c^\varepsilon(t)}\big(x - x^\varepsilon(t)\big) + \varepsilon\eta^\varepsilon\big(t, x - x^\varepsilon(t)\big) \tag{2.1}$$

for some semi-martingale processes $c^\varepsilon(t)$, $x^\varepsilon(t)$ with values in $\mathbb{R}^{+*}$ and $\mathbb{R}$ respectively, and η^ε with values in $H^1(\mathbb{R})$. In order to keep $|c^\varepsilon(t) - c_0|$, and $|\varepsilon\eta^\varepsilon|_{H^1}$ small, we will require the orthogonality conditions

$$\int_{\mathbb{R}} \eta^\varepsilon(t,x)\varphi_{c_0}(x)\,\mathrm{d}x = (\eta^\varepsilon, \varphi_{c_0}) = 0, \quad \text{a.s., } t \leqslant \tau^\varepsilon, \tag{2.2}$$

and

$$\int_{\mathbb{R}} \eta^\varepsilon(t,x)\partial_x\varphi_{c_0}(x)\,\mathrm{d}x = (\eta^\varepsilon, \partial_x\varphi_{c_0}) = 0, \quad \text{a.s., } t \leqslant \tau^\varepsilon. \tag{2.3}$$

The precise statement is the following result, proved in Section 3.

Theorem 2.1. *Assume $\phi \in \mathcal{L}_2^1$ and let $c_0 > 0$ be fixed. For $\varepsilon > 0$, let $u^\varepsilon(t,x)$, as defined above, be the solution of* (1.1) *with $u(0,x) = \varphi_{c_0}(x)$. Then there exists $\alpha_0 > 0$ such that, for each α, $0 < \alpha \leqslant \alpha_0$, there is a stopping time $\tau_\alpha^\varepsilon > 0$ a.s. and there are semi-martingale processes $c^\varepsilon(t)$ and $x^\varepsilon(t)$, defined a.s. for $t \leqslant \tau_\alpha^\varepsilon$, with values respectively in $\mathbb{R}^{+*}$ and $\mathbb{R}$, so that if we set $\varepsilon\eta^\varepsilon(t) = u^\varepsilon(t, \cdot + x^\varepsilon(t)) - \varphi_{c^\varepsilon(t)}$, then* (2.2) *and* (2.3) *hold. Moreover, a.s. for $t \leqslant \tau_\alpha^\varepsilon$,*

$$\big|\varepsilon\eta^\varepsilon(t)\big|_{H^1(\mathbb{R})} \leqslant \alpha \tag{2.4}$$

and

$$\big|c^\varepsilon(t) - c_0\big| \leqslant \alpha. \tag{2.5}$$

In addition, there is a constant $C > 0$, such that for any $T > 0$ and any $\alpha \leqslant \alpha_0$, there is a $\varepsilon_0 > 0$, with, for each $\varepsilon < \varepsilon_0$,

$$\mathbb{P}(\tau_\alpha^\varepsilon \leqslant T) \leqslant \frac{C}{\alpha^4}\varepsilon^2 T \|\phi\|_{\mathcal{L}_2^1}. \tag{2.6}$$

Remark 2.2. The processes $c^\varepsilon(t)$ and $x^\varepsilon(t)$, and therefore $\eta^\varepsilon(t)$, depend a priori on α. However, we did not reflect this dependence in the notations, since we will see that

$$c_{\alpha_1}^\varepsilon(t) = c_{\alpha_2}^\varepsilon(t), \quad \text{a.s. for } t \leqslant \tau_{\alpha_1}^\varepsilon \wedge \tau_{\alpha_2}^\varepsilon$$

and the same is true for $x^\varepsilon(t)$, with obvious notations.

Remark 2.3. Estimate (2.6) implies that for any $\alpha \leqslant \alpha_0$, τ_α^ε goes to infinity in probability as ε goes to zero; this would have also been the case however if we had simply written

$$u^\varepsilon(t,x) = \varphi_{c_0}(x - c_0 t) + \varepsilon\tilde{\eta}^\varepsilon(t, x - c_0 t)$$

and defined $\tilde{\tau}_\alpha^\varepsilon$ as

$$\tilde{\tau}_\alpha^\varepsilon = \inf\big\{t \in \mathbb{R}^+, \big|u^\varepsilon(t, x + c_0 t) - \varphi_{c_0}\big|_{H^1} \geqslant \alpha\big\}.$$

Indeed, it is not difficult to prove, using the same arguments as in [6], Section 3.3, that for any $T > 0$, $u^\varepsilon(t, \cdot + c_0 t)$ converges almost surely to φ_{c_0} in $C([0,T]; H^1(\mathbb{R}))$ as ε goes to zero. However, in this case, since the secular modes are not eliminated, the remaining term $\varepsilon\tilde{\eta}^\varepsilon$ remains small on a much shorter time interval. Indeed, keeping in mind the case of a finite dimensional linear system with nonpositive spectrum and such that 0 is a degenerate double eigenvalue, we expect that this time interval is of order $\varepsilon^{-2/3}$. On the other hand, (2.6) shows that with the use of the modulation, $\varepsilon\eta^\varepsilon$ is small on a time interval of the order of ε^{-2}. Moreover, it is not clear that we could prove a better estimate than

$$\mathbb{P}(\tilde{\tau}_\alpha^\varepsilon \leqslant T) \leqslant C_\alpha \varepsilon^2 T \mathrm{e}^{L_\alpha T} \|\phi\|_{\mathcal{L}_2^1},$$

yielding a time of order $\ln(1/\varepsilon)$.

Remark 2.4. Note that there is no uniqueness of the "main part of the solution", and accordingly of the modulation parameters. We have to choose some specific condition on the remaining part, in addition with the fact that it has to stay small as long as possible. This latter condition is in general ensured – in the case $\varepsilon = 0$ and perturbations of the initial data – by choosing the modulation parameters in such a way that the secular modes associated with the linearized operator around the initial solitary wave are eliminated (see [23]). The linearized operator around the modulated solitary wave may also be used (see [18]). There is some choice. However, in the case of an additive perturbation as given in (1.1), we cannot expect to be able to predict the asymptotic behavior of the solution as time goes to infinity (note that some energy is continuously injected in the system in that case). Hence, in order to minimize the computations, we choose the simplest orthogonality conditions which ensure that the remaining part stays small as long as possible, i.e. we require that this remaining part is orthogonal in $L^2(\mathbb{R})$ to both φ_{c_0} and $\partial_x\varphi_{c_0}$.

Remark 2.5. As was mentioned in Remark 2.4, we are not able to predict the asymptotic behavior in time of the solution of (1.1). This is the reason why we do not make use of a monotonicity formula, as is done in [18] where the H^1 asymptotic stability is proved for the KdV equation (see also [19], Lemma 3). Indeed, it does not seem that the use of such a formula would simplify our proof of the exit time estimate, which is essentially based on the coercivity of some Lyapunov functional under our orthogonality conditions. However, we hope to be able to use such a monotonicity formula in future works, in particular in order to study the case of multi-soliton solutions (see [20] for the deterministic case) or to study more precisely the asymptotic behavior in time of Eq. (2.7) below.

We now turn to analyze the behavior of η^ε, and of the modulation parameters x^ε and c^ε as ε goes to zero.

Theorem 2.6. *Let $\phi \in \mathcal{L}_2^2$, and assume moreover that there is a sequence $\gamma = (\gamma_n)_{n\in\mathbb{N}}$ of real positive numbers with $\lim_{n\to\infty}\gamma_n = +\infty$, such that ϕ is Hilbert–Schmidt from $L^2(\mathbb{R})$ into X_γ. Fix $c_0 > 0$ and let η^ε, x^ε, c^ε, for $\varepsilon > 0$ be given by Theorem 2.1, with $\alpha \leqslant \alpha_0$ fixed. Then for any $T > 0$, the process $(\eta^\varepsilon(t))_{t\in[0,T]}$ converges in probability, as ε goes to zero, to a Gaussian process η satisfying the additive linear equation*

$$\begin{aligned}\mathrm{d}\eta = {} & \partial_x L_{c_0}\eta\,\mathrm{d}t + |\partial_x\varphi_{c_0}|_{L^2}^{-2}\big(\eta, L_{c_0}\partial_x^2\varphi_{c_0}\big)\partial_x\varphi_{c_0}\,\mathrm{d}t - |\partial_x\varphi_{c_0}|_{L^2}^{-2}\big((\mathcal{T}_{c_0t}\phi)\,\mathrm{d}\widetilde{W}, \partial_x\varphi_{c_0}\big)\partial_x\varphi_{c_0} \\ & - (\varphi_{c_0}, \partial_c\varphi_{c_0})^{-1}\big((\mathcal{T}_{c_0t}\phi)\,\mathrm{d}\widetilde{W}, \varphi_{c_0}\big)\partial_c\varphi_{c_0} + (\mathcal{T}_{c_0t}\phi)\,\mathrm{d}\widetilde{W}\end{aligned} \tag{2.7}$$

and with $\eta(0) = 0$. Here L_{c_0} is the unbounded operator on $L^2(\mathbb{R})$, with domain $D(L_{c_0}) = H^2(\mathbb{R})$ defined by (1.10). The convergence holds in probability in $C([0,T]; H^s_{\mathrm{loc}})$ for any $s < 1$, and in $L^2(\Omega; L^\infty(0,T;H^1))$ weak star.

The above process η satisfies for any $T > 0$ the estimate

$$\mathbb{E}\Big(\sup_{t\leqslant T}\big|\eta(t)\big|_{H^1}^2\Big) \leqslant C(1+T)\|\phi\|_{\mathcal{L}_2^1}^2 \tag{2.8}$$

for some constant $C > 0$, depending only on c_0.

Moreover, the modulation parameters may be written, for $t \leqslant \tau^\varepsilon$, as

$$\mathrm{d}x^\varepsilon = c^\varepsilon\,\mathrm{d}t + \varepsilon y^\varepsilon\,\mathrm{d}t + \varepsilon(z^\varepsilon, \mathrm{d}W) \tag{2.9}$$

and

$$\mathrm{d}c^\varepsilon = \varepsilon a^\varepsilon\,\mathrm{d}t + \varepsilon(b^\varepsilon, \mathrm{d}W) \tag{2.10}$$

for some adapted processes y^ε, a^ε, with values in $\mathbb{R}$, and predictable processes z^ε and b^ε with values in $L^2(\mathbb{R})$ satisfying: as ε goes to zero, a^ε converges to 0 in probability in $C(\mathbb{R}^+)$, while y^ε converges in probability to $|\partial_x\varphi_{c_0}|_{L^2}^{-2}(\eta(t), L_{c_0}\partial_x^2\varphi_{c_0})$ in $C(\mathbb{R}^+)$, η being as above. On the other hand, $\phi^* z^\varepsilon$ converges in probability in $C(\mathbb{R}^+; L^2(\mathbb{R}))$ to $-|\partial_x\varphi_{c_0}|_{L^2}^{-2}(\mathcal{T}_{c_0t}\phi)^*\partial_x\varphi_{c_0}$ and $\phi^* b^\varepsilon$ converges in probability in $C(\mathbb{R}^+; L^2(\mathbb{R}))$ to $(\partial_c\varphi_{c_0}, \varphi_{c_0})^{-1}(\mathcal{T}_{c_0t}\phi)^*\varphi_{c_0}$.

Remark 2.7. The estimate (2.8) seems to be optimal, as long as we deal with the energy space $H^1(\mathbb{R})$. However it is rather unsatisfactory, since it does not reflect the fact that we have got rid in some sense of the secular modes, and a uniform estimate would be expected. It appears that such an estimate probably holds under additional "localization" assumptions on ϕ, i.e. assuming e.g. that the process $(W(t))_{t\geqslant 0}$ lives in a space of functions with exponential decay on the right. That will be the object of further studies.

Remark 2.8. Theorem 2.6 implies that at first order in ε, i.e. neglecting all the terms which are $o(\varepsilon)$ as ε goes to zero, the equations for the modulation parameters may formally be written as

$$\begin{cases} \mathrm{d}x^\varepsilon = c_0\,\mathrm{d}t + \varepsilon y\,\mathrm{d}t + \varepsilon B_1\,\mathrm{d}t + \varepsilon\,\mathrm{d}B_2, \\ \mathrm{d}c^\varepsilon = \varepsilon\,\mathrm{d}B_1. \end{cases}$$

Here, y is the real valued centered Gaussian process given by

$$y = |\partial_x\varphi_{c_0}|^{-2}\big(\eta, L_{c_0}\partial_x^2\varphi_{c_0}\big),$$

η being the solution of (2.7) with $\eta(0)=0$, and (B_1, B_2) is a $\mathbb{R}^2$-valued centered Gaussian martingale given by

$$B_1(t) = (\partial_c\varphi_{c_0}, \varphi_{c_0})^{-1}\int_0^t \big(\mathcal{T}_{-c_0 s}\varphi_{c_0}, \mathrm{d}W(s)\big),$$

and

$$B_2(t) = -|\partial_x\varphi_{c_0}|_{L^2}^{-2}\int_0^t \big(\mathcal{T}_{-c_0 s}\partial_x\varphi_{c_0}, \mathrm{d}W(s)\big).$$

Note that

$$\mathbb{E}\big(B_1^2(t)\big) = (\partial_c\varphi_{c_0}, \varphi_{c_0})^{-2}\int_0^t |\phi^*\mathcal{T}_{-c_0 s}\varphi_{c_0}|_{L^2}^2\,\mathrm{d}s$$

and

$$\mathbb{E}\big(B_2^2(t)\big) = |\partial_x\varphi_{c_0}|_{L^2}^{-4}\int_0^t |\phi^*\mathcal{T}_{-c_0 s}\partial_x\varphi_{c_0}|_{L^2}^2\,\mathrm{d}s,$$

and that in the case where ϕ is nondegenerate – in the sense that the null-space of ϕ^* is reduced to $\{0\}$ – B_1 and B_2 are time changed Brownian motions. Due to the spatial correlation of the noise, i.e. to the presence of the operator ϕ, B_1 and B_2 are not independent in general. Indeed, their correlation function is given by

$$\mathbb{E}\big(B_1(t)B_2(s)\big) = -(\partial_c\varphi_{c_0}, \varphi_{c_0})^{-1}|\partial_x\varphi_{c_0}|_{L^2}^{-2}\int_0^{t\wedge s} (\phi^*\mathcal{T}_{-c_0\sigma}\varphi_{c_0}, \phi^*\mathcal{T}_{-c_0\sigma}\partial_x\varphi_{c_0})\,\mathrm{d}\sigma.$$

They would have been independent in the case of a space–time white noise – i.e. the case $\phi = \mathrm{id}$ – which however does not satisfy the assumptions of Theorems 2.1 and 2.6.

The next section is devoted to the proof of Theorem 2.1. We first prove with the use of an implicit function theorem the existence of a stopping time τ_α^ε, for $\alpha \leqslant \alpha_0$, such that the decomposition (2.1) holds for $t \leqslant \tau_\alpha^\varepsilon$, with x^ε and c^ε semi-martingales and η^ε satisfying (2.2) and (2.3). We then estimate $|\varepsilon\eta^\varepsilon(t\wedge\tau_\alpha^\varepsilon)|_{H^1}$ in order to prove (2.6). For that purpose, we make use of the Lyapunov functional Q_{c_0} (see (1.9)), and in particular of the fact, mentioned in Section 1, that $Q''_{c_0}(\varphi_{c_0})$ is a positive operator when restricted to the orthogonal of $\operatorname{span}\{\varphi_{c_0}, \partial_x\varphi_{c_0}\}$.

In Section 4, we first derive the equation for η^ε, by using the Itô and Itô–Wentzell formulae. We then deduce the modulation equations, i.e. the equations for the parameters y^ε, z^ε, a^ε, b^ε arising in the expressions (2.9) and (2.10) of c^ε and x^ε as semi-martingale processes. This allows us, at the end of Section 4, to estimate these parameters in terms of $|\eta^\varepsilon|_{L^2}$.

Coming back, in Section 5, to the equation for η^ε, and making use of the latter bounds on the modulation parameters, several estimates on η^ε are derived. The aim is of course to prove that the family composed with η^ε and the modulation parameters, for all $\varepsilon > 0$, is tight in some adequate function space. Actually, the technical part of the proof of these estimates, which relies mainly on the application of the Itô formula to different functionals of η^ε, is postponed to the appendix, Section 7.

Finally, a compactness method is applied in Section 6 to get the existence of a limit and the limit equation.

3. Modulation and estimate on the exit time

The following lemma will be useful for the proof of Theorem 2.1. It simply follows from an application of the Itô formula, and the same smoothing procedure as in [5].

Lemma 3.1. *Let u^ε be the solution of* (1.1) *given by Theorem* 1.1*, with $u^\varepsilon(0,x) = \varphi_{c_0}(x)$ and assume that $\phi \in \mathcal{L}_2^1$; then for any stopping time τ we have*

$$m\big(u^\varepsilon(\tau)\big) = m(\varphi_{c_0}) + \varepsilon \int_0^\tau \big(u^\varepsilon(s), \mathrm{d}W(s)\big) + \frac{\varepsilon^2}{2}\tau\|\phi\|^2_{\mathcal{L}_2^0}, \quad a.s. \tag{3.1}$$

and

$$\begin{aligned} H\big(u^\varepsilon(\tau)\big) = H(\varphi_{c_0}) &+ \varepsilon \int_0^\tau \big(\partial_x u^\varepsilon(s), \partial_x \,\mathrm{d}W(s)\big) - \varepsilon \int_0^\tau \big(\big(u^\varepsilon(s)\big)^2, \mathrm{d}W(s)\big) \\ &+ \frac{\varepsilon^2}{2}\tau\|\phi\|^2_{\mathcal{L}_2^1} - \varepsilon^2 \sum_{k\in\mathbb{N}} \int_0^\tau \big(u^\varepsilon(s)\phi e_k, \phi e_k\big)\,\mathrm{d}s \quad a.s. \end{aligned} \tag{3.2}$$

We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Under the assumptions of Theorem 2.1, we denote, for α with $0 < \alpha < c_0/4$, by $B_{\varphi_{c_0}}(2\alpha)$ the ball in $H^1(\mathbb{R})$ of center φ_{c_0} and radius 2α. We then consider the mapping

$$\begin{aligned} \mathcal{I}: (c_0 - 2\alpha, c_0 + 2\alpha) \times (-2\alpha, 2\alpha) \times B_{\varphi_{c_0}}(2\alpha) &\to \mathbb{R} \times \mathbb{R}, \\ (c, x_0, u) &\mapsto (\mathcal{I}_1, \mathcal{I}_2) \end{aligned}$$

defined by

$$\mathcal{I}_1(c, x_0, u) = \int_{\mathbb{R}} \big(u(x + x_0) - \varphi_c(x)\big)\varphi_{c_0}(x)\,\mathrm{d}x$$

and

$$\mathcal{I}_2(c, x_0, u) = \int_{\mathbb{R}} \big(u(x + x_0) - \varphi_c(x)\big)\partial_x\varphi_{c_0}(x)\,\mathrm{d}x.$$

Clearly, $\mathcal{I}$ is a C^2 mapping of its arguments – note that $\varphi_c(x)$ is an infinitely smooth function of both c and x. Moreover $\mathcal{I}(c_0, 0, \varphi_{c_0}) = 0$, and it follows from (1.6) that

$$\partial_c \mathcal{I}_1(c_0, 0, \varphi_{c_0}) = -(\varphi_{c_0}, \partial_c\varphi_{c_0}) = -\frac{3}{4c_0}|\varphi_{c_0}|^2_{L^2} < 0,$$

and

$$\partial_{x_0}\mathcal{I}_2(c_0, 0, \varphi_{c_0}) = -|\partial_x\varphi_{c_0}|^2_{L^2} < 0.$$

Hence, we may apply, for $\alpha \leqslant \alpha_0$ where α_0 is sufficiently small, the implicit function theorem, to get the existence of a C^2 mapping $(c(u), x(u))$ defined for $u \in B_{\varphi_{c_0}}(2\alpha)$, such that

$$\mathcal{I}_1\big(c(u), x(u), u\big) = \mathcal{I}_2\big(c(u), x(u), u\big) = 0.$$

Moreover, reducing again α if necessary, we may apply the implicit function theorem uniformly around the points $(c, 0, u_0)$ satisfying

$$\mathcal{I}(c, 0, u_0) = 0, \quad |c - c_0| < \alpha, \quad \text{and} \quad |u_0 - \varphi_{c_0}|_{H^1} < \alpha.$$

Applying this with $u = u^\varepsilon(t)$, we get the existence of $c^\varepsilon(t)$ and $x^\varepsilon(t)$ such that (2.2) and (2.3) hold, with $\varepsilon\eta(t) = u^\varepsilon(t, \cdot + x^\varepsilon(t)) - \varphi_{c^\varepsilon(t)}$. Note that the H^1-valued process u^ε is a semi-martingale with values in $H^{-2}(\mathbb{R})$. Since the functional $\mathcal{I}$ defined above is clearly a C^2 functional of u on $H^{-2}(\mathbb{R})$, it follows that locally in time, the processes x^ε and c^ε are given by a deterministic C^2 function of $u^\varepsilon \in H^{-2}$. The Itô formula then implies that c^ε and x^ε are semi-martingale processes. Moreover, since we clearly have $\mathcal{I}(c^\varepsilon(t), 0, u^\varepsilon(t, \cdot + x^\varepsilon(t))) = 0$, the existence of $x^\varepsilon(t)$ and $c^\varepsilon(t)$ holds as long as

$$\big|c^\varepsilon(t) - c_0\big| < \alpha \quad \text{and} \quad \big|u^\varepsilon\big(t, \cdot + x^\varepsilon(t)\big) - \varphi_{c_0}\big|_{H^1} < \alpha.$$

Let us denote by $\bar{\tau}^\varepsilon_\alpha$ the stopping time

$$\bar{\tau}^\varepsilon_\alpha = \inf\big\{t \geqslant 0,\ \big|c^\varepsilon(t) - c_0\big| \geqslant \alpha \text{ or } \big|u^\varepsilon\big(t, \cdot + x^\varepsilon(t)\big) - \varphi_{c_0}\big|_{H^1} \geqslant \alpha\big\},$$

so that $c^\varepsilon(t)$ and $x^\varepsilon(t)$ are defined for $t \leqslant \bar{\tau}^\varepsilon_\alpha$; let us also denote by τ^ε_β, for $\beta > 0$, the stopping time

$$\tau^\varepsilon_\beta = \inf\big\{t \geqslant 0,\ \big|c^\varepsilon(t) - c_0\big| \geqslant \beta \text{ or } \big|u^\varepsilon\big(t, \cdot + x^\varepsilon(t)\big) - \varphi_{c^\varepsilon(t)}\big|_{H^1} \geqslant \beta\big\}.$$

Since, as long as $\big|c^\varepsilon(t) - c_0\big| \leqslant \alpha \leqslant \alpha_0$, the inequality $|\varphi_{c^\varepsilon(t)} - \varphi_{c_0}|_{H^1} \leqslant C\alpha$ holds, with a constant C depending only on c_0 and α_0, it follows obviously that

$$\tau^\varepsilon_\alpha \leqslant \bar{\tau}^\varepsilon_{(C+1)\alpha} \leqslant \tau^\varepsilon_{(C+1)^2\alpha}.$$

Hence, decreasing again α_0, the processes $x^\varepsilon(t)$ and $c^\varepsilon(t)$ are defined for all $t \leqslant \tau^\varepsilon_{\alpha_0}$, and satisfy for all $t \leqslant \tau^\varepsilon_\alpha$, $\alpha \leqslant \alpha_0$, (2.4) and (2.5) in addition with (2.2) and (2.3).

Thus it remains only to prove the estimate (2.6). This is actually the most technical part of the proof. We will make use of the functional Q_{c_0} defined by (1.9). Note that Q_{c_0} is a C^2 functional of $u \in H^1(\mathbb{R})$ and that $Q'_{c_0}(\varphi_{c_0}) = 0$ by (1.7). Moreover, it is well known (see e.g. [14]) that there is a constant $\nu > 0$, depending only on c_0, such that for any $v \in H^1(\mathbb{R})$ with $(v, \varphi_{c_0}) = (v, \partial_x\varphi_{c_o}) = 0$, we have

$$\big(Q''_{c_0}(\varphi_{c_0})v, v\big) \geqslant \nu|v|^2_{H^1}, \tag{3.3}$$

where $Q''_{c_0}(\varphi_{c_0}) = L_{c_0} \in \mathcal{L}(H^1(\mathbb{R}); H^{-1}(\mathbb{R}))$. Now, we may write, almost surely for $t \leqslant \tau^\varepsilon_\alpha$:

$$\begin{aligned} &Q_{c_0}\big(u^\varepsilon\big(t, \cdot + x^\varepsilon(t)\big)\big) - Q_{c_0}(\varphi_{c^\varepsilon(t)}) \\ &\quad = \big(Q'_{c_0}\big(\varphi_{c^\varepsilon(t)}\big), \varepsilon\eta^\varepsilon(t)\big) + \big(Q''_{c_0}(\varphi_{c^\varepsilon(t)})\varepsilon\eta^\varepsilon(t), \varepsilon\eta^\varepsilon(t)\big) + \mathrm{o}\big(\big|\varepsilon\eta^\varepsilon(t)\big|^2_{H^1}\big). \end{aligned} \tag{3.4}$$

Note that $\mathrm{o}(|\varepsilon\eta^\varepsilon(t)|^2_{H^1})$ is uniform in ω, ε and t, since $Q'_{c_0}(\varphi_c)$ and $Q''_{c_0}(\varphi_c)$ depend continuously on c, and since $|c^\varepsilon(t) - c_0| \leqslant \alpha$ and $|u^\varepsilon(t, \cdot + x^\varepsilon(t)) - \varphi_{c^\varepsilon(t)}|_{H^1} = |\varepsilon\eta^\varepsilon(t)|_{H^1} \leqslant \alpha$ for $t \leqslant \tau^\varepsilon_\alpha$.

We then assume that α_0 has been chosen small enough so that the last term in (3.4) is almost surely less than $\frac{\nu}{4}|\varepsilon\eta^\varepsilon(t)|^2_{H^1}$ for all $t \leqslant \tau^\varepsilon_\alpha$.

Note that there is a constant C, depending only on c_0 and α_0 such that the inequality

$$\big\|Q''_{c_0}(\varphi_{c^\varepsilon(t)}) - Q''_{c_0}(\varphi_{c_0})\big\|_{\mathcal{L}(H^1;H^{-1})} \leqslant 2|\varphi_{c^\varepsilon(t)} - \varphi_{c_0}|_{L^2} \leqslant C\big|c^\varepsilon(t) - c_0\big|$$

holds, for all $t \leqslant \tau^\varepsilon_\alpha$, and thus

$$\big(Q''_{c_0}(\varphi_{c^\varepsilon})\varepsilon\eta^\varepsilon, \varepsilon\eta^\varepsilon\big) \geqslant \nu|\varepsilon\eta^\varepsilon|^2_{H^1} - C|c^\varepsilon - c_0||\varepsilon\eta^\varepsilon|^2_{H^1}.$$

On the other hand, since $Q'_{c^\varepsilon}(\varphi_{c^\varepsilon}) = 0$ by (1.7),

$$\big|Q'_{c_0}(\varphi_{c^\varepsilon})(\varepsilon\eta^\varepsilon)\big| = \big|\big((c_0 - c^\varepsilon)\varphi_{c^\varepsilon}, \varepsilon\eta^\varepsilon\big)\big| \leqslant \frac{\nu}{2}|\varepsilon\eta^\varepsilon|^2_{H^1} + C|c_0 - c^\varepsilon|^2;$$

it then follows from (3.4) that for all $\alpha \leqslant \alpha_0$, and for all $t \leqslant \tau_\alpha^\varepsilon$, we have a.s.

$$Q_{c_0}\big(u^\varepsilon\big(t,\cdot+x^\varepsilon(t)\big)\big) - Q_{c_0}(\varphi_{c^\varepsilon}) \geqslant \frac{\nu}{4}\big|\varepsilon\eta^\varepsilon(t)\big|_{H^1}^2 - C\big|c^\varepsilon(t)-c_0\big|^2 \tag{3.5}$$

for a constant C depending only on α_0 and c_0.

We now make use of Lemma 3.1 to deduce from (3.5) that a.s. for all t, denoting for simplicity $\tau = \tau_\alpha^\varepsilon \wedge t$:

$$\begin{aligned}
\big|\varepsilon\eta^\varepsilon(\tau)\big|_{H^1}^2 &\leqslant \frac{4}{\nu}\big[Q_{c_0}\big(u^\varepsilon\big(\tau,\cdot+x^\varepsilon(\tau)\big)\big) - Q_{c_0}(\varphi_{c^\varepsilon(\tau)})\big] + C\big|c^\varepsilon(\tau)-c_0\big|^2 \\
&\leqslant \frac{4}{\nu}\big[Q_{c_0}\big(u^\varepsilon(\tau)\big) - Q_{c_0}(\varphi_{c^\varepsilon(\tau)})\big] + C\big|c^\varepsilon(\tau)-c_0\big|^2 \\
&\leqslant \frac{4}{\nu}\Bigg[Q_{c_0}(\varphi_{c_0}) - Q_{c_0}(\varphi_{c^\varepsilon(\tau)}) + \varepsilon\int_0^\tau \big(\partial_x u^\varepsilon(s), \partial_x\,\mathrm{d}W(s)\big) \\
&\quad - \varepsilon\int_0^\tau \big(\big(u^\varepsilon(s)\big)^2, \mathrm{d}W(s)\big) + c_0\varepsilon\int_0^\tau \big(u^\varepsilon(s), \mathrm{d}W(s)\big) + \frac{\varepsilon^2}{2}\tau\|\phi\|_{\mathcal{L}_2^1}^2 \\
&\quad + \varepsilon^2\|\phi\|_{\mathcal{L}_2^1}^2 \int_0^\tau \big|u^\varepsilon(s)\big|_{L^2}\,\mathrm{d}s + \frac{c_0}{2}\varepsilon^2\tau\|\phi\|_{\mathcal{L}_2^0}^2\Bigg] + C\big|c^\varepsilon(\tau)-c_0\big|^2.
\end{aligned} \tag{3.6}$$

We now estimate $|c^\varepsilon(\tau)-c_0|^2$. On the one hand, the orthogonality condition $(\eta^\varepsilon, \varphi_{c_0}) = 0$ implies

$$\begin{aligned}
\big|u^\varepsilon(\tau)\big|_{L^2}^2 &= \big|u^\varepsilon\big(\tau,\cdot+x^\varepsilon(\tau)\big)\big|_{L^2}^2 = \big|\varepsilon\eta^\varepsilon(\tau) + \varphi_{c^\varepsilon(\tau)}\big|_{L^2}^2 \\
&= \big|\varepsilon\eta^\varepsilon(\tau)\big|_{L^2}^2 + |\varphi_{c^\varepsilon(\tau)}|_{L^2}^2 + 2\big(\varepsilon\eta^\varepsilon(\tau), \varphi_{c^\varepsilon(\tau)} - \varphi_{c_0}\big)
\end{aligned}$$

and on the other hand, from Lemma 3.1

$$\big|u^\varepsilon(\tau)\big|_{L^2}^2 = |\varphi_{c_0}|_{L^2}^2 + 2\varepsilon\int_0^\tau \big(u^\varepsilon(s), \mathrm{d}W(s)\big) + \varepsilon^2\tau\|\phi\|_{\mathcal{L}_2^0}^2.$$

It thus follows from the equality of the two terms that for some constants C and μ, depending only on c_0 and α_0,

$$\begin{aligned}
\mu\big|c^\varepsilon(\tau)-c_0\big| &\leqslant \big||\varphi_{c_0}|_{L^2}^2 - |\varphi_{c^\varepsilon(\tau)}|_{L^2}^2\big| \\
&\leqslant \big|\varepsilon\eta^\varepsilon(\tau)\big|_{L^2}^2 + 2\big|\varepsilon\eta^\varepsilon(\tau)\big|_{H^1}|\varphi_{c^\varepsilon(\tau)} - \varphi_{c_0}|_{L^2} + 2\varepsilon\left|\int_0^\tau \big(u^\varepsilon(s), \mathrm{d}W(s)\big)\right| + \varepsilon^2\tau\|\phi\|_{\mathcal{L}_2^0}^2 \\
&\leqslant \big|\varepsilon\eta^\varepsilon(\tau)\big|_{L^2}^2 + C\alpha\big|c^\varepsilon(\tau)-c_0\big| + 2\varepsilon\left|\int_0^\tau \big(u^\varepsilon(s), \mathrm{d}W(s)\big)\right| + \varepsilon^2\tau\|\phi\|_{\mathcal{L}_2^0}^2.
\end{aligned}$$

Hence, choosing again α_0 sufficiently small, we get

$$\big|c^\varepsilon(\tau)-c_0\big|^2 \leqslant C\left[\big|\varepsilon\eta^\varepsilon(\tau)\big|_{L^2}^4 + 4\varepsilon^2\left|\int_0^\tau \big(u^\varepsilon(s), \mathrm{d}W(s)\big)\right|^2 + \varepsilon^4\tau^2\|\phi\|_{\mathcal{L}_2^0}^4\right]. \tag{3.7}$$

Inserting (3.7) in the right-hand side of (3.6), and using in addition that, because $Q_{c_0}'(\varphi_{c_0}) = 0$,

$$\big|Q_{c_0}(\varphi_{c_0}) - Q_{c_0}(\varphi_{c^\varepsilon(\tau)})\big| \leqslant C|\varphi_{c_0} - \varphi_{c^\varepsilon(\tau)}|_{H^1}^2 \leqslant C\big|c_0 - c^\varepsilon(\tau)\big|^2$$

for some constant $C(c_0, \alpha_0)$, we obtain

$$
\begin{aligned}
\left|\varepsilon\eta^\varepsilon(\tau)\right|_{H^1}^2 \leqslant C\Bigg[&\left|\varepsilon\eta^\varepsilon(\tau)\right|_{L^2}^4 + 4\varepsilon^2\left|\int_0^\tau \left(u^\varepsilon(s), \mathrm{d}W(s)\right)\right|^2 + \varepsilon^4\tau^2\|\phi\|_{\mathcal{L}_2^0}^4 + \varepsilon\int_0^\tau \left(\partial_x u^\varepsilon(s), \partial_x \,\mathrm{d}W(s)\right) \\
&- \varepsilon\int_0^\tau \left(\left(u^\varepsilon(s)\right)^2, \mathrm{d}W(s)\right) + c_0\varepsilon\int_0^\tau \Big(u^\varepsilon(s), \mathrm{d}W(s)\Big) + \frac{\varepsilon^2}{2}\tau\|\phi\|_{\mathcal{L}_2^1}^2 \\
&+ \varepsilon^2\|\phi\|_{\mathcal{L}_2^1}^2\int_0^\tau \left|u^\varepsilon(s)\right|_{L^2}\mathrm{d}s + c_0\frac{\varepsilon^2}{2}\tau\|\phi\|_{\mathcal{L}_2^0}^2\Bigg].
\end{aligned} \tag{3.8}
$$

Again, the constant C in the right-hand side of (3.8) only depends on c_0 and α_0.

Let us now fix $T > 0$ and set

$$\Omega_1^{T,\varepsilon,\alpha} = \left\{\omega \in \Omega,\ \tau_\alpha^\varepsilon(\omega) \leqslant T,\ \text{and } \left|\varepsilon\eta^\varepsilon(\tau_\alpha^\varepsilon)\right|_{H^1} = \alpha\right\}$$

and

$$\Omega_2^{T,\varepsilon,\alpha} = \left\{\omega \in \Omega,\ \tau_\alpha^\varepsilon(\omega) \leqslant T,\ \text{and } \left|c^\varepsilon(\tau_\alpha^\varepsilon) - c_0\right| = \alpha\right\}$$

so that

$$\mathbb{P}\left(\tau_\alpha^\varepsilon(\omega) \leqslant T\right) \leqslant \mathbb{P}\left(\Omega_1^{T,\varepsilon,\alpha}\right) + \mathbb{P}\left(\Omega_2^{T,\varepsilon,\alpha}\right).$$

Let $\alpha_1 > 0$ small enough so that $C\alpha_1^2 \leqslant 1/2$, where $C = C(\alpha_0, c_0)$ is the constant appearing in the right-hand side of (3.8). Multiplying both sides of (3.8) by $\mathbb{1}_{\Omega_1^{T,\varepsilon,\alpha}}$, assuming $\alpha \leqslant \alpha_1$ and taking the expectation with $\tau = \tau_\alpha^\varepsilon \wedge T$, we easily get

$$
\begin{aligned}
\frac{\alpha^2}{2}\mathbb{P}\left(\Omega_1^{T,\varepsilon,\alpha}\right) \leqslant C\Bigg[&4\varepsilon^2\mathbb{E}\left|\int_0^\tau \left(u^\varepsilon(s), \mathrm{d}W(s)\right)\mathbb{1}_{\Omega_1^{T,\varepsilon,\alpha}}\right|^2 + \varepsilon^4T^2\|\phi\|_{\mathcal{L}_2^0}^4\mathbb{P}\left(\Omega_1^{T,\varepsilon,\alpha}\right) \\
&+ \varepsilon\mathbb{E}\left(\int_0^\tau \left(\partial_x u^\varepsilon(s), \partial_x \,\mathrm{d}W(s)\right)\mathbb{1}_{\Omega_1^{T,\varepsilon,\alpha}}\right) - \varepsilon\mathbb{E}\left(\int_0^\tau \left(\left(u^\varepsilon(s)\right)^2, \mathrm{d}W(s)\right)\mathbb{1}_{\Omega_1^{T,\varepsilon,\alpha}}\right) \\
&+ c_0\varepsilon\mathbb{E}\left(\int_0^\tau \left(u^\varepsilon(s), \mathrm{d}W(s)\right)\mathbb{1}_{\Omega_1^{T,\varepsilon,\alpha}}\right) + \frac{\varepsilon^2}{2}T\|\phi\|_{\mathcal{L}_2^1}^2\mathbb{P}\left(\Omega_1^{T,\varepsilon,\alpha}\right) \\
&+ \varepsilon^2\|\phi\|_{\mathcal{L}_2^1}^2\mathbb{E}\left(\int_0^\tau \left|u^\varepsilon(s)\right|_{L^2}\mathrm{d}s\mathbb{1}_{\Omega_1^{T,\varepsilon,\alpha}}\right) + c_0\frac{\varepsilon^2}{2}T\|\phi\|_{\mathcal{L}_2^0}^2\mathbb{P}\left(\Omega_1^{T,\varepsilon,\alpha}\right)\Bigg].
\end{aligned}
$$

Now, using the Cauchy–Schwarz inequality in the right-hand side above, together with the fact that $|u^\varepsilon(s)|_{H^1}^2 \leqslant C$ a.s. for $s \in [0, \tau_\alpha^\varepsilon \wedge T]$, where C only depends on c_0 and α_0, and the Bürkholder–Davis–Gundy inequality, we get

$$
\begin{aligned}
\frac{\alpha^2}{2}\mathbb{P}\left(\Omega_1^{T,\varepsilon,\alpha}\right) \leqslant{}& C_1\Bigg[4\varepsilon^2\left(\mathbb{E}\left(\int_0^\tau \left(u^\varepsilon(s), \mathrm{d}W(s)\right)\right)^4\right)^{1/2} + \varepsilon\left(\mathbb{E}\left(\int_0^\tau \left(\partial_x u^\varepsilon(s), \partial_x \,\mathrm{d}W(s)\right)\right)^2\right)^{1/2} \\
&+ \varepsilon\left(\mathbb{E}\left(\int_0^\tau \left(\left(u^\varepsilon(s)\right)^2, \mathrm{d}W(s)\right)\right)^2\right)^{1/2} + c_0\varepsilon\left(\mathbb{E}\left(\int_0^\tau \left(u^\varepsilon(s), \mathrm{d}W(s)\right)\right)^2\right)^{1/2}\Bigg]\mathbb{P}\left(\Omega_1^{T,\varepsilon,\alpha}\right)^{1/2} \\
&+ \left(C_2\varepsilon^4T^2\|\phi\|_{\mathcal{L}_2^0}^4 + C_3\varepsilon^2T\|\phi\|_{\mathcal{L}_2^1}^2\right)\mathbb{P}\left(\Omega_1^{T,\varepsilon,\alpha}\right) \\
\leqslant{}& C_1'\varepsilon\sqrt{T}\,\|\phi\|_{\mathcal{L}_2^1}\left(1 + \varepsilon\sqrt{T}\,\|\phi\|_{\mathcal{L}_2^1}\right)\mathbb{P}\left(\Omega_1^{T,\varepsilon,\alpha}\right)^{1/2} + C_2'\varepsilon^2T\|\phi\|_{\mathcal{L}_2^1}^2\left(1 + \varepsilon^2T\|\phi\|_{\mathcal{L}_2^1}^2\right)\mathbb{P}\left(\Omega_1^{T,\varepsilon,\alpha}\right)
\end{aligned}
$$

and it follows, if we choose $\varepsilon \leqslant \varepsilon_0$ with

$$C_2' \varepsilon_0^2 T \|\phi\|_{\mathcal{L}_2^1}^2 \big(1 + \varepsilon_0^2 T \|\phi\|_{\mathcal{L}_2^1}^2\big) \leqslant \frac{\alpha^2}{4}, \tag{3.9}$$

that

$$\mathbb{P}\big(\Omega_1^{T,\varepsilon,\alpha}\big) \leqslant \frac{C}{\alpha^4} \varepsilon^2 T \|\phi\|_{\mathcal{L}_2^1}^2. \tag{3.10}$$

It remains to estimate $\mathbb{P}(\Omega_2^{T,\varepsilon,\alpha})$. Coming back to (3.7) and using the same arguments as above, we easily get

$$\alpha^2 \mathbb{P}\big(\Omega_2^{T,\varepsilon,\alpha}\big) \leqslant C\big[\alpha^4 \mathbb{P}\big(\Omega_2^{T,\varepsilon,\alpha}\big) + \varepsilon^4 T^2 \|\phi\|_{\mathcal{L}_2^0}^4 \mathbb{P}\big(\Omega_2^{T,\varepsilon,\alpha}\big) + \varepsilon^2 T \|\phi\|_{\mathcal{L}_2^0}^2 \mathbb{P}\big(\Omega_2^{T,\varepsilon,\alpha}\big)^{1/2}\big];$$

therefore, if $\alpha \leqslant \alpha_1$ and ε is small enough, we have

$$\mathbb{P}\big(\Omega_2^{T,\varepsilon,\alpha}\big) \leqslant \frac{C}{\alpha^4} \varepsilon^4 T^2 \|\phi\|_{\mathcal{L}_2^0}^4. \tag{3.11}$$

(2.6) follows from (3.10) and (3.11), for $\alpha \leqslant \alpha_1$ and ε small enough with respect to α and T.

4. Modulation equations

We now fix α in such a way that the conclusion of Theorem 2.1 holds, and we turn to derive the equations for the modulation parameters x^ε and c^ε, and for η^ε. These modulation equations will allow us to obtain estimates on these parameters in the next section. As α is from now on fixed, we write τ^ε for τ_α^ε in all what follows.

We know from Theorem 2.1 that x^ε and c^ε are semi-martingale processes, adapted to the filtration generated by the Wiener process $(W(t))_{t\geqslant 0}$. We may thus write a priori the equations for x^ε and c^ε in the form (2.9) and (2.10), where y^ε and a^ε are real valued adapted processes with paths in $L^1(0,\tau^\varepsilon)$ a.s., and z^ε and b^ε are $L^2(\mathbb{R})$-valued predictable processes, with paths in $L^2(0,\tau^\varepsilon; L^2(\mathbb{R}))$ a.s.

Lemma 4.1. *With the above notations, η^ε satisfies the equation*

$$\begin{aligned} \mathrm{d}\eta^\varepsilon = {} & \partial_x L_{c_0} \eta^\varepsilon \,\mathrm{d}t + (y^\varepsilon \partial_x \varphi_{c^\varepsilon} - a^\varepsilon \partial_c \varphi_{c^\varepsilon})\,\mathrm{d}t + (c^\varepsilon - c_0 + \varepsilon y^\varepsilon)\partial_x \eta^\varepsilon \,\mathrm{d}t - \varepsilon \partial_x\big((\eta^\varepsilon)^2\big)\,\mathrm{d}t \\ & - 2\partial_x\big((\varphi_{c^\varepsilon} - \varphi_{c_0})\eta^\varepsilon\big)\,\mathrm{d}t + (\partial_x \varphi_{c^\varepsilon})(z^\varepsilon, \mathrm{d}W) - (\partial_c \varphi_{c^\varepsilon})(b^\varepsilon, \mathrm{d}W) \\ & + \varepsilon \partial_x \eta^\varepsilon (z^\varepsilon, \mathrm{d}W) + (\mathrm{d}W)\big(t, \cdot + x^\varepsilon(t)\big) + \varepsilon \sum_{\ell\in\mathbb{N}} \mathcal{T}_{x^\varepsilon}(\partial_x \phi e_\ell)(z^\varepsilon, \phi e_\ell)\,\mathrm{d}t \\ & + \frac{\varepsilon}{2}\big(\partial_x^2 \varphi_{c^\varepsilon} |\phi^* z^\varepsilon|_{L^2}^2 - \partial_c^2 \varphi_{c^\varepsilon} |\phi^* b^\varepsilon|_{L^2}^2\big)\,\mathrm{d}t + \frac{1}{2}\varepsilon^2 \partial_x^2 \eta^\varepsilon |\phi^* z^\varepsilon|_{L^2}^2 \,\mathrm{d}t. \end{aligned} \tag{4.1}$$

Proof. We first perform formal computations, after what we explain how they can be justified. We apply the Itô–Wentzell formula to $u^\varepsilon(t, x + x^\varepsilon(t))$, using the fact that u^ε satisfies Eq. (1.1) and x^ε satisfies Eq. (2.9). We then formally get

$$\begin{aligned} d\big(u^\varepsilon\big(t, x + x^\varepsilon(t)\big)\big) = {} & -\partial_x^3 u^\varepsilon\big(t, x + x^\varepsilon(t)\big)\,\mathrm{d}t - \partial_x\Big(\big(u^\varepsilon\big(t, x + x^\varepsilon(t)\big)\big)^2\Big)\,\mathrm{d}t + \varepsilon(\mathrm{d}W)\big(t, x + x^\varepsilon(t)\big) \\ & + \varepsilon(\partial_x u^\varepsilon)\big(t, x + x^\varepsilon(t)\big) y^\varepsilon(t)\,\mathrm{d}t + c^\varepsilon(t)(\partial_x u^\varepsilon)\big(t, x + x^\varepsilon(t)\big)\,\mathrm{d}t \\ & + \varepsilon(\partial_x u^\varepsilon)\big(t, x + x^\varepsilon(t)\big)\big(z^\varepsilon(t), \mathrm{d}W(t)\big) + \frac{\varepsilon^2}{2}\big(\partial_x^2 u^\varepsilon\big)\big(t, x + x^\varepsilon(t)\big)\big|\phi^* z^\varepsilon(t)\big|_{L^2}^2 \,\mathrm{d}t \\ & + \varepsilon^2 \sum_{\ell\in\mathbb{N}} (\partial_x \phi e_\ell)\big(x + x^\varepsilon(t)\big)\big(z^\varepsilon(t), \phi e_\ell\big)\,\mathrm{d}t, \end{aligned} \tag{4.2}$$

where

$$(\mathrm{d}W)\big(t, x + x^\varepsilon(t)\big) = \sum_{\ell\in\mathbb{N}} (\phi e_\ell)\big(x + x^\varepsilon(t)\big)\,\mathrm{d}\beta_\ell(t) = \sum_{\ell\in\mathbb{N}} (\mathcal{T}_{x^\varepsilon(t)} \phi e_\ell)(x)\,\mathrm{d}\beta_\ell(t).$$

Using now the (standard) Itô formula and Eq. (2.10), we have

$$\begin{aligned}\mathrm{d}(\varphi_{c^\varepsilon(t)}) &= \partial_c\varphi_{c^\varepsilon(t)}\,\mathrm{d}c^\varepsilon(t) + \frac{\varepsilon^2}{2}\partial_c^2\varphi_{c^\varepsilon(t)}|\phi^* b^\varepsilon|_{L^2}^2\,\mathrm{d}t \\ &= \varepsilon\partial_c\varphi_{c^\varepsilon(t)}a^\varepsilon(t)\,\mathrm{d}t + \varepsilon\partial_c\varphi_{c^\varepsilon(t)}\big(b^\varepsilon(t),\mathrm{d}W(t)\big) + \frac{\varepsilon^2}{2}\partial_c^2\varphi_{c^\varepsilon(t)}|\phi^* b^\varepsilon|_{L^2}^2\,\mathrm{d}t. \end{aligned}\tag{4.3}$$

Replacing then in (4.2) $u^\varepsilon(t, x + x^\varepsilon(t))$ by $\varphi_{c^\varepsilon(t)}(x) + \varepsilon\eta^\varepsilon(t, x)$, using (4.3), (1.7) and (1.10), we deduce Eq. (4.1) for η^ε, where we have used

$$\partial_x L_{c^\varepsilon} u = \partial_x L_{c_0} u + (c^\varepsilon - c_0)(\partial_x u) - 2\partial_x\big((\varphi_{c^\varepsilon} - \varphi_{c_0})u\big).$$

The above computations may be justified as follows: consider a sequence ϕ^n of linear operators in $\mathcal{L}_2^4$, converging to ϕ in $\mathcal{L}_2^1$. It is not difficult to see that there is a unique global solution u_n^ε of (1.1), with $u_n^\varepsilon(0) = \varphi_{c_0}$, with paths a.s. in $C(\mathbb{R}^+; H^4(\mathbb{R}))$, and that u_n^ε converges to u^ε a.s. in $C(\mathbb{R}^+; H^1)$ as n goes to infinity. Moreover, taking α smaller if necessary, all the arguments in the proof of Theorem 2.1 apply uniformly in n, for n large enough. It means that we may write, for α small enough, and for any $n \geqslant n_0(\alpha)$,

$$u_n^\varepsilon\big(t, x + x_n^\varepsilon(t)\big) = \varphi_{c_n^\varepsilon(t)}(x) + \varepsilon\eta_n^\varepsilon(t, x)$$

with $(\eta_n^\varepsilon, \varphi_{c_0}) = (\eta_n^\varepsilon, \partial_x\varphi_{c_0}) = 0$, and this almost surely for $t \leqslant \tau^\varepsilon$ not depending on n. All the above arguments are justified for a fixed n, since all the terms arising in the equation for u_n^ε are continuous in both the t and x variables; we then argue as above on u_n^ε instead of u^ε, and take the limit as n goes to infinity in the final equations. □

We now derive the equations for the modulation parameters y^ε, a^ε, z^ε and b^ε. We set for convenience in all what follows

$$Z_\ell^\varepsilon(t) = \begin{pmatrix} (z^\varepsilon(t), \phi e_\ell) \\ (b^\varepsilon(t), \phi e_\ell) \end{pmatrix}, \quad \ell \in \mathbb{N}, \quad \text{and} \quad Y^\varepsilon(t) = \begin{pmatrix} y^\varepsilon(t) \\ a^\varepsilon(t) \end{pmatrix}. \tag{4.4}$$

Lemma 4.2. *The modulation parameters satisfy the system of equations*

$$A^\varepsilon(t) Z_\ell^\varepsilon(t) = F_\ell^\varepsilon(t), \quad \forall \ell \in \mathbb{N} \tag{4.5}$$

and

$$A^\varepsilon(t) Y^\varepsilon(t) = G^\varepsilon(t), \quad \forall t \leqslant \tau^\varepsilon, \tag{4.6}$$

where Z_ℓ^ε and Y^ε are defined in (4.4)*, A^ε is defined by*

$$A^\varepsilon(t) = \begin{pmatrix} (\partial_x\varphi_{c^\varepsilon} + \varepsilon\partial_x\eta^\varepsilon, \partial_x\varphi_{c_0}) & -(\partial_c\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0}) \\ -(\partial_x\varphi_{c^\varepsilon}, \varphi_{c_0}) & (\partial_c\varphi_{c^\varepsilon}, \varphi_{c_0}) \end{pmatrix}, \tag{4.7}$$

F^ε is given by

$$F_\ell^\varepsilon(t) = \begin{pmatrix} -(\mathcal{T}_{x^\varepsilon}(\phi e_\ell), \partial_x\varphi_{c_0}) \\ (\mathcal{T}_{x^\varepsilon}(\phi e_\ell), \varphi_{c_0}) \end{pmatrix} \tag{4.8}$$

and

$$G^\varepsilon(t) = \begin{pmatrix} G_1^\varepsilon(t) \\ G_2^\varepsilon(t) \end{pmatrix},$$

with

$$\begin{aligned} G_1^\varepsilon(t) &= \big(\eta^\varepsilon, L_{c_0}\partial_x^2\varphi_{c_0}\big) + (c^\varepsilon - c_0)\big(\eta^\varepsilon, \partial_x^2\varphi_{c_0}\big) + \varepsilon\big(\partial_x\big((\eta^\varepsilon)^2\big), \partial_x\varphi_{c_0}\big) + 2\big(\partial_x\big((\varphi_{c^\varepsilon} - \varphi_{c_0})\eta^\varepsilon\big), \partial_x\varphi_{c_0}\big) \\ &\quad - \varepsilon\sum_{\ell\in\mathbb{N}}\big(\mathcal{T}_{x^\varepsilon}(\partial_x\phi e_\ell), \partial_x\varphi_{c_0}\big)(z^\varepsilon, \phi e_\ell) - \frac{\varepsilon}{2}\big(\partial_x^2\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0}\big)|\phi^* z^\varepsilon|_{L^2}^2 + \frac{\varepsilon}{2}\big(\partial_c^2\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0}\big)|\phi^* b^\varepsilon|_{L^2}^2 \\ &\quad - \frac{1}{2}\varepsilon^2\big(\eta^\varepsilon, \partial_x^3\varphi_{c_0}\big)|\phi^* z^\varepsilon|_{L^2}^2 \end{aligned}\tag{4.9}$$

and

$$G_2^\varepsilon(t) = -\varepsilon\big(\partial_x\big((\eta^\varepsilon)^2\big), \varphi_{c_0}\big) - 2\big(\partial_x\big((\varphi_{c^\varepsilon} - \varphi_{c_0})\eta^\varepsilon\big), \varphi_{c_0}\big) + \varepsilon\sum_{\ell\in\mathbb{N}}\big(\mathcal{T}_{x^\varepsilon}(\partial_x\phi e_\ell), \varphi_{c_0}\big)(z^\varepsilon, \phi e_\ell)$$
$$+ \frac{\varepsilon}{2}\big(\partial_x^2\varphi_{c^\varepsilon}, \varphi_{c_0}\big)|\phi^* z^\varepsilon|_{L^2}^2 - \frac{\varepsilon}{2}\big(\partial_c^2\varphi_{c^\varepsilon}, \varphi_{c_0}\big)|\phi^* b^\varepsilon|_{L^2}^2 + \frac{1}{2}\varepsilon^2\big(\eta^\varepsilon, \partial_x^2\varphi_{c_0}\big)|\phi^* z^\varepsilon|_{L^2}^2. \tag{4.10}$$

Proof. We take the L^2 inner product of Eq. (4.1) with φ_{c_0}, and make use of the orthogonality conditions (2.2) and (2.3) together with the fact that $L_{c_0}\partial_x\varphi_{c_0} = 0$, as can be seen easily from (1.7). This leads to

$$\begin{aligned}0 &= \mathrm{d}(\eta^\varepsilon, \varphi_{c_0}) = (\mathrm{d}\eta^\varepsilon, \varphi_{c_0})\\
&= (y^\varepsilon\partial_x\varphi_{c^\varepsilon} - a^\varepsilon\partial_c\varphi_{c^\varepsilon}, \varphi_{c_0})\,\mathrm{d}t - \varepsilon\big(\partial_x\big((\eta^\varepsilon)^2\big), \varphi_{c_0}\big)\,\mathrm{d}t - 2\big(\partial_x((\varphi_{c^\varepsilon} - \varphi_{c_0})\eta^\varepsilon), \varphi_{c_0}\big)\,\mathrm{d}t + (\partial_x\varphi_{c^\varepsilon}, \varphi_{c_0})(z^\varepsilon, \mathrm{d}W)\\
&\quad - (\partial_c\varphi_{c^\varepsilon}, \varphi_{c_0})(b^\varepsilon, \mathrm{d}W) + \big((\mathrm{d}W)(t, \cdot + x^\varepsilon), \varphi_{c_0}\big) + \varepsilon\sum_{\ell\in\mathbb{N}}\big(\mathcal{T}_{x^\varepsilon}(\partial_x\phi e_\ell), \varphi_{c_0}\big)(z^\varepsilon, \phi e_\ell)\,\mathrm{d}t\\
&\quad + \frac{\varepsilon}{2}\big(\partial_x^2\varphi_{c^\varepsilon}, \varphi_{c_0}\big)|\phi^* z^\varepsilon|_{L^2}^2\,\mathrm{d}t - \frac{\varepsilon}{2}\big(\partial_c^2\varphi_{c^\varepsilon}, \varphi_{c_0}\big)|\phi^* b^\varepsilon|_{L^2}^2\,\mathrm{d}t + \frac{1}{2}\varepsilon^2\big(\partial_x^2\eta^\varepsilon, \varphi_{c_0}\big)|\phi^* z^\varepsilon|_{L^2}^2\,\mathrm{d}t.\end{aligned} \tag{4.11}$$

In the same way, taking the inner product of (4.1) with $\partial_x\varphi_{c_0}$, and making use of (2.3), it comes out that

$$\begin{aligned}0 &= \mathrm{d}(\eta^\varepsilon, \partial_x\varphi_{c_0}) = (\mathrm{d}\eta^\varepsilon, \partial_x\varphi_{c_0})\\
&= -\big(\eta^\varepsilon, L_{c_0}\partial_x^2\varphi_{c_0}\big)\,\mathrm{d}t + (y^\varepsilon\partial_x\varphi_{c^\varepsilon} - a^\varepsilon\partial_c\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0})\,\mathrm{d}t + (c^\varepsilon - c_0 + \varepsilon y^\varepsilon)(\partial_x\eta^\varepsilon, \partial_x\varphi_{c_0})\,\mathrm{d}t\\
&\quad - \varepsilon\big(\partial_x\big((\eta^\varepsilon)^2\big), \partial_x\varphi_{c_0}\big)\,\mathrm{d}t - 2\big(\partial_x\big((\varphi_{c^\varepsilon} - \varphi_{c_0})\eta^\varepsilon\big), \partial_x\varphi_{c_0}\big)\,\mathrm{d}t + (\partial_x\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0})(z^\varepsilon, \mathrm{d}W)\\
&\quad - (\partial_c\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0})(b^\varepsilon, \mathrm{d}W) + \varepsilon(\partial_x\eta^\varepsilon, \partial_x\varphi_{c_0})(z^\varepsilon, \mathrm{d}W) + \big((\mathrm{d}W)(t, \cdot + x^\varepsilon), \partial_x\varphi_{c_0}\big)\\
&\quad + \varepsilon\sum_{\ell\in\mathbb{N}}\big(\mathcal{T}_{x^\varepsilon}(\partial_x\phi e_\ell), \partial_x\varphi_{c_0}\big)(z^\varepsilon, \phi e_\ell)\,\mathrm{d}t + \frac{\varepsilon}{2}\big(\partial_x^2\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0}\big)|\phi^* z^\varepsilon|_{L^2}^2\,\mathrm{d}t - \frac{\varepsilon}{2}\big(\partial_c^2\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0}\big)|\phi^* b^\varepsilon|_{L^2}^2\,\mathrm{d}t\\
&\quad + \frac{1}{2}\varepsilon^2\big(\partial_x^2\eta^\varepsilon, \partial_x\varphi_{c_0}\big)|\phi^* z^\varepsilon|_{L^2}^2\,\mathrm{d}t.\end{aligned} \tag{4.12}$$

We may now write that both the drift and martingale part of the right-hand side in Eqs. (4.11) and (4.12) are identically equal to zero. The identification of the martingale parts in both equations gives, for each $\ell \in \mathbb{N}$ the system of equations

$$\begin{cases}(\partial_x\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0})(z^\varepsilon, \phi e_\ell) - (\partial_c\varphi_{c^\varepsilon}, \partial_x\varphi_{c_0})(b^\varepsilon, \phi e_\ell) + (\varepsilon\partial_x\eta^\varepsilon, \partial_x\varphi_{c_0})(z^\varepsilon, \phi e_\ell) = -\big(\mathcal{T}_{x^\varepsilon}(\phi e_\ell), \partial_x\varphi_{c_0}\big),\\
(\partial_c\varphi_{c^\varepsilon}, \varphi_{c_0})(b^\varepsilon, \phi e_\ell) - (\partial_x\varphi_{c^\varepsilon}, \varphi_{c_0})(z^\varepsilon, \phi e_\ell) = (\mathcal{T}_{x^\varepsilon}(\phi e_\ell), \varphi_{c_0}),\end{cases}$$

which is exactly (4.5). The identification of the drift parts gives (4.6). □

We deduce from the modulation equations given in Lemma 4.2 the following estimates for the modulation parameters.

Corollary 4.3. *Under the assumptions of Theorem* 2.1*, there is a $\alpha_1 > 0$ such that for $\alpha \leqslant \alpha_1$, there is a constant $C(c_0, \alpha)$ with*

$$\big|\phi^* z^\varepsilon(t)\big|_{L^2} + \big|\phi^* b^\varepsilon(t)\big|_{L^2} \leqslant C(c_0, \alpha)\|\phi\|_{\mathcal{L}_2^0}, \quad \text{for all } t \leqslant \tau^\varepsilon. \tag{4.13}$$

Moreover, there are constants C_1 and C_2, depending only on c_0, α and $\|\phi\|_{\mathcal{L}_2^1}$ such that

$$\big|a^\varepsilon(t)\big| + \big|y^\varepsilon(t)\big| \leqslant C_1\big|\eta^\varepsilon(t)\big|_{L^2} + \varepsilon C_2, \quad \text{a.s. for } t \leqslant \tau^\varepsilon. \tag{4.14}$$

Proof. Note that, almost surely for $t \leqslant \tau^\varepsilon$, $A^\varepsilon(t) = A_0 + \mathrm{O}(|c^\varepsilon - c_0| + |\varepsilon\eta^\varepsilon|_{H^1})$ with

$$A_0 = \begin{pmatrix}|\partial_x\varphi_{c_0}|_{L^2}^2 & 0\\ 0 & (\varphi_{c_0}, \partial_c\varphi_{c_0})\end{pmatrix},$$

and with $\mathrm{O}(|c^\varepsilon - c_0| + |\varepsilon\eta^\varepsilon|_{H^1})$ uniform in ε, t and ω, as long as $t \leqslant \tau^\varepsilon$. Hence, choosing $\alpha \leqslant \alpha_1$ smaller if necessary (depending only on c_0), it follows that setting

$$\tilde{A}^\varepsilon(t) = A_0 + \mathbb{1}_{[0,\tau^\varepsilon)}(t)\big(A^\varepsilon(t) - A_0\big), \tag{4.15}$$

the matrix $\tilde{A}^\varepsilon(t)$ is invertible, for all t, and for a.e. $\omega \in \Omega$,

$$\left\| \left(\tilde{A}^\varepsilon(t)\right)^{-1} \right\| \leqslant C(c_0, \alpha).$$

We deduce that for $\varepsilon \leqslant \varepsilon_0$, $t \leqslant \tau^\varepsilon$, (4.5) may be solved by $Z_\ell^\varepsilon(t) = (\tilde{A}^\varepsilon(t))^{-1} F_\ell^\varepsilon(t)$, leading to the estimate

$$\left|\left(z^\varepsilon(t), \phi e_\ell\right)\right| + \left|\left(b^\varepsilon(t), \phi e_\ell\right)\right| \leqslant C(c_0, \alpha)\left|F_\ell^\varepsilon(t)\right| \leqslant C(c_0, \alpha)|\phi e_\ell|_{L^2}|\varphi_{c_0}|_{H^1} \tag{4.16}$$

for all $\ell \in \mathbb{N}$ and all $t \leqslant \tau^\varepsilon$; (4.13) follows from the Parseval theorem.

To prove the estimate on the drift part, we note that for $\alpha \leqslant \alpha_1$, the estimate (4.13) easily leads to

$$\left|G_1^\varepsilon(t)\right| + \left|G_2^\varepsilon(t)\right| \leqslant C_1\left|\eta^\varepsilon(t)\right|_{L^2} + \varepsilon C_2, \quad \text{a.s. for } t \leqslant \tau^\varepsilon,$$

where C_1 and C_2 are constants depending only on c_0, α and $\|\phi\|_{\mathcal{L}_2^1}$. In view of (4.6), and the above arguments on A^ε, (4.14) follows, with possibly different constants. □

Under the more restrictive assumptions of Theorem 2.6, we get the following estimates on z^ε and b^ε, which will be useful for the tightness of the family.

Corollary 4.4. *Under the assumptions of Theorem* 2.6*, and if $\alpha \leqslant \alpha_1$, there is a constant $C(c_0, \alpha)$ with*

$$\sup_{t \leqslant \tau^\varepsilon} \left(\left|\phi^* z^\varepsilon(t)\right|_{X_\gamma} + \left|\phi^* b^\varepsilon(t)\right|_{X_\gamma}\right) \leqslant C(c_0, \alpha)|\varphi_{c_0}|_{H^1}\|\phi\|_{\mathcal{L}_2(L^2; X_\gamma)}. \tag{4.17}$$

Proof. If $\gamma = (\gamma_n)_{n \in \mathbb{N}}$ is a sequence of real positive numbers such that ϕ is Hilbert–Schmidt from $L^2(\mathbb{R})$ into X_γ, where X_γ is defined as in (1.11), then we may write using the Parseval identity and (4.16)

$$|\phi^* z^\varepsilon|_{X_\gamma}^2 + |\phi^* b^\varepsilon|_{X_\gamma}^2 = \sum_{\ell \in \mathbb{N}} \gamma_\ell(\phi e_\ell, z^\varepsilon)^2 + \gamma_\ell(\phi e_\ell, b^\varepsilon)^2 \leqslant C^2(c_0, \alpha)|\varphi_{c_0}|_{H^1}^2 \sum_{\ell \in \mathbb{N}} \gamma_\ell|\phi e_\ell|_{L^2}^2.$$

Hence, (4.17) follows. □

5. Estimates on the remainder term, and tightness

In this section, we list some estimates on the remainder term η^ε defined in the preceding sections. These estimates will allow us to apply a compactness method, and pass to the limit as ε goes to zero. This will be done in Section 6.

Again, here, $\alpha \leqslant \alpha_0$ is fixed. Moreover, we assume from now on that $\phi \in \mathcal{L}_2^2$. We may write down Eq. (4.1) for η^ε in the equivalent form

$$\begin{aligned} \mathrm{d}\eta^\varepsilon = {} & \partial_x L_{c^\varepsilon}\eta^\varepsilon\,\mathrm{d}t + (y^\varepsilon\partial_x\varphi_{c^\varepsilon} - a^\varepsilon\partial_c\varphi_{c^\varepsilon})\,\mathrm{d}t + \varepsilon(\partial_x\eta^\varepsilon)y^\varepsilon\,\mathrm{d}t - \varepsilon\partial_x\left((\eta^\varepsilon)^2\right)\mathrm{d}t + \partial_x\varphi_{c^\varepsilon}(z^\varepsilon, \mathrm{d}W) \\ & - \partial_c\varphi_{c^\varepsilon}(b^\varepsilon, \mathrm{d}W) + \varepsilon\partial_x\eta^\varepsilon(z^\varepsilon, \mathrm{d}W) + (\mathrm{d}W)(t, \cdot + x^\varepsilon) + \varepsilon\sum_{\ell \in \mathbb{N}} \mathcal{T}_{x^\varepsilon}(\partial_x\phi e_\ell)(z^\varepsilon, \phi e_\ell)\,\mathrm{d}t \\ & + \frac{\varepsilon}{2}\left(\partial_x^2\varphi_{c^\varepsilon}|\phi^* z^\varepsilon|_{L^2}^2 - \partial_c^2\varphi_{c^\varepsilon}|\phi^* b^\varepsilon|_{L^2}^2\right)\mathrm{d}t + \frac{1}{2}\varepsilon^2\partial_x^2\eta^\varepsilon|\phi^* z^\varepsilon|_{L^2}^2\,\mathrm{d}t. \end{aligned} \tag{5.1}$$

Some of the estimates listed in the present section are obtained with the use of Eq. (5.1), applying the Itô formula to different functionals of η^ε. The details of the proofs are rather technical, and are postponed to Appendix A.

Applying the Itô formula to $|\eta^\varepsilon|_{L^2}^2$, we first get the following estimate (see Appendix A for the proof).

Lemma 5.1. *For any positive T, there is a constant $C(T)$ depending only on T, α, c_0 and $\|\phi\|_{\mathcal{L}_2^1}$ such that*

$$\mathbb{E}\Big(\sup_{t \in [0, \tau^\varepsilon \wedge T]} \left|\eta^\varepsilon(t)\right|_{L^2}^2\Big) \leqslant C(T). \tag{5.2}$$

In the same way, applying the Itô formula to $|\partial_x\eta^\varepsilon|_{L^2}^2$, we obtain (see Appendix A)

Lemma 5.2. *For any positive T, there is a constant $C(T)$ depending only on T, α, c_0 and $\|\phi\|_{\mathcal{L}_2^2}$ such that*

$$\mathbb{E}\Big(\sup_{t\in[0,\tau^\varepsilon\wedge T]}\big|\eta^\varepsilon(t)\big|_{H^1}^2\Big)\leqslant C(T). \tag{5.3}$$

Remark 5.3. Some of the terms arising in the course of the proof of Lemma 5.2 need the use of auxiliary estimates, like estimates on $\mathbb{E}(\sup_{t\leqslant\tau^\varepsilon\wedge T}|\varepsilon\partial_x^2\eta^\varepsilon(t)|_{L^2}^4)$. These estimates are stated and proved in Appendix A, and are responsible for the restriction on the regularity of ϕ, i.e. the fact that we require $\phi\in\mathcal{L}_2^2$ and not only $\phi\in\mathcal{L}_2^1$.

In order to proceed with the compactness method, we need estimates on the moduli of continuity. Note that the family $(\eta^\varepsilon, y^\varepsilon, a^\varepsilon, z^\varepsilon, b^\varepsilon)=(\eta^\varepsilon, Y^\varepsilon, Z^\varepsilon)$ is a priori only defined for $t\leqslant\tau^\varepsilon$. We define it for $t\in\mathbb{R}^+$ by simply setting $\eta^\varepsilon(t)=\eta^\varepsilon(\tau^\varepsilon)$ for $t\geqslant\tau^\varepsilon$ and the same for Y^ε and Z^ε.

The next estimate is a simple consequence of Lemma 5.2 and Eq. (5.1).

Lemma 5.4. *For any positive T and any real number β with $0\leqslant\beta<1/2$, there is a constant $C(T,\beta)$ depending only on T,β,α,c_0 and $\|\phi\|_{\mathcal{L}_2^2}$ such that*

$$\mathbb{E}\big(\big|\eta^\varepsilon(t)\big|_{C^\beta([0,T];H^{-2}(\mathbb{R}))}^2\big)\leqslant C(T,\beta). \tag{5.4}$$

The tightness of the laws of Z^ε in $[C([0,T];L^2(\mathbb{R}))]^2$ and of Y^ε in $[C([0,T];\mathbb{R})]^2$ will be obtained thanks to (4.17) and the following estimates.

Lemma 5.5. *For any positive T and any real number β with $0\leqslant\beta<1/2$, there is a constant $C(T,\beta)$ depending only on T,β,α,c_0 and $\|\phi\|_{\mathcal{L}_2^2}$ such that*

$$\mathbb{E}\big(|\phi^*z^\varepsilon|_{C^\beta([0,T];L^2)}+|\phi^*b^\varepsilon|_{C^\beta([0,T];L^2)}\big)\leqslant C(T,\beta) \tag{5.5}$$

and

$$\mathbb{E}\big(|a^\varepsilon|_{C^\beta([0,T])}+|y^\varepsilon|_{C^\beta([0,T])}\big)\leqslant C(T,\beta). \tag{5.6}$$

Proof of Lemma 5.5. Let $T>0$ and β, with $0\leqslant\beta<1/2$ be fixed. The integrated form of Eq. (2.10), together with estimates (4.13), (4.14) and the definition of τ^ε easily produce

$$\mathbb{E}\big(|c^\varepsilon|_{C^\beta([0,T\wedge\tau^\varepsilon])}\big)\leqslant C(T,\beta). \tag{5.7}$$

In view of Eq. (2.9), and using again (4.13) and (4.14), it comes

$$\mathbb{E}\big(|x^\varepsilon|_{C^\beta([0,T\wedge\tau^\varepsilon])}\big)\leqslant C(T,\beta).$$

We then turn to Eq. (4.5) for

$$Z_\ell^\varepsilon=\begin{pmatrix}(z^\varepsilon,\phi e_\ell)\\(b^\varepsilon,\phi e_\ell)\end{pmatrix}$$

and we estimate

$$\mathbb{E}\Big(\sup_{s,t\leqslant\tau^\varepsilon\wedge T}\big|F_\ell^\varepsilon(t)-F_\ell^\varepsilon(s)\big|^2\Big)\leqslant C\mathbb{E}\Big(\sup_{s,t\leqslant\tau^\varepsilon\wedge T}|\mathcal{T}_{x^\varepsilon(t)}\phi e_\ell-\mathcal{T}_{x^\varepsilon(s)}\phi e_\ell|_{L^2}^2\Big)\big(|\varphi_{c_0}|_{L^2}^2+|\partial_x\varphi_{c_0}|_{L^2}^2\big).$$

From Eq. (2.9) for x^ε and the Itô formula, if $s<t$,

$$\phi e_\ell\big(\cdot+x^\varepsilon(t)\big)-\phi e_\ell\big(\cdot+x^\varepsilon(s)\big)=\int_s^t(\partial_x\phi e_\ell)\big(\cdot+x^\varepsilon(\sigma)\big)\,\mathrm{d}x^\varepsilon(\sigma)+\frac{1}{2}\varepsilon^2\int_s^t(\partial_x^2\phi e_\ell)\big(\cdot+x^\varepsilon(\sigma)\big)\big|\phi^*z^\varepsilon(\sigma)\big|_{L^2}^2\,\mathrm{d}\sigma$$

so that using Eq. (2.9), and estimates (4.13) and (4.14) again, one obtains

$$\mathbb{E}\Big(\sup_{s,t\leqslant\tau^\varepsilon\wedge T}\big|\phi e_\ell\big(\cdot+x^\varepsilon(t)\big)-\phi e_\ell\big(\cdot+x^\varepsilon(s)\big)\big|_{L^2}^2\Big)\leqslant C(\alpha,\beta,c_0)|t-s|^{2\beta}|\phi e_\ell|_{H^2}^2 \tag{5.8}$$

from which

$$\mathbb{E}\Big(\sup_{s,t\leqslant\tau^\varepsilon\wedge T}\big|F_\ell^\varepsilon(t)-F_\ell^\varepsilon(s)\big|^2\Big)\leqslant C(T,\alpha,\beta,c_0)|t-s|^{2\beta}|\phi e_\ell|_{H^2}^2 \tag{5.9}$$

follows. In the same way,

$$\begin{aligned}\mathbb{E}\Big(\sup_{s,t\leqslant T}\big|\big(\tilde A^\varepsilon\big)^{-1}(t)-\big(\tilde A^\varepsilon\big)^{-1}(s)\big|^2\Big)&\leqslant\mathbb{E}\Big(\sup_{s,t\leqslant T}\big|\big(\tilde A^\varepsilon\big)^{-1}(t)\big(\tilde A^\varepsilon(t)-\tilde A^\varepsilon(s)\big)\big(\tilde A^\varepsilon\big)^{-1}(s)\big|^2\Big)\\&\leqslant C\mathbb{E}\Big(\sup_{s,t\leqslant\tau^\varepsilon\wedge T}\big|\tilde A^\varepsilon(t)-\tilde A^\varepsilon(s)\big|^2\Big)\\&\leqslant C(T,\alpha,\beta,c_0)|t-s|^{2\beta}\end{aligned}$$

since $(\tilde A^\varepsilon)^{-1}$ is uniformly bounded in t, ω, and ε for $t\geqslant 0$. This together with (4.5) and (5.9) imply (5.5).

The proof of (5.6) requires estimates on $G^\varepsilon(t)$ in $C^\beta([0,T])$. From (4.9),

$$\begin{aligned}\big|G_1^\varepsilon(t)-G_1^\varepsilon(s)\big|\leqslant C\big(\alpha,c_0,\|\phi\|_{\mathcal{L}_2^2}\big)\Big\{&\big|\eta^\varepsilon(t)-\eta^\varepsilon(s)\big|_{H^{-2}}+\big|c^\varepsilon(t)-c^\varepsilon(s)\big|\Big(1+\sup_{t\leqslant\tau^\varepsilon}\big|\eta^\varepsilon(t)\big|_{L^2}\Big)\\&+\sum_{\ell\in\mathbb{N}}\big|\mathcal{T}_{x^\varepsilon(t)}(\partial_x\phi e_\ell)-\mathcal{T}_{x^\varepsilon(s)}(\partial_x\phi e_\ell)\big|_{L^2}\big|\big(z^\varepsilon(t),\phi e_\ell\big)\big|+\big|\phi^*z^\varepsilon(t)-\phi^*z^\varepsilon(s)\big|_{L^2}\\&+\big|\phi^*b^\varepsilon(t)-\phi^*b^\varepsilon(s)\big|_{L^2}+\varepsilon\big|\big((\eta^\varepsilon)^2(t)-(\eta^\varepsilon)^2(s),\partial_x\varphi_{c_0}\big)\big|\Big\}\end{aligned}$$

and the last term is bounded above by

$$C\Big(\sup_{t\leqslant\tau^\varepsilon}\big|\varepsilon\eta^\varepsilon(t)\big|_{H^2}\Big)\big|\partial_x^2\varphi_{c_0}\big|_{H^2}\big|\eta^\varepsilon(t)-\eta^\varepsilon(s)\big|_{H^{-2}};$$

hence estimates (5.2), (5.4), (5.7), (5.8), (5.5) and Lemma A.2 together with the Cauchy–Schwarz inequality lead to

$$\mathbb{E}\Big(\sup_{s,t\leqslant\tau^\varepsilon\wedge T}\big|G_1^\varepsilon(t)-G_1^\varepsilon(s)\big|\Big)\leqslant C\big(T,\alpha,\beta,c_0,\|\phi\|_{\mathcal{L}_2^2}\big)|t-s|^\beta.$$

Moreover, the same estimate holds with G_2^ε replacing G_1^ε, and (5.6) follows, with the help of the above argument on $(\tilde A^\varepsilon)^{-1}(t)$.

Setting now, for all $t\geqslant 0$,

$$c^\varepsilon(t)=c_0+\varepsilon\int_0^t a^\varepsilon(s)\,\mathrm{d}s+\varepsilon\int_0^t\big(b^\varepsilon(s),\,\mathrm{d}W(s)\big)$$

and

$$x^\varepsilon(t)=\int_0^t c^\varepsilon(s)\,\mathrm{d}s+\varepsilon\int_0^t y^\varepsilon(s)\,\mathrm{d}s+\varepsilon\int_0^t\big(z^\varepsilon(s),\mathrm{d}W(s)\big),$$

collecting all the lemmas in Section 5, and using in addition the well known identity

$$\mathbb{E}\big(\big|W(t)-W(s)\big|_{H^2}^2\big)=|t-s|\|\phi\|_{\mathcal{L}_2^2}^2$$

together with the compactness of the embedding of $H^2(\mathbb{R})$ into $H^1_{\mathrm{loc}}(\mathbb{R})$, we get the following corollary.

Corollary 5.6. *For any $T>0$ and any s with $0\leqslant s<1$, the family $(\eta^\varepsilon,a^\varepsilon,y^\varepsilon,\phi^*b^\varepsilon,\phi^*z^\varepsilon,c^\varepsilon,x^\varepsilon,W)_{0<\varepsilon\leqslant\varepsilon_0}$ is tight in $C([0,T];H^s_{\mathrm{loc}}(\mathbb{R}))\times(C([0,T]))^2\times(C([0,T];L^2(\mathbb{R})))^2\times(C([0,T]))^2\times C([0,T];H^1_{\mathrm{loc}}(\mathbb{R}))$.*

6. Passage to the limit and conclusion

We now end the proof of Theorem 2.6. Fix $s<1$ and $T>0$; let $X^\varepsilon=(\eta^\varepsilon,a^\varepsilon,y^\varepsilon,\phi^*b^\varepsilon,\phi^*z^\varepsilon,c^\varepsilon,x^\varepsilon)$, and consider a pair of sub-sequences $(X^{\varepsilon_{k_n}},X^{\varepsilon_{p_n}})_{n\in\mathbb{N}}$ of the family $(X^\varepsilon)_{0<\varepsilon\leqslant\varepsilon_0}$ with $\lim_{n\to\infty}\varepsilon_{k_n}=0$ and $\lim_{n\to\infty}\varepsilon_{p_n}=0$. We infer from the preceding section, Prokhorov and Skorokhod theorems that there is a sub-sequence of $(X^{\varepsilon_{k_n}},X^{\varepsilon_{p_n}},W)_{n\in\mathbb{N}}$, still denoted $(X^{\varepsilon_{k_n}},X^{\varepsilon_{p_n}},W)_{n\in\mathbb{N}}$, a probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$, and random variables $(\widetilde{X}_1^n,\widetilde{X}_2^n,\widetilde{W}^n)$, $n\in\mathbb{N}$ and $(\widetilde{X}_1,\widetilde{X}_2,\widetilde{W})$, with values in $[C([0,T];H^s_{\mathrm{loc}})\times(C([0,T]))^2\times(C([0,T];L^2))^2\times(C([0,T]))^2]^2\times C([0,T];H^1_{\mathrm{loc}})$ such that for any $n\in\mathbb{N}$,

$$\mathcal{L}\big(\widetilde{X}_1^n,\widetilde{X}_2^n,\widetilde{W}^n\big)=\mathcal{L}\big(X^{\varepsilon_{k_n}},X^{\varepsilon_{p_n}},W\big)$$

and such that for $j=1,2$,

$$\widetilde{X}_j^n\to\widetilde{X}_j\quad\text{as } n\to\infty,\ \widetilde{\mathbb{P}}\text{ a.s in } C\big([0,T];H^s_{\mathrm{loc}}\big)\times\big(C\big([0,T]\big)\big)^2\times\big(C\big([0,T];L^2\big)\big)^2\times\big(C\big([0,T]\big)\big)^2$$

and

$$\widetilde{W}^n\to\widetilde{W}\quad\text{as } n\to\infty,\ \widetilde{\mathbb{P}}\text{ a.s in } C\big([0,T];H^1_{\mathrm{loc}}\big).$$

We denote by $(\tilde\eta_j^n,\tilde a_j^n,\tilde y_j^n,\tilde b_j^n,\tilde z_j^n,\tilde c_j^n,\tilde x_j^n)$ the components of $\widetilde{X}_j^n$ and by $(\tilde\eta_j,\tilde a_j,\tilde y_j,\tilde b_j,\tilde z_j,\tilde c_j,\tilde x_j)$ those of $\widetilde{X}_j$, for $j=1,2$. We also define

$$\widetilde{\mathcal{F}}_t=\sigma\big\{\widetilde{X}_j(s),\widetilde{W}_j(s),0\leqslant s\leqslant t,\ j=1,2\big\}$$

and

$$\widetilde{\mathcal{F}}_t^n=\sigma\big\{\widetilde{X}_j^n(s),\widetilde{W}_j^n(s),0\leqslant s\leqslant t,\ j=1,2\big\};$$

it is easily seen that $\widetilde{W}$ and $\widetilde{W}^n$ are Wiener processes associated respectively with $(\widetilde{\mathcal{F}}_t)_{t\geqslant0}$ and $(\widetilde{\mathcal{F}}_t^n)_{t\geqslant0}$, with covariance operator $\phi\phi^*$, and that we may thus write $\widetilde{W}^n=\phi\widetilde{W}_c^n$ and $\widetilde{W}=\phi\widetilde{W}_c$ where $\widetilde{W}_c^n$ and $\widetilde{W}_c$ are cylindrical Wiener processes on the probability space $(\widetilde{\Omega},\widetilde{\mathcal{F}},\widetilde{\mathbb{P}})$.

It is easily checked that, for $j=1,2$, $\tilde c_j^n$ and $\tilde x_j^n$ satisfy respectively

$$\tilde c_j^n(t)=c_0+\varepsilon_j^n\int_0^t\tilde a_j^n(s)\,\mathrm{d}s+\varepsilon_j^n\int_0^t\big(\tilde b_j^n(s),\mathrm{d}\widetilde{W}^n(s)\big)$$

and

$$\tilde x_j^n(t)=\int_0^t\tilde c_j^n(s)\,\mathrm{d}s+\varepsilon_j^n\int_0^t\tilde y_j^n(s)\,\mathrm{d}s+\varepsilon\int_0^t\big(\tilde z_j^n(s),\mathrm{d}\widetilde{W}^n(s)\big),$$

where $\varepsilon_j^n=\varepsilon^{k_n}$ if $j=1$ and $\varepsilon_j^n=\varepsilon^{p_n}$ if $j=2$; taking the limit in $C([0,T])$as n goes to infinity, we easily deduce

$$\tilde c_j(t)=c_0\quad\text{a.s. for any } t\in[0,T]\text{ and } j=1,2\tag{6.1}$$

and

$$\tilde x_j(t)=\int_0^t\tilde c_j(s)\,\mathrm{d}s=c_0t\quad\text{a.s. for any } t\in[0,T],\ j=1,2.\tag{6.2}$$

Moreover, setting for $j=1,2$

$$\tilde\tau_j^n=\inf\big\{t\leqslant T,\ \big|\tilde c_j^n-c_0\big|\geqslant\alpha\text{ or }\big|\varepsilon_j^n\tilde\eta_j^n\big|_{H^1}\geqslant\alpha\big\},$$

then for $j=1,2$, Theorem 2.1 implies

$$\widetilde{\mathbb{P}}\big(\tilde\tau_j^n\geqslant T\big)=\mathbb{P}\big(\tau^{\varepsilon_j^n}\geqslant T\big)\leqslant C(\alpha,c_0)\big(\varepsilon_j^n\big)^2T\|\phi\|^2_{\mathcal{L}_2^1}$$

which goes to zero as n goes to infinity; hence, extracting again a sub-sequence if necessary, we have

$$\tilde\tau_j^n \to T \quad \text{a.s. as } n\to\infty \text{ for } j=1,2. \tag{6.3}$$

Also, for $j=1,2$ and $t\leqslant \tilde\tau_j^n$, one may easily verify that $\tilde\eta_j^n$ satisfies the following equation:

$$\begin{aligned} \mathrm{d}\tilde\eta_j^n &= \partial_x L_{c_0}\tilde\eta_j^n\,\mathrm{d}t + \big(\tilde y_j^n\partial_x\varphi_{\tilde c_j^n} - \tilde a_j^n\partial_c\varphi_{\tilde c_j^n}\big)\,\mathrm{d}t + \big(\tilde c_j^n - c_0 + \varepsilon_j^n\tilde y_j^n\big)\,\mathrm{d}t - \varepsilon_j^n\partial_x\big((\tilde\eta_j^n)^2\big)\,\mathrm{d}t \\ &\quad - 2\partial_x\big((\varphi_{\tilde c_j^n}-\varphi_{c_0})\tilde\eta_j^n\big)\,\mathrm{d}t + \varepsilon_j^n\sum_{\ell\in\mathbb{N}}\mathcal{T}_{\tilde x_j^n}(\partial_x\phi e_\ell)\big(\tilde z_j^n, e_\ell\big)\,\mathrm{d}t + \frac{\varepsilon_j^n}{2}\big(\partial_x^2\varphi_{\tilde c_j^n}\big|\tilde z_j^n\big|_{L^2}^2 - \partial_c^2\varphi_{\tilde c_j^n}\big|\tilde b_j^n\big|_{L^2}^2\big)\,\mathrm{d}t \\ &\quad + \frac{1}{2}\big(\varepsilon_j^n\big)^2\partial_x^2\tilde\eta_j^n\big|\tilde z_j^n\big|_{L^2}^2\,\mathrm{d}t + \psi\big(\tilde x_j^n,\tilde c_j^n,\tilde\eta_j^n\big)\,\mathrm{d}\widetilde W_c^n, \end{aligned} \tag{6.4}$$

where we have set, for $v\in L^2(\mathbb{R})$,

$$\psi\big(\tilde x_j^n,\tilde c_j^n,\tilde\eta_j^n\big)v = \big(\tilde z_j^n, v\big)\partial_x\varphi_{\tilde c_j^n} - \big(\tilde b_j^n, v\big)\partial_c\varphi_{\tilde c_j^n} + \big(\tilde z_j^n, v\big)\varepsilon_j^n\partial_x\tilde\eta_j^n + \mathcal{T}_{\tilde x_j^n}(\phi v).$$

Note that $(\mathbb{1}_{[0,\tilde\tau_j^n)}\tilde x_j^n)_{n\in\mathbb{N}}$ and $(\mathbb{1}_{[0,\tilde\tau_j^n)}\tilde c_j^n)_{n\in\mathbb{N}}$ are bounded sequences in $L^p(\Omega; C([0,T]))$ for any $p\geqslant 2$, while $(\varepsilon_j^n\tilde\eta_j^n)_{n\in\mathbb{N}}$ is a bounded sequence in $L^4(\Omega; C([0,T];H^2))$ by Lemma A.2; this implies, for $j=1,2$, that $\mathbb{1}_{[0,\tilde\tau_j^n)}\psi(\tilde x_j^n,\tilde c_j^n,\tilde\eta_j^n)$, which converges to $\tilde\psi_j$ defined by

$$\tilde\psi_j v = (\tilde z_j, v)\partial_x\varphi_{c_0} - (\tilde b_j, v)\partial_c\varphi_{c_0} + \mathcal{T}_{c_0t}(\phi v)$$

a.s. in $C([0,T];\mathcal{L}_2^0)$, also converges in $L^2(\Omega; C([0,T];\mathcal{L}_2^0))$ by equi-integrability. Using this and taking the limit as n goes to infinity in Eq. (6.4), we deduce that $\tilde\eta_j$ satisfies, for $j=1,2$ and $t\leqslant T$,

$$\mathrm{d}\tilde\eta_j = \partial_x L_{c_0}\tilde\eta_j\,\mathrm{d}t + (\tilde y_j\partial_x\varphi_{c_0} - \tilde a_j\partial_c\varphi_{c_0})\partial_x\tilde\eta_j\,\mathrm{d}t + \partial_x\varphi_{c_0}\big(\tilde z_j,\mathrm{d}\widetilde W_c\big) - \partial_c\varphi_{c_0}\big(\tilde b_j,\mathrm{d}\widetilde W_c\big) + (\mathcal{T}_{c_0t}\phi)\,\mathrm{d}\widetilde W_c. \tag{6.5}$$

Moreover, taking the limit in the equations for $\tilde y_j^n, \tilde a_j^n, \tilde b_j^n, \tilde z_j^n$, which are the same as (4.5) and (4.6), we obtain

$$\tilde y_j = |\partial_x\varphi_{c_0}|_{L^2}^{-2}\big(\tilde\eta_j, L_{c_0}\partial_x^2\varphi_{c_0}\big) \quad \text{and} \quad \tilde a_j = 0,\ j=1,2, \tag{6.6}$$

and on the other hand

$$\begin{cases} (\tilde z_j, e_\ell) = -|\partial_x\varphi_{c_0}|_{L^2}^{-2}\big(\mathcal{T}_{c_0t}(\phi e_\ell), \partial_x\varphi_{c_0}\big), \\ (\tilde b_j, e_\ell) = (\varphi_{c_0}, \partial_c\varphi_{c_0})^{-1}\big(\mathcal{T}_{c_0t}(\phi e_\ell), \varphi_{c_0}\big). \end{cases} \tag{6.7}$$

Thus, $\tilde\eta_j$ satisfies the equivalent of Eq. (2.7). Moreover, we deduce from Lemma 5.2 that

$$\mathbb{E}\Big(\sup_{t\in[0,T]}\big|\tilde\eta_j^n(t)\big|_{H^1}^2\Big)\leqslant C(T),$$

where $C(T)$ is the constant appearing in Lemma 5.2. Therefore, $\tilde\eta_j^n$ tends to $\tilde\eta_j$ in $L^2(\Omega; L^\infty(0,T;H^1(\mathbb{R})))$ weak star.

Now, it is not difficult to see that (2.7) has a unique solution with paths in $L^1(0,T;H^1(\mathbb{R}))$, a.s. such that $\eta(0)=0$. We then make use of the following lemma, which was first applied by Gyöngy and Krylov in [15].

Lemma 6.1. *Let Z_n be a sequence of random elements in a Polish space E equipped with the Borel σ-algebra. Then Z_n converges in probability to an E-valued random element if and only if for every pair of sub-sequences $(Z_{\varphi(n)}, Z_{\psi(n)})$, there is a sub-sequence of $(Z_{\varphi(n)}, Z_{\psi(n)})$ which converges in law to a random element supported on the diagonal $\{(x,y)\in E\times E,\ x=y\}$.*

We deduce that, for any $s<1$, the whole family η^ε converges in probability in $C([0,T];H^s_{\mathrm{loc}}(\mathbb{R}))$, and weakly in $L^\infty(0,T;H^1(\mathbb{R}))$ to a process η satisfying Eq. (2.7) and $\eta(0)=0$. In addition, a^ε converges to 0 and y^ε converges to $|\partial_x\varphi_{c_0}|_{L^2}^{-2}(\eta, L_{c_0}\partial_x^2\varphi_{c_0})$ in probability in $C([0,T])$, while $\phi^* z^\varepsilon$ converges to $-|\partial_x\varphi_{c_0}|_{L^2}^{-2}(\mathcal{T}_{c_0t}\phi)^*\partial_x\varphi_{c_0}$ and $\phi^* b^\varepsilon$ converges to $(\varphi_{c_0}, \partial_c\varphi_{c_0})^{-1}(\mathcal{T}_{c_0t}\phi)^*\varphi_{c_0}$ in probability in $C([0,T];L^2)$. Those convergence also hold in $L^2(\Omega)$ by equi-integrability.

To end the proof of Theorem 2.6, it only remains to prove estimate (2.8). Note that η has paths in $C[0,T];H^1(\mathbb{R}))$ a.s. We apply the Itô formula – the next computations may be justified with the help of the usual smoothing procedure, see [5] – to the functional

$$(\eta, L_{c_0}\eta) = \int_{\mathbb{R}} |\partial_x \eta|^2 \,\mathrm{d}x + c_0 \int_{\mathbb{R}} \eta^2 \,\mathrm{d}x - 2\int_{\mathbb{R}} \varphi_{c_0}\eta \,\mathrm{d}x.$$

It follows

$$\mathrm{d}(\eta, L_{c_0}\eta) = 2(L_{c_0}\eta,\ \mathrm{d}\eta) + \sum_{k\in\mathbb{N}} \big(L_{c_0}\psi(t)e_k, \psi(t)e_k\big)\,\mathrm{d}t, \tag{6.8}$$

where the operator $\psi(t)$ acting on $L^2(\mathbb{R})$ is defined as

$$\psi(t)e_k = -|\partial_x\varphi_{c_0}|_{L^2}^{-2}\big(\mathcal{T}_{c_0t}(\phi e_k), \partial_x\varphi_{c_0}\big)\partial_x\varphi_{c_0} - (\varphi_{c_0}, \partial_c\varphi_{c_0})^{-1}\big(\mathcal{T}_{c_0t}(\phi e_k), \varphi_{c_0}\big)\partial_c\varphi_{c_0} + \mathcal{T}_{c_0t}(\phi e_k).$$

It is easily checked, using (2.7), the self-adjointness of L_{c_0} and the fact that $L_{c_0}\partial_x\varphi_{c_0} = 0$ while $L_{c_0}\partial_c\varphi_{c_0} = -\varphi_{c_0}$, that

$$(\mathrm{d}\eta, L_{c_0}\eta) = (\varphi_{c_0}, \partial_c\varphi_{c_0})^{-1}\big((\mathcal{T}_{c_0t}\phi)\,\mathrm{d}W, \varphi_{c_0}\big)(\varphi_{c_0}, \eta) + \big((\mathcal{T}_{c_0t}\phi)\,\mathrm{d}W, L_{c_0}\eta\big). \tag{6.9}$$

Since the above remarks also lead to

$$\begin{aligned}\sum_{k\in\mathbb{N}}\big(L_{c_0}\psi(t)e_k, \psi(t)e_k\big) &= (\varphi_{c_0}, \partial_c\varphi_{c_0})^{-1}\sum_{k\in\mathbb{N}}\big(\mathcal{T}_{c_0t}(\phi e_k), \varphi_{c_0}\big)^2 + \sum_{k\in\mathbb{N}}\big(L_{c_0}\mathcal{T}_{c_0t}(\phi e_k), \mathcal{T}_{c_0t}(\phi e_k)\big)\\ &\leqslant C\|\phi\|^2_{\mathcal{L}_2^1}\end{aligned} \tag{6.10}$$

with a constant C only depending on c_0, integrating (6.8), using (6.9) and taking the expectation yields:

$$\mathbb{E}\big(L_{c_0}\eta(t), \eta(t)\big) \leqslant \mathbb{E}\big(L_{c_0}\eta(0), \eta(0)\big) + C\|\phi\|^2_{\mathcal{L}_2^1}t \leqslant C\|\phi\|^2_{\mathcal{L}_2^1}t \tag{6.11}$$

since $\eta(0) = 0$. On the other hand, from (3.3),

$$\mathbb{E}\big(L_{c_0}\eta(t), \eta(t)\big) \geqslant \nu\mathbb{E}\big(\big|\eta(t)\big|^2_{H^1}\big);$$

indeed, it is easily checked from (2.7) that $(\eta, \varphi_{c_0}) = (\eta, \partial_x\varphi_{c_0}) = 0$. We deduce

$$\mathbb{E}\big(\big|\eta(t)\big|^2_{H^1}\big) \leqslant C\|\phi\|^2_{\mathcal{L}_2^1}t, \quad \forall t \geqslant 0. \tag{6.12}$$

Coming back to (6.9), integrating in time and using Cauchy–Schwarz's and Doob's inequalities, we get

$$\begin{aligned}\mathbb{E}\Bigg(\sup_{t\leqslant T}\int_0^t\big(L_{c_0}\eta(s), \mathrm{d}\eta(s)\big)\Bigg) &\leqslant (\varphi_{c_0}, \partial_c\varphi_{c_0})^{-1}\Bigg\{\mathbb{E}\sup_{t\leqslant T}\Bigg[\int_0^t(\partial_c\varphi_{c_0}, L_{c_0}\eta)\big(\varphi_{c_0}, (\mathcal{T}_{c_0s}\phi)\,\mathrm{d}W(s)\big)\Bigg]^2\Bigg\}^{1/2}\\ &\quad + \Bigg\{\mathbb{E}\sup_{t\leqslant T}\Bigg[\int_0^t\big((\mathcal{T}_{c_0s}\phi)\,\mathrm{d}W(s), L_{c_0}\eta(s)\big)\Bigg]^2\Bigg\}^{1/2}\\ &\leqslant 2(\varphi_{c_0}, \partial_c\varphi_{c_0})^{-1}\Bigg\{\sum_{k\in\mathbb{N}}\int_0^T\mathbb{E}\big(\varphi_{c_0}, \eta(s)\big)^2\big(\mathcal{T}_{c_0s}(\phi e_k), \varphi_{c_0}\big)^2\,\mathrm{d}s\Bigg\}^{1/2}\\ &\quad + 2\Bigg\{\sum_{k\in\mathbb{N}}\int_0^T\mathbb{E}\big(\mathcal{T}_{c_0s}(\phi e_k), L_{c_0}\eta\big)^2\,\mathrm{d}s\Bigg\}^{1/2}\\ &\leqslant C\|\phi\|^2_{\mathcal{L}_2^1}\Bigg(\mathbb{E}\int_0^T\big|\eta(t)\big|^2_{H^1}\,\mathrm{d}t\Bigg)^{1/2}\end{aligned}$$

with a constant C depending only on c_0. We conclude thanks to (6.8), (6.10), (6.12) and another use of (3.3).

This ends the proof of Theorem 2.6.

Appendix A

We begin with the proof of Lemma 5.1. As mentioned in Section 5, we use Eq. (5.1) and apply the Itô formula to the functional $|\eta^\varepsilon(t)|_{L^2}^2$.

Proof of Lemma 5.1. Again, the computations below should be justified. However, this can be done in the same way as it was described for the justification of the application of the Itô–Wentzell formula in Section 4.

Hence, a formal application of the Itô formula to $|\eta^\varepsilon(t)|_{L^2}^2$, where η^ε satisfies Eq. (5.1), gives

$$\begin{aligned} \mathrm{d}|\eta^\varepsilon|_{L^2}^2 &= 2(\eta^\varepsilon, \mathrm{d}\eta^\varepsilon) + \mathrm{tr}\big(\psi^\varepsilon(\psi^\varepsilon)^*\big)\,\mathrm{d}t \\ &= 2(\eta^\varepsilon, \partial_x L_{c^\varepsilon}\eta^\varepsilon)\,\mathrm{d}t + 2(\eta^\varepsilon, y^\varepsilon\partial_x\varphi_{c^\varepsilon} - a^\varepsilon\partial_c\varphi_{c^\varepsilon})\,\mathrm{d}t + 2(\partial_x\varphi_{c^\varepsilon}, \eta^\varepsilon)(z^\varepsilon, \mathrm{d}W) - 2(\partial_c\varphi_{c^\varepsilon}, \eta^\varepsilon)(b^\varepsilon, \mathrm{d}W) \\ &\quad + 2\big(\eta^\varepsilon, (\mathrm{d}W)(\cdot + x^\varepsilon)\big) + 2\varepsilon\sum_{\ell\in\mathbb{N}}\big(\mathcal{T}_{x^\varepsilon}(\partial_x\phi e_\ell), \eta^\varepsilon\big)(z^\varepsilon, \phi e_\ell)\,\mathrm{d}t + \varepsilon\big(\partial_x^2\varphi_{c^\varepsilon}, \eta^\varepsilon\big)|\phi^* z^\varepsilon|_{L^2}^2\,\mathrm{d}t \\ &\quad - \varepsilon\big(\partial_c^2\varphi_{c^\varepsilon}, \eta^\varepsilon\big)|\phi^* b^\varepsilon|_{L^2}^2\,\mathrm{d}t - \varepsilon^2|\partial_x\eta^\varepsilon|_{L^2}^2|\phi^* z^\varepsilon|_{L^2}^2\,\mathrm{d}t + \sum_{\ell\in\mathbb{N}}|\psi^\varepsilon e_\ell|_{L^2}^2\,\mathrm{d}t, \end{aligned} \tag{A.1}$$

where we have set

$$\psi^\varepsilon e_\ell = \partial_x\varphi_{c^\varepsilon}(z^\varepsilon, \phi e_\ell) - \partial_c\varphi_{c^\varepsilon}(b^\varepsilon, \phi e_\ell) + \varepsilon\partial_x\eta^\varepsilon(z^\varepsilon, \phi e_\ell) + \mathcal{T}_{x^\varepsilon}(\phi e_\ell), \tag{A.2}$$

and we have used $(\partial_x\eta^\varepsilon, \eta^\varepsilon) = 0$.

Integration of (A.1) between 0 and $\tau = \tau^\varepsilon \wedge t$, with $\eta^\varepsilon(0) = 0$, and the use of (1.10) lead to the bound (after several integrations by parts)

$$\begin{aligned} \mathbb{E}\big(\big|\eta^\varepsilon(t)\big|_{L^2}^2\big) \leqslant C\mathbb{E}\int_0^\tau \big\{ &|\partial_x\varphi_{c^\varepsilon}|_{L^\infty}\big|\eta^\varepsilon(s)\big|_{L^2}^2 + \big|y^\varepsilon(s)\big|\big|\eta^\varepsilon(s)\big|_{L^2}|\partial_x\varphi_{c^\varepsilon}|_{L^2} + \big|a^\varepsilon(s)\big|\big|\eta^\varepsilon(s)\big|_{L^2}|\partial_c\varphi_{c^\varepsilon}|_{L^2} \\ &+ \varepsilon\big|\eta^\varepsilon(s)\big|_{L^2}\|\partial_x\phi\|_{\mathcal{L}_2^0}\big|\phi^* z^\varepsilon(s)\big|_{L^2} + \varepsilon\big|\partial_x^2\varphi_{c^\varepsilon}\big|_{L^2}|\eta^\varepsilon(s)|_{L^2}\big|\phi^* z^\varepsilon(s)\big|_{L^2}^2 \\ &+ \varepsilon\big|\partial_c^2\varphi_{c^\varepsilon}\big|_{L^2}\big|\eta^\varepsilon(s)\big|_{L^2}\big|\phi^* b^\varepsilon(s)\big|_{L^2}^2 + |\partial_x\varphi_{c^\varepsilon}|_{L^2}^2\big|\phi^* z^\varepsilon(s)\big|_{L^2}^2 + |\partial_c\varphi_{c^\varepsilon}|_{L^2}^2\big|\phi^* b^\varepsilon(s)\big|_{L^2}^2 + \|\phi\|_{\mathcal{L}_2^0}^2 \\ &+ \varepsilon|\partial_x\partial_c\varphi_{c^\varepsilon}|_{L^2}\big|\eta^\varepsilon(s)\big|_{L^2}\big|\phi^* z^\varepsilon(s)\big|_{L^2}\big|\phi^* b^\varepsilon(s)\big|_{L^2}\big\}\,\mathrm{d}s; \end{aligned} \tag{A.3}$$

the constant C appearing in the right-hand side above is an absolute constant.

Now, it follows easily from (A.3), (4.13), (4.14) and the fact that $|\partial_x\varphi_{c^\varepsilon}|_{L^\infty}$, $|\partial_x\varphi_{c^\varepsilon}|_{H^1}$ and $|\partial_c\varphi_{c^\varepsilon}|_{H^1}$ are uniformly bounded, for $t \in [0, \tau^\varepsilon)$, by a constant depending only on c_0 and α, that

$$\mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon)}(t)\big|\eta^\varepsilon(t)\big|_{L^2}^2\big) \leqslant C\big(c_0, \alpha, \|\phi\|_{\mathcal{L}_2^1}\big)\int_0^t \mathbb{E}\big(1 + \mathbb{1}_{[0,\tau^\varepsilon)}(s)\big|\eta^\varepsilon(s)\big|_{L^2}^2\big)\,\mathrm{d}s$$

and thus

$$\sup_{t\in[0,T]} \mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon)}(t)\big|\eta^\varepsilon(t)\big|_{L^2}^2\big) \leqslant C\big(c_0, \alpha, \|\phi\|_{\mathcal{L}_2^1}, T\big). \tag{A.4}$$

We now come back to (A.1) to estimate the martingale part. We clearly have

$$\begin{aligned} \mathbb{E}\left(\sup_{t\in[0,\tau^\varepsilon\wedge T]}\left|\int_0^t (\partial_x\varphi_{c^\varepsilon}, \eta^\varepsilon)(z^\varepsilon, \mathrm{d}W)\right|\right) &\leqslant \mathbb{E}\left(\sup_{t\in[0,\tau^\varepsilon\wedge T]}\left|\int_0^t (\partial_x\varphi_{c^\varepsilon}, \eta^\varepsilon)(z^\varepsilon, \mathrm{d}W)\right|^2\right)^{1/2} \\ &\leqslant C\left\{\int_0^T \mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon)}(s)|\partial_x\varphi_{c^\varepsilon}|_{L^\infty}^2\big|\eta^\varepsilon(s)\big|_{L^2}^2\big|\phi^* z^\varepsilon(s)\big|_{L^2}^2\,\mathrm{d}s\big)\right\}^{1/2} \\ &\leqslant C\big(c_0, \alpha, \|\phi\|_{\mathcal{L}_2^0}\big)\left\{\int_0^T \mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon)}(s)\big|\eta^\varepsilon(s)\big|_{L^2}^2\big)\,\mathrm{d}s\right\}^{1/2} \end{aligned} \tag{A.5}$$

and in the same way,

$$\mathbb{E}\left(\sup_{t\in[0,\tau^\varepsilon\wedge T]}\left|\int_0^t(\partial_c\varphi_{c^\varepsilon},\eta^\varepsilon)(b^\varepsilon,\mathrm{d}W)\right|\right)\leqslant C\big(c_0,\alpha,\|\phi\|_{\mathcal{L}_2^0}\big)\left\{\int_0^T\mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon)}(s)\big|\eta^\varepsilon(s)\big|_{L^2}^2\big)\,\mathrm{d}s\right\}^{1/2}. \tag{A.6}$$

At last,

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,\tau^\varepsilon\wedge T]}\left|\int_0^t\big(\eta^\varepsilon(s),(\mathrm{d}W)\big(\cdot+x^\varepsilon(s)\big)\big)\right|\right)&\leqslant\mathbb{E}\left(\sup_{t\in[0,\tau^\varepsilon\wedge T]}\left|\sum_{\ell\in\mathbb{N}}\int_0^t\big(\eta^\varepsilon(s),\mathcal{T}_{x^\varepsilon(t)}(\phi e_\ell)\big)\,\mathrm{d}\beta_\ell(s)\right|^2\right)^{1/2}\\&\leqslant C\left(\sum_{\ell\in\mathbb{N}}\mathbb{E}\int_0^{\tau^\varepsilon\wedge T}\big|\eta^\varepsilon(s)\big|_{L^2}^2|\phi e_\ell|_{L^2}^2\,\mathrm{d}s\right)^{1/2}\\&\leqslant C\left(\int_0^T\mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon]}(s)\big|\eta^\varepsilon(s)\big|_{L^2}^2\big)\,\mathrm{d}s\right)^{1/2}\|\phi\|_{\mathcal{L}_2^0}^2.\end{aligned} \tag{A.7}$$

The conclusion of Lemma 5.1 follows then from (A.1), with the help of estimates (A.5), (A.6) and (A.7). □

We now turn to the proof of Lemma 5.2. In the course of the proof, we will need some additional estimates on η^ε, which we state and prove now.

Lemma A.1. *Let $\phi\in\mathcal{L}_2^1$ and let η^ε, satisfying Eq.* (5.1)*, be given by Theorem* 2.1*. Then for any $T>0$, there is a constant C depending only on c_0, α, $\|\phi\|_{\mathcal{L}_2^1}$ and T such that*

$$\mathbb{E}\Big(\sup_{t\leqslant\tau^\varepsilon\wedge T}\big|\eta^\varepsilon(t)\big|_{L^2}^4\Big)\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^1}\big).$$

Proof. Again, the proof is performed by applying the Itô formula, this time to $F(\eta^\varepsilon)$ with $F(u)=|u|_{L^2}^4$, and using the same estimates as in the proof of Lemma 5.1, together with the martingale inequality given by Theorem 3.14 in [9]. We leave the details to the reader. □

The next lemma is the most technical to prove, and is also the reason why we need the regularity assumption $\phi\in\mathcal{L}_2^2$.

Lemma A.2. *Assume now that $\phi\in\mathcal{L}_2^2$ and let again η^ε satisfying Eq.* (5.1) *be given by Theorem* 2.1*. Let $T>0$, then there is a constant $C(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^2})$ such that*

$$\mathbb{E}\Big(\sup_{t\leqslant\tau^\varepsilon\wedge T}\big|\varepsilon\partial_x^2\eta^\varepsilon(t)\big|_{L^2}^4\Big)\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^2}\big). \tag{A.8}$$

Proof. We will use the Itô formula again, but this time we will need an higher order invariant of the homogeneous KdV equation (1.5). Namely, we make use of $\mathcal{E}(u)$ defined for $u\in H^2(\mathbb{R})$ by

$$\mathcal{E}(u)=\frac{9}{5}\int_{\mathbb{R}}\big(\partial_x^2u\big)^2\,\mathrm{d}x-3\int_{\mathbb{R}}(\partial_xu)^2u\,\mathrm{d}x+\frac{1}{4}\int_{\mathbb{R}}u^4\,\mathrm{d}x. \tag{A.9}$$

We first prove the estimate

$$\mathbb{E}\Big(\sup_{t\leqslant\tau^\varepsilon\wedge T}\big|\varepsilon\partial_x^2\eta^\varepsilon(t)\big|_{L^2}^2\Big)\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^2}\big). \tag{A.10}$$

Again, we proceed formally and a smoothing procedure is required to justify the following computations. Note that for $u, v \in H^2(\mathbb{R})$, we have

$$\mathcal{E}'(u) = \frac{18}{5}\partial_x^4 u + 3(\partial_x u)^2 + 6\big(\partial_x^2 u\big)u + u^3$$

and

$$\mathcal{E}''(u)\cdot v = \frac{18}{5}\partial_x^4 v + 6u\partial_x^2 v + 6(\partial_x u)(\partial_x v) + 6\big(\partial_x^2 u\big)v + 3u^2 v,$$

and that since $\mathcal{E}(u)$ is an invariant for the KdV equation (1.5), we have, at least formally,

$$\big(\mathcal{E}'(u), \partial_x^3 u + \partial_x\big(u^2\big)\big) = 0.$$

Now, applying the Itô formula to $\mathcal{E}(u^\varepsilon)$, where u^ε is the solution of (1.1) with $u^\varepsilon(0) = \varphi_{c_0}$, and using the preceding remarks leads to

$$\begin{aligned}
\mathrm{d}\mathcal{E}(u^\varepsilon) &= -\varepsilon\big(\mathcal{E}'(u^\varepsilon), \mathrm{d}W\big) + \frac{1}{2}\operatorname{tr}\big(\mathcal{E}''(u^\varepsilon)\phi\phi^*\big)\,\mathrm{d}t \\
&= -\frac{18}{5}\varepsilon\big(\partial_x^4 u^\varepsilon, \mathrm{d}W\big) - 3\varepsilon\big((\partial_x u^\varepsilon)^2, \mathrm{d}W\big) - 6\varepsilon\big(u^\varepsilon\partial_x^2 u^\varepsilon, \mathrm{d}W\big) - \varepsilon\big((u^\varepsilon)^3, \mathrm{d}W\big) \\
&\quad + \frac{1}{2}\varepsilon^2\sum_{\ell\in\mathbb{N}}\bigg\{\frac{18}{5}\big(\partial_x^4\phi e_\ell, \phi e_\ell\big) + 6\big(u^\varepsilon\partial_x^2\phi e_\ell, \phi e_\ell\big) + 6\big((\partial_x u^\varepsilon)(\partial_x\phi e_\ell), \phi e_\ell\big) \\
&\quad + 6\big(\big(\partial_x^2 u^\varepsilon\big)\phi e_\ell, \phi e_\ell\big) + 3\big((u^\varepsilon)^2\phi e_\ell, \phi e_\ell\big)\bigg\}.
\end{aligned} \tag{A.11}$$

Next, we integrate (A.11) between 0 and $\tau\wedge t$ with $\tau = \tau^\varepsilon \wedge \tau_R$ and

$$\tau_R = \inf\big\{t \geqslant 0,\ \big|u^\varepsilon(t)\big|_{H^2} \geqslant R\big\}. \tag{A.12}$$

We then need to estimate the martingales. We first have, using integrations by parts and Doob's inequality,

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{t\in[0,\tau\wedge T]}\bigg(\int_0^{\tau\wedge t}\big(\partial_x^4 u^\varepsilon, \mathrm{d}W\big)\bigg)^2\bigg) \\
&\quad\leqslant 4\mathbb{E}\bigg(\int_0^{\tau\wedge t}\sum_{\ell\in\mathbb{N}}\big(\partial_x^2 u^\varepsilon, \partial_x^2\phi e_\ell\big)^2\,\mathrm{d}s\bigg) \leqslant C\big(T, \|\phi\|_{\mathcal{L}_2^2}\big)\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2 u^\varepsilon\big|_{L^2}^2\Big).
\end{aligned} \tag{A.13}$$

In the same way, but using the martingale inequality of Theorem 3.14 in [9],

$$\mathbb{E}\bigg(\sup_{t\leqslant\tau\wedge T}\int_0^{\tau\wedge t}\big((\partial_x u^\varepsilon)^2, \mathrm{d}W\big)\bigg) \leqslant C\big(T, \|\phi\|_{\mathcal{L}_2^1}\big)\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}|\partial_x u^\varepsilon|_{L^2}^2\Big) \tag{A.14}$$

and

$$\mathbb{E}\bigg(\sup_{t\leqslant\tau\wedge T}\int_0^{\tau\wedge t}\big(\big(\partial_x^2 u^\varepsilon\big)u^\varepsilon, \mathrm{d}W\big)\bigg) \leqslant C\big(T, \|\phi\|_{\mathcal{L}_2^1}\big)\Big(\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2 u^\varepsilon(t)\big|_{L^2}^2\Big)\Big)^{1/2}\Big(\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|u^\varepsilon(t)\big|_{L^2}^2\Big)\Big)^{1/2}. \tag{A.15}$$

Finally, using again Theorem 3.14 in [9]:

$$\begin{aligned}
\mathbb{E}\bigg(\sup_{t\leqslant\tau\wedge T}\int_0^{\tau\wedge t}\big((u^\varepsilon)^3, \mathrm{d}W\big)\bigg) &\leqslant 3\|\phi\|_{\mathcal{L}_2^1}\mathbb{E}\bigg(\bigg(\int_0^{\tau\wedge T}\big|u^\varepsilon(s)\big|_{H^1}^6\,\mathrm{d}s\bigg)^{1/2}\bigg) \\
&\leqslant C\big(T, \|\phi\|_{\mathcal{L}_2^1}\big)\mathbb{E}\bigg(\sup_{t\leqslant\tau\wedge T}\big|u^\varepsilon(t)\big|_{H^1}^3\bigg).
\end{aligned} \tag{A.16}$$

The deterministic terms are easily estimated as follows:

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leqslant\tau\wedge T}\sum_{k\in\mathbb{N}}\int_0^{\tau\wedge t}\big(u^\varepsilon(s)\partial_x^2(\phi e_k),\phi e_k\big)\,\mathrm{d}s\right)&\leqslant\mathbb{E}\left(\int_0^{\tau\wedge T}\sum_{k\in\mathbb{N}}\big|\big(u^\varepsilon(s)\partial_x^2\phi e_k,\phi e_k\big)\big|\,\mathrm{d}s\right)\\&\leqslant C\|\phi\|^2_{\mathcal{L}_2^2}\mathbb{E}\left(\int_0^{\tau\wedge T}\big|u^\varepsilon(s)\big|_{L^2}\,\mathrm{d}s\right)\\&\leqslant C\big(T,\|\phi\|_{\mathcal{L}_2^2}\big)\Big(\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|u^\varepsilon(t)\big|^2_{L^2}\Big)\Big)^{1/2};\end{aligned}\tag{A.17}$$

the expression

$$\mathbb{E}\left(\int_0^{\tau\wedge T}\sum_{k\in\mathbb{N}}\big|\big(\partial_x u^\varepsilon(s)\partial_x(\phi e_k),\phi e_k\big)\big|\,\mathrm{d}s\right)$$

is estimated in the same way, after an integration by parts, an the same is true for

$$\mathbb{E}\left(\int_0^{\tau\wedge T}\sum_{k\in\mathbb{N}}\big|\big(\partial_x^2 u^\varepsilon(s)\phi e_k,\phi e_k\big)\big|\,\mathrm{d}s\right).$$

Finally,

$$\mathbb{E}\left(\int_0^{\tau\wedge T}\big|\big((u^\varepsilon)^2(s)\phi e_k,\phi e_k\big)\big|\,\mathrm{d}s\right)\leqslant C\big(T,\|\phi\|_{\mathcal{L}_2^1}\big)\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|u^\varepsilon(t)\big|^2_{L^2}\Big).\tag{A.18}$$

Collecting (A.13)–(A.18), we deduce from (A.11) that

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\mathcal{E}\big(u^\varepsilon(t)\big)\Big)\leqslant C\big(T,\|\phi\|^2_{\mathcal{L}_2^2}\big)\Big\{&1+\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|u^\varepsilon(t)\big|^3_{H^1}\Big)\\&+\Big(\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|^2_{L^2}\Big)\Big)^{1/2}\Big[1+\Big(\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|u^\varepsilon(t)\big|^2_{L^2}\Big)\Big)^{1/2}\Big]\Big\}.\end{aligned}\tag{A.19}$$

On the other hand, the expression (A.9) for $\mathcal{E}(u)$ shows that there is a constant $C>0$, such that for any $u\in H^2(\mathbb{R})$,

$$\big|\partial_x^2 u\big|^2_{L^2}\leqslant\frac{5}{9}\mathcal{E}(u)+C\big(1+|u|^4_{H^1}\big)\tag{A.20}$$

and this together with (A.19) implies

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|^2_{L^2}\Big)&\leqslant\frac{5}{9}\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\mathcal{E}\big(u^\varepsilon(t)\big)\Big)+C\Big(1+\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|u^\varepsilon(t)\big|^4_{H^1}\Big)\Big)\\&\leqslant\frac{1}{2}\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}|\partial_x^2u^\varepsilon(t)|^2_{L^2}\Big)+C\big(T,\|\phi\|_{\mathcal{L}_2^2}\big)\Big(1+\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|u^\varepsilon(t)\big|^4_{H^1}\Big)\Big).\end{aligned}\tag{A.21}$$

Next, we use the decomposition

$$u^\varepsilon(t)=\varphi_{c^\varepsilon(t)}\big(x-x^\varepsilon(t)\big)+\varepsilon\eta^\varepsilon\big(t,x-x^\varepsilon(t)\big)$$

to deduce that

$$\begin{aligned}\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\varepsilon\partial_x^2\eta^\varepsilon(t)\big|^2_{L^2}\Big)&\leqslant 2\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|^2_{L^2}\Big)+2\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2\varphi_{c^\varepsilon(t)}\big|^2_{L^2}\Big)\\&\leqslant 2\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|^2_{L^2}\Big)+C(\alpha,c_0)\end{aligned}$$

and since on the other hand

$$\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|u^\varepsilon(t)\big|_{H^1}^4\Big)\leqslant C\Big(\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\varphi_{c^\varepsilon(t)}\big|_{H^1}^4\Big)+\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\varepsilon\eta^\varepsilon(t)\big|_{H^1}^4\Big)\Big)\leqslant C(\alpha,c_0)$$

it follows from (A.21) that

$$\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\varepsilon\partial_x^2\eta^\varepsilon(t)\big|_{L^2}^2\Big)\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^2}^2\big);$$

recalling that $\tau=\tau^\varepsilon\wedge\tau_R$, (A.10) is obtained by letting R go to infinity.

We now prove (A.8). We apply again the Itô formula, this time to $\mathcal{E}^2(u^\varepsilon)$. This gives, using Eq. (A.11):

$$\mathrm{d}\big(\mathcal{E}^2(u^\varepsilon)\big)=2\mathcal{E}(u^\varepsilon)\,\mathrm{d}\mathcal{E}(u^\varepsilon)+\sum_{\ell\in\mathbb{N}}(\tilde\psi^\varepsilon e_\ell)^2\,\mathrm{d}t \tag{A.22}$$

with

$$\tilde\psi^\varepsilon e_\ell=-\frac{18}{5}\varepsilon\big(\partial_x^4u^\varepsilon,\phi e_\ell\big)-3\varepsilon\big((\partial_xu^\varepsilon)^2,\phi e_\ell\big)-6\varepsilon\big((\partial_x^2u^\varepsilon)u^\varepsilon,\phi e_\ell\big)-\varepsilon\big((u^\varepsilon)^3,\phi e_\ell\big).$$

We integrate again (A.22) between 0 and $\tau=\tau^\varepsilon\wedge\tau_R$, where τ_R is defined as in (A.12) and take the expectation. Next, we estimate the martingales. By Theorem 3.14 in [9],

$$\begin{aligned}\mathbb{E}\left(\sup_{t\leqslant\tau\wedge T}\frac{36}{5}\varepsilon\int_0^{\tau\wedge t}\mathcal{E}\big(u^\varepsilon(s)\big)\big(\partial_x^4u^\varepsilon(s),\mathrm{d}W(s)\big)\right)&\leqslant C\varepsilon\mathbb{E}\left(\left(\sum_{k\in\mathbb{N}}\int_0^{\tau\wedge T}\mathcal{E}^2\big(u^\varepsilon(s)\big)\big(\partial_x^2u^\varepsilon(s),\partial_x^2\phi e_k\big)^2\,\mathrm{d}s\right)^{1/2}\right)\\&\leqslant C\varepsilon\|\phi\|_{\mathcal{L}_2^2}\mathbb{E}\left(\left(\int_0^{\tau\wedge T}\mathcal{E}^2\big(u^\varepsilon(s)\big)\big|\partial_x^2u^\varepsilon(s)\big|_{L^2}^2\,\mathrm{d}s\right)^{1/2}\right).\end{aligned}$$

Now, since for $u\in H^2(\mathbb{R})$,

$$\mathcal{E}(u)\leqslant\frac{9}{5}\big|\partial_x^2u\big|_{L^2}^2+C\big(1+|u|_{H^1}^4\big),$$

the preceding term is bounded above by

$$\begin{aligned}&C\|\phi\|_{\mathcal{L}_2^2}\left\{\mathbb{E}\left(\left(\int_0^{\tau\wedge t}\big|\partial_x^2u^\varepsilon(s)\big|_{L^2}^6\,\mathrm{d}s\right)^{1/2}\right)+\mathbb{E}\left(\left(\int_0^{\tau\wedge T}\big|\partial_x^2u^\varepsilon\big|_{L^2}^2\big(1+\big|u^\varepsilon(s)\big|_{H^1}^8\big)\,\mathrm{d}s\right)^{1/2}\right)\right\}\\&\quad\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^2}\big)\left\{1+\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}^2\Big)+\Big(\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}^4\Big)\Big)^{3/4}\right\},\end{aligned}\tag{A.23}$$

where we have used the fact that

$$\sup_{t\leqslant\tau\wedge T}\big|u^\varepsilon(t)\big|_{H^1}\leqslant C(c_0,\alpha). \tag{A.24}$$

We prove in the same way that

$$\begin{aligned}&\mathbb{E}\left(\sup_{t\leqslant\tau\wedge T}\int_0^{\tau\wedge t}\mathcal{E}\big(u^\varepsilon(s)\big)\big(\big(\partial_xu^\varepsilon(s)\big)^2,\mathrm{d}W(s)\big)\right)\\&\quad\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^1}\big)\Big(1+\Big(\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}^4\Big)\Big)^{1/2}\Big)\end{aligned}\tag{A.25}$$

and

$$\mathbb{E}\left(\sup_{t\leqslant\tau\wedge T}\int_0^{\tau\wedge t}\mathcal{E}\big(u^\varepsilon(s)\big)\big(u^\varepsilon(s)\partial_x^2u^\varepsilon(s),\mathrm{d}W(s)\big)\right)$$
$$\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^1}\big)\Big\{1+\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}^2\Big)+\Big(\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}^4\Big)\Big)^{3/4}\Big\}. \tag{A.26}$$

Lastly, with the same arguments

$$\mathbb{E}\left(\sup_{t\leqslant\tau\wedge T}\int_0^{\tau\wedge t}\mathcal{E}\big(u^\varepsilon(s)\big)\big((u^\varepsilon)^3(s),\mathrm{d}W(s)\big)\right)\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^1}\big)\Big(1+\Big(\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}^4\Big)\Big)^{3/4}\Big). \tag{A.27}$$

As for the deterministic integrals arising in (A.22),

$$\mathbb{E}\left(\sup_{t\leqslant\tau\wedge T}\sum_{k\in\mathbb{N}}\int_0^{\tau\wedge t}\mathcal{E}\big(u^\varepsilon(s)\big)\big(\partial_x^4\phi e_k,\phi e_k\big)\,\mathrm{d}s\right)\leqslant C\|\phi\|_{\mathcal{L}_2^2}^2\mathbb{E}\left(\int_0^{\tau\wedge T}\big|\mathcal{E}\big(u^\varepsilon(s)\big)\big|\,\mathrm{d}s\right)\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^2}\big) \tag{A.28}$$

by (A.10), and the same is true for

$$\mathbb{E}\left(\sup_{t\leqslant\tau\wedge T}\sum_{k\in\mathbb{N}}\int_0^{\tau\wedge t}\mathcal{E}\big(u^\varepsilon(s)\big)\big(u^\varepsilon(s)\partial_x^2\phi e_k,\phi e_k\big)\,\mathrm{d}s\right)$$

since (A.24) holds. The terms

$$\mathbb{E}\left(\sup_{t\leqslant\tau\wedge T}\sum_{k\in\mathbb{N}}\int_0^{\tau\wedge t}\mathcal{E}\big(u^\varepsilon(s)\big)\big(\partial_xu^\varepsilon(s)\partial_x\phi e_k,\phi e_k\big)\,\mathrm{d}s\right)$$

and

$$\mathbb{E}\left(\sup_{t\leqslant\tau\wedge T}\sum_{k\in\mathbb{N}}\int_0^{\tau\wedge t}\mathcal{E}\big(u^\varepsilon(s)\big)\big(\big(u^\varepsilon\big)^2(s)\phi e_k,\phi e_k\big)\,\mathrm{d}s\right)$$

are estimated in the same way and

$$\mathbb{E}\left(\sup_{t\leqslant\tau\wedge T}\sum_{k\in\mathbb{N}}\int_0^{\tau\wedge t}\mathcal{E}\big(u^\varepsilon(s)\big)\big(\partial_x^2u^\varepsilon(s)\phi e_k,\phi e_k\big)\,\mathrm{d}s\right)$$
$$\leqslant C\big(T,\|\phi\|_{\mathcal{L}_2^2}\big)\Big\{\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}^3\Big)+\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}\big|u^\varepsilon(t)\big|_{H^1}^4\Big)+1\Big\}$$
$$\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^2}\big)\Big\{1+\Big(\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}^4\Big)\Big)^{3/4}+\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}^2\Big)\Big\}. \tag{A.29}$$

Finally, the terms arising from the Itô correction in (A.22) are bounded above, arguing as before, by the right-hand side of (A.29).

Collecting (A.23), (A.25)–(A.29) yields

$$\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\mathcal{E}^2\big(u^\varepsilon(t)\big)\Big)\leqslant\frac{1}{2}\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}^4\Big)+C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^2}\big)\Big(1+\mathbb{E}\Big(\sup_{t\leqslant\tau\wedge T}\big|\partial_x^2u^\varepsilon(t)\big|_{L^2}^2\Big)\Big).$$

We conclude thanks to the fact that for $u\in H^2(\mathbb{R})$

$$\big|\partial_x^2u\big|_{L^2}^4\leqslant\frac{50}{81}\mathcal{E}^2(u)+C\big(1+|u|_{H^1}^8\big),$$

that the same estimate is true for $\mathbb{E}(\sup_{t\leqslant\tau\wedge T}|\partial_x^2 u^\varepsilon(t)|_{L^2}^4)$, and the lemma follows, after using again the decomposition

$$u^\varepsilon(t)=\varphi_{c^\varepsilon(t)}\big(x-x^\varepsilon(t)\big)+\varepsilon\eta^\varepsilon\big(t,x-x^\varepsilon(t)\big)$$

and letting R go to infinity. □

We are now in position to prove Lemma 5.2.

Proof of Lemma 5.2. We apply the Itô formula to $|\partial_x\eta^\varepsilon(t)|_{L^2}^2$ where η^ε satisfies Eq. (5.1). We get

$$\begin{aligned}
\mathrm{d}|\partial_x\eta^\varepsilon|_{L^2}^2 &= -2\big(\partial_x^2\eta^\varepsilon,\mathrm{d}\eta^\varepsilon\big)+\sum_{\ell\in\mathbb{N}}|\partial_x\psi^\varepsilon e_\ell|_{L^2}^2\,\mathrm{d}t\\
&= -2\big(\partial_x^2\eta^\varepsilon,\partial_x L_{c^\varepsilon}\eta^\varepsilon\big)-2\big(\partial_x^2\eta^\varepsilon,y^\varepsilon\partial_x\varphi_{c^\varepsilon}-a^\varepsilon\partial_c\varphi_{c^\varepsilon}\big)\,\mathrm{d}t+2\varepsilon\big(\partial_x^2\eta^\varepsilon,\partial_x(\eta^\varepsilon)^2\big)\,\mathrm{d}t\\
&\quad-2\big(\partial_x^2\eta^\varepsilon,\partial_x\varphi_{c^\varepsilon}\big)\big(z^\varepsilon,\mathrm{d}W\big)+2\big(\partial_x^2\eta^\varepsilon,\partial_c\varphi_{c^\varepsilon}\big)\big(b^\varepsilon,\mathrm{d}W\big)-2\big(\partial_x^2\eta^\varepsilon,(\mathrm{d}W)(t,\cdot+x^\varepsilon)\big)\\
&\quad-2\varepsilon\sum_{\ell\in\mathbb{N}}\big(\partial_x^2\eta^\varepsilon,\mathcal{T}_{x^\varepsilon}(\partial_x\phi e_\ell)\big)\big(z^\varepsilon,\phi e_\ell\big)\,\mathrm{d}t-\varepsilon\big(\partial_x^2\eta^\varepsilon,\partial_x^2\varphi_{c^\varepsilon}\big)|\phi^*z^\varepsilon|_{L^2}^2\,\mathrm{d}t\\
&\quad+\varepsilon\big(\partial_x^2\eta^\varepsilon,\partial_c^2\varphi_{c^\varepsilon}\big)|\phi^*b^\varepsilon|_{L^2}^2\,\mathrm{d}t-\varepsilon^2\big|\partial_x^2\eta^\varepsilon\big|_{L^2}^2|\phi^*z^\varepsilon|_{L^2}^2\,\mathrm{d}t+\sum_{\ell\in\mathbb{N}}|\partial_x\psi^\varepsilon e_\ell|_{L^2}^2\,\mathrm{d}t,
\end{aligned}\tag{A.30}$$

where ψ^ε is defined as in (A.2), and

$$\partial_x^2 L_{c^\varepsilon}=-\partial_x^4+c^\varepsilon\partial_x^2-2\varphi_{c^\varepsilon}\partial_x^2-4(\partial_x\varphi_{c^\varepsilon})\partial_x-2\big(\partial_x^2\varphi_{c^\varepsilon}\big).$$

Next, we integrate (A.30) between 0 and $\tau^\varepsilon\wedge t$, and take the expectation; using several integrations by parts, it is not difficult to see that the resulting expression is bounded above by

$$\begin{aligned}
&C\mathbb{E}\int_0^{\tau^\varepsilon\wedge t}\Big(|\partial_x^2\varphi_{c^\varepsilon}|_{L^\infty}\big|\eta^\varepsilon(s)\big|_{H^1}^2+\big|y^\varepsilon(s)\big|\big|\partial_x\eta^\varepsilon(s)\big|_{L^2}\big|\partial_x^2\varphi_{c^\varepsilon}\big|_{L^2}+\big|a^\varepsilon(s)\big|\big|\partial_x\eta^\varepsilon(s)\big|_{L^2}|\partial_x\partial_c\varphi_{c^\varepsilon}|_{L^2}\\
&\quad+\big|\varepsilon\partial_x^2\eta^\varepsilon\big|_{L^2}|\eta^\varepsilon|_{L^2}|\partial_x\eta^\varepsilon|_{L^2}+\varepsilon\|\phi\|_{\mathcal{L}_2^2}|\partial_x\eta^\varepsilon|_{L^2}|\phi^*z^\varepsilon|_{L^2}+\varepsilon\big|\partial_x^3\varphi_{c^\varepsilon}\big|_{L^2}|\partial_x\eta^\varepsilon|_{L^2}|\phi^*z^\varepsilon|_{L^2}^2\\
&\quad+\varepsilon\big|\partial_x\partial_c^2\varphi_{c^\varepsilon}\big|_{L^2}|\partial_x\eta^\varepsilon|_{L^2}|\phi^*b^\varepsilon|_{L^2}^2+\|\phi\|_{\mathcal{L}_2^1}^2+\varepsilon|\partial_x\eta^\varepsilon|_{L^2}\big|\partial_x^2\partial_c\varphi_{c^\varepsilon}\big|_{L^2}|\phi^*z^\varepsilon|_{L^2}|\phi^*b^\varepsilon|_{L^2}\Big)\,\mathrm{d}s.
\end{aligned}$$

Using then (4.13), (4.14) and the fact that $|\partial_x^2\varphi_{c^\varepsilon}|_{H^1}$, $|\partial_x\partial_c\varphi_{c^\varepsilon}|_{H^1}$ and $|\partial_x\partial_c^2\varphi_{c^\varepsilon}|_{L^2}$ are bounded uniformly for $t\leqslant\tau^\varepsilon$ by a constant depending only on c_0 and α, we deduce that for $t\leqslant T$,

$$\begin{aligned}
\mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon]}(t)\big|\partial_x\eta^\varepsilon(t)\big|_{L^2}^2\big)&\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^2}\big)\Bigg\{1+\int_0^t\mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon]}(s)\big|\eta^\varepsilon(s)\big|_{L^2}^2\big)\,\mathrm{d}s\\
&\quad+\int_0^t\mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon]}(s)\big|\varepsilon\partial_x^2\eta^\varepsilon(s)\big|_{L^2}^4\big)\mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon]}(s)\big|\eta^\varepsilon(s)\big|_{L^2}^4\big)\mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon]}(s)\big|\partial_x\eta^\varepsilon(s)\big|_{L^2}^2\big)\,\mathrm{d}s\Bigg\}.
\end{aligned}$$

By Lemmas 5.1, A.1, A.2 and the Gronwall Lemma, we obtain

$$\sup_{t\in[0,T]}\mathbb{E}\big(\mathbb{1}_{[0,\tau^\varepsilon]}(t)\big|\eta^\varepsilon(t)\big|_{H^1}^2\big)\leqslant C\big(T,c_0,\alpha,\|\phi\|_{\mathcal{L}_2^2}\big).$$

The martingale part is estimated in the same way as in the proof of Lemma 5.1, using integrations by parts. □

References

[1] F. Abdullaev, S. Darmanyan, P. Khabibullaev, Optical Solitons, Springer Series in Nonlinear Dynamics, Springer-Verlag, Berlin, 1993.
[2] M.J. Ablowitz, H. Segur, Solitons and the Inverse Scattering Transform, SIAM, Philadelphia, 1981.
[3] G.G. Bass, Y.S. Kivshar, V.V. Konotop, Y.A. Sinitsyn, Dynamics of solitons under random perturbations, Phys. Rep. 157 (1988) 63–181.
[4] T.B. Benjamin, The stability of solitary waves, Proc. Roy. Soc. London Ser. A 328 (1972) 153–183.

[5] A. de Bouard, A. Debussche, On the stochastic Korteweg–de Vries equation, J. Funct. Anal. 154 (1998) 215–251.
[6] A. de Bouard, A. Debussche, The stochastic nonlinear Schrödinger equation in H^1, Stochastic Anal. Appl. 21 (2003) 97–126.
[7] A. de Bouard, A. Debussche, Y. Tsutsumi, White noise driven Korteweg–de Vries equation, J. Funct. Anal. 169 (1999) 532–558.
[8] A. de Bouard, A. Debussche, Y. Tsutsumi, Periodic solutions of the Korteweg–de Vries equation driven by white noise, SIAM J. Math. Anal., in press.
[9] G. Da Prato, J. Zabczyk, Stochastic equations in infinite dimensions, in: Encyclopedia of Mathematics and its Application, Cambridge University Press, Cambridge, 1992.
[10] A. Debussche, J. Printems, Numerical simulation of the stochastic Korteweg–de Vries equation, Phys. D 134 (1999) 200–226.
[11] G.E. Falkovich, I. Kolokolov, V. Lebedev, S.K. Turitsyn, Statistics of soliton-bearing systems with additive noise, Phys. Rev. E 63 (2001) 025601(R).
[12] M.I. Freidlin, A.D. Wentzell, Random Perturbations of Dynamical Systems, Comprehensive Studies in Mathematics, Springer-Verlag, Berlin, 1984.
[13] J. Garnier, Long-time dynamics of Korteweg–de Vries solitons driven by random perturbations, J. Statist. Phys. 105 (2001) 789–833.
[14] M. Grillakis, J. Shatah, W. Strauss, Stability theory of solitary waves in the presence of symmetry I, J. Funct. Anal. 74 (1987) 160–197.
[15] I. Gyöngy, N.V. Krylov, Existence of strong solutions for Itô's stochastic equations via approximations, Probab. Theory Related Fields 105 (1996) 143–158.
[16] R. Herman, The stochastic, damped Korteweg–de Vries equation, J. Phys. A. 23 (1990) 1063–1084.
[17] V.V. Konotop, L. Vasquez, Nonlinear Random Waves, World Scientific, Singapore, 1994.
[18] Y. Martel, F. Merle, Asymptotic stability of solitons for subcritical generalized KdV equations, Arch. Ration. Mech. Anal. 157 (2001) 219–254.
[19] Y. Martel, F. Merle, Asymptotic stability of solitons of the subcritical gKdV equations revisited, Nonlinearity 18 (2005) 55–80.
[20] Y. Martel, F. Merle, T.P. Tsai, Stability and asymptotic stability in the energy space of the sum of N solitons for subcritical gKdV equations, Comm. Math. Phys. 231 (2002) 347–373.
[21] R.M. Miura, The Korteweg–de Vries equation: a survey of results, SIAM Rev. 18 (1976) 412–459.
[22] R.L. Pego, M.I. Weinstein, Eigenvalues, and instabilities of solitary waves, Philos. Trans. Roy. Soc. London Ser. A 340 (1992) 47–94.
[23] R.L. Pego, M.I. Weinstein, Asymptotic stability of solitary waves, Comm. Math. Phys. 164 (1994) 305–349.
[24] J. Printems, Aspects Théoriques et numériques de l'équation de Korteweg–de Vries stochastique, thèse, Université Paris-Sud, Orsay, France, 1998.
[25] M. Scalerandi, A. Romano, C.A. Condat, Korteweg–de Vries solitons under additive stochastic perturbations, Phys. Rev. E 58 (1998) 4166–4173.
[26] R. Temam, Sur un problème non linéaire, J. Math. Pures Appl. 48 (1969) 159–172.
[27] M. Wadati, Stochastic Korteweg–de Vries equation, J. Phys. Soc. Japan 52 (1983) 2642–2648.
[28] M. Wadati, Y. Akutsu, Stochastic Korteweg–de Vries equation with and without damping, J. Phys. Soc. Japan 53 (1984) 3342–3350.