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Ann. I. H. Poincaré – AN 34 (2017) 69–88

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

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Moderate solutions of semilinear elliptic equations with Hardy potential

Moshe Marcus [a,*], Phuoc-Tai Nguyen [a,b]

[a] *Department of Mathematics, Technion, Haifa 32000, Israel*
[b] *Departamento de Matemática, Pontificia Universidad Católica de Chile, Santiago, Chile*

Received 21 July 2014; received in revised form 14 September 2015; accepted 22 October 2015

Available online 28 October 2015

Abstract

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. We study positive solutions of equation (E) $-L_\mu u+u^q=0$ in Ω where $L_\mu=\Delta+\frac{\mu}{\delta^2}$, $0<\mu$, $q>1$ and $\delta(x)=\operatorname{dist}(x,\partial\Omega)$. A positive solution of (E) is moderate if it is dominated by an L_μ-harmonic function. If $\mu<C_H(\Omega)$ (the Hardy constant for Ω) every positive L_μ-harmonic function can be represented in terms of a finite measure on $\partial\Omega$ via the Martin representation theorem. However the classical measure boundary trace of any such solution is zero. We introduce a notion of normalized boundary trace by which we obtain a complete classification of the positive moderate solutions of (E) in the subcritical case, $1<q<q_{\mu,c}$. (The critical value depends only on N and μ.) For $q\geq q_{\mu,c}$ there exists no moderate solution with an isolated singularity on the boundary. The normalized boundary trace and associated boundary value problems are also discussed in detail for the linear operator L_μ. These results form the basis for the study of the nonlinear problem.

MSC: 35J60; 35J75; 35J10

Keywords: Hardy potential; Martin kernel; Moderate solutions; Normalized boundary trace; Critical exponent; Removable singularities

1. Introduction

In this paper, we investigate boundary value problem with measure data for the following equation

$$-\Delta u-\frac{\mu}{\delta^2}u+u^q=0 \tag{1.1}$$

in a C^2 bounded domain Ω, where $q>1$, $\mu\in\mathbb{R}$ and $\delta(x)=\operatorname{dist}(x,\partial\Omega)$. This problem is naturally linked to the theory of linear Schrödinger equations $-L^V u=0$ where $L^V:=\Delta+V$ and the potential V satisfies $|V|\leq c\delta^{-2}$. Such equations have been studied in numerous papers (see [1,2] and the references therein).

* Corresponding author.
E-mail addresses: marcusm@math.technion.ac.il (M. Marcus), nguyenphuoctai.hcmup@gmail.com (P.-T. Nguyen).

http://dx.doi.org/10.1016/j.anihpc.2015.10.001

Put

$$L_\mu := \Delta + \frac{\mu}{\delta^2}. \tag{1.2}$$

A solution $u \in L^1_{loc}(\Omega)$ of the equation $-L_\mu u = 0$ is called an L_μ*-harmonic function*. Similarly, if

$$-L_\mu u \geq 0 \quad \text{or} \quad -L_\mu u \leq 0$$

we say that u is L_μ*-superharmonic* or L_μ*-subharmonic* respectively. If $\mu = 0$ we shall just use the terms harmonic, superharmonic, subharmonic.

Some problems involving equations (1.1) and (1.2) with $\mu < 1/4$ were studied by Bandle, Moroz and Reichel [4]. They derived estimates of local L_μ-subharmonic and superharmonic functions and applied these results to study conditions for existence or nonexistence of large solutions of (1.1). They also showed that the classical Keller–Osserman estimate [14,24] remains valid for (1.1).

The condition $\mu < \frac{1}{4}$ is related to Hardy's inequality. Denote by $C_H(\Omega)$ the best constant in Hardy's inequality, i.e.,

$$C_H(\Omega) = \inf_{H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega (u/\delta)^2 dx}. \tag{1.3}$$

By Marcus, Mizel and Pinchover [17], $C_H(\Omega) \in (0, \frac{1}{4}]$ and $C_H(\Omega) = \frac{1}{4}$ when Ω is convex. Furthermore the infimum is achieved if and only if $C_H(\Omega) < 1/4$. By Brezis and Marcus [7], for every $\mu < 1/4$ there exists a unique number $\lambda_{\mu,1}$ such that

$$\mu = \inf_{H^1_0(\Omega)} \frac{\int_\Omega (|\nabla u|^2 - \lambda_{\mu,1} u^2) dx}{\int_\Omega (u/\delta)^2 dx}$$

and the infimum is achieved. Thus $\lambda_{\mu,1}$ is an eigenvalue of $-L_\mu$ and, by [7, Lemma 2.1], it is a simple eigenvalue. We denote by $\varphi_{\mu,1}$ the corresponding positive eigenfunction normalized by $\int_\Omega (\varphi^2_{\mu,1}/\delta^2) dx = 1$.

The mapping $[1/4, \infty) \ni \mu \mapsto \lambda_{\mu,1}$ is strictly decreasing. Therefore if $\mu < C_H(\Omega)$ then $\lambda_{\mu,1} > 0$. Consequently, in this case, $\varphi_{\mu,1}$ is a positive supersolution of $-L_\mu$. This fact and a classical result of Ancona [2] imply that for every $y \in \partial\Omega$, there exists a positive L_μ-harmonic function in Ω which vanishes on $\partial\Omega \setminus \{y\}$ and is unique up to a constant. Denote this function by $K^\Omega_\mu(\cdot, y)$, normalized by setting it equal to 1 at a fixed reference point $x_0 \in \Omega$. The function $(x, y) \mapsto K^\Omega_\mu(x, y)$, $(x, y) \in \Omega \times \partial\Omega$, is the L_μ*-Martin kernel* in Ω relative to x_0. Further, by [2]:

Representation Theorem. *For every $\nu \in \mathfrak{M}^+(\partial\Omega)$ the function*

$$\mathbb{K}^\Omega_\mu[\nu](x) := \int_{\partial\Omega} K^\Omega_\mu(x, y) d\nu(y) \quad \forall x \in \Omega \tag{1.4}$$

is L_μ-harmonic, i.e., $L_\mu \mathbb{K}^\Omega_\mu[\nu] = 0$. Conversely, for every positive L_μ-harmonic function u there exists a unique measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that $u = \mathbb{K}^\Omega_\mu[\nu]$.

This theorem implies that – in the present case – the L_μ-Martin boundary of Ω coincides with the Euclidean boundary. (For the general definition of Martin boundary see, e.g. [1]. However this notion will not be used here beyond the representation theorem stated above.) The measure ν such that $u = \mathbb{K}^\Omega_\mu[\nu]$ is called the L_μ*-boundary measure of* u. If $\mu = 0$, ν is equivalent to the classical measure boundary trace of u (see Definition 1.1). But if $0 < \mu < C_H(\Omega)$, it can be shown that, for every $\nu \in \mathfrak{M}^+(\partial\Omega)$, the measure boundary trace of $\mathbb{K}^\Omega_\mu[\nu]$ is zero (see Corollary 2.11 below).

In the case $\mu = 0$, the boundary value problem

$$\begin{aligned} -\Delta u + |u|^{q-1}u &= 0 \quad \text{in } \Omega \\ u &= \nu \quad \text{on } \partial\Omega \end{aligned} \tag{1.5}$$

where $q > 1$ and ν is either a finite measure or a positive (possibly unbounded) measure, has been studied by numerous authors. Following Brezis [6], if ν is a finite measure, a weak solution of (1.5) is defined as follows: u is a solution of the problem if u and $\delta|u|^q$ are integrable in Ω and

$$\int_\Omega (-u\Delta\zeta + |u|^{q-1}u\zeta)dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\nu \quad \forall\zeta \in C_0^2(\overline{\Omega}) \tag{1.6}$$

where $\mathbf{n}$ is the outer unit normal on $\partial\Omega$. Brezis proved that, if a solution exists then it is unique. Gmira and Véron [13] showed that there exists a critical exponent, $q_c := \frac{N+1}{N-1}$, such that if $1 < q < q_c$, (1.6) has a weak solution for every finite measure ν but, if $q \geq q_c$ there exists no positive solution with isolated point singularity.

Marcus and Véron [20] proved that every positive solution of the equation

$$-\Delta u + u^q = 0 \tag{1.7}$$

possesses a boundary trace given by a positive measure ν, not necessarily bounded. In the subcritical case the blow-up set of the trace is a closed set. Furthermore they showed that, in this case, for every such positive measure ν, the boundary value problem (1.5) has a unique solution.

In the case $q = 2$, $N = 2$ this result was previously proved by Le Gall [15] using a probabilistic definition of the boundary trace.

In the supercritical case the problem turned out to be much more challenging. It was studied by several authors using various techniques. The problem was studied by Le Gall, Dynkin, Kuznetsov, Mselati a.o. employing mainly probabilistic methods. Consequently the results applied only to $1 < q \leq 2$. In parallel it was studied by Marcus and Veron employing purely analytic methods that were not subject to the restriction $q \leq 2$. A complete classification of the positive solutions of (1.5) in terms of their behavior at the boundary was provided by Mselati [18] for $q = 2$, by Dynkin [11] for $q_c \leq q \leq 2$ and finally by Marcus [16] for every $q \geq q_c$. For details and related results we refer the reader to [23,22,21,3,10] and the references therein.

In the case of equation (1.1) one is faced by the problem that, according to the classical definition of measure boundary trace, *every positive L_μ-harmonic function has measure boundary trace zero*. Therefore, in order to classify the positive solutions of (1.1) in terms of their behavior at the boundary, it is necessary to introduce a different notion of trace. As in the study of (1.7), we first consider the question of boundary trace for positive L_μ-harmonic or superharmonic functions.

We recall the classical definition of measure boundary trace.

Definition 1.1. (i) A sequence $\{D_n\}$ is a C^2 *exhaustion* of Ω if for every n, D_n is of class C^2, $\overline{D}_n \subset D_{n+1}$ and $\cup_n D_n = \Omega$. If the domains are uniformly of class C^2 we say that $\{D_n\}$ is a uniform C^2 exhaustion.

(ii) Let $u \in W^{1,p}_{loc}(\Omega)$ for some $p > 1$. We say that u possesses a *measure boundary trace* on $\partial\Omega$ if there exists a finite measure ν on $\partial\Omega$ such that, for every uniform C^2 exhaustion $\{D_n\}$ and every $\varphi \in C(\overline{\Omega})$,

$$\lim_{n\to\infty} \int_{\partial D_n} u|_{\partial D_n} \varphi dS = \int_{\partial\Omega} \varphi d\nu.$$

Here $u|_{D_n}$ denotes the Sobolev trace. The measure boundary trace of u is denoted by $\mathrm{tr}\,(u)$.

For $\beta > 0$, denote

$$\Omega_\beta = \{x \in \Omega : \delta(x) < \beta\}, \; D_\beta = \{x \in \Omega : \delta(x) > \beta\}, \; \Sigma_\beta = \{x \in \Omega : \delta(x) = \beta\}.$$

Put

$$\alpha_\pm := \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu}. \tag{1.8}$$

It can be shown (see Corollary 2.11 below) that the classical *measure boundary trace* of $\mathbb{K}^\Omega_\mu[\nu]$ is zero but there exist constants C_1, C_2 such that, for every $\nu \in \mathfrak{M}(\partial\Omega)$,

$$C_1 \|\nu\|_{\mathfrak{M}(\partial\Omega)} \leq \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} \mathbb{K}_\mu^\Omega[\nu](x) dS_x \leq C_2 \|\nu\|_{\mathfrak{M}(\partial\Omega)} \tag{1.9}$$

for all $\beta \in (0, \beta_0)$ where $\beta_0 > 0$ depends only on Ω. In view of this we introduce the following definition of trace.

Definition 1.2. A positive function u possesses a *normalized boundary trace* if there exists a measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that

$$\lim_{\beta \to 0} \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} |u - \mathbb{K}_\mu^\Omega[\nu]| dS_x = 0. \tag{1.10}$$

The normalized boundary trace will be denoted by $\mathrm{tr}^*(u)$.

Remark. The notion of normalized boundary trace is well defined. Indeed, suppose that ν and ν' satisfy (1.10). Put $v = (\mathbb{K}_\mu^\Omega[\nu - \nu'])_+$ then v is a nonnegative L_μ-subharmonic function, $v \leq \mathbb{K}[\nu + \nu']$ and $\mathrm{tr}^*(v) = 0$. By Proposition 2.14, $v = 0$, i.e., $\mathbb{K}_\mu^\Omega[\nu - \nu'] \leq 0$. By interchanging the roles of ν and ν', we deduce that $\mathbb{K}_\mu^\Omega[\nu' - \nu] \leq 0$. Thus $\nu = \nu'$.

Denote by G_μ^Ω the Green function of $-L_\mu$ in Ω and, for every positive Radon measure τ in Ω, put

$$\mathbb{G}_\mu^\Omega[\tau](x) := \int_\Omega G_\mu^\Omega(x, y) d\tau(y).$$

Denote by $\mathfrak{M}_f(\Omega)$, f a positive Borel function in Ω, the space of Radon measures τ on Ω satisfying $\int_\Omega f\, d|\tau| < \infty$ and by $\mathfrak{M}_f^+(\Omega)$ the positive cone of this space.

If τ is a positive measure such that $\mathbb{G}_\mu^\Omega[\tau](x) < \infty$ for some $x \in \Omega$ then $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$ and $\mathbb{G}_\mu^\Omega[\tau]$ is finite everywhere in Ω. The underlying reason for this is the behavior of the Green function at the boundary: for every $\beta > 0$ there exists c_β such that

$$c_\beta^{-1} \delta(x)^{\alpha_+} \leq G_\mu^\Omega(x, y) \leq c_\beta \delta(x)^{\alpha_+} \quad \forall x \in \Omega_{\beta/2},\ y \in D_\beta.$$

For details see Section 2.2 below.

We begin with the study of the linear boundary value problem,

$$\begin{aligned} -L_\mu u &= \tau \quad \text{in } \Omega \\ \mathrm{tr}^*(u) &= \nu, \end{aligned} \tag{1.11}$$

where $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega)$. As usual we look for solutions $u \in L^1_{loc}(\Omega)$ and the equation is understood in the sense of distributions. The representation theorem implies that if $\tau = 0$ the problem has a unique solution, $u = \mathbb{K}_\mu^\Omega[\nu]$.

We list below our main results regarding this problem.

Proposition I.

(i) *If u is a non-negative L_μ-harmonic function and $\mathrm{tr}^*(u) = 0$ then $u = 0$.*
(ii) *If $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega)$ then $\mathbb{G}_\mu^\Omega[\tau]$ has normalized trace zero. Thus $\mathbb{G}_\mu^\Omega[\tau]$ is a solution of (1.11) with $\nu = 0$.*
(iii) *Let u be a positive L_μ-subharmonic function. If u is dominated by an L_μ-superharmonic function then $L_\mu u \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega)$ and u has a normalized boundary trace. In this case $\mathrm{tr}^*(u) = 0$ if and only if $u \equiv 0$.*
(iv) *Let u be a positive L_μ-superharmonic function. Then there exist $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega)$ such that*

$$u = \mathbb{G}_\mu^\Omega[\tau] + \mathbb{K}_\mu^\Omega[\nu]. \tag{1.12}$$

In particular, u is an L_μ-potential (i.e., u does not dominate any positive L_μ-harmonic function) if and only if $\mathrm{tr}^*(u) = 0$.
(v) *For every $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega)$, problem (1.11) has a unique solution. The solution is given by (1.12).*

Next we study the nonlinear boundary value problem,

$$\begin{aligned} -L_\mu u + u^q &= 0 \quad \text{in } \Omega \\ \operatorname{tr}^*(u) &= \nu \end{aligned} \tag{1.13}$$

where $\nu \in \mathfrak{M}^+(\partial\Omega)$.

Definition 1.3. (i) A positive solution of (1.1) is L_μ*-moderate* if it is dominated by an L_μ-harmonic function.
(ii) A positive function $u \in L^q_{loc}(\Omega)$ is a *(weak) solution of* (1.13) if it satisfies the equation (in the sense of distributions) and has normalized boundary trace ν.

Definition 1.4. Put

$$X(\Omega) = \{\zeta \in C^2(\Omega) : \delta^{\alpha_-} L_\mu \zeta \in L^\infty(\Omega),\ \delta^{-\alpha_+}\zeta \in L^\infty(\Omega)\}.$$

A function $\zeta \in X(\Omega)$ is called an *admissible test function* for (1.13).

Following are our main results concerning the nonlinear problem (1.13). Theorems A–D apply to arbitrary exponent $q > 1$.

Theorem A. *Assume that* $0 < \mu < C_H(\Omega)$, $q > 1$. *Let u be a positive solution of* (1.1). *Then the following statements are equivalent:*
(i) *u is L_μ-moderate.*
(ii) *u admits a normalized boundary trace $\nu \in \mathfrak{M}^+(\partial\Omega)$. In other words, u is a solution of* (1.13).
(iii) *$u \in L^q_{\delta^{\alpha_+}}(\Omega)$ and*

$$u + \mathbb{G}^\Omega_\mu[u^q] = \mathbb{K}^\Omega_\mu[\nu] \tag{1.14}$$

where $\nu = \operatorname{tr}^*(u)$.

Furthermore, a positive function u is a solution of (1.13) *if and only if $u/\delta^{\alpha_-} \in L^1(\Omega)$, $\delta^{\alpha_+}u^q \in L^1(\Omega)$ and*

$$\int_\Omega (-uL_\mu\zeta + u^q\zeta)dx = -\int_\Omega \mathbb{K}^\Omega_\mu[\nu]L_\mu\zeta\, dx \quad \forall \zeta \in X(\Omega). \tag{1.15}$$

Theorem B. *Assume* $0 < \mu < C_H(\Omega)$, $q > 1$.

I. Uniqueness. *For every $\nu \in \mathfrak{M}^+(\partial\Omega)$, there exists at most one positive solution of* (1.13).

II. Monotonicity. *Assume $\nu_i \in \mathfrak{M}^+(\partial\Omega)$, $i = 1, 2$. Let u_{ν_i} be the unique solution of* (1.13) *with ν replaced by ν_i, $i = 1, 2$. If $\nu_1 \le \nu_2$ then $u_{\nu_1} \le u_{\nu_2}$.*

III. A-priori estimate. *There exists a positive constant $c = c(N, \mu, \Omega)$ such that every positive solution u of* (1.13) *satisfies,*

$$\|u\|_{L^1_{\delta^{-\alpha_-}}(\Omega)} + \|u\|_{L^q_{\delta^{\alpha_+}}(\Omega)} \le c\,\|\nu\|_{\mathfrak{M}(\partial\Omega)}. \tag{1.16}$$

Theorem C. *Assume* $0 < \mu < C_H(\Omega)$, $q > 1$. *If $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\mathbb{K}^\Omega_\mu[\nu] \in L^q_{\delta^{\alpha_+}}(\Omega)$ then there exists a unique solution of the boundary value problem* (1.13).

Corollary C1. *For every positive function $f \in L^1(\partial\Omega)$* (1.13) *with $\nu = f$ admits a unique positive solution.*

Theorem D. *Assume* $0 < \mu < C_H(\Omega)$, $q > 1$. *If u is a positive solution of* (1.13) *then*

$$\lim_{x\to y} \frac{u(x)}{\mathbb{K}^\Omega_\mu[\nu](x)} = 1 \quad \textit{non-tangentially},\ \nu\textit{-a.e. on } \partial\Omega. \tag{1.17}$$

Let

$$q_{\mu,c} := \frac{N+\alpha_+}{N-1-\alpha_-}. \tag{1.18}$$

In the next two results we show, among other things, that $q_{\mu,c}$ is the *critical exponent* for (1.13). This means that, if $1 < q < q_{\mu,c}$ then problem (1.13) has a unique solution for every measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ but, if $q \geq q_{\mu,c}$ then the problem has no solution for some measures ν, e.g. Dirac measure.

In Theorem E we consider the subcritical case $1 < q < q_{\mu,c}$ and in Theorem F the supercritical case.

Theorem E. *Assume $0 < \mu < C_H(\Omega)$ and $1 < q < q_{\mu,c}$. Then:*

I. EXISTENCE AND UNIQUENESS. *For every $\nu \in \mathfrak{M}^+(\partial\Omega)$ (1.13) admits a unique positive solution u_ν.*

II. STABILITY. *If $\{\nu_n\}$ is a sequence of measures in $\mathfrak{M}^+(\partial\Omega)$ weakly convergent to $\nu \in \mathfrak{M}^+(\partial\Omega)$ then $u_{\nu_n} \to u_\nu$ in $L^1_{\delta^{-\alpha_-}}(\Omega)$ and in $L^q_{\delta^{\alpha_+}}(\Omega)$.*

III. LOCAL BEHAVIOR. *Let $\nu = k\delta_y$, where $k > 0$ and δ_y is the Dirac measure concentrated at $y \in \partial\Omega$. Then, under the assumptions of Theorem E, the unique solution of (1.13), denoted by $u_{k\delta_y}$, satisfies*

$$\lim_{x\to y} \frac{u_{k\delta_y}(x)}{K_\mu^\Omega(x,y)} = k. \tag{1.19}$$

Remark. Note that in part III we have 'uniform convergence' not just 'non-tangential convergence' as in Theorem D.

Theorem F. *Assume $0 < \mu < C_H(\Omega)$ and $q \geq q_{\mu,c}$. Then for every $k > 0$ and $y \in \partial\Omega$, there is no positive solution of (1.1) with normalized boundary trace $k\delta_y$.*

In the first part of the paper we study properties of positive L_μ-harmonic functions and the boundary value problem (1.11). In the second part, these results are applied to a study of the corresponding boundary value problem for the nonlinear equation (1.1). These results yield a complete classification of the positive moderate solutions of (1.1) in the subcritical case. They also provide a framework for the study of positive solutions of (1.1) that may blow up at some parts of the boundary. The existence of such solutions in the subcritical case has been studied (by different methods) in [5]. The boundary trace for positive non-moderate solutions and corresponding boundary value problems will be treated in a forthcoming paper.

The main ingredients used in this paper are: the Representation Theorem previously stated and other basic results of potential theory (see [1]), a sharp estimate of the Green kernel of $-L_\mu$ due to Filippas, Moschini and Tertikas [9], estimates for convolutions in weak L^p spaces (see [23, Section 2.3.2]) and the comparison principle obtained in [4].

2. The linear equation

Throughout this paper we assume that $0 < \mu < C_H(\Omega)$.

2.1. Some potential theoretic results

We denote by $\mathfrak{M}_{\delta^\alpha}(\Omega)$, $\alpha \in \mathbb{R}$, the space of Radon measures τ on Ω satisfying $\int_\Omega \delta^\alpha(x)d|\tau| < \infty$ and by $\mathfrak{M}^+_{\delta^\alpha}(\Omega)$ the positive cone of $\mathfrak{M}_{\delta^\alpha}(\Omega)$. When $\alpha = 0$, we use the notation $\mathfrak{M}(\Omega)$ and $\mathfrak{M}^+(\Omega)$. We also denote by $\mathfrak{M}(\partial\Omega)$ the space of finite Radon measures on $\partial\Omega$ and by $\mathfrak{M}^+(\partial\Omega)$ the positive cone of $\mathfrak{M}(\partial\Omega)$.

Let D be a C^2 domain such that $D \Subset \Omega$ and $h \in L^1(\partial\Omega)$. Denote by $\mathbb{S}_\mu(D,h)$ the solution of the problem

$$\begin{cases} -L_\mu u = 0 & \text{in } D \\ \quad\ \ u = h & \text{on } \partial D. \end{cases} \tag{2.1}$$

Lemma 2.1. *Let u be L_μ-superharmonic in Ω and D be a C^2 domain such that $D \Subset \Omega$. Then $u \geq \mathbb{S}_\mu(D,u)$ a.e. in D.*

Proof. Since u is L_μ-superharmonic in Ω, there exists $\tau \in \mathfrak{M}^+(\Omega)$ such that $-L_\mu u = \tau$. Let v be the solution of

$$\begin{cases} -L_\mu v = \tau & \text{in } D \\ \quad v = 0 & \text{on } \partial D \end{cases} \tag{2.2}$$

and put $w = \mathbb{S}_\mu(D, u)$. Then $w \geq 0$ and $u|_D = v + w \geq v$. □

Lemma 2.2. *Let u be a nonnegative L_μ-superharmonic and $\{D_n\}$ be a C^2 exhaustion of Ω. Then*

$$\hat{u} := \lim_{n\to\infty} \mathbb{S}_\mu(D_n, u)$$

exists and is the largest L_μ-harmonic function dominated by u.

Proof. By Lemma 2.1, $\mathbb{S}_\mu(D_n, u) \leq u|_{D_n}$, hence the sequence $\{\mathbb{S}_\mu(D_n, u)\}$ is decreasing. Consequently, $\hat{u}$ exists and is an L_μ-harmonic function dominated by u. Next, if v is an L_μ-harmonic function dominated by u then $v \leq \mathbb{S}_\mu(D_n, u)$ for every $n \in \mathbb{N}$. Letting $n \to \infty$ yields $v \leq \hat{u}$. □

Definition 2.3. A nonnegative L_μ-superharmonic function is called an *L_μ-potential* if its largest L_μ-harmonic minorant is zero.

As a consequence of Lemma 2.2, we obtain

Lemma 2.4. *Let u_p be a nonnegative L_μ-superharmonic function in Ω. If for some C^2 exhaustion $\{D_n\}$ of Ω,*

$$\lim_{n\to\infty} \mathbb{S}_\mu(D_n, u_p) = 0, \tag{2.3}$$

then u_p is an L_μ-potential in Ω. Conversely, if u_p is an L_μ-potential, then (2.3) holds for every C^2 exhaustion $\{D_n\}$ of Ω.

For easy reference we quote below the Riesz decomposition theorem (see [1]).

Theorem 2.5. *Every nonnegative L_μ-superharmonic function u in Ω can be written in a unique way in the form $u = u_p + u_h$ where u_p is an L_μ-potential and u_h is a nonnegative L_μ-harmonic function in Ω.*

The next result is a consequence of the Fatou convergence theorem [1, Theorem 1.8] and the following well-known fact: if a function satisfies the local Harnack inequality, fine convergence at the boundary (in the sense of [1]) implies non-tangential convergence.

Theorem 2.6. *Let u_p be a positive L_μ-potential and u be a positive L_μ-harmonic function. Assume that $\frac{u_p}{u}$ satisfies the Harnack inequality. Then*

$$\lim_{x\to y} \frac{u_p(x)}{u(x)} = 0 \quad \textit{non-tangentially, } \nu\textit{-a.e. on } \partial\Omega$$

where ν is the L_μ-boundary measure of u.

2.2. The action of the Green and Martin kernels on spaces of measures

From [2], for every $y \in \partial\Omega$, there exists a positive L_μ-harmonic function in Ω which vanishes on $\partial\Omega \setminus \{y\}$. When normalized, this function is unique. We choose a fixed reference point x_0 in Ω and denote by $K^\Omega_{\mu,y}$ this L_μ-harmonic function, normalized by $K^\Omega_{\mu,y}(x_0) = 1$. The function $K^\Omega_\mu(\cdot, y) = K^\Omega_{\mu,y}(\cdot)$ is the L_μ-Martin kernel in Ω, normalized at x_0.

For $\nu \in \mathfrak{M}(\partial\Omega)$ denote

$$\mathbb{K}^\Omega_\mu[\nu](x) = \int_{\partial\Omega} K^\Omega_\mu(x, y) d\nu(y).$$

In what follows the notation $f \sim g$ means: there exists a positive constant c such that $c^{-1} f < g < cf$ in the domain of the two functions or in a specified subset of this domain. Of course, in the latter case, the constant depends on the subset.

Let G_μ^Ω be the Green kernel for the operator L_μ in $\Omega \times \Omega$. Fix a point $x_0 \in \Omega$. It is well known that the function $x \mapsto G_\mu^\Omega(x, x_0)$ behaves like the first eigenfunction $\varphi_{\mu,1}(x)$ near the boundary, i.e., $G_\mu^\Omega(\cdot, x_0) \sim \varphi_{\mu,1}$ in Ω_β, $(0 < \beta < \delta(x_0))$.

By [19, Lemmas 5,1, 5.2] (see also [8, Lemma 7] for an alternative proof)

$$c^{-1}\delta(x)^{\alpha_+} \le \varphi_{\mu,1}(x) \le c\delta(x)^{\alpha_+}. \tag{2.4}$$

Thus, if $0 < \beta < \delta(x_0)$,

$$c_\beta^{-1}\delta(x)^{\alpha_+} \le G_\mu^\Omega(x, x_0) \le c_\beta \delta(x)^{\alpha_+} \quad \forall x \in \Omega_\beta. \tag{2.5}$$

Therefore, if $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$ then

$$\mathbb{G}_\mu^\Omega[\tau](x) := \int_\Omega G_\mu^\Omega(x, y) d\tau(y) < \infty \quad \text{a.e. in } \Omega.$$

Indeed, by (2.5) and the symmetry of the Green kernel, for every $x \in \Omega$, the integral over $\Omega_{\delta(x)/2}$ is finite. For $y \in D_{\delta(x)/4}$, $G_\mu^\Omega(x, y) \le c|x-y|^{2-N}$. Therefore the integral is finite over this set as well. Inequality (2.5) also implies that, if τ is a positive Radon measure in Ω and $\mathbb{G}_\mu^\Omega[\tau](x) < \infty$ for some point $x \in \Omega$ then $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$ and $\mathbb{G}_\mu^\Omega[\tau]$ is finite everywhere in Ω.

By [9, Theorem 4.11], for every $x, y \in \Omega$, $x \ne y$,

$$G_\mu^\Omega(x, y) \sim \min\left\{|x-y|^{2-N}, \delta(x)^{\alpha_+}\delta(y)^{\alpha_+}|x-y|^{2\alpha_- - N}\right\} \tag{2.6}$$

Since

$$K_\mu^\Omega(x, y) := \lim_{z \to y} \frac{G_\mu^\Omega(x, z)}{G_\mu^\Omega(x_0, z)} \quad \forall x \in \Omega$$

it follows from (2.6) that

$$K_\mu^\Omega(x, y) \sim \delta(x)^{\alpha_+}|x-y|^{2\alpha_- - N} \quad \forall x \in \Omega, y \in \partial\Omega. \tag{2.7}$$

Let $G^\Omega = G_0^\Omega$ and $P^\Omega = P_0^\Omega$ denote the Green and Poisson kernels of $-\Delta$ in Ω. Then, by (2.7)

$$\frac{K_\mu^\Omega(x, y)}{\delta(x)^{\alpha_-}} \sim \frac{\delta(x)}{|x-y|^N}\left(\frac{|x-y|}{\delta(x)}\right)^{2\alpha_-} \sim P^\Omega(x, y)\left(\frac{|x-y|}{\delta(x)}\right)^{2\alpha_-}. \tag{2.8}$$

Denote $L_w^p(\Omega; \tau)$, $1 \le p < \infty$, $\tau \in \mathfrak{M}^+(\Omega)$, the weak L^p space defined as follows: a measurable function f in Ω belongs to this space if there exists a constant c such that

$$\lambda_f(a; \tau) := \tau(\{x \in \Omega : |f(x)| > a\}) \le ca^{-p}, \quad \forall a > 0. \tag{2.9}$$

The function λ_f is called the distribution function of f (relative to τ). For $p \ge 1$, denote

$$L_w^p(\Omega; \tau) = \{f \text{ Borel measurable} : \sup_{a>0} a^p \lambda_f(a; \tau) < \infty\}$$

and

$$\|f\|^*_{L_w^p(\Omega;\tau)} = (\sup_{a>0} a^p \lambda_f(a; \tau))^{\frac{1}{p}}. \tag{2.10}$$

This expression is not a norm, but for $p > 1$, it is equivalent to the norm

$$\|f\|_{L_w^p(\Omega;\tau)} = \sup\left\{\frac{\int_\omega |f| d\tau}{\tau(\omega)^{1/p'}} : \omega \subset \Omega, \omega \text{ measurable }, 0 < \tau(\omega)\right\}. \tag{2.11}$$

More precisely,

$$\|f\|^*_{L^p_w(\Omega;\tau)} \le \|f\|_{L^p_w(\Omega;\tau)} \le \frac{p}{p-1}\,\|f\|^*_{L^p_w(\Omega;\tau)}. \tag{2.12}$$

Notice that, for every $\alpha > -1$,

$$L^p_w(\Omega;\delta^\alpha dx) \subset L^r_{\delta^\alpha}(\Omega) \quad \forall r \in [1,p).$$

For every $x \in \partial\Omega$, denote by $\mathbf{n}_x$ the outward unit normal vector to $\partial\Omega$ at x.

The following is a well-known geometric property of C^2 domains.

Proposition 2.7. *There exists $\beta_0 > 0$ such that*

(i) For every point $x \in \overline{\Omega}_{\beta_0}$, there exists a unique point $\sigma_x \in \partial\Omega$ such that $|x-\sigma_x| = \delta(x)$. This implies $x = \sigma_x - \delta(x)\mathbf{n}_{\sigma_x}$.

(ii) The mappings $x \mapsto \delta(x)$ and $x \mapsto \sigma_x$ belong to $C^2(\overline{\Omega}_{\beta_0})$ and $C^1(\overline{\Omega}_{\beta_0})$ respectively. Furthermore, $\lim_{x\to\sigma(x)} \nabla\delta(x) = -\mathbf{n}_x$.

By combining (2.6), (2.7) and [23, Lemma 2.3.2], we obtain

Proposition 2.8. *There exist constants c_i depending only on N, μ, β, Ω such that,*

$$\left\|\mathbb{G}^\Omega_\mu[\tau]\right\|_{L_w^{\frac{N+\beta}{N-2}}(\Omega,\delta^\beta)} \le c_1\,\|\tau\|_{\mathfrak{M}(\Omega)}, \quad \forall \tau \in \mathfrak{M}(\Omega),\ \beta > -1, \tag{2.13}$$

$$\left\|\mathbb{G}^\Omega_\mu[\tau]\right\|_{L_w^{\frac{N+\beta}{N-2\alpha_-}}(\Omega,\delta^{\beta-\alpha_+})} \le c_1\,\|\tau\|_{\mathfrak{M}_{\delta^{\alpha_+}}(\Omega)}, \quad \forall \tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega),\ \beta > -2\alpha_-, \tag{2.14}$$

$$\left\|\mathbb{K}^\Omega_\mu[\nu]\right\|_{L_w^{\frac{N+\beta}{N-1-\alpha_-}}(\Omega,\delta^\beta)} \le c_2\,\|\nu\|_{\mathfrak{M}(\partial\Omega)}, \quad \forall \nu \in \mathfrak{M}(\partial\Omega),\ \beta > -1. \tag{2.15}$$

Proof. We assume that τ is positive; otherwise we replace τ by $|\tau|$. We consider τ as a positive measure in $\mathbb{R}^N$ by extending τ by zero outside of Ω. For $a \in (0,N)$, denote $\Gamma_a(x) = |x|^{a-N}$. By [23, inequality (2.3.17)],

$$\|\Gamma_a * \tau\|_{L_w^{\frac{N+\beta}{N-a}}(\Omega,\delta^\beta)} \le c\,\|\tau\|_{\mathfrak{M}(\Omega)} \quad \forall \beta > \max\{-1,-a\} \tag{2.16}$$

where $c = c(N,a,\beta,diam(\Omega))$. By (2.6),

$$G^\Omega_\mu(x,y) \le c\min\{\Gamma_2(x-y),\, \delta(x)^{\alpha_+}\delta(y)^{\alpha_+}\Gamma_{2\alpha_-}(x-y)\}.$$

Hence, by (2.16),

$$\begin{aligned}
\left\|\mathbb{G}^\Omega_\mu[\tau]\right\|_{L_w^{\frac{N+\beta}{N-2}}(\Omega,\delta^\beta)} &\le c\,\|\Gamma_2 * \tau\|_{L_w^{\frac{N+\beta}{N-2}}(\Omega,\delta^\beta)} \\
&\le c'\,\|\tau\|_{\mathfrak{M}(\Omega)} \quad \forall \beta > -1, \\
\left\|\mathbb{G}^\Omega_\mu[\tau]\right\|_{L_w^{\frac{N+\beta}{N-2\alpha_-}}(\Omega,\delta^{\beta-\alpha_+})} &\le c\left\|\Gamma_{2\alpha_-} * (\delta^{\alpha_+}\tau)\right\|_{L_w^{\frac{N+\beta}{N-2\alpha_-}}(\Omega,\delta^\beta)} \\
&\le c\,\|\tau\|_{\mathfrak{M}_{\delta^{\alpha_+}}(\Omega)} \quad \forall \beta > -2\alpha_-.
\end{aligned}$$

Next we extend ν by zero outside $\partial\Omega$ and observe that, by (2.7), $K^\Omega_\mu(x,y) \le c\Gamma_{1+\alpha_-}(x-y)$. Hence $\mathbb{K}^\Omega_\mu[\nu] \le c\Gamma_{1+\alpha_-} * \nu$ and by (2.16),

$$\left\|\mathbb{K}^\Omega_\mu[\nu]\right\|_{L_w^{\frac{N+\beta}{N-1-\alpha_-}}(\Omega,\delta^\beta)} \le c\left\|\Gamma_{1+\alpha_-} * \nu\right\|_{L_w^{\frac{N+\beta}{N-1-\alpha_-}}(\Omega,\delta^\beta)} \le c\,\|\nu\|_{\mathfrak{M}(\partial\Omega)} \quad \forall \beta > -1. \qquad \square$$

Corollary 2.9. *Let $\beta > -1$.*

(i) If $\{\nu_n\} \subset \mathfrak{M}^+(\partial\Omega)$ converges weakly to $\nu \in \mathfrak{M}^+(\partial\Omega)$ then $\{\mathbb{K}^\Omega_\mu[\nu_n]\}$ converges to $\mathbb{K}^\Omega_\mu[\nu]$ in $L^p_{\delta^\beta}(\Omega)$ for every p such that $1 \le p < \frac{N+\beta}{N-1-\alpha_-}$.

(ii) If $\{\tau_n\} \subset \mathfrak{M}^+(\Omega)$ converges weakly (relative to $C_0(\bar{\Omega})$) to $\tau \in \mathfrak{M}^+(\Omega)$ then $\{\mathbb{G}^\Omega_\mu[\tau_n]\}$ converges to $\mathbb{G}^\Omega_\mu[\tau]$ in $L^p_{\delta^\beta}(\Omega)$ for every p such that $1 \le p < \frac{N+\beta}{N-2}$.

Proof. We prove the first statement. The second is proved in a similar way.

Since $K_\mu^\Omega(x,.)\in C(\partial\Omega)$ for every $x\in\Omega$, $\{\mathbb{K}_\mu^\Omega[\nu_n]\}$ converges to $\mathbb{K}_\mu^\Omega[\nu]$ every where in Ω. By Holder inequality and (2.15), we deduce that $\{(\mathbb{K}_\mu^\Omega[\nu_n])^p\}$ is equi-integrable w.r.t. $\delta^\beta dx$ for any $1\le p<\frac{N+\beta}{N-1-\alpha_-}$. By Vitali's theorem, $\mathbb{K}_\mu^\Omega[\nu_n]\to\mathbb{K}_\mu^\Omega[\nu]$ in $L^p_{\delta^\beta}(\Omega)$. $\square$

2.3. *Estimates related to the normalized trace*

Proposition 2.10. *There exist positive constants C_1, C_2 such that, for every $\beta\in(0,\beta_0)$,*

$$C_1\beta^{\alpha_-}\le\int_{\Sigma_\beta}K_\mu^\Omega(x,y)dS_x\le C_2\beta^{\alpha_-}\quad\forall y\in\partial\Omega.\tag{2.17}$$

The constants C_1, C_2 depend on N,Ω,μ but not on y.

Furthermore, for every $r_0>0$,

$$\lim_{\beta\to0}\frac{1}{\beta^{\alpha_-}}\int_{\Sigma_\beta\setminus B_{r_0}(y)}K_\mu^\Omega(x,y)dS_x=0\quad\forall y\in\partial\Omega.\tag{2.18}$$

For r_0 fixed, the rate of convergence is independent of y.

Proof. By (2.7),

$$\frac{1}{\beta^{\alpha_-}}\int_{\Sigma_\beta\setminus B_{r_0}(y)}K_\mu^\Omega(x,y)dS_x\le c\beta^{\alpha_+-\alpha_-}.\tag{2.19}$$

This implies (2.18).

For the next estimate it is convenient to assume that the coordinates are placed so that $y=0$ and the tangent hyperplane to $\partial\Omega$ at 0 is $x_N=0$ with the x_N axis pointing into the domain. For $x\in\mathbb{R}^N$ put $x'=(x_1,\cdots,x_{N-1})$. Pick $r_0\in(0,\beta_0)$ sufficiently small (depending only on the C^2 characteristic of Ω) so that

$$\frac{1}{2}(|x'|^2+\delta(x)^2)\le|x|^2\quad\forall x\in\Omega\cap B_{r_0}(0).$$

Then, if $x\in\Sigma_\beta\cap B_{r_0}(0)=:\Sigma_{\beta,0}$,

$$\frac{1}{4}(|x'|+\beta)\le|x|.$$

This inequality and (2.7) imply,

$$\begin{aligned}\int_{\Sigma_{\beta,0}}K_\mu^\Omega(x,0)dS_x&\le c_0\beta^{\alpha_+}\int_{\Sigma_{\beta,0}}(|x'|+\beta)^{2\alpha_--N}dS_x\\&\le c_1\beta^{\alpha_+}\int_{|x'|<r_0}(|x'|+\beta)^{2\alpha_--N}dx'\\&\le c_2\beta^{\alpha_+}\int_0^{r_0}(t+\beta)^{2\alpha_--2}dt\\&<c_2\beta^{\alpha_-}\int_1^\infty\tau^{-2\alpha_+}d\tau=\frac{c_2}{2\alpha_+-1}\beta^{\alpha_-}.\end{aligned}$$

Thus, for $\beta<r_0$,

$$\frac{1}{\beta^{\alpha_-}}\int_{\Sigma_{\beta,0}}K_\mu^\Omega(x,0)dS_x\le\frac{c_2}{2\alpha_+-1}.\tag{2.20}$$

Estimates (2.19) and (2.20) imply the second estimate in (2.17). The first estimate in (2.17) follows from (2.8). □

Since (2.17) holds uniformly w.r. to $y \in \partial\Omega$, an application of Fubini's yields the following.

Corollary 2.11. *For every $\nu \in \mathfrak{M}^+(\partial\Omega)$,*

$$C_1 \|\nu\|_{\mathfrak{M}(\partial\Omega)} \le \liminf_{\beta\to 0} \int_{\Sigma_\beta} \frac{\mathbb{K}_\mu^\Omega[\nu]}{\delta(x)^{\alpha_-}} dS_x \le \limsup_{\beta\to 0} \int_{\Sigma_\beta} \frac{\mathbb{K}_\mu^\Omega[\nu]}{\delta(x)^{\alpha_-}} dS_x \le C_2 \|\nu\|_{\mathfrak{M}(\partial\Omega)} \tag{2.21}$$

with C_1, C_2 as in (2.17).

Proposition 2.12. *If $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$ then*

$$\mathrm{tr}^*(\mathbb{G}_\mu^\Omega[\tau]) = 0 \tag{2.22}$$

and, for $0 < \beta < \beta_0$,

$$\frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau] dS_x \le c \int_\Omega \delta^{\alpha_+} d|\tau|, \tag{2.23}$$

where c is a constant depending on μ, Ω.

Proof. We may assume that $\tau > 0$. Denote $v := \mathbb{G}_\mu^\Omega[\tau]$. We start with the proof of (2.23).

By Fubini's theorem and (2.6),

$$\int_{\Sigma_\beta} v dS_x \le c\Big(\int_\Omega \int_{\Sigma_\beta \cap B_{\frac{\beta}{2}}(y)} |x-y|^{2-N} dS_x\, d\tau(y) + \beta^{\alpha_+} \int_\Omega \int_{\Sigma_\beta \setminus B_{\frac{\beta}{2}}(y)} |x-y|^{2\alpha_- - N} dS_x\, \delta^{\alpha_+}(y) d\tau(y)\Big) = I_1(\beta) + I_2(\beta).$$

Note that, if $x \in \Sigma_\beta$ and $|x-y| \le \beta/2$ then $\beta/2 \le \delta(y) \le 3\beta/2$. Therefore

$$\begin{aligned} I_1(\beta) &\le c_1 \int_{\Sigma_\beta \cap B_{\frac{\beta}{2}}(y)} |x-y|^{2-\alpha_+ - N} dS_x \int_\Omega \delta(y)^{\alpha_+} d\tau(y) \\ &\le c_1' \int_0^{\beta/2} r^{2-\alpha_+ - N} r^{N-2} dr \int_\Omega \delta(y)^{\alpha_+} d\tau(y) \\ &\le c_1'' \beta^{\alpha_-} \int_\Omega \delta(y)^{\alpha_+} d\tau(y) \end{aligned}$$

and

$$I_2(\beta) \le c_2 \beta^{\alpha_+} \int_{\beta/2}^\infty r^{2\alpha_- - N} r^{N-2} dr \int_\Omega \delta(y)^{\alpha_+} d\tau = c_2' \beta^{\alpha_-} \int_\Omega \delta(y)^{\alpha_+} d\tau.$$

This implies (2.23).

Given $\epsilon \in (0, \|\tau\|_{\mathfrak{M}_{\delta^{\alpha_+}}(\Omega)})$ and $\beta_1 \in (0, \beta_0)$ put $\tau_1 = \tau \chi_{\bar{D}_{\beta_1}}$ and $\tau_2 = \tau - \tau_1$. Pick $\beta_1 = \beta_1(\epsilon)$ such that

$$\int_{\Omega_{\beta_1}} \delta(y)^{\alpha_+}\, d\tau \le \epsilon. \tag{2.24}$$

Thus the choice of β_1 depends on the rate at which $\int_{\Omega_\beta} \delta^{\alpha_+}\, d\tau$ tends to zero as $\beta \to 0$.

Put $v_i = \mathbb{G}_\mu^\Omega[\tau_i]$. Then, for $0 < \beta < \beta_1/2$,

$$\int_{\Sigma_\beta} v_1\, dS_x \le c_3 \beta^{\alpha_+} \beta_1^{2\alpha_- - N} \int_\Omega \delta^{\alpha_+}(y) d\tau_1(y).$$

Thus,

$$\lim_{\beta\to 0} \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} v_1\, dS_x = 0. \tag{2.25}$$

On the other hand, by (2.23) and (2.24),

$$\frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} v_2\, dS_x \le c\epsilon \quad \forall \beta < \beta_0. \tag{2.26}$$

This implies that $\mathrm{tr}^*(v) = 0$. $\square$

It is well-known that u is an L_μ-potential if and only if there exists a positive measure τ in Ω such that $u = \mathbb{G}_\mu^\Omega[\tau]$ (see e.g. [1, Theorem 12]). The estimate (2.6) implies that if $\mathbb{G}_\mu^\Omega[\tau] \not\equiv \infty$ then $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$. Therefore as a consequence of the previous proposition:

Corollary 2.13. *A positive L_μ-superharmonic function u is a potential if and only if* $\mathrm{tr}^*(u) = 0$.

Remark. Let $D \Subset \Omega$ be a C^2 domain and denote by G_μ^D and P_μ^D the Green and Poisson kernels of L_μ in D. (To avoid misunderstanding we point out that, in the formula defining L_μ, $\delta(x)$ denotes, as before, the distance from x to $\partial\Omega$, not to ∂D.) As every positive L_μ harmonic function has measure boundary trace zero, there is no Poisson kernel for L_μ in Ω. However, L_μ has a Poisson kernel in every C^2 domain D strictly contained in Ω. This follows from the fact that the Green kernel G_μ^D exists and behaves like G_0^D.

Proposition 2.14. *Let w be a non-negative L_μ-subharmonic function. If w is dominated by an L_μ-superharmonic function then $L_\mu w \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega)$ and w has a normalized boundary trace $\nu \in \mathfrak{M}^+(\partial\Omega)$. If, in addition,* $\mathrm{tr}^*(w) = 0$ *then* $w = 0$.

Proof. The first assumption implies that there exists a positive Radon measure λ in Ω such that $-L_\mu w = -\lambda$.

First assume that $\lambda \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$. Then $v := w + \mathbb{G}_\mu^\Omega[\lambda]$ is a non-negative L_μ-harmonic function and consequently, by the representation theorem, $v = \mathbb{K}_\mu^\Omega[\nu]$ for some $\nu \in \mathfrak{M}^+(\partial\Omega)$. By Proposition 2.12, $\mathrm{tr}^*(w) = \nu$. If $\nu = 0$ then $v = 0$ and therefore $w = 0$. Now let us drop the assumption on λ.

Let v_β be the unique solution of the boundary value problem,

$$-L_\mu v_\beta = -\lambda_\beta \;\text{ in }\; D_\beta, \quad v_\beta = h_\beta \;\text{ on }\; \partial D_\beta$$

where λ_β is the restriction of λ to D_β and h_β is the restriction of w to ∂D_β. (The uniqueness follows from [4, Lemma 2.3].) The uniqueness implies that $v_\beta = w\lfloor_{D_\beta}$. By assumption there exists a positive L_μ-superharmonic function, say V, such that $w \le V$. Hence

$$w + \mathbb{G}_\mu^{D_\beta}[\lambda_\beta] = \mathbb{P}_\mu^{D_\beta}[h_\beta] \le \mathbb{P}_\mu^{D_\beta}[V\lfloor_{\partial D_\beta}] \le V.$$

This implies that $\mathbb{G}_\mu^\Omega[\lambda] = \lim_{\beta\to 0} \mathbb{G}_\mu^{D_\beta}[\lambda_\beta] < \infty$. For fixed $x \in \Omega$, $G_\mu^\Omega(x, y) \sim \delta(y)^{\alpha_+}$. Therefore the finiteness of $\mathbb{G}_\mu^\Omega[\lambda]$ implies that $\lambda \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$. By the first part of the proof w has a normalized trace. $\square$

Remark. See Proposition 2.20 below for a complementary result.

2.4. Test functions

Denote

$$X(\Omega)=\{\zeta\in C^2(\Omega):\delta^{\alpha_-}L_\mu\zeta\in L^\infty(\Omega),\,\delta^{-\alpha_+}\zeta\in L^\infty(\Omega)\}.$$

Proposition 2.15. *For any $\zeta\in X(\Omega)$, $\delta^{\alpha_-}|\nabla\zeta|\in L^\infty(\Omega)$.*

Proof. Let $\zeta\in X(\Omega)$ then there exist a positive constant c_1 and a function $f\in L^\infty(\Omega)$ such that $|\zeta|\le c_1\delta^{\alpha_+}$ and

$$-L_\mu\zeta=\delta^{-\alpha_-}f.$$

Take arbitrary point $x_*\in\Omega_{\beta_0}$ and put $d_*=\frac{1}{2}\delta(x_*)$, $y_*=\frac{1}{d_*}x_*$, $\zeta_*(y)=\zeta(d_*y)$ for $y\in\frac{1}{d_*}\Omega_{d_*}$. Note that if $x\in B_{d_*}(x_*)$ then $y=\frac{1}{d_*}x\in B_1(y_*)$ and $1\le\operatorname{dist}(y,\partial(\frac{1}{d_*}\Omega_{d_*}))\le 3$. In $B_1(y_*)$,

$$-\Delta\zeta_*-\frac{\mu}{\operatorname{dist}(y,\partial(\frac{1}{d_*}\Omega_{d_*}))^2}\zeta_*=d_*^{2-\alpha_-}\operatorname{dist}(y,\partial(\frac{1}{d_*}\Omega_{d_*}))^{-\alpha_-}f(d_*y).$$

By local estimate for elliptic equations [12, Theorem 8.32], there exists a positive constant $c_2=c_2(N,\mu)$ such that

$$\max_{B_{\frac{1}{2}}(y_*)}|\nabla\zeta_*|\le c_2[\max_{B_1(y_*)}|\zeta_*|+\max_{B_1(y_*)}(d_*^{2-\alpha_-}\operatorname{dist}(y,\partial(\frac{1}{d_*}\Omega_{d_*}))^{-\alpha_-}|f(d_*y)|].$$

This implies

$$d_*|\nabla\zeta(x_*)|\le c_3(\delta(x_*)^{\alpha_+}+\|f\|_{L^\infty(\Omega)}\delta(x_*)^{2-\alpha_-}),$$

where $c_3=c_3(N,\mu,c_1)$. Therefore

$$|\nabla\zeta(x)|\le c_4\delta(x)^{\alpha_+-1}\quad\forall x\in\Omega_{\beta_0}$$

where $c_4=c_4(N,\mu,c_1,\|f\|_{L^\infty(\Omega)})$. Thus $\delta^{-\alpha_-}|\nabla\zeta|\in L^\infty(\Omega)$. □

Definition 2.16. Let $x_0\in\Omega$ and denote $\tilde\beta(x_0)=\min(\beta_0,\frac{1}{2}\delta(x_0))$. We say that $\tilde G_\mu^\Omega$ is a *proper regularization* of G_μ^Ω relative to x_0 if $\tilde G_\mu^\Omega(x)=G_\mu^\Omega(x_0,x)$ for $x\in\overline{\Omega}_{\tilde\beta(x_0)}$, $\tilde G_\mu^\Omega\in C^2(\Omega)\cap C(\overline{\Omega})$ and $\tilde G_\mu^\Omega\ge 0$ in Ω. Similarly $\tilde\delta$ is a *proper regularization* of δ relative to x_0 if $\tilde\delta(x)=\delta(x)$ for $x\in\overline{\Omega}_{\tilde\beta(x_0)}$, $\tilde\delta\in C^2(\overline{\Omega})$ and $\tilde\delta\ge 0$ in Ω.

Remark. Using (2.6) and (2.4), it is easily verified that the functions $\varphi_{\mu,1}$, $\mathbb{G}_\mu^\Omega[\eta]$ (for $\eta\in L^\infty(\Omega)$), $\tilde G_\mu^\Omega$ and $\tilde\delta^{\alpha_+}$ belong to $X(\Omega)$. Moreover, using Proposition 2.15, one obtains,

$$\zeta\in X(\Omega)\quad\text{and}\quad h\in C^2(\bar\Omega)\Longrightarrow h\zeta\in X(\Omega).$$

In the proofs of the next two propositions we use the following construction. Let $D\Subset\Omega$ be a C^2 domain. The Green function for $-L_\mu$ in D is denoted by G_μ^D. (To avoid misunderstanding we point out that, in the formula defining L_μ, $\delta(x)$ denotes, as before, the distance from x to $\partial\Omega$, not to ∂D.) Given $x_0\in\Omega$ we construct a family of functions $\mathcal{G}(x_0)=\{\tilde G_\mu^{D_\beta}:0<\beta<\frac{1}{2}\tilde\beta(x_0)\}$ such that, for each β, $\tilde G_\mu^{D_\beta}$ is a proper regularization of $G_\mu^{D_\beta}(x_0,\cdot)$ in D_β and $\mathcal{G}(x_0)$ has the following properties:

- For every $\beta\in(0,\frac{1}{2}\tilde\beta(x_0))$, $\tilde G_\mu^{D_\beta}\in C^2(\overline{D}_\beta)$, $\tilde G_\mu^{D_\beta}\ge 0$ and $\tilde G_\mu^{D_\beta}(x)=G_\mu^{D_\beta}(x_0,x)$ for $x\in D_\beta\setminus D_{\tilde\beta(x_0)}$.
- The sequences $\{\tilde G_\mu^{D_\beta}\}$ and $\{L_\mu\tilde G_\mu^{D_\beta}\}$ converge to $\tilde G_\mu^\Omega$ and $L_\mu\tilde G_\mu^\Omega$ respectively, as $\beta\to 0$, a.e. in Ω.
- $\left\|\tilde G_\mu^{D_\beta}+|L_\mu\tilde G_\mu^{D_\beta}|\right\|_{L^\infty(D_\beta)}\le M_{x_0}$ where M_{x_0} is a positive constant independent of β.

$\mathcal{G}(x_0)$ will be called a *uniform regularization* of $\{G_\mu^{D_\beta}\}$.

For any function $h\in C^2(\partial\Omega)$, we say that $\tilde h$ is an *admissible extension* of h relative to x_0 in $\overline{\Omega}$ if $\tilde h(x)=h(\sigma(x))$ for $x\in\Omega_{\tilde\beta(x_0)}$ and $\tilde h\in C^2(\overline{\Omega})$.

2.5. Nonhomogeneous linear equations

Here we discuss the boundary value problem (1.11) in Ω.

Lemma 2.17. *Let $u \in L^1_{loc}(\Omega)$ be a positive solution (in the sense of distributions) of equation*

$$-L_\mu u = \tau \tag{2.27}$$

in Ω where τ is a non-negative Radon measure.

If $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$ then

$$-\int_\Omega \mathbb{G}_\mu^\Omega[\tau] L_\mu \zeta \, dx = \int_\Omega \zeta \, d\tau \quad \forall \zeta \in X(\Omega). \tag{2.28}$$

Proof. We may assume that τ is positive. By Proposition 2.12, $\mathrm{tr}^*(\mathbb{G}_\mu^\Omega[\tau]) = 0$. Therefore, given $\varepsilon > 0$, there exists $\bar\beta = \bar\beta(\varepsilon) < \frac{1}{2}\beta_0$ such that,

$$\frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau] \, dS_x < \varepsilon \quad \text{and} \quad \int_{\Omega_\beta} \delta^{\alpha_+} d\tau < \varepsilon \quad \forall \beta \in (0, \bar\beta]. \tag{2.29}$$

Let

$$I(\beta) := \int_{D_\beta} \mathbb{G}_\mu^\Omega[\tau] L_\mu \zeta \, dx + \int_{D_\beta} \zeta \, d\tau.$$

To prove (2.28) we show that

$$\lim_{\beta \to 0} I(\beta) = 0. \tag{2.30}$$

Put

$$\tau_1 := \chi_{\bar D_{\bar\beta}} \tau, \quad \tau_2 := \chi_{\Omega_{\bar\beta}} \tau$$

and, for $0 < \beta < \bar\beta$,

$$I_k(\beta) := \int_{D_\beta} \mathbb{G}_\mu^\Omega[\tau_k] L_\mu \zeta \, dx + \int_{D_\beta} \zeta \, d\tau_k, \quad k = 1, 2.$$

As $|\zeta| \le c\delta^{\alpha_+}$ and $|L_\mu \zeta| \le \frac{c}{\delta^{\alpha_-}}$, (2.29) implies,

$$|I_2(\beta)| \le c\varepsilon \quad \forall \beta \in (0, \bar\beta). \tag{2.31}$$

For every $\beta \in (0, \bar\beta)$,

$$-\int_{D_\beta} \mathbb{G}_\mu^\Omega[\tau_1] L_\mu \zeta \, dx = \int_{D_\beta} \zeta \, d\tau_1 + \int_{\Sigma_\beta} \frac{\partial \mathbb{G}_\mu^\Omega[\tau_1]}{\partial \mathbf{n}} \zeta \, dS_x - \int_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau_1] \frac{\partial \zeta}{\partial \mathbf{n}} dS_x.$$

Thus

$$I_1(\beta) = -\int_{\Sigma_\beta} \frac{\partial \mathbb{G}_\mu^\Omega[\tau_1]}{\partial \mathbf{n}} \zeta \, dS_x + \int_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau_1] \frac{\partial \zeta}{\partial \mathbf{n}} dS_x =: I_{1,1}(\beta) + I_{1,2}(\beta).$$

By Proposition 2.15 and (2.29),

$$|I_{1,2}(\beta)| \le c\varepsilon \quad \forall \beta \in (0, \bar\beta). \tag{2.32}$$

Next we estimate $I_{1,1}(\beta)$ for $\beta \in (0, \bar{\beta}/2)$. By Fubini,

$$I_{1,1}(\beta) = -\int_{\Sigma_\beta} \frac{\partial}{\partial \mathbf{n}_x} \int_{D_{\bar{\beta}}} G_\mu^\Omega(x,y) d\tau_1(y) \zeta(x) dS_x$$
$$= -\int_{D_{\bar{\beta}}} \int_{\Sigma_\beta} \frac{\partial G_\mu^\Omega(x,y)}{\partial \mathbf{n}_x} \zeta(x)\, dS_x d\tau_1(y).$$

For every $y \in D_{\bar{\beta}}$ the function $x \mapsto G_\mu^\Omega(x,y)$ is L_μ-harmonic in $\Omega_{\bar{\beta}}$. By local elliptic estimates, for every $\xi \in \Sigma_\beta$,

$$\sup_{x \in B_{\beta/4}(\xi)} |\nabla_x G_\mu^\Omega(x,y)| \le c\beta^{-1} \sup_{x \in B_{\beta/2}(\xi)} G_\mu^\Omega(x,y).$$

By Harnack's inequality,

$$\sup_{x \in B_{\beta/2}(\xi)} G_\mu^\Omega(x,y) \le c' \inf_{x \in B_{\beta/2}(\xi)} G_\mu^\Omega(x,y).$$

The constants c, c' are independent of $\beta \in (0, \bar{\beta}/2)$, $y \in D_{\bar{\beta}}$ and $\xi \in \Sigma_\beta$. Therefore we obtain,

$$|\nabla_x G_\mu^\Omega(x,y)| \le C\beta^{-1} G_\mu^\Omega(x,y) \quad \forall x \in \Sigma_\beta,\ \forall y \in D_{\bar{\beta}},\ \forall \beta \in (0, \bar{\beta}/2). \tag{2.33}$$

Hence,

$$|I_{1,1}(\beta)| \le C\beta^{-1} \int_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau_1] |\zeta|\, dS_x.$$

As $|\zeta(x)| \le c\delta(x)^{\alpha_+}$ it follows that,

$$|I_{1,1}(\beta)| \le C \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau_1] dS_x.$$

Therefore, by (2.29),

$$|I_{1,1}(\beta)| \le C'\varepsilon \quad \forall \beta \in (0, \bar{\beta}/2). \tag{2.34}$$

Finally (2.30) follows from (2.31), (2.32) and (2.34). □

Theorem 2.18. *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$. Then:*

(i) Problem (1.11) has a unique solution. The solution is given by

$$u = \mathbb{G}_\mu^\Omega[\tau] + \mathbb{K}_\mu^\Omega[\nu]. \tag{2.35}$$

(ii) There exists a positive constant $c = c(N, \mu, \Omega)$ such that

$$\|u\|_{L^1_{\delta^{-\alpha_-}}(\Omega)} \le c(\|\tau\|_{\mathfrak{M}_{\delta^{\alpha_+}}(\Omega)} + \|\nu\|_{\mathfrak{M}(\partial\Omega)}). \tag{2.36}$$

(iii) u is a solution of (1.11) if and only if $u \in L^1_{\delta^{-\alpha_-}}(\Omega)$ and

$$-\int_\Omega u L_\mu \zeta\, dx = \int_\Omega \zeta\, d\tau - \int_\Omega \mathbb{K}_\mu^\Omega[\nu] L_\mu \zeta\, dx \quad \forall \zeta \in X(\Omega). \tag{2.37}$$

Proof. (i) Proposition 2.12 implies that (2.35) is a solution of (1.11).

If u and u' are two solutions of (1.11) then $v := (u - u')_+$ is a nonnegative L_μ-subharmonic function such that $\operatorname{tr}^*(v) = 0$ and $v \le 2\mathbb{G}_\mu^\Omega[|\tau|]$ which is a positive L_μ-superharmonic function. By Proposition 2.14, $v \equiv 0$ and hence $u \le u'$ in Ω. Similarly $u' \le u$, so that $u = u'$.

(ii) In view of (2.14) and (2.15), (2.36) is an immediate consequence of (2.35).

(iii) Let u be the solution of (1.11). By (2.36), $u \in L^1_{\delta^{-\alpha_-}}(\Omega)$ and by Lemma 2.17 and (2.35), u satisfies (2.37).

Conversely, suppose that $u \in L^1_{\delta^{-\alpha_-}}(\Omega)$ and satisfies (2.37). We show that u is a solution of (1.11) or, equivalently, of (2.35).

By (2.37) with $\zeta \in C_c^\infty(\Omega)$, u is a solution (in the sense of distributions) of the equation in (1.11). It remains to show that $\mathrm{tr}^*(u) = \nu$. Put $U = u - \mathbb{G}_\mu^\Omega[\tau] - \mathbb{K}_\mu^\Omega[\nu]$ and note that, as $-L_\mu u = \tau$, U is L_μ-harmonic.

Let $z \in \Omega$ and let $\mathcal{G}(z)$ be a uniform regularization of $\{G_\mu^{D_\beta} : 0 < \beta < \frac{1}{2}\tilde{\beta}(z)\}$ (see Section 2.4). Then, for every $\beta \in (0, \frac{1}{2}\tilde{\beta}(z))$, $\tilde{G}_\mu^{D_\beta} \in C_0^2(\overline{D}_\beta)$. Recall that $\tilde{G}_\mu^{D_\beta}(x) = G_\mu^{D_\beta}(z,x)$. Therefore, as $\frac{\partial G_\mu^{D_\beta}(z,x)}{\partial \mathbf{n}_x} = P_\mu^{D_\beta}(z,x)$, $x \in \Sigma_\beta$, we obtain

$$- \int_{D_\beta} U(x) L_\mu \tilde{G}_\mu^{D_\beta}(x)dx = \int_{\Sigma_\beta} U(x) P_\mu^{D_\beta}(z,x) dS_x = U(z). \tag{2.38}$$

The second equality is a consequence of the fact that U is L_μ-harmonic. But $L_\mu \tilde{G}_\mu^{D_\beta}(x) \to L_\mu \tilde{G}_\mu^\Omega(z,x)$ pointwise and the sequence $\{L_\mu \tilde{G}_\mu^{D_\beta}\}$ is bounded by a constant M_z. We observe that $U \in L^1(\Omega)$; in fact by assumption $u \in L^1_{\delta^{-\alpha_-}}(\Omega)$ and therefore, by Proposition 2.8, $U \in L^1_{\delta^{-\alpha_-}}(\Omega)$. Consequently, by (2.38),

$$U(z) = - \int_\Omega U(x) L_\mu \tilde{G}_\mu^\Omega(z,x)dx.$$

Since $G_\mu^\Omega(z,\cdot) \in X(\Omega)$, by (2.37) the right hand side vanishes. Thus U vanishes in Ω, i.e., u satisfies (2.35). □

Corollary 2.19. *Let u be a positive L_μ superharmonic function. Then there exist $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega)$ such that (1.12) holds.*

Proof. By the Riesz decomposition theorem u can be written in the form $u = u_p + u_h$ where u_p is an L_μ-potential and u_h is a non-negative L_μ-harmonic function. Therefore there exists $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that $u_h = \mathbb{K}_\mu^\Omega[\nu]$. Since u_p is an L_μ-potential there exists a positive Radon measure τ such that $u_p = \mathbb{G}_\mu^\Omega[\tau]$ (see e.g. [1, Theorem 12]). This necessarily implies that $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$. □

Proposition 2.20. *Let w be a non-negative L_μ-subharmonic function. If w has a normalized boundary trace then it is dominated by an L_μ-harmonic function.*

Proof. There exist a positive Radon measure τ in Ω and a measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that

$$-L_\mu w = -\tau \quad \text{in } \Omega, \quad \mathrm{tr}^*(w) = \nu.$$

Let u_β be the solution of

$$-L_\mu u = -\tau_\beta \quad \text{in } D_\beta, \quad u = \mathbb{K}_\mu^\Omega[\nu] \quad \text{on } \Sigma_\beta$$

where $\tau_\beta := \tau \chi_{D_\beta}$. Then,

$$u_\beta + \mathbb{G}_\mu^{D_\beta}[\tau_\beta] = \mathbb{K}_\mu^\Omega[\nu].$$

Letting $\beta \to 0$ we obtain,

$$\mathbb{G}_\mu^\Omega[\tau] \le \mathbb{K}_\mu^\Omega[\nu].$$

Hence $\tau \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega)$ and consequently

$$w + \mathbb{G}_\mu^\Omega[\tau] = \mathbb{K}_\mu^\Omega[\nu]. \qquad \square \tag{2.39}$$

3. The nonlinear equation

In this section, we consider the nonlinear equation

$$-L_\mu u + u^q = 0 \tag{3.1}$$

in Ω with $0 < \mu < C_H(\Omega)$ and $q > 1$.

Proof of Theorem A. Since u is a positive solution of (1.1), u is L_μ-subharmonic. Assuming (i), u is dominated by an L_μ-harmonic function. Therefore, by Proposition 2.14, (i) $\Longrightarrow$ (ii) and $u \in L^q_{\delta^{\alpha_+}}(\Omega)$. On the other hand, by Proposition 2.20 (ii) $\Longrightarrow$ (i).

As mentioned above, (i) implies that $u \in L^q_{\delta^{\alpha_+}}(\Omega)$ and that there exists $\nu \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\partial\Omega)$ such that $\mathrm{tr}^*(u) = \nu$. Therefore, by Theorem 2.18, (1.14) is a consequence of (2.37). Thus (i) $\Longrightarrow$ (iii).

Finally, the implication (iii) $\Longrightarrow$ (i) is obvious.

It remains to prove the last assertion. If u is a positive solution of (1.13) then, by (iii), $u \in L^q_{\delta^{\alpha_+}}(\Omega)$ and (1.15) follows from Theorem 2.18.

Conversely, assume that $\delta^{\alpha_+}u^q$, $u/\delta^{\alpha_-} \in L^1(\Omega)$ and (1.15) holds. Then, by (1.15) with $\zeta \in C_c^\infty(\Omega)$, u is a solution of (1.1). Taking $\zeta_f = \mathbb{G}_\mu^\Omega[f]$ where $f \in C_c(\Omega)$ and $f \ge 0$ we obtain

$$\int_\Omega (\mathbb{K}_\mu^\Omega[\nu] - u) f \, dx = \int_\Omega u^q \zeta_f \, dx < \infty.$$

This implies $u \le \mathbb{K}_\mu^\Omega[\nu]$, i.e., u is L_μ-moderate. Therefore by (i), u is a solution of (1.13). □

Proof of Theorem B.

Uniqueness. Let u_1 and u_2 be two positive solutions of (1.13). Then $v := (u_1 - u_2)_+$ is a subsolution of (1.1) and therefore an L_μ-subharmonic function. Furthermore, by (iii) in Theorem A, $u_1, u_2 \in L^q_{\delta^{\alpha_+}}(\Omega)$ and $v \le \mathbb{G}_\mu^\Omega[u_1^q + u_2^q] =: \bar{v}$. Obviously $\bar{v}$ is L_μ superharmonic and $\mathrm{tr}^*(v) = 0$. Therefore, by Proposition 2.14, $v = 0$. Thus $u_1 \le u_2$ and similarly $u_2 \le u_1$.

Monotonicity. As before, $v := (u_1 - u_2)_+$ is L_μ-subharmonic and it is dominated by an L_μ-superharmonic function. Since $\nu_1 \le \nu_2$, $\mathrm{tr}^*(v) = 0$. Hence by Proposition 2.14, $v = 0$.

A-priori estimate. Suppose that u is a positive solution of (1.13). Then (1.15) with $\zeta = \mathbb{G}_\mu^\Omega[1]$ implies (1.16). (Recall that $\mathbb{G}_\mu^\Omega[1] \sim \delta^{\alpha_+}$.) □

For the proof of the next theorem we need

Lemma 3.1. *Let $D \Subset \Omega$ be a C^2 domain and $q > 1$. If h is a positive function in $L^1(\partial D)$ then there exists a unique solution of the boundary value problem,*

$$\begin{aligned} -L_\mu u + u^q &= 0 \quad \text{in } D \\ u &= h \quad \text{on } \partial D. \end{aligned} \tag{3.2}$$

Proof. First assume that h is bounded. Let P_μ^D denote the Poisson kernel of $-L_\mu$ in D and put $u_0 := \mathbb{P}_\mu^D[h]$. Thus u_0 is bounded. We show that there exists a non-increasing sequence of positive functions $\{u_n\}_1^\infty$, dominated by u_0, such that u_n is the solution of the boundary value problem,

$$\begin{aligned} -\Delta v + v^q &= \frac{\mu}{\delta^2} u_{n-1} \quad \text{in } D \\ v &= h \quad \text{on } \partial D \quad n = 1, 2, \dots \end{aligned} \tag{3.3}$$

As usual δ denotes the distance to $\partial\Omega$, not to ∂D. For $n = 1$, u_0 is a supersolution of the problem and, obviously $v = 0$ is a subsolution. Consequently there exists a unique solution u_1. By induction, for $n > 1$,

$$-\Delta u_{n-1} + u_{n-1}^q = \frac{\mu}{\delta^2} u_{n-2} \ge \frac{\mu}{\delta^2} u_{n-1}.$$

Thus $v = u_{n-1}$ is a supersolution of (3.3) and it is bounded. It follows that there exists $0 \le u_n \le u_{n-1}$ such that

$$-\Delta u_n + u_n^q = \frac{\mu}{\delta^2} u_{n-1} \text{ in } D, \quad u_n = h \text{ on } \partial D.$$

As the sequence is monotone we conclude that $u = \lim u_n$ is a solution of (3.2).

If $h \in L^1(\partial D)$, we approximate it by a monotone increasing sequence of non-negative bounded functions $\{h_k\}$. If v_k is the solution of (3.2) with h replaced by h_k then $\{v_k\}$ increases (by the comparison principle [4, Lemma 3.2]) and $v = \lim v_k$ is a solution of (3.2).

Uniqueness follows by the comparison principle. □

Proof of Theorem C. Put $u_0 := \mathbb{K}_\mu^\Omega[\nu]$ and $h_\beta := u_0 \lfloor_{\Sigma_\beta}$. Let u_β be the solution of (3.2) with h replaced by h_β, $\beta \in (0, \beta_0)$. Since u_0 is a supersolution of (1.1) it follows that $\{u_\beta\}$ decreases as $\beta \downarrow 0$. Therefore $u := \lim_{\beta \to 0} u_\beta$ is a solution of (1.1).

We claim that $\operatorname{tr}^*(u) = \nu$. Indeed,

$$u_\beta + \mathbb{G}_\mu^{D_\beta}[u_\beta^q] = \mathbb{P}_\mu^{D_\beta}[h_\beta] = u_0. \tag{3.4}$$

Furthermore, in D_β, $u_\beta \le u_0 \in L^q_{\delta^{\alpha_+}}(\Omega)$. Therefore

$$\mathbb{G}_\mu^{D_\beta}[u_\beta^q] \to \mathbb{G}_\mu^\Omega[u^q].$$

Hence, by (3.4),

$$u + \mathbb{G}_\mu^\Omega[u^q] = u_0 = \mathbb{K}_\mu^\Omega[\nu].$$

By Proposition 2.12, $\operatorname{tr}^*(u) = \nu$.

By Theorem B the solution is unique. □

Proof of Corollary C1. By the previous theorem, if $\nu = f$ where f is a positive bounded function then (1.13) has a solution. If $0 \le f \in L^1(\Omega)$ then it is the limit of an increasing sequence of such functions. Therefore, once again problem (1.13) with $\nu = f$ has a solution.

Proof of Theorem D. Put $v = \mathbb{K}_\mu^\Omega[\nu] - u$. By the comparison principle $v \ge 0$. Clearly v is L_μ-superharmonic in Ω and, by definition $\operatorname{tr}^*(v) = 0$. By Proposition I(iv) v is an L_μ potential. Consequently, by Theorem 2.6,

$$\lim_{x \to y} \frac{v(x)}{\mathbb{K}_\mu^\Omega[\nu]} = 0 \quad \text{non-tangentially, } \nu \text{ a.e. on } \partial\Omega.$$

This implies (1.17). □

Proof of Theorem E. By Proposition 2.8, specifically inequality (2.15), $\mathbb{K}_\mu^\Omega[\nu] \in L^q_{\delta^{\alpha_+}}(\Omega)$ for every $q \in (1, q_{\mu,c})$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$. Therefore the first assertion of the theorem is a consequence of Theorem C.

We turn to the proof of stability. Put $v_n = \mathbb{K}_\mu^\Omega[\nu_n]$. By Proposition 2.8, $\{v_n\}$ is bounded in $L^q_{\delta^{\alpha_+}}(\Omega)$ for every $q \in (1, q_{\mu,c})$ and in $L^p_{\delta^{-\alpha_-}}(\Omega)$ for every $p \in (1, \frac{N-\alpha_-}{N-1-\alpha_-})$. In addition $v_n \to v$ pointwise in Ω. This implies that $\{v_n^q \delta^{\alpha_+}\}$ and $\{v_n/\delta^{\alpha_-}\}$ are uniformly integrable in Ω. Since $u_{\nu_n} \le v_n$ it follows that this conclusion applies also to $\{u_{\nu_n}\}$.

By the extension of the Keller–Osserman inequality due to [4], the sequence $\{u_{\nu_n}\}$ is uniformly bounded in every compact subset of Ω. Therefore, by a standard argument, we can extract a subsequence, still denoted by $\{u_{\nu_n}\}$ that converges pointwise to a solution u of (1.1). In view of the uniform convergence mentioned above we conclude that

$$u_{\nu_n} \to u \quad \text{in } L^q_{\delta^{\alpha_+}}(\Omega) \text{ and in } L^1_{\delta^{-\alpha_-}}(\Omega).$$

By Theorem A,

$$u_{\nu_n} + \mathbb{G}_\mu^\Omega[u_{\nu_n}^q] = \mathbb{K}_\mu^\Omega[\nu_n].$$

In view of the previous observations, passing to the limit as $n \to \infty$, we obtain,

$$u + \mathbb{G}_\mu^\Omega[u^q] = \mathbb{K}_\mu^\Omega[\nu].$$

Again by Theorem A it follows that u is the (unique) solution of (1.13). Because of the uniqueness we conclude that the entire sequence $\{u_{\nu_n}\}$ (not just a subsequence) converges to u as stated in assertion II. of the theorem.

Finally we prove assertion III. By Theorem A

$$u_{k\delta_y} + \mathbb{G}_\mu^\Omega[u_{k\delta_y}^q] = kK_\mu^\Omega(\cdot, y). \tag{3.5}$$

Combining (2.7), (2.6) and the fact $u_{k\delta_y} \le kK_\mu^\Omega(\cdot, y)$, we obtain

$$\frac{\mathbb{G}_\mu^\Omega[u_{k\delta_y}^q](x)}{K_\mu^\Omega(x,y)} \le k^q \frac{\mathbb{G}_\mu^\Omega[(K_\mu^\Omega(., y)^q](x)}{K_\mu^\Omega(x,y)} \le ck^q |x-y|^{N+\alpha_+ - q(N-1-\alpha_-)}.$$

Since $1 < q < q_{\mu,c}$, it follows that

$$\lim_{x\to y} \frac{\mathbb{G}_\mu^\Omega[u_{k\delta_y}^q](x)}{K_\mu^\Omega(x,y)} = 0.$$

Therefore, by (3.5), we obtain (1.19). □

Proof of Theorem F. Let $y \in \partial\Omega$. By negation, assume that there exists a positive solution u of (1.13) with $\nu = k\delta_y$ for some $k > 0$. By Theorem A, $u \le k\mathbb{K}_\mu^\Omega(., y)$ and $u \in L^q_{\delta^{\alpha_+}}(\Omega)$. Let $\gamma \in (0, 1)$ and denote $C_\gamma(y) = \{x \in \Omega : \gamma|x-y| \le \delta(x)\}$. By Theorem D,

$$\lim_{x \in C_\gamma(y), x\to y} \frac{u(x)}{K_\mu^\Omega(x,y)} = k.$$

This implies that there exist positive numbers r_0, c such that

$$u(x) \ge cK_\mu^\Omega(x,y) \quad \forall x \in C_\gamma(y) \cap B_{r_0}(y). \tag{3.6}$$

By (2.7),

$$\begin{aligned} J_\gamma &:= \int_{C_\gamma(y)\cap B_{r_0}(y)} (K_\mu^\Omega(x,y))^q \delta(x)^{\alpha_+} dx \\ &\ge c' \int_{C_\gamma(y)\cap B_{r_0}(y)} \delta(x)^{\alpha_+(q+1)} |x-y|^{(2\alpha_- - N)q} dx \\ &\ge c'\gamma^{\alpha_+(q+1)} \int_{C_\gamma(y)\cap B_{r_0}(y)} |x-y|^{\alpha_+ - q(N-1-\alpha_-)} dx. \end{aligned}$$

Since $q \ge q_{\mu,c}$ the last integral is divergent. But (3.6) and the fact that $u \in L^q_{\delta^{\alpha_+}}(\Omega)$ imply that $J_\gamma < \infty$. We reached a contradiction. □

Conflict of interest statement

No conflict of interest.

Acknowledgements

This research was supported by the Israel Science Foundation founded by the Israel Academy of Sciences and Humanities, through grant 91/10. Part of this research was carried out, by the first author, during a visit at the Isaac Newton Institute, Cambridge as part of the FRB program. He wishes to thank the institute for providing a pleasant and stimulating atmosphere. The second author was also partially supported by a Technion fellowship.

The authors wish to thank Professor Pinchover for many useful discussions.

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