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Nonexistence of local solutions to semilinear partial differential inequalities ☆

Non existence de solutions locales des inégalités semi-linéaires aux dérivées partielles

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Abstract

We investigate nonexistence of local nonnegative solutions for some semilinear elliptic and parabolic inequalities with first order terms and singular coefficients. In contrast with previous investigations of the subject, no use of the maximum principle is made. Instead, a direct bootstrap argument is used, which relies on a suitable choice of the functions used to test the differential inequalities.

Résumé

Nous étudions la non existence de solutions locales non négatives pour quelques iné galités elliptiques et paraboliques avec termes du premier ordre et coefficients singuliers. Au lieu d e méthodes faisant appel au pricipe du maximum, on utilise une méthode directe de "bootstrap" basée sur un choix convenable des fonctions test.

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1. Introduction

In this paper we investigate *nonexistence of local* nonnegative *solutions* – namely, nonexistence of nonnegative solutions in any neighbourhood of the origin – for semilinear elliptic inequalities of the following type:

$$\begin{cases} -\Delta u \geqslant \lambda |x|^{-\mu}(x, \nabla u) + |x|^{-\alpha} u^q & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 3$ is a bounded smooth domain which contains the origin, $q > 1$ and λ, μ, α are real parameters; by $(\cdot, \cdot)$ we denote the scalar product in $\mathbb{R}^n$. We also investigate the related phenomenon of *instantaneous blow-up* of local (nonnegative) solutions to semilinear parabolic inequalities of the following types:

$$\begin{cases} u_t - \Delta u \geqslant \lambda |x|^{-\mu}(x, \nabla u) + |x|^{-\alpha} u^q & \text{in } Q := \Omega \times (0, T], \\ u \geqslant 0 & \text{in } Q. \end{cases} \tag{1.2}$$

Our motivation comes from paper [1], where nonexistence of local nonnegative solutions was investigated for the elliptic problem:

$$\begin{cases} -\Delta u \geqslant |x|^{-\beta} u^q & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega \end{cases} \tag{1.3}$$

for $q > 1$ and $\beta \geqslant 2$ (this is a particular case of problem (1.1) with $\lambda = 0$ and $\alpha = \beta$). Moreover, instantaneous blow-up of local nonnegative solutions was proved for the companion parabolic inequality (namely, for problem (1.2) with $\lambda = 0$ and $\alpha = \beta$). The investigation in [1] was motivated by the "failure" of the Implicit Function Theorem pointed out in [3] (related results are in [2,4,12]). On the other hand, elliptic equations with singular coefficients and/or solutions have been widely investigated with independent motivations, often of geometrical nature (e.g., see [6,8,11,13]). Also instantaneous blow-up phenomena are currently investigated (in particular, see [5]).

It seems worth investigating how the presence of *first order singular terms* affects the situation already known for problem (1.3). In this connection, let us first consider problem (1.1) with $\mu = 2$ and the equality sign, namely:

$$\begin{cases} -\Delta u = \lambda |x|^{-2}(x, \nabla u) + |x|^{-\alpha} u^q & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega \end{cases} \tag{1.4}$$

(we refer the reader to [21] for an exhaustive analysis of this case). Any classical solution of this problem in $\Omega \setminus \{0\}$ satisfies there the equation

$$-\operatorname{div}\big(|x|^{\lambda} \nabla u\big) = |x|^{\lambda - \alpha} u^q; \tag{1.5}$$

the latter is a prototype for a class of degenerate elliptic equations, arising in physical problems related with anisotropic continuous media (see [7]). It is worth observing that, if we introduce the new unknown function $v := |x|^{\lambda/2} u$, problem (1.4) reads:

$$\begin{cases} -\Delta v - c|x|^{-2} v = |x|^{-\alpha - \frac{q-1}{2}\lambda} v^q & \text{in } \Omega, \\ v \geqslant 0 & \text{in } \Omega, \end{cases} \tag{1.6}$$

where

$$c = c(n, \lambda) := -\frac{\lambda}{4}\big[\lambda + 2(n-2)\big]$$

(in particular, for $\lambda \in (2(2-n), 0)$ there holds $0 < c \leqslant c_0$, $c_0 := (n-2)^2/4$ denoting the best constant in the Hardy inequality). Thus the above change of unknown establishes a useful link between problem (1.4) and semilinear problems involving an inverse square potential (in particular, see [9,22,23]).

As a further motivation to investigate problem (1.1), observe that a natural generalization of (1.3) suggests to study nonexistence of nonnegative solutions to the inequality

$$-\Delta_g u \geqslant |x|^{-\beta} u^q \tag{1.7}$$

on a Riemannian manifold $(\mathcal{M}, g)$ of dimension $n \geqslant 3$ containing the origin (as usual, Δ_g denotes the Laplace–Beltrami operator on $(\mathcal{M}, g)$). Assume for simplicity the metric g to be conformally flat, i.e., $g = \phi^{4/(n-2)}\delta$, where δ denotes the Euclidean metric. Since in a system of local coordinates $(x_1, \dots, x_n) \equiv x$ at a point $p \in M$ there holds:

$$\Delta_g = \frac{1}{\sqrt{|g|}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x_j}\right)$$

(where $|g| := \det(g_{ij})$ and $(g^{ij}) := (g_{ij})^{-1}$), from (1.7) we obtain the problem

$$\begin{cases} -\Delta u - \frac{2}{\phi}(\nabla\phi, \nabla u) \geqslant |x|^{-\beta}[\phi(x)]^{\frac{4}{n-2}}u^q & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega. \end{cases} \tag{1.8}$$

Choosing $\phi(x) = |x|^{\lambda/2}$ we get:

$$\begin{cases} -\Delta u - \lambda|x|^{-2}(x, \nabla u) \geqslant |x|^{-\beta+\frac{2\lambda}{n-2}}u^q & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega \end{cases} \tag{1.9}$$

– namely, problem (1.1) with $\mu = 2$ and $\alpha = \beta - \frac{2\lambda}{n-2}$. We shall prove in the following that nonexistence of local nonnegative solutions to problem (1.9) occurs for any $\beta > 0$; this is at variance from the situation encountered in [1], where $\lambda = 0$ and nonexistence only occurs for $\alpha = \beta \geqslant 2$ (see Theorem 2.3(a), (b) below).

More generally, local nonexistence results are proved in Theorem 2.3 for any value of the parameter μ in the original problem (1.1). For $\mu \leqslant 2$ nonexistence is only proved if $\alpha \geqslant 2$ (see Theorem 2.3(a), (b)); in this case nonexistence of local solutions depends on the singularity of the source term as in [1]. On the other hand, if $\mu > \frac{2q}{q-1}$ nonexistence can be proved also for $\alpha < 0$ (see Theorem 2.3(c)); in this case the coefficient of the source term is regular in Ω, thus the nonexistence result depends on the singularity of the first order term. Let us mention that first order singular terms also influence existence of solutions of the Dirichlet problem for the equation corresponding to the inequality in (1.1) (see [16,17]).

It seems now natural to investigate, along with the elliptic problem (1.1), the parabolic problem (1.2). In particular, we study the parabolic counterpart of the elliptic problem (1.9), namely:

$$\begin{cases} u_t - \Delta u - \lambda|x|^{-2}(x, \nabla u) \geqslant |x|^{-\beta+\frac{2\lambda}{n-2}}u^q & \text{in } \Omega, \\ u \geqslant 0 & \text{in } \Omega. \end{cases} \tag{1.10}$$

In analogy with the situation outlined before for the elliptic case, we prove instantaneous blow-up of local nonnegative solutions to problem (1.10) for any $\beta > 0$ (for a suitable class of initial data; see Theorem 2.7(b) below), at variance from the situation encountered in [1]. More general instantaneous blow-up results are proved in Theorem 2.7 for the original problem (1.2).

Let us add some remarks concerning our method of proof. A typical nonexistence result in [1] for problem (1.3) is about the so-called *very weak solutions*; these are defined testing the differential inequality only against functions with compact support in Ω and vanishing in a neighbourhhod of the origin (see the analogous Definition 2.1 below). After proving a *removable singularity result* at the origin, such solutions can be extended to *weak solutions*; then by the maximum principle nonexistence follows in any neighbourhhod of the origin. A similar argument can be used in the parabolic case.

In fact, it is the behaviour of the solution near the isolated singularity, as allowed by the differential inequality itself, which determines local existence or nonexistence (for semilinear elliptic equations this has been known for a long time; in particular, see [10]). Instead of proving a removable singularity result, then using the maximum principle for the extended solutions, we take advantage of the local behaviour of solutions near the origin by a direct bootstrap argument, which relies on a proper choice of the test functions. This approach – which does not make use of the maximum principle – allows us to deal not only with problems (1.1)–(1.2), but also with hyperbolic inequalities (see [19]). Let us mention that the same approach was used elsewhere (e.g., see [14,18]; generalizations can be found in [15]).

2. Mathematical background and results

Solutions to problem (1.1) are meant in the following sense.

Definition 2.1. By a *solution* to problem (1.1) in Ω we mean any function $u \in H^1_{\mathrm{loc}}(\Omega \setminus \{0\}) \cap L^q_{\mathrm{loc}}(\Omega \setminus \{0\})$ such that:

(i) $u \geqslant 0$ almost everywhere in Ω;
(ii) for any test function $\zeta \in C_0^\infty(\Omega \setminus \{0\})$, $\zeta \geqslant 0$ there holds:

$$\int_\Omega (\nabla u, \nabla \zeta) + \lambda \int_\Omega u \operatorname{div}\bigl(|x|^{-\mu} x \zeta\bigr) \geqslant \int_\Omega |x|^{-\alpha} u^q \zeta. \tag{2.1}$$

Concerning solutions to problem (1.2), we make similarly the following definitions.

Definition 2.2. By a *solution* to problem (1.2) in $Q := \Omega \times (0, T]$ we mean any function $u \in C([0, T]; H^1_{\mathrm{loc}}(\Omega \setminus \{0\}) \cap L^q_{\mathrm{loc}}((\Omega \setminus \{0\}) \times (0, T))$ such that:

(i) $u \geqslant 0$ almost everywhere in Q;
(ii) for any test function $\zeta \in C^{\infty,1}_{x,t}((\Omega \setminus \{0\}) \times [0, T])$, $\zeta \geqslant 0$, $\zeta(\cdot, t) \in C_0^\infty(\Omega \setminus \{0\})$ $(t \in [0, T])$, $\zeta(\cdot, T) = 0$ there holds:

$$\iint_Q (\nabla u, \nabla \zeta) + \lambda \iint_Q u \operatorname{div}\bigl(|x|^{-\mu} x \zeta\bigr) \geqslant \iint_Q |x|^{-\alpha} u^q \zeta + \int_\Omega u(x, 0)\zeta(x, 0) + \iint_Q u \zeta_t. \tag{2.2}$$

Concerning the elliptic problem (1.1) the following nonexistence results will be proved.

Theorem 2.3. *Let either of the following assumptions be satisfied*:

(a) $\mu < 2$, $\alpha \geqslant 2$;
(b) $\mu = 2$ *and either* $\alpha > 2$, $\lambda \geqslant 2 - n$, *or* $\alpha = 2$, $\lambda > 2 - n$;
(c) $\mu > 2$, $\alpha > \mu + (2 - \mu)q$ *and* $\lambda > 0$.

Then the only solution to problem (1.1) *in any neighbourhood* $\Omega_1 \subseteq \Omega$ *containing the origin is trivial.*

Remark 2.4. If the equality sign holds in problem (1.1), $\mu \geqslant 2$, $\alpha > 2$ and $1 < q < \frac{n+2}{n-2}$, then Corollary 3.2 in [10] can be applied to obtain the same result of Theorem 2.3 above. In the same situation, if $\alpha = 2$ and $1 < q < \frac{n+2}{n-2}$, then Theorem 3.1 in [10] can be applied; thus the solution is bounded at the origin and by [20] any possible singularity at the origin is removable.

Concerning the removability of singularity at the origin of solutions to problem (1.1), the following result will be proved.

Proposition 2.5. *Let* u *be a solution to problem* (1.1) *in some neighbourhood* $\Omega_1 \subseteq \Omega$ *containing the origin.*
(i) *Let either assumption* (a) *or assumption* (b) *of Theorem* 2.3 *be satisfied. Then*

$$\int_{B_\eta} |x|^{-\alpha+2-n} u^q < \infty, \qquad \int_{B_\eta} |x|^{1-n} |\nabla u| < \infty, \tag{2.3}$$

where $B_\eta := \{x \in \mathbb{R}^n \mid |x| < \eta\}$.

(ii) *Let assumption* (c) *of Theorem* 2.3 *be satisfied. Then*

$$\int_{B_\eta} |x|^{-\alpha+\mu-n} u^q < \infty, \qquad \int_{B_\eta} |x|^{3-\mu-n} |\nabla u| < \infty. \tag{2.4}$$

Proposition 2.6. *Let* u *be any solution to problem* (1.1) *in some neighbourhood* $\Omega_1 \subseteq \Omega$ *containing the origin. Let either assumption of Theorem* 2.3 *be satisfied*; *if assumption* (c) *holds, assume* $\mu < n$. *Then*

(i) $|x|^{-\alpha} u^q \in L^1_{\mathrm{loc}}(\Omega)$, $|x|^{-1}|\nabla u| \in L^1_{\mathrm{loc}}(\Omega)$, $|x|^{1-\mu}|\nabla u| \in L^1_{\mathrm{loc}}(\Omega)$;
(ii) *for any test function* $\zeta \in C_0^\infty(\Omega)$, $\zeta \geqslant 0$ *there holds*:

$$\int_\Omega (\nabla u, \nabla \zeta) \geqslant \lambda \int_\Omega |x|^{-\mu} (x, \nabla u)\zeta + \int_\Omega |x|^{-\alpha} u^q \zeta. \tag{2.5}$$

In connection with the above proposition, let us notice that solutions to problem (1.1) defined in Definition 2.1 are analogous to the very weak solutions in [1]. In the same spirit, any function $u \in L^q_{\mathrm{loc}}(\Omega)$, $u \geqslant 0$ a.e. in Ω satisfying the properties (i)–(ii) of Proposition 2.6 can be defined to be a weak solution to problem (1.1) (see [1]).

Concerning the parabolic problem (1.2), the following instantaneous blow-up result will be proved.

Theorem 2.7. *Let either assumption of Theorem* 2.3 *be satisfied. Moreover, let*

$$\liminf_{x \to 0} |x|^{-\gamma} u(x, 0) > 0 \tag{2.6}$$

for some $\gamma < 0$. *Then the only solution to problem* (1.2) *in any cylinder* $\Omega_1 \times (0, \tau]$, $\Omega_1 \subseteq \Omega$ *containing the origin and* $\tau \in (0, T)$, *is trivial.*

Remark 2.8. With the exception of the case $\alpha = 2$, Theorem 2.7 still holds if we assume $\gamma < \frac{\alpha}{q-1}$ in (2.6) (see part (iii) of the proof of Theorem 2.7).

3. Elliptic inequalities: Proofs

The proof of Theorem 2.3 makes use of a proper choice of the test function ζ in inequality (2.1). For this purpose some preliminary remarks are needed.

Let $\Omega_1 \subseteq \Omega$ be any neighbourhood containing the origin, $0 < \varepsilon < \eta$, $\eta > 2\varepsilon$ so small that $A_{\varepsilon,\eta} := \{x \in \mathbb{R}^n \mid \varepsilon < |x| < \eta\} \subseteq \Omega_1 \setminus \{0\}$. For $r \in [\varepsilon, \eta]$ define $\phi_0(r) := r^\sigma - \eta^\sigma$, where $\sigma < 0$ will be fixed later. Define also

$$\phi_1(r) := \bar{\phi}(r/\varepsilon) \quad (r \in [\varepsilon, \eta]),$$

where $\bar{\phi} \in C^\infty([0, \eta/\varepsilon])$ is nondecreasing, such that

$$\bar{\phi}(s) := \begin{cases} 0 & \text{if } s \in (0, 1), \\ 1 & \text{if } s \in (2, \eta/\varepsilon). \end{cases}$$

Finally, set

$$\bar{\zeta}(r) := r^\rho \phi_0(r) \phi_1(r) \quad (r \in [\varepsilon, \eta]),$$

where ρ is a real parameter to be chosen later.

Some relevant properties of the function $\bar{\zeta}$ are the content of the following lemma.

Lemma 3.1. (i) *There holds*:

$$\bar{\zeta}(\varepsilon) = \bar{\zeta}(\eta) = 0; \qquad \frac{d\bar{\zeta}}{dr}(\varepsilon) \geqslant 0, \quad \frac{d\bar{\zeta}}{dr}(\eta) \leqslant 0.$$

(ii) *There exists a sequence* $\{\zeta_k\} \subseteq C_0^\infty(A_{\varepsilon,\eta})$, $\zeta_k \geqslant 0$ *for any* k, *such that* $\zeta_k \to \tilde{\zeta}$ *in* $W_0^{1,p}(A_{\varepsilon,\eta})$ $(p \in (1,\infty))$, *where*

$$\tilde{\zeta}(x) := \bar{\zeta}(|x|) \quad (x \in \bar{A}_{\varepsilon,\eta}). \tag{3.1}$$

Proof. Claim (i) follows immediately from the definition of $\bar{\zeta}$. Concerning (ii), observe that $\tilde{\zeta} \in C_0(\bar{A}_{\varepsilon,\eta}) \cap W^{1,p}(A_{\varepsilon,\eta})$, thus $\tilde{\zeta} \in W_0^{1,p}(A_{\varepsilon,\eta})$ for any $p \in (1,\infty)$; then the conclusion follows.

Proposition 3.2. *Let* u *be a solution to problem* (1.1) *in some neighbourhood* $\Omega_1 \subseteq \Omega$ *containing the origin. Then for any* $0 < \varepsilon < \eta$, η *sufficiently small there holds*:

$$\int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \leqslant -\int\limits_{A_{\varepsilon,\eta}} |x|^{-(n-1)} \frac{d\psi}{dr}(|x|) u, \tag{3.2}$$

where

$$\psi(r) := r^{n-1} \frac{d\bar{\zeta}}{dr}(r) - \lambda r^{n-\mu} \bar{\zeta}(r) \quad (r \in [\varepsilon, \eta]). \tag{3.3}$$

Proof. Set $\zeta = \zeta_k$ in inequality (2.1), with ζ_k as in Lemma 3.1(ii). Letting $k \to \infty$ we obtain

$$\int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \leqslant \int\limits_{A_{\varepsilon,\eta}} (\nabla u, \nabla \tilde{\zeta}) + \lambda \int\limits_{A_{\varepsilon,\eta}} u \operatorname{div}\bigl(|x|^{-\mu} x \zeta\bigr).$$

Due to Lemma 3.1(ii), we obtain

$$\int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \leqslant -\int\limits_{A_{\varepsilon,\eta}} \bigl\{\Delta\tilde{\zeta} - \lambda \operatorname{div}\bigl(|x|^{-\mu} x \zeta\bigr)\bigr\} u.$$

An elementary calculation shows that

$$\Delta\tilde{\zeta} - \lambda \operatorname{div}\bigl(|x|^{-\mu} x \zeta\bigr) = |x|^{-(n-1)} \frac{d\psi}{dr}(|x|);$$

hence the conclusion follows.

Let us observe that the function ψ defined in (3.3) reads:

$$\psi = \phi_1 \psi_1 + r^{n-1+\rho} \phi_0 \frac{d\phi_1}{dr}, \tag{3.4}$$

where

$$\psi_1 = \psi_1(r) := r^{n-1} \frac{d}{dr}\bigl[r^\rho \phi_0(r)\bigr] - \lambda r^{n-\mu+\rho} \phi_0(r) \quad (r \in [\varepsilon, \eta]).$$

The following technical lemma plays an important role in the sequel; its proof is postponed until the end of this section.

Lemma 3.3. *Let either of the following assumptions be satisfied*:

(i) $\mu < 2$, $\rho \leqslant 2-n$, $\sigma < 0$;
(ii) $\mu < 2$, $\rho \in (2-n, 0)$, $\rho \leqslant 4-n-\mu$, $\sigma = -\rho + 2 - n$;
(iii) $\mu = 2$, $\rho \leqslant 2-n \leqslant \lambda$, $\sigma < 0$;
(iv) $\mu = 2$, $\rho = \lambda > 2-n$, $\sigma = -\rho + 2 - n$;
(v) $\mu > 2$, $\rho \leqslant \mu - n$, $\lambda > 0$, $\sigma < 0$.

Then there exists $\eta_0 > 0$ (*depending on* $n, \lambda, \mu, \rho, \sigma$) *such that for any* $\eta < \eta_0$ *there holds*:

$$\frac{d\psi_1}{dr} \geqslant 0 \quad \text{in } (\varepsilon, \eta). \tag{3.5}$$

Proposition 3.4. *Let* u *be a solution to problem* (1.1) *in some neighbourhood* $\Omega_1 \subseteq \Omega$ *containing the origin. Let either assumption of Lemma* 3.3 *be satisfied and* $\eta < \eta_0$. *Then for any* $\varepsilon > 0$ *sufficiently small there holds*:

$$\int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \leqslant \bar{C}\varepsilon^{\theta} \tag{3.6}$$

for some constant $\bar{C} > 0$, *where*

$$\theta := n - \alpha + \rho + \sigma + (\alpha - 2)\frac{q}{q-1}. \tag{3.7}$$

Proof. (i) Let u be any solution to problem (1.1) in a neighbourhood $\Omega_1 \subseteq \Omega$ containing the origin. Due to the choice $\eta < \eta_0$ and to Lemma 3.3, from inequality (3.2) we obtain easily

$$\int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \leqslant -\int\limits_{A_{\varepsilon,\eta}} |x|^{-(n-1)} \chi(|x|) u, \tag{3.8}$$

where

$$\chi(r) := \frac{d\phi_1}{dr}\psi_1 + \frac{d}{dr}\left[r^{n-1+\rho}\phi_0 \frac{d\phi_1}{dr}\right] \quad (r \in [\varepsilon, \eta]). \tag{3.9}$$

Using Hölder inequality, the right-hand side of inequality (3.8) can be estimated as follows:

$$\left| \int\limits_{A_{\varepsilon,\eta}} |x|^{-(n-1)} \chi(|x|) u \right| \leqslant \left\{ \int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \right\}^{\frac{1}{q}} \left\{ \int\limits_{A_{\varepsilon,\eta}} |x|^{-(n-1)q'} \left[|x|^{-\alpha}\tilde{\zeta}(x)\right]^{-(q'-1)} \chi(|x|)^{q'} \right\}^{\frac{1}{q'}}, \tag{3.10}$$

where $q' := \frac{q}{q-1}$. From (3.8) and (3.10) we obtain

$$\int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \leqslant C \int\limits_{\varepsilon}^{2\varepsilon} \left[r^{n-\alpha-1}\bar{\zeta}(r)\right]^{-\frac{1}{q-1}} \chi(r)^{\frac{q}{q-1}}\, dr \tag{3.11}$$

(where $C > 0$ only depends on n), since $\chi \equiv 0$ in $(\varepsilon, 2\varepsilon)$ by definition of ϕ_1.

(ii) Let us now estimate the integral in the right-hand side of inequality (3.11). To this purpose, set $s := \frac{r}{\varepsilon} \in [1, 2]$. It is easily seen that

$$\bar{\zeta}(\varepsilon s) \geqslant c_1 \varepsilon^{\rho+\sigma} \bar{\phi}(s),$$
$$\chi(\varepsilon s) \leqslant c_2 \varepsilon^{n-3+\rho+\sigma} \left[\bar{\phi}'(s) + \bar{\phi}''(s)\right],$$

for some $c_1, c_2 > 0$ and any $s \in [1, 2]$; here use of the equalities

$$\phi_1(\varepsilon s) = \bar{\phi}(s); \qquad \frac{d\phi_1}{dr}(\varepsilon s) = \frac{\bar{\phi}'(s)}{\varepsilon}; \qquad \frac{d^2\phi_1}{dr^2}(\varepsilon s) = \frac{\bar{\phi}''(s)}{\varepsilon^2}$$

has been made. Moreover, choosing $\bar{\phi}(s) = \mathrm{O}((s-1)^\gamma)$ with $\gamma > \max\{2, \frac{q+1}{q-1}\}$ as $s \to 1^+$, there holds:

$$\int_1^2 \left[\frac{\bar{\phi}'(s)^q}{\bar{\phi}(s)}\right]^{\frac{1}{q-1}} < \infty, \qquad \int_1^2 \left[\frac{\bar{\phi}''(s)^q}{\bar{\phi}(s)}\right]^{\frac{1}{q-1}} < \infty.$$

It follows that

$$\int_\varepsilon^{2\varepsilon} \left[r^{n-\alpha-1}\bar{\zeta}(r)\right]^{-\frac{1}{q-1}} \chi(r)^{\frac{q}{q-1}}\, dr \leqslant \tilde{C}\varepsilon^\theta \tag{3.12}$$

for some $\tilde{C} > 0$, where

$$\theta := (n-3+\rho+\sigma)\frac{q}{q-1} - (n-\alpha-1+\rho+\sigma)\frac{1}{q-1} + 1 = n-\alpha+\rho+\sigma+(\alpha-2)\frac{q}{q-1}.$$

Then by inequalities (3.11)–(3.12) the conclusion follows.

Now we can prove Theorem 2.3.

Proof of Theorem 2.3. Assume that a nontrivial solution to problem (1.1) exists in some neighbourhood $\Omega_1 \subseteq \Omega$ of the origin. The result will follow, if we prove that under the present hypotheses: (α) either assumption of Lemma 3.3 is satisfied, thus inequality (3.6) holds; (β) from inequality (3.6) a contradiction with the assumption $u \neq 0$ follows.

(i) Let assumption (a) be satisfied with $\alpha > 2$. We claim that in this case the parameters ρ, σ of the test function $\tilde{\zeta}$ can be chosen so that assumption (i) of Lemma 3.3 is satisfied and moreover $\theta > 0$. If so, use of Proposition 3.4 can be made; hence by inequality (3.6) for any $\eta < \eta_0$ there holds:

$$\int_{B_\eta} |x|^{-\alpha} u^q \tilde{\zeta}(x) = \lim_{\varepsilon \to 0^+} \int_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) = 0.$$

Then the conclusion follows.

To prove the above claim, observe that the requirement $\theta > 0$ reads:

$$\rho + \sigma > \alpha - n - (\alpha-2)\frac{q}{q-1}$$

(see (3.7)). On the other hand, choosing ρ, σ as in Lemma 3.3(i) gives

$$\rho + \sigma < 2 - n.$$

The above inequalities are compatible since $\alpha > 2$, thus the claim follows. This completes the proof in the present case.

(ii) Let assumption (a) be satisfied with $\alpha = 2$. In this case choosing the parameters ρ, σ as in Lemma 3.3(i) would give $\theta < 0$; hence a more convenient choice is that of Lemma 3.3(ii), which gives $\theta = 0$. Now inequality (3.6) reads

$$\int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \leqslant \bar{C}. \tag{3.13}$$

On the other hand, from inequalities (3.10) and (3.12) (with $\theta = 0$) we obtain:

$$\left| \int\limits_{A_{\varepsilon,\eta}} |x|^{-(n-1)} \chi(|x|) u \right| = \left| \int\limits_{A_{\varepsilon,2\varepsilon}} |x|^{-(n-1)} \chi(|x|) u \right| \leqslant \tilde{C}^{\frac{1}{q'}} \left\{ \int\limits_{A_{\varepsilon,2\varepsilon}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \right\}^{\frac{1}{q}} \tag{3.14}$$

(recall that $\chi \equiv 0$ in $(\varepsilon, 2\varepsilon)$).

Now observe that the left-hand side of inequality (3.13) monotonically increases as $\varepsilon \to 0^+$ (in fact, $\bar{\zeta}(r) = r^\rho \phi_0(r) \bar{\phi}(\frac{r}{\varepsilon})$ is a decreasing function of ε since $\bar{\phi}' \geqslant 0$). Then by monotone convergence and inequality (3.13) there holds:

$$\lim_{\varepsilon \to 0^+} \int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \leqslant \bar{C}.$$

This implies:

$$\lim_{\varepsilon \to 0^+} \int\limits_{A_{\varepsilon,2\varepsilon}} |x|^{-\alpha} u^q \tilde{\zeta}(x) = 0;$$

hence from inequality (3.14) the conclusion follows in this case, too.

(iii) Let assumption (b) be satisfied. Since $\mu = 2$, we can use Lemma 3.3(iii), (iv) in the present case. We choose the parameters ρ, σ of the test function $\tilde{\zeta}$ as in Lemma 3.3(iii) if $\alpha > 2$, or as in Lemma 3.3(iv) if $\alpha = 2$. Arguing as in (i)–(ii) above the conclusion follows.

(iv) Let assumption (c) be satisfied. In this case use can be made of Lemma 3.3(v). As in part (i) above, requiring $\theta > 0$ gives the compatibility condition

$$\mu - n > \alpha - n - (\alpha - 2)\frac{q}{q-1},$$

which can be satisfied since $\alpha > \mu + (2 - \mu)q$; then the conclusion follows. This completes the proof.

The main point in the proof of Theorem 2.3 was showing that the assumptions of Lemma 3.3 concerning the parameters ρ, σ could be satisfied, so as to make use of inequality (3.6). The same argument gives the proof of Proposition 2.5.

Proof of Proposition 2.5. (a) Suppose first $\mu = 2$; in this case the differential inequality in (1.1) can be rewritten as follows:

$$-|x|^{-\lambda} \operatorname{div}\{|x|^\lambda \nabla u\} \geqslant |x|^{-\alpha} u^q, \tag{3.15}$$

hence

$$\int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \leqslant \int\limits_{A_{\varepsilon,\eta}} |x|^\lambda \nabla u \nabla\big[|x|^{-\lambda} \tilde{\zeta}(x)\big]. \tag{3.16}$$

Due to the definition of $\tilde{\zeta}$, it is easily seen that

$$\nabla\big[|x|^{-\lambda} \tilde{\zeta}(x)\big] \leqslant |x|^{-\lambda+\rho+\sigma-1};$$

choosing the parameters as in Lemma 3.3(iii), (iv) gives $\rho + \sigma \leqslant 2 - n$, whence the second inequality in (2.3) follows. The proof of the first one is similar, due to inequality (3.16). This completes the proof of (2.3) when $\mu = 2$.

(b) Suppose now $\mu \neq 2$; in this case the differential inequality in (1.1) reads:

$$-e^{-\frac{\lambda}{2-\mu}|x|^{2-\mu}} \operatorname{div}\left\{e^{\frac{\lambda}{2-\mu}|x|^{2-\mu}} \nabla u\right\} \geqslant |x|^{-\alpha} u^q, \tag{3.17}$$

hence

$$\int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q \tilde{\zeta}(x) \leqslant \int\limits_{A_{\varepsilon,\eta}} e^{\frac{\lambda}{2-\mu}|x|^{2-\mu}} \nabla u \nabla\left[e^{-\frac{\lambda}{2-\mu}|x|^{2-\mu}} \tilde{\zeta}(x)\right]. \tag{3.18}$$

It is easily seen that

$$\nabla\left[e^{-\frac{\lambda}{2-\mu}|x|^{2-\mu}} \tilde{\zeta}(x)\right] \leqslant e^{-\frac{\lambda}{2-\mu}|x|^{2-\mu}} \max\left\{|x|^{\rho+\sigma-1}, |x|^{\rho+\sigma+1-\mu}\right\}.$$

If $\mu < 2$, choosing the parameters as in Lemma 3.3(i), (ii) gives $\rho + \sigma \leqslant 2 - n$, whence the second inequality in (2.3) follows. The proof of the first one is similar, due to inequality (3.18). On the other hand, if $\mu > 2$ the choice of Lemma 3.3(v) gives $\rho + \sigma \leqslant \mu - n$, whence the second inequality in (2.4) follows. Again, the first one follows similarly by inequality (3.18). This completes the proof.

Proof of Proposition 2.6. Claim (i) follows easily from inequalities (2.3)–(2.4) under the present assumptions. Concerning (ii), for any test function $\zeta \in C_0^\infty(\Omega)$, $\zeta \geqslant 0$ set

$$\zeta_\varepsilon(x) := \zeta(x)\bar{\phi}\left(\frac{|x|}{\varepsilon}\right) \quad (\varepsilon > 0),$$

the function $\bar{\phi}$ being as above. Then $\zeta_\varepsilon \in C_0^\infty(\Omega \setminus \{0\})$, $\zeta_\varepsilon \geqslant 0$ and $\zeta_\varepsilon \to \zeta$ a.e. in Ω as $\varepsilon \to 0^+$. Moreover, by Definition 2.1 there holds:

$$\int\limits_\Omega (\nabla u, \nabla \zeta_\varepsilon) + \lambda \int\limits_\Omega u \operatorname{div}\left(|x|^{-\mu} x \zeta_\varepsilon\right) \geqslant \int\limits_\Omega |x|^{-\alpha} u^q \zeta_\varepsilon. \tag{3.19}$$

Due to (i), it is immediately seen that

$$\int\limits_\Omega |x|^{-\alpha} u^q \zeta_\varepsilon \to \int\limits_\Omega |x|^{-\alpha} u^q \zeta,$$

$$\int\limits_\Omega u \operatorname{div}\left(|x|^{-\mu} x \zeta_\varepsilon\right) \to \int\limits_\Omega u \operatorname{div}\left(|x|^{-\mu} x \zeta\right)$$

as $\varepsilon \to 0^+$. On the other hand,

$$\int\limits_\Omega (\nabla u, \nabla \zeta_\varepsilon) = \int\limits_\Omega (\nabla u, \nabla \zeta)\bar{\phi}\left(\frac{|x|}{\varepsilon}\right) + \frac{1}{\varepsilon} \int\limits_{A_{\varepsilon,2\varepsilon}} |x|^{-1}(x, \nabla u)\bar{\phi}'\left(\frac{|x|}{\varepsilon}\right)\zeta.$$

Since $|x|^{-1}|\nabla u| \in L^1_{\mathrm{loc}}(\Omega)$ by (i), there holds:

$$\frac{1}{\varepsilon}\left| \int\limits_{A_{\varepsilon,2\varepsilon}} |x|^{-1}(x, \nabla u)\bar{\phi}'\left(\frac{|x|}{\varepsilon}\right)\zeta \right| \leqslant C \int\limits_{A_{\varepsilon,2\varepsilon}} |x|^{-1}|\nabla u| \to 0$$

as $\varepsilon \to 0^+$. Letting $\varepsilon \to 0^+$ in inequality (3.19) the conclusion follows.

Let us finally prove Lemma 3.3.

Proof of Lemma 3.3. Observe first that

$$\frac{d\psi_1}{dr} = r^{n-3+\rho+\sigma}\,\psi_0\left(\frac{r}{\eta}\right),$$

where

$$\psi_0(s) := (\rho+\sigma)(n-2+\rho+\sigma) - \rho(n-2+\rho)s^{-\sigma} - \lambda\big[(n-\mu+\rho+\sigma) - (n-\mu+\rho)s^{-\sigma}\big]\eta^{2-\mu}s^{2-\mu} \quad (s\in[0,1]),$$

as an elementary calculation shows.

(i) Since $\rho \leqslant 2-n \leqslant 0$ and $\sigma < 0$, there holds:

$$(\rho+\sigma)(n-2+\rho+\sigma) - \rho(n-2+\rho)s^{-\sigma} \geqslant (\rho+\sigma)(n-2+\rho+\sigma) - \rho(n-2+\rho) > 0;$$

in fact, it is easily seen that the function $f(s) := s(n-2+s)$ is strictly increasing in the interval $[\rho+\sigma, \rho]$. Since by assumption $2-\mu > 0$, choosing η sufficiently small proves the claim.

(ii) In this case $n-2+\rho+\sigma = 0$ and $-\rho(n-2+\rho) > 0$; since $2-\mu+\sigma = 4-n-\mu-\rho \geqslant 0$, the claim follows as in (i).

(iii) In this case

$$\psi_0(s) := (\rho+\sigma-\lambda)(n-2+\rho+\sigma) - (\rho-\lambda)(n-2+\rho)s^{-\sigma}.$$

A slight change of the argument used in (i) proves the claim.

(iv) In this case $\psi_0 \equiv 0$ in the interval $[0,1]$.

(v) Rewrite ψ_0 as follows:

$$\psi_0(s) = \eta^{2-\mu}s^{2-\mu}\big\{-\lambda\big[(n-\mu+\rho+\sigma) - (n-\mu+\rho)s^{-\sigma}\big] + \big[(\rho+\sigma)(n-2+\rho+\sigma) - \rho(n-2+\rho)s^{-\sigma}\big]\eta^{\mu-2}s^{\mu-2}\big\} \quad (s\in[0,1]).$$

Since $-(n-\mu+\rho) \geqslant 0$, $\lambda > 0$ and $\sigma < 0$, there holds:

$$-\lambda\big[(n-\mu+\rho+\sigma) - (n-\mu+\rho)s^{-\sigma}\big] \geqslant \lambda|\sigma| > 0.$$

Since $\mu - 2 > 0$ we can argue as in (i); hence the conclusion follows.

4. Parabolic inequalities: Proofs

As in the elliptic case, the proof of Theorem 2.7 relies on a proper choice of the test function in inequality (2.2). This gives the following result, analogous to Proposition 3.2.

Proposition 4.1. *Let u be a solution to problem* (1.2) *in some cylinder $\Omega_1\times(0,\tau]\subseteq Q$, Ω_1 containing the origin and $\tau\in(0,T)$. Then for any $0<\varepsilon<\eta$, η sufficiently small and any $\tau\in(0,T)$ there holds*:

$$\int_0^\tau (\tau-t)^\beta\,dt\int_{A_{\varepsilon,\eta}} |x|^{-\alpha}u^q(x,t)\tilde\zeta(x) \leqslant -\int_0^\tau (\tau-t)^\beta\,dt\int_{A_{\varepsilon,\eta}} |x|^{-(n-1)}\frac{d\psi}{dr}(|x|)u(x,t) + \beta\int_0^\tau (\tau-t)^{\beta-1}\,dt\int_{A_{\varepsilon,\eta}} u(x,t)\tilde\zeta(x) - \tau^\beta\int_{A_{\varepsilon,\eta}} u(x,0)\tilde\zeta(x), \tag{4.1}$$

where $\beta > \max\{1, \frac{1}{q-1}\}$ and $\tilde\zeta$ and ψ are the functions defined in (3.1), *respectively in* (3.3).

Proof. Let $\tau \in (0,T)$; set

$$\hat{\phi}(t) := \begin{cases} (\tau - t)^{\beta} & \text{if } t \in (0,\tau), \\ 0 & \text{if } t \in (\tau, T), \end{cases}$$

with $\beta > \max\{1, \frac{1}{q-1}\}$. Let $\{\zeta_k\} \subseteq C_0^{\infty}(A_{\varepsilon,\eta})$ be the approximating sequence in Lemma 3.1(ii); set $\zeta(x,t) = \zeta_k(x)\hat{\phi}(t)$ in inequality (2.2). Letting $k \to \infty$ we obtain easily:

$$\begin{aligned} &\int_0^{\tau} (\tau - t)^{\beta}\, dt \int_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q(x,t)\tilde{\zeta}(x) \\ &\quad \leqslant \int_0^{\tau} (\tau - t)^{\beta}\, dt \Big(\int_{A_{\varepsilon,\eta}} \big(\nabla u(x,t), \nabla\tilde{\zeta}\big) + \lambda \int_{A_{\varepsilon,\eta}} u(x,t)\, \mathrm{div}\big(|x|^{-\mu} x\zeta\big) \Big) \\ &\qquad + \beta \int_0^{\tau} (\tau - t)^{\beta-1}\, dt \int_{A_{\varepsilon,\eta}} u(x,t)\tilde{\zeta}(x) - \tau^{\beta} \int_{A_{\varepsilon,\eta}} u(x,0)\tilde{\zeta}(x). \end{aligned}$$

On the other hand, as in the proof of Proposition 3.2 for any $t \in (0,\tau)$ there holds:

$$\begin{aligned} &\int_{A_{\varepsilon,\eta}} \big(\nabla u(x,t), \nabla\tilde{\zeta}\big) + \lambda \int_{A_{\varepsilon,\eta}} u(x,t)\, \mathrm{div}\big(|x|^{-\mu} x\zeta\big) \\ &\quad \leqslant - \int_{A_{\varepsilon,\eta}} u(x,t)\big\{\Delta\tilde{\zeta} - \lambda\, \mathrm{div}\big(|x|^{-\mu} x\zeta\big)\big\} = - \int_{A_{\varepsilon,\eta}} |x|^{-(n-1)} \frac{d\psi}{dr}(|x|) u(x,t). \end{aligned}$$

Then from the above inequalities the conclusion follows.

Proposition 4.2. *Let u be a solution to problem* (1.2) *in some cylinder $\Omega_1 \times (0,\tau] \subseteq Q$, Ω_1 containing the origin and $\tau \in (0,T)$. Let the assumptions of Lemma* 3.3 *be satisfied and $\eta < \eta_0$ accordingly. Moreover, let*

$$\rho + \sigma > -\frac{\alpha}{q-1} - n. \tag{4.2}$$

Then for any $\varepsilon > 0$ sufficiently small and any $\tau \in (0,T)$ there holds:

$$\begin{aligned} &\int_0^{\tau} (\tau - t)^{\beta}\, dt \Big\{ \int_{A_{\varepsilon,\eta}} u(x,t)\tilde{\zeta}(x) \Big\}^q \\ &\quad \leqslant M\tau^{\beta} C_1(\varepsilon,\eta) \Big\{ C_1(\varepsilon,\eta)^{\frac{1}{q-1}} \tau^{-\frac{1}{q-1}} + C_2(\varepsilon,\eta)\tau - \int_{A_{\varepsilon,\eta}} u(x,0)\tilde{\zeta}(x) \Big\}, \end{aligned} \tag{4.3}$$

for some $M = M(\beta,q) > 0$, where

$$C_1(\varepsilon,\eta) := \Big\{ \int_{\varepsilon}^{\eta} r^{\frac{\alpha}{q-1}+n-1} \bar{\zeta}(r)\, dr \Big\}^{q-1}, \tag{4.4}$$

$$C_2(\varepsilon,\eta) := \int_{A_{\varepsilon,\eta}} |x|^{-(n-1)q'} \big[|x|^{-\alpha}\tilde{\zeta}(x)\big]^{-(q'-1)} \chi(|x|)^{q'}\, dx. \tag{4.5}$$

Proof. (i) Observe that for any $t \in (0,\tau)$

$$\int\limits_{A_{\varepsilon,\eta}} u(x,t)\tilde{\zeta}(x) \leqslant \left\{ \int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q(x,t)\tilde{\zeta}(x) \right\}^{\frac{1}{q}} \left\{ \int\limits_{A_{\varepsilon,\eta}} |x|^{\frac{\alpha}{q-1}} \tilde{\zeta}(x) \right\}^{\frac{1}{q'}}$$

$$= \left\{ \int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q(x,t)\tilde{\zeta}(x) \right\}^{\frac{1}{q}} \left\{ \int\limits_{\varepsilon}^{\eta} r^{\frac{\alpha}{q-1}+n-1} \bar{\zeta}(r)\, dr \right\}^{\frac{1}{q'}}.$$

Set

$$v(t) := \int\limits_{A_{\varepsilon,\eta}} u(x,t)\tilde{\zeta}(x) \quad (t \in (0,\tau));$$

then by definition (4.4) the above inequality reads:

$$v^q(t) \leqslant C_1(\varepsilon,\eta) \int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q(x,t)\tilde{\zeta}(x) \tag{4.6}$$

for any $t \in (0,\tau)$.

(ii) Due to Lemma 3.3 and the choice $\eta < \eta_0$, for any $t \in (0,\tau)$ we have:

$$-\int\limits_{A_{\varepsilon,\eta}} |x|^{-(n-1)} \frac{d\psi}{dr}(|x|) u(x,t) \leqslant -\int\limits_{A_{\varepsilon,\eta}} |x|^{-(n-1)} \chi(|x|) u(x,t),$$

where χ is the function defined in (3.9). Using inequality (3.10) and Young inequality we obtain:

$$\left| \int\limits_{A_{\varepsilon,\eta}} |x|^{-(n-1)} \chi(|x|) u(x,t) \right| \leqslant \frac{1}{q} \left| \int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q(x,t)\tilde{\zeta}(x) \right| + \frac{1}{q'} C_2(\varepsilon,\eta) \tag{4.7}$$

for any $t \in (0,\tau)$ (see definition (4.5)). Then from the above inequality and (4.1) we get easily:

$$\int\limits_0^{\tau} (\tau - t)^{\beta}\, dt \int\limits_{A_{\varepsilon,\eta}} |x|^{-\alpha} u^q(x,t)\tilde{\zeta}(x)$$

$$\leqslant \frac{\tau^{\beta+1}}{\beta+1} C_2(\varepsilon,\eta) + \beta q' \int\limits_0^{\tau} (\tau - t)^{\beta-1}\, dt \int\limits_{A_{\varepsilon,\eta}} u(x,t)\tilde{\zeta}(x) - q'\tau^{\beta} \int\limits_{A_{\varepsilon,\eta}} u(x,0)\tilde{\zeta}(x) \quad (t \in (0,\tau)). \tag{4.8}$$

Hence by inequality (4.6) and the definition of v there holds:

$$\int\limits_0^{\tau} (\tau - t)^{\beta} v^q(t)\, dt \leqslant C_1(\varepsilon,\eta) \left\{ \frac{\tau^{\beta+1}}{\beta+1} C_2(\varepsilon,\eta) + \beta q' \int\limits_0^{\tau} (\tau - t)^{\beta-1} v(t)\, dt - q'\tau^{\beta} v(0) \right\}. \tag{4.9}$$

(iii) Due to Young inequality, it is easily seen that:

$$\beta q' C_1(\varepsilon,\eta) \int\limits_0^{\tau} (\tau - t)^{\beta-1} v(t)\, dt \leqslant \frac{1}{q} \int\limits_0^{\tau} (\tau - t)^{\beta} v^q(t)\, dt + \frac{\beta^{q'}(q')^{q'-1}}{\beta - q' + 1} C_1^{q'}(\varepsilon,\eta) \tau^{\beta-q'+1}$$

(here use of the assumption $\beta > \frac{1}{q-1}$ has been made). From the above inequality and inequality (4.9) we obtain:

$$\left(1-\frac{1}{q}\right)\int_0^\tau (\tau-t)^\beta v^q(t)\,dt \leqslant \frac{\tau^{\beta+1}}{\beta+1}C_1(\varepsilon,\eta)C_2(\varepsilon,\eta) + \frac{\beta^{q'}(q')^{q'-1}}{\beta-q'+1}C_1^{q'}(\varepsilon,\eta)\tau^{\beta-q'+1} - q'C_1(\varepsilon,\eta)\tau^\beta v(0).$$

Then the conclusion follows.

Remark 4.3. Observe that assumption (4.2) of Proposition 4.2 is compatible with the assumptions made in Lemma 3.3(i), (iv) if $\alpha > -2(q-1)$, or respectively with those of Lemma 3.3(v) if $\alpha > -\mu(q-1)$.

Now we can prove Theorem 2.7.

Proof of Theorem 2.7. The idea of the proof is the same as for Theorem 2.3, making use of inequality (4.3) instead of inequality (3.6). Let us suppose that assumption (a) of Theorem 2.3 is satisfied, the proof being the same in the remaining cases (b)–(c).

(i) Observe first that, due to assumption (2.6), there exist $k > 0$ and $\eta_1 > 0$ such that for any $|x| < \eta < \eta_1$ there holds: $u(x,0) \geqslant k|x|^\gamma$. Hence

$$\int_{B_\eta} |x|^{\rho+\sigma}\left[1-\left(\frac{\eta}{|x|}\right)^\sigma\right]u(x,0)\,dx \geqslant k\int_0^\eta r^{\gamma+\rho+\sigma+n-1}\left[1-\left(\frac{\eta}{r}\right)^\sigma\right]dr.$$

If $\gamma \leqslant -2$, the integral in the right-hand side of the above inequality diverges. On the other hand, if $\gamma > -2$ we obtain:

$$\int_{B_\eta} |x|^{\rho+\sigma}\left[1-\left(\frac{\eta}{|x|}\right)^\sigma\right]u(x,0)\,dx \geqslant K\eta^{\gamma+\rho+\sigma+n}$$

for some $K > 0$.

(ii) Let us take the limit of inequality (4.3) as $\varepsilon \to 0^+$. To this purpose, observe that for any $t \in [0,\tau]$:

$$\lim_{\varepsilon\to 0^+}\int_{A_{\varepsilon,\eta}} u(x,t)\tilde{\zeta}(x) = \int_{B_\eta} |x|^{\rho+\sigma}\left[1-\left(\frac{\eta}{|x|}\right)^\sigma\right]u(x,t)$$

by monotonicity, due to the choice of the function $\tilde{\zeta}$.

Concerning the coefficient $C_1(\varepsilon,\eta)$ we have (see definition (4.4)):

$$C_1(\varepsilon,\eta) \leqslant \left\{\int_\varepsilon^\eta r^{\frac{\alpha}{q-1}+\rho+\sigma+n-1}\,dr\right\}^{q-1},$$

thus by monotonicity

$$\lim_{\varepsilon\to 0^+} C_1(\varepsilon,\eta) \leqslant L\eta^{\alpha+(\rho+\sigma+n)(q-1)}$$

for some $L > 0$, provided that condition (4.2) is satisfied.

Finally, from the proof of Proposition 3.4 we obtain (see (3.10), (3.12)):

$$C_2(\varepsilon,\eta) = C_2(\varepsilon,2\varepsilon) \leqslant \bar{C}\varepsilon^\theta$$

for some constant $\bar{C} > 0$, where

$$\theta := n - \alpha + \rho + \sigma + (\alpha - 2)\frac{q}{q-1}.$$

(iii) Let us first assume $\mu < 2, \alpha > 2$. As in part (i) of the proof of Theorem 2.3, in this case we can choose the parameters ρ, σ so that $\theta > 0$ and condition (4.2) is satisfied (see Remark 4.3). Taking the limit of inequality (4.3) as $\varepsilon \to 0^+$ gives (see (i)–(ii) above):

$$\int_0^\tau (\tau - t)^\beta \, dt \left\{ \int_{B_\eta} |x|^{\rho+\sigma} \left[1 - \left(\frac{\eta}{|x|} \right)^\sigma \right] u(x,t)\, dx \right\}^q$$
$$\leqslant M \tau^\beta \eta^{\frac{\alpha}{q-1}+\rho+\sigma+n} \left\{ \tau^{-\frac{1}{q-1}} - K \eta^{\gamma - \frac{\alpha}{q-1}} \right\} \tag{4.10}$$

for any $\tau \in (0, T)$, if $\gamma > -2$. In this case the right-hand side of the above inequality is negative for any $\tau > \tau_* = \tau_*(\eta) := K^{-(q-1)} \eta^{\alpha - \gamma(q-1)}$; since $\tau_*(\eta) \to 0^+$ as $\eta \to 0^+$, the conclusion follows in this case. On the other hand, if $\gamma \leqslant -2$ the right-hand side of inequality (4.3) tends to $-\infty$ as $\varepsilon \to 0^+$, thus a contradiction follows in this case, too. This proves the result in the case $\alpha > 2$.

(iv) Finally, let $\mu < 2, \alpha = 2$. As in the proof of Lemma 3.3(ii), (iv) we choose the parameters ρ, σ so that $\rho + \sigma + n = 2$, thus $\theta = 0$ (clearly, condition (4.2) is satisfied by this choice). Taking the limit of inequality (4.3) as $\varepsilon \to 0^+$ now gives:

$$\int_0^\tau (\tau - t)^\beta \, dt \left\{ \int_{B_\eta} |x|^{\rho+\sigma} \left[1 - \left(\frac{\eta}{|x|} \right)^\sigma \right] u(x,t)\, dx \right\}^q \leqslant M \tau^\beta \left\{ f(\eta, \tau) - K \eta^{\gamma+2} \right\}, \tag{4.11}$$

where

$$f(\eta, \tau) := \eta^{\frac{2q}{q-1}} \tau^{-\frac{1}{q-1}} + \bar{C}\tau.$$

It is easily seen that the function $f(\eta, \cdot)$ has a unique minimum $\tau_* = \tau_*(\eta) := [(q-1)\bar{C}]^{-\frac{q-1}{q}} \eta^2$ in $[0, T]$; moreover, $f(\eta, \tau_*) = q\bar{C}\tau_*$. Then by inequality (4.11) there holds:

$$\int_0^\tau (\tau - t)^\beta \, dt \left\{ \int_{B_\eta} |x|^{\rho+\sigma} \left[1 - \left(\frac{\eta}{|x|} \right)^\sigma \right] u(x,t) \right\}^q \leqslant M' \tau^{\beta+2} \{ \bar{C} - K \eta^\gamma \} \tag{4.12}$$

for some $M' > 0$. Since $\gamma < 0$ and $\tau_*(\eta) \to 0^+$ as $\eta \to 0^+$, the conclusion follows also in this case. This completes the proof.

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