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The preventive effect of the convection and of the diffusion in the blow-up phenomenon for parabolic equations

L'effet préventif de la convection et de la diffusion dans les phénomènes d'explosion pour des équations paraboliques

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Abstract

The goal of this paper is to investigate the role of the gradient term and of the diffusion coefficient in the preventing of the blow-up of the solution for semilinear and quasilinear parabolic problems.

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Résumé

Le but de cet article est de voir comment le terme de gradient et le coefficient de diffusion empêchent l'explosion des solutions des problèmes semilinéaires et quasilinéaires paraboliques.

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1. Introduction and main results

In the present paper we consider the following convective diffusion equations

$$u_t + \mathbf{a}(t,\mathbf{x}) \cdot \nabla u = \Delta u + \lambda u^p + f(t,\mathbf{x}), \tag{0.1}$$

$$u_t + a_i(t,\mathbf{x}) u_{x_i}^{q_i} = \Delta u + \lambda u^p + f(t,\mathbf{x}), \tag{0.2}$$

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and equation

$$u_t + a(t,\mathbf{x})|\nabla u|^q = \Delta u + \lambda u^p + f(t,\mathbf{x}) \tag{0.3}$$

in the domain $Q_T = \Omega \times (0,T)$, $\Omega \subset \mathbf{R}^n$ coupled with conditions

$$u(0,\mathbf{x}) = u_0(\mathbf{x}), \quad u(t,\mathbf{x})|_{S_T} = 0, \quad S_T = \partial\Omega \times (0,T). \tag{0.4}$$

Here λ and q are positive constants, q_i and p are positive integer, $a_i(t,\mathbf{x})$, $a(t,\mathbf{x})$ and $f(t,\mathbf{x})$ are given functions, $\mathbf{a}(t,\mathbf{x}) = (a_1(t,\mathbf{x}),\ldots,a_n(t,\mathbf{x}))$, $\mathbf{a}(t,\mathbf{x})\cdot\nabla u = a_i u_{x_i}$ and

$$a_i u_{x_i}^{q_i} = \sum_{i=1}^{n} a_i u_{x_i}^{q_i}.$$

Without loss of generality assume that domain Ω lies in the strip $|x_1| \leqslant l_1$. Suppose that

$$u_0(\mathbf{x})|_{\partial\Omega} = 0 \quad \text{and} \quad \max\left|u_{0x_1}(\mathbf{x})\right| \leqslant K_1, \tag{0.5}$$

where K_1 is some positive constant.

Let us formulate the results concerning the preventive effect of the gradient terms.

Theorem 1. *Assume that*

$$a_1(t,\mathbf{x}) \geqslant \lambda(2l_1)^p K_1^{p-1} + \frac{\max_{Q_T}|f(t,\mathbf{x})|}{K_1}$$

or

$$a_1(t,\mathbf{x}) \leqslant -\lambda(2l_1)^p K_1^{p-1} - \frac{\max_{Q_T}|f(t,\mathbf{x})|}{K_1}.$$

If condition (0.5) *is fulfilled then the solution of problem* (0.1), (0.4) *remains bounded for all* $t > 0$ *and*

$$\max_{Q_T}\left|u(t,\mathbf{x})\right| \leqslant 2K_1 l_1. \tag{0.6}$$

Theorem 2. *Suppose that* (0.5) *is fulfilled.*

(i) *If* q_1 *is odd and*

$$a_1(t,\mathbf{x}) \geqslant \lambda(2l_1)^p K_1^{p-q_1} + \frac{\max_{Q_T}|f(t,\mathbf{x})|}{K_1^{q_1}}$$

or

$$a_1(t,\mathbf{x}) \leqslant -\lambda(2l_1)^p K_1^{p-q_1} - \frac{\max_{Q_T}|f(t,\mathbf{x})|}{K_1^{q_1}},$$

then the solution of problem (0.2), (0.4) *remains bounded for all* $t > 0$ *and* (0.6) *holds.*

If in addition $q_1 > p$, *then estimate* (0.6) *holds with* $a_1(t,\mathbf{x}) \geqslant \alpha$ *or* $a_1(t,\mathbf{x}) \leqslant -\alpha$ *for any strictly positive number* α.

(ii) *If* q_1 *and* p *are even and*

$$a_1(t,\mathbf{x}) \geqslant \lambda(2l_1)^p K_1^{p-q_1} + \frac{\max_{Q_T} f}{K_1^{q_1}}, \quad f \geqslant 0,\ u_0 \geqslant 0,$$

then the solution of problem (0.2), (0.4) *remains bounded for all* $t > 0$ *and*

$$0 \leqslant u(t,\mathbf{x}) \leqslant 2K_1 l_1. \tag{0.7}$$

If in addition $q_1 > p$, *then estimate* (0.7) *is valid for* $a_1(t,\mathbf{x}) > \alpha$ *for any positive* α.

Theorem 3. *Suppose that p is even and*

$$a(t,\mathbf{x}) \geqslant \lambda(2l_1)^p K_1^{p-q} + \frac{\max_{Q_T} f(t,\mathbf{x})}{K_1^q}, \quad f \geqslant 0,\ u_0 \geqslant 0.$$

If (0.5) *is fulfilled, then the solution of problem* (0.3), (0.4) *remains bounded for all $t > 0$ and* (0.7) *holds.*

For $q > p$ this estimate holds with $a(t,\mathbf{x}) > \alpha$ for any positive α.

Let us give a simple physical interpretation of Theorem 1. Let $u(t,\mathbf{x})$ be temperature and $\mathbf{a} = (a_1, a_2, a_3)$ velocity field. If the velocity is big enough at least in one direction (for example x_1 or $-x_1$), then the convective transfer (from the left if a_1 is positive or from the right if negative) in this direction brings sufficient cold substance from the boundary, so as not to allow the term u^p to blow-up the temperature.

Eq.(0.3) without gradient term (i.e. when $a \equiv 0$) was investigated by many authors and there is extensive literature on this subject (see, for example, [12] and the references there). It is well known that the phenomenon of blowing up of the solution may occur in this case, i.e. $|u(t,\mathbf{x}^*)| \to +\infty$ when $t \to t^*$ at least for one $\mathbf{x}^* \in \Omega$. In [3] the authors introduce the gradient term $|\nabla u|^q$ in order to investigate the effect of this term on global existence or non-existence of the solution of the Dirichlet problem. Later the influence of the gradient term in the blow up phenomenon for Eq. (0.3) was studied in [4–8,14–16]. The main issue of these works was to determine for which p and q the blow-up in finite time occurs and for which the solution remains bounded. Roughly speaking it turns out that blow-up in finite time may occur if and only if $p > q$. For more details see [15]. So we can conclude that the gradient term controls the source term in the sense that there is no blow up if $p < q$. From Theorem 3 of the present paper it follows that if the coefficient a is big enough, then the gradient term controls the source term even in the case when the power of the gradient term is less then the power of the source term $p \geqslant q$. If $p < q$, then Theorem 3 guarantees the boundedness of the solution for any positive a. Moreover, as it follows from Theorems 1, 2 the presence of the derivative only in one direction x_{i_0} can "hold" the maximum of the solution if the coefficient a_{i_0} is big enough. The interpretation presented above gives some idea why this happens. Application of Eq. (0.3) was given in [13].

Convective diffusion equations with blow up term u^p were studied in [1,2,10]. Different cases of blow up of the solution were investigated there.

The prevent of blow-up of the solution can be also obtained by taking sufficiently big diffusion (heat conductivity) coefficient at least in one direction. To demonstrate this we consider the following equation

$$u_t = \kappa u_{x_1x_1} + \Delta' u + \lambda u^p + f(t,\mathbf{x}), \tag{0.8}$$

coupled with conditions (0.4), here $\Delta' u \equiv \sum_{i=2}^n u_{x_i x_i}$.

Theorem 4. *Suppose that*

$$\kappa \geqslant 3\lambda l_1^{p+1}(2K_1)^{p-1} + \frac{3l_1}{2K_1}\max_{Q_T}\bigl|f(t,\mathbf{x})\bigr|.$$

Then the solution of problem (0.8), (0.4) *is bounded for all $t > 0$ and*

$$\max_{Q_T}\bigl|u(t,\mathbf{x})\bigr| \leqslant 2K_1 l_1.$$

The physical interpretation of Theorem 4 is that if the heat conductivity at least in one direction is sufficiently big, then the heat flow through the boundary in that direction is big enough to prevent the unbounded growth of the temperature. Let us mention here that for the equation

$$u_t = \kappa |u|^p u_{x_1x_1} + \Delta' u + \lambda u^p + f(t,\mathbf{x}) \tag{0.9}$$

the boundedness of the solution of problem (0.9), (0.4) for any positive κ follows immediately from [19]. Here combining the method proposed in [19] with the proof of Theorems 1–3 of the present paper we obtain similar result for problem (0.8), (0.4).

We would like to emphasize that all results of the present paper mentioned above can be easily extended to the case where the term λu^p is substituted by function $Q(u)$. The restrictions on $Q(u)$ are formulated in Section 4, in particular besides λu^p we can take $Q(u) = e^u$.

The paper is organized as follows. In Section 2 we give the proofs of Theorems 1–3. Section 3 deals with problem (0.8), (0.4). In Section 4 we substitute λu^p by $Q(u)$. In the last section we discuss the existence theorems. For problems (0.1), (0.4) and (0.8), (0.4) the existence of a classical solution follows from the L_∞ estimate of the solution under some assumptions on the smoothness of the coefficients. The same is valid for problems (0.2), (0.4) and (0.3), (0.4), if $q_i \leqslant 2$ and $q \leqslant 2$ correspondingly (see [9,11]). We formulate conditions which guarantee the existence in the special case of classical solution for $q_i > 2$ for problem (0.2), (0.4). Concerning the global solvability of problem (0.3), (0.4) for $q > 2$ we refer to [8,14,16].

2. Proof of Theorem 1–3

Proof of Theorem 1. Suppose first that

$$a_1(t,\mathbf{x}) > \lambda(2l_1)^p K_1^{p-1} + \frac{\max_{Q_T}|f(t,\mathbf{x})|}{K_1}. \tag{1.1}$$

Introduce the auxiliary equation:

$$u_t - \Delta u = -\mathbf{a}(t,\mathbf{x})\cdot\nabla u + f_p(u) + f(t,\mathbf{x}) \quad \text{in } Q_T, \tag{1.2}$$

where

$$f_p(z) = \begin{cases} \lambda z^p, & \text{for } |z| \leqslant 2K_1 l_1, \\ \lambda(2K_1 l_1)^p, & \text{for } z > 2K_1 l_1, \\ \lambda(-2K_1 l_1)^p, & \text{for } z < -2K_1 l_1. \end{cases}$$

The goal is to obtain the estimate $|u(t,\mathbf{x})| \leqslant 2K_1 l_1$ for the solution of problem (1.2), (0.4). If such estimate takes place, then Eqs. (1.2) and (0.1) coincide and as a consequence the solution of problem (0.1), (0.4) will be bounded.

Consider function $v(t,\mathbf{x}) \equiv u(t,\mathbf{x}) - h(x_1)$, where $h(x_1) = K_1(l_1 + x_1)$. One can easily see that

$$v_t - \Delta v = -\mathbf{a}(t,\mathbf{x})\cdot\nabla u + f_p(u) + f(t,\mathbf{x}). \tag{1.3}$$

If the function $v(t,\mathbf{x})$ attains maximum at the point $N \in \bar{Q}_T \setminus \Gamma$ (Γ is the parabolic boundary of the domain Q_T), then at this point we have $\nabla v = 0$, i.e. $u_{x_1} = h' = K_1$, $u_{x_i} = 0$, for $i \geqslant 2$ and hence

$$\begin{aligned} v_t - \Delta v|_N &= -a_1(N)K_1 + f_p\big(u(N)\big) + f(N) \\ &< -\lambda(2l_1)^p K_1^p - \max_{Q_T}\big|f(t,\mathbf{x})\big| + \lambda(2K_1 l_1)^p + f(N) \leqslant 0. \end{aligned}$$

This contradicts the assumption that at N we have maximum of v. Here we use the inequality $f_p(u) \leqslant \lambda(2K_1 l_1)^p$. On Γ the function $v(t,\mathbf{x})$ is non-positive. In fact, due to (0.5), $v(0,\mathbf{x}) = u_0(\mathbf{x}) - K_1(l_1 + x_1) \leqslant 0$ and from (0.4) $v|_{S_T} = -h|_{S_T} \leqslant 0$. Hence $v \leqslant 0$ in $\bar{Q}_T$ and

$$u(t,\mathbf{x}) \leqslant K_1(l_1 + x_1) \leqslant 2K_1 l_1.$$

Now let us consider function $\tilde{v}(t,\mathbf{x}) \equiv u(t,\mathbf{x}) + h(x_1)$. We have

$$\tilde{v}_t - \Delta\tilde{v} = -\mathbf{a}(t,\mathbf{x})\cdot\nabla u + f_p(u) + f(t,\mathbf{x}).$$

If the function $\tilde v(t,\mathbf{x})$ attains minimum at the point $N_1 \in \bar Q_T \setminus \Gamma$ then at this point we have $\nabla \tilde v = 0$, i.e. $u_{x_1} = -h' = -K_1$, $u_{x_i} = 0$ for $i \geqslant 2$ and hence

$$\begin{aligned}\tilde v_t - \Delta \tilde v|_{N_1} &= a(N_1)K_1 + f_p\big(u(N_1)\big) + f(N_1)\\ &> \lambda(2l_1)^p K_1^p + \max_{Q_T}\big|f(t,\mathbf{x})\big| - \lambda(2K_1 l_1)^p + f(N_1) \geqslant 0.\end{aligned}$$

This contradicts the assumption that $\tilde v$ attains minimum at N_1. Here we use the inequality $f_p(u) \geqslant -\lambda(2K_1 l_1)^p$. It is clear that on Γ the function $\tilde v$ is non-negative. In fact, due to (0.5) $\tilde v(0,\mathbf{x}) = u_0(\mathbf{x}) + K_1(l_1 + x_1) \geqslant 0$ and from (0.4) $\tilde v|_{S_T} = h|_{S_T} \geqslant 0$, hence $\tilde v \geqslant 0$ in $\bar Q_T$ and

$$u(t,x) \geqslant -K(l_1 + x_1) \geqslant -2K_1 l_1.$$

For $a_1 > \lambda(2l_1)^p K_1^{p-1} + K_1^{-1}\max_{Q_T}|f|$ Theorem 1 is proved.

Suppose now that $a_1 \geqslant \lambda(2l_1)^p K_1^{p-1} + K_1^{-1}\max_{Q_T}|f|$. Substitute function v in (1.3) by $v_1 = v\mathrm{e}^{-t}$. For v_1 we have

$$v_{1t} + v_1 - \Delta v_1 = \mathrm{e}^{-t}\big(-\mathbf{a}(t,\mathbf{x})\cdot\nabla u + f_p(u) + f(t,\mathbf{x})\big).$$

Function v_1 cannot attain positive maximum at the point $N \in \bar Q_T \setminus \Gamma$ because in this point the left side is strictly positive and the right side non-positive. On Γ the function v_1 is non-positive, hence $v_1 \leqslant 0$ in $\bar Q_T$ and $u \leqslant K_1(l_1 + x_1) \leqslant 2K_1 l_1$.

Similarly considering $\tilde v_1 = \tilde v\mathrm{e}^{-t}$ instead of $\tilde v$ we obtain the needed estimate from below.

The case $a_1(t,\mathbf{x}) \leqslant -\lambda(2l_1)^p K_1^{p-1} - K_1^{-1}\max_{Q_T}|f|$ can be treated in the same way with the only difference in the choice of the barrier. Here instead of $h(x_1) \equiv K_1(l_1 + x_1)$ we must take $\tilde h(x_1) \equiv K_1(l_1 - x_1)$.

Theorem 1 is proved. □

Proof of Theorem 2. If q_1 is odd, then the proof of this theorem is similar to the previous one. In fact, consider auxiliary equation

$$u_t - \Delta u = -a_i(t,\mathbf{x})u_{x_i}^{q_i} + f_p(u) + f(t,\mathbf{x}) \quad \text{in } Q_T, \tag{1.4}$$

where f_p is the same as in the proof of Theorem 1. For the function $v(t,\mathbf{x}) \equiv u(t,\mathbf{x}) - h(x_1)$, where $h(x_1) = K_1(l_1 + x_1)$ we have

$$v_t - \Delta v = -a_i(t,\mathbf{x})u_{x_i}^{q_i} + f_p(u) + f(t,\mathbf{x}).$$

If the function $v(t,\mathbf{x})$ attains maximum at the point $N \in \bar Q_T \setminus \Gamma$, then at this point we have $\nabla v = 0$, i.e. $u_{x_1} = h' = K_1$, $u_{x_i} = 0$, for $i \geqslant 2$ and hence

$$\begin{aligned}v_t - \Delta v|_N &= -a_1(N)K_1^{q_1} + f_p\big(u(N)\big) + f(N)\\ &< -\lambda(2l_1)^p K_1^p - \max_{Q_T}\big|f(t,\mathbf{x})\big| + \lambda(2K_1 l_1)^p + f(N) \leqslant 0.\end{aligned}$$

This contradicts the assumption that at N we have maximum of v. Taking into account that on Γ the function $v(t,\mathbf{x})$ is non-positive we conclude that $v(t,\mathbf{x}) \leqslant 0$ in $\bar Q_T$ and hence

$$u(t,\mathbf{x}) \leqslant K_1(l_1 + x_1) \leqslant 2K_1 l_1.$$

Function $\tilde v(t,\mathbf{x}) \equiv u(t,\mathbf{x}) + h(x_1)$ satisfies the equation

$$\tilde v_t - \Delta \tilde v = -a_i(t,\mathbf{x})u_{x_i}^{q_i} + f_p(u) + f(t,\mathbf{x}).$$

If $\tilde v(t,\mathbf{x})$ attains minimum at the point $N_1 \in \bar Q_T \setminus \Gamma$, then at this point we have $\nabla \tilde v = 0$, i.e. $u_{x_1} = -h' = -K_1$, $u_{x_i} = 0$ for $i \geqslant 2$ and hence

$$\tilde{v}_t - \Delta\tilde{v}|_{N_1} = a_1(N_1)K_1^{q_1} + f_p\big(u(N_1)\big) + f(N_1)$$
$$> \lambda(2l_1)^p K_1^p + \max_{Q_T}\big|f(t,\mathbf{x})\big| - \lambda(2K_1 l_1)^p + f(N_1) \geqslant 0.$$

This contradicts the assumption that $\tilde{v}(t,\mathbf{x})$ attains minimum at N_1. Taking into account that on Γ the function $\tilde{v}(t,\mathbf{x})$ is non-negative we conclude that $\tilde{v} \geqslant 0$ in $\bar{Q}_T$ and consequently

$$u(t,x) \geqslant -K_1(l_1 + x_1) \geqslant -2K_1 l_1.$$

For even q_1 the estimate $u(t,\mathbf{x}) \leqslant 2K_1 l_1$ can be obtained similarly. In order to obtain the estimate from the below we need to assume that p is even number too. For even p one can easily see that the solution of problem (1.4), (0.4) cannot attain negative minimum in $\bar{Q}_T \setminus \Gamma$, and due to the fact that $u_0(\mathbf{x}) \geqslant 0$ we obtain the needed estimate from the below.

If $q_1 > p$, then for any $\alpha > 0$ we can select K_1 sufficiently big (without changing $u_0(\mathbf{x})$) so that

$$\frac{\lambda(2l_1)^p}{K_1^{q_1-p}} + \frac{\max_{Q_T}|f(t,\mathbf{x})|}{K_1^{q_1}} \leqslant \alpha.$$

Theorem 2 is proved. □

Proof of Theorem 3. It is similar to the proof of Theorem 2. In fact, consider auxiliary equation

$$u_t - \Delta u = -a(t,\mathbf{x})|\nabla u|^q + f_p(u) + f(t,\mathbf{x}) \quad \text{in } Q_T, \tag{1.5}$$

where f_p is the same as in the proof of Theorem 1. For the function $v(t,\mathbf{x}) \equiv u(t,\mathbf{x}) - h(x_1)$, where $h(x_1) = K_1(l_1 + x_1)$ we have

$$v_t - \Delta v = -a(t,\mathbf{x})|\nabla u|^q + f_p(u) + f(t,\mathbf{x}).$$

At the point $N \in \bar{Q}_T \setminus \Gamma$ of maximum of v similarly to the previous cases we have

$$v_t - \Delta v|_N = -a(N)K_1^q + f_p\big(u(N)\big) + f(N) \tag{1.6}$$
$$< -\lambda(2l_1)^p K_1^p - \max_{Q_T}\big|f(t,\mathbf{x})\big| + \lambda(2K_1 l_1)^p + f(N) \leqslant 0. \tag{1.7}$$

This contradicts the assumption that at N we have maximum of v. Taking into account that on Γ the function v is non-positive we conclude that $v \leqslant 0$ in $\bar{Q}_T$ and hence $u \leqslant 2K_1 l_1$.

For even p one can easily see that the solution of problem (1.5), (0.4) cannot attain negative minimum in $\bar{Q}_T \setminus \Gamma$ and due to the fact that $u_0(\mathbf{x}) \geqslant 0$ we obtain the needed estimate from the below.

If $q > p$, then as in previous case for any $\alpha > 0$ we can select K_1 sufficiently big such that $\lambda(2l_1)^p K_1^{p-q} \leqslant \alpha$.

Theorem 3 is proved. □

3. The influence of the diffusion

Proof of Theorem 4. Suppose that

$$\kappa > 3\lambda l_1^{p+1}(2K_1)^{p-1} + \frac{3l_1}{2K_1}\max_{Q_T}|f|.$$

Construct the auxiliary problem using the cut function f_p introduced in Section 2. Consider equation

$$u_t - \kappa u_{x_1x_1} - \Delta' u = f_p(u) + f(t,\mathbf{x}), \tag{3.1}$$

coupled with conditions (0.4). As in the proof of Theorem 1, here the goal is to establish the estimate $|u| \leqslant 2K_1 l_1$ for the solution of problem (3.1), (0.4). Such estimate implies the coincidence of Eqs. (3.1) and (0.8) and consequently the estimate $|u| \leqslant 2K_1 l_1$ is valid for the solution of problem (0.8), (0.6) as well.

Introduce the barrier function $h(x_1)$:

$$h(x_1) \equiv -\frac{K_1}{3l_1}x_1^2 + \frac{5}{3}K_1x_1 + 2K_1l_1.$$

Such choice is stipulated by the necessity the barrier function to satisfy the following properties:

$$h(-l_1) = 0, \quad h(0) = 2K_1l_1, \quad h'(x_1) \geqslant K_1, \quad -\kappa h'' > \lambda(2K_1l_1)^p + \sup|f|.$$

For $v(t,\mathbf{x}) \equiv u(t,\mathbf{x}) - h(x_1)$ we have

$$v_t - \kappa v_{x_1x_1} - \Delta' v = f_p(u) + f(t,\mathbf{x}) + \kappa h''(x_1) = f_p(u) + f(t,\mathbf{x}) - \kappa\frac{2K_1}{3l_1} \tag{3.2}$$

$$< f_p(u) + f(t,\mathbf{x}) - \lambda(2K_1l_1)^p - \max\bigl|f(t,\mathbf{x})\bigr| \leqslant 0. \tag{3.3}$$

Thus function v cannot attain maximum in $Q_T \setminus \Gamma$. On Γ we have $v \leqslant 0$. Hence $u(t,\mathbf{x}) \leqslant h(x_1)$.

Similarly we prove that $\tilde{v} \equiv u - h(-x_1) \leqslant 0$, i.e. $u(t,\mathbf{x}) \leqslant h(-x_1)$. Consequently

$$u(t,\mathbf{x}) \leqslant h(0) = 2K_1l_1.$$

Taking functions $u(t,\mathbf{x}) + h(x_1)$ and $u(t,\mathbf{x}) + h(-x_1)$ instead of $u(t,\mathbf{x}) - h(x_1)$ and $u(t,\mathbf{x}) - h(-x_1)$ respectively in the same manner we conclude that

$$u(t,\mathbf{x}) \geqslant -h(0) = -2K_1l_1.$$

Similarly to the proof of Theorem 1, introducing function $v_1 = v\mathrm{e}^{-t}$, we can prove that Theorem 4 is valid with $\kappa \geqslant 3l_1^{p+1}(2K_1)^{p-1} + 3l_1(2K_1)^{-1}\max_{Q_T}|f|$.

Theorem 4 is proved. □

4. Arbitrary source

In this section we consider arbitrary function $Q(u)$ instead of λu^p. Assume that Q satisfies the following inequality

$$-Q(2l_1K_1) \leqslant Q(z) \leqslant Q(2l_1K_1) \quad \text{for } |z| \leqslant 2l_1K_1. \tag{4.1}$$

We restrict ourselves by the generalization of Theorem 1. Other results can be generalized exactly in the same manner.

Consider equation

$$u_t + \mathbf{a}(t,\mathbf{x}) \cdot \nabla u = \Delta u + Q(u) + f(t,\mathbf{x}), \tag{4.2}$$

coupled with conditions (0.4).

Theorem 5. *Assume that Q satisfies* (4.1) *and*

$$a_1(t,\mathbf{x}) \geqslant Q(2l_1K_1)K_1^{-1} + \frac{\max_{Q_T}|f(t,\mathbf{x})|}{K_1}$$

or

$$a_1(t,\mathbf{x}) \leqslant -Q(2l_1K_1)K_1^{-1} - \frac{\max_{Q_T}|f(t,\mathbf{x})|}{K_1}.$$

Then the solution of problem (4.2), (0.4) *remains bounded for all $t > 0$ and*

$$\max_{Q_T}\bigl|u(t,\mathbf{x})\bigr| \leqslant 2K_1l_1.$$

Proof. Without loss of generality assume that

$$a_1(t,\mathbf{x}) > Q(2l_1K_1)K_1^{-1} + \frac{\max_{Q_T}|f(t,\mathbf{x})|}{K_1}.$$

The only difference in comparison with Theorem 1 is in constructing of the cut function. Introduce the auxiliary equation:

$$u_t - \Delta u = -\mathbf{a}(t,\mathbf{x})\cdot\nabla u + f_Q(u) + f(t,\mathbf{x}) \quad \text{in } Q_T, \tag{4.3}$$

where

$$f_Q(z) = \begin{cases} Q(z), & \text{for } |z| \leqslant 2K_1l_1, \\ Q(2K_1l_1), & \text{for } z > 2K_1l_1, \\ Q(-2K_1l_1), & \text{for } z < -2K_1l_1. \end{cases}$$

Exactly in the same manner as in the proof of Theorem 1 we show that for the solution of problem (4.3), (0.4) the estimate $|u(t,\mathbf{x})| \leqslant 2K_1l_1$ is valid. Consequently (4.3) and (4.2) coincide and as a consequence the solution of problem (4.2), (0.4) is bounded.

The case $a_1(t,\mathbf{x}) \leqslant -Q(u)K_1^{-1} - K_1^{-1}\max_{Q_T}|f|$ can be treated in the same way. □

Consider now equation

$$u_t = \kappa u_{x_1x_1} + \Delta' u + Q(u) + f(t,\mathbf{x}) \tag{4.4}$$

coupled with conditions (0.4). Analogously we can prove the following theorem.

Theorem 6. *Assume that Q satisfies* (4.1) *and*

$$\kappa \geqslant \frac{3l_1}{2K_1}Q(2K_1l_1) + \frac{3l_1}{2K_1}\max_{Q_T}\bigl|f(t,\mathbf{x})\bigr|.$$

Then the solution of problem (4.4), (0.4) *remains bounded for all $t > 0$ and*

$$\max_{Q_T}\bigl|u(t,\mathbf{x})\bigr| \leqslant 2K_1l_1.$$

5. Remarks on the existence of the solution

As it was already mentioned in the introduction, for problem (0.1), (0.4) as well as for (0.2), (0.4) if $q_i \leqslant 2$ and (0.3), (0.4) if $q \leqslant 2$ the existence of a classical solution follows from the L_∞ estimate of the solution under some assumptions on the smoothness of the coefficients. Namely functions $a_i(t,\mathbf{x})$ and $f(t,\mathbf{x})$ must be Hölder continuous functions (see [9,11]). The a priori estimate of the gradient which is the key step when proving the existence theorem generally speaking fails when $q_i > 2$ and $q > 2$. Recently there appeared several papers where the condition on no more than quadratic growth of the gradient term is generalized (see [17,18,20]). Unfortunately, for the Dirichlet problem, Eqs. (0.2) and (0.3) do not satisfy assumptions of these papers and we cannot use the approach suggested there. Let us apply here the classical approach that goes back to S.N. Bernstein and involves the preliminary boundary estimates, differentiation of the equation with respect to x_i, $i = 1,\dots,n$, followed by multiplication by u_{x_i} and summation over i. The maximum principle is then applied to the resulting equation for the function $w = |\nabla u|^2$.

Consider problem (0.2), (0.4) in domain $P_T = (0,T)\times P$, where $P = \{\mathbf{x}\colon |x_i| < l_i, i = 1,\dots,n\}$. Suppose that p and $q_i > 2$ are even and

$$a_i \geqslant \lambda(2l_i)^p K_i^{p-q_i} + \frac{\max_{Q_T} f}{K_i}, \quad f \geqslant 0,\ u_0 \geqslant 0,$$

where $|u_{0x_i}(\mathbf{x})| \leqslant K_i$, $i = 1, \ldots, n$. From Theorem 2 we conclude that in P_T the following estimates hold

$$0 \leqslant u(t,\mathbf{x}) \leqslant K_i(l_i + x_i), \quad 0 \leqslant u(t,\mathbf{x}) \leqslant K_i(l_i - x_i). \tag{5.3}$$

These inequalities imply the boundary gradient estimates:

$$\left|u_{x_i}(t,\mathbf{x})\right|\Big|_{x_i=\pm l_i} \leqslant K_i.$$

Having these boundary gradient estimates we can easily apply the Bernstein's approach if $a_i = a_i(t)$, $i = 1, \ldots, n$ to obtain the apriori estimate of $|\nabla u|$ which in turn implies the existence of the classical solution under the smoothness assumptions on a_i and f. For more details see [9,11].

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