Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Ann. I. H. Poincaré – AN 23 (2006) 839–848
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

On a Liouville phenomenon for entire weak supersolutions of elliptic partial differential equations

Autour d'un phénomène de Liouville pour les sursolutions entières faibles d'équations aux derivées partielles elliptiques

Vasilii V. Kurta

American Mathematical Society (Mathematical Reviews), 416 Fourth Street, P.O. Box 8604, Ann Arbor, MI 48107-8604, USA

Received 25 October 2004; received in revised form 20 July 2005; accepted 6 December 2005

Available online 7 July 2006

In fond memory of Professor Heinz Bauer

Abstract

We study a new Liouville-type phenomenon for entire weak supersolutions of elliptic partial differential equations of the form $A(u) = 0$ on $\mathbb{R}^n$, $n \geqslant 2$. Typical examples of the operator $A(u)$ are the p-Laplacian for $p > 1$, the mean curvature operator, and their well-known modifications.

Résumé

Ce travail est consacré à l'étude d'un nouveau phénomène de type de Liouville pour les sursolutions entières faibles d'équations aux derivées partielles elliptiques de la forme $A(u) = 0$ sur $\mathbb{R}^n$, $n \geqslant 2$. Des exemples typiques de l'opérateur $A(u)$ sont le p-laplacien pour $p > 1$, l'opérateur de courbure moyenne, et leurs modifications bien connues.

MSC: 35J60; 31C45; 31B05; 35R45

Keywords: Elliptic; Entire; Liouville; Partial differential equation; Quasilinear; Supersolution; Weak

1. Introduction

Liouville's well-known theorem says that any superharmonic function on $\mathbb{R}^2$ bounded below by a constant is itself a constant. On the other hand it is also well known that for $n \geqslant 3$ there exist non-constant superharmonic functions on $\mathbb{R}^n$ bounded below by a constant. The purpose of this work is to determine for $n \geqslant 3$ the 'sharp distance at infinity' between the non-constant superharmonic functions on $\mathbb{R}^n$ bounded below by a constant and this constant itself in

E-mail address: vvk@ams.org (V.V. Kurta).

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2005.12.001

the form of a theorem of Liouville type and to characterize basic properties of quasilinear elliptic partial differential operators which make it possible to obtain such a theorem for supersolutions of quasilinear elliptic partial differential equations of the form

$$A(u)=0 \tag{1}$$

on $\mathbb{R}^n$, $n\geqslant 2$. Typical examples of the operator $A(u)$ are the p-Laplacian

$$\Delta_p(u):=\operatorname{div}\bigl(|\nabla u|^{p-2}\nabla u\bigr),\quad p>1, \tag{2}$$

its well-known modification (see, e.g., [8, p. 155])

$$\tilde{\Delta}_p(u):=\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(\left|\frac{\partial u}{\partial x_i}\right|^{p-2}\frac{\partial u}{\partial x_i}\right),\quad p>1, \tag{3}$$

the mean curvature operator

$$\Xi(u):=\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right), \tag{4}$$

and its well-known modifications.

Note that a Liouville theorem for solutions of linear uniformly elliptic second-order partial differential equations on $\mathbb{R}^n$, $n>2$, was first obtained, as a direct consequence of a Harnack inequality, in [1] under some continuity assumptions on the coefficients of the equations and in [12] without continuity assumptions on the coefficients of the equations. In the case of quasilinear uniformly elliptic second-order partial differential equations on $\mathbb{R}^n$, $n\geqslant 2$, a Liouville theorem is a direct consequence of a Harnack inequality first obtained in [14]. Note also that a Liouville theorem for mappings of $\mathbb{R}^n$, $n>2$, with bounded distortion was first obtained in [13] by using the same Harnack inequality from [14]. Finally, in the case of linear uniformly elliptic second-order partial differential equations on $\mathbb{R}^2$, a Liouville theorem is a direct consequence of a Harnack inequality first obtained in [7].

2. Definitions

Let $A(u)$ be a differential operator defined formally by

$$A(u)=\sum_{i=1}^n\frac{\mathrm{d}}{\mathrm{d}x_i}A_i(x,u,\nabla u). \tag{5}$$

Here and in what follows, $n\geqslant 2$. We assume that the functions $A_i(x,\eta,\xi)$, $i=1,\dots,n$, satisfy the usual Carathéodory conditions on $\mathbb{R}^n\times\mathbb{R}^1\times\mathbb{R}^n$; namely, they are continuous in η and ξ for almost all $x\in\mathbb{R}^n$ and measurable in x for any $\eta\in\mathbb{R}^1$ and $\xi\in\mathbb{R}^n$.

Definition 1. Let $\alpha>1$ be a given number. The operator $A(u)$ given by (5) belongs to the class $\mathcal{A}(\alpha)$ if for all $\eta\in\mathbb{R}^1$, all $\xi,\psi\in\mathbb{R}^n$, and almost all $x\in\mathbb{R}^n$ the following two inequalities hold:

$$0\leqslant\sum_{i=1}^n\xi_iA_i(x,\eta,\xi), \tag{6}$$

with equality only if $\xi=0$, and

$$\left|\sum_{i=1}^n\psi_iA_i(x,\eta,\xi)\right|^\alpha\leqslant\mathcal{K}|\psi|^\alpha\left(\sum_{i=1}^n\xi_iA_i(x,\eta,\xi)\right)^{\alpha-1}, \tag{7}$$

with $\mathcal{K}$ a certain positive constant.

It is easy to see that condition (7) is fulfilled whenever the inequality

$$\left(\sum_{i=1}^nA_i^2(x,\eta,\xi)\right)^{\alpha/2}\leqslant\mathcal{K}\left(\sum_{i=1}^n\xi_iA_i(x,\eta,\xi)\right)^{\alpha-1} \tag{8}$$

holds for all $\eta \in \mathbb{R}^1$, all $\xi, \psi \in \mathbb{R}^n$, and almost all $x \in \mathbb{R}^n$. Hence, the operator $A(u)$ given by (5) and satisfying conditions (6) and (8) belongs to the class $\mathcal{A}(\alpha)$.

Remark 1. Conditions (7) and (8) on the behavior of the coefficients of partial differential operators were introduced in [10].

It is not difficult to verify that for any given $p > 1$ the differential operators (2) and (3) as well as the differential operator $A(u)$ given by (5) and satisfying the well-known growth conditions

$$\left(\sum_{i=1}^{n} A_i^2(x, \eta, \xi)\right)^{1/2} \leqslant \mathcal{K}_1 |\xi|^{p-1} \tag{9}$$

and

$$|\xi|^p \leqslant \mathcal{K}_2 \sum_{i=1}^{n} \xi_i A_i(x, \eta, \xi), \tag{10}$$

with $\mathcal{K}_1, \mathcal{K}_2$ positive constants, belong to the class $\mathcal{A}(\alpha)$ with $\alpha = p$.

It is also easy to see that linear divergent elliptic partial differential operators of the form

$$L := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j}\left(a_{ij}(x) \frac{\partial}{\partial x_i}\right) \tag{11}$$

with $a_{ij}(x)$ measurable bounded coefficients and with the (possibly non-uniformly) positive-definite quadratic form

$$\sum_{i,j=1}^{n} a_{ij}(x) \xi_i \xi_j \tag{12}$$

belong to the class $\mathcal{A}(\alpha)$ with $\alpha = 2$ but do not satisfy condition (10) for any fixed $p > 1$.

In connection with this we give another example of an operator that belongs to the class $\mathcal{A}(\alpha)$ with a certain $\alpha > 1$ but does not satisfy condition (10). Let $a(x, \eta, \xi)$ be a positive bounded function that satisfies the Carathéodory conditions on $\mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n$. It is easy to see that for a given $p > 1$ the weighted p-Laplacian

$$\bar{\Delta}_p(u) := \operatorname{div}\bigl(a(x, u, \nabla u) |\nabla u|^{p-2} \nabla u\bigr) \tag{13}$$

belongs to the class $\mathcal{A}(\alpha)$ with $\alpha = p$ but does not satisfy condition (10) for any fixed $p > 1$ if the function $a(x, \eta, \xi)$ is only assumed to be positive.

It can happen that an operator $A(u)$ given by (5) belongs simultaneously to several different classes $\mathcal{A}(\alpha)$. For example, the mean curvature operator $\Xi(u)$ given by (4) belongs to the classes $\mathcal{A}(\alpha)$ for all $1 < \alpha \leqslant 2$; similarly its modification for $p \geqslant 2$,

$$\Xi_p(u) := \operatorname{div}\left(\frac{|\nabla u|^{p-2} \nabla u}{\sqrt{1 + |\nabla u|^2}}\right), \tag{14}$$

belongs to the classes $\mathcal{A}(\alpha)$ for all $\alpha \in (p-1, p]$ and $p \geqslant 2$. Obviously, operators given by (4) and (14) do not satisfy conditions (9)–(10) for any fixed $p \geqslant 1$.

Definition 2. Let $\alpha > 1$ be a given number, and let the operator $A(u)$ given by (5) belong to the class $\mathcal{A}(\alpha)$. A measurable function $u : \mathbb{R}^n \to \mathbb{R}^1$ is called an entire weak supersolution of Eq. (1) on $\mathbb{R}^n$ if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $|\nabla u| \in L^\alpha_{\mathrm{loc}}(\mathbb{R}^n)$, and the integral inequality

$$\int_{\mathbb{R}^n} \sum_{i=1}^{n} \varphi_{x_i} A_i(x, u, \nabla u)\, \mathrm{d}x \geqslant 0 \tag{15}$$

holds for every non-negative function $\varphi \in W^{1,\alpha}(\mathbb{R}^n)$ with compact support.

3. Results

Theorem 1. *Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $\alpha \geqslant n$. Let the operator $A(u)$ given by* (5) *belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of* (1) *on $\mathbb{R}^n$ bounded below by a constant. Then $u(x)$ is a constant on $\mathbb{R}^n$.*

Theorem 2. *Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by* (5) *belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of* (1) *on $\mathbb{R}^n$ bounded below by a constant c and such that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Then either $u(x) = c$ on $\mathbb{R}^n$ or the relation*

$$\liminf_{r\to+\infty}\Big[\sup_{r\leqslant|x|\leqslant 2r}\big(u(x)-c\big)\Big]r^{\frac{n-\alpha}{\alpha-1-\nu}} = +\infty \tag{16}$$

holds with any fixed $\nu \in (0, \alpha - 1)$.

Theorem 3. *Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by* (5) *belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of* (1) *on $\mathbb{R}^n$ bounded below by a constant c. Then either $u(x) = c$ on $\mathbb{R}^n$ or the relation*

$$\liminf_{r\to+\infty} r^{-\alpha}\int_{r\leqslant|x|\leqslant 2r}\big(u(x)-c\big)^{\alpha-1-\nu}\,\mathrm{d}x = +\infty \tag{17}$$

holds with any fixed $\nu \in (0, \alpha - 1)$.

Due to the arbitrariness of the constant c in Theorems 2 and 3, the statements of these theorems can be reformulated in a slightly different form.

Theorem 2′. *Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by* (5) *belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of* (1) *on $\mathbb{R}^n$ bounded below by a constant and such that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Then either $u(x)$ is a constant on $\mathbb{R}^n$ or relation* (16) *holds with any fixed real number c such that $u(x) \geqslant c$ on $\mathbb{R}^n$ and any fixed $\nu \in (0, \alpha - 1)$.*

Theorem 3′. *Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by* (5) *belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of* (1) *on $\mathbb{R}^n$ bounded below by a constant. Then either $u(x)$ is a constant on $\mathbb{R}^n$ or relation* (17) *holds with any fixed real number c such that $u(x) \geqslant c$ on $\mathbb{R}^n$ and any fixed $\nu \in (0, \alpha - 1)$.*

Remark 2. It is important to note that for any given $n \geqslant 2$ and $\alpha > 1$ such that $n > \alpha$ the function

$$u(x) = \big(1 + |x|^{\frac{\alpha}{\alpha-1}}\big)^{\frac{\alpha-n}{\alpha}} \tag{18}$$

is an entire weak supersolution of the equation

$$\Delta_p(u) = 0 \tag{19}$$

with $p = \alpha$ that is bounded below and is such that relations (16) and (17) hold with any fixed $\nu \in (0, \alpha - 1)$ and, at the same time, the relations

$$\lim_{r\to+\infty}\Big[\sup_{r\leqslant|x|\leqslant 2r}\big(u(x)-0\big)\Big]r^{\frac{n-\alpha}{\alpha-1}} = C_1 \tag{20}$$

and

$$\lim_{r\to+\infty} r^{-\alpha}\int_{r\leqslant|x|\leqslant 2r}\big(u(x)-0\big)^{\alpha-1}\,\mathrm{d}x = C_2, \tag{21}$$

with C_1, C_2 certain positive constants, also hold.

Remark 3. The results of this work were announced in [5]. To prove these results we further develop an approach that was proposed for solving similar problems in [6].

Remark 4. The results of Theorem 1 are new only for $\alpha = n$. Similar results to those of Theorem 1 for entire weak continuous supersolutions of (1) on $\mathbb{R}^n$ for $\alpha = n$ were first obtained in [11]. For $\alpha > n$, the results of Theorem 1 for entire weak supersolutions of (1) on $\mathbb{R}^n$, which in this case are continuous on $\mathbb{R}^n$ by the well-known Sobolev imbedding theory, were also first obtained in [11]. Here, we give a new proof of these results from [11] by developing an approach from [6] which does not explicitly use the continuity of entire weak supersolutions of (1) on $\mathbb{R}^n$.

Remark 5. In the case when $\alpha = p$ and $A(u) = \Delta_p(u)$, Theorem 1 coincides with well-known Liouville-type theorems for entire superharmonic and p-superharmonic functions locally bounded on $\mathbb{R}^n$ (see, e.g., [2, p. 68] and [3, p. 179]). Also, in this case, the results of Theorems 2 and 3 correlate well with certain results in the theory of entire superharmonic and p-superharmonic functions (see, e.g., [2, pp. 131, 139] and [3, pp. 133, 135]).

4. Proofs

Proof of Theorem 2. The statement of Theorem 2 follows immediately from Theorem 3. In fact, let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by (5) belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of (1) on $\mathbb{R}^n$ bounded below by a constant c, i.e., $u(x) \geqslant c$ on $\mathbb{R}^n$, and such that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$. Hence, by Theorem 3, either $u(x) = c$ on $\mathbb{R}^n$ or relation (17) holds with any fixed $\nu \in (0, \alpha - 1)$. Further, via the trivial inequality

$$r^{-\alpha} \int\limits_{r \leqslant |x| \leqslant 2r} \big(u(x) - c\big)^{\alpha-1-\nu} \, \mathrm{d}x \leqslant r^{-\alpha} \Big[\sup_{r \leqslant |x| \leqslant 2r} \big(u(x) - c\big)^{\alpha-1-\nu} \Big] \int\limits_{r \leqslant |x| \leqslant 2r} \mathrm{d}x, \tag{22}$$

which obviously holds for any $r > 0$, it follows from (17) that

$$\liminf_{r \to +\infty} \Big[\sup_{r \leqslant |x| \leqslant 2r} \big(u(x) - c\big)^{\alpha-1-\nu} \Big] r^{n-\alpha} = +\infty. \tag{23}$$

Then, since

$$\sup_{r \leqslant |x| \leqslant 2r} \big(u(x) - c\big)^{\alpha-1-\nu} \leqslant \Big[\sup_{r \leqslant |x| \leqslant 2r} \big(u(x) - c\big) \Big]^{\alpha-1-\nu} \tag{24}$$

and

$$\Big[\sup_{r \leqslant |x| \leqslant 2r} \big(u(x) - c\big) \Big]^{\alpha-1-\nu} r^{n-\alpha} = \Big(\Big[\sup_{r \leqslant |x| \leqslant 2r} \big(u(x) - c\big) \Big] r^{\frac{n-\alpha}{\alpha-1-\nu}} \Big)^{\alpha-1-\nu}, \tag{25}$$

the validity of (16) follows immediately from that of (23) and (25). □

In what follows, a 'smooth' function is a C^∞-function on $\mathbb{R}^n$, $B(r)$ is an open ball on $\mathbb{R}^n$ of radius $r > 0$ centered at the origin, and $\overline{B(r)}$ is the closure of $B(r)$.

Proof of Theorem 3. Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $n > \alpha$. Let the operator $A(u)$ given by (5) belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of (1) on $\mathbb{R}^n$ bounded below by a constant c, i.e., $u(x) \geqslant c$ on $\mathbb{R}^n$. Let r and ε be positive numbers, and let $\zeta : \mathbb{R}^n \to [0, 1]$ be a smooth function which equals 1 on $\overline{B(r)}$ and 0 outside $B(2r)$. Substituting, without loss of generality, $\varphi(x) = (u(x) - c + \varepsilon)^{-\nu} \zeta^\alpha(x)$ as a test function in inequality (15), where $\nu \in (0, \alpha - 1)$ is arbitrary, and integrating by parts, we find

$$\begin{aligned} &\alpha \int\limits_{B(2r) \setminus B(r)} \sum_{i=1}^n \zeta_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu} \zeta^{\alpha-1} \, \mathrm{d}x \\ &\quad \geqslant \nu \int\limits_{B(2r)} \sum_{i=1}^n u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu-1} \zeta^\alpha \, \mathrm{d}x. \end{aligned} \tag{26}$$

Estimating the left-hand side of (26) by using condition (7) on the coefficients of the operator $A(u)$, we have

$$\alpha\mathcal{K}^{1/\alpha}\int\limits_{B(2r)\setminus B(r)}\left(\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)\right)^{(\alpha-1)/\alpha}|\nabla\zeta|(u-c+\varepsilon)^{-\nu}\zeta^{\alpha-1}\,\mathrm{d}x$$
$$\geqslant\left|\alpha\int\limits_{B(2r)\setminus B(r)}\sum_{i=1}^{n}\zeta_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu}\zeta^{\alpha-1}\,\mathrm{d}x\right|. \tag{27}$$

Further, estimating the left-hand side of (27) by Hölder's inequality, we arrive at

$$\alpha\mathcal{K}^{1/\alpha}\left(\int\limits_{B(2r)\setminus B(r)}|\nabla\zeta|^{\alpha}(u-c+\varepsilon)^{\alpha-1-\nu}\,\mathrm{d}x\right)^{1/\alpha}$$
$$\times\left(\int\limits_{B(2r)\setminus B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\zeta^{\alpha}\,\mathrm{d}x\right)^{(\alpha-1)/\alpha}$$
$$\geqslant\left|\alpha\int\limits_{B(2r)\setminus B(r)}\sum_{i=1}^{n}\zeta_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu}\zeta^{\alpha-1}\,\mathrm{d}x\right|. \tag{28}$$

In turn, (26) and (28) imply the inequality

$$\alpha\mathcal{K}^{1/\alpha}\left(\int\limits_{B(2r)\setminus B(r)}|\nabla\zeta|^{\alpha}(u-c+\varepsilon)^{\alpha-1-\nu}\,\mathrm{d}x\right)^{1/\alpha}$$
$$\times\left(\int\limits_{B(2r)\setminus B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\zeta^{\alpha}\,\mathrm{d}x\right)^{(\alpha-1)/\alpha}$$
$$\geqslant\nu\int\limits_{B(2r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\zeta^{\alpha}\,\mathrm{d}x \tag{29}$$

and, therefore, the inequality

$$\alpha^{\alpha}\mathcal{K}\int\limits_{B(2r)\setminus B(r)}|\nabla\zeta|^{\alpha}(u-c+\varepsilon)^{\alpha-1-\nu}\,\mathrm{d}x\geqslant\nu^{\alpha}\int\limits_{B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x. \tag{30}$$

It is easy to see that the right-hand side of (30) increases monotonically if $\varepsilon>0$ decreases strongly monotonically to zero. Therefore, it follows from (30) that the inequality

$$\alpha^{\alpha}\mathcal{K}\int\limits_{B(2r)\setminus B(r)}|\nabla\zeta|^{\alpha}(u-c+\varepsilon)^{-\nu+\alpha-1}\,\mathrm{d}x\geqslant\nu^{\alpha}\int\limits_{B(r)}\sum_{i=1}^{n}u_{x_i}A_i(x,u,\nabla u)(u-c+\delta)^{-\nu-1}\,\mathrm{d}x \tag{31}$$

holds with any $\delta>0$ and any $\varepsilon\in(0,\delta]$. Since for any sequence $\varepsilon_k>0$ monotonically decreasing to zero as $k\to+\infty$ the sequence of functions

$$\Phi_k(x):=|\nabla\zeta|^{\alpha}(u-c+\varepsilon_k)^{\alpha-1-\nu} \tag{32}$$

measurable on $\mathbb{R}^n$ converges a.e. on $\mathbb{R}^n$ to the function

$$\Phi(x):=|\nabla\zeta|^{\alpha}(u-c)^{\alpha-1-\nu} \tag{33}$$

measurable on $\mathbb{R}^n$, since for sufficiently large k

$$\left|\Phi_k(x)\right|\leqslant|\nabla\zeta|^{\alpha}(u-c+1)^{\alpha-1-\nu} \tag{34}$$

on $\mathbb{R}^n$, and since the function

$$|\nabla\zeta|^\alpha (u-c+1)^{\alpha-1-\nu} \tag{35}$$

is locally integrable on $\mathbb{R}^n$, then, by Lebesgue's theorem (see, e.g., [4, p. 303]), for $\varepsilon = \varepsilon_k > 0$ monotonically decreasing to zero we can pass to the limit as $k \to +\infty$ on the left-hand side of (31). As a result, we obtain the inequality

$$\alpha^\alpha \mathcal{K} \int\limits_{B(2r)\setminus B(r)} |\nabla\zeta|^\alpha (u-c)^{\alpha-1-\nu}\,\mathrm{d}x \geqslant \nu^\alpha \int\limits_{B(r)} \sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u)(u-c+\delta)^{-\nu-1}\,\mathrm{d}x, \tag{36}$$

which holds with any $\delta > 0$. Then, for any $r > 0$ and any sequence $\varepsilon_k > 0$ monotonically decreasing to zero as $k \to +\infty$, it follows from (36), by letting $\delta = \varepsilon_k$ and

$$\Psi_k(x) := \sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon_k)^{-\nu-1}, \tag{37}$$

that the sequence of integrals

$$\int\limits_{B(r)} \Psi_k(x)\,\mathrm{d}x \tag{38}$$

is bounded above by the positive constant

$$c_1 = \mathcal{K}\left(\frac{\alpha}{\nu}\right)^\alpha \int\limits_{B(2r)\setminus B(r)} |\nabla\zeta|^\alpha (u-c)^{\alpha-1-\nu}\,\mathrm{d}x, \tag{39}$$

which does not depend on ε_k. Hence, since

$$\Psi_1(x) \leqslant \Psi_2(x) \leqslant \cdots \leqslant \Psi_k(x) \leqslant \cdots \tag{40}$$

on $\mathbb{R}^n$, then by Beppo Levi's theorem (see, e.g., [4, p. 305]), for any $r > 0$ there exists a function $\Theta_r : B(r) \to \mathbb{R}^1$ integrable on $B(r)$ and such that the sequence of functions $\Psi_k(x)$ converges a.e. to $\Theta_r(x)$ on $B(r)$ and

$$\lim_{k\to+\infty} \int\limits_{B(r)} \Psi_k(x)\,\mathrm{d}x = \int\limits_{B(r)} \Theta_r(x)\,\mathrm{d}x. \tag{41}$$

Further, it is easy to see that the family of functions $\{\Theta_r\}_{r>0}$ uniquely determines a function $\Psi : \mathbb{R}^n \to \mathbb{R}^1$ which is non-negative, measurable, locally integrable on $\mathbb{R}^n$ and is such that $\Psi(x) = \Theta_r(x)$ on $B(r)$ for all $r > 0$. Therefore, the sequence of functions $\Psi_k(x)$ given by (37) converges a.e. to $\Psi(x)$ on $\mathbb{R}^n$ for any sequence $\varepsilon_k > 0$ monotonically decreasing to zero as $k \to +\infty$. Then, by choosing $\delta = \varepsilon_k$ in (36), where the sequence $\varepsilon_k > 0$ converges monotonically to zero as $k \to +\infty$, and passing to the limit on the right-hand side of (36), we find, due to (41), the inequality

$$\alpha^\alpha \mathcal{K} \int\limits_{B(2r)\setminus B(r)} |\nabla\zeta|^\alpha (u-c)^{\alpha-1-\nu}\,\mathrm{d}x \geqslant \nu^\alpha \int\limits_{B(r)} \Psi(x)\,\mathrm{d}x. \tag{42}$$

We divide the rest of the proof into three cases according to the behavior of the right-hand side of (42), which can monotonically approach zero, $+\infty$, or some positive number I as r strongly monotonically approaches $+\infty$.

If the right-hand side of (42) approaches zero as $r \to +\infty$, then, due to the non-negativity of the function $\Psi(x)$, we have that $\Psi(x) = 0$ on $\mathbb{R}^n$. Further, since by (37) and (40) the inequality

$$\sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon_k)^{-\nu-1} \leqslant \Psi(x) \tag{43}$$

holds on $\mathbb{R}^n$ for any sequence $\varepsilon_k > 0$ monotonically decreasing to zero as $k \to +\infty$, then, again, due to the non-negativity of the left-hand side of (43), we obtain that

$$\sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon_k)^{-\nu-1} = 0 \tag{44}$$

on $\mathbb{R}^n$. Hence, by condition (6) on the coefficients of the operator $A(u)$, the supersolution $u(x) = \text{const.}$ on $\mathbb{R}^n$, and, therefore, either $u(x) = c$ on $\mathbb{R}^n$ or relation (17) holds with any fixed $\nu \in (0, \alpha - 1)$.

If the right-hand side of (42) approaches $+\infty$ as $r \to +\infty$, then, due to monotonicity, (42) yields that

$$\liminf_{r\to+\infty} \int\limits_{B(2r)\setminus B(r)} |\nabla\zeta|^\alpha (u-c)^{\alpha-1-\nu}\,\mathrm{d}x = +\infty. \tag{45}$$

Finally, if the right-hand side of (42) monotonically approaches a certain positive number I as r approaches $+\infty$, i.e.,

$$\lim_{r\to+\infty} \nu^\alpha \int\limits_{B(r)} \Psi(x)\,\mathrm{d}x = I > 0, \tag{46}$$

we again consider inequality (29), just noting here that, due to monotonicity,

$$\int\limits_{B(2r_k)\setminus B(r_k)} \Psi(x)\,\mathrm{d}x \to 0 \tag{47}$$

for any sequence $r_k > 0$ such that $r_k \to +\infty$. First, we have from (29) the inequality

$$\alpha\mathcal{K}^{1/\alpha}\left(\int\limits_{B(2r)\setminus B(r)} |\nabla\zeta|^\alpha (u-c+\varepsilon)^{\alpha-1-\nu}\,\mathrm{d}x\right)^{1/\alpha} \times \left(\int\limits_{B(2r)\setminus B(r)} \sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x\right)^{(\alpha-1)/\alpha} \geqslant \nu \int\limits_{B(r)} \sum_{i=1}^n u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x. \tag{48}$$

In (48), let $\varepsilon = \varepsilon_k > 0$ converge monotonically to zero as $k \to +\infty$. Then, by Lebesgue's theorem (see, e.g., [4, p. 303]), we can pass to the limit on both sides of (48). Namely, we know from the above that for any sequence $\varepsilon_k > 0$ monotonically decreasing to zero as $k \to +\infty$ the sequences of functions $\Phi_k(x)$ and $\Psi_k(x)$ measurable and locally integrable on $\mathbb{R}^n$ and given, respectively, by (32) and (37), converge a.e. on $\mathbb{R}^n$, respectively, to the functions $\Phi(x)$ and $\Psi(x)$ measurable and locally integrable on $\mathbb{R}^n$. Further, arguing as above and letting $\varepsilon = \varepsilon_k > 0$ monotonically decrease to zero as $k \to +\infty$, by Lebesgue's theorem (see, e.g., [4, p. 303]) we can pass to the limit on both sides of (48). As a result, we arrive at the inequality

$$\alpha\mathcal{K}^{1/\alpha}\left(\int\limits_{B(2r)\setminus B(r)} |\nabla\zeta|^\alpha (u-c)^{\alpha-1-\nu}\,\mathrm{d}x\right)^{1/\alpha}\left(\int\limits_{B(2r)\setminus B(r)} \Psi(x)\,\mathrm{d}x\right)^{(\alpha-1)/\alpha} \geqslant \nu \int\limits_{B(r)} \Psi(x)\,\mathrm{d}x. \tag{49}$$

In (49), for $r = r_k > 0$ monotonically increasing to $+\infty$, by passing to the limit as $r_k \to +\infty$, we obtain from (46), (47), and (49) that

$$\lim_{r_k\to+\infty} \int\limits_{B(2r_k)\setminus B(r_k)} |\nabla\zeta|^\alpha (u-c)^{\alpha-1-\nu}\,\mathrm{d}x = +\infty. \tag{50}$$

Thus, due to the arbitrariness in the choice of the sequence r_k in (50), we again arrive at relation (45).

Now, without loss of generality, we choose in (45) the function $\zeta(x)$ in the form $\zeta(x) = \psi(|x|/(2r))$, where $\psi : [0, +\infty) \to [0, 1]$ is a smooth function that equals 1 on $[0, 1/2]$ and 0 on $[1, +\infty)$ and is such that the inequality

$$|\nabla\zeta| \leqslant c_2 r^{-1} \tag{51}$$

holds on $\mathbb{R}^n$ with a certain positive constant c_2 for an arbitrary $r > 0$. Relation (17) then follows immediately from (45) and (51). □

Proof of Theorem 1. Let $n \geqslant 2$ and $\alpha > 1$ be given numbers such that $\alpha \geqslant n$. Let the operator $A(u)$ given by (5) belong to the class $\mathcal{A}(\alpha)$, and let $u(x)$ be an entire weak supersolution of (1) on $\mathbb{R}^n$ bounded below by a constant c, i.e., $u(x) \geqslant c$ on $\mathbb{R}^n$. Let r, R, and ε be positive numbers such that $R > r$, and let $\zeta : \mathbb{R}^n \to [0,1]$ be a smooth function which equals 1 on $\overline{B(r)}$ and 0 outside $B(R)$. Substituting, without loss of generality, $\varphi(x) = (u(x) - c + \varepsilon)^{-\nu}\zeta^{\alpha}(x)$ as a test function in inequality (15), where $\nu > \alpha - 1$ is an arbitrary positive number, and integrating by parts, we have the inequality

$$\alpha \int_{B(R)\setminus B(r)} \sum_{i=1}^{n} \zeta_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu}\zeta^{\alpha-1}\,\mathrm{d}x \geqslant \nu \int_{B(R)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\zeta^{\alpha}\,\mathrm{d}x. \tag{52}$$

Further, we repeat the proof of Theorem 3 word for word from (26) to (30). As a result, we arrive at the inequality

$$\alpha^{\alpha}\mathcal{K} \int_{B(R)\setminus B(r)} |\nabla\zeta|^{\alpha}(u-c+\varepsilon)^{\alpha-1-\nu}\,\mathrm{d}x \geqslant \nu^{\alpha} \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x. \tag{53}$$

It follows immediately from (53) that the inequality

$$\alpha^{\alpha}\varepsilon^{\alpha-1-\nu}\mathcal{K} \int_{B(R)\setminus B(r)} |\nabla\zeta|^{\alpha}\,\mathrm{d}x \geqslant \nu^{\alpha} \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x \tag{54}$$

holds with any fixed $\varepsilon > 0$ and $\nu > \alpha - 1$.

Now, first let $\alpha > n$. In (54), choosing $R = 2r$ and the function $\zeta(x)$ in the form $\zeta(x) = \psi(|x|/R)$, where $\psi : [0,+\infty) \to [0,1]$ is a smooth function that equals 1 on $[0,1/2]$ and 0 on $[1,+\infty)$ and is such that the inequality (51) holds on $\mathbb{R}^n$ with a certain positive constant c_2 for an arbitrary $R > 0$, we obtain from (51) and (54) the inequality

$$c_3 r^{n-\alpha} \geqslant \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x, \tag{55}$$

which holds with a certain positive constant c_3 that does not depend on r. Passing to the limit as $r \to +\infty$ in (55), we find, due to the non-negativity of the integrand, that the equality

$$\sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1} = 0 \tag{56}$$

holds on $\mathbb{R}^n$, and, therefore, by condition (6) on the coefficients of the operator $A(u)$, that $u(x) = \text{const.}$ on $\mathbb{R}^n$.

If $\alpha = n$, we choose in (54) the function $\zeta(x)$ in the form $\zeta(x) = \psi(\frac{\ln(|x|/r)}{\ln(R/r)})$ with arbitrary $R > r > 1$, where $\psi : [-\infty,+\infty) \to [0,1]$ is a smooth function which equals 1 on $[-\infty,0]$ and 0 on $[1,+\infty)$. It is not difficult to understand (see, e.g., [9, p. 12]) that the inequality

$$\left|\nabla\zeta(x)\right| \leqslant \frac{c_4}{|x|\ln(R/r)} \tag{57}$$

holds on $\mathbb{R}^n$ with a certain positive constant c_4 for arbitrary $R > r > 1$. It then follows from (54) and (57) that the inequality

$$c_5 \int_{B(R)\setminus B(r)} \left(|x|\ln(R/r)\right)^{-n}\,\mathrm{d}x \geqslant \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x \tag{58}$$

holds, and, therefore, so does the inequality

$$c_6\left(\ln(R/r)\right)^{-n+1} \geqslant \int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x,u,\nabla u)(u-c+\varepsilon)^{-\nu-1}\,\mathrm{d}x \tag{59}$$

with arbitrary $R > r > 1$ and certain positive constants c_5 and c_6 that do not depend on R. Passing to the limit as $R \to +\infty$ in (59), we find that the equality

$$\int_{B(r)} \sum_{i=1}^{n} u_{x_i} A_i(x, u, \nabla u)(u - c + \varepsilon)^{-\nu-1}\, \mathrm{d}x = 0 \tag{60}$$

holds with an arbitrary $r > 1$. Passing to the limit as $r \to +\infty$ in (60), we again obtain, due to the non-negativity of the integrand in (60) and by condition (6), that $u(x) = \text{const.}$ on $\mathbb{R}^n$. □

Acknowledgements

The author is grateful to the referee for useful comments and suggestions for improving the presentation of the results.

References

[1] D. Gilbarg, J. Serrin, On isolated singularities of solutions of second order elliptic differential equations, J. Analyse Math. 4 (1955/56) 309–340.
[2] W.K. Hayman, P.B. Kennedy, Subharmonic Functions, Academic Press, 1976 (284 p.).
[3] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Clarendon Press, 1993 (363 p.).
[4] A.N. Kolmogorov, S.V. Fomin, Introductory Real Analysis, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1970 (403 p.).
[5] V.V. Kurta, About a Liouville phenomenon, C. R. Math. Acad. Sci. Paris, Ser. I 338 (2004) 19–22.
[6] V.V. Kurta, Some problems of qualitative theory for nonlinear second-order equations, Doctoral Dissert., Steklov Math. Inst., Moscow, 1994 (323 p.).
[7] L. Lichtenstein, Beiträge zur Theorie der linearen partiellen Differentialgleichungen zweiter Ordnung vom elliptischen Typus. Unendliche Folgen positiver Lösungen, Rend. Circ. Mat. Palermo 33 (1912) 201–211.
[8] J.-L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Gauthier-Villars, Paris, 1969 (554 p.).
[9] V.G. Maz'ya, T.O. Shaposhnikova, Theory of Multipliers in Spaces of Differentiable Functions, Pitman, Boston, MA, 1985 (Advanced Publishing Program, 344 p.).
[10] V.M. Miklyukov, Capacity and a generalized maximum principle for quasilinear equations of elliptic type, Dokl. Akad. Nauk SSSR 250 (1980) 1318–1320.
[11] V.M. Miklyukov, Asymptotic properties of subsolutions of quasilinear equations of elliptic type and mappings with bounded distortion, Mat. Sb. 111 (153) (1980) 42–66.
[12] J. Moser, On Harnack's theorem for elliptic differential equations, Comm. Pure Appl. Math. 14 (1961) 577–591.
[13] Yu.G. Reshetnyak, Mappings with bounded distortion as extremals of integrals of Dirichlet type, Sibirsk. Mat. Zh. 9 (1968) 652–666.
[14] J. Serrin, Local behavior of solutions of quasi-linear equations, Acta Math. 111 (1964) 247–302.