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Multi solitary waves for nonlinear Schrödinger equations

Ondes solitaires multiples des équations de Schrödinger nonlinéaires

Yvan Martel [a,*], Frank Merle [b,1]

[a] *Département de Mathématiques, Université de Versailles-Saint-Quentin-en-Yvelines, 45, av. des Etats-Unis, 78035 Versailles cedex, France*
[b] *Département de Mathématiques, Université de Cergy-Pontoise, 2, av. Adolphe-Chauvin, 95302 Cergy-Pontoise cedex, France*

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Abstract

We consider the nonlinear Schrödinger equation in $\mathbb{R}^d$ for any $d \geqslant 1$, with a nonlinearity such that solitary waves exist and are stable. Let $R_k(t,x)$ be K arbitrarily given solitary waves of the equation with different speeds $v_1, v_2, \ldots, v_K$. In this paper, we prove that there exists a solution $u(t)$ of the equation such that

$$\lim_{t\to+\infty}\left\|u(t)-\sum_{k=1}^{K}R_k(t)\right\|_{H^1}=0.$$

Résumé

On considère l'équation de Schrödinger nonlinéaire dans $\mathbb{R}^d$ pour tout $d \geqslant 1$, avec une nonlinéarité telle que des ondes solitaires existent et sont stables. Soit $R_k(t,x)$, K ondes solitaires de l'équation données arbitrairement, avec des vitesses différentes $v_1, v_2, \ldots, v_K$. Dans ce papier, on démontre qu'il existe une solution $u(t)$ de l'équation telle que

$$\lim_{t\to+\infty}\left\|u(t)-\sum_{k=1}^{K}R_k(t)\right\|_{H^1}=0.$$

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* Corresponding author.
E-mail addresses: martel@math.uvsq.fr (Y. Martel), Frank.Merle@math.u-cergy.fr (F. Merle).

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1. Introduction

We consider the nonlinear Schrödinger equations in $\mathbb{R}^d$, for any $d \geqslant 1$:

$$\begin{cases} \mathrm{i}\partial_t u = -\Delta u - |u|^{p-1}u, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) = u_0. \end{cases} \tag{1}$$

Recall first that Ginibre and Velo [5] proved that Eq. (1) is locally well-posed in $H^1(\mathbb{R}^d)$ for $1 < p < (d+2)/(d-2)$: for any $u_0 \in H^1(\mathbb{R}^d)$, there exist $T > 0$ and a unique maximal solution $u \in C([0,T), H^1(\mathbb{R}^d))$ of (1) on $[0,T)$. Moreover, either $T = +\infty$ or $T < +\infty$ and then $\lim_{t \to T} \|\nabla u(t)\|_{L^2} = +\infty$. It is also well known that H^1 solutions of (1) satisfy the following three conservation laws: for all $t \in [0,T)$,

- L^2-norm:

$$\int \left|u(t,x)\right|^2 \mathrm{d}x = \int \left|u_0(x)\right|^2 \mathrm{d}x; \tag{2}$$

- Energy:

$$E\big(u(t)\big) = \frac{1}{2}\int \left|\nabla u(t,x)\right|^2 \mathrm{d}x - \frac{1}{p+1}\int \left|u(t,x)\right|^{p+1} \mathrm{d}x = E(u_0); \tag{3}$$

- Momentum:

$$\operatorname{Im}\int \nabla u(t,x)\bar{u}(t,x)\,\mathrm{d}x = \operatorname{Im}\int \nabla u_0(x)\bar{u}_0(x)\,\mathrm{d}x. \tag{4}$$

In particular, from the energy and mass conservations and the following Gagliardo–Nirenberg inequality in $\mathbb{R}^d$:

$$\forall v \in H^1\big(\mathbb{R}^d\big), \quad \int |v|^{p+1} \leqslant C\left(\int |\nabla v|^2\right)^{d(p-1)/4}\left(\int |v|^2\right)^{1+(2-d)(p-1)/4}, \tag{5}$$

it follows that for $1 < p < 1 + 4/d$, any H^1 solution of (1) is global and uniformly bounded in H^1.

Recall that Eq. (1) admits the following symmetries:

- Space–time translation invariance: if $u(t,x)$ satisfies (1), then for any $t_0, x_0 \in \mathbb{R}$, $w(t,x) = u(t-t_0, x-x_0)$ also satisfies (1).
- Phase invariance: if $u(t,x)$ satisfies (1), then for any $\gamma_0 \in \mathbb{R}$, $w(t,x) = u(t,x)\,\mathrm{e}^{\mathrm{i}\gamma_0}$ also satisfies (1).
- Galilean invariance: if $u(t,x)$ satisfies (1), then for any $v_0 \in \mathbb{R}$,

$$w(t,x) = u(t, x - v_0 t)\,\mathrm{e}^{\mathrm{i}\frac{v_0}{2}(x - \frac{v_0}{2}t)} \tag{6}$$

  also satisfies (1).

We now consider solitary waves of (1)

$$u(t,x) = \mathrm{e}^{\mathrm{i}\omega_0 t} Q_{\omega_0}(x), \tag{7}$$

for $\omega_0 > 0$, where $Q_{\omega_0} \in H^1(\mathbb{R}^d)$ is solution of

$$\Delta Q_{\omega_0} + Q_{\omega_0}^p = \omega_0 Q_{\omega_0}, \quad Q_{\omega_0} > 0. \tag{8}$$

Recall that such positive solution of (8) exists and is unique up to translations (see [1,4] and [6]), moreover, it is the solution of a variational problem. We call Q_{ω_0} the solution of (8) which is radially symmetric. By the symmetries of Eq. (1), for any $v_0 \in \mathbb{R}^d$, $x_0 \in \mathbb{R}^d$ and $\gamma_0 \in \mathbb{R}$,

$$u(t,x) = Q_{\omega_0}(x - x_0 - v_0 t)\,\mathrm{e}^{\mathrm{i}(\frac{1}{2}v_0 \cdot x - \frac{1}{4}|v_0|^2 t + \omega_0 t + \gamma_0)}$$

is also a solitary wave of (1), moving on the line $x = x_0 + v_0 t$.

Using the concentration-compactness method, Cazenave and Lions [2] proved that these solitary waves are stable when $1 < p < 1 + 4/d$, i.e. when the nonlinearity has a subcritical growth. Weinstein [16] proved the same result by a different approach based on the expansion of conservation laws.

In this paper, we assume that $1 < p < 1 + 4/d$, so that the solitary waves are stable, and we prove the following result.

Theorem 1 *(Existence of multi solitary waves for the subcritical NLSE). Let*

$$1 < p < 1 + 4/d. \tag{9}$$

Let $K \in \mathbb{N}^*$. For any $k \in \{1, \dots, K\}$, let $\omega_k^0 > 0$, $v_k \in \mathbb{R}^d$, $x_k^0 \in \mathbb{R}^d$ and $\gamma_k^0 \in \mathbb{R}$. Assume that

$$\text{for any } k \neq k', \quad v_k \neq v_{k'}. \tag{10}$$

Let

$$R_k(t,x) = Q_{\omega_k^0}\big(x - x_k^0 - v_k t\big)\,\mathrm{e}^{\mathrm{i}(\frac{1}{2}v_k\cdot x - \frac{1}{4}|v_k|^2 t + \omega_k^0 t + \gamma_k^0)}. \tag{11}$$

Then, there exists an H^1 solution $U(t)$ of (1) *such that,*

$$\text{for all } t \geqslant 0, \quad \left\| U(t) - \sum_{k=1}^{K} R_k(t) \right\|_{H^1} \leqslant C\,\mathrm{e}^{-\theta_0 t}, \tag{12}$$

for some $\theta_0 > 0$ and $C > 0$.

Such solutions for the nonlinear Schrödinger equations correspond to an exceptional behavior. Indeed, $U(t)$ as constructed in Theorem 1 is a nondispersive solution in the sense that by strong H^1 convergence:

$$\int U^2(t) = \sum_{k=1}^{K} \int R_k^2(t) \quad \text{and} \quad E(U(t)) = \sum_{k=1}^{K} E(R_{k(t)}).$$

This means that all the mass and energy available in the solution is used for the solitary waves (in general, part of the L^2 norm is spread out by the dispersive effect of the equation as time goes on).

Note also that the nonlinear Schrödinger equation being time reversible (if $u(t,x)$ is solution then $\bar{u}(-t,x)$ is also solution), the result of Theorem 1 could be stated in a similar way for $t \to -\infty$.

Comments on Theorem 1.

1. *Integrable case*. Solutions such as in Theorem 1 were known to exist for the integrable case: $d = 1$ and $p = 3$:

   $$\mathrm{i}\partial_t u + \partial_x^2 u + |u|^2 u = 0,$$

   see Zakharov and Shabat [17] for a derivation of their explicit expression. Moreover, these solutions have very special properties: they describe the perfect interaction between several solitary waves. In the nonintegrable cases, the only known property of the solution $U(t)$ constructed in Theorem 1 concerns $t \to +\infty$, and we do not know what happens to the K solitary waves backwards in time.
2. *Assumptions*. The only assumption in Theorem 1 is the subcriticality of p which implies that the solitary waves are nonlinearly stable. For the critical case $p = 1 + \frac{4}{d}$, the result of Theorem 1 was proved by Merle [13] in 1990. It was obtained as a consequence of a blow up result and the conformal invariance. Note that the compactness argument used in the present paper for the proof of Theorem 1 is similar to the main argument of the proof of the existence result in [13].
   We conjecture that Theorem 1 is also true for $p \in (1 + \frac{4}{d}, \frac{d+2}{d-2})$, for which the solitary waves are actually unstable.
3. *Generalized KdV equations*. An existence result similar to Theorem 1 was proved by Martel [8] for the subcritical and critical generalized KdV equations, using tools from Martel and Merle (see [9] and references therein) and Martel, Merle and Tsai [10]. Note that for the generalized KdV equations, it is also proved in [8] that being given the parameters of K solitons, the solution converging to the sum of these K solitons in H^1 is unique in a large class. Note finally in the case of the gKdV equations that the stability and asymptotic stability of multi-soliton solutions follow from [10].

4. *Stability of K solitary waves*. We point out that Martel, Merle and Tsai [11] have proved stability results for the sum of several solitary waves of nonlinear Schrödinger equations, which imply stability of the family of the multi solitary waves in the energy space H^1. However, the results in [11] are restricted to $d = 1$, 2 or 3, under a flatness condition of f near 0 which does not allow the pure power case. It is also required that the relative speeds $v_k - v_{k'}$ are large enough. The general stability result (i.e. under the assumptions of Theorem 1) is an open problem.
5. *Uniqueness*. For the nonlinear Schrödinger equations, the general uniqueness problem in Theorem 1 (i.e. in the class of H^1 solutions such that the quantity in (12) goes to zero) is open, unlike for the gKdV equations.

The proof of Theorem 1 depends on some calculations developed in [11]. To keep the present paper self-contained, we have reproduced these calculations in Appendix A. However, we point out that even if initially we use the same calculations, a large part of the proof of Theorem 1 is different and in fact more elementary than the proof in [11].

1.1. Case of a general nonlinearity

We extend Theorem 1 to the nonlinear Schrödinger equation in $\mathbb{R}^d$ with a general nonlinearity:

$$\begin{cases} \mathrm{i}\partial_t u = -\Delta u - f\big(|u|^2\big)u, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0) = u_0, \end{cases} \tag{13}$$

with f of class C^1 satisfying $f(0) = 0$ and

$$\forall s \geqslant 1, \quad \big|f'(s^2)\big| < Cs^{p-2}, \quad \text{for some } p < \frac{d+2}{d-2}. \tag{14}$$

It is well known that Eq. (13) has properties similar to (1): local H^1 well-posedness [5], conservation laws and symmetries (except that the scaling invariance is no longer true). Moreover if for $\omega_0 > 0$, Q_{ω_0} is solution of the following elliptic problem:

$$\Delta Q_{\omega_0} + f\big(Q_{\omega_0}^2\big)Q_{\omega_0} = \omega_0 Q_{\omega_0}, \quad Q_{\omega_0} > 0, \tag{15}$$

then

$$u(t,x) = Q_{\omega_0}(x - x_0 - v_0 t)\,\mathrm{e}^{\mathrm{i}(\frac{1}{2}v_0\cdot x - \frac{1}{4}|v_0|^2 t + \omega_0 t + \gamma_0)}$$

is solution of (13) where $v_0 \in \mathbb{R}^d$, $x_0 \in \mathbb{R}^d$ and $\gamma_0 \in \mathbb{R}$.

The stability problem for one solitary wave solution is solved in a similar way (see [2,16]). From [16], a natural assumption for nonlinear stability is the existence of $\lambda > 0$ such that for any real-valued function $\eta \in H^1$:

$$(\eta, Q_\omega) = (\eta, \nabla Q_\omega) = 0 \Rightarrow \int \big\{|\nabla\eta|^2 + \omega|\eta|^2 - \big(f\big(Q_\omega^2\big) + 2Q_\omega^2 f'\big(Q_\omega^2\big)\big)|\eta|^2\big\} \geqslant \lambda\|\eta\|_{H^1}^2. \tag{16}$$

The result for Eq. (13) is the following.

Theorem 2. *Assume* $f(0) = 0$ *and* (14). *Let* $K \in \mathbb{N}^*$. *For any* $k \in \{1, \ldots, K\}$, *let* $\omega_k^0 > 0$, $v_k \in \mathbb{R}^d$, $x_k^0 \in \mathbb{R}^d$ *and* $\gamma_k^0 \in \mathbb{R}$. *Assume that*

$$\text{for any } k \neq k', \quad v_k \neq v_{k'}. \tag{17}$$

Assume that for all $k \in \{1, \ldots, K\}$, ω_k^0 *is such that an* H^1 *positive solution* $Q_{\omega_k^0}$ *of* (15) *exists and satisfies* (16) *for some* $\lambda_k > 0$. *Let*

$$R_k(t,x) = Q_{\omega_k^0}\big(x - x_k^0 - v_k t\big)\,\mathrm{e}^{\mathrm{i}(\frac{1}{2}v_k\cdot x - \frac{1}{4}|v_k|^2 t + \omega_k^0 t + \gamma_k^0)}. \tag{18}$$

Then, there exists an H^1 *solution* $U(t)$ *of* (13) *such that,*

$$\text{for all } t \geqslant 0, \quad \left\| U(t) - \sum_{k=1}^{K} R_k(t) \right\|_{H^1} \leqslant C\,\mathrm{e}^{-\theta_0 t}, \tag{19}$$

for some $\theta_0 > 0$ *and* $C > 0$.

Remark. For the pure power case, Theorem 2 is equivalent to Theorem 1, since by Maris [7] and McLeod [12], (16) is satisfied for $f(s^2)=s^{p-1}$ if and only if $1<p<1+4/d$ (see below Lemma 6(ii)).

The proof of Theorem 2 is similar to the one of Theorem 1, up to slight modifications that we will give through the paper when necessary.

2. Construction of the solution assuming uniform estimates

In this section, we prove Theorem 1 assuming the main uniform estimates to be proved in the next section. The proof of Theorem 2 is the same, up to a slight modification.

Let $R_k(t)$ be K solitary waves of Eq. (1) as defined in Theorem 1:

$$R_k(t,x)=Q_{\omega_k^0}\big(x-x_k^0-v_k t\big)\,\mathrm{e}^{\mathrm{i}(\frac{1}{2}v_k\cdot x-\frac{1}{4}|v_k|^2t+\omega_k^0 t+\gamma_k^0)}, \tag{20}$$

and let

$$R(t)=\sum_{k=1}^{K}R_k(t). \tag{21}$$

The construction of a solution $U(t)$ satisfying the conclusion of Theorem 1 is based on an asymptotic argument. Let $(T_n)_{n\geqslant 1}$ be an increasing sequence of $\mathbb{R}^+$ with $\lim_{n\to+\infty}T_n=+\infty$. For all $n\geqslant 1$, we consider u_n the unique global H^1 solution of

$$\begin{cases}\mathrm{i}\,\partial_t u_n=-\Delta u_n-|u_n|^{p-1}u_n, & (t,x)\in\mathbb{R}\times\mathbb{R}^d,\\ u_n(T_n)=R(T_n).\end{cases} \tag{22}$$

We claim the following result, which is the key point of the proof of Theorem 1.

Proposition 1 *(Uniform estimates). There exist $T_0>0$, $C_0>0$, $\theta_0>0$ such that, for all $n\geqslant 1$,*

$$\forall t\in[T_0,T_n],\quad \big\|u_n(t)-R(t)\big\|_{H^1}\leqslant C_0\,\mathrm{e}^{-\theta_0 t}. \tag{23}$$

Let us prove Theorem 1 assuming Proposition 1. Proposition 1 is proved in Section 3. To simplify the notation, assume

$$T_0=0.$$

We first claim a global bound on the sequence (u_n).

Lemma 1. *There exists $C\geqslant 0$ such that for any $n\geqslant 1$, for any $t\in[0,T_n]$,*

$$\big\|u_n(t)\big\|_{H^1}\leqslant C.$$

Proof. It is a consequence of (23) and the fact that the H^1 norm of $R(t)$ is uniformly bounded. □

Next, we claim a strong compactness result in $L^2(\mathbb{R}^d)$.

Lemma 2. *There exist $U_0\in H^1(\mathbb{R}^d)$ and a subsequence $(u_{\phi(n)})$ of (u_n) such that*

$$u_{\phi(n)}(0)\to U_0\quad \text{in } L^2\big(\mathbb{R}^d\big) \text{ as } n\to+\infty. \tag{24}$$

Proof of Lemma 2. First, we claim the following:

$$\forall\epsilon_0>0,\ \exists K_0=K_0(\epsilon_0)>0,\ \text{such that } \forall n\geqslant 1,\qquad \int_{|x|>K_0}\big|u_n(0,x)\big|^2\,\mathrm{d}x<\epsilon_0. \tag{25}$$

To prove (25), fix $\epsilon_0 > 0$, and let $t_0 \geqslant 0$ be such that $C_0^2 \mathrm{e}^{-2\theta_0 t_0} < \epsilon_0$, where θ_0 and C_0 appear in the statement of Proposition 1. By Proposition 1, we have, for n large enough

$$\int \left|u_n(t_0) - R(t_0)\right|^2 \leqslant C_0^2 \mathrm{e}^{-2\theta_0 t_0} < \epsilon_0.$$

Fix also $K_1 > 0$ such that

$$\int_{|x|>K_1} \left|R(t_0)\right|^2 < \epsilon_0.$$

It follows that

$$\int_{|x|>K_1} \left|u_n(t_0)\right|^2 < 4\epsilon_0.$$

Consider now a C^1 cut-off function $g:\mathbb{R} \to [0,1]$ such that

$$g \equiv 0 \quad \text{on } (-\infty, 1]; \qquad 0 < g' < 2 \quad \text{on } (1,2); \qquad g = 1 \quad \text{on } [2,+\infty).$$

For $\gamma_0 > 0$ to be fixed later, we have by direct calculations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \left|u_n(t)\right|^2 g\left(\frac{|x|-K_1}{\gamma_0}\right) = -\frac{1}{\gamma_0}\operatorname{Im}\int u\left(\nabla\bar{u}\cdot\frac{x}{|x|}\right)g'\left(\frac{|x|-K_1}{\gamma_0}\right),$$

and so

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int \left|u_n(t)\right|^2 g\left(\frac{|x|-K_1}{\gamma_0}\right)\right| \leqslant \frac{2}{\gamma_0}\sup_{t\geqslant 0}\left\|u_n(t)\right\|_{H^1}^2.$$

By Lemma 1, $u_n(t)$ is bounded in H^1 independently of n and t. Thus, we can choose $\gamma_0 > 0$ independent of n such that $\gamma_0 \geqslant \frac{2}{\epsilon_0} t_0 \sup_{t\geqslant 0}\|u_n(t)\|_{H^1}^2$. Then, we find

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int \left|u_n(t)\right|^2 g\left(\frac{|x|-K_1}{\gamma_0}\right)\right| \leqslant \frac{\epsilon_0}{t_0}.$$

By integration between 0 and t_0,

$$\int \left|u_n(0)\right|^2 g\left(\frac{|x|-K_1}{\gamma_0}\right) - \int \left|u_n(t_0)\right|^2 g\left(\frac{|x|-K_1}{\gamma_0}\right) \leqslant \int_0^{t_0}\left|\frac{\mathrm{d}}{\mathrm{d}t}\int \left|u_n(t)\right|^2 g\left(\frac{|x|-K_1}{\gamma_0}\right)\right| \leqslant \epsilon_0.$$

By the properties of g we conclude:

$$\int_{|x|>2\gamma_0+K_1} \left|u_n(0)\right|^2 \leqslant \int \left|u_n(0)\right|^2 g\left(\frac{|x|-K_1}{\gamma_0}\right) \leqslant \int \left|u_n(t_0)\right|^2 g\left(\frac{|x|-K_1}{\gamma_0}\right) + \epsilon_0 \leqslant \int_{|x|>K_1} \left|u_n(t_0)\right|^2 + \epsilon_0 \leqslant 5\epsilon_0.$$

Thus, (25) is proved.

Since $\|u_n(0)\|_{H^1} < C$, there exists a subsequence of (u_n), which we denote by $(u_{\phi(n)})$, and $U_0 \in H^1(\mathbb{R}^d)$ such that

$$u_{\phi(n)}(0) \to U_0 \quad \text{in } L^2_{\mathrm{loc}} \text{ as } n \to +\infty,$$

and by (25), we conclude that $u_{\phi(n)}(0) \to U_0$ in $L^2(\mathbb{R}^d)$ as $n \to +\infty$. Thus Lemma 2 is proved. □

Let us continue the proof of Theorem 1. We consider the global H^1 solution $U(t)$ of

$$\begin{cases} \mathrm{i}\,\partial_t U = -\Delta U - |U|^{p-1}U, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d, \\ U(0) = U_0. \end{cases} \tag{26}$$

Fix $t \geqslant 0$. For n large enough, we have $T_n > t$ and by continuous dependence of the solution of (1) upon the initial data in $L^2(\mathbb{R}^d)$, we have

$$u_{\phi(n)}(t) \to U(t) \quad \text{in } L^2\left(\mathbb{R}^d\right) \text{ as } n \to +\infty \tag{27}$$

(see global well-posedness results and continuity results for the subcritical Schrödinger equation in $L^2(\mathbb{R}^d)$ by Tsutsumi [14]).

Since $(u_{\phi(n)}(t) - R(t))$ converges strongly to $U(t) - R(t)$ in $L^2(\mathbb{R}^d)$ and $u_{\phi(n)}(t) - R(t)$ is uniformly bounded in $H^1(\mathbb{R}^d)$, it follows that

$$u_{\phi(n)}(t) - R(t) \rightharpoonup U(t) - R(t) \quad \text{in } H^1(\mathbb{R}).$$

By (23), and property of the weak convergence, we obtain

$$\forall t \geqslant 0, \quad \|U(t) - R(t)\|_{H^1} \leqslant C_0 \,\mathrm{e}^{-\theta_0 t}. \tag{28}$$

Therefore, Theorem 1 is proved.

The previous argument has to be slightly adapted for the proof of Theorem 2. Indeed, well-posedness in L^2 for the Schrödinger equation is proved only for the subcritical pure power case in [14], i.e. $f(s^2) = s^{p-1}$, for $1 < p < 1+4/d$. For Eq. (13), under assumption (14), local well-posedness in H^{s_0} for some $0 \leqslant s_0 < 1$ (depending on the power p in (14)) was proved by Cazenave and Weissler in [3] (see Theorems 1.1 and 1.2 and property (4.1) page 823 in [3]). We use the space H^{s_0} with $0 \leqslant s_0 < 1$ instead of L^2 in the previous argument. Note that by interpolation $u_n(0)$ converges to $U(0)$ in H^{s_0} strong as $n \to +\infty$, and blow up for $t \geqslant 0$ is not possible by Lemma 1.

3. Proof of the uniform estimates

Proposition 1 is a consequence of the following result.

Proposition 2 *(Reduction of the proof). There exist $A_0 > 0$, $\theta_0 > 0$, $T_0 > 0$, $N_0 > 0$ such that for all $n \geqslant N_0$, for all $t^* \in [T_0, T_n]$, if*

$$\forall t \in [t^*, T_n], \quad \|u_n(t) - R(t)\|_{H^1} \leqslant A_0 \,\mathrm{e}^{-\theta_0 t}, \tag{29}$$

then

$$\forall t \in [t^*, T_n], \quad \|u_n(t) - R(t)\|_{H^1} \leqslant \frac{A_0}{2} \,\mathrm{e}^{-\theta_0 t}. \tag{30}$$

Let us check by an usual continuity argument in H^1 that Proposition 2 implies Proposition 1. Recall that for all $n \geqslant 1$, the map $t \mapsto u_n(t) \in H^1(\mathbb{R})$ is continuous. Let A_0, T_0 and N_0 be defined by Proposition 2. Let $n \geqslant N_0$. Since $u_n(T_n) \equiv R(T_n)$, there exists $\tau_1 > 0$ small so that (29) is true on $[T_n - \tau_1, T_n]$. We define

$$t^* = \inf\bigl\{t \in [T_0, T_n] \text{ such that for all } t' \in [t, T_n], \text{(29) holds}\bigr\}.$$

We claim that $t^* = T_0$. Indeed, if $t^* > T_0$, then by Proposition 2, (30) holds on $[t^*, T_n]$. Thus, by continuity, there exists $\tau_2 > 0$ small such that (30) holds on $[t^* - \tau_2, T_n]$ with $2A_0/3$ instead of $A_0/2$. But this contradicts the definition of t^*. Thus, $t^* = T_0$.

Therefore, for any $n \geqslant N_0$, (23) holds on $[T_0, T_n]$ with $C_0 = A_0$. By possibly taking a larger value of C_0 and T_0, the same estimate holds for any $n \geqslant 1$ and for any $T_0 \leqslant t \leqslant T_n$. Thus Proposition 1 is proved, assuming Proposition 2. The rest of this section is devoted to the proof of Proposition 2.

Proof of Proposition 2. Before starting the proof, note that since $T_n \to +\infty$ as $n \to +\infty$, and T_0 is fixed, Proposition 2 is similar to a stability result in large time. However, it is simpler to prove Proposition 2 than a general stability result for multi solitary waves solutions (not available at this point, see [11]), since the perturbation with respect to an exact solution is only due to the interaction term between the various solitary waves which is of size $\frac{C}{t}\,\mathrm{e}^{-\theta t}$, whereas in the context of a stability result one has to control larger extra terms.

First, we claim the following result.

Claim 1. *Let (v_k) be K vectors of $\mathbb{R}^d$ such that for any $k \neq k'$, $v_k \neq v_{k'}$. Then, there exists an orthonormal basis $(e_1, \ldots, e_d)$ of $\mathbb{R}^d$ such that for any $k \neq k'$, $(v_k, e_1) \neq (v_{k'}, e_1)$.*

Proof of Claim 1. This is an elementary geometrical property of $\mathbb{R}^d$. Let $k, k' = 1, \ldots, K$, $k \neq k'$. The set of vectors $e_1 \in \mathbb{R}^d$ with $|e_1| = 1$ satisfying $(v_k - v_{k'}, e_1) = 0$ is of Lebesgue measure 0 on the sphere. Therefore, we can pick up a vector $e_1 \in \mathbb{R}^d$ with $|e_1| = 1$ and satisfying, for any $k \neq k'$, the condition $(v_k - v_{k'}, e_1) \neq 0$. Then, we consider any orthonormal basis of the form $(e_1, \ldots, e_K)$. □

Without any restriction, we can assume that the direction e_1 given by Claim 1 is x_1, since Eq. (1) is invariant by rotation. Therefore, we may assume that for any $k \neq k'$, $v_{k,1} \neq v_{k',1}$. We suppose in fact that

$$v_{1,1} < v_{2,1} < \cdots < v_{K,1}. \tag{31}$$

From now on, we set $\theta_0 > 0$ such that

$$\sqrt{\theta_0} = \frac{1}{16} \min\Big(v_{2,1} - v_{1,1}, \ldots, v_{K,1} - v_{k-1,1}, \sqrt{\omega_1^0}, \ldots, \sqrt{\omega_K^0}\Big). \tag{32}$$

Let $A_0 > 1$ and $T_0 > 0$ large enough to be defined later. Let N_0 be such that $T_{N_0} > T_0$. We denote u_n simply by u. We assume (29) on $[t^*, T_n]$, for some $t^* \in [T_0, T_n]$.

1. Decomposition of u_n. The first step is to modulate the scaling, phase and translation parameters in the decomposition of the solution in a sum of solitary waves to obtain orthogonality conditions. The following lemma, based on the implicit function theorem, is standard (see for example [11], Lemma 2.4 and Corollary 3).

Lemma 3. *There exists $C_1 > 0$ such that if T_0 is large enough, then there exist unique C^1 functions $\omega_k : [t^*, T_n] \to (0, +\infty)$, $x_k : [t^*, T_n] \to \mathbb{R}^d$, $\gamma_k : [t^*, T_n] \to \mathbb{R}$, such that if we set*

$$\varepsilon(t, x) = u(t, x) - \widetilde{R}(t, x), \tag{33}$$

where

$$\widetilde{R}(t) = \sum_{k=1}^{K} \widetilde{R}_k(t), \quad \widetilde{R}_k(t, x) = Q_{\omega_k(t)}\big(\cdot - x_k^0 - v_k t - x_k(t)\big)\, \mathrm{e}^{\mathrm{i}(\frac{1}{2} v_k \cdot x + \delta_k(t))},$$

and

$$\delta_k(t) = -\frac{1}{4}|v_k|^2 t + \omega_k^0 t + \gamma_k(t),$$

then $\varepsilon(t)$ satisfies, for all $k = 1, \ldots, K$, and for all $t \in [t^*, T_n]$,

$$\operatorname{Re} \int \widetilde{R}_k(t)\bar{\varepsilon}(t) = \operatorname{Im} \int \widetilde{R}_k(t)\bar{\varepsilon}(t) = \operatorname{Re} \int \nabla \widetilde{R}_k(t)\bar{\varepsilon}(t) = 0. \tag{34}$$

Moreover, for all $t \in [t^*, T_n]$,

$$\big\|\varepsilon(t)\big\|_{H^1} + \sum_{k=1}^{K} \big|\omega_k(t) - \omega_k^0\big| \leqslant C_1 A_0\, \mathrm{e}^{-\theta_0 t}, \tag{35}$$

and for all $k = 1, \ldots, K$,

$$\big|\dot{\omega}_k(t)\big|^2 + \big|\dot{x}_k(t)\big|^2 + \big|\dot{\gamma}_k(t) - \big(\omega_k(t) - \omega_k^0\big)\big|^2 \leqslant C_1 \big\|\varepsilon(t)\big\|_{H^1}^2 + C_1\, \mathrm{e}^{-2\theta_0 t}. \tag{36}$$

Note that by $u(T_n) = R(T_n)$ and uniqueness of the decomposition at time $t = T_n$, we necessarily have

$$\varepsilon(T_n) \equiv 0, \quad \widetilde{R}(T_n) \equiv R(T_n), \quad \omega_k(T_n) = \omega_k^0, \quad x_k(T_n) = 0, \quad \gamma_k(T_n) = \gamma_k^0. \tag{37}$$

2. Control of local quantities. We introduce cut-off functions adapted to the solution $u(t)$. Let $\psi(x)$ be a C^3 function such that

$$0 \leqslant \psi \leqslant 1 \quad \text{on } \mathbb{R}, \qquad \psi(x) = 0 \quad \text{for } x \leqslant -1, \qquad \psi(x) = 1 \quad \text{for } x > 1, \qquad \psi' \geqslant 0 \quad \text{on } \mathbb{R}, \tag{38}$$

and satisfying, for some constant $C > 0$,

$$\big(\psi'(x)\big)^2 \leqslant C\psi(x), \qquad \big(\psi''(x)\big)^2 \leqslant C\psi'(x) \quad \text{for all } x \in \mathbb{R}.$$

For this, consider $\psi(x) = \frac{1}{16}(1+x)^4$ for $x \in [-1, 0]$ close to -1, and similarly at $x = 1$.

For all $k = 2, \dots, K$, let

$$\sigma_k = \frac{1}{2}(v_{k-1,1} + v_{k,1}).$$

For $L > 0$ large enough to be fixed later, for any $k = 2, \dots, K-1$, let

$$\begin{aligned} &\varphi_k(t,x) = \psi\left(\frac{x_1 - \sigma_k t}{L}\right) - \psi\left(\frac{x_1 - \sigma_{k+1} t}{L}\right), \\ &\varphi_1(t,x) = 1 - \psi\left(\frac{x_1 - \sigma_2 t}{L}\right), \qquad \varphi_K(t,x) = \psi\left(\frac{x_1 - \sigma_K t}{L}\right), \end{aligned} \tag{39}$$

and finally, set for all $k = 1, \dots, K$:

$$\mathcal{I}_k(t) = \int \left|u(t,x)\right|^2 \varphi_k(t,x)\,\mathrm{d}x, \qquad \mathcal{M}_k(t) = \operatorname{Im} \int \nabla u(t,x) \bar{u}(t,x) \varphi_k(t,x)\,\mathrm{d}x. \tag{40}$$

The quantities $\mathcal{I}_k(t)$ and $\mathcal{M}_k(t)$ are local versions of the L^2 norm and momentum. Ordering the $v_{j,1}$ as in (31) was useful to split the various solitary waves using only the coordinate x_1.

We claim the following result on $\mathcal{I}_k(t)$ and $\mathcal{M}_k(t)$.

Lemma 4. *Let $L > 0$. There exists $C_2 > 0$ such that if L and T_0 are large enough, then for all $k = 2, \dots, K$, for all $t \in [t^*, T_n]$,*

$$\left|\mathcal{I}_k(T_n) - \mathcal{I}_k(t)\right| + \left|\mathcal{M}_k(T_n) - \mathcal{M}_k(t)\right| \leqslant \frac{C_2 A_0^2}{L} \mathrm{e}^{-2\theta_0 t}.$$

Remark. The factor $1/L$ that appears in the above estimate is fundamental in the proof of the uniform estimates. Indeed, by taking L large enough (and thus T_0 large enough), everything happens as if the quantities $\mathcal{I}_k(t)$ and $\mathcal{M}_k(t)$ were constant in time. *We thus obtain $2K$ almost invariant quantities together with two invariant quantities* (L^2 norm and momentum). Since $\mathcal{I}_k(t)$ and $\mathcal{M}_k(t)$ are local versions of the L^2 mass and of the momentum around each solitary wave, and since these two quantities are involved in the proof of the stability of one soliton, it is clear that Lemma 4 is fundamental to the uniform estimates.

Before proving Lemma 4, we recall standard Virial identities for Eq. (1) which follow from direct calculations (similar results hold for (13)). For the reader's convenience, these calculations are reproduced in Appendix A.

Claim 2. *Let $z(t)$ be an H^1 solution of* (1). *Let $\phi : x_1 \in \mathbb{R} \mapsto \phi(x_1)$ be a C^3 real-valued function of one variable such that ϕ, ϕ' and ϕ''' are bounded. Then, for all $t \in \mathbb{R}$,*

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^d} |z|^2 \phi(x_1) = \operatorname{Im} \int_{\mathbb{R}^d} \partial_{x_1} z \bar{z} \phi'(x_1), \\ &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Im} \int_{\mathbb{R}^d} \partial_{x_1} z \bar{z} \phi(x_1) = \int_{\mathbb{R}^d} |\partial_{x_1} z|^2 \phi'(x_1) - \frac{1}{4} \int_{\mathbb{R}^d} |z|^2 \phi'''(x_1) - \frac{p-1}{2(p+1)} \int_{\mathbb{R}^d} |z|^{p+1} \phi'(x_1), \end{aligned}$$

and for $j = 2, \dots, d$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \operatorname{Im} \int_{\mathbb{R}^d} \partial_{x_j} z \bar{z} \phi(x_1) = \operatorname{Re} \int_{\mathbb{R}^d} \partial_{x_j} z \partial_{x_1} \bar{z} \phi'(x_1).$$

Proof of Lemma 4. By Claim 2, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int |u|^2 \psi\left(\frac{x_1 - \sigma_k t}{L}\right) = \frac{1}{L} \operatorname{Im} \int \partial_{x_1} u \bar{u} \psi'\left(\frac{x_1 - \sigma_k t}{L}\right) - \frac{\sigma_k}{2L} \int |u|^2 \psi'\left(\frac{x_1 - \sigma_k t}{L}\right).$$

Thus, by the properties of ψ,

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\int |u|^2\psi\left(\frac{x_1-\sigma_k t}{L}\right)\right| \leqslant \frac{C}{L}\int_{\Omega_1}\left(|\partial_{x_1}u|^2+|u|^2\right), \tag{41}$$

where we have set:

$$\Omega_1=\Omega_1(t)=\left]-L+\sigma_k t, L+\sigma_k t\right[\times\mathbb{R}^{d-1}.$$

Similarly, by Claim 2, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\int \partial_{x_1}u\bar{u}\psi\left(\frac{x_1-\sigma_k t}{L}\right)=\frac{1}{L}\int\left(|\partial_{x_1}u|^2-\frac{p-1}{2(p+1)}|u|^{p+1}\right)\psi'\left(\frac{x_1-\sigma_k t}{L}\right)$$
$$-\frac{1}{4L^3}\int |u|^2\psi'''\left(\frac{x_1-\sigma_k t}{L}\right)-\frac{\sigma_k}{2L}\operatorname{Im}\int \partial_{x_1}u\bar{u}\psi'\left(\frac{x_1-\sigma_k t}{L}\right).$$

Thus, we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\int \partial_{x_1}u\bar{u}\psi\left(\frac{x_1-\sigma_k t}{L}\right)\right| \leqslant \frac{C}{L}\int_{\Omega_1}\left(|\nabla u|^2+|u|^2+|u|^{p+1}\right).$$

Now by the Sobolev inequality applied to $u(x)h(x_1-\sigma_k t)$ where $h=h(x_1)$ is a C^1 function such that $h(x_1)=1$ for $|x_1|<L$ and $h(x_1)=0$ for $|x_1|>L+1$, we have

$$\int_{\Omega_1}|u|^{p+1}\leqslant C\left(\int_{\widetilde{\Omega}_1}\left(|\nabla u|^2+|u|^2\right)\right)^{(p+1)/2},$$

where

$$\widetilde{\Omega}_1(t)=\left]-(L+1)+\sigma_k t,(L+1)+\sigma_k t\right[\times\mathbb{R}^{d-1}.$$

Thus, we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\int \partial_{x_1}u\bar{u}\psi\left(\frac{x_1-\sigma_k t}{L}\right)\right| \leqslant \frac{C}{L}\int_{\widetilde{\Omega}_1}\left(|\nabla u|^2+|u|^2\right)+\frac{C}{L}\left(\int_{\widetilde{\Omega}_1}\left(|\nabla u|^2+|u|^2\right)\right)^{(p+1)/2}. \tag{42}$$

Finally, again by Claim 2, we find for $j=2,\dots,d$,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\int \partial_{x_j}u\bar{u}\psi\left(\frac{x_1-\sigma_k t}{L}\right)=\frac{1}{L}\operatorname{Re}\int \partial_{x_j}u\partial_{x_1}\bar{u}\psi'\left(\frac{x_1-\sigma_k t}{L}\right)+\frac{\sigma_k}{2L}\operatorname{Im}\int \partial_{x_j}u\bar{u}\psi'\left(\frac{x_1-\sigma_k t}{L}\right),$$

and so, for $j=2,\dots,d$, we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}\int \partial_{x_j}u\bar{u}\psi\left(\frac{x_1-\sigma_k t}{L}\right)\right| \leqslant \frac{C}{L}\int_{\Omega_1}\left(|\nabla u|^2+|u|^2\right). \tag{43}$$

Next, note that by $u(t)=R(t)+(u(t)-R(t))$, we have

$$\int_{\widetilde{\Omega}_1}\left(\left|\nabla u(t)\right|^2+\left|u(t)\right|^2\right)\leqslant 2\int_{\widetilde{\Omega}_1}\left(\left|\nabla R(t)\right|^2+\left|R(t)\right|^2\right)+2\left\|u(t)-R(t)\right\|_{H^1}^2.$$

By (29), we have $\|u(t)-R(t)\|_{H^1}^2\leqslant A_0^2\,\mathrm{e}^{-2\theta_0 t}$.

Recall that by standard ODE arguments ([1], pp. 329–330), Q_ω has exponential decay properties:

$$\left|\nabla Q_\omega(x)\right|+\left|Q_\omega(x)\right|\leqslant C\,\mathrm{e}^{-\frac{\sqrt{\omega}}{2}|x|}. \tag{44}$$

Thus, by the definition of σ_k and θ_0, we obtain

$$\int_{\widetilde{\Omega}_1} \left(\left|\nabla R(t)\right|^2 + \left|R(t)\right|^2\right) \leqslant C\,\mathrm{e}^{-8\sqrt{\theta_0}(\sqrt{\theta_0}t-L)} \leqslant C\,\mathrm{e}^{-4\theta_0 t},$$

by taking T_0 and L such that $\sqrt{\theta_0}T_0 \geqslant 2L$. Therefore, from (41)–(43) and the definition of $\mathcal{I}_k(t)$ and $\mathcal{M}_k(t)$, and taking $A_0\,\mathrm{e}^{-\theta_0 T_0}$ small enough, we obtain

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{I}_k(t)\right| + \left|\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{M}_k(t)\right| \leqslant \frac{CA_0^2}{L}\,\mathrm{e}^{-2\theta_0 t}. \tag{45}$$

Note that for $\mathcal{I}_1(t)$ and $\mathcal{M}_1(t)$ we have also used the conservation of mass and momentum.

The result now follows by integrating (45) between t and T_n.

3. Control of the variation of $\omega_k(t)$.

Claim 3. *For any $t \in [t^*, T_n]$,*

$$\left|\omega_k(t) - \omega_k^0\right| \leqslant C\left\|\varepsilon(t)\right\|_{L^2}^2 + C\left(\frac{A_0^2}{L} + 1\right)\mathrm{e}^{-2\theta_0 t}.$$

Claim 3 states that the variation of the scaling of the solitary waves from t to T_n is quadratic in $\varepsilon(t)$. This information is crucial for the rest of the proof. The result is due to the choice of the orthogonality conditions $\operatorname{Re}\int \widetilde{R}_k(t)\bar{\varepsilon}(t) = 0$, and to the almost conservation of the local mass around each solitary waves in the sense of Lemma 4 (in particular, we do not use estimate (36) on $\dot{\omega}_k(t)$).

Proof of Claim 3. By expanding $u(t) = \widetilde{R}(t) + \varepsilon(t)$, using the orthogonality $\int \varepsilon(t)\widetilde{R}_k(t) = 0$, the property of support of φ_k and the exponential decay of each $Q_{\omega_k(t)}$, we have

$$\mathcal{I}_k(t) = \int \left|u(t)\right|^2\varphi_k(t) = \int Q_{\omega_k(t)}^2 + \int \left|\varepsilon(t)\right|^2\varphi_k(t) + \mathrm{O}\left(\mathrm{e}^{-2\theta_0 t}\right)$$

(see similar calculations in Appendix A, proof of Lemma 6(i)). By Lemma 4, we have

$$\left|\mathcal{I}_k(t) - \mathcal{I}_k(T_n)\right| \leqslant \frac{CA_0^2}{L}\,\mathrm{e}^{-2\theta_0 t}.$$

Thus, from $\omega_k(T_n) = \omega_k^0$ ((37)) and $\varepsilon(T_n) \equiv 0$, it follows that

$$\left|\int Q_{\omega_k(t)}^2 - \int Q_{\omega_k^0}^2\right| \leqslant C\left\|\varepsilon(t)\right\|_{L^2}^2 + C\left(\frac{A_0^2}{L} + 1\right)\mathrm{e}^{-2\theta_0 t}.$$

Since by explicit calculations $\int Q_\omega^2 = \omega^{\frac{2}{p-1}-\frac{d}{2}}\int Q^2$ (for Theorem 1), we have

$$\begin{aligned}\int Q_{\omega_k(t)}^2 - \int Q_{\omega_k^0}^2 &= \left(\omega_k^{\frac{2}{p-1}-\frac{d}{2}}(t) - \left(\omega_k^0\right)^{\frac{2}{p-1}-\frac{d}{2}}\right)\int Q^2 \\ &= \left(\frac{2}{p-1} - \frac{d}{2}\right)\left(\omega_k^0\right)^{\frac{2}{p-1}-\frac{d}{2}-1}\left(\omega_k(t) - \omega_k^0\right)\int Q^2 + \mathrm{O}\left(\omega_k(t) - \omega_k^0\right)^{1+\beta_0},\end{aligned}$$

where $\beta_0 > 0$ and $\frac{2}{p-1} - \frac{d}{2} > 0$ by the subcriticality assumption on p.

For $\omega_k(t) - \omega_k^0$ small enough (by (35)), we obtain Claim 3. $\square$

The same argument applies to the proof of Theorem 2 since (16) implies that at $\omega = \omega_k^0$:

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\int Q_w^2 > 0$$

(see Weinstein [16]).

4. Control of $\|\varepsilon(t)\|_{H^1}$ by energy method. After controlling the variation of the scaling (and thus the size) of the solitary waves in the previous point, we only have to control the size of $\varepsilon(t)$ in H^1 before completing the proof. This is the object of the next lemma.

Lemma 5. *For any $t \in [t^*, T_n]$,*

$$\|\varepsilon(t)\|_{H^1}^2 + |\omega_k(t) - \omega_k^0| + |x_k(t)|^2 + |\gamma_k(t) - \gamma_k^0|^2 \leqslant C\left(\frac{A_0^2}{L} + 1\right) e^{-2\theta_0 t}.$$

The proof of the control of $\varepsilon(t)$ in Lemma 5 uses the previous estimates and almost conservation laws (Lemma 4) as well as another functional related to local L^2 norm, local momentum and energy.

Indeed, we define

$$\mathcal{J}(t) = \sum_{k=1}^{K} \left\{ \left(\omega_k(0) + \frac{|v_k|^2}{4}\right) \mathcal{I}_k(t) - v_k \cdot \mathcal{M}_k(t) \right\}, \tag{46}$$

and we set

$$\mathcal{G}(t) = E(u(t)) + \mathcal{J}(t). \tag{47}$$

Let us recall from [11] why such a functional is a natural object to study the stability problem.

When proving the stability of one solitary wave

$$R_0(t,x) = Q_{\omega(t)}(x - x(t))\, e^{i(\frac{1}{2} v_0 x + \gamma(t))},$$

one usually uses the Galilean transformation (6) to restrict to the case $v_0 = 0$. Then the functional used to study the stability problem is

$$E(u(t)) + \omega(0) \int |u(t)|^2.$$

If there are more than one solitary wave it is not possible to assume that the speeds are all 0. Thus, it is convenient to know that in the case of one solitary wave, the complete functional (i.e. with v_0 not necessarily zero) is:

$$E(u(t)) + \left(\omega(0) + \frac{|v_0|^2}{4}\right) \int |u(t)|^2 - v_0 \cdot \operatorname{Im} \int \partial_x u(t) \bar{u}(t)$$

(we refer to Section 2.3 in [11] for this observation). Since the coefficients in front of the L^2 norm and in front of the momentum depend on the parameters of the solitary wave, it is necessary in the case of several solitary waves to localize these functionals. One can see by (40) that the expression of $\mathcal{G}(t)$ in (47) is natural since locally around the k-th solitary wave, $\mathcal{G}(t)$ behaves as the functional for one solitary with the suitable parameters.

We now gather in the next lemma two technical results proved for $d = 1$ in [11] (Proposition 4.2 and Lemma 4.1 in [11]).

Lemma 6.

(i) *Expansion of $\mathcal{G}$ with respect to parameters. For all $t \in [t^*, T_n]$, we have*

$$\mathcal{G}(t) = \sum_{k=1}^{K} \left\{ E(Q_{\omega_k^0}) + \omega_k^0 \int Q_{\omega_k^0}^2 \right\} + H(\varepsilon(t), \varepsilon(t)) + \sum_{k=1}^{K} O(|\omega_k(t) - \omega_k^0|^2) + \|\varepsilon(t)\|_{H^1}^2 \beta(\|\varepsilon(t)\|_{H^1}) + O(e^{-2\theta_0 t}), \tag{48}$$

with $\beta(\epsilon) \to 0$ as $\epsilon \to 0$, where

$$H(\varepsilon,\varepsilon)=\int|\nabla\varepsilon|^2-\sum_{k=1}^{K}\int\big(f\big(|R_k|^2\big)|\varepsilon|^2+2f'\big(|R_k|^2\big)\big[\mathrm{Re}\big(\overline{R}_k\varepsilon\big)\big]^2\big)$$
$$+\sum_{k=1}^{K}\left\{\left(\omega_k(t)+\frac{|v_k|^2}{4}\right)\int|\varepsilon|^2\varphi_k(t)-v_k\cdot\left(\mathrm{Im}\int\nabla\varepsilon\bar{\varepsilon}\varphi_k(t)\right)\right\}.$$

(ii) *Coercivity of H. There exists $\lambda>0$ such that, for all $t\in[t^*,T_n]$,*

$$H\big(\varepsilon(t),\varepsilon(t)\big)\geqslant\lambda\big\|\varepsilon(t)\big\|_{H^1}^2.$$

Property (i) corresponds to the expansion $u(t)=\widetilde{R}(t)+\varepsilon(t)$ in the definition of $\mathcal{G}(t)$. The crucial property of this expansion is the lack of first order terms. Indeed, the first term in $\varepsilon(t)$ is quadratic $(H(\varepsilon,\varepsilon))$ and $|\omega_k(t)-\omega_k^0|$ also appears with power two. For the reader's convenience, we give the proof of (i) in $\mathbb{R}^d$ in Appendix A for the case of a general nonlinearity f.

Remark. Property (ii) is a standard property. Recall that Lemma 6(ii) is proved in [11] provided that (16) holds for any solitary wave $Q_{\omega_k^0}$, $k=1,\dots,K$. The fact that (16) is true for any ground state of (8) in the subcritical pure power case is due to Weinstein [15] for $d=1$ and $d=3$. Later, it was extended to any dimension by Mariş (see Section 2 of [7]) under some condition on the nonlinearity. From a result of MacLeod (see page 504 of [12]), the pure power case $f(s^2)=s^{p-1}$ satisfies the desired condition for $1<p<1+4/d$. Then, we conclude by a localization argument.

Let us now sketch the proof of Lemma 5. Since the functional $\mathcal{G}(t)$ contains conserved quantities $(E(t))$ and almost conserved quantities (Lemma 4), it is almost conserved. Thus property (i) together with the control of the variation of $\omega_k(t)$ in Claim 3 and property (ii) implies easily a control on $\varepsilon(t)$.

Proof of Lemma 5. Note that by Lemma 4 and the definition of $\mathcal{J}$, we have, for all $t\in[t^*,T_n]$,

$$\big|\mathcal{J}(T_n)-\mathcal{J}(t)\big|\leqslant\frac{CA_0^2}{L}\,\mathrm{e}^{-2\theta_0 t}. \tag{49}$$

From the conservation of $E(u(t))$ and (49), we obtain, for any $t\in[t^*,T_n]$,

$$\mathcal{G}(t)\leqslant\mathcal{G}(T_n)+\frac{CA_0^2}{L}\,\mathrm{e}^{-2\theta_0 t}.$$

Thus, by Lemma 6(i), $w_k(T_n)=w_k^0$ and $\varepsilon(T_n)\equiv0$ (see (37)), it follows that

$$H\big(\varepsilon(t),\varepsilon(t)\big)\leqslant C\big|\omega_k(t)-\omega_k^0\big|^2+C\big\|\varepsilon(t)\big\|_{H^1}^2\beta\big(\big\|\varepsilon(t)\big\|_{H^1}\big)+C\left(\frac{A_0^2}{L}+1\right)\mathrm{e}^{-2\theta_0 t},$$

where $\beta(\epsilon)\to0$ as $\epsilon\to0$. By Lemma 6(ii) and Claim 3, we obtain

$$\lambda\big\|\varepsilon(t)\big\|_{H^1}^2\leqslant C\big\|\varepsilon(t)\big\|_{H^1}^2\beta\big(\big\|\varepsilon(t)\big\|_{H^1}\big)+C\left(\frac{A_0^2}{L}+1\right)\mathrm{e}^{-2\theta_0 t},$$

and thus for $\|\varepsilon(t)\|_{H^1}$ small enough,

$$\big\|\varepsilon(t)\big\|_{H^1}^2\leqslant C\left(\frac{A_0^2}{L}+1\right)\mathrm{e}^{-2\theta_0 t},$$

where C is independent of A_0. Note that $\|\varepsilon(t)\|_{H^1}$ small is implied by (35) and taking $A_0\,\mathrm{e}^{-\theta_0 T_0}$ small enough.

The control of $|\omega_k(t)-\omega_k^0|$ follows from Claim 3. Next, we use (36):

$$\big|\dot{x}_k(t)\big|+\big|\dot{\gamma}_k(t)\big|\leqslant C\big\|\varepsilon(t)\big\|_{L^2}+C\,\mathrm{e}^{-\theta_0 t}+\big|\omega_k(t)-\omega_k^0\big|\leqslant C\sqrt{\frac{A_0^2}{L}+1}\,\mathrm{e}^{-\theta_0 t}.$$

By integration between T_n and $t \in [t^*, T_n]$, and (37), we obtain:

$$\left|x_k(t)\right|^2 + \left|\gamma_k(t) - \gamma_k^0\right|^2 \leqslant C\left(\frac{A_0^2}{L} + 1\right) \mathrm{e}^{-2\theta_0 t},$$

and thus Lemma 5 is proved. □

5. Conclusion of the proof. By Lemma 5, for all $t \in [t^*, T_n]$,

$$\left\|R(t) - \widetilde{R}(t)\right\|_{H^1}^2 \leqslant C \sum_{k=1}^{K} \left(\left|x_k(t)\right|^2 + \left|\omega_k(t) - \omega_k^0\right|^2 + \left|\gamma_k(t) - \gamma_k^0\right|^2\right) \leqslant C\left(\frac{A_0^2}{L} + 1\right) \mathrm{e}^{-2\theta_0 t},$$

and thus

$$\left\|u(t) - R(t)\right\|_{H^1}^2 \leqslant 2\left\|\varepsilon(t)\right\|_{H^1}^2 + 2\left\|\widetilde{R}(t) - R(t)\right\|_{H^1}^2 \leqslant C\left(\frac{A_0^2}{L} + 1\right) \mathrm{e}^{-2\theta_0 t},$$

where $C > 0$ does not depend on A_0. Choose now

$$A_0^2 > 32C \quad \text{and} \quad L = A_0^2,$$

and T_0 large enough. It follows that:

$$\left\|u(t) - R(t)\right\|_{H^1}^2 \leqslant 2C\,\mathrm{e}^{-2\theta_0 t} \leqslant \frac{A_0^2}{16}\,\mathrm{e}^{-2\theta_0 t}.$$

Therefore, the conclusion is that for any $t \in [t^*, T_n]$, $\|u_n(t) - R(t)\|_{H^1} \leqslant \frac{A_0}{4}\,\mathrm{e}^{-\theta_0 t}$, which completes the proof of Proposition 2.

Appendix A

Proof of Claim 2. Let $z(t)$ be an H^1 solution of (13) that we can approach by more regular solutions to justify the calculations.

The first identity concerns $\int |z(t,x)|^2 \phi(x_1)\,\mathrm{d}x$. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int |z|^2 \phi(x_1) = 2\,\mathrm{Re} \int \partial_t z \bar{z} \phi(x_1) = -2\,\mathrm{Im} \int \left(\Delta z + f\left(|z|^2\right) z\right) \bar{z} \phi(x_1) = 2\,\mathrm{Im} \int \partial_{x_1} z \bar{z} \phi'(x_1),$$

by integration by parts, all the others terms being real-valued.

For the second identity, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Im} \int \partial_{x_1} z \bar{z} \phi(x_1) = 2\,\mathrm{Im} \int \partial_{x_1} z \partial_t \bar{z} \phi(x_1) + \mathrm{Im} \int \partial_t \bar{z} z\, \phi'(x_1) = \mathrm{Im} \int \partial_t \bar{z}\left(2\partial_{x_1} z \phi(x_1) + z\phi'(x_1)\right),$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Im} \int \partial_{x_1} z \bar{z} \phi(x_1) = \mathrm{Im}\left[-\mathrm{i} \int \left(\Delta \bar{z} + f\left(|z|^2\right)\bar{z}\right)\left(2\partial_{x_1} z \phi(x_1) + z\phi'(x_1)\right)\right].$$

We have

$$\begin{aligned}\mathrm{Re} \int \partial_{x_1}^2 \bar{z}\left(2\partial_{x_1} z \phi(x_1) + z\phi'(x_1)\right) &= -2 \int |\partial_{x_1} z|^2 \phi'(x_1) - \mathrm{Re} \int \partial_{x_1} \bar{z} z \phi''(x_1) \\ &= -2 \int |\partial_{x_1} z|^2 \phi'(x_1) + \frac{1}{2} \int |z|^2 \phi'''(x_1)\end{aligned}$$

and for $j = 2, \dots, d$, by integration by parts, since $\phi(x_1)$ does not depend on x_j,

$$\mathrm{Re} \int \partial_{x_j}^2 \bar{z}\left(2\partial_{x_1} z \phi(x_1) + z\phi'(x_1)\right) = -\int \partial_{x_1}\left(|\partial_{x_j} z|^2\right)\phi(x_1) - \int |\partial_{x_j} z|^2 \phi'(x_1) = 0.$$

Let $F(s) = \int_0^s f(s')\,\mathrm{d}s'$. Then, finally, by integrating by parts,

$$\operatorname{Im}\left[\mathrm{i}\int f\left(|z|^2\right)\bar{z}\left(2\partial_{x_1}z\phi(x_1)+z\phi'(x_1)\right)\right]=\int\left\{f\left(|z|^2\right)z^2-F\left(|z|^2\right)\right\}\phi'(x_1).$$

The third identity is similar and easier. □

Proof of Lemma 6(i). A first remark is that for $\omega > 0$ close to $\omega_0 > 0$, we have

$$\left|E(Q_{\omega_0})+\omega_0\int Q_{\omega_0}^2-E(Q_\omega)-\omega_0\int Q_\omega^2\right|\leqslant C|\omega_0-\omega|^2. \tag{50}$$

We refer to Weinstein [16], Section 2, Eq. (2.5) for this property.

Note now that by the definition of φ_k in (39), we have $1=\sum_{k=1}^K\varphi_k$. Thus,

$$\mathcal{G}(t)=\sum_{k=1}^K\int\left\{|\nabla u|^2-F\left(|u|^2\right)-\left(\omega_k(0)+\frac{|v_k|^2}{4}\right)|u|^2-v_k\cdot\operatorname{Im}(\nabla u\bar{u})\right\}\varphi_k(t).$$

Expanding $u(t)=R(t)+\varepsilon(t)$ in the expression of $E(u(t))$, we obtain

$$E\big(u(t)\big)=E\big(R(t)\big)-2\operatorname{Re}\int\left(\Delta\overline{R}+f\left(|R|^2\right)\overline{R}\right)\varepsilon+\int|\nabla\varepsilon|^2-\int\left\{f\left(|R|^2\right)|\varepsilon|^2+2f'\left(|R|^2\right)\left[\operatorname{Re}\left(\overline{R}\varepsilon\right)\right]^2\right\}$$
$$+\left\|\varepsilon(t)\right\|_{H^1}^2\beta\left(\left\|\varepsilon(t)\right\|_{H^1}\right).$$

Note that the centers of $R_k(t)$ and $R_{k-1}(t)$ are located at a distance larger than $4\theta_0 t$, and since the R_k are exponentially decaying (see (44)), we have for $k\neq k'$,

$$\int|R_kR_{k'}|+\int|\nabla R_kR_{k'}|+\int|\nabla R_k\nabla R_{k'}|<C\,\mathrm{e}^{-2\theta_0 t}. \tag{51}$$

Note also that since $f(0)=0$, we have $|F(s)|<Cs^2$ and $|f(s)|<Cs$ in a neighborhood of zero. Thus:

$$E\big(u(t)\big)=\sum_{k=1}^K\left\{E\big(R_k(t)\big)-2\operatorname{Re}\int\left(\Delta\overline{R}_k+f\left(|R_k|^2\right)\overline{R}_k\right)\varepsilon\right\}+\mathrm{O}\left(\mathrm{e}^{-2\theta_0 t}\right)+\int|\nabla\varepsilon|^2$$
$$-\sum_{k=1}^K\left\{\int\left\{f\left(|R_k|^2\right)|\varepsilon|^2+2f'\left(|R_k|^2\right)\left[\operatorname{Re}\left(\overline{R}_k\varepsilon\right)\right]^2\right\}\right\}+\left\|\varepsilon(t)\right\|_{H^1}^2\beta\left(\left\|\varepsilon(t)\right\|_{H^1}\right).$$

We turn now to $\mathcal{J}(t)$. Recall that

$$\mathcal{J}(t)=\sum_{k=1}^K\left\{\left(\omega_k(0)+\frac{|v_k|^2}{4}\right)\int\left|u(t)\right|^2\varphi_k(t)-v_k\cdot\operatorname{Im}\int\nabla u(t)\bar{u}(t)\varphi_k(t)\right\}. \tag{52}$$

For the first term, we have:

$$\int\left|u(t)\right|^2\varphi_k(t)=\int\left|R(t)\right|^2\varphi_k(t)+\int\left|\varepsilon(t)\right|^2\varphi_k(t)+2\operatorname{Re}\int R(t)\varepsilon(t)\varphi_k(t).$$

By the properties of φ_k and R,

$$\int\left|R(t)\right|^2\varphi_k(t)=\int\left|R_k(t)\right|^2+\mathrm{O}\left(\mathrm{e}^{-2\theta_0 t}\right),$$

and

$$\operatorname{Re}\int R(t)\varepsilon(t)\varphi_k(t)=\operatorname{Re}\int R_k(t)\varepsilon(t)+\mathrm{O}\left(\mathrm{e}^{-2\theta_0 t}\right)=\mathrm{O}\left(\mathrm{e}^{-2\theta_0 t}\right),$$

by the orthogonality conditions on $\varepsilon(t)$.

For the second term in (52), we have, by similar arguments and integration by parts:

$$\operatorname{Im}\int\nabla u\bar{u}\varphi_k(t)=\operatorname{Im}\int\nabla R_k\overline{R}_k-\operatorname{Im}\int\overline{R}_k\varepsilon\varphi_k'(t)-2\operatorname{Im}\int\nabla\overline{R}_k\varepsilon+\operatorname{Im}\int\nabla\varepsilon\bar{\varepsilon}\varphi_k(t)+\mathrm{O}\left(\mathrm{e}^{-2\theta_0 t}\right).$$

By the properties of R_k and φ_k', we have $|\int \overline{R}_k \varepsilon \varphi_k'(t)| \leqslant C\,\mathrm{e}^{-2\theta_0 t}$, and so

$$\operatorname{Im}\int \nabla u \bar{u} \varphi_k(t) = \operatorname{Im}\int \nabla R_k \overline{R}_k - 2\operatorname{Im}\int \nabla \overline{R}_k \varepsilon + \operatorname{Im}\int \nabla \varepsilon \bar{\varepsilon} \varphi_k(t) + \mathrm{O}\big(\mathrm{e}^{-2\theta_0 t}\big).$$

Gathering these calculations, we obtain finally for $\mathcal{J}(t)$:

$$\mathcal{J}(t) = \sum_{k=1}^{K} \left(\omega_k(0) + \frac{|v_k|^2}{4}\right)\left\{\int \big|R_k(t)\big|^2 + 2\operatorname{Re}\int R_k(t)\varepsilon(t) + \int \big|\varepsilon(t)\big|^2 \varphi_k(t)\right\}$$
$$- v_k \cdot \left\{\operatorname{Im}\int \nabla R_k \overline{R}_k - 2\operatorname{Im}\int \nabla \overline{R}_k \varepsilon + \operatorname{Im}\int \nabla \varepsilon \bar{\varepsilon} \varphi_k(t)\right\} + \mathrm{O}\big(\mathrm{e}^{-2\theta_0 t}\big).$$

By the equation of R_k, and the orthogonality conditions on $\varepsilon(t)$, we have

$$-2\operatorname{Re}\int \big(\Delta \overline{R}_k + f\big(|R_k|^2\big)\overline{R}_k\big)\varepsilon + 2\left(\omega_k(0) + \frac{|v_k|^2}{4}\right)\operatorname{Re}\int \overline{R}_k \varepsilon + 2v_k \cdot \operatorname{Im}\int \nabla \overline{R}_k \varepsilon = 0,$$

which means that the terms of order 1 in ε all disappear when we sum $E(u(t))$ and $\mathcal{J}(t)$.

Therefore, with the definition of $H(\varepsilon, \varepsilon)$ we obtain

$$\mathcal{G}(t) = \sum_{k=1}^{K}\left\{E(Q_{\omega_k(t)}) + \omega_k^0 \int Q_{\omega_{k(t)}}^2\right\} + H\big(\varepsilon(t), \varepsilon(t)\big) + \big\|\varepsilon(t)\big\|_{H^1}^2 \beta\big(\big\|\varepsilon(t)\big\|_{H^1}\big) + \mathrm{O}\big(\mathrm{e}^{-2\theta_0 t}\big).$$

The proof of Lemma 6(i) is complete using (50). □

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