

Ground states of nonlinear Schrödinger equations with potentials

Solutions d'énergie minimale des équations de Schrödinger non-linéaires avec potentiel

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Received 1 September 2005; accepted 31 January 2006

Available online 7 July 2006

Abstract

In this paper we study the nonlinear Schrödinger equation:

$$\begin{cases} -\Delta u + V(x)u = f(x, u), \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

We give general conditions which assure the existence of *ground state solutions*. Under a Nehari type condition, we show that the standard Ambrosetti–Rabinowitz super-linear condition can be replaced by a more natural super-quadratic condition.

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Résumé

Dans cet article nous étudions l'équation non-linéaire de Schrödinger :

$$\begin{cases} -\Delta u + V(x)u = f(x, u), \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

Nous donnons les conditions générales qui garantissent l'existence de solutions d'énergie minimale. Sous une condition de type Nehari, nous démontrons que la condition super-linéaire d'Ambrosetti–Rabinowitz peut être remplacée par une condition super-quadratique plus naturelle.

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MSC: 35B05; 35J60

Keywords: Nonlinear Schrödinger equations; Ground state solutions; The Ambrosetti–Rabinowitz condition

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¹ Supported in part by NNSF of China (10161010), Fujian Provincial Natural Science Foundation of China (A0410015), and the Ky and Yu-Fen Fan Fund from AMS.

1. Introduction

We study the nonlinear Schrödinger equation with potentials:

$$\begin{cases} -\Delta u + V(x)u = f(x, u), \\ u \in H^1(\mathbb{R}^N). \end{cases} \quad (1.1)$$

We are concerned with the existence of *ground state solutions*, i.e., solutions corresponding to the least positive critical value of the variational functional:

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx - \int_{\mathbb{R}^N} F(x, u) dx,$$

where $F(x, u) = \int_0^u f(x, t) dt$.

To establish the existence of ground states, usually besides the growth condition on the nonlinearity and a Nehari type condition, the following superlinear condition due to Ambrosetti–Rabinowitz (e.g., [2,12]) is assumed:

(AR) There is $\mu > 2$ such that for $u \neq 0$ and $x \in \mathbb{R}^N$,

$$0 < \mu F(x, u) \leq u f(x, u),$$

where $F(x, u) = \int_0^u f(x, t) dt$.

This condition implies that for some $C > 0$, $F(x, u) \geq C|u|^\mu$.

In this paper we show that a weaker and more natural version suffices to assure the existence of a *ground state solution*. Instead of (AR) we assume the following super-quadratic condition

(SQ) $\lim_{|u| \rightarrow \infty} \frac{F(x, u)}{u^2} = \infty$, uniformly in x .

We always assume $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V(x) > 0$. We consider two cases of the potentials, one is periodic, i.e., the x -dependence is periodic; the other is when V has a bounded potential well in the sense that $\lim_{|x| \rightarrow \infty} V(x)$ exists and is equal to $\sup_{\mathbb{R}^N} V$. The results will be stated and proved in Sections 2 and 3.

We would like to mention earlier results on existence of entire solutions of Schrödinger type equations with or without potentials which was studied in [3,4,9,10] (see references therein). In recent years there have been intensive studies on semiclassical states for nonlinear Schrödinger equations for which in Eq. (1.1) there is a small parameter corresponding to the Plank constant. We refer [1] for references in this direction. Our results do not require smallness of such a parameter. A recent result in [5] is in similar spirit of our Theorem 3.1; but the conditions in [5] and ours are mutually non-inclusive and the methods are different.

For (1.1) in bounded domains or if the potential function $V(x)$ possesses certain compactness condition, one can prove (1.1) have certain solutions. In [8] Liu and Wang first used (SQ) to get the bounds of minimizing sequences on the Nehari manifold, and under coercive condition of $V(x)$ they proved the existence of three solutions: one positive, one negative, and one sign-changing. The results in this current paper are natural generalizations of that in [8] to noncompact cases. In the two cases we do not have compact embedding, which is the main difficulty in this paper. We shall make use of a combination of the techniques in [8,7] with applications of the concentration-compactness principle of Lions [6,11,12].

2. The periodic case

We consider weak solutions of

$$\begin{cases} -\Delta u + V(x)u = f(x, u), \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

We need the following assumptions:

(V₁) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V(x) \geq V_0 > 0$. $V(x)$ is 1-periodic in each of x_1, x_2, \dots, x_N .

(f₁) $f(x, t) \in C^1$ is 1-periodic in each of x_1, x_2, \dots, x_N , f_t is a Caratheodory function and there exists $C > 0$, such that

$$|f_t(x, t)| \leq C(1 + |t|^{2^*-2}), \quad \lim_{|t| \rightarrow \infty} \frac{|f(x, t)|}{|t|^{2^*-1}} = 0, \quad \text{uniformly in } x \in \mathbb{R}^N.$$

(f₂) $f(x, t) = o(|t|)$, as $|t| \rightarrow 0$, uniformly in x .

(f₃) $\lim_{|t| \rightarrow \infty} \frac{F(x, t)}{t^2} = \infty$, uniformly in x .

(f₄) $\frac{f(x, t)}{|t|}$ is strictly increasing in t .

Here $2^* = \frac{2N}{N-2}$ for $N \geq 3$. For $N = 1, 2$ we assume there is $q > 2$ in the place of 2^* in (f₁). We work in Hilbert space $X = \{u \in H^1(\mathbb{R}^N); \int_{\mathbb{R}^N} V(x)u^2 dx < \infty\}$, with norm $\|u\|^2 = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx$. The functional associated with Eq. (1.1) is

$$\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx - \int_{\mathbb{R}^N} F(x, u) dx, \quad u \in X.$$

Define

$$\gamma(u) = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) dx - \int_{\mathbb{R}^N} f(x, u)u dx.$$

Theorem 2.1. Under assumptions (V₁), (f₁)–(f₄) Eq. (1.1) has a weak solution $u \in X$, such that $\Phi(u) = c > 0$, c is defined as

$$c = \inf_{\mathcal{N}} \Phi(u),$$

where $\mathcal{N} = \{u \in X: u \neq 0, \gamma(u) = 0\}$.

First we need a few lemmas.

Lemma 2.2. Let (u_n) be a minimizing sequence for c . Then

- (i) There is $\beta > 0$ such that $\liminf_{n \rightarrow \infty} \|u_n\| \geq \beta$.
- (ii) (u_n) is bounded in X .
- (iii) For a subsequence, up to translations, u_n converges weakly to $u \neq 0$.

Proof. (i) The proof is similar to the case with (AR) satisfied. We omit its proofs (see [12]).

(ii) Let (u_n) be a minimizing sequence of c . If (u_n) is not bounded, we define $v_n = u_n / \|u_n\|$, so $\|v_n\| = 1$. Passing to a subsequence, we may assume, $v_n \rightharpoonup v$ in X , $v_n \rightarrow v$ in $L^p_{loc}(\mathbb{R}^N)$, $2 \leq p < 2^*$, $v_n \rightarrow v$ a.e. on \mathbb{R}^N .

If $v \neq 0$, we have

$$\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{u_n^2} v_n^2 dx = \frac{c + o(1)}{\|u_n\|^2} > 0.$$

By Fadou’s lemma and (f₃) we have a contradiction as follows,

$$\frac{1}{2} \geq \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} \frac{F(x, u_n)}{u_n^2} v_n^2 dx \geq \int_{\mathbb{R}^N} \liminf_{n \rightarrow \infty} \frac{F(x, u_n)}{u_n^2} v_n^2 dx = \infty.$$

If $v = 0$, we take $y_n = (y_n^1, y_n^2, \dots, y_n^N) \in \mathbb{N}^N$ with all y_n^i ($1 \leq i \leq N$) being integers. Define translations of v_n by $w_n(x) = v_n(x + y_n)$. Since $V(x)$ and $f(x, u)$ are periodic, we have $\|w_n\| = \|v_n\| = 1$, $|w_n|_p = |v_n|_p$, and $\Phi(w_n) = \Phi(v_n)$. Passing to a subsequence, we have $w_n \rightharpoonup w$ in $H^1(\mathbb{R}^N)$, $w_n \rightarrow w$ in $L^p_{loc}(\mathbb{R}^N)$, $2 \leq p < 2^*$, $w_n \rightarrow w$ a.e.

on \mathbb{R}^N . If there exist y_n , such that $w_n \rightharpoonup w \neq 0$, we will get a contradiction as the case of $v \neq 0$. If for any y_n , $w_n \rightharpoonup 0$, we will get a contradiction by proving $v_n \rightarrow 0$ in $L^p(\mathbb{R}^N)$. In this case, we claim for all $p \in (2, 2^*)$,

$$\limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_2(y)} |v_n|^p dx = 0.$$

If this is not true, there exists $p \in (2, 2^*)$, $\delta > 0$,

$$\limsup_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_2(y)} |v_n|^p dx \geq \delta > 0,$$

then there exists $z_n \in \mathbb{R}^N$ such that, $\lim_{n \rightarrow \infty} \int_{B_2(z_n)} |v_n|^p dx \geq \delta/2 > 0$. We can choose $y_n \in \mathbb{N}^N \in B_2(z_n)$ such that $B_1(y_n) \subset B_2(z_n)$ and

$$\lim_{n \rightarrow \infty} \int_{B_1(y_n)} |v_n|^p dx \geq \frac{\delta}{2} > 0,$$

we have $\lim_{n \rightarrow \infty} \int_{B_1(0)} |w_n|^p dx \geq \delta/2 > 0$, that is $w_n \rightharpoonup w \neq 0$, a contradiction.

By Lions Lemma (cf. [12, Lemma 1.21]), we get $v_n \rightarrow 0$ in $L^p(\mathbb{R}^N)$, $p \in (2, 2^*)$. Fix $p \in (2, 2^*)$. By (f_1) and (f_2) , for any $\varepsilon > 0$ there is $C_\varepsilon > 0$ such that $|f(x, u)| \leq \varepsilon(|u| + |u|^{2^*-1}) + C_\varepsilon|u|^{p-1}$. Then $|F(x, u)| \leq \varepsilon(|u|^2 + |u|^{2^*}) + C_\varepsilon|u|^p$. Then fixing an $R > \sqrt{2c}$, using Lebesgue Dominated Convergence theorem, we have

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(x, Rv_n) dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(x, Rv_n) dx = 0.$$

Since by (f_4) , $\Phi(tu_n) \leq \Phi(u_n)$ for $t \geq 0$ we thus have

$$c + o(1) = \Phi(u_n) \geq \Phi(Rv_n) = \frac{1}{2}R^2 - \int_{\mathbb{R}^N} F(x, Rv_n) dx,$$

which is a contradiction. Thus (u_n) is bounded.

(iii) We can assume u_n weakly converges to u . To show $u \neq 0$, again we define translations of u_n as above, assume $y_n = (y_n^1, y_n^2, \dots, y_n^N) \in \mathbb{N}^N$, with all y_n^i ($1 \leq i \leq N$) being integers. $u_n^{y_n} = u_n(x + y_n)$ are all possible translation of u_n . If for some $y_n \subset \mathbb{N}^N$, $u_n^{y_n} \rightharpoonup u \neq 0$ we are done. If for any $y_n \subset \mathbb{N}^N$, $u_n^{y_n} \rightharpoonup 0$, by similar argument as above we can prove $u_n \rightarrow 0$ in $L^p(\mathbb{R}^N)$, $p \in (2, 2^*)$. Then as $n \rightarrow \infty$, $\int_{\mathbb{R}^N} u_n f(x, u_n) dx \rightarrow 0$. Thus by (i) we have a contradiction:

$$0 < \beta \leq \|u_n\|^2 = \int_{\mathbb{R}^N} u_n f(x, u_n) dx \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad \square$$

Lemma 2.3. For each $u \in X \setminus \{0\}$, there exists unique $t = t(u) > 0$, such that $tu \in \mathcal{N}$.

This is similar to the case of assuming (AR), we omit it.

Lemma 2.4. Let $(u_n) \subset X$ be a sequence such that $\gamma(u_n) \rightarrow 0$ and $\int_{\mathbb{R}^N} f(x, u_n)u_n \rightarrow a > 0$ as $n \rightarrow \infty$. Then exist $t_n > 0$ such that $t_n u_n \in \mathcal{N}$, $t_n \rightarrow 1$, as $n \rightarrow \infty$.

Proof. Since $u_n \neq 0$, by Lemma 2.3, there exists only one $t_n > 0$, such that $t_n u_n \in \mathcal{N}$, i.e.

$$t_n^2 \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)|u_n|^2) dx - \int_{\mathbb{R}^N} f(x, t_n u_n)t_n u_n dx = 0.$$

By (f_1) and (f_2) , $|f(x, u)u| \leq \varepsilon(|u|^2 + |u|^{2^*}) + C_\varepsilon|u|^p$, we see t_n cannot go zero, that is $t_n \geq t_0 > 0$. By (f_4) , $f(x, u)u \geq 2F(x, u)$. If $t_n \rightarrow \infty$, we get

$$a + o(1) = \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)|u_n|^2) dx = \int_{\mathbb{R}^N} \frac{f(x, t_n u_n)t_n u_n}{t_n^2} dx \geq 2 \int_{\mathbb{R}^N} \frac{F(x, t_n u_n)}{t_n^2 u_n^2} u_n^2 dx.$$

By the condition, up to translations, $u_n \rightarrow u \neq 0$ a.e. in \mathbb{R}^N . We have

$$\int_{\mathbb{R}^N} \frac{F(x, t_n u_n)}{t_n^2 u_n^2} u_n^2 dx \rightarrow +\infty, \quad \text{as } n \rightarrow \infty$$

a contradiction. Thus $0 < t_0 \leq t_n \leq C$. Assume $t_n \rightarrow T$, now we claim $T = 1$. Since $t_n u_n \in \mathcal{N}$, by $\gamma(u_n) \rightarrow 0$ we have

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)|u_n|^2) dx = \int_{\mathbb{R}^N} f(x, u_n) u_n dx + o(1).$$

Since $t_n \rightarrow T$, by (f_1) and (f_3)

$$T^2 \int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)|u_n|^2) dx - \int_{\mathbb{R}^N} f(x, T u_n) T u_n dx = o(1),$$

that is

$$o(1) = \int_{\mathbb{R}^N} \frac{f(x, T u_n)}{T u_n} u_n^2 - \frac{f(x, u_n)}{u_n} u_n^2 dx = \int_{\mathbb{R}^N} \left(\frac{f(x, T u_n)}{T u_n} - \frac{f(x, u_n)}{u_n} \right) u_n^2 dx.$$

For a subsequence $u_n \rightarrow u$ in $L^p_{\text{loc}}(\mathbb{R}^N)$ $2 \leq p < 2^*$. Up to translations, we may assume $u \neq 0$. Then by (f_4) and Fatou’s lemma

$$\int_{\mathbb{R}^N} \left(\frac{f(x, T u)}{T u} - \frac{f(x, u)}{u} \right) u^2 dx = 0,$$

by (f_4) we have $T = 1$. \square

Next we construct a special minimizing sequence along which $\int_{\mathbb{R}^N} F(x, u)$ is weakly continuous. Consider Eq. (1.1) on $B_R(0)$,

$$\begin{cases} -\Delta u + V(x)u = f(x, u), & \text{in } B_R(0), \\ u = 0, & \text{on } \partial B_R(0). \end{cases} \tag{2.1}$$

We can similarly define \mathcal{N}_R, c_R . By the result of [8], c_R is achieved by a positive solution of (2.1) called u_R . It is easy to check that $c_R > c$ and $c_R \rightarrow c$ as $R \rightarrow \infty$. This implies (u_R) as $R \rightarrow \infty$ minimizes c . Let $R_n \rightarrow \infty, u_n := u_{R_n}$. Fix $p \in (2, 2^*)$.

Lemma 2.5.

- (i) $\int_{\mathbb{R}^N} |u_n|^p \rightarrow A > 0$.
- (ii) There exist $x_n \in \mathbb{R}^N$ such that $\forall \varepsilon > 0, \exists R > 0, \liminf \int_{B_R(x_n)} |u_n|^p \geq A - \varepsilon$.

Proof. (i) follows from $\gamma(u_n) = 0$ and the fact that for any $\varepsilon > 0$ there is $C_\varepsilon > 0, |f(x, u)| \leq \varepsilon(|u| + |u|^{2^*-1}) + C_\varepsilon |u|^{p-1}$.

For (ii) we apply the concentration compactness principle to $\int_{\mathbb{R}^N} |u_n|^p$. Then there exist $\alpha \in (0, 1), (x_n) \subset \mathbb{R}^N, \forall \varepsilon > 0, \exists R > 0, \forall r > R, r' > R$, have

$$\liminf \int_{B_r(x_n)} |u_n|^p \geq \alpha A - \varepsilon, \quad \liminf \int_{B_{r'}^c(x_n)} |u_n|^p \geq (1 - \alpha)A - \varepsilon.$$

Next we claim $\alpha = 1$. Choose $\varepsilon_n \rightarrow 0, r_n \rightarrow \infty, r'_n = 4r_n$. Let ξ be a cut-off function such that $\xi(s) = 0$, for $s \leq 1$ or $s \geq 4, \xi(s) = 1$, for $2 \leq s \leq 3$, and $|\xi'(s)| \leq 2$. Take $\phi(x) = \xi(|x - x_n|/r_n) u_n$. Using equation

$$\int_{B_{R_n}} (\nabla u_n \nabla \phi + V(x) u_n \phi - f(x, u_n) \phi) dx = 0,$$

we have,

$$\int_{B_{3r_n}(x_n) \setminus B_{2r_n}(x_n)} (|\nabla u_n|^2 + V(x)|u_n|^2) \, dx + \int_{B_{3r_n}(x_n) \setminus B_{2r_n}(x_n)} f(x, u_n)u_n \, dx = o(1).$$

Take another cut-off function η such that $\eta(s) = 1$, for $s \leq 2$, $\eta(s) = 0$, for $s \geq 3$, and $|\eta'(s)| \leq 2$, for $2 \leq s \leq 3$. Set

$$w_n(x) = \eta\left(\frac{|x - x_n|}{r_n}\right)u_n, \quad v_n(x) = \left(1 - \eta\left(\frac{|x - x_n|}{r_n}\right)\right)u_n(x).$$

Using equation as above we have

$$\Phi(u_n) = \Phi(w_n) + \Phi(v_n) + o(1),$$

and

$$\int_{\mathbb{R}^n} |w_n|^p \geq \alpha A - \varepsilon_n, \quad \int_{\mathbb{R}^n} |v_n|^p \geq (1 - \alpha)A - \varepsilon_n.$$

Finally using w_n to test the equation for (u_n) we get

$$\gamma(w_n) = \langle \Phi'(u_n), w_n \rangle + o(1) = o(1).$$

Similarly $\gamma(v_n) = o(1)$, by Lemma 2.4, $\exists t_n \rightarrow 1, s_n \rightarrow 1$, such that $t_n w_n \in \mathcal{N}, s_n v_n \in \mathcal{N}$. Then

$$c + o(1) = \Phi(u_n) = \Phi(w_n) + \Phi(v_n) + o(1) = \Phi(t_n w_n) + \Phi(s_n v_n) + o(1) \geq 2c + o(1),$$

which is a contradiction. Thus $\alpha = 1$. \square

Proof of Theorem 2.1. Let $(u_n) \subset \mathcal{N}$ be the minimizing sequence for c given above. By Lemma 2.2 (u_n) is bounded in X and weak convergent to $u \neq 0$. By Lemma 2.5, $-\int_{\mathbb{R}^N} F(x, u_n)$ is weakly continuous. Using the weakly lower semi-continuity we have $\Phi(u) \leq c$. If $u \in \mathcal{N}$ we have $\Phi(u) = c$. If $u \notin \mathcal{N}$, by Lemma 2.5, there is $t > 0$ such that $tu_n \in \mathcal{N}$. Then

$$c \leq \Phi(tu) \leq \liminf_{n \rightarrow \infty} \Phi(tu_n) \leq \liminf_{n \rightarrow \infty} \Phi(u_n) = c.$$

Since \mathcal{N} is smooth, the minimizer is a critical point of Φ . \square

3. The potential well case

We consider weak solutions of

$$\begin{cases} -\Delta u + V(x)u = f(u), \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{3.1}$$

for the case where potential function $V(x)$ has a bounded potential well. Since the nonlinearity is autonomous, the conditions on f needs modified slightly. More precisely, we make the following assumptions.

(V₂) $0 < \inf_{\mathbb{R}^N} V(x) \leq \lim_{|x| \rightarrow \infty} V(x) = \sup_{\mathbb{R}^N} V(x) < \infty$.

(f₁) $f(t) \in C^1$. f_t is a Caratheodory function and there exists $C > 0$, s. t.

$$|f_t(t)| \leq C(1 + |t|^{2^*-2}), \quad \lim_{|t| \rightarrow \infty} \frac{|f(t)|}{|t|^{2^*-1}} = 0.$$

(f₂) $f(t) = o(|t|)$, as $|t| \rightarrow 0$.

(f₃) $\lim_{|t| \rightarrow \infty} \frac{F(t)}{t^2} = \infty$.

(f₄) $\frac{f(t)}{|t|}$ is strictly increasing in t .

Theorem 3.1. Under assumptions (V_2) , (f_1) – (f_4) Eq. (3.1) has a weak solution $u \in X$, such that $\Phi(u) = c > 0$, c is defined as

$$c = \inf_{\mathcal{N}} \Phi(u),$$

where $\mathcal{N} = \{u \in X: u \neq 0, \gamma(u) = 0\}$.

In this section we denote $V_\infty = \lim_{|x| \rightarrow \infty} V(x)$. There is an associated problem

$$\begin{cases} -\Delta u + V_\infty u = f(u), \\ u \in H^1(\mathbb{R}^N). \end{cases}$$

We define the energy functional Φ_∞ by replacing V with V_∞ , $c_\infty = \inf_{\mathcal{N}_\infty} \Phi_\infty(u)$, here $\mathcal{N}_\infty = \{u \in X/\{0\}: \langle \Phi'_\infty(u), u \rangle = 0\}$. Since V_∞ is a constant, by Theorem 2.1, $c_\infty > 0$ is achieved at some $u_\infty \in \mathcal{N}_\infty$.

Lemma 3.2. $0 < c < c_\infty$.

Proof. It is easy to see $c > 0$. Let u_∞ be the minimizer of c_∞ . Then $\gamma(u_\infty) < 0$, and there is $t > 0$ such that $tu_\infty \in \mathcal{N}$. We have

$$c \leq \Phi(tu_\infty) < \Phi_\infty(tu_\infty) \leq \Phi_\infty(u_\infty) = c_\infty. \quad \square$$

We note that with minor changes Lemma 2.3 and 2.4 still hold.

Lemma 3.3. Let (u_n) be a minimizing sequence for c . Then

- (i) There is $\beta > 0$ such that $\liminf_{n \rightarrow \infty} \|u_n\| \geq \beta$.
- (ii) (u_n) is bounded in X .
- (iii) For a subsequence, u_n converges weakly to $u \neq 0$.

Proof. (i) The same as Lemma 2.2.

(ii) If not, define $v_n = u_n/\|u_n\|$. Passing to a subsequence, we may assume, $v_n \rightharpoonup v$ in X . If $v_n \rightarrow 0$ in $L^q(\mathbb{R}^N)$ for $2 \leq q < 2^*$, we use the Lebesgue Dominated Convergence theorem to get for any $R > 0$ fixed, $\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} F(Rv_n) dx = 0$. Therefore a contradiction by choosing a large $R > 0$ in $\Phi(u_n) \geq \Phi(Rv_n) = \frac{1}{2}R^2 - \int_{\mathbb{R}^N} F(Rv_n) dx$. Thus by the concentration compactness principle there are $y_n \in \mathbb{R}^N$ such that $w_n(x) = v_n(y_n + x) \rightarrow w \neq 0$. Then the proof follows from the arguments in Lemma 2.2(ii). Thus (u_n) is bounded.

(iii) We can assume $u_n \rightharpoonup u$ in X , $u_n \rightarrow u$ in $L^p_{loc}(\mathbb{R}^N)$. If $u = 0$, we have $\int_{\mathbb{R}^N} (V(x) - V_\infty)|u_n|^2 dx \rightarrow 0$, as $n \rightarrow \infty$. Thus we have $c + o(1) = \Phi_\infty(u_n) + o(1)$. Similarly we have $\gamma(u_n) = 0$, $\gamma_\infty(u_n) = o(1)$. By Lemma 2.4 there exist $t_n \rightarrow 1$ such that $t_n u_n \in \mathcal{N}_\infty$. Then we have $c + o(1) = \Phi_\infty(u_n) + o(1) = \Phi_\infty(t_n u_n) + o(1) \geq c_\infty + o(1)$, a contradiction with Lemma 3.2. \square

Next consider Eq. (3.1) on $B_R(0)$,

$$\begin{cases} -\Delta u + V(x)u = f(u), & \text{in } B_R(0), \\ u = 0, & \text{on } \partial B_R(0) \end{cases} \tag{3.2}$$

we can similarly define $\mathcal{N}_R = \mathcal{N} \cap H^1_0(B_R)$, c_R . By the result of [8], c_R is achieved by a positive solution called u_R . It is easy to check that $c_R > c$ and $c_R \rightarrow c$ as $R \rightarrow \infty$.

Lemma 3.4. Let $u_R \in \mathcal{N}_R$ be a minimizer of c_R . Assume for a subsequence $R_n \rightarrow \infty$, $\int_{B_{R_n}} |u_n|^p \rightarrow A \in (0, \infty)$, where $u_n = u_{R_n}$. Then there exists $(y_n) \subset \mathbb{R}^N$ s.t. for any $\varepsilon > 0$, exists $r_\varepsilon > 0$, for all $r \geq r_\varepsilon$,

$$\liminf_{n \rightarrow \infty} \int_{B_r(y_n)} |u_n|^p \geq A - \varepsilon.$$

Proof. Note that u_R satisfies (3.3) for all $\varphi \in H_0^1(B_R)$,

$$\int_{B_{R_n}} (\nabla u_R \nabla \varphi + V(x) u_R \varphi) dx - \int_{\mathbb{R}^N} f(u_R) \varphi dx = 0. \quad (3.3)$$

Since (u_n) is bounded in $H^1(\mathbb{R}^N)$, by using the concentration compactness principle, exists $\alpha \in (0, 1]$ and $(y_n) \subset \mathbb{R}^N$ s.t. for any $\varepsilon > 0$, there exists $r_\varepsilon > 0$, for all $r' \geq r \geq r_\varepsilon$,

$$\liminf_{n \rightarrow \infty} \int_{B_r(y_n)} |u_n|^p \geq \alpha A - \varepsilon, \quad (3.4)$$

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_{r'}(y_n)} |u_n|^p \geq (1 - \alpha)A - \varepsilon. \quad (3.5)$$

Now suppose $\alpha < 1$, then following exactly the same construction as in Lemma 2.5 we have two sequences w_n and v_n satisfying

$$\liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |w_n|^p \geq \alpha A, \quad \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^N} |v_n|^p \geq (1 - \alpha)A, \quad \Phi(u_n) = \Phi(w_n) + \Phi(v_n) + o(1).$$

Moreover, if we take $\varphi = w_n$, by (3.3)

$$\gamma(w_n) = \langle \Phi'(u_n), w_n \rangle + o(1) = o(1).$$

Similarly, $\gamma(v_n) = o(1)$. By Lemma 2.4, there exist $t_n \rightarrow 1$, $s_n \rightarrow 1$, s.t.

$$\tilde{w}_n = t_n w_n \in \mathcal{N}, \quad \tilde{v}_n = s_n v_n \in \mathcal{N}.$$

If (y_n) is bounded, then $\liminf_{n \rightarrow \infty} \Phi(\tilde{w}_n) \geq c$ and $\liminf_{n \rightarrow \infty} \Phi(\tilde{v}_n) \geq c_\infty$. If (y_n) is unbounded, then

$$\liminf_{n \rightarrow \infty} \Phi(\tilde{w}_n) \geq c_\infty \quad \text{and} \quad \liminf_{n \rightarrow \infty} \Phi(\tilde{v}_n) \geq c.$$

Altogether, we have

$$\Phi(u_n) = \Phi(w_n) + \Phi(v_n) + o(1) = \Phi(t_n w_n) + \Phi(s_n v_n) + o(1),$$

and

$$\liminf \Phi(u_n) \geq \liminf \Phi(t_n w_n) + \liminf \Phi(s_n v_n) \geq c + c_\infty.$$

A contradiction, so we have $\alpha = 1$. \square

Proof of Theorem 3.1. Let $(u_n) \subset \mathcal{N}$ be the minimizing sequence for c given in Lemma 3.4. Let $A = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |u_n|^p dx$. By Lemma 3.3, (u_n) is bounded in X and weakly converges to $u \neq 0$. By Lemma 3.4, there exists $(y_n) \subset \mathbb{R}^N$ s.t. $\forall \varepsilon > 0, \exists r > 0$,

$$\liminf_{n \rightarrow \infty} \int_{B_r(y_n)} |u_n|^p \geq A - \varepsilon.$$

Then (y_n) must be bounded. Otherwise, $\gamma_\infty(u_n) = \gamma(u_n) + o(1)$. We find $t_n \rightarrow 1$, s.t. $\gamma(t_n u_n) = 0$. Then we have

$$c_\infty \leq \liminf \Phi_\infty(t_n u_n) = \liminf \Phi_\infty(u_n) = \liminf \Phi(u_n) = c$$

a contraction with $c < c_\infty$. Now, when (y_n) is bounded, we have $u_n \rightarrow u$ in $L^p(\mathbb{R}^N)$. This gives that along this sequence $\Phi(u_n)$ is weakly lower semi-continuous, we have

$$c = \inf_{\mathcal{N}} \Phi(u) \leq \Phi(u) \leq \liminf \Phi(u_n) = c. \quad \square$$

Remark 3.5. Though we assume in this section f depends only on t , looking at the proofs we see the arguments can be used with little changes to deal with the following case: $f = b(x)f(t)$ with b satisfying $b \in C^1(\mathbb{R}^N, \mathbb{R})$, $b_1 \leq b(x) \leq b_2$ for some $b_1, b_2 > 0$, and $b(x) \geq \inf_{\mathbb{R}^N} b(x) = \lim_{|x| \rightarrow \infty} b(x)$. The precisely statement is the same.

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