On quasiconvex hulls in symmetric 2 × 2 matrices

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Abstract

In this paper we study the quasiconvex hull of compact sets of symmetric 2×2 matrices. We are interested in situations where the quasiconvex hull can be separated into smaller independent pieces. Our main result is a geometric criterion which is sufficient for the quasiconvex hull of the union of two compact sets $K_1 \cup K_2$ to separate in the sense that $(K_1 \cup K_2)^{\mathrm{qc}} = K_1^{\mathrm{qc}} \cup K_2^{\mathrm{qc}}$. The key point in the proof is a kind of directional maximum principle for second order elliptic equations in the plane in non-divergence form with measurable coefficients.
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Résumé

On étudie les enveloppes quasiconvexes des ensembles compacts de matrices symétriques 2×2. On s'intéresse aux situations où l'enveloppe quasiconvexe se laisse séparer dans des morceaux indépendants plus petits. Le résultat principal est un critère géométrique suffisant pour l'enveloppe quasiconvexe d'une union de deux ensembles compacts $K_1 \cup K_2$ pour se séparer comme $(K_1 \cup K_2)^{\mathrm{qc}} = K_1^{\mathrm{qc}} \cup K_2^{\mathrm{qc}}$. Le point essentiel dans la preuve est un principe du maximum directionnel pour les équations elliptiques de deuxième ordre dans le plan sous la forme non-divergence avec des coefficients mesurables.
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1. Introduction

A central issue in the calculus of variations is the study of approximate and exact (Lipschitz) solutions to differential inclusions of the form

$$\nabla u(x) \in K \quad \text{for almost every } x \in \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^n$ is a simply connected domain, $K \subset \mathbb{R}^{m \times n}$ is a prescribed (compact) set of matrices and $u : \Omega \to \mathbb{R}^m$. A typical problem concerning approximate solutions can be described as follows. Suppose $\{u_j\}$ is a sequence of uniformly Lipschitz functions such that $\mathrm{dist}(\nabla u_j, K) \to 0$ in $L^1(\Omega)$ and $u_j \to u$ uniformly. In general the limit u need not be a solution of (1) due to the presence of rapid oscillations in the sequence. It is of importance, both theoretically and practically (in connection with variational approaches to material microstructure, see for example in

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[5,9,18]) to characterize the set of possible values of the limit gradient $\nabla u(x)$ and more specifically to find conditions on the set K which ensure that the limit is a solution.

It is well known that oscillations in a sequence of gradients $\{\nabla u_j\}$ can be characterized in terms of the gradient Young measure $\{\nu_x\}_{x \in \Omega}$ generated by the sequence, and the gradient of the limit is given by the formula $\nabla u(x) = \bar{\nu}_x$. Here $\bar{\nu}_x$ denotes the barycenter (or center of mass) of the probability measure ν_x for each $x \in \Omega$. In terms of $\{\nu_x\}_{x \in \Omega}$ the issue is to characterize the set of barycenters $\bar{\nu}_x$ for gradient Young measures such that $\text{supp}\,\nu_x \subset K$. An important tool here is localization in the sense that for almost every $x \in \Omega$ the measure ν_x coincides with a *homogeneous* gradient Young measure (see Section 2). Accordingly, the set of possible values of ∇u for limits of approximate solutions to (1) is given by the quasiconvex hull K^{qc}, defined as

$$K^{qc} = \{\bar{\nu}: \ \nu \text{ is a homogeneous gradient Young measure, } \text{supp}\,\nu \subset K\}. \tag{2}$$

Thus the problem formulated above amounts to finding conditions on K which ensure that $K^{qc} = K$.

The first basic technique for estimating the quasiconvex hull K^{qc} is to use the fact that the rank-one convex hull K^{rc} and the polyconvex hull K^{pc} provide an inner and outer estimate respectively. More precisely,

$$K^{rc} \subset K^{qc} \subset K^{pc}.$$

For certain sets with high symmetry $K^{pc} = K^{rc}$ (see [8,10]), which implies also equality for the quasiconvex hull. However, in general neither equality holds, although it remains an outstanding open problem whether $K^{rc} = K^{qc}$ in the space of 2×2 matrices. By using the characterization of the quasiconvex hull via duality (see (11) in Section 2), in principle further restrictions on K^{qc} can be obtained from explicit quasiconvex functions which are not polyconvex, but in practice such examples are in very short supply (see [13,20]). Another approach is via separation of homogeneous gradient Young measures, as follows.

Definition 1. Two disjoint sets $U_1, U_2 \subset \mathbb{R}^{m \times n}$ are said to be separating for homogeneous gradient Young measures[1] if whenever ν is a homogeneous gradient Young measure with $\text{supp}\,\nu \subset U_1 \cup U_2$, then $\text{supp}\,\nu \subset U_1$ or $\text{supp}\,\nu \subset U_2$.

Thus, if it is possible to find separating sets U_1, U_2 such that $K = K_1 \cup K_2$ with $K_i \subset U_i$, then from (2) it follows that

$$K^{qc} = K_1^{qc} \cup K_2^{qc}.$$

This enables one to reduce the calculation of K^{qc} to the two smaller problems of calculating K_1^{qc} and K_2^{qc}, where the rank-one convex and the polyconvex hulls will provide better estimates. A standard example of separating sets occurs in the space of symmetric 2×2 matrices $\mathbb{R}^{2 \times 2}_{sym}$:

Example 1. Let $U_1, U_2 \subset \mathbb{R}^{2 \times 2}_{sym}$ be given by

$$U_1 = \{\text{positive definite matrices}\}, \qquad U_2 = \{\text{negative definite matrices}\}.$$

Then U_1, U_2 are separating for homogeneous gradient Young measures.

We recall the argument for the convenience of the reader. Note that for symmetric 2×2 matrices $\{\det > 0\} = U_1 \cup U_2$. The main point is to use V. Šverák's examples of quasiconvex functions (see [20])

$$F_l(A) = \begin{cases} |\det A| & \text{if the index of } A \text{ is } l, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

The index of A is the number of negative eigenvalues. If ν is a homogeneous gradient Young measure with $\text{supp}\,\nu \subset U_1 \cup U_2 = \{\det > 0\}$, then $\det \bar{\nu} = \int \det d\nu > 0$ (because det is quasi-affine, see Section 2) and therefore the matrix $\bar{\nu}$ is either positive definite or negative definite. Let us assume that $\bar{\nu}$ is positive definite (i.e. $\bar{\nu} \in U_1$). Since $F_0 - \det$ is also quasiconvex, we have

$$0 = F_0(\bar{\nu}) - \det(\bar{\nu}) \leqslant \int F_0 - \det d\nu. \tag{4}$$

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[1] Such sets are called homogeneously incompatible in [4].

On the other hand $F_0 - \det \leqslant 0$ on $U_1 \cup U_2$, in particular $F_0 - \det \leqslant 0$ on supp ν. Thus from (4) we deduce that necessarily $F_0 - \det \equiv 0$ on supp ν and this implies that supp $\nu \subset U_1$. For the case when $\bar\nu \in U_2$, we use F_2 instead of F_0 in the same argument.

In this paper we are concerned with the case which is in some sense complementary to Example 1, when

$$K_1 \cup K_2 \subset \{\det < 0\}.$$

The main difference is that in $\mathbb{R}^{2\times2}_{\mathrm{sym}}$ the set $\{\det > 0\}$ has two disjoint components whereas $\{\det < 0\}$ is connected. Indeed, even when considering the rank-one convex hulls, additional conditions are needed for separation. In [21] it was proved (Theorem 4) that if K_1, K_2 are sign-separated in the sense that $\det(A - B) < 0$ whenever $A \in K_1$ and $B \in K_2$, then they are separating for the rank-one convex hull if and only if the following condition holds:

Condition 1. There exists a curve $\gamma : \mathbb{R} \to \mathbb{R}^{2\times2}_{\mathrm{sym}}$ such that the set

$$\mathcal{U} := \big\{A: \det(A - B) < 0 \text{ for all } B \in \gamma\big\}$$

consists of at least two connected components and K_1, K_2 lie in different components.

To explain geometrically what this condition means, let $\Lambda = \{A \in \mathbb{R}^{2\times2}_{\mathrm{sym}}: \det A \geqslant 0\}$ and let $\Lambda_B = B + \Lambda = \{B + A: A \in \Lambda\}$. We note that Λ is a double-sided (solid) cone[2] with vertex at the origin. Then the set $\mathcal{U}$ is simply the complement of the set $\bigcup_{B\in\gamma} \Lambda_B$. In other words the sets K_1 and K_2 are separating for the rank-one convex hull if and only if it is possible for the cone Λ to "travel through the middle" in such a way that it doesn't touch K_1 or K_2.

The question arises whether two sets are separating for homogeneous gradient Young measures under Condition 1. We give a positive answer in the simplest case, when γ is a straight line.

Theorem 1. *Let ν be a homogeneous gradient Young measure with* supp $\nu \subset K_1 \cup K_2 \subset \mathbb{R}^{2\times2}_{\mathrm{sym}}$ *and suppose that there exists $B \in \mathbb{R}^{2\times2}_{\mathrm{sym}}$ with $\det B < 0$ such that*

$$K_1 \cup K_2 \subset \mathcal{U}_B := \big\{A: \det(A - tB) < 0 \text{ for all } t \in \mathbb{R}\big\} \tag{5}$$

with K_1 and K_2 contained in different connected components of $\mathcal{U}_B$. Then

$$\text{supp}\,\nu \subset K_1 \quad or \quad \text{supp}\,\nu \subset K_2.$$

Previously results concerning the separation of gradient Young measures were obtained in [4,16,19,23]. It should be pointed out however, that these results deal mainly with separation of non-homogeneous gradient Young measures, in the sense that if $\{\nu_x\}_{x\in\Omega}$ is a gradient Young measure with supp $\nu_x \subset K_1 \cup K_2$ for almost every $x \in \Omega$, then supp $\nu_x \subset K_1$ for almost every $x \in \Omega$ or supp $\nu_x \subset K_2$ for almost every $x \in \Omega$. This notion of separation is a lot stronger than separation in the sense of Definition 1, in particular it immediately implies separation for homogeneous gradient Young measures. The difference between the two notions is that in addition to restricting oscillations (i.e. preventing sequences from oscillating rapidly between K_1 and K_2), separation for non-homogeneous gradient Young measures amounts to a certain regularity in the form of control on the size of oscillations of the gradient. In fact the proofs in [16,23] rely on using elliptic regularity to obtain bounds on the size of oscillations.

In contrast, our motivation is to understand the relationship between the rank-one convex and the quasiconvex hull in $\mathbb{R}^{2\times2}$, and more specifically whether sets which are separating for the rank-one convex hull are also separating for the quasiconvex hull. This requires us to deal with large sets where it is not possible to prove *a priori* bounds on the size of oscillations, because the sets themselves could contain arbitrarily large rank-one lines. Indeed, our proof is based on a maximum principle type argument, taylored to prevent oscillations *between* K_1 and K_2. On the other hand it is not clear whether in general two disjoint compact sets K_1, K_2 satisfying the condition of the theorem would be in fact separating for non-homogeneous gradient Young measures.

Before setting out to prove Theorem 1 we show that with a simple change of variables it can be reduced to a special case. To this end note that

$$\det(A - tB) = \det A - t\,\mathrm{trace}(A\,\mathrm{cof}\,B) + t^2 \det B,$$

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[2] In coordinates $\left(\begin{smallmatrix} z+x & y \\ y & z-x \end{smallmatrix}\right)$ one has $\Lambda = \{z^2 \geqslant x^2 + y^2\}$.

hence (5) only makes sense for B with $\det B \leqslant 0$. If $\det B = 0$ then

$$\mathcal{U}_B = \big\{A \colon \det A < 0 \text{ and } \operatorname{trace}(A \operatorname{cof} B) = 0\big\},$$

and for subspaces with only one rank-one line it is well known that rank-one convexity and quasiconvexity coincide (see Theorem 4 in [17]). So the only non-trivial case is when $\det B < 0$.

Secondly, let $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. If $\det B < 0$, then there exists an orthogonal matrix P such that PBP^{T} is diagonal, say $PBP^{\mathrm{T}} = \left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{smallmatrix}\right)$ for some $\lambda_1, \lambda_2 > 0$. Therefore if

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & 0 \\ 0 & \frac{1}{\sqrt{\lambda_2}} \end{pmatrix} P,$$

then $QBQ^{\mathrm{T}} = J$. Hence by considering $\tilde{K}_i = QK_iQ^{\mathrm{T}}$ we may assume without loss of generality that $B = J$. Then $\mathcal{U}_B$ becomes

$$\mathcal{U}_J = \big\{A \colon |a_{11} + a_{22}| < |a_{11} - a_{22}|\big\}. \tag{6}$$

Since K_1 and K_2 are compact, if $K_1 \cup K_2 \subset \mathcal{U}_J$, then there exists $0 < k < 1$ such that $K_1 \cup K_2 \subset E_k$, which is defined below. The key observation is that this puts us in an elliptic setting where on the one hand we can use elliptic operators to project the generating sequence of the Young measure, and on the other hand apply maximum principle type arguments.

Definition 2. For $k > 0$ let

$$E_k = \big\{A \in \mathbb{R}^{2 \times 2}_{\mathrm{sym}} \colon |a_{11} + a_{22}| \leqslant k|a_{11} - a_{22}|\big\},$$

and let

$$E_k^{\pm} = \big\{A \in \mathbb{R}^{2 \times 2}_{\mathrm{sym}} \colon |a_{11} + a_{22}| \leqslant \pm k(a_{11} - a_{22})\big\}.$$

From the foregoing discussion it follows that Theorem 1 is implied by

Theorem 2. *Suppose ν is a homogeneous gradient Young measure with*

$$\operatorname{supp} \nu \subset E_k$$

for some $k \in (0, 1)$. Then

$$\operatorname{supp} \nu \subset E_k^+ \quad or \quad \operatorname{supp} \nu \subset E_k^-.$$

As an application of Theorem 1 we obtain equality of hulls for sets of the form

$$K = \left\{\begin{pmatrix} x & y \\ y & z \end{pmatrix} \colon |x| = a, |y| = b, |z| = c\right\} \tag{7}$$

for any $a, b, c > 0$. This set first appeared in [11] and the various semiconvex hulls have been calculated in [12] Section 2.1. In particular it is shown in [12] Theorem 2.1.1 that $K^{\mathrm{rc}} = K^{\mathrm{qc}}$ if $ac - b^2 \geqslant 0$, but the expression for the quasiconvex hull in the case $ac - b^2 < 0$ is left open (see Remark 2.1.3. in [12]). Using Theorem 1 we obtain

Corollary 1. *Let K be the set in (7) with $ac - b^2 < 0$. Then*

$$K^{\mathrm{qc}} = K^{\mathrm{rc}} = \big\{A \in K^c \colon |y| = b\big\}. \tag{8}$$

Proof. Let $B = \left(\begin{smallmatrix} a & 0 \\ 0 & -c \end{smallmatrix}\right)$, and consider the corresponding set $\mathcal{U}_B$. A simple calculation shows that

$$\mathcal{U}_B = \left\{\begin{pmatrix} x & y \\ y & z \end{pmatrix} \colon xz + \frac{1}{4ac}(cx - az)^2 - y^2 < 0\right\}.$$

We claim that $K \subset \mathcal{U}_B$. If $|x| = a$, $|y| = b$, $|z| = c$ with x and z having the same sign, then $xz + \frac{1}{4ac}(cx - az)^2 - y^2 = ac - b^2 < 0$, and if x and z have opposite signs, then $xz + \frac{1}{4ac}(cx - az)^2 - y^2 = -ac + ac - b^2 = -b^2 < 0$. Thus $K \subset \mathcal{U}_B$. Moreover, $\mathcal{U}_B$ consists of two connected components

$$\mathcal{U}_B = \mathcal{U}_1 \cup \mathcal{U}_2,$$

with $\mathcal{U}_i = \{A \in \mathcal{U}_B \colon (-1)^i A_{12} < 0\}$, and similarly $K = K_1 \cup K_2$ with $K_i = \{A \in K \colon (-1)^i A_{12} < 0\} \subset \mathcal{U}_i$. Hence Theorem 1 implies that $K^{\mathrm{qc}} = K_1^{\mathrm{qc}} \cup K_2^{\mathrm{qc}}$. But it is not difficult to see that $K_i^{\mathrm{rc}} = K_i^{\mathrm{qc}} = K_i^c$, from which (8) follows. □

During the referee process it was brought to the author's attention that in parallel with the current work B. Bojarski, L. D'Onofrio, T. Iwaniec and C. Sbordone, in connection with G-closure problems for first order elliptic operators in the plane, obtained a stronger version of our Theorem 3 in [6] (Corollary 7.1), using the theory of quasiconformal mappings. With the appropriate interpretation, the combination of their result with the techniques in Section 4 leads to the strengthening of Theorem 1 in the sense that separation of homogeneous gradient Young measures holds for sets K_1, K_2 in the *full space* $\mathbb{R}^{2\times 2}$ provided that the condition (5) applies.

In Section 3 we prove a kind of directional maximum principle for solutions to certain elliptic equations in non-divergence form. This is really the main point. In Section 4 we give the proof of Theorem 2. Finally, in Appendix A we provide the necessary L^p estimates for elliptic equations in non-divergence form, as done in [2].

2. Preliminaries

Throughout the paper we write $\mathbb{R}^{m\times n}$ for the space of $m \times n$ matrices, and $\mathbb{R}^{n\times n}_{\mathrm{sym}}$ for the space of $n \times n$ symmetric matrices. For functions $u \colon \mathbb{R}^n \to \mathbb{R}^m$ we denote by ∇u the full gradient, i.e. the matrix $\nabla u(x) = (\partial_i u^j(x))_{i=1\ldots n,\, j=1\ldots m}$, and for functions $u \colon \mathbb{R}^n \to \mathbb{R}$ we denote by $D^2 u$ the Hessian matrix,

$$D^2 u(x) = \left(\partial_i \partial_j u(x)\right)_{i,j=1\ldots n}.$$

Let $K \subset \mathbb{R}^{m\times n}$ be a compact set and let $u_j \colon \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ be a sequence of uniformly Lipschitz functions such that $\mathrm{dist}(\nabla u_j, K) \to 0$ strongly in $L^1(\Omega)$. The technical tool to describe possible oscillations in the sequence of gradients $\{\nabla u_j\}$ is the Young measure $\{\nu_x\}_{x\in\Omega}$ generated by the sequence (see e.g. [3,22]). Specifically, $\{\nu_x\}_{x\in\Omega}$ is a family of probability measures on $\mathbb{R}^{m\times n}$, depending measurably on $x \in \Omega$, such that $\mathrm{supp}\, \nu_x \subset K$ and for every $f \in C_0(\mathbb{R}^{m\times n})$

$$f(\nabla u_j) \overset{*}{\rightharpoonup} \int_{\mathbb{R}^{m\times n}} f(A)\, \mathrm{d}\nu_x(A) \quad \text{in } L^\infty(\Omega). \tag{9}$$

A function $f \colon \mathbb{R}^{m\times n} \to \mathbb{R}$ is said to be quasiconvex if for all $A \in \mathbb{R}^{m\times n}$

$$\int_\Omega f(A + \nabla \eta) - f(A)\, \mathrm{d}x \geqslant 0 \quad \text{for all } \eta \in C_0^\infty(\Omega, \mathbb{R}^m).[3]$$

D. Kinderlehrer and P. Pedregal proved in [15] that homogeneous gradient Young measures are in duality with quasiconvex functions via Jensen's inequality: a (compactly supported) probability measure μ on $\mathbb{R}^{m\times n}$ is a homogeneous gradient Young measure if and only if

$$f(\bar{\mu}) \leqslant \int f(A)\, \mathrm{d}\mu(A) \quad \text{for all } f \colon \mathbb{R}^{m\times n} \to \mathbb{R} \text{ quasiconvex}. \tag{10}$$

In particular if μ is a compactly supported probability measure with $\bar{\mu} = A$ such that (10) holds, then for any simply connected Lipschitz domain $\Omega \subset \mathbb{R}^n$ there exists a sequence of uniformly Lipschitz functions $u_j \colon \Omega \to \mathbb{R}^m$ such that $u_j(x) = Ax$ on $\partial\Omega$ and $\{\nabla u_j\}$ generates the Young measure μ in the sense of (9). Duality leads to an equivalent definition of quasiconvex hull:

$$K^{\mathrm{qc}} = \left\{ A \colon f(A) \leqslant \sup_K f \text{ for all quasiconvex } f \colon \mathbb{R}^{m\times n} \to \mathbb{R} \right\}. \tag{11}$$

---

[3] Here Ω is any simply connected Lipschitz domain. It can be easily shown that the definition is independent of the specific choice of domain.

Furthermore, a family of uniformly compactly supported probability measures $\{\nu_x\}_{x \in \Omega}$ (depending measurably on x) is a gradient Young measure if and only if there exists a Lipschitz function u such that $Du(x) = \bar{\nu}_x$ and ν_x is a homogeneous gradient Young measure for almost every $x \in \Omega$. For further information concerning the general theory of gradient Young measures as well as extensions to sequences of Sobolev maps, we refer the reader to [18].

Polyconvex functions are an important subclass of quasiconvex functions. A function $f : \mathbb{R}^{m \times n} \to \mathbb{R}$ is said to be polyconvex if there exists a convex function g such that $f(A) = g(M(A))$, where $M(A)$ denotes the vector of all minors of $A \in \mathbb{R}^{m \times n}$. Functions which are linear combinations of minors are quasi-affine in the sense that $L(\bar{\nu}) = \int L \, d\nu$ for all homogeneous gradient Young measures. For 2×2 matrices, the case of interest in this paper, the only quasi-affine function is the determinant $A \mapsto \det A$. A function is said to be rank-one convex if $t \mapsto f(A + tB)$ is convex whenever $B \in \mathbb{R}^{m \times n}$ is a matrix of rank 1. The rank-one convex hull K^{rc} and the polyconvex hull K^{pc} of a compact set K can be defined in the same way as (11) with rank-one convex and polyconvex functions instead of quasiconvex functions. From the characterization (10) it follows that the classes of rank-one-, quasi- and polyconvex functions are closed under taking convex combinations, multiplying by a positive constant, and taking pointwise suprema.

In this paper we deal with compact sets of symmetric matrices. By employing the Hodge decomposition in L^2, one can show (see e.g. [20]) that homogeneous gradient Young measures supported on symmetric matrices can in fact be generated by the second derivatives of scalar valued functions. More precisely, if ν is a homogeneous gradient Young measure such that $\mathrm{supp}\, \nu \subset \mathbb{R}^{n \times n}_{\mathrm{sym}}$ and $A = \bar{\nu}$, then for any simply connected Lipschitz domain $\Omega \subset \mathbb{R}^n$ there exists a sequence of functions $u_j : \Omega \to \mathbb{R}$ such that $u_j(x) = \frac{1}{2} x \cdot Ax$ on $\partial \Omega$, $\|D^2 u_j\|_{L^p(\Omega)} \leqslant c(p)$ for all $p < \infty$ and $\{D^2 u_j\}$ generates the Young measure ν. Functions defined only on symmetric matrices are said to be quasiconvex if Jensen's inequality (10) holds for all homogeneous gradient Young measures which are supported on symmetric matrices.

3. A directional maximum principle

Theorem 3. *Let $\mu : Q \to \mathbb{R}$ be measurable with $|\mu| \leqslant k < 1$ and suppose $u \in W^{2,p}(Q)$ for some $p > 2$ satisfies*

$$
\begin{aligned}
& L_\mu u = (1 + \mu)\partial_x^2 u + (1 - \mu)\partial_y^2 u = 0 \quad \text{in } Q, \\
& u = u_0 \quad \text{on } \partial Q
\end{aligned}
\tag{12}
$$

where u_0 is a quadratic form such that

$$
\begin{aligned}
& x \mapsto u_0(x, y) \text{ is convex,} \\
& y \mapsto u_0(x, y) \text{ is concave.}
\end{aligned}
\tag{13}
$$

Then

$$
\partial_x^2 u \geqslant 0 \geqslant \partial_y^2 u \quad \text{a.e. in } Q.
\tag{14}
$$

Proof. The idea of the proof is simple: assuming for a moment that u is smooth for which $\partial_y^2 u$ is not always non-positive, we consider the test-function w we get by taking for each fixed x the concave hull of $y \mapsto u(x, y)$. By assumption $w \neq u$. In the contact set (where $w = u$) $D^2 w = D^2 u$ almost everywhere, hence $L_\mu w = 0$. Moreover in regions where $w > u$, by definition $y \mapsto w(x, y)$ is affine, hence $\partial_y^2 w = 0$. If we can prove that in addition $\partial_x^2 w \geqslant 0$, then $L_\mu w \geqslant 0$. But then w is a subsolution, hence $w \leqslant u$ which by definition of w implies that $w = u$, resulting in a contradiction.

We will first prove the result in the case when $\mu \in C^\alpha(\overline{Q})$ and then pass to the general case using uniform $W^{2,p}$ estimates.

The case when $\mu \in C^\alpha(\overline{Q})$

Standard interior estimates (e.g. Theorem 9.19 in [14]) and Sobolev embedding imply that $u \in C^{2,\alpha}(Q) \cap C^1(\overline{Q})$. Let $M = \|Du\|_{C^0(\overline{Q})}$.

Let $w : Q \to \mathbb{R}$ be defined as follows. For each fixed $x \in [0, 1]$ let $y \mapsto w(x, y)$ be the concave hull of $y \mapsto u(x, y)$ (i.e. the smallest concave function which lies above u) and let $\Gamma = \{u = w\}$ be the contact set. Since $y \mapsto u_0(x, y)$ is concave, $u = w$ for $x = 0, 1$. Furthermore, since u is Lipschitz in $\overline{Q}$,

$$
y \mapsto My(1 - y) + (1 - y)u_0(x, 0) + yu_0(x, 1)
\tag{15}
$$

is a concave upper bound to $y \mapsto u(x, y)$, hence $u = w$ for $y = 0, 1$. Thus

$$\partial Q \subset \Gamma.$$

Our aim is to prove that w is a viscosity subsolution to (12).

Step 1: w is Lipschitz continuous in $\overline{Q}$. Since $y \mapsto u(x, y)$ is C^1, $y \mapsto w(x, y)$ is C^1 for each x with

$$|\partial_y w(x, y)| \leqslant M.$$

Let $(x_0, y_0) \in Q$ and consider

$$l(y) := w(x_0, y_0) + \partial_y w(x_0, y_0) \cdot (y - y_0). \tag{16}$$

Then $u(x_0, y) \leqslant l(y)$, hence $u(x, y) \leqslant l(y) + M|x - x_0|$. But then by the definition of w

$$u(x, y) \leqslant w(x, y) \leqslant l(y) + M|x - x_0|. \tag{17}$$

If $(x_0, y_0) \in \Gamma$ then (17) implies

$$|w(x, y) - w(x_0, y_0)| \leqslant M(|x - x_0| + |y - y_0|). \tag{18}$$

On the other hand if $(x_0, y_0) \notin \Gamma$ then $w(x_0, y) \equiv l(y)$ in some non-empty interval (y_1, y_2) containing y_0 such that $(x_0, y_i) \in \Gamma$ for $i = 1, 2$. For any $\lambda \in (0, 1)$ let $y = \lambda y_1 + (1 - \lambda) y_2$. By concavity of $y \mapsto w(x, y)$ and linearity of $y \mapsto w(x_0, y)$ we have

$$\begin{aligned}
w(x, y) - w(x_0, y_0) &= w(x, y) - w(x_0, y) + w(x_0, y) - w(x_0, y_0) \\
&\geqslant \lambda w(x, y_1) + (1 - \lambda) w(x, y_2) - \lambda w(x_0, y_1) - (1 - \lambda) w(x_0, y_2) + l(y) - l(y_0) \\
&\geqslant \lambda(u(x, y_1) - u(x_0, y_1)) + (1 - \lambda)(u(x, y_2) - u(x_0, y_2)) + l(y) - l(y_0) \\
&\geqslant -M(|x - x_0| + |y - y_0|).
\end{aligned} \tag{19}$$

Combining (19) with (17) yields (18) also in this case. Finally we consider the case when $(x_0, y_0) \in \partial Q$. If $y_0 = 0$ or $y_0 = 1$, (18) follows immediately from (15) (recall that $u = w$ on ∂Q). For the case $x_0 = 0$ or $x_0 = 1$ note that

$$v(x, y) = (1 - x) u_0(0, y) + x u_0(1, y)$$

satisfies $Lv \leqslant 0$ in Q and $u \leqslant v$ on ∂Q, hence $u \leqslant v$ in Q by the maximum principle. But since $y \mapsto v(x, y)$ is concave, we deduce that $w \leqslant v$ in Q. Therefore

$$u \leqslant w \leqslant v \quad \text{in } Q,$$

and (18) follows.

Step 2: w is a viscosity subsolution. Fix $(x_0, y_0) \in Q$ and let $\phi \in C^2(Q)$ be such that $w - \phi$ has a local maximum at (x_0, y_0) and $w(x_0, y_0) = \phi(x_0, y_0)$. We need to show that

$$L_\mu \phi(x_0, y_0) \geqslant 0$$

(see Proposition 2.4 in [7]). If $(x_0, y_0) \in \Gamma \cap Q$ then on the one hand

$$\phi(x, y) \geqslant w(x, y) \geqslant u(x, y)$$

locally near (x_0, y_0), and on the other hand

$$\phi(x_0, y_0) = w(x_0, y_0) = u(x_0, y_0).$$

Since $L_\mu u = 0$, the maximum principle implies $L_\mu \phi(x_0, y_0) \geqslant 0$.

Now assume that $(x_0, y_0) \in \{u < w\}$. As in (16), there exists $y_1 < y_0 < y_2$ such that $w(x_0, y) \equiv l(y)$ in $[y_1, y_2]$ and $(x_0, y_i) \in \Gamma$ for $i = 1, 2$. Therefore necessarily $\partial_y^2 \phi(x_0, y_0) \geqslant 0$. Suppose that $\partial_x^2 \phi(x_0, y_0) < 0$. Let $\lambda \in (0, 1)$ be such that $y_0 = \lambda y_1 + (1 - \lambda) y_2$. By linearity of $y \mapsto w(x_0, y)$

$$\phi(x_0, y_0) = w(x_0, y_0) = \lambda w(x_0, y_1) + (1 - \lambda) w(x_0, y_2),$$

and by concavity

$$w(x_0 + t, y_0) \geqslant \lambda w(x_0 + t, y_1) + (1 - \lambda) w(x_0 + t, y_2)$$

whenever $0 < x_0 + t < 1$. Since (x_0, y_0) is a local maximum for $w - \phi$,

$$w(x_0 + t, y_0) + w(x_0 - t, y_0) \leqslant 2w(x_0, y_0) - ct^2$$

for some $c > 0$ and all sufficiently small $|t|$. Combining these gives

$$
\begin{aligned}
\lambda\big(u(x_0 - t, y_1) &+ u(x_0 + t, y_1)\big) + (1 - \lambda)\big(u(x_0 - t, y_2) + u(x_0 + t, y_2)\big) \\
&\leqslant \lambda\big(w(x_0 - t, y_1) + w(x_0 + t, y_1)\big) + (1 - \lambda)\big(w(x_0 - t, y_2) + w(x_0 + t, y_2)\big) \\
&\leqslant w(x_0 - t, y_0) + w(x_0 + t, y_0) \\
&\leqslant 2w(x_0, y_0) - ct^2 \\
&= 2\lambda w(x_0, y_1) + 2(1 - \lambda)w(x_0, y_2) - ct^2 \\
&= 2\lambda u(x_0, y_1) + 2(1 - \lambda)u(x_0, y_2) - ct^2
\end{aligned}
$$

for small $|t|$. Rearranging terms and letting $t \to 0$ yields (since $u \in C^2$)

$$\lambda \partial_x^2 u(x_0, y_1) + (1 - \lambda)\partial_x^2 u(x_0, y_2) \leqslant -c < 0. \tag{20}$$

Suppose first that $0 < y_1 < y_2 < 1$. Then $\partial_y^2 u(x_0, y_i) \leqslant 0$ for $i = 1, 2$ because $y \mapsto u(x_0, y) - l(y)$ achieves its maximum at y_1 and y_2. Therefore $\partial_x^2 u(x_0, y_i) \geqslant 0$ for $i = 1, 2$ because $u \in C^2(Q)$ solves (12) pointwise. If $y_1 = 0$ or $y_2 = 1$ then (13) directly implies that $\partial_x^2 u(x_0, y_i) \geqslant 0$. In any case we obtain a contradiction with (20). Hence necessarily $\partial_x^2 \phi(x_0, y_0) \geqslant 0$, and so

$$L_\mu \phi(x_0, y_0) = (1 + \mu)\partial_x^2 \phi(x_0, y_0) + (1 - \mu)\partial_y \phi(x_0, y_0) \geqslant 0.$$

We have shown that w is a viscosity subsolution to (12) such that $u = w$ on ∂Q. The maximum principle (Theorem 3.2 in [7]) implies that $u \geqslant w$ in Q. But then $u = w$ in Q and hence $\partial_x^2 u \geqslant 0 \geqslant \partial_y^2 u$ in Q as required.

The case when μ is measurable

Let $\mu_j \in C^\alpha(\overline{\Omega})$ be a sequence such that $|\mu_j| \leqslant k$ and

$$
\begin{aligned}
\mu_j &\to \mu \quad \text{strongly in } L^q \text{ for all } q < \infty, \\
\mu_j &\stackrel{*}{\rightharpoonup} \mu \quad \text{weakly* in } L^\infty.
\end{aligned}
\tag{21}
$$

Fix $q \in (2, \min(p_k, p))$, where p_k is given in Proposition 1 in Appendix A, and let $u_j \in W^{2,q}(Q)$ be the solution of

$$
\begin{aligned}
L_{\mu_j} u_j &= 0 \quad \text{in } Q, \\
u_j &= u_0 \quad \text{on } \partial Q.
\end{aligned}
\tag{22}
$$

The proof above applies to each u_j, showing that

$$\partial_x^2 u_j \geqslant 0 \geqslant \partial_y^2 u_j \quad \text{in } Q \tag{23}$$

for each j. Proposition 1 implies that u_j is bounded uniformly in $W^{2,q}(Q)$, hence upto taking a subsequence we may assume that

$$u_j \rightharpoonup \tilde{u} \quad \text{weakly in } W^{2,q} \tag{24}$$

for some $\tilde{u} \in W^{2,q}(Q)$. Combining (21) and (24) yields

$$L_{\mu_j} u_j \rightharpoonup L_\mu \tilde{u} \quad \text{in } L^1,$$

therefore $L_\mu \tilde{u} = 0$ and hence $\tilde{u} = u$. But then passing to the limit in (23) yields

$$\partial_x^2 u \geqslant 0 \geqslant \partial_y^2 u \quad \text{a.e. in } Q. \qquad \square$$

4. Proof of Theorem 2

Proof. First we show that $\bar{v} \in E_k$. We use the coordinates

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}.$$

Let

$$f(A) := \left((a_{11} + a_{22})^2 - k^2 (a_{11} - a_{22})^2 \right)_+,$$

where $(\cdot)_+$ denotes the positive part. Then $f \geqslant 0$ and $\{f = 0\} = E_k$, so it suffices to show that f is polyconvex. For this notice that

$$g(A) := a_{11} a_{22} = \sup_{B \in l} \det(A - B),$$

where $l = \{B \in \mathbb{R}^{2 \times 2}_{\mathrm{sym}} \colon b_{11} = b_{22} = 0\}$, is polyconvex because is it a pointwise supremum of polyconvex functions. Therefore

$$f(A) = \left(\frac{1}{4} (1 - k^2)(\operatorname{tr} A)^2 + k^2 g(A) \right)_+$$

is also polyconvex.

Let $u_0 : \mathbb{R}^2 \to \mathbb{R}$ be the quadratic form with $D^2 u_0 = \bar{v}$ and assume without loss of generality that $\bar{v} \in E_k^+$. As explained in Section 2 we may assume that v is generated by a sequence $\{D^2 u_j\}$ where $u_j : Q \to \mathbb{R}$ are such that $u_j = u_0$ on ∂Q and $\|u_j\|_{W^{2,p}(Q)} \leqslant C(p)$ for each $1 \leqslant p < \infty$.

Now we "project" $D^2 u_j$ down to E_k using the method of [1]. Let

$$h(A) = \left(|a_{11} + a_{22}| - k|a_{11} - a_{22}| \right)_+.$$

For each j there exists a measurable function $\mu_j : Q \to \mathbb{R}$ with $|\mu_j| \leqslant k$ such that

$$h\left(D^2 u_j(x, y) \right) = \left| \left(1 + \mu_j(x, y) \right) \partial_x^2 u_j(x, y) + \left(1 - \mu_j(x, y) \right) \partial_y^2 u_j(x, y) \right|. \tag{25}$$

Such a function μ_j can be explicitly constructed as follows. Let $\pi : \mathbb{R}^{2 \times 2}_{\mathrm{sym}} \to \mathbb{R}$ be defined as

$$\pi(A) = \begin{cases} -\max\left\{ -k, \min\left\{ k, \frac{a_{11} + a_{22}}{a_{11} - a_{22}} \right\} \right\} & \text{if } a_{11} \neq a_{22}, \\ -k & \text{if } a_{11} = a_{22}, \end{cases}$$

so that $\pi(A)$ is lower semicontinuous and $|\pi(A)| \leqslant k$ for all $A \in \mathbb{R}^{2 \times 2}_{\mathrm{sym}}$. Let $A \in \mathbb{R}^{2 \times 2}_{\mathrm{sym}}$ such that $a_{11} \neq a_{22}$ and $A \notin E_k$. If $(a_{11} + a_{22})/(a_{11} - a_{22}) > k$, then either $a_{11} + a_{22} > 0$ and $a_{11} - a_{22} > 0$, or both are negative. Hence

$$a_{11} + a_{22} + \pi(A)(a_{11} - a_{22}) = \begin{cases} |a_{11} + a_{22}| - k|a_{11} - a_{22}| & \text{if both are positive,} \\ -|a_{11} + a_{22}| + k|a_{11} - a_{22}| & \text{if both are negative.} \end{cases}$$

A similar argument applies if $(a_{11} + a_{22})/(a_{11} - a_{22}) < -k$, thus in summary if $a_{11} \neq a_{22}$ and $A \notin E_k$ then $|a_{11} + a_{22} + \pi(A)(a_{11} - a_{22})| = ||a_{11} + a_{22}| - k|a_{11} - a_{22}|| = h(A)$. If $a_{11} = a_{22}$, then again

$$\left| a_{11} + a_{22} + \pi(A)(a_{11} - a_{22}) \right| = |a_{11} + a_{22}| = h(A).$$

Finally if $a_{11} \neq a_{22}$ and $A \in E_k$, then $a_{11} + a_{22} + \pi(A)(a_{11} - a_{22}) = 0$, again verifying the identity

$$h(A) = \left| a_{11} + a_{22} + \pi(A)(a_{11} - a_{22}) \right|.$$

Therefore it suffices to define $\mu_j(x, y) = \pi(D^2 u_j(x, y))$ to obtain (25). The measurability of μ_j follows from measurability of $D^2 u_j$ and the lower semicontinuity of π.

Since $\operatorname{supp} v \subset \{h = 0\}$ and $D^2 u_j$ is uniformly bounded in L^p for any $p < \infty$,

$$\|L_{\mu_j} u_j\|_{L^p(Q)}^p = \int_Q h^p \left(D^2 u_j \right) \mathrm{d}x \to 0 \quad \text{as } j \to \infty,$$

where

$$L_\mu u = (1 + \mu)\partial_x^2 u + (1 - \mu)\partial_y^2 u.$$

Fix $p \in (2, p_k)$, where p_k is given in Proposition 1, and let $v_j \in W^{2,p}(Q)$ be the solution of the Dirichlet problem

$$\begin{aligned} L_{\mu_j} v_j &= L_{\mu_j} u_j \quad \text{in } Q, \\ v_j &= 0 \quad \text{on } \partial Q. \end{aligned} \tag{26}$$

Let $w_j = u_j - v_j$. By Proposition 1 there exists a constant $C = C(p, k)$ such that

$$\left\| D^2 u_j - D^2 w_j \right\|_{L^p(Q)} \leqslant C \| L_{\mu_j} u_j \|_{L^p(Q)} \to 0 \quad \text{as } j \to \infty,$$

hence $D^2 w_j$ also generates the gradient Young measure ν. Moreover

$$\begin{aligned} L_{\mu_j} w_j &= 0 \quad \text{in } Q, \\ w_j &= u_0 \quad \text{on } \partial Q. \end{aligned} \tag{27}$$

Since we assumed that $\bar{\nu} \in E_k^+$, condition (13) in Theorem 3 is satisfied. Hence the theorem implies that

$$D^2 w_j \in E_k^+ \quad \text{a.e.,}$$

and thus $\operatorname{supp} \nu \subset E_k^+$. $\square$

Appendix A. L^p estimates for elliptic equations

In this section we provide the global $W^{2,p}$ estimates for the Dirichlet problem. These estimates were obtained in by K. Astala, T. Iwaniec and G. Martin in [2], where the connection is made with Beltrami equations in the plane. In fact, this connection leads to the identification of the optimal exponents $q_k^* = 1 + k < 2 < p_k^* = 1 + \frac{1}{k}$ where a priori estimates of the type (A.2) hold. In order to keep this paper self-contained we show how the $W^{2,p}$ estimates can be obtained for *some* range of exponents $q_k < 2 < p_k$ in a fairly elementary way. This is sufficient for our purposes because we are interested in homogeneous gradient Young measures with compact support. It should be pointed out however that due to the results in [2] the separation results in this paper (in particular Theorem 2) also hold for homogeneous gradient Young measures generated by sequences of gradients $\{\nabla u_j\}$ uniformly bounded in L^p for some $p > q_k^*$.

Our proof is inspired by the method in [2], and we do not claim any originality in this section. Recall that

$$L_\mu = (1 + \mu)\partial_x^2 + (1 - \mu)\partial_y^2,$$

where $|\mu| \leqslant k < 1$.

Proposition 1. *For any $k < 1$ there exist two exponents $q_k < 2 < p_k$ such that for $f \in L^p(Q)$ with $p \in (q_k, p_k)$ the equation*

$$L_\mu u = f \tag{A.1}$$

has a unique solution $u \in W^{2,p}(Q)$ vanishing on ∂Q and for some constant $C = C(p, k)$

$$\left\| D^2 u \right\|_{L^p(Q)} \leqslant C \| f \|_{L^p(Q)}. \tag{A.2}$$

Proof. First we derive an a priori estimate in the whole space $\mathbb{R}^2$. To this end let $u \in C_c^\infty(\mathbb{R}^2)$ be the solution of (A.1) for some μ. Let $T : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ be defined by $Tw = (\partial_x^2 - \partial_y^2)\Delta^{-1} w$. Here Δ^{-1} denotes the inverse of the operator $\Delta : W^{2,2}(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. Thus T is a singular integral operator with symbol $(\xi_1^2 - \xi_2^2)/|\xi|^2$, and operator norm $\|T\|_{L^p \to L^p}$ equal to 1 for $p = 2$ and continuous in p. Furthermore, with $w = \Delta u$ Eq. (A.1) can be rewritten as

$$(I + \mu T)w = f.$$

Therefore, since $|\mu| \leqslant k$, there exists $q_k < 2 < p_k$ such that for $p \in (q_k, p_k)$ we have $\|\mu T\|_{L^p \to L^p} < 1$, and thus the operator $I + \mu T$ is invertible in $L^p(\mathbb{R}^2)$. In particular we find

$$\|w\|_{L^p(\mathbb{R}^2)} \leqslant C(p, k)\|f\|_{L^p(\mathbb{R}^2)}$$

for $p \in (q_k, p_k)$. Combined with the inequality $\|D^2 u\|_{L^p(\mathbb{R}^2)} \leqslant C(p)\|\Delta u\|_{L^p(\mathbb{R}^2)}$ we obtain for $p \in (q_k, p_k)$ the a priori estimate

$$\|D^2 u\|_{L^p(\mathbb{R}^2)} \leqslant C(p, k)\|f\|_{L^p(\mathbb{R}^2)}. \tag{A.3}$$

To obtain the estimate (A.2) for solutions of the Dirichlet problem in $Q = [0, 1]^2$ we first define a periodic extension as follows. Let $u \in C_0^\infty(Q)$ be a solution to (A.1) and let u^* be defined in such a way that

$$u^*(x, y) = -u^*(2k - x, y) \quad \text{and} \quad u^*(x, y) = -u^*(x, 2k - y) \tag{A.4}$$

for all $k \in \mathbb{Z}$ and $u^* = u$ in Q. This can be achieved for example by first defining u^* on $[-1, 1]^2$ as

$$u^*(x, y) = \begin{cases} u(x, y), & Q, \\ -u(-x, y), & (-1, 0) + Q, \\ -u(x, -y), & (0, -1) + Q, \\ u(-x, -y), & (-1, -1) + Q, \end{cases}$$

and then extending to $\mathbb{R}^2$ periodically. From (A.4) it is easy to see that $u^* \in W_{\text{loc}}^{2,p}(\mathbb{R}^2)$ with

$$\|u^*\|_{W^{2,p}((k,l)+Q)} = \|u\|_{W^{2,p}(Q)}.$$

Indeed, by periodicity it suffices to show that $u^* \in W_{\text{loc}}^{2,p}((-1, 1)^2)$. To this end let $\phi \in C_0^\infty((-1, 1)^2)$. We need to show that

$$\left| \int Du^* D\phi \right| \leqslant C\|u\|_{W^{2,p}(Q)}\|\phi\|_{L^q}, \tag{A.5}$$

where $p^{-1} + q^{-1} = 1$. For any $\varepsilon > 0$ let $\eta : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\eta(t) = 1$ for $|t| > \varepsilon$, $\eta(t) = 0$ for $|t| < \varepsilon/2$ and $|\eta'(t)| \leqslant C\varepsilon^{-1}$. Then

$$\int \eta(x)\eta(y)Du^* D\phi = \int Du^* D\big(\phi\eta(x)\eta(y)\big) - \int \eta(y)\phi\partial_x u^*\eta'(x) - \int \eta(x)\phi\partial_y u^*\eta'(y). \tag{A.6}$$

Now

$$\left| \int \eta(y)\phi\partial_x u^*\eta'(x) \right| = \left| \int_{\varepsilon/2 < x < \varepsilon} \eta(y)\phi(x, y)\partial_x u^*(x, y)\eta'(x) - \int_{-\varepsilon < x < -\varepsilon/2} \eta(y)\phi(x, y)\partial_x u^*(-x, y)\eta'(-x) \right|$$

$$= \left| \int_{\varepsilon/2 < x < \varepsilon} \eta(y)\big(\phi(x, y) - \phi(-x, y)\big)\partial_x u^*(x, y)\eta'(x) \right|$$

$$\leqslant C\|\phi\|_{C^1} \int_{(\varepsilon/2, \varepsilon) \times [0,1]} \big|\partial_x u(x, y)\big| \to 0 \quad \text{as } \varepsilon \to 0.$$

Similarly $|\int \eta(x)\phi\partial_y u^*\eta'(y)| \to 0$ as $\varepsilon \to 0$. Furthermore

$$\int Du^* D\big(\phi\eta(x)\eta(y)\big) \leqslant 4\|D^2 u\|_{L^p(Q)}\|\phi\|_{L^q(Q)}.$$

Hence (A.6) implies that

$$\left| \int \eta(x)\eta(y)Du^* D\phi \right| \leqslant 4\|D^2 u\|_{L^p(Q)}\|\phi\|_{L^q(Q)} + \mathrm{o}(\varepsilon).$$

From this we obtain (A.5) by letting $\varepsilon \to 0$.

Coming back to the equation we see that $L_{\mu^*}u^* = f^*$, where μ^* and f^* are defined in a similar way from μ and f respectively (in particular $|\mu^*| \leqslant k$). Let ϕ_n be a smooth cut-off function such that $\phi_n(z) = 1$ for $z \in [-n, n]^2$ and $\phi_n(z) = 0$ for $z \notin [-(n+1), n+1]^2$. We have the usual estimates $|\nabla \phi_n| \leqslant Cn^{-1}$ and $|D^2 \phi_n| \leqslant Cn^{-2}$. By considering $L_\mu(\phi_n u^*)$ and using (A.3) we have for $p \in (q_k, p_k)$

$$\left\| D^2 u^* \right\|_{L^p([-n,n]^2)} \leqslant C(p, k) \left(\| f^* \|_{L^p([-(n+1),n+1]^2)} + \frac{1}{n} \| u^* \|_{W^{1,p}([-(n+1),n+1]^2)} \right).$$

But as $\| D^2 u^* \|_{L^p([-n,n]^2)} = n^2 \| D^2 u^* \|_{L^p([-1,1]^2)}$, after dividing by n^2 and letting $n \to \infty$ we obtain

$$\left\| D^2 u^* \right\|_{L^p([-1,1])} \leqslant C(p, k) \| f^* \|_{L^p([-1,1])}.$$

The estimate (A.2) follows from this using the definition of u^* and f^*. With the a priori estimate (A.2) at our disposal the existence of a unique solution of the Dirichlet problem now follows from the usual continuity method. $\quad\square$

Acknowledgements

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