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Isolated periodic minima are unstable

Les minima periodiques isolés sont instables

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Abstract

A classical result, studied, among others, by Carathéodory [C. Carathéodory, Calculus of Variations and Partial Differential Equations of the First Order, Chelsea, New York, 1989], says that, at least generically, periodic minimizers are hyperbolic, and consequently, unstable as solutions of the associated Euler–Lagrange equation. A new version of this fact, also valid in the nonhyperbolic case, is given.

Résumé

Un résultat classique, étudié, entre autres, par Carathéodory [C. Carathéodory, Calculus of Variations and Partial Differential Equations of the First Order, Chelsea, New York, 1989], dit que, au moins génériquement, les minimiseurs périodiques sont hyperboliques et par conséquent, instables comme solutions de l'équation d'Euler–Lagrange associée. Une nouvelle version de ce fait, aussi valable dans le cas nonhyperbolique, est donnée.

Keywords: Instability; Periodic minimizers; Parabolic case

1. Introduction

Consider the standard *periodic* minimization problem

$$\min_{x\in\mathcal{C}^1(\mathbb{R}/T\mathbb{Z})}\left[\int_0^T L\big(t,x(t),x'(t)\big)\,\mathrm{d}t\right]. \tag{1}$$

It is well known that every (local) solution of this problem must be a T-periodic solution of the associated Euler–Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}t}L_p(t,x,x') = L_x(t,x,x'). \tag{2}$$

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Problem (1) was the issue of Chapter 17 of [1]. In a result which is attributed to Poincaré, it is shown that if the T-periodic curve x_* is a (local) solution, then it is also a (local) solution of the subharmonic minimization problem

$$\min_{x\in\mathcal{C}^1(\mathbb{R}/nT\mathbb{Z})}\left[\int_0^{nT} L\big(t,x(t),x'(t)\big)\,\mathrm{d}t\right] \tag{3}$$

for any $n\in\mathbb{N}$. Using classical arguments which go back to Jacobi, it is then easy to prove (see Appendix: 'Stability of periodic minimals' to Chapter 2 of [4]), that the linear, variational equation associated to (2) at $x = x_*$ must be disconjugate (nonzero solutions cannot vanish more than once), and in particular, the associated characteristic (Floquet) multipliers μ_1, μ_2, are real positive. They should, moreover, verify $\mu_1\mu_2 = 1$, and thus, generically we expect either $0 < \mu_1 < 1 < \mu_2$ or $0 < \mu_2 < 1 < \mu_1$, so that, by Lyapunov First method, the solution x_* of the minimization problem (1) is unstable. To get a feeling of what may happen in the parabolic case ($\mu_1 = \mu_2 = 1$), consider the autonomous Newtonian equation

$$\ddot{x} + V'(x) = 0,$$

where the $\mathcal{C}^2$ potential $V:\mathbb{R}\to\mathbb{R}$ verifies $V'(0) = V''(0) = 0$, so that $x_* \equiv 0$ is a parabolic equilibrium. Observe that the associated action functional attains a (local) minimum at x_* if and only if the potential V attains a (local) maximum at 0. On the other hand, conservation of energy along trajectories means that the solution x of this equation with the initial condition $x(0) = 0$, $x'(0) = v_0 > 0$, verifies

$$\frac{1}{2}x'(t)^2 + V\big(x(t)\big) = \frac{1}{2}v_0^2 + V(0),$$

for any time t, so that $|x'(t)| \geqslant v_0$ whenever $V(x(t)) \leqslant V(0)$. In particular, if V has a local maximum at 0, the solution $x_* \equiv 0$ is unstable. The question immediately arises of whether a solution x_* of the more general minimization problem (1) should be unstable as a solution of (2).

This problem is not new. Dancer and Ortega [2] showed that any stable isolated T-periodic solution of (2) has fixed point index 1 (and then, it cannot be a minimizer), thus extending Carathéodory's Theorem on the instability of periodic minimizers from the hyperbolic to the (possibly parabolic) isolated case. The result was completed by Ortega in an analytic setting: in [7] he proved that fixed points of area-preserving, analytic mappings of the plane are either unstable or isolated. When applied to the Poincaré mapping associated to (2), this gives rise to the following result: Assume that the Lagrangian $L = L(t,x,p)$ verifies the usual convexity condition with respect to p and is analytic in the state variables x, p. Then, any T-periodic solution of the Euler Lagrange equation (2) is either unstable or isolated. Together with [2], this result completed the proof of the instability of periodic minimizers in the analytic case. Finally, a different, elementary proof of the instability of periodic minimizers (for the Newtonian Lagrangian $L(t,x,p) = p^2/2 - V(t,x)$), was given in [8], both for the isolated and analytic cases.

In these papers, instability is understood as the logical negation of Lyapunov stability, a concept which will be referred to as 'Lyapunov instability' in what follows. However, a stronger notion of instability is considered by Siegel and Moser in Chapter III of [9]. Let the topological space X, the open set $U \subset X$, the topological embedding $P: U \to X$, and the fixed point $u_* = P(u_*)$, be given. We shall say that u_* is *SM-unstable* if there exists a neighborhood $\mathcal{U}_*$ of u_* such that for each point $u_0 \neq u_*$ belonging to $\mathcal{U}_*$ there exists some (past or future) iterate $u_n = P^n(u_0)$ ($n\in\mathbb{Z}$), such that $u_r = P^r(u_0) \in \mathcal{U}_*$ for any integer r between 0 and n, but $u_n \notin \mathcal{U}_*$. Since periodic solutions correspond to fixed points of the associated Poincaré mapping, this concept can be immediately translated to T-periodic solutions of (T-periodic in time) second order ordinary differential equations such as (2). Thus, we shall say that the T-periodic solution x_* is SM-unstable if there exists some $\rho > 0$ such that the only (not necessarily periodic) solution x of the equation which is defined on the whole real line and verifies

$$\big|x(t) - x_*(t)\big| < \rho, \quad \big|x'(t) - x_*'(t)\big| < \rho \quad \text{for any } t\in\mathbb{R},$$

is $x = x_*$.

Now, it follows immediately from Hartman–Grobman Theorem that hyperbolic periodic solutions are SM-unstable, giving rise to the following question:

Is it true that any (possibly parabolic) periodic minimizer is SM-unstable?

This paper is devoted to show that the answer to the question above is 'yes' provided that the minimizer is isolated as a T-periodic solution of the Euler–Lagrange equation (2). To state our main result, let the Lagrangian $L : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ be T-periodic in the first variable t and have class $\mathcal{C}^{0,2}(\mathbb{R} \times \mathbb{R}^2)$. A generic element of the real line $\mathbb{R}$ will be denoted as t, while a generic element of the plane $\mathbb{R}^2$ will be written as (x, p). We assume the usual convexity condition of L with respect to the third variable p,

$$L_{pp}(t, x, p) > 0, \quad (t, x, p) \in \mathbb{R} \times \mathbb{R}^2. \tag{4}$$

Consider next the action functional $\mathcal{A} : \mathcal{C}^1(\mathbb{R}/T\mathbb{Z}) \to \mathbb{R}$ defined by

$$\mathcal{A}[x] := \int_0^T L\big(t, x(t), x'(t)\big)\, \mathrm{d}t, \quad x \in \mathcal{C}^1(\mathbb{R}/T\mathbb{Z}), \tag{5}$$

and assume that it attains a local minimum at the periodic curve $x_* \in \mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$, i.e., there exists some $\rho > 0$ such that $\mathcal{A}[x_*] \leqslant \mathcal{A}[x]$ for any $x \in \mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ with $\|x - x_*\|_\infty < \rho$ and $\|x' - x'_*\|_\infty < \rho$.

Theorem 1.1. *Assume the above. Assume also that x_* is isolated in the set of T-periodic solutions of the Euler–Lagrange equation* (2) *when this set is endowed with the $\mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ topology. Then, x_* is SM-unstable.*

Observe that nonisolated periodic solutions cannot be SM-unstable, as it follows easily from the definition. We do not know whether nonisolated periodic minima should still be Lyapunov unstable. We observe also that Theorem 1.1 is not directly generalizable to the vector case $x \in \mathbb{R}^N$, since an example of a two-dimensional elliptic minimizer was given in Chapter 17 of [1]. However, recent works by Offin [5] and Offin and Skoczylas [6], have shown that, in certain vector cases with symmetries, minimizers are either parabolic or hyperbolic.

In many situations, it is usual to look for minimizers in the Sobolev space $H^1(\mathbb{R}/T\mathbb{Z})$ of T-periodic functions rather than in $\mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$. We observe, however, that the $\mathcal{C}^{0,2}(\mathbb{R} \times \mathbb{R}^2)$ regularity of L together with assumption (4), imply that any $H^1(\mathbb{R}/T\mathbb{Z})$ solution of (2) must be a $\mathcal{C}^1$ function. Consequently, any $H^1(\mathbb{R}/T\mathbb{Z})$ (local) minimizer of $\mathcal{A}$ is indeed a $\mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ (local) minimizer. Moreover, continuous dependence of solutions of (2) with respect to the initial conditions means that x^* is a $H^1(\mathbb{R}/T\mathbb{Z})$-isolated critical point of $\mathcal{A}$ if and only if it is a $\mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$-isolated critical point and indeed, if and only if $x^*(0)$ is an isolated fixed point of the Poincaré mapping P.

The choice of the term 'SM-instability' might seem odd at first glance because, with this definition, taken from [9], asymptotically stable fixed points may be SM-unstable. However, the concept of asymptotic stability is strange to the worlds of area-preserving maps and Hamiltonian systems, where past and future (Lyapunov) stability are equivalent concepts, and moreover, equivalent to the existence of a basis of invariant neighborhoods of the fixed point, or periodic solution, under consideration. This means that, for fixed points of area-preserving maps, or periodic solutions of Hamiltonian systems, *SM-instability implies Lyapunov's*. We close this work with Section 5, where we give an example of an area-preserving, $\mathcal{C}^\infty$ diffeomorphism $P : \mathbb{R}^2 \to \mathbb{R}^2$ with a Lyapunov unstable but not SM-unstable, isolated fixed point. Sections 2 and 3 are quite standard, and prepare the framework needed in Section 4, where we will show Theorem 1.1 in the parabolic case, which is the nonclassical one.

In this paper, we denote by $\mathbb{R}/T\mathbb{Z}$ the quotient space of the real line where two numbers are identified whenever they differ by an integer multiple of T. Accordingly, $\mathcal{C}(\mathbb{R}/T\mathbb{Z})$ and $\mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ denote respectively the functional spaces of continuous and continuously derivable T-periodic curves. Somewhere later in the paper we will need to use periodic functions with zero mean on each period, and we will consider the corresponding spaces $\widetilde{\mathcal{C}}(\mathbb{R}/T\mathbb{Z})$ and $\widetilde{\mathcal{C}}^1(\mathbb{R}/T\mathbb{Z})$. As it is usual, $\mathcal{C}^1[0, T]$ will be used to denote the space of (not necessarily periodic) continuously derivable functions which are defined on the compact interval $[0, T]$.

2. On some second-order differential equations which are not in normal form

From now on, let the $\mathcal{C}^{0,2}(\mathbb{R} \times \mathbb{R}^2)$ Lagrangian $L = L(t, x, p)$ be T-periodic in time, i.e., $L(t, x, p) = L(t + T, x, p)$ for any $(t, x, p) \in \mathbb{R} \times \mathbb{R}^2$, and verify the convexity assumption (4). For the reader's convenience, we reproduce here the associated Euler–Lagrange equation, which was already given at the beginning of this paper:

$$\frac{\mathrm{d}}{\mathrm{d}t} L_p(t, x, x') = L_x(t, x, x'). \tag{2}$$

In case L_p is a $\mathcal{C}^1$ function on t, x, and p, the chain rule can be used to call solution of (2) to any $\mathcal{C}^2$ function x such that the equality

$$L_{pp}\big(t,x(t),x'(t)\big)x''(t)+L_{xp}\big(t,x(t),x'(t)\big)x'(t)+L_{tp}\big(t,x(t),x'(t)\big)=L_x\big(t,x(t),x'(t)\big) \tag{6}$$

holds at each instant t. Observe, however, that since the expression above involves the function L_{tp}, it makes no sense if we just assume L to be smooth on the state variables (x,p).

This motivated Moser [4] to consider an alternative concept of solution, which we present next. The (not necessarily periodic) $\mathcal{C}^1$ function x, which is assumed to be defined on some nontrivial interval of the real line, will be said to be a solution of (2) if the associated momentum $L_p(\cdot,x(\cdot),x'(\cdot))$ has still class $\mathcal{C}^1$, its derivative being $L_x(\cdot,x(\cdot),x'(\cdot))$. Equivalently, if the integro-differential equation

$$L_p\big(t_1,x(t_1),x'(t_1)\big)-L_p\big(t_0,x(t_0),x'(t_0)\big)=\int_{t_0}^{t_1}L_x\big(t,x(t),x'(t)\big)\,\mathrm{d}t,$$

holds for any times $t_0<t_1$ in the domain of x.

An easy consequence of the implicit function theorem is that, in the above mentioned case of L_p being a $\mathcal{C}^1$ function in all three variables, this is equivalent to x being a $\mathcal{C}^2$ curve for which (6) holds. On the other hand, the classical results of Calculus of Variations show that our assumption on the Lagrangian $L=L(t,x,p)$ to be a $\mathcal{C}^{0,2}(\mathbb{R}\times\mathbb{R}^2)$ function implies that the associated action functional $\mathcal{A}$, defined as in (5), has class $\mathcal{C}^2$, the function $x\in\mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ being a critical point if and only if it solves (2).

Our first task will be to check that the usual results on the existence and uniqueness of solutions of initial value problems associated to second order ODEs, continue to hold for equations such as (2). Precisely:

Lemma 2.1. *For each initial condition*

$$x(t_0)=x_0,\quad x'(t_0)=p_0, \tag{7}$$

there exists an unique solution $x=X(\cdot,t_0,x_0,p_0)$ *of Eq.* (2), *defined on some maximal open interval* $]\omega_-(t_0,x_0,p_0),\omega_+(t_0,x_0,p_0)[$ *containing* t_0, *and verifying* (7). *Here,* $-\infty\leqslant\omega_-(t_0,x_0,p_0)<t_0<\omega_+(t_0,x_0,p_0)\leqslant+\infty$, *and, moreover,*

(a) *if* $\omega_-(t_0,x_0,p_0)>-\infty$ (*resp.,* $\omega_+(t_0,x_0,p_0)<+\infty$), *there exists a sequence* $\{t_n\}_n\to\omega_-(t_0,x_0,p_0)$ (*resp.,* $\{t_n\}_n\to\omega_+(t_0,x_0,p_0)$), *in the interval* $]\omega_-(t_0,x_0,p_0),\omega_+(t_0,x_0,p_0)[$, *such that*

$$\big|X(t_n,t_0,x_0,p_0)\big|+\big|X_t(t_n,t_0,x_0,p_0)\big|\to\infty;$$

(b) *the set*

$$D:=\big\{(t,t_0,x_0,p_0)\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}^2\colon\ \omega_-(t_0,x_0,p_0)<t<\omega_+(t_0,x_0,p_0)\big\},$$

is open in $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^2$, *and the 'resolvent mapping'* $X:D\to\mathbb{R}$, *defined as above, is continuous.*

Proof. It will be convenient to rewrite the second-order differential equation (2) as a suitable first-order system in $\mathbb{R}^2$. With this aim, we remember assumption (4), which implies that L_p is strictly increasing with respect to p, and we call $S=S(t,x,q)$ its partial inverse, i.e.,

$$S\big(t,x,L_p(t,x,p)\big)=p,\quad (t,x,p)\in\mathbb{R}\times\mathbb{R}^2.$$

The function S is defined in some open subset of $\mathbb{R}\times\mathbb{R}^2$ and has class $\mathcal{C}^{0,1}$ there. Next, let the $\mathcal{C}^1$ curve $x:I\to\mathbb{R}$ be given, and let $q=q(t)$ be given by

$$q(t)=L_p\big(t,x(t),x'(t)\big),\quad t\in I.$$

We observe that x is a solution of (2) if and only if (x,q) solves the system

$$\begin{pmatrix}x\\q\end{pmatrix}'=\begin{pmatrix}S(t,x,q)\\L_x(t,x,S(t,x,q))\end{pmatrix}, \tag{8}$$

while the initial condition (7) becomes, in the new variables, $x(t_0) = x_0$, $q(t_0) = L_p(t_0, x_0, p_0)$. The result follows immediately from the standard theorems on the existence and uniqueness, as well as continuous dependence, of solutions of initial value problems associated to equations which are smooth in the state variables. □

We turn next our attention to linear equations of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\alpha(t)\xi' + \beta(t)\xi\big] = \beta(t)\xi' + \gamma(t)\xi, \tag{9}$$

where the curves $\alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}$ are continuous, T-periodic, and $\alpha > 0$. We remark that this is the Euler–Lagrange equation associated to the quadratic Lagrangian

$$\mathfrak{L}^0(t, \xi, \zeta) = \frac{\alpha(t)}{2}\zeta^2 + \beta(t)\zeta\xi + \frac{\gamma(t)}{2}\xi^2, \quad (t, \xi, \zeta) \in \mathbb{R} \times \mathbb{R}^2.$$

Observe that the partial derivative $\mathfrak{L}^0_\zeta$ of $\mathfrak{L}^0$ with respect to ζ is linear on the state variables ξ, ζ. Then, the same happens with its partial inverse with respect to ζ, and the associated first-order system, constructed as in (8), is linear. Consequently, initial value problems associated to (9) are uniquely and globally solvable.

This fact allows us to consider the *monodromy matrix* associated to this equation. It is defined as the real 2×2 matrix whose columns are $(\xi_1(T), \xi_1'(T))$ and $(\xi_2(T), \xi_2'(T))$, the functions ξ_1 and ξ_2 being the solutions of (9) verifying the initial conditions $\xi(0) = 1$, $\xi'(0) = 0$, and $\xi(0) = 0$, $\xi'(0) = 1$ respectively. The reason why we are particularly interested in this matrix is given next. To state it, let us call D_0 the open subset of $\mathbb{R}^2$ defined by $D_0 := \{(x_0, p_0) \in \mathbb{R}^2\colon\ (T, 0, x_0, p_0) \in D\}$, and let $P : D_0 \to \mathbb{R}^2$ denote the Poincaré mapping associated to Eq. (2), caring each initial condition $(x_0, p_0) \in D_0$ into $(X(T, 0, x_0, p_0), X_t(T, 0, x_0, p_0))$.

Lemma 2.2. *$P \in \mathcal{C}^1(D_0)$, its derivative at the fixed point $(x_0, p_0) = P(x_0, p_0)$ being the monodromy matrix associated to* (9) *for*

$$\alpha(t) = L_{pp}\big(t, x(t), x'(t)\big), \quad \beta(t) = L_{xp}\big(t, x(t), x'(t)\big), \quad \gamma(t) = L_{xx}\big(t, x(t), x'(t)\big), \tag{10}$$

where $x(t) = X(t, 0, x_0, p_0)$ for any $t \in \mathbb{R}$.

The proof of this result follows from similar arguments to those used in the proof of Lemma 2.1. To complete this section, we recall the reader's attention on a fact which will be needed in the proof of Theorem 1.1. It can be stated as follows: inside the space $\mathcal{C}^1[0, T]$ of continuously derivable functions on $[0, T]$, the set of solutions of Eq. (2) is closed by uniform convergence. Precisely,

Lemma 2.3. *Let the sequence $\{x_n\}_n \subset \mathcal{C}^1[0, T]$ converge uniformly to the $\mathcal{C}^1[0, T]$ function $x_* : [0, T] \to \mathbb{R}$. Assume that x_n is a solution of* (2) *for any $n \in \mathbb{N}$. Then, x_* itself solves* (2).

Proof. For each natural number $n \in \mathbb{N}$, Lagrange's mean value theorem, when applied to the $\mathcal{C}^1[0, T]$ function $x_n - x_*$, implies the existence of some point $t_n \in [0, T]$ such that

$$x_n'(t_n) - x_*'(t_n) = \frac{(x_n(T) - x_*(T)) - (x_n(0) - x_*(0))}{T - 0}.$$

Since $x_n(0) \to x_*(0)$ and $x_n(T) \to x_*(T)$, it follows from the above expression that $x_n'(t_n) - x_*'(t_n) \to 0$ as $n \to +\infty$. Now, the sequence $\{t_n\}_n$ is contained on $[0, T]$, so that it has a convergent subsequence $\{t_{n_r}\}_r \to t_*$. Observe that

$$x_{n_r}(t_{n_r}) \to x_*(t_*), \qquad x_{n_r}'(t_{n_r}) \to x_*'(t_*),$$

as $r \to +\infty$. The continuous dependence of the solutions on the initial conditions, as established in Lemma 2.1(b), implies that $x_*(t) = X(t, t_*, x_*(t_*), x_*'(t_*))$ for any $t \in [0, T] \cap\,]\omega_-(t_*, x_*(t_*), x_*'(t_*)), \omega_+(t_*, x_*(t_*), x_*'(t_*))[$. Now, Lemma 2.1(a) may be used to obtain that the interval $[0, T]$ must be contained inside $]\omega_-(t_*, x_*(t_*), x_*'(t_*)),$ $\omega_+(t_*, x_*(t_*), x_*'(t_*))[$, so that $x_*(t) = X(t, t_*, x_*(t_*), x_*'(t_*))$ for any $t \in [0, T]$. We conclude that x_* is a solution of (2). □

3. The nondegenerate case

Lemma 2.2 implies that the (possibly complex) eigenvalues of the monodromy matrix associated to linear equations such as (9) (also called the *Floquet multipliers* of this equation), play a key role in the study of the stability of periodic solutions. Namely, as a consequence of the Hartman–Grobman theorem (see [3], Chapter IX, Lemma 8.1), the point $(x_0, p_0) = P(x_0, p_0)$ will be SM-unstable for the Poincaré matrix P associated to (2) provided that neither of the two Floquet multipliers μ_1, μ_2 of (9) (the continuous functions α, β, γ being given as in (10) for $x(t) = X(t, 0, x_0, p_0)$), belongs to the unit circumference on the complex plane. In this case, the linear equation (9) is called *hyperbolic* and (x_0, p_0) is called a *hyperbolic fixed point*.

Now, let the continuous and T-periodic curves $\alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}$ with $\alpha > 0$ be arbitrary, and consider, for each $\lambda \in \mathbb{R}$, the linear equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\alpha(t)\xi' + \beta(t)\xi\big] = \beta(t)\xi' + \gamma(t)\xi - \lambda\xi, \tag{11}$$

which is actually the Euler–Lagrange equation associated to the quadratic Lagrangian

$$\mathfrak{L}^\lambda(t, \xi, \zeta) = \frac{\alpha(t)}{2}\zeta^2 + \beta(t)\zeta\xi + \frac{\gamma(t)}{2}\xi^2 - \frac{\lambda}{2}\xi^2, \quad (t, \xi, \zeta) \in \mathbb{R} \times \mathbb{R}^2.$$

Observe that, if $\lambda < 0$ is small enough, the action functional $\mathfrak{A}^\lambda$ associated to $\mathfrak{L}^\lambda$ is coercive on the Sobolev space $H^1(\mathbb{R}/T\mathbb{Z})$ of T-periodic functions. Thus, well-known arguments show the existence of a sequence $\lambda_n \to +\infty$ of eigenvalues of (11) with $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, and such that (11) has a nontrivial T-periodic solution if and only if $\lambda \in \{\lambda_n \colon n \in \mathbb{N}\}$.

On the other hand, we can continuously associate, to each $\lambda \in \mathbb{R}$, the two Floquet multipliers $\mu_1^\lambda, \mu_2^\lambda$ of Eq. (11). These verify $\mu_1^\lambda + \mu_2^\lambda \in \mathbb{R}$, since this is the trace of the associated monodromy matrix, and $\mu_1^\lambda \mu_2^\lambda = 1$, since this is its determinant. We deduce in particular that either $\mu_1^\lambda, \mu_2^\lambda$ both belong to the unit circumference of complex numbers with modulus 1, or otherwise, they must be both real.

In Lemma 3.1 below we will show that it is the latter case the one which holds when $\lambda < \lambda_1$. Our proof will use the well-known concept of *recurrence* for solutions of periodic differential equations, which we recall briefly next. Let the T-periodic in time Lagrangian $L : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ and the $\mathcal{C}^1$ solution $x : I \to \mathbb{R}$ of the Euler–Lagrange equation (2) be given. x is said to be recurrent in the future if the interval I where it is defined is not bounded from above, and moreover, there exists some $\omega_* \in I$ and some sequence $\{p_n\}_n \to +\infty$ of integers such that

$$\lim_{n\to+\infty} x(\omega_* + p_n T) = x(\omega_*), \qquad \lim_{n\to+\infty} x'(\omega_* + p_n T) = x'(\omega_*).$$

Observe that continuous dependence on the initial conditions means that the concept above does not depend on the particular choice of the initial instant ω_*. It is, moreover, equivalent to the existence of a sequence $\{p_n\}_n \to +\infty$ of integers such that

$$\lim_{n\to+\infty} x(p_n T + t) = x(t),$$

uniformly with respect to t belonging to compact subintervals of I.

Lemma 3.1. *Assume that* $\lambda < \lambda_1$. *Then, the Floquet multipliers* μ_1^λ, μ_2^λ *are real, and either* $0 < \mu_1^\lambda < 1 < \mu_2^\lambda$ *or* $0 < \mu_2^\lambda < 1 < \mu_2^\lambda$.

Proof. Observe first that $|\mu_1^\lambda| = 1 = |\mu_2^\lambda|$ implies the existence of nonzero solutions of the linear equation (11) which are recurrent in the future and may be chosen real. Consequently, in order to show that $\mu_1^\lambda, \mu_2^\lambda$ are real when $\lambda < \lambda_1$ it will suffice to prove that no nontrivial recurrent solutions of (11) can exist. We call φ_1 a generator of the (one-dimensional) eigenspace associated to the first eigenvalue λ_1. This is a nowhere-vanishing, T-periodic function, and, after possibly replacing φ by $-\varphi$, we may assume that it is positive everywhere. Assume, by contradiction, that $\xi : \mathbb{R} \to \mathbb{R}$ is a recurrent in the future, nontrivial solution of (11) for some $\lambda < \lambda_1$. Then, we may find a sequence $\{p_n\}_n \to +\infty$ such that $\lim_{n\to+\infty} \xi(p_n T + t) = \xi(t)$ uniformly with respect to $t \in [0, T]$. On the other hand, φ is T-periodic, and we deduce that

$$\lim_{n\to+\infty} \frac{\xi(p_n T + t)}{\varphi(p_n T + t)} = \frac{\xi(t)}{\varphi(t)},$$

uniformly with respect to $t \in [0, T]$. We deduce the existence of some time $t_0 \in \mathbb{R}$ where ξ/φ attains either a positive local maximum or a negative local minimum. Replacing, if necessary, ξ by $-\xi$, we may assume that it is a positive local maximum, and we observe that this implies the function $\eta := (\xi(t_0)/\varphi(t_0))\varphi - \xi$, which verifies

$$\eta(t_0) = \eta'(t_0) = 0, \tag{12}$$

to attain a local minimum at $t = t_0$. On the other hand, it follows easily from the definitions that η is a solution of the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\alpha(t)\eta' + \beta(t)\eta\big] = \beta(t)\eta' + \gamma(t)\eta - \lambda\eta + (\lambda - \lambda_1)\big(\xi(t_0)/\varphi(t_0)\big)\varphi(t),$$

so that, in view of (12) above, at time $t = t_0$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_0}\big[\alpha(t)\eta'(t) + \beta(t)\eta(t)\big] = (\lambda - \lambda_1)\big(\xi(t_0)/\varphi(t_0)\big)\varphi(t_0) < 0. \tag{13}$$

However,

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\bigg|_{t=t_0}\big(\beta(t)\eta(t)\big) = \lim_{t\to t_0}\frac{\beta(t)\eta(t)}{t - t_0} = \beta(t_0)\eta'(t_0) = 0,$$

and inequality (13) becomes $(\mathrm{d}/\mathrm{d}t)|_{t=t_0}(\alpha(t)\eta'(t)) < 0$. It follows that $\alpha(t)\eta'(t)$ is a strictly decreasing function of t in a neighborhood of $t = t_0$. Moreover, it vanishes at $t = t_0$, and we deduce the existence of some $\epsilon > 0$ such that $\alpha(t)\eta'(t) > 0$ for any $t \in]t_0 - \epsilon, t_0[$ and $\alpha(t)\eta'(t) < 0$ for any $t \in]t_0, t_0 + \epsilon[$. Since $\alpha > 0$ by assumption, we deduce that $\eta'(t)$ itself must be positive for any $t \in]t_0 - \epsilon, t_0[$ and negative for any $t \in]t_0, t_0 + \epsilon[$. Consequently, η attains a (strict) local maximum at $t = t_0$, a contradiction.

Thus, the Floquet multipliers $\mu_1^\lambda, \mu_2^\lambda$ are real when $\lambda < \lambda_1$, and it only remains to see that they are positive. With this aim, observe first that, since $\mu_1^\lambda\mu_2^\lambda = 1$ for any λ, they cannot vanish. On the other hand, being the roots of the characteristic polynomial of the associated monodromy matrix, they depend continuously on λ, and consequently, they converge to 1 as $\lambda \to \lambda_1$. The result follows. □

We have already observed that if $\lambda < 0$ is small enough, the action functional $\mathfrak{A}_\lambda$ associated to the quadratic Lagrangian $\mathfrak{L}_\lambda$ is coercive on $H^1(\mathbb{R}/T\mathbb{Z})$. This result can be seen as a consequence of the so-called *Wirtinger's inequality*, which states that

$$\mathfrak{A}^\lambda[\xi] \geqslant \frac{\lambda_1 - \lambda}{2}\|\xi\|^2_{L_2(0,T)}, \quad \xi \in H^1(\mathbb{R}/T\mathbb{Z}). \tag{14}$$

Remark that the coefficient $(\lambda_1 - \lambda)/2$ is optimal, since the equality is attained at $\xi = \varphi_1$, the first eigenfunction. We arrive to the following result, which summarizes the main conclusions of this section.

Proposition 3.2. *Assume that the action function $\mathcal{A}$, defined as in* (5)*, attains a local minimum at $x_* \in \mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$. Consider the functions $\alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}$ defined as in* (10) *for $x = x_*$. The following hold*:

(i) λ_1*, the first periodic eigenvalue of* (11)*, is greater or equal than zero.*
(ii) *If $\lambda_1 > 0$, then x_* is SM-unstable.*

Proof. If $\mathcal{A}$ attains a local minimum at $x = x_*$, we should have not only $\mathcal{A}'[x_*] = 0$, but also that the bilinear form $\mathcal{A}''[x_*]$ is positive semidefinite on $\mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$. This means that $\mathcal{A}''[x_*](\xi, \xi) = 2\mathfrak{A}_0(\xi) \geqslant 0$ for any $\xi \in \mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$, and in particular, $2\mathfrak{A}_0(\varphi_1) = \lambda_1\|\varphi_1\|^2_{L_2(0,T)} \geqslant 0$. Consequently, $\lambda_1 \geqslant 0$, showing (i).

To see (ii), apply Lemma 3.1 with $\lambda = 0$ to obtain that, if $\lambda_1 > 0$, the linear problem (9) is hyperbolic. Then, the argument given at the beginning of this section, based on Hartman–Grobman theorem, shows that x_* is SM-unstable. This shows the result. □

We remark that Lemma 3.1 is a classical fact whose proof was usually given as follows: If $\lambda < \lambda_1$, inequality (14) implies that $\mathfrak{A}^\lambda$ attains its minimum at $\xi = 0$, but then, Theorem 1.3.1 of [4] states that no nonzero solution of (11) can vanish twice, and in particular, the Floquet eigenvalues $\mu_1^\lambda, \mu_2^\lambda$ must be real. Here, we have given a slightly different proof containing, in a simpler case, some of the main arguments of the proof of Theorem 1.1 which will be our goal next.

4. The degenerate case

Thus, it only remains to consider what happens when $\lambda_1 = 0$, i.e., if α, β, γ are given as in (10) for $x = x_*$, the linear equation (9) has a T-periodic solution $\varphi_1 : \mathbb{R} \to \mathbb{R}$ with $\varphi_1(t) > 0$ for any $t \in \mathbb{R}$. We call this the 'degenerate case', since the quadratic functional $\mathfrak{A}_0[\xi] = (1/2)\mathcal{A}''[x_*](\xi, \xi)$ happens to be only positive semidefinite on the Sobolev space $H^1(\mathbb{R}/T\mathbb{Z})$. It can be also referred to as the 'parabolic case', since, with the notation of the previous section, it implies that $\mu_1^0 = \mu_2^0 = 1$.

Let us introduce some new notation here. We will denote by $\pi : \mathcal{C}(\mathbb{R}/T\mathbb{Z}) \to \mathbb{R}$ the linear projection carrying each continuous and T-periodic function f into its mean value, i.e.,

$$\pi[f] := \frac{1}{T} \int_0^T f(t)\, \mathrm{d}t,$$

and we consider the Banach spaces

$$\widetilde{\mathcal{C}}(\mathbb{R}/T\mathbb{Z}) := \big\{ f \in \mathcal{C}(\mathbb{R}/T\mathbb{Z}) \colon \pi[f] = 0 \big\}, \qquad \widetilde{\mathcal{C}}^1(\mathbb{R}/T\mathbb{Z}) := \widetilde{\mathcal{C}}(\mathbb{R}/T\mathbb{Z}) \cap \mathcal{C}^1(\mathbb{R}/T\mathbb{Z}),$$

of T-periodic functions with zero mean, which are respectively endowed with the usual $\mathcal{C}(\mathbb{R}/T\mathbb{Z})$ and $\mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ topologies. The topological isomorphism $\mathcal{I} : \widetilde{\mathcal{C}}(\mathbb{R}/T\mathbb{Z}) \to \widetilde{\mathcal{C}}^1(\mathbb{R}/T\mathbb{Z})$ maps each function to its unique primitive with zero mean. Finally, we call $P, Q : \mathcal{C}^1(\mathbb{R}/T\mathbb{Z}) \to \mathcal{C}(\mathbb{R}/T\mathbb{Z})$ the Nemytskii operators associated to L_x and L_p respectively, i.e.,

$$P[x] = L_x(t, x, x'), \qquad Q[x] = L_p(t, x, x').$$

It is well known that both P and Q are $\mathcal{C}^1$ operators, and

$$P'[x]u = \gamma(t)u + \beta(t)u', \qquad Q'[x]u = \beta(t)u + \alpha(t)u',$$

for any $x, u \in C^1(\mathbb{R}/T\mathbb{Z})$, the T-periodic curves $\alpha, \beta, \gamma : \mathbb{R} \to \mathbb{R}$ being defined as in (10).

We observe next that a T-periodic solution of (2) can be equivalently characterized as a function $x \in \mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ such that $P[x] \in \widetilde{\mathcal{C}}(\mathbb{R}/T\mathbb{Z})$ and $Q[x] - \mathcal{I}[P[x]]$ is constant. In another words, the T-periodic function $x \in \mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ is a solution of (2) if and only if it solves the following Lyapunov–Schmidt type system

$$(\mathrm{Id} - \pi)\big[Q[x] - \mathcal{I}\big[(\mathrm{Id} - \pi)P[x]\big]\big] = 0, \tag{15}$$
$$\pi\big[P[x]\big] = 0, \tag{16}$$

where Id stands for the identity mapping. In our following result, we use the implicit function theorem to find a continuum of solutions of (15) emanating from x_*.

Proposition 4.1. *There exists some $\epsilon > 0$ and a $\mathcal{C}^1$ curve $\mathcal{X} :]-\epsilon, \epsilon[\to \mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ such that*

(i) $\mathcal{X}(0) = x_*$.
(ii) $\mathcal{X}(\bar{x})$ *is a solution of* (15) *for any* $\bar{x} \in]-\epsilon, \epsilon[$.
(iii) $\mathcal{X}'(\bar{x}) > 0$ *for any* $\bar{x} \in]-\epsilon, \epsilon[$.

Proof. We consider the operator $T : \mathbb{R} \times \widetilde{\mathcal{C}}^1(\mathbb{R}/T\mathbb{Z}) \to \widetilde{\mathcal{C}}(\mathbb{R}/T\mathbb{Z})$ defined by

$$T(\bar{x}, \tilde{x}) := (\mathrm{Id} - \pi)\big[Q[x_* + \bar{x} + \tilde{x}] - \mathcal{I}\big[(\mathrm{Id} - \pi)P[x_* + \bar{x} + \tilde{x}]\big]\big],$$

so that Eq. (15), for $x = x_* + \bar{x} + \tilde{x}$, becomes

$$T(\bar{x}, \tilde{x}) = 0.$$

Observe that, x_* being a solution of (15), (16), $T(0,0)=0$. On the other hand, T is a $\mathcal{C}^1$ operator, and, moreover, for any $u \in \widetilde{\mathcal{C}}^1(\mathbb{R}/T\mathbb{Z})$ we have:

$$T_{\tilde{x}}(0,0)u = (\mathrm{Id}-\pi)\big[\alpha(t)u' + \beta(t)u - \mathcal{I}\big[(\mathrm{Id}-\pi)\big(\beta(t)u' + \gamma(t)u\big)\big]\big],$$

the T-periodic functions α, β, γ being given as in (10) for $x = x_*$. We want to show that $T_{\tilde{x}}(0,0)$ is a topological isomorphism from $\widetilde{\mathcal{C}}^1(\mathbb{R}/T\mathbb{Z})$ into $\widetilde{\mathcal{C}}(\mathbb{R}/T\mathbb{Z})$, and with this aim, we observe that this linear operator may be decomposed as $T_{\tilde{x}}(0,0) = \mathcal{R} + \mathcal{K}$, the operators $\mathcal{R}, \mathcal{K} : \widetilde{\mathcal{C}}^1(\mathbb{R}/T\mathbb{Z}) \to \widetilde{\mathcal{C}}(\mathbb{R}/T\mathbb{Z})$ being defined by

$$\mathcal{R}[u] = (\mathrm{Id}-\pi)\big[\alpha(t)u'\big], \qquad \mathcal{K}[u] = (\mathrm{Id}-\pi)\big[\beta(t)u - \mathcal{I}\big[(\mathrm{Id}-\pi)\big(\beta(t)u' + \gamma(t)u\big)\big]\big].$$

We remark that $\mathcal{R}$ itself is a topological isomorphism, its inverse being given by

$$\mathcal{R}^{-1}[v] = \mathcal{I}\left[\frac{v}{\alpha(t)} - \frac{\pi[v/\alpha]/\pi[1/\alpha]}{\alpha(t)}\right], \quad v \in \widetilde{\mathcal{C}}(\mathbb{R}/T\mathbb{Z}),$$

while $\mathcal{K}$ is a compact operator. Thus, $T_{\tilde{x}}(0,0)$ is a compact perturbation of a topological isomorphism, and consequently, a Fredholm operator with zero index. It means that, in order to show that $T_{\tilde{x}}(0,0)$ is itself a topological isomorphism, it suffices to prove that it has only trivial kernel. Thus, we take some $\widetilde{\mathcal{C}}^1(\mathbb{R}/T\mathbb{Z})$ function $\xi \in \ker T_{\tilde{x}}(0,0)$: it is then immediate to check that it must be a solution of the linear equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\big[\alpha(t)\xi' + \beta(t)\xi\big] = \beta(t)\xi' + \gamma(t)\xi + C, \tag{17}$$

for some constant $C \in \mathbb{R}$. Multiply both sides above by φ_1 and integrate by parts on $[0,T]$ to check that $C=0$, and then, ξ must be a scalar multiple of φ_1, but since it has zero mean value, it must be the zero function, i.e., $\ker T_{\tilde{x}}(0,0) = 0$.

This means that $T_{\tilde{x}}(0,0) : \widetilde{\mathcal{C}}^1(\mathbb{R}/T\mathbb{Z}) \to \widetilde{\mathcal{C}}(\mathbb{R}/T\mathbb{Z})$ is a topological isomorphism, as claimed. The implicit function theorem implies now the existence of some $\epsilon > 0$ and a $\mathcal{C}^1$ curve $\widetilde{X} :]-\epsilon, \epsilon[\to \widetilde{\mathcal{C}}^1(\mathbb{R}/T\mathbb{Z})$ such that

($\tilde{\text{i}}$) $\widetilde{X}(0) = 0$,
($\widetilde{\text{ii}}$) $T(\bar{x}, \widetilde{X}(\bar{x})) = 0$, $\bar{x} \in]-\epsilon, \epsilon[$,

and we consider the $\mathcal{C}^1$ curve $\mathcal{X} :]-\epsilon, \epsilon[\to \mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ defined by

$$\mathcal{X}(\bar{x}) := \bar{x} + \tilde{x} + \widetilde{X}(\bar{x}), \quad \bar{x} \in]-\epsilon, \epsilon[.$$

Now, (i) and (ii) follow immediately from ($\tilde{\text{i}}$) and ($\widetilde{\text{ii}}$). In order to check (iii), we see that, after possibly replacing ϵ by a smaller number, it suffices to check that $\mathcal{X}'(0) = 1 + \widetilde{X}'(0) > 0$ on $\mathbb{R}$. With this aim, we derivate in equality ($\widetilde{\text{ii}}$), to get that $T_{\bar{x}}(0,0) + T_{\tilde{x}}(0,0)\widetilde{X}'(0) = 0$, or, what is the same, $\xi = 1 + \widetilde{X}'(0)$ is a T-periodic solution of (17) for some $C \in \mathbb{R}$. We deduce that $C = 0$ and, moreover, $1 + \widetilde{X}'(0) = \mu\varphi_1$ for some $\mu \in \mathbb{R}$. It means that $\pi[1 + \widetilde{X}'(0)] = 1 = \mu\pi[\varphi]$, and then, $\mu > 0$. Consequently, $1 + \widetilde{X}'(0) = \mu\varphi_1 > 0$ on $\mathbb{R}$. The result follows. □

Let us consider now the $\mathcal{C}^1$ function $f :]-\epsilon, \epsilon[\to \mathbb{R}$ defined by

$$f(\bar{x}) := \pi\big[P\big[\mathcal{X}(\bar{x})\big]\big], \quad \bar{x} \in]-\epsilon, \epsilon[,$$

so that (ii) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}L_p\big(t, \mathcal{X}(\bar{x})(t), \mathcal{X}(\bar{x})'(t)\big) = L_x\big(t, \mathcal{X}(\bar{x})(t), \mathcal{X}(\bar{x})'(t)\big) - f(\bar{x}), \quad t \in \mathbb{R}, \tag{18}$$

for any $\bar{x} \in]-\epsilon, \epsilon[$. Consequently, $\mathcal{X}(\bar{x})$ is a T-periodic solution of (2) if and only if $f(\bar{x}) = 0$. In particular, $f(0) = 0$ and, the T-periodic solution x_* being isolated, we deduce that $f(\bar{x}) \neq 0$ whenever $\bar{x} \neq 0$ is close enough to zero. Thus, after possibly replacing ϵ by a smaller positive number, we may assume that $f(\bar{x}) \neq 0$ for any $\bar{x} \in]-\epsilon, \epsilon[\setminus\{0\}$. Our next result uses the minimizing property of x_* to find the sign of $f(\bar{x})$ for $\bar{x} \neq 0$.

Lemma 4.2. *$f(\bar{x}) < 0$ if $-\epsilon < \bar{x} < 0$ and $f(\bar{x}) > 0$ if $0 < \bar{x} < \epsilon$.*

Proof. We consider the $\mathcal{C}^1$ function $F :]-\epsilon, \epsilon[\to \mathbb{R}$ defined by $F(\bar{x}) := \mathcal{A}[\mathcal{X}(\bar{x})]$. The functional $\mathcal{A}$ attaining a local minimum at x_*, the function F attains a local minimum at $\bar{x} = 0$. Moreover,

$$F'(\bar{x}) = \mathcal{A}'\big[\mathcal{X}(\bar{x})\big]\mathcal{X}'(\bar{x})$$
$$= \int_0^T L_x\big(t, \mathcal{X}(\bar{x})(t), \mathcal{X}(\bar{x})'(t)\big)\mathcal{X}'(\bar{x})(t)\,\mathrm{d}t + \int_0^T L_p\big(t, \mathcal{X}(\bar{x})(t), \mathcal{X}(\bar{x})'(t)\big)\mathcal{X}'(\bar{x})'(t)\,\mathrm{d}t,$$

so that, integration by parts together with (18) gives

$$\frac{1}{T}F'(\bar{x}) = \frac{1}{T}\int_0^T f(\bar{x})\mathcal{X}'(\bar{x})(t)\,\mathrm{d}t = \frac{f(\bar{x})}{T}\int_0^T \mathcal{X}'(\bar{x})(t)\,\mathrm{d}t = f(\bar{x}), \quad \bar{x} \in]-\epsilon, \epsilon[. \tag{19}$$

We deduce that $F'(0) = 0$ and $F'(\bar{x}) \neq 0$ for any $\bar{x} \in]-\epsilon, \epsilon[$, $\bar{x} \neq 0$. Moreover, F attains a local minimum at $\bar{x} = 0$, and, accordingly, $F'(\bar{x}) < 0$ if $-\epsilon < \bar{x} < 0$ and $F'(\bar{x}) > 0$ if $0 < \bar{x} < \epsilon$. In view of (19), this implies the result. □

Next, we consider the mapping between cylinders $\psi : (\mathbb{R}/T\mathbb{Z}) \times]-\epsilon, \epsilon[\to (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}$ defined by

$$\psi(t, \bar{x}) := \big(t, \mathcal{X}(\bar{x})(t)\big), \quad (t, \bar{x}) \in \mathbb{R} \times]-\epsilon, \epsilon[.$$

Item (iii) implies that it is a $\mathcal{C}^1$ diffeomorphism onto its image, which is an open subset of $(\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}$. Moreover, $\psi(\cdot, 0) = \mathcal{X}(0) = x_*$. Consequently, there exists some positive number $\rho > 0$ such that the set $\{(t, x) \in (\mathbb{R}/T\mathbb{Z}) \times \mathbb{R}\colon |x - x_*(t)| < \rho\}$ is contained in $\psi((\mathbb{R}/T\mathbb{Z})\times]-\epsilon/2, \epsilon/2[)$.

Proof of Theorem 1.1. In order to establish the main result of this paper, let the solution $x : \mathbb{R} \to \mathbb{R}$ of (2) verify $|x(t) - x_*(t)| < \rho$ for any $t \in \mathbb{R}$. By definition of ρ, $(t, x(t)) \in \psi(\mathbb{R}\times]-\epsilon/2, \epsilon/2[)$ for any $t \in \mathbb{R}$, and thus, there exists a $\mathcal{C}^1$ curve $y : \mathbb{R} \to \mathbb{R}$ with $|y(t)| < \epsilon/2$ and $(t, x(t)) = \psi(t, y(t))$, or, what is the same, $x(t) = \mathcal{X}(y(t))(t)$ for any $t \in \mathbb{R}$. We want to prove that $x = x_*$, or, equivalently, that $y \equiv 0$. The next result is a first step in this direction.

Lemma 4.3. *The function y does not have positive local maxima or negative local minima.*

Proof. Using a contradiction argument, assume for instance that y attains a positive local maximum $y(t_1) > 0$ at the point $t_1 \in \mathbb{R}$. Let us call $x_1 := \mathcal{X}(y(t_1))$, which is a $\mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ function satisfying the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}L_p\big(t, x_1(t), x_1'(t)\big) = L_x\big(t, x_1(t), x_1'(t)\big) - f\big(y(t_1)\big) < L_x\big(t, x_1(t), x_1'(t)\big), \tag{20}$$

for any $t \in \mathbb{R}$. At the instant $t = t_1$, $x_1(t_1) = x(t_1)$, $x_1'(t_1) = x'(t_1)$, and the combination of (20) and (2) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_1}\big[L_p\big(t, x_1(t), x_1'(t)\big) - L_p\big(t, x(t), x'(t)\big)\big] < L_x\big(t_1, x_1(t_1), x_1'(t_1)\big) - L_x\big(t_1, x(t_1), x'(t_1)\big) = 0,$$

which can also be rewritten in the following way

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_1}\left[z(t)\int_0^1 L_{xp}\big(t, (1-\lambda)x(t) + \lambda x_1(t), (1-\lambda)x'(t) + \lambda x_1'(t)\big)\,\mathrm{d}\lambda\right]$$
$$+ \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=t_1}\left[z'(t)\int_0^1 L_{pp}\big(t, (1-\lambda)x(t) + \lambda x_1(t), (1-\lambda)x'(t) + \lambda x_1'(t)\big)\,\mathrm{d}\lambda\right] < 0 \tag{21}$$

where $z = x_1 - x$. Observe that z is a $\mathcal{C}^1$ curve, and, the function y attaining a local maximum at $t = t_1$, it follows that z attains a local minimum at the same time. In particular,

$$z(t_1) = 0 = z'(t_1).$$

Note that this implies that the first of summand in expression (21) vanishes. Indeed, it is the derivative of the product of two continuous functions of time, the first of which vanishes, together with its derivative, at the point under consideration. We deduce that the second summand is negative, and consequently, the expression inside the brackets is a strictly decreasing function of t in a neighborhood of $t = t_1$. However, it vanishes at time t_1, and thus, it must be possible to find some $\epsilon > 0$ such that it is positive whenever $t_1 - \epsilon < t < t_1$ and negative whenever $t_1 < t < t_1 + \epsilon$. Now, $L_{pp} > 0$ by assumption (4), and we deduce that

$$z'(t) > 0 \quad \text{if } t_1 - \epsilon < t < t_1, \qquad z'(t) > 0 \quad \text{if } t_1 - \epsilon < t < t_1,$$

contradicting the fact that z attains a local minimum at $t = t_1$. □

Lemma 4.3 above implies that either the function y is monotonous on $\mathbb{R}$, or otherwise there exists some $\omega_* \in \mathbb{R}$ such that y is monotonous on $]-\infty, \omega_*[$ and also on $]\omega_*, +\infty[$. In any case, remembering that now that $y(\mathbb{R}) \subset\,]-\epsilon/2, \epsilon/2[$, we deduce the existence of limits $L_\pm \in [-\epsilon/2, \epsilon/2]$ such that $y(t) \to L_\pm$ as $t \to \pm\infty$.

Lemma 4.4. $L_\pm = 0$*, i.e., x is homoclinic a x_*.*

Proof. Let us show that $L_+ = 0$; an analogous argument can be used to see that $L_- = 0$. Indeed, if we call $x_+ := \mathcal{X}(L_+)$, which is a $\mathcal{C}^1(\mathbb{R}/T\mathbb{Z})$ curve, then $x(t) - x_+(t) = \mathcal{X}(y(t))(t) - \mathcal{X}(L_+)(t) \to 0$ as $t \to +\infty$. Since x_+ is T-periodic, we deduce that

$$\lim_{n\to+\infty} \left[x(nT + t) - x_+(t)\right] = 0,$$

uniformly with respect to $t \in [0, T]$. However, the function x being a solution of (2), the functional sequence $t \mapsto x(nT + t)$ is made of solutions of the same equation. We may use Lemma 2.3 to deduce that x_+ itself solves (2), which implies that $L_+ = 0$. □

We have shown y to be a continuous function on $\mathbb{R}$ which tends to 0 at $\pm\infty$, but does not have positive maxima or negative minima. Thus, $y \equiv 0$. The proof of Theorem 1.1 is complete. □

5. A Lyapunov-unstable equilibrium which is not SM-unstable

In this section we construct an example of a 2π-periodic in time Newtonian potential $V = V(t, x)$ such that the equilibrium $x_* \equiv 0$, which is the only 2π-periodic solution of the equation $x'' + V_x(t, x) = 0$, is unstable in the Lyapunov sense but not SM-unstable. Our example is $\mathcal{C}^\infty$ with respect to the state variable x; hence, so is its associated Poincaré mapping P. But we do not know how to build an analytic example.

To start with, let the $\mathcal{C}^\infty$ potential $U : \mathbb{R} \to \mathbb{R}$ verify

$$U(0) = 0 = \max_{\mathbb{R}} U, \tag{22}$$

and consider the associated Newtonian equation $x'' = -U'(x)$. Choose some solution $x : I \subset \mathbb{R} \to \mathbb{R}$ of this equation and assume that x changes sign, i.e., there are instants $t_0 < t_1$ of time such that $x(t_0)x(t_1) < 0$.

Proposition 5.1. *Under the above, x is monotonous, and, moreover, there exists some $\epsilon > 0$ such that, either $x'(t) \geqslant \epsilon$ for any $t \in \mathbb{R}$, or $x'(t) \leqslant -\epsilon$ for any $t \in \mathbb{R}$.*

Proof. Our equation being autonomous and without friction, time is reversible, i.e., $x(-t)$ is a solution for any solution x. Thus, we may assume that $x(t_0) < 0 < x(t_1)$, and we are going to show that $x'(t) > 0$ for any $t \in \mathbb{R}$.

To see this, we use conservation of energy, i.e., the quantity $\mathcal{E} \equiv (1/2)x'(t)^2 + U(x(t))$ does not depend on t. On the other hand, x being a continuous function which changes sign, Bolzano's Theorem provides the existence of some instant $t_0 < t_* < t_1$ of time such that $x(t_*) = 0$. By uniqueness of solutions to initial value problems, $x'(t_*) \neq 0$, and we deduce that $\mathcal{E} = (1/2)x'(t_*)^2 + U(x(t_*)) > U(0) = 0$; our solution has positive energy. It implies in particular that x' cannot vanish at any time, since $x'(t) = 0$ would imply $\mathcal{E} = U(x(t)) \leqslant 0$. Thus, x' does not change sign, and we deduce that $x'(t) > 0$ for any $t \in \mathbb{R}$. Moreover, $(1/2)x'(t)^2 = \mathcal{E} - U(x(t)) \geqslant \mathcal{E}$, and we deduce that $x'(t) \geqslant \epsilon := \sqrt{2\mathcal{E}}$ for any $t \in \mathbb{R}$. □

A immediate consequence of Proposition 5.1 is the following

Corollary 5.2. *Let the $\mathcal{C}^\infty$ potential U verify* (22)*, and let $T > 0$ be arbitrary. Then, the only solution of the antiperiodic boundary value problem*

$$x'' = -U'(x), \qquad x(0) = -x(T), \quad x'(0) = -x'(T),$$

is $x \equiv 0$.

Now, let the $\mathcal{C}^\infty$ potential $U : \mathbb{R} \to \mathbb{R}$ verify (22), and moreover:

(a) $U(x) = U(-x)\ \forall x \in \mathbb{R}$.
(b) There exists a sequence $\{x_n\}_n \to 0$ of positive numbers with $U'(x_n) = 0$ for any $n \in \mathbb{N}$.
(c) There exists a constant $C > 0$ such that $U(x) \geqslant -C$ for any $x \in \mathbb{R}$.

The role of assumption (c) is to guarantee that all solutions of the equation $x'' + U'(x) = 0$ are defined on the whole real line. Let us fix some $T > 0$ and call $Q : \mathbb{R}^2 \to \mathbb{R}^2$ the Poincaré mapping (period T), associated to this equation.

On the other hand, (a) implies that if x is a solution of $x'' + U'(x) = 0$, then $-x$ is another one. Consequently,

$$Q(-x, -y) = -Q(x, y) \quad \text{for any } (x, y) \in \mathbb{R}^2. \tag{23}$$

Finally, assumption (b) has the following consequence:

$$(x_n, 0) \text{ is a fixed point of } Q \text{ for any } n \in \mathbb{N}. \tag{24}$$

Theorem 5.3. *Denote $P := -Q$, which is an area-preserving, $\mathcal{C}^\infty$ diffeomorphism of $\mathbb{R}^2$ into itself. The following hold*:

(i) $(0, 0)$ *is the only fixed point of P.*
(ii) $(0, 0)$ *is Lyapunov unstable*; *indeed, for any $v \neq 0$, $\|P^n(0, v)\| \to \infty$ as $n \to +\infty$.*
(iii) $(x_n, 0)$ *is a fixed point of P^2 for any $n \in \mathbb{N}$*; *in particular,* $(0, 0)$ *is not SM-unstable.*

Proof. A fixed point of P is an element $(x, v) \in \mathbb{R}^2$ such that $Q(x, v) = (-x, -v)$. Consequently, (i) follows immediately from Corollary 5.2 above. On the other hand, as a consequence of (23), $P^{2n} = Q^{2n}$, while $P^{2n-1} = -Q^{2n-1}$, and, together with Proposition 5.1, this means that the abscissa component of $P^{2n}(0, v)$ converges to $+\infty$, while that of $P^{2n-1}(0, v)$ converges to $-\infty$ as $n \to +\infty$, implying (ii). Finally, $(x_n, 0)$ being a fixed point of Q, it is also a fixed point of $Q^2 = P^2$ for any $n \in \mathbb{N}$, so that $(0, 0)$ is not SM-unstable. □

To complete the argument, we observe that P is indeed the Poincaré mapping (period 2π) associated to the equation $x'' + V'(t, x) = 0$ when the 2π-periodic potential $V = V(t, x)$ is defined by

$$V(t, x) = U(x) \quad \text{if } t \in [0, \pi[, \qquad V(t, x) = -x \quad \text{if } t \in [\pi, 2\pi[,$$

on $[0, 2\pi[\times\mathbb{R}$, and then extended periodically in time to the whole plane $\mathbb{R}^2$.

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