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Nonoccurrence of the Lavrentiev phenomenon for nonconvex variational problems

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Abstract

In this paper we establish nonoccurrence of the Lavrentiev phenomenon for two classes of nonconvex variational problems. For the first class of integrands we show the existence of a minimizing sequence of Lipschitzian functions while for the second class we establish that an infimum on the full admissible class is equal to the infimum on a set of Lipschitzian functions with the same Lipschitzian constant.
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Résumé

Dans cet article nous démontrons que le phénomène de Lavrentiev pour deux classes de problèmes variationnels non convexes ne peut avoir lieu. Nous montrons, pour la première classe d'intégrands, l'existence d'une suite minimisante de fonctions lipschtziennes, alors que pour la seconde classe, nous démontrons qu'un infimum sur toute la classe admissible est égale a l'infimum sur un ensemble de fonctions lipschtziennes, avec la même constante de Lipschtiz.
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1. Introduction

The Lavrentiev phenomenon in the calculus of variations was discovered in 1926 by M. Lavrentiev in [9]. There it was shown that it is possible for the variational integral of a two-point Lagrange problem, which is sequentially weakly lower semicontinuous on the admissible class of absolutely continuous functions, to possess an infimum on the dense subclass of C^1 admissible functions that is strictly greater than its minimum value on the admissible

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class. Since this seminal work the Lavrentiev phenomenon is of great interest. See, for instance, [1–4,6–8,10–12] and the references mentioned there. Mania [11] simplified the original example of Lavrentiev. Ball and Mizel [3,4] demonstrated that the Lavrentiev phenomenon can occur with fully regular integrands. Nonoccurrence of the Lavrentiev phenomenon was studied in [1,2,7,8,10,12]. Clarke and Vinter [7] showed that the Lavrentiev phenomenon cannot occur when a variational integrand $f(t,x,u)$ is independent of t. Sychev and Mizel [12] considered a class of integrands $f(t,x,u)$ which are convex with respect to the last variable. For this class of integrands they established that the Lavrentiev phenomenon does not occur. In this paper we establish nonoccurrence of Lavrentiev phenomenon for two classes of nonconvex nonautonomous variational problems with integrands $f(t,x,u)$. For the first class of integrands we show the existence of a minimizing sequence of Lipschitzian functions while for the second class we establish that an infimum on the full admissible class is equal to the infimum on a set of Lipschitzian functions with the same Lipschitzian constant.

Assume that $(X, \|\cdot\|)$ is a Banach space. Let $-\infty < \tau_1 < \tau_2 < \infty$. Denote by $W^{1,1}(\tau_1, \tau_2; X)$ the set of all functions $x : [\tau_1, \tau_2] \to X$ for which there exists a Bochner integrable function $u : [\tau_1, \tau_2] \to X$ such that

$$x(t) = x(\tau_1) + \int_{\tau_1}^{t} u(s)\,\mathrm{d}s, \quad t \in (\tau_1, \tau_2]$$

(see, e.g., [5]). It is known that if $x \in W^{1,1}(\tau_1, \tau_2; X)$, then this equation defines a unique Bochner integrable function u which is called the derivative of x and is denoted by x'.

We denote by $\mathrm{mes}(\Omega)$ the Lebesgue measure of a Lebesgue measurable set $\Omega \subset \mathbb{R}^1$.

Let $a, b \in \mathbb{R}^1$ satisfy $a < b$. Suppose that $f : [a,b] \times X \times X \to \mathbb{R}^1$ is a continuous function such that the following assumptions hold:

(A1) $$f(t,x,u) \geqslant \phi\big(\|u\|\big) \quad \text{for all } (t,x,u) \in [a,b] \times X \times X, \tag{1.1}$$

where $\phi : [0,\infty) \to [0,\infty)$ is an increasing function such that

$$\lim_{t\to\infty} \phi(t)/t = \infty; \tag{1.2}$$

(A2) for each $M, \epsilon > 0$ there exist $\Gamma, \delta > 0$ such that

$$\big|f(t,x_1,u) - f(t,x_2,u)\big| \leqslant \epsilon \max\big\{f(t,x_1,u), f(t,x_2,u)\big\} \tag{1.3}$$

for each $t \in [a,b]$, each $u \in X$ satisfying $\|u\| \geqslant \Gamma$ and each $x_1, x_2 \in X$ satisfying

$$\|x_1 - x_2\| \leqslant \delta, \quad \|x_1\|, \|x_2\| \leqslant M; \tag{1.4}$$

(A3) for each $M, \epsilon > 0$ there exists $\delta > 0$ such that

$$\big|f(t,x_1,y_1) - f(t,x_2,y_2)\big| \leqslant \epsilon$$

for each $t \in [a,b]$ and each $x_1, x_2, y_1, y_2 \in X$ satisfying

$$\|x_i\|, \|y_i\| \leqslant M, \quad i = 1, 2,$$

and

$$\|x_1 - x_2\|, \|y_1 - y_2\| \leqslant \delta.$$

Remark 1.1. If $X = \mathbb{R}^n$, then (A3) follows from the continuity of f.

Let $z_1, z_2 \in X$. Denote by $\mathcal{B}$ the set of all functions $v \in W^{1,1}(a,b;X)$ such that $v(a) = z_1$, $v(b) = z_2$. Denote by $\mathcal{B}_L$ the set of all $v \in \mathcal{B}$ for which there is $M_v > 0$ such that

$$\big\|v'(t)\big\| \leqslant M_v \quad \text{for almost every } t \in [a,b]. \tag{1.5}$$

Clearly for each $v \in \mathcal{B}$ the function $f(t, v(t), v'(t))$, $t \in [a, b]$, is measurable. We consider the variational problem

$$I(v) := \int_a^b f\big(t, v(t), v'(t)\big)\,\mathrm{d}t \to \min, \quad v \in \mathcal{B}, \tag{1.6}$$

and establish the following result.

Theorem 1.1.

$$\inf\big\{I(v):\ v \in \mathcal{B}\big\} = \inf\big\{I(v):\ v \in \mathcal{B}_L\big\}.$$

Theorem 1.1 is proved in Section 3.

It is not difficult to see that the following propositions hold.

Propsotion 1.1. *Let $\phi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\lim_{t\to\infty} \phi(t)/t = \infty$, $g : [a, b] \times X \to \mathbb{R}^1$ be a continuous function such that*

$$g(t, u) \geqslant \phi\big(\|u\|\big) \quad \textit{for all } (t, u) \in [a, b] \times X$$

and let $h : [a, b] \times X \to [0, \infty)$ be a continuous function. Assume that for $\xi = g, h$ the following property holds:

(A4) *For each $M, \epsilon > 0$ there exists $\delta > 0$ such that*

$$\big|\xi(t, x_1) - \xi(t, x_2)\big| \leqslant \epsilon$$

for each $t \in [a, b]$ and each $x_1, x_2 \in X$ satisfying

$$\|x_i\| \leqslant M, \quad i = 1, 2, \qquad \|x_1 - x_2\| \leqslant \delta.$$

Then (A1)–(A3) *hold with the function*

$$f(t, x, u) = h(t, x) + g(t, u), \quad (t, x, u) \in [a, b] \times X \times X.$$

Propsotion 1.2. *Let $\phi : [0, \infty) \to [0, \infty)$ be an increasing function such that $\lim_{t\to\infty} \phi(t)/t = \infty$, $g : [a, b] \times X \to \mathbb{R}^1$ be a continuous function such that*

$$g(t, u) \geqslant \phi\big(\|u\|\big) \quad \textit{for all } (t, u) \in [a, b] \times X$$

and let $h : [a, b] \times X \to [0, \infty)$ be a continuous function such that

$$\inf\big\{h(t, x):\ (t, x) \in [a, b] \times X\big\} > 0.$$

Assume that (A4) *holds with $\xi = g, h$. Then the function $f(t, x, u) = g(t, u)h(t, x)$, $(t, x, u) \in [a, b] \times X \times X$ satisfies* (A1)–(A3).

Corollary 1.1. *Let $X = \mathbb{R}^n$, $\phi : [0, \infty) \to [0, \infty)$ be an increasing function such that*

$$\lim_{t\to\infty} \phi(t)/t = \infty,$$

$g : [a, b] \times X \to \mathbb{R}^1$ be a continuous function such that

$$g(t, u) \geqslant \phi\big(\|u\|\big) \quad \textit{for all } (t, u) \in [a, b] \times X, \tag{1.7}$$

let $h : [a, b] \times X \to [0, \infty)$ be a continuous function such that

$$\inf\big\{h(t, x):\ (t, x) \in [a, b] \times X\big\} > 0 \tag{1.8}$$

and let

$$f(t,x,u)=g(t,u)h(t,x), \quad (t,x,u)\in[a,b]\times X\times X. \tag{1.9}$$

Then

$$\inf\{I(v):\ v\in\mathcal{B}\}=\inf\{I(v):\ v\in\mathcal{B}_L\}. \tag{1.10}$$

It should be mentioned that there are many examples of integrands of the form (1.9) for which the Lavrentiev phenomenon occurs. Corollary 1.1 shows that if such integrands satisfy inequalities (1.7) and (1.8), then the Lavrentiev phenomenon does not occur.

Now we present out second main result.

Let $a,b\in\mathbb{R}^1$, $a<b$. Suppose that $f:[a,b]\times X\times X\to\mathbb{R}^1$ is a continuous function which satisfies the following assumptions:

(B1) There is an increasing function $\phi:[0,\infty)\to[0,\infty)$ such that

$$f(t,x,u)\geqslant\phi(\|u\|) \quad \text{for all } (t,x,u)\in[a,b]\times X\times X, \tag{1.11}$$

$$\lim_{t\to\infty}\phi(t)/t=\infty. \tag{1.12}$$

(B2) For each $M>0$ there exist positive numbers δ, L and an integrable nonnegative scalar function $\psi_M(t)$, $t\in[a,b]$, such that for each $t\in[a,b]$, each $u\in X$ and each $x_1,x_2\in X$ satisfying

$$\|x_1\|,\|x_2\|\leqslant M, \qquad \|x_1-x_2\|\leqslant\delta$$

the following inequality holds*:*

$$\left|f(t,x_1,u)-f(t,x_2,u)\right|\leqslant\|x_1-x_2\|L\big(f(t,x_1,u)+\psi_M(t)\big).$$

(B3) For each $M>0$ there is $L>0$ such that for each $t\in[a,b]$ and each $x_1,x_2,u_1,u_2\in X$ satisfying $\|x_i\|,\|u_i\|\leqslant M$, $i=1,2$, the following inequality holds:

$$\left|f(t,x_1,u_1)-f(t,x_2,u_2)\right|\leqslant L\big(\|x_1-x_2\|+\|u_1-u_2\|\big).$$

Remark 1.2. It is not difficult to see that if (B2) holds with each ψ_M bounded, then f satisfies (A1)–(A3).

For each $z_1,z_2\in X$ denote by $\mathcal{A}(z_1,z_2)$ the set of all $x\in W^{1,1}(a,b;X)$ such that $x(a)=z_1$, $x(b)=z_2$.

For each $x\in\mathcal{A}$ set

$$I(x)=\int_a^b f\big(t,x(t),x'(t)\big)\,\mathrm{d}t. \tag{1.13}$$

The next theorem is our second main result.

Theorem 1.2. *Let $M>0$. Then there exists $K>0$ such that for each $z_1,z_2\in X$ satisfying $\|z_1\|,\|z_2\|\leqslant M$ and each $x(\cdot)\in\mathcal{A}(z_1,z_2)$ the following assertion holds*:

If $\mathrm{mes}\{t\in[a,b]:\ \|x'(t)\|>K\}>0$, *then there exists $y\in\mathcal{A}(z_1,z_2)$ such that $I(y)<I(x)$ and $\|y'(t)\|\leqslant K$ for almost every* $t\in[a,b]$.

Remark 1.3. (B3) implies that f is bounded on any bounded subset of $[a,b]\times X\times X$.

It is not difficult to see that the following proposition holds.

Propsotion 1.3. *Let* $\phi:[0,\infty)\to[0,\infty)$ *be an increasing function such that* $\lim_{t\to\infty}\phi(t)/t=\infty$, $g:[a,b]\times X\to\mathbb{R}^1$ *be a continuous function such that*

$$g(t,u)\geqslant\phi\big(\|u\|\big)\quad\textit{for all }(t,u)\in[a,b]\times X$$

and let $h:[a,b]\times X\to[0,\infty)$ *be a continuous function such that*

$$\inf\big\{h(t,x):\ (t,x)\in[a,b]\times X\big\}>0.$$

Assume that for $\xi=g,h$ *the following property holds*:

For each $M>0$ *there is* $L>0$ *such that for each* $t\in[a,b]$ *and each* $x_1,x_2,u_1,u_2\in X$ *satisfying* $\|x_i\|,\|u_i\|\leqslant M$, $i=1,2$, *the following inequality holds*:

$$\big|\xi(t,x_1)-\xi(t,x_2)\big|\leqslant L\|x_1-x_2\|.$$

Then (B1)–(B3) *hold with the function*

$$f(t,x,u)=h(t,x)g(t,u),\quad (t,x,u)\in[a,b]\times X\times X.$$

For each $z_1,z_2\in X$ set

$$U(z_1,z_2)=\inf\big\{I(x):\ x\in\mathcal{A}(z_1,z_2)\big\}. \tag{1.14}$$

It is easy to see that $U(z_1,z_2)$ is finite for each $z_1,z_2\in X$.

The paper is organized as follows. Section 2 contains auxiliary results for Theorem 1.1 which is proved in Section 3. In Section 4 we discuss some properties of integrands which satisfy assumptions (A1)–(A3). Our second main result (Theorem 1.2) is proved in Section 6. Section 5 contains auxiliary results for Theorem 1.2.

2. Auxiliary results

Set

$$M_0=\inf\big\{I(v):\ v\in\mathcal{B}\big\}. \tag{2.1}$$

Clearly M_0 is a finite number.

Lemma 2.1. *There exists a number* $M_1>0$ *such that for each* $v\in\mathcal{B}$ *satisfying* $I(v)\leqslant M_0+2$ *the following inequality holds*:

$$\big\|v(t)\big\|\leqslant M_1\quad\textit{for all }t\in[a,b]. \tag{2.2}$$

Proof. Relation (1.2) implies that there is $c_0\geqslant 1$ such that

$$\phi(t)\geqslant t\quad\text{for all }t\geqslant c_0. \tag{2.3}$$

Assume that $v\in\mathcal{B}$ satisfies

$$I(v)\leqslant M_0+2. \tag{2.4}$$

Let $\tau\in(a,b]$ and

$$E_1=\big\{t\in[a,\tau]:\ \big\|v'(t)\big\|\geqslant c_0\big\},\qquad E_2=[a,\tau]\setminus E_1. \tag{2.5}$$

By (2.5) and the definition of $\mathcal{B}$

$$\begin{aligned}\|v(\tau)\| &= \left\|v(a)+\int_a^\tau v'(t)\,\mathrm{d}t\right\| \leqslant \|v(a)\| + \int_a^\tau \|v'(t)\|\,\mathrm{d}t\\ &\leqslant \|z_1\| + \int_{E_1} \|v'(t)\|\,\mathrm{d}t + \int_{E_2} \|v'(t)\|\,\mathrm{d}t\\ &\leqslant \|z_1| + c_0\,\mathrm{mes}(E_2) + \int_{E_1} \|v'(t)\|\,\mathrm{d}t\\ &\leqslant \|z_1\| + c_0(b-a) + \int_{E_1} \|v'(t)\|\,\mathrm{d}t. \end{aligned} \tag{2.6}$$

We estimate $\int_{E_1} \|v'(t)\|\,\mathrm{d}t$. It follows from (2.5), (2.3) and (1.1) that for all $t \in E_1$

$$\|v'(t)\| \leqslant \phi(\|v'(t)\|) \leqslant f(t, v(t), v'(t)).$$

Together with (1.1), (1.6), (2.5) and (2.4) this inequality implies that

$$\int_{E_1} \|v'(t)\|\,\mathrm{d}t \leqslant \int_{E_1} f(t, v(t), v'(t))\,\mathrm{d}t \leqslant I(v) \leqslant M_0 + 2.$$

Combined with (2.6) this inequality implies that

$$\|v(\tau)\| \leqslant \|z_1\| + c_0(b-a) + M_0 + 2.$$

Thus the inequality (2.2) holds with

$$M_1 = \|z_1\| + c_0(b-a) + M_0 + 2.$$

Lemma 2.1 is proved. □

Lemma 2.2. *Let $\epsilon, M > 0$. Then there exist $\Gamma, \delta > 0$ such that*

$$|f(t, x_1, u) - f(t, x_2, u)| \leqslant \epsilon \min\{f(t, x_1, u), f(t, x_2, u)\} \tag{2.7}$$

for each $t \in [a, b]$, each $u \in X$ satisfying $\|u\| \geqslant \Gamma$ and each $x_1, x_2 \in X$ satisfying

$$\|x_1\|, \|x_2\| \leqslant M, \qquad \|x_1 - x_2\| \leqslant \delta. \tag{2.8}$$

Proof. Choose a number $\epsilon_0 \in (0, 1)$ such that

$$\epsilon_0(1-\epsilon_0)^{-1} < \epsilon. \tag{2.9}$$

By (A2) there exist $\Gamma, \delta > 0$ such that

$$|f(t, x_1, u) - f(t, x_2, u)| \leqslant \epsilon_0 \max\{f(t, x_1, u), f(t, x_2, u)\} \tag{2.10}$$

for each $t \in [a, b]$, each $u \in X$ satisfying $\|u\| \geqslant \Gamma$ and each $x_1, x_2 \in X$ satisfying (2.8).

Assume that $t \in [a, b]$, $u \in X$ satisfy $\|u\| \geqslant \Gamma$ and $x_1, x_2 \in X$ satisfy (2.8). Then (2.10) is true. We show that (2.7) holds.

We may assume without loss of generality that

$$f(t, x_2, u) \geqslant f(t, x_1, u). \tag{2.11}$$

In view of (2.10) and (2.11)

$$f(t, x_2, u) - f(t, x_1, u) \leqslant \epsilon_0 f(t, x_2, u)$$

and

$$f(t,x_1,u) \leqslant f(t,x_2,u) \leqslant (1-\epsilon_0)^{-1} f(t,x_1,u). \tag{2.12}$$

Combined with (2.10), (2.11) and (2.9) the inequality (2.12) implies that

$$\begin{aligned}\bigl|f(t,x_1,u)-f(t,x_2,u)\bigr| &\leqslant \epsilon_0 f(t,x_2,u) \leqslant \epsilon_0(1-\epsilon_0)^{-1} f(t,x_1,u)\\ &\leqslant \epsilon f(t,x_1,u) = \epsilon \min\bigl\{f(t,x_1,u),\, f(t,x_2,u)\bigr\}.\end{aligned}$$

This completes the proof of Lemma 2.2. □

3. Proof of Theorem 1.1

Set

$$M_0 = \inf\bigl\{I(v):\ v \in \mathcal{B}\bigr\}. \tag{3.1}$$

Clearly M_0 is a finite number. Let $\epsilon \in (0,1)$. In order to prove the theorem it is sufficient to show that for each $v \in \mathcal{B}$ satisfying $I(v) \leqslant M_0 + 1$ there is $u \in \mathcal{B}_L$ such that $I(u) \leqslant I(v) + \epsilon$.

By Lemma 2.1 there is $M_1 > 0$ such that

$$\bigl\|v(t)\bigr\| \leqslant M_1, \quad t \in [a,b], \tag{3.2}$$

for all $v \in \mathcal{B}$ satisfying $I(v) \leqslant M_0 + 2$.

Choose a positive number ϵ_0 such that

$$8\epsilon_0(M_0+4) < \epsilon \tag{3.3}$$

and a positive number γ_0 such that

$$\gamma_0 < 1 \quad \text{and} \quad 8\gamma_0(M_0+2) < b-a. \tag{3.4}$$

Relation (1.2) implies that there is $N > 1$ such that

$$\phi(t)/t \geqslant \gamma_0^{-1} \quad \text{for all } t \geqslant N. \tag{3.5}$$

In view of Lemma 2.2 there are

$$\delta_0 \in (0,1), \qquad N_0 > N \tag{3.6}$$

such that for each $t \in [a,b]$, each $y \in X$ satisfying $\|y\| \geqslant N_0$ and each $x_1, x_2 \in X$ satisfying

$$\|x_1\|, \|x_2\| \leqslant M_1 + 2, \qquad \|x_1 - x_2\| \leqslant \delta_0 \tag{3.7}$$

the following inequality holds:

$$\bigl|f(t,x_1,y)-f(t,x_2,y)\bigr| \leqslant \epsilon_0 \min\bigl\{f(t,x_1,y),\, f(t,x_2,y)\bigr\}. \tag{3.8}$$

By (A3) there exists

$$\delta_1 \in (0,\delta_0) \tag{3.9}$$

such that

$$\bigl|f(t,x_1,y_1)-f(t,x_2,y_2)\bigr| \leqslant \bigl(8(b-a+1)\bigr)^{-1}\epsilon \tag{3.10}$$

for each $t \in [a,b]$ and each $x_1, x_2, y_1, y_2 \in X$ satisfying

$$\|x_1\|, \|x_2\| \leqslant M_1 + 2, \qquad \|y_1\|, \|y_2\| \leqslant N_0 + 1, \qquad \|x_1 - x_2\|, \|y_1 - y_2\| \leqslant \delta_1. \tag{3.11}$$

It follows from (A3) that we can choose

$$M_2 > \sup\bigl\{f(t,y,0)\colon\ t\in[a,b],\ \ y\in X \text{ and } \|y\|\leqslant M_1+1\bigr\}. \tag{3.12}$$

Choose a positive number γ_1 such that

$$8\gamma_1(M_0+M_1+4) < \delta_1\min\{1,b-a\}. \tag{3.13}$$

By (1.2) there is a number N_1 such that

$$N_1 > N_0+M_2+4 \quad\text{and}\quad \phi(t)/t\geqslant\gamma_1^{-1} \quad \text{for all } t\geqslant N_1. \tag{3.14}$$

Assume that

$$v\in\mathcal{B} \quad\text{and}\quad I(v)\leqslant M_0+2. \tag{3.15}$$

It follows from (3.15) and the choice of M_1 that the inequality (3.2) holds. Set

$$\begin{aligned} &E_1=\bigl\{t\in[a,b]\colon\ \bigl\|v'(t)\bigr\|\geqslant N_1\bigr\},\\ &E_2=\bigl\{t\in[a,b]\colon\ \bigl\|v'(t)\bigr\|\leqslant N_0\bigr\}, \qquad E_3=[a,b]\setminus(E_1\cup E_2). \end{aligned}\tag{3.16}$$

(3.16), (3.14), (1.1), (1.6) and (3.15) imply that

$$\begin{aligned} \biggl\|\int_{E_1} v'(t)\,\mathrm{d}t\biggr\| &\leqslant \int_{E_1}\bigl\|v'(t)\bigr\|\,\mathrm{d}t \leqslant \int_{E_1}\gamma_1\phi\bigl(\|v'(t)\|\bigr)\,\mathrm{d}t\\ &\leqslant \gamma_1\int_{E_1} f\bigl(t,v(t),v'(t)\bigr)\,\mathrm{d}t \leqslant \gamma_1 I(v)\leqslant\gamma_1(M_0+2). \end{aligned}\tag{3.17}$$

Set

$$h_0=\int_{E_1} v'(t)\,\mathrm{d}t. \tag{3.18}$$

By (3.17) and (3.18)

$$\|h_0\|\leqslant\gamma_1(M_0+2). \tag{3.19}$$

Now we estimate $\mathrm{mes}(E_2)$. It follows from (3.16), the choice of N (see (3.5)), (3.6), (1.1), (1.6) and (3.15) that

$$\begin{aligned} \mathrm{mes}(E_1\cup E_3) &\leqslant N_0^{-1}\int_{E_1\cup E_3}\bigl\|v'(t)\bigr\|\,\mathrm{d}t \leqslant \gamma_0 N_0^{-1}\int_{E_1\cup E_3}\phi\bigl(\bigl\|v'(t)\bigr\|\bigr)\,\mathrm{d}t\\ &\leqslant \gamma_0 N_0^{-1}\int_{E_1\cup E_3} f\bigl(t,v(t),v'(t)\bigr)\,\mathrm{d}t \leqslant \gamma_0 N_0^{-1} I(v)\leqslant\gamma_0 I(v)\leqslant\gamma_0(M_0+2). \end{aligned}$$

Combined with (3.16) and (3.4) this inequality implies that

$$\mathrm{mes}(E_2)\geqslant b-a-\gamma_0(M_0+2)\geqslant(3/4)(b-a). \tag{3.20}$$

Define a measurable function $\xi:[a,b]\to X$ by

$$\begin{aligned} &\xi(t)=0, \quad t\in E_1, \qquad \xi(t)=v'(t), \quad t\in E_3,\\ &\xi(t)=v'(t)+\bigl(\mathrm{mes}(E_2)\bigr)^{-1}h_0, \quad t\in E_2. \end{aligned}\tag{3.21}$$

Clearly the function ξ is Bochner integrable. It follows from (3.16), (3.21), (3.18) and (3.15) that

$$\int_a^b \xi(t)\,\mathrm{d}t = \int_{E_1} \xi(t)\,\mathrm{d}t + \int_{E_2} \xi(t)\,\mathrm{d}t + \int_{E_3} \xi(t)\,\mathrm{d}t = \int_{E_2} \xi(t)\,\mathrm{d}t + \int_{E_3} \xi(t)\,\mathrm{d}t$$
$$= \int_{E_2} v'(t)\,\mathrm{d}t + h_0 + \int_{E_3} v'(t)\,\mathrm{d}t, \tag{3.22}$$
$$\sum_{i=1}^{3} \int_{E_i} v'(t)\,\mathrm{d}t = \int_a^b v'(t)\,\mathrm{d}t = z_2 - z_1.$$

Define a function $u : [a, b] \to X$ by

$$u(\tau) = \int_0^\tau \xi(t)\,\mathrm{d}t + z_1, \quad \tau \in [a, b]. \tag{3.23}$$

In view of (3.23), (3.22), (3.21) and (3.16) $u \in \mathcal{B}_L$.

Now we show that

$$\|u(t) - v(t)\| \leqslant \delta_1 \quad \text{for all } t \in [a, b]. \tag{3.24}$$

Let $s \in (a, b]$. By (3.15), (3.23), (3.16), (3.21), (3.17) and (3.19)

$$\|v(s) - u(s)\| = \left\| \int_a^s \left[v'(t) - \xi(t)\right]\mathrm{d}t \right\|$$
$$\leqslant \left\| \int_{[a,s]\cap E_1} \left[v'(t) - \xi(t)\right]\mathrm{d}t \right\| + \left\| \int_{[a,s]\cap E_2} \left[v'(t) - \xi(t)\right]\mathrm{d}t \right\| + \left\| \int_{[a,s]\cap E_3} \left[v'(t) - \xi(t)\right]\mathrm{d}t \right\|$$
$$\leqslant \int_{E_1} \|v'(t)\|\,\mathrm{d}t + \|h_0\| \leqslant 2\gamma_1(M_0 + 2).$$

Thus for each $s \in [a, b]$

$$\|v(s) - u(s)\| \leqslant 2\gamma_1(M_0 + 2). \tag{3.25}$$

Combined with (3.13) this inequality implies (3.24). It follows from (3.24), (3.2), (3.9) and (3.6) that

$$\|u(t)\| \leqslant M_1 + 1 \quad \text{for all } t \in [a, b]. \tag{3.26}$$

We estimate $I(u) - I(v)$. In view of (1.6) and (3.16)

$$I(u) - I(v) = \sum_{i=1}^{3} \int_{E_i} \left[f\big(t, u(t), u'(t)\big) - f\big(t, v(t), v'(t)\big)\right]\mathrm{d}t. \tag{3.27}$$

By (3.23), (3.21), (3.26) and (3.12) for almost every $t \in E_1$

$$f\big(t, u(t), u'(t)\big) = f\big(t, u(t), \xi(t)\big) = f\big(t, u(t), 0\big) < M_2. \tag{3.28}$$

(1.1), (3.16), (3.14) and (3.13) imply that for almost every $t \in E_1$

$$f\big(t, v(t), v'(t)\big) \geqslant \phi\big(\|v'(t)\|\big) \geqslant N_1 > M_2 + 4.$$

Combined with (3.28) this inequality implies that

$$\int_{E_1} \left[f\big(t, u(t), u'(t)\big) - f\big(t, v(t), v'(t)\big) \right] \mathrm{d}t \leqslant 0. \tag{3.29}$$

Let $t \in E_2$ and $v'(t), u'(t)$ exist. It follows from (3.16) that

$$\|v'(t)\| \leqslant N_0. \tag{3.30}$$

(3.23) and (3.21) imply that

$$u'(t) = \xi(t) = v'(t) + \big(\operatorname{mes}(E_2)\big)^{-1} h_0.$$

Together with (3.19), (3.20) and (3.13) this equality implies that

$$\|u'(t) - v'(t)\| = \big(\operatorname{mes}(E_2)\big)^{-1} \|h_0\| \leqslant \gamma_1 (M_0 + 2) 2 (b - a)^{-1} < \frac{\delta_1}{4}. \tag{3.31}$$

(3.31), (3.30), (3.9) and (3.6) imply that

$$\|u'(t)\| \leqslant N_0 + 1. \tag{3.32}$$

By (3.2), (3.26), (3.30) (3.32), (3.31), (3.24) and the choice of δ_1 (see (3.10), (3.11))

$$\big| f\big(t, v(t), v'(t)\big) - f\big(t, u(t), u'(t)\big) \big| \leqslant \big(8(b - a + 1)\big)^{-1} \epsilon.$$

Since this inequality holds for almost every $t \in E_2$ we obtain that

$$\left| \int_{E_2} \left[f\big(t, u(t), u'(t)\big) - f\big(t, v(t), v'(t)\big) \right] \mathrm{d}t \right| \leqslant \frac{\epsilon}{8}. \tag{3.33}$$

Let $t \in E_3$ and $u'(t)$ and $v'(t)$ exist. By (3.16) and (3.14)

$$\|v'(t)\| \geqslant N_0. \tag{3.34}$$

In view of (3.23) and (3.21)

$$\big| f\big(t, v(t), v'(t)\big) - f\big(t, u(t), u'(t)\big) \big| = \big| f\big(t, v(t), v'(t)\big) - f\big(t, u(t), v'(t)\big) \big|.$$

It follows from this equality, (3.2), (3.26), (3.24), (3.9), (3.34) and the choice of δ_0, N_0 (see (3.6)–(3.8)) that

$$\big| f\big(t, v(t), v'(t)\big) - f\big(t, u(t), u'(t)\big) \big| \leqslant \epsilon_0 f\big(t, v(t), v'(t)\big).$$

By this inequality which holds for almost every $t \in E_3$, (3.15) and (3.3)

$$\left| \int_{E_3} \left[f\big(t, u(t), u'(t)\big) - f\big(t, v(t), v'(t)\big) \right] \mathrm{d}t \right| \leqslant \int_{E_3} \epsilon_0 f\big(t, v(t), v'(t)\big) \mathrm{d}t \leqslant \epsilon_0 I(v) \leqslant \epsilon_0 (M_0 + 2) < \frac{\epsilon}{8}.$$

Combined with (3.29), (3.33) and (3.27) this inequality implies that $I(u) - I(v) \leqslant \epsilon/2$. This completes the proof of Theorem 1.1. □

4. Properties of integrands which satisfy (A1)–(A3)

Let $(X, \|\cdot\|)$ be a Banach space and let $a, b \in \mathbb{R}^1$ be such that $a < b$.

Propsotion 4.1. *Assume that* $g:[a,b]\times X\times X\to\mathbb{R}^1$ *is a continuous function,* (A1)–(A3) *hold with* $f=g$ *and* $h:[a,b]\times X\times X\to[0,\infty)$ *is a continuous function such that* (A3) *holds with* $f=h$ *and for each bounded subset* $D\subset[a,b]\times X$

$$\lim_{\|u\|\to\infty}\big[h(t,x,u)/g(t,x,u)\big]=0 \quad \textit{uniformly for } (t,x)\in D. \tag{4.1}$$

Then (A1)–(A3) *hold with* $f=g+h$.

Proof. Clearly (A1) and (A3) hold with $f=g+h$. We show that (A2) holds with $f=g+h$.

Let $M,\epsilon>0$. By Lemma 2.2 there exist Γ_0, $\delta>0$ such that for each $t\in[a,b]$ and each $x_1,x_2,u\in X$ satisfying

$$\|u\|\geqslant\Gamma_0,\quad \|x_1\|,\|x_2\|\leqslant M,\quad \|x_1-x_2\|\leqslant\delta \tag{4.2}$$

the following inequality holds:

$$\big|g(t,x_1,u)-g(t,x_2,u)\big|\leqslant(\epsilon/8)\min\big\{g(t,x_1,u),g(t,x_2,u)\big\}. \tag{4.3}$$

In view of (4.1) there is

$$\Gamma>\Gamma_0 \tag{4.4}$$

such that for each $t\in[a,b]$ and each $x,u\in X$ satisfying

$$\|x\|\leqslant M,\qquad \|u\|\geqslant\Gamma \tag{4.5}$$

the following inequality holds:

$$h(t,x,u)\leqslant(\epsilon/8)g(t,x,u). \tag{4.6}$$

Assume now that $t\in[a,b]$ and $x_1,x_2,u\in X$ satisfy

$$\|x_1\|,\|x_2\|\leqslant M,\quad \|x_1-x_2\|\leqslant\delta,\quad \|u\|\geqslant\Gamma. \tag{4.7}$$

By (4.7), (4.4) and the choice of Γ_0, δ the inequality (4.3) holds. It follows from (4.7) and the choice of Γ (see (4.4)–(4.6)) that

$$h(t,x_i,u)\leqslant(\epsilon/8)g(t,x_i,u),\quad i=1,2. \tag{4.8}$$

(4.3) implies that

$$\begin{aligned}\big|(g+h)(t,x_2,u)-(g+h)(t,x_1,u)\big| &\leqslant \big|g(t,x_1,u)-g(t,x_2,u)\big|+\big|h(t,x_1,u)-h(t,x_2,u)\big|\\ &\leqslant(\epsilon/8)\min\big\{g(t,x_1,u),g(t,x_2,u)\big\}+\big|h(t,x_1,u)-h(t,x_2,u)\big|.\end{aligned} \tag{4.9}$$

We may assume without loss of generality that

$$h(t,x_2,u)\geqslant h(t,x_1,u). \tag{4.10}$$

In view of (4.10), (4.9) and (4.8)

$$\big|(g+h)(t,x_2,u)-(g+h)(t,x_1,u)\big|\leqslant(\epsilon/8)g(t,x_2,u)+h(t,x_2,u)\leqslant(\epsilon/8)g(t,x_2,u)+(\epsilon/8)g(t,x_2,u).$$

Hence (A2) holds with $f=g+h$. This completes the proof of Proposition 4.1. □

It is easy to see that if $g:[a,b]\times X\times X\to\mathbb{R}^1$ is a continuous function, (A1)–(A3) hold with $f=g$ and $\lambda>0$, then (A1)–(A3) hold with $f=\lambda g$.

Propsotion 4.2. *Let* $g_1,g_2:[a,b]\times X\times X\to\mathbb{R}^1$ *be a continuous function such that* (A1)–(A3) *hold with* $f=g_1,g_2$. *Then* (A1)–(A3) *hold with* $f=g_1+g_2$.

Proof. Clearly (A1) and (A3) hold with $f = g_1 + g_2$. Let us show that (A2) holds with $f = g_1 + g_2$.

Let $\epsilon, M > 0$. By Lemma 2.2 there are $\Gamma, \delta > 0$ such that for each $t \in [a, b]$, each $u \in X$ satisfying $\|u\| \geqslant \Gamma$ and each $x_1, x_2 \in X$ satisfying

$$\|x_1\|, \|x_2\| \leqslant M, \qquad \|x_1 - x_2\| \leqslant \delta \tag{4.11}$$

we have

$$\left|g_i(t, x_1, u) - g_i(t, x_2, u)\right| \leqslant \epsilon \min\left\{g_i(t, x_1, u), g_2(t, x_2, u)\right\}, \quad i = 1, 2. \tag{4.12}$$

Let $t \in [a, b]$, $u \in X$, $\|u\| \geqslant \Gamma$, $x_1, x_2 \in X$ satisfy (4.11). Then (4.12) holds and

$$\left|(g_1 + g_2)(t, x_1, u) - (g_1 + g_2)(t, x_2, u)\right| \leqslant \epsilon g_1(t, x_1, u) + \epsilon g_2(t, x_1, u).$$

Hence (A2) holds with $f = g_1 + g_2$. This completes the proof of Proposition 4.2. □

5. Auxiliary results for Theorem 1.2

For each $z_1, z_2 \in X$ set

$$U(z_1, z_2) = \inf\left\{I(x):\ x \in \mathcal{A}(z_1, z_2)\right\}.$$

It is easy to see that $U(z_1, z_2)$ is finite for each $z_1, z_2 \in X$.

Lemma 5.1. *Let $M > 0$. Then there is $M_1 > 0$ such that*

$$U(z_1, z_2) \leqslant M_1 \quad \textit{for each } z_1, z_2 \in X \textit{ satisfying } \|z_1\|, \|z_2\| \leqslant M.$$

Proof. Set

$$M_1 = \sup\left\{f(s, z, u):\ s \in [a, b],\ z, u \in X \text{ and } \|z\|, \|u\| \leqslant 2M\left(1 + (b - a)^{-1}\right)\right\}(b - a). \tag{5.1}$$

By Remark 1.3 M_1 is finite. Assume that $z_1, z_2 \in X$ and

$$\|z_1\|, \|z_2\| \leqslant M. \tag{5.2}$$

Define $x \in \mathcal{A}(z_1, z_2)$ by

$$x(t) = z_1 + (t - a)(b - a)^{-1}(z_2 - z_1), \quad t \in [a, b]. \tag{5.3}$$

Clearly

$$\|x(t)\| \leqslant M \quad \text{for all } t \in [a, b]. \tag{5.4}$$

It is easy to see that for all $t \in [a, b]$

$$\left\|x'(t)\right\| \leqslant (b - a)^{-1}\|z_2 - z_1\| \leqslant 2M(b - a)^{-1}. \tag{5.5}$$

(5.4), (5.5) and (5.1) imply that for all $t \in [a, b]$

$$f\left(t, x(t), x'(t)\right) \leqslant M_1/(b - a).$$

This inequality implies that

$$U(z_1, z_2) \leqslant I(x) \leqslant M_1.$$

Lemma 5.1 is proved. □

Lemma 5.2. *Let $M > 0$. Then there is $M_0 > 0$ such that for each $z_1, z_2 \in X$ satisfying $\|z_1\|, \|z_2\| \leqslant M$ and each $x \in \mathcal{A}(z_1, z_2)$ satisfying $I(x) \leqslant U(z_1, z_2) + 1$ the inequality $\|x(t)\| \leqslant M_0$ holds for all $t \in [a, b]$.*

Proof. By Lemma 5.1 there is $M_1 > 0$ such that

$$U(z_1, z_2) \leqslant M_1 \quad \text{for each } z_1, z_2 \in X \text{ satisfying } \|z_1\|, \|z_2\| \leqslant M. \tag{5.6}$$

(1.12) implies that there is $c_0 \geqslant 1$ such that

$$\phi(t) \geqslant t \quad \text{for all } t \geqslant c_0. \tag{5.7}$$

Set

$$M_0 = M + M_1 + 1 + c_0(b - a). \tag{5.8}$$

Assume that $z_1, z_2 \in X$, $x \in \mathcal{A}(z_1, z_2)$,

$$\|z_1\|, \|z_2\| \leqslant M, \qquad I(x) \leqslant U(z_1, z_2) + 1. \tag{5.9}$$

In view of (5.9) and the choice of M_1 (see (5.6))

$$U(z_1, z_2) \leqslant M_1 \quad \text{and} \quad I(x) \leqslant M_1 + 1. \tag{5.10}$$

Let $\tau \in (a, b]$ and set

$$E_1 = \bigl\{t \in [a, \tau]\colon \ \bigl\|x'(t)\bigr\| \geqslant c_0\bigr\}, \qquad E_2 = [a, \tau] \setminus E_1. \tag{5.11}$$

By (5.11) and (5.9)

$$\begin{aligned} \bigl\|x(\tau)\bigr\| &= \left\| x(a) + \int_a^\tau x'(t)\,\mathrm{d}t \right\| \leqslant \bigl\|x(a)\bigr\| + \int_a^\tau \bigl\|x'(t)\bigr\|\,\mathrm{d}t \leqslant \|z_1\| + \int_{E_1} \bigl\|x'(t)\bigr\|\,\mathrm{d}t + \int_{E_2} \bigl\|x'(t)\bigr\|\,\mathrm{d}t \\ &\leqslant M + c_0 \operatorname{mes}(E_2) + \int_{E_1} \bigl\|x'(t)\bigr\|\,\mathrm{d}t \leqslant (b - a)c_0 + M + \int_{E_1} \bigl\|x'(t)\bigr\|\,\mathrm{d}t. \end{aligned} \tag{5.12}$$

It follows from (5.11), (5.7), (1.11) and (5.10) that

$$\int_{E_1} \bigl\|x'(t)\bigr\|\,\mathrm{d}t \leqslant \int_{E_1} \phi\bigl(\bigl\|x'(t)\bigr\|\bigr)\,\mathrm{d}t \leqslant \int_a^b \phi\bigl(\bigl\|x'(t)\bigr\|\bigr)\,\mathrm{d}t \leqslant I(x) \leqslant M_1 + 1.$$

Combined with (5.12) and (5.8) this implies that

$$\bigl\|x(\tau)\bigr\| \leqslant (b - a)c_0 + M + M_1 + 1 = M_0.$$

This completes the proof of Lemma 5.2. □

6. Proof of Theorem 1.2

Let $M > 0$. By Lemma 5.1 there is $M_1 > 0$ such that

$$U(z_1, z_2) \leqslant M_1 \quad \text{for each } z_1, z_2 \in X \text{ satisfying } \|z_1\|, \|z_2\| \leqslant M. \tag{6.1}$$

In view of Lemma 5.2 there is $M_0 > 0$ such that for each $z_1, z_2 \in X$ and each $x \in \mathcal{A}(z_1, z_2)$ satisfying

$$\|z_1\|, \|z_2\| \leqslant M, \qquad I(x) \leqslant U(z_1, z_2) + 1 \tag{6.2}$$

the following inequality holds:

$$\|x(t)\| \leqslant M_0, \quad t \in [a,b]. \tag{6.3}$$

By (B2) there are $\delta_0, L_0 > 0$ and an integrable scalar function $\psi_0(t) \geqslant 0$, $t \in [a,b]$ such that for each $t \in [a,b]$, each $u \in X$ and each $x_1, x_2 \in X$ satisfying

$$\|x_1\|, \|x_2\| \leqslant M_0 + 4, \qquad \|x_1 - x_2\| \leqslant \delta_0 \tag{6.4}$$

the following inequality holds:

$$\left| f(t,x_1,u) - f(t,x_2,u)\right| \leqslant \|x_1 - x_2\| L_0\big(f(t,x_1,u) + \psi_0(t)\big). \tag{6.5}$$

Choose a positive number γ_0 such that

$$\gamma_0 < 1 \quad \text{and} \quad 8\gamma_0(M_1+1) < b-a. \tag{6.6}$$

(1.12) implies that there is $K_0 > 1$ such that

$$\phi(t)/t \geqslant \gamma_0^{-1} \quad \text{for all } t \geqslant K_0. \tag{6.7}$$

Set

$$\Delta_0 = \sup\big\{ f(t,z,0)\colon\ t \in [a,b],\ z \in X,\ \|z\| \leqslant M_0 + 2\big\}. \tag{6.8}$$

Remark 1.3 implies that Δ_0 is finite.

It follows from (B3) that there is $L_1 > 1$ such that for each $t \in [a,b]$ and each $x_1, x_2, u_1, u_2 \in X$ satisfying

$$\|x_1\|, \|x_2\|, \|u_1\|, \|u_2\| \leqslant K_0 + M_0 + 2 \tag{6.9}$$

the following inequality holds:

$$\left| f(t,x_1,u_1) - f(t,x_2,u_2)\right| \leqslant L_1\big(\|x_1 - x_2\| + \|u_1 - u_2\|\big). \tag{6.10}$$

Choose a number $\gamma_1 \in (0,1)$ such that

$$2\gamma_1(M_1+2) < \big(\min\{1, b-a\}\big)\min\{1, \delta_0/4\}, \tag{6.11}$$

$$\gamma_1 < \left(16L_1\left(b-a+1+\int_a^b \psi_0(t)\,\mathrm{d}t\right) + 64 + 16L_0\left(b-a+1+M_1+\int_a^b \psi_0(t)\,\mathrm{d}t\right)\right)^{-1}. \tag{6.12}$$

By (1.12) there is a number $K > 0$ such that

$$K > 8\Delta_0 + K_0 + 2, \tag{6.13}$$

$$\phi(t)/t \geqslant \gamma_1^{-1} \quad \text{for all } t \geqslant K. \tag{6.14}$$

Assume that

$$z_1, z_2 \in X, \quad \|z_1\|, \|z_2\| \leqslant M, \tag{6.15}$$

$$x \in \mathcal{A}(z_1, z_2), \tag{6.16}$$

$$\operatorname{mes}\big\{t \in [a,b]\colon\ \|x'(t)\| > K\big\} > 0. \tag{6.17}$$

We show that there is $u \in \mathcal{A}(z_1, z_2)$ such that $I(u) < I(x)$ and $\|u'(t)\| \leqslant K$ for almost every $t \in [a,b]$.

We may assume without loss of generality that

$$I(x) \leqslant U(z_1, z_2) + 1. \tag{6.18}$$

(6.18), (6.1) and (6.15) imply that

$$I(x) \leqslant M_1 + 1. \tag{6.19}$$

In view of (6.15), (6.16), (6.18) and the choice of M_0 (see (6.2), (6.3))

$$\|x(t)\| \leqslant M_0,\ t \in [a,b]. \tag{6.20}$$

Set

$$E_1 = \{t \in [a,b]\colon\ \|x'(t)\| \geqslant K\}, \quad E_2 = \{t \in [a,b]\colon\ \|x'(t)\| \leqslant K_0\}, \quad E_3 = [a,b] \setminus (E_1 \cup E_2). \tag{6.21}$$

Set

$$d = \int_{E_1} \|x'(t)\|\,\mathrm{d}t, \tag{6.22}$$

$$h_0 = \int_{E_1} x'(t)\,\mathrm{d}t. \tag{6.23}$$

It follows from (6.22), (6.21) and (6.17) that

$$d > 0. \tag{6.24}$$

Clearly

$$\|h_0\| \leqslant d. \tag{6.25}$$

By (6.22), (6.21), (6.14), (1.11) and (6.19)

$$d = \int_{E_1} \|x'(t)\|\,\mathrm{d}t \leqslant \int_{E_1} \gamma_1 \phi\big(\|x'(t)\|\big)\,\mathrm{d}t \leqslant \gamma_1 \int_a^b \phi\big(\|x'(t)\|\big)\,\mathrm{d}t$$
$$\leqslant \gamma_1 \int_a^b f\big(t, x(t), x'(t)\big)\,\mathrm{d}t \leqslant \gamma_1(M_1+1). \tag{6.26}$$

Now we estimate $\mathrm{mes}(E_2)$. It follows from (6.21), (6.7), (6.13), (1.11) and (6.19) that

$$\mathrm{mes}(E_1 \cup E_3) \leqslant K_0^{-1} \int_{E_1 \cup E_3} \|x'(t)\|\,\mathrm{d}t \leqslant K_0^{-1} \int_{E_1 \cup E_3} \gamma_0 \phi\big(\|x'(t)\|\big)\,\mathrm{d}t \leqslant \gamma_0 K_0^{-1} \int_a^b \phi\big(\|x'(t)\|\big)\,\mathrm{d}t$$
$$\leqslant \gamma_0 \int_a^b \phi\big(\|x'(t)\|\big)\,\mathrm{d}t \leqslant \gamma_0 \int_a^b f\big(t, x(t), x'(t)\big)\,\mathrm{d}t \leqslant \gamma_0(M_1+1). \tag{6.27}$$

Together with (6.21) this inequality implies that

$$\mathrm{mes}(E_2) \geqslant b - a - \gamma_0(M_1+1). \tag{6.28}$$

(6.28) and (6.6) imply that

$$\mathrm{mes}(E_2) \geqslant 3(b-a)/4. \tag{6.29}$$

Define a measurable function $\xi : [a,b] \to X$ by

$$\xi(t) = 0, \quad t \in E_1, \quad \xi(t) = x'(t), \quad t \in E_3, \quad \xi(t) = x'(t) + \big(\mathrm{mes}(E_2)\big)^{-1} h_0, \quad t \in E_2. \tag{6.30}$$

Clearly ξ is a Bochner integrable function. It follows from (6.30), (6.21), (6.23) and (6.16) that

$$\int_a^b \xi(t)\,dt = \sum_{i=1}^{3}\int_{E_i} \xi(t)\,dt = \int_{E_2} \xi(t)\,dt + \int_{E_3} \xi(t)\,dt$$
$$\int_{E_2} x'(t)\,dt + h_0 + \int_{E_3} x'(t)\,dt = \int_a^b x'(t)\,dt = z_2 - z_1. \tag{6.31}$$

Define a function $u:[a,b]\to X$ by

$$u(\tau) = z_1 + \int_a^\tau \xi(t)\,dt, \quad \tau\in[a,b]. \tag{6.32}$$

By (6.31)

$$u\in\mathcal{A}(z_1,z_2). \tag{6.33}$$

In view of (6.32), (6.30), (6.29) and (6.25) for almost every $t\in E_2$

$$\|x'(t)-u'(t)\| = \|x'(t)-\xi(t)\| = \big(\mathrm{mes}(E_2)\big)^{-1}\|h_0\| \leqslant 2(b-a)^{-1}d. \tag{6.34}$$

Combined with (6.26), (6.21) and (6.11) this relation implies that for almost every $t\in E_2$

$$\|u'(t)\| \leqslant \|x'(t)\| + 2(b-a)^{-1}d \leqslant \|x'(t)\| + 2(b-a)^{-1}\gamma_1(M_1+1) \leqslant K_0+1. \tag{6.35}$$

(6.35), (6.13), (6.21), (6.30) and (6.32) imply that

$$\|u'(t)\|\leqslant K \quad \text{for almost every } t\in[a,b]. \tag{6.36}$$

We show that $I(u) < I(x)$.

Let $s\in(a,b]$. It follows from (6.32), (6.16), (6.21), (6.30), (6.25) and (6.22) that

$$\begin{aligned}\|x(s)-u(s)\| &= \left\|\int_a^s \big[x'(t)-u'(t)\big]\,dt\right\| = \left\|\int_a^s \big[x'(t)-\xi(t)\big]\,dt\right\| \\ &\leqslant \left\|\int_{[a,s]\cap E_1} \big[x'(t)-\xi(t)\big]\,dt\right\| + \left\|\int_{[a,s]\cap E_2} \big[x'(t)-\xi(t)\big]\,dt\right\| + \left\|\int_{[a,s]\cap E_3} \big[x'(t)-\xi(t)\big]\,dt\right\| \\ &\leqslant \int_{E_1}\|x'(t)\|\,dt + \|h_0\| \leqslant 2d.\end{aligned}$$

Therefore

$$\|x(s)-u(s)\|\leqslant 2d \quad \text{for all } s\in[a,b]. \tag{6.37}$$

In view of (6.21)

$$I(u)-I(x) = \sum_{i=1}^{3}\int_{E_i}\big[f\big(t,u(t),u'(t)\big) - f\big(t,x(t),x'(t)\big)\,dt\big]. \tag{6.38}$$

By (6.32), (6.30), (6.37), (6.20), (6.26) and (6.11) for almost every $t\in E_1$

$$\begin{aligned}f\big(t,u(t),u'(t)\big) = f\big(t,u(t),0\big) &\leqslant \sup\big\{f(t,z,0):\ z\in X,\ \|z\|\leqslant M_0+2d\big\} \\ &\leqslant \sup\big\{f(t,z,0):\ z\in X,\ \|z\|\leqslant M_0+2\big\}.\end{aligned}$$

Combined with (6.8) this inequality implies that for almost every $t \in E_1$

$$f\big(t, u(t), u'(t)\big) \leqslant \Delta_0. \tag{6.39}$$

It follows from (1.11), (6.21), (6.14) and (6.13) that for almost every $t \in E_1$

$$f\big(t, x(t), x'(t)\big) \geqslant \phi\big(\|x'(t)\|\big) \geqslant \|x'(t)\| \geqslant K > 8\Delta_0.$$

Together with (6.39) this inequality implies that for almost every $t \in E_1$

$$f\big(t, x(t), x'(t)\big) - f\big(t, u(t), u'(t)\big) \geqslant 3f\big(t, x(t), x'(t)\big)/4. \tag{6.40}$$

(6.40) implies that

$$\int\limits_{E_1} \big[f\big(t, u(t), u'(t)\big) - f\big(t, x(t), x'(t)\big)\big]\,\mathrm{d}t \leqslant -\frac{3}{4}\int\limits_{E_1} f\big(t, x(t), x'(t)\big)\,\mathrm{d}t. \tag{6.41}$$

By (6.20), (6.37), (6.26) and (6.11) for all $t \in [a, b]$

$$\|u(t)\| \leqslant \|x(t)\| + 2d \leqslant M_0 + 2\gamma_1(M_1 + 1) \leqslant M_0 + 1. \tag{6.42}$$

It follows from (6.20), (6.42), (6.35), (6.21) and the choice L_1 (see (6.9), (6.10)) that for almost every $t \in E_2$

$$\big|f\big(t, x(t), x'(t)\big) - f\big(t, u(t), u'(t)\big)\big| \leqslant L_1\big(\|x(t) - u(t)\| + \|x'(t) - u'(t)\|\big).$$

Combined with (6.37) and (6.34) this inequality implies that for almost every $t \in E_2$

$$\big|f\big(t, x(t), x'(t)\big) - f\big(t, u(t), u'(t)\big)\big| \leqslant L_1\big(2d + 2(b - a)^{-1}d\big).$$

Therefore

$$\left|\int\limits_{E_2} \big[f\big(t, x(t), x'(t)\big) - f\big(t, u(t), u'(t)\big)\big]\,\mathrm{d}t\right| \leqslant 2dL_1(1 + b - a). \tag{6.43}$$

(6.37), (6.11) and (6.26) imply that for all $t \in [a, b]$

$$\|x(t) - u(t)\| \leqslant 2d \leqslant 2\gamma_1(M_1 + 1) < \delta_0/4. \tag{6.44}$$

By (6.44), (6.42), (6.20), (6.30), the choice of δ_0, L_0, ψ_0 (see (6.4), (6.5), (6.30)) and (6.32) for all $t \in E_3$

$$\begin{aligned}\big|f\big(t, x(t), x'(t)\big) - f\big(t, u(t), u'(t)\big)\big| &= \big|f\big(t, x(t), x'(t)\big) - f\big(t, u(t), x'(t)\big)\big|\\ &\leqslant \|x(t) - u(t)\| L_0\big(f\big(t, x(t), x'(t)\big) + \psi_0(t)\big).\end{aligned}$$

Together with (6.37) this inequality implies that for almost all $t \in E_3$

$$\big|f\big(t, x(t), x'(t)\big) - f\big(t, u(t), u'(t)\big)\big| \leqslant 2dL_0\big(f\big(t, x(t), x'(t)\big) + \psi_0(t)\big).$$

Therefore combining with (6.19) this implies that

$$\begin{aligned}\left|\int\limits_{E_3} \big[f\big(t, x(t), x'(t)\big) - f\big(t, u(t), u'(t)\big)\big]\,\mathrm{d}t\right| &\leqslant 2dL_0 \int\limits_{E_3} \big(f\big(t, x(t), x'(t)\big) + \psi_0(t)\big)\,\mathrm{d}t\\ &\leqslant 2dL_0\left(I(x) + \int_a^b \psi_0(t)\,\mathrm{d}t\right)\\ &\leqslant 2dL_0\left(M_1 + 1 + \int_a^b \psi_0(t)\,\mathrm{d}t\right).\end{aligned} \tag{6.45}$$

(6.38), (6.41), (6.43) and (6.45) imply that

$$I(u) - I(x) \leqslant -(3/4)\int_{E_1} f\big(t, x(t), x'(t)\big)\,\mathrm{d}t + 2dL_1(1+b-a) + 2dL_0\left(M_1 + 1 + \int_a^b \psi_0(t)\,\mathrm{d}t\right). \tag{6.46}$$

It follows from (1.11), (6.21) and (6.14) that for all $t \in E_1$

$$f\big(t, x(t), x'(t)\big) \geqslant \phi\big(\|x'(t)\|\big) \geqslant \gamma_1^{-1}\|x'(t)\|.$$

Combined with (6.22) and (6.12) this inequality implies that

$$\int_{E_1} f\big(t, x(t), x'(t)\big)\,\mathrm{d}t \geqslant \gamma_1^{-1}\int_{E_1} \|x'(t)\|\,\mathrm{d}t = \gamma_1^{-1}d$$

$$> d\left(16L_1(b-a+1) + 64 + 16L_0\left(b-a+1+M_1+\int_a^b \psi_0(t)\,\mathrm{d}t\right)\right).$$

Together with (6.46) this implies that $I(u) - I(x) < 0$. Theorem 1.2 is proved. □

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