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Analysis of miscible displacement through porous media with vanishing molecular diffusion and singular wells ☆

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Abstract

This article proves the existence of solutions to a model of incompressible miscible displacement through a porous medium, with zero molecular diffusion and modelling wells by spatial measures. We obtain the solution by passing to the limit on problems indexed by vanishing molecular diffusion coefficients. The proof employs cutoff functions to excise the supports of the measures and the discontinuities in the permeability tensor, thus enabling compensated compactness arguments used by Y. Amirat and A. Ziani for the analysis of the problem with L^2 wells (Amirat and Ziani, 2004 [1]). We give a novel treatment of the diffusion–dispersion term, which requires delicate use of the Aubin–Simon lemma to ensure the strong convergence of the pressure gradient, owing to the troublesome lower-order terms introduced by the localisation procedure.

Keywords: Flow in porous medium; Elliptic–parabolic system; Degenerate equations; Existence; Measure data; Vanishing diffusion

1. Introduction

1.1. The miscible displacement problem

We study the single-phase, miscible displacement of one incompressible fluid by another through a porous medium, as occurs in enhanced oil recovery processes. Neglecting gravity, the model reads [10,18]

$$\left.\begin{aligned} \mathbf{u}(x,t) &= -\frac{\mathbf{K}(x)}{\mu(c(x,t))}\nabla p(x,t) \\ \operatorname{div}\mathbf{u}(x,t) &= (q^I - q^P)(x,t) \end{aligned}\right\}, \quad (x,t) \in \Omega \times (0,T), \tag{1.1a}$$

$$\Phi(x)\partial_t c(x,t) - \operatorname{div}\big(\mathbf{D}(x,\mathbf{u}(x,t))\nabla c - c\mathbf{u}\big)(x,t) + (q^P c)(x,t) = (q^I \hat{c})(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1.1b}$$

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subject to the no-flow boundary conditions

$$\mathbf{u}(x,t)\cdot\mathbf{n}=0, \quad (x,t)\in\partial\Omega\times(0,T), \text{ and} \tag{1.1c}$$

$$\mathbf{D}(x,\mathbf{u}(x,t))\nabla c(x,t)\cdot\mathbf{n}=0, \quad (x,t)\in\partial\Omega\times(0,T), \tag{1.1d}$$

the initial condition

$$c(x,0)=c_0(x), \quad x\in\Omega, \tag{1.1e}$$

and a normalisation condition to eliminate arbitrary constants in the solution p of the elliptic equation (1.1a):

$$\int_\Omega p(x,t)\,\mathrm{d}x=0 \quad \text{for all } t\in(0,T). \tag{1.1f}$$

The unknowns of the system are the pressure p and Darcy velocity $\mathbf{u}$ of the fluid mixture, and the concentration c of one of the components in the fluid mixture. The reservoir is represented by Ω, a bounded connected open subset of $\mathbb{R}^d$, $d=2$ or 3, and the recovery process occurs over the time interval $(0,T)$. The reservoir-dependent quantities of porosity and absolute permeability are Φ and $\mathbf{K}$, respectively. We denote by q^I and q^P the sums of injection well source terms and production well sink terms (henceforth collectively referred to as source terms), respectively, and write $\hat{c}$ for the concentration of the injected fluid.

The coefficient $\mathbf{D}$ in (1.1b) is the diffusion–dispersion tensor, derived by Peaceman [17] as

$$\mathbf{D}(x,\mathbf{u})=\Phi(x)\bigg(d_m\mathbf{I}+|\mathbf{u}|\Big(d_l E(\mathbf{u})+d_t(\mathbf{I}-E(\mathbf{u}))\Big)\bigg), \tag{1.1g}$$

where

$$E(\mathbf{u})=\left(\frac{\mathbf{u}_i\mathbf{u}_j}{|\mathbf{u}|^2}\right)_{1\le i,j\le d} \tag{1.1h}$$

is the projection in the direction of flow. The constants d_m, d_l and d_t are the molecular diffusion coefficient and the longitudinal and transverse mechanical dispersion coefficients, respectively. After Koval [16] (see also [5,20]), the concentration-dependent viscosity μ of the fluid mixture often assumes the form

$$\mu(c)=\mu(0)\left(1+\left(M^{1/4}-1\right)c\right)^{-4} \quad \text{for } c\in[0,1], \tag{1.1i}$$

where the mobility ratio $M:=\frac{\mu(0)}{\mu(1)}>1$. Finally, the boundary condition (1.1c) enforces a compatibility condition upon the source terms:

$$\int_\Omega q^I(x,t)\,\mathrm{d}x=\int_\Omega q^P(x,t)\,\mathrm{d}x \quad \text{for all } t\in(0,T). \tag{1.1j}$$

1.2. Principal contributions

Our main result, Theorem 2.2, is the existence of weak solutions to (1.1) when $d_m=0$ and q^I and q^P are modelled spatially as bounded, nonnegative Radon measures on Ω. Indeed, the novelty of this article is the presence of both these features simultaneously; Amirat and Ziani [1] analyse the system as $d_m\to 0$ with $q^I, q^P\in L^\infty(0,T;L^2(\Omega))$, and our previous work [9] establishes existence for $d_m>0$ and measure source terms. Fabrie and Gallouët [11] assume that the diffusion–dispersion tensor is uniformly bounded to address the latter scenario. The first existence result for (1.1) as written above is due to Feng [12], focussing mostly on the two-dimensional problem with sources in $L^\infty(0,T;L^2(\Omega))$. The subsequent analysis of Chen and Ewing [5] is valid for very general boundary conditions in three dimensions, but assumes $d_m>0$ and regular source terms. Uniqueness is known for "strong" solutions [12], but appears to be open for weak solutions even with $d_m>0$ fixed [1,5,12].

We prove Theorem 2.2 by passing to the limit as $d_m\to 0$ on a sequence of problems with measure source terms defined in Section 3. In further contrast to Amirat–Ziani who take $\Phi\equiv 1$ and $\mathbf{K}$ continuous, we only assume that the porosity is bounded, and we allow for discontinuous permeabilities of the kind that one expects in practice [6].

Working in such a low-regularity environment leads to the challenge of identifying the limits of the nonlinear terms $-\frac{\mathbf{K}}{\mu(c)}\nabla p$ and $\mathbf{D}(\cdot,\mathbf{u})\nabla c$ as $d_m \to 0$. For this task we use smooth cutoff functions — first appearing in Section 4.1 — to excise both the supports of the measures and the discontinuities in $\mathbf{K}$, thereby localising the problem to where the data is smooth enough for us to employ a compensated compactness-type lemma (Lemma B.1).

This localisation procedure nonetheless introduces problems of its own in the form of lower-order terms that inhibit a straightforward proof of strong convergence of the pressure gradients, as is the case for L^2 sources. We handle these lower-order terms by exploiting the uniqueness of the solution to the elliptic problem in combination with careful use of the Aubin–Simon compactness lemma to first prove strong convergence of the pressure itself in Section 4.4.

Strong convergence of the pressure gradients (and then the Darcy velocities) is crucial for our treatment of the diffusion–dispersion term $\mathbf{D}(\cdot,\mathbf{u})\nabla c$ in Section 4.5, which we believe is also novel. In particular, we fill a gap in the work of Amirat–Ziani by giving meaning to ∇c in the limit as $d_m \to 0$. When the molecular diffusion is neglected, the concentration gradient is only well-defined as a function in non-stagnant zones of the reservoir; that is, where $\mathbf{u} \neq \mathbf{0}$. We introduce a new notion in Section 2.2 that resolves this difficulty.

1.3. Why vanishing molecular diffusion and singular wells?

The interest in studying (1.1) with $d_m = 0$ is twofold. In practice, the mechanical dispersion coefficients will be at least an order of magnitude larger than d_m, so the effects of molecular diffusion are negligible compared to those of mechanical dispersion [2,19,24]. Moreover, in practical simulations of (1.1) the mesh size is such that the effects of molecular diffusion are dominated by numerical diffusion, so d_m is often neglected from the simulation [20,21].

Scale differences motivate the decision to model q^I and q^P as measures. The diameter of typical reservoir ($\sim 10^3$ m) is several orders of magnitude larger than that of a typical wellbore ($\sim 10^{-1}$ m). At field scale the wells are thus effectively point (resp. line) sources in two (resp. three) dimensional models.

2. Assumptions and main result

2.1. Assumptions on the data

We make the following assumptions on the data:

$$T \in \mathbb{R}_+^* \text{ and } \Omega \text{ is a bounded, connected, open subset of } \mathbb{R}^d,\ d \le 3, \\ \text{with a Lipschitz continuous boundary.} \tag{2.1a}$$

Writing $\mathfrak{D}_\mathbf{K}$ for the closure of the set of discontinuities of $\mathbf{K}$, we assume that $\mathfrak{D}_\mathbf{K}$ has zero Lebesgue measure (in practice, $\mathfrak{D}_\mathbf{K}$ is contained in a finite union of hypersurfaces). Write $\mathcal{S}_d(\mathbb{R})$ for the set of $d \times d$ symmetric matrices. The permeability satisfies

$$\mathbf{K} : \Omega \to \mathcal{S}_d(\mathbb{R}) \text{ is locally Lipschitz continuous on } \Omega \setminus \mathfrak{D}_\mathbf{K}, \text{ and } \exists k_* > 0 \text{ such that,} \\ \text{for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^d,\ k_*|\xi|^2 \le \mathbf{K}(x)\xi\cdot\xi \le k_*^{-1}|\xi|^2. \tag{2.1b}$$

The porosity Φ is such that

$$\Phi \in L^\infty(\Omega) \text{ and there exists } \phi_* > 0 \text{ such that for a.e. } x \in \Omega,\ \phi_* \le \Phi(x) \le \phi_*^{-1}. \tag{2.1c}$$

Particularly important to our analysis are the assumptions on the viscosity:

$$\mu \in C^2([0,1];(0,\infty)) \text{ is such that } \mu'' > 0 \text{ and } (1/\mu)'' > 0. \\ \text{We write } \mu_* \text{ and } \mu^* \text{ for the minimum and maximum of } \mu, \text{ respectively.} \tag{2.1d}$$

This implies the strict convexity of μ and $1/\mu$. Note that the form (1.1i) satisfies (2.1d). By setting $d_m = 0$ in (1.1g), we introduce the mechanical dispersion tensor

$$\mathbf{D}_\circ(x,\mathbf{u}) = \Phi(x)|\mathbf{u}|\Big(d_l E(\mathbf{u}) + d_t(\mathbf{I} - E(\mathbf{u}))\Big), \tag{2.1e}$$

and note that it satisfies

$$\begin{aligned}&\mathbf{D}_\circ : \Omega \times \mathbb{R}^d \to \mathcal{S}_d(\mathbb{R}) \text{ is a Carathéodory function such that for a.e. } x \in \Omega \text{ and for all } \zeta, \xi \in \mathbb{R}^d,\\ &\quad \mathbf{D}_\circ(x,\zeta)\xi\cdot\xi \geq \phi_* \min(d_l, d_t)|\zeta||\xi|^2 \text{ and } |\mathbf{D}_\circ(x,\zeta)\xi| \leq \phi_*^{-1}\max(d_l,d_t)|\zeta||\xi|.\end{aligned} \tag{2.1f}$$

The injected and initial concentration are such that

$$\hat{c} \in L^\infty(0,T;C(\overline{\Omega})) \text{ satisfies } 0 \leq \hat{c}(x,t) \leq 1 \text{ for a.e. } (x,t) \in \Omega \times (0,T), \tag{2.1g}$$

$$c_0 \in L^\infty(\Omega) \text{ satisfies } 0 \leq c_0(x) \leq 1 \text{ for a.e. } x \in \Omega. \tag{2.1h}$$

The source terms are such that

$$\begin{aligned}&q^I = a\nu \text{ and } q^P = b\nu, \text{ where}\\ &\quad a, b \in L^\infty(0,T;C(\overline{\Omega})) \text{ are nonnegative on } \Omega \times (0,T),\\ &\quad \nu \in \mathcal{M}_+(\Omega) \cap (W^{1,\ell}(\Omega))' \text{ for all } \ell > 2,\\ &\quad \text{and supp}\,(\nu) \text{ has zero Lebesgue measure.}\end{aligned} \tag{2.1i}$$

Here $\mathcal{M}_+(\Omega)$ is the set of bounded nonnegative Radon measures on Ω. The compatibility condition imposed by (1.1c) becomes

$$\int_\Omega a(x,t)\,\mathrm{d}\nu(x) = \int_\Omega b(x,t)\,\mathrm{d}\nu(x) \quad \forall t \in (0,T). \tag{2.1j}$$

Remark 2.1. We impose the condition $\nu \in (W^{1,\ell}(\Omega))'$ for all $\ell > 2$ in order to employ a sharp uniqueness result for the elliptic equation with measure data. This uniqueness result — which compensates for the absence of estimates on $\partial_t p$ — is instrumental to establishing the strong convergence of the pressure. This $(W^{1,\ell}(\Omega))'$ regularity is satisfied by all measures in two dimensions, and by all measures that can reasonably be used to model wells in three dimensions; see [11].

For a topological vector space $X(\Omega)$ of functions on Ω, we write $(X(\Omega))'$ for its topological dual. When writing the duality pairing $\langle\cdot,\cdot\rangle_{(X(\Omega))',X(\Omega)}$, we omit the spaces if they are clear from the context. When $z \in (1,\infty)$ is a Lebesgue exponent, we write $z' = \frac{z}{z-1}$ for its conjugate. We denote by $W^{1,z}_\star(\Omega)$ those elements of $W^{1,z}(\Omega)$ whose integral over Ω vanishes. For $k \in \mathbb{R}$ and $g : \Omega \to \mathbb{R}$, we denote by $\{g = k\}$ the level set $\{x \in \Omega \,|\, g(x) = k\}$; similarly for sublevel sets $\{g \leq k\}$, $\{g < k\}$ and superlevel sets $\{g \geq k\}$, $\{g > k\}$. When a constant appears in an estimate we track only its relevant dependencies. In particular, we do not indicate dependencies with respect to ϕ_*, d_l, d_t, T, Ω, k_*, μ_*, μ^* or $\hat{c}$, as these quantities remain constant throughout the paper. When stating that a certain constant depends only on some quantity X, it is implicitly understood that this dependency is nondecreasing.

Before detailing our results, we must first introduce a new concept that is key to our notion of solution when $d_m = 0$.

2.2. *The concentration gradient in the absence of molecular diffusion*

Consider $d_m = \varepsilon > 0$. Write $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$ for the corresponding solution to (1.1) (the existence of which we discuss shortly), and $\mathbf{D}_\varepsilon(\cdot, \mathbf{u}_\varepsilon)$ the corresponding diffusion–dispersion tensor. A straightforward computation using the definition (1.1g) shows that

$$\int_0^T\int_\Omega \mathbf{D}_\varepsilon(\cdot,\mathbf{u}_\varepsilon)\nabla c_\varepsilon\cdot\nabla c_\varepsilon \geq \varepsilon\int_0^T\int_\Omega |\nabla c_\varepsilon|^2 + \min(d_l,d_t)\int_0^T\int_\Omega |\mathbf{u}_\varepsilon||\nabla c_\varepsilon|^2.$$

Thus, in order to obtain estimates on ∇c_ε as $\varepsilon \to 0$, it seems necessary to first restrict attention to regions where $|\mathbf{u}_\varepsilon| > \eta > 0$. This leads to the following definition, which we use in the treatment of the diffusion–dispersion term to give meaning to ∇c in the limit as $d_m \to 0$.

Definition 2.1. Let $f, v \in L^2(0,T;L^2(\Omega))$, with $v \geq 0$. We say that f has a $\{v>0\}$-*gradient* if

- there are sequences $(f_\varepsilon)_{\varepsilon>0}$ in $L^2(0,T;H^1(\Omega))$ and $(v_\varepsilon)_{\varepsilon>0}$ in $L^1(\Omega\times(0,T))$ such that as $\varepsilon\to 0$,

$$f_\varepsilon \rightharpoonup f \quad \text{weakly in } L^2(0,T;L^2(\Omega)),$$
$$v_\varepsilon \to v \quad \text{a.e. on } \Omega\times(0,T);$$

- there is a sequence $(\eta_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ with $\eta_i \to 0^+$ as $i\to\infty$ such that for every $i\in\mathbb{N}$, $\mathrm{meas}(\{v=\eta_i\})=0$, and for some function $\chi_{\eta_i}\in L^2(0,T;L^2(\Omega)^d)$,

$$\mathbf{1}_{\{v_\varepsilon>\eta_i\}}\nabla f_\varepsilon \rightharpoonup \chi_{\eta_i} \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d) \text{ as } \varepsilon\to 0.$$

We then denote $\nabla_{\{v>\eta_i\}}f := \chi_{\eta_i}$ the $\{v>\eta_i\}$-*gradient of* f and define the $\{v>0\}$-*gradient of* f as the function $\nabla_{\{v>0\}}f$ satisfying

$$\nabla_{\{v>0\}}f = \begin{cases} \nabla_{\{v>\eta_i\}}f & \text{on } \{v>\eta_i\} \quad \forall i\in\mathbb{N}, \\ 0 & \text{on } \{v=0\}. \end{cases}$$

Appendix A establishes some important properties that this construction satisfies.

Remark 2.2. If f is a regular function then $\nabla_{\{v>0\}}f = \nabla f$ on $\{v>0\}$.

2.3. Main result

The principal contribution of this article is the following existence result.

Theorem 2.2. *Under Hypotheses (2.1), there exists a weak solution* $(p,\mathbf{u},c)$ *to (1.1) with* $d_m=0$ *in the following sense:*

$$\begin{aligned} &c\in L^\infty(\Omega\times(0,T)),\ 0\leq c(x,t)\leq 1 \quad \textit{for a.e. } (x,t)\in\Omega\times(0,T),\\ &\quad c\in L^\infty(0,T;L^1(\Omega,\nu)),\ 0\leq c(x,t)\leq 1 \quad \textit{for } \nu\textit{-a.e. } x\in\Omega, \textit{ for a.e. } t\in(0,T), \end{aligned} \tag{2.2a}$$

$$\Phi\partial_t c\in L^2(0,T;(W^{1,s}(\Omega))') \quad \forall s>2d, \tag{2.2b}$$

$$\Phi c\in C([0,T];(W^{1,s}(\Omega))'), \quad \Phi c(\cdot,0)=\Phi c_0 \textit{ in } (W^{1,s}(\Omega))' \quad \forall s>2d, \tag{2.2c}$$

$$\begin{aligned} &c \textit{ has a } \{|\mathbf{u}|>0\}\textit{-gradient, and}\\ &\quad \mathbf{D}_\circ(\cdot,\mathbf{u})\nabla_{\{|\mathbf{u}|>0\}}c\in L^2(0,T;L^r(\Omega)^d) \quad \forall r<\frac{2d}{2d-1}, \end{aligned} \tag{2.2d}$$

$$p\in L^\infty(0,T;W^{1,q}_\star(\Omega)), \quad \mathbf{u}\in L^\infty(0,T;L^q(\Omega)^d) \quad \forall q<\frac{d}{d-1}, \tag{2.2e}$$

$$\begin{aligned} &\int_0^T \langle \Phi\partial_t c(\cdot,t),\varphi(\cdot,t)\rangle\,\mathrm{d}t + \int_0^T\int_\Omega \mathbf{D}_\circ(x,\mathbf{u}(x,t))\nabla_{\{|\mathbf{u}|>0\}}c(x,t)\cdot\nabla\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad -\int_0^T\int_\Omega c(x,t)\mathbf{u}(x,t)\cdot\nabla\varphi(x,t)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega c(x,t)\varphi(x,t)b(x,t)\,\mathrm{d}\nu(x)\,\mathrm{d}t\\ &= \int_0^T\int_\Omega \hat{c}(x,t)\varphi(x,t)a(x,t)\,\mathrm{d}\nu(x)\,\mathrm{d}t \quad \forall\varphi\in\bigcup_{s>2d}L^2(0,T;W^{1,s}(\Omega)), \end{aligned} \tag{2.2f}$$

$$\mathbf{u}(x,t) = -\frac{\mathbf{K}(x)}{\mu(c(x,t))}\nabla p(x,t),$$
$$-\int_0^T\int_\Omega \mathbf{u}(x,t)\cdot\nabla\psi(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega (a-b)\,(x,t)\psi(x,t)\,\mathrm{d}\nu(x)\,\mathrm{d}t \quad \forall\psi\in\bigcup_{q>d}L^1(0,T;W^{1,q}(\Omega)). \tag{2.2g}$$

To reiterate, the duality product in the first term of (2.2f) is between $W^{1,s}(\Omega)$ and its dual.

Remark 2.3. Following Remark 2.2, if c is regular then $\nabla_{\{v>0\}}c$ can be replaced with ∇c in (2.2f).

3. Approximate problems and associated estimates

We obtain the solution $(p,\mathbf{u},c)$ to (2.2) by passing to the limit on approximate problems defined below. Let $\varepsilon>0$. Replace the molecular diffusion coefficient d_m in (1.1g) with ε to obtain a family of diffusion–dispersion tensors:

$$\mathbf{D}_\varepsilon(x,\mathbf{u}) := \Phi(x)\Big(\varepsilon\mathbf{I} + |\mathbf{u}|\Big(d_l E(\mathbf{u}) + d_t(\mathbf{I}-E(\mathbf{u}))\Big)\Big). \tag{3.1}$$

Then for almost every $x\in\Omega$, for all $\xi,\zeta\in\mathbb{R}^d$,

$$\mathbf{D}_\varepsilon(x,\zeta)\xi\cdot\xi \ge \phi_*(\varepsilon+\min(d_l,d_t)|\zeta|)|\xi|^2, \tag{3.2}$$
$$|\mathbf{D}_\varepsilon(x,\zeta)| \le \phi_*^{-1}(\varepsilon+\max(d_l,d_t)|\zeta|). \tag{3.3}$$

Moreover, writing $\mathbf{D}_\varepsilon^{1/2}$ for the square-root of $\mathbf{D}_\varepsilon$ (which is well-defined since $\mathbf{D}_\varepsilon$ is positive-definite), one can show that

$$|\mathbf{D}_\varepsilon^{1/2}(\cdot,\zeta)| \le \phi_*^{-1/2}(\varepsilon+\max(d_l,d_t)|\zeta|)^{1/2}. \tag{3.4}$$

In order to define our approximate problems, we need access to the solution when the source terms are regular and the molecular diffusion is fixed. To this end, replace $\mathbf{D}$ by $\mathbf{D}_\varepsilon$ in (1.1) and fix both ε and $\nu_n\in L^2(\Omega)$ (where $n\in\mathbb{N}$ will vary in subsequent notions of solution). Then Feng [12] and Chen and Ewing [5] show that there exists a weak solution $(p_\varepsilon^n,\mathbf{u}_\varepsilon^n,c_\varepsilon^n)$ to (1.1) satisfying

$$\begin{aligned}
&c_\varepsilon^n\in L^2(0,T;H^1(\Omega)),\quad 0\le c_\varepsilon^n(x,t)\le 1 \quad \text{for a.e. } (x,t)\in\Omega\times(0,T),\\
&\Phi\partial_t c_\varepsilon^n\in L^2(0,T;(W^{1,4}(\Omega))'),\\
&\Phi c_\varepsilon^n\in C([0,T];(W^{1,4}(\Omega))'),\quad \Phi c_\varepsilon^n(\cdot,0)=\Phi c_0 \text{ in } (W^{1,4}(\Omega))',\\
&\mathbf{D}_\varepsilon(\cdot,\mathbf{u}_\varepsilon^n)\nabla c_\varepsilon^n\in L^2(0,T;L^{4/3}(\Omega)^d),\\
&p_\varepsilon^n\in L^\infty(0,T;H^1_\star(\Omega)),\quad \mathbf{u}_\varepsilon^n\in L^\infty(0,T;L^2(\Omega)^d),\\
&(p_\varepsilon^n,\mathbf{u}_\varepsilon^n,c_\varepsilon^n) \text{ satisfies (2.2f) for all } \varphi\in L^2(0,T;W^{1,4}(\Omega))\\
&\quad \text{with } \mathbf{D}_\circ \text{ and } \nabla_{\{|\mathbf{u}|>0\}}c \text{ replaced by } \mathbf{D}_\varepsilon \text{ and } \nabla c_\varepsilon^n, \text{ respectively},\\
&(p_\varepsilon^n,\mathbf{u}_\varepsilon^n,c_\varepsilon^n) \text{ satisfies (2.2g) for all } \psi\in L^1(0,T;H^1(\Omega)).
\end{aligned} \tag{3.5}$$

Keeping $\mathbf{D}_\varepsilon$ (with ε fixed), consider now $\nu\in\mathcal{M}_+(\Omega)$. Our previous work [9] shows that for every $\varepsilon>0$, there exists a solution $(p_\varepsilon,\mathbf{u}_\varepsilon,c_\varepsilon)$ to (1.1) in the following sense:

$$\begin{aligned}
&c_\varepsilon\in L^2(0,T;H^1(\Omega)),\quad 0\le c_\varepsilon(x,t)\le 1 \quad \text{for a.e. } (x,t)\in\Omega\times(0,T),\\
&c_\varepsilon\in L^\infty(0,T;L^1(\Omega,\nu)),\quad 0\le c_\varepsilon(x,t)\le 1 \quad \text{for } \nu\text{-a.e. } x\in\Omega, \text{ for a.e. } t\in(0,T),
\end{aligned} \tag{3.6a}$$
$$\Phi\partial_t c_\varepsilon\in L^2(0,T;(W^{1,s}(\Omega))')\quad \forall s>2d, \tag{3.6b}$$
$$\Phi c_\varepsilon\in C([0,T];(W^{1,s}(\Omega))'),\quad \Phi c_\varepsilon(\cdot,0)=\Phi c_0 \text{ in } (W^{1,s}(\Omega))'\quad \forall s>2d, \tag{3.6c}$$
$$\mathbf{D}_\varepsilon(\cdot,\mathbf{u}_\varepsilon)\nabla c_\varepsilon\in L^2(0,T;L^r(\Omega)^d)\quad \forall r<\frac{2d}{2d-1}, \tag{3.6d}$$

$$p_\varepsilon \in L^\infty(0,T;W^{1,q}_\star(\Omega)),\quad \mathbf{u}_\varepsilon \in L^\infty(0,T;L^q(\Omega)^d)\quad \forall q<\frac{d}{d-1}, \tag{3.6e}$$

$(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$ satisfies (2.2f) with $\mathbf{D}_\circ$ and $\nabla_{\{|\mathbf{u}|>0\}}c$ replaced by $\mathbf{D}_\varepsilon$ and ∇c_ε, respectively, (3.6f)

$(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$ satisfies (2.2g). (3.6g)

Remark 3.1. Standard arguments show that the integral relation in (3.6g) is equivalent to

$$-\int_\Omega \mathbf{u}_\varepsilon(x,t)\cdot\nabla\psi(x)\,\mathrm{d}x = \int_\Omega (a-b)(x,t)\psi(x)\,\mathrm{d}\nu(x),\quad \text{for a.e. } t\in(0,T),\ \forall\psi\in\bigcup_{q>d}W^{1,q}(\Omega). \tag{3.7}$$

We are now ready to define precisely the approximate problems that we work with in the subsequent analysis. The following two definitions provide the details.

Definition 3.1 *(Solution-by-truncation to (3.5)).* Assume (2.1). Fix $\nu_n \in L^2(\Omega)$, $\varepsilon>0$ and take $k\in\mathbb{N}$. Define the truncated tensor, for $(x,\zeta)\in\Omega\times\mathbb{R}^d$, by

$$\mathbf{D}^k_\varepsilon(x,\zeta) = \mathbf{D}_\varepsilon\left(x, \min(|\zeta|,k)\frac{\zeta}{|\zeta|}\right). \tag{3.8}$$

Then a *solution-by-truncation to* (3.5) is a triple $(p^n_\varepsilon, \mathbf{u}^n_\varepsilon, c^n_\varepsilon)$ that satisfies (3.5) and such that, for some solution $(p^{n,k}_\varepsilon, \mathbf{u}^{n,k}_\varepsilon, c^{n,k}_\varepsilon)$ to (3.5) with $\mathbf{D}_\varepsilon$ replaced by $\mathbf{D}^k_\varepsilon$, along a subsequence as $k\to\infty$,

$$\begin{aligned} &p^{n,k}_\varepsilon \to p^n_\varepsilon \text{ strongly in } L^2(0,T;H^1(\Omega)),\\ &\mathbf{u}^{n,k}_\varepsilon \to \mathbf{u}^n_\varepsilon \text{ strongly in } L^2(0,T;L^2(\Omega)^d), \text{ and}\\ &c^{n,k}_\varepsilon \to c^n_\varepsilon \text{ a.e. on } \Omega\times(0,T) \text{ and weakly in } L^2(0,T;H^1(\Omega)). \end{aligned} \tag{3.9}$$

Remark 3.2. Our previous work [9, Section 3.3] establishes the existence of a solution-by-truncation to (3.5). The interest in considering $\nu_n\in L^2(\Omega)$ and a truncated (and therefore bounded) diffusion–dispersion tensor is twofold. It enables us to consider test functions $\varphi\in L^2(0,T;H^1(\Omega))$ for the concentration equation, so that $\varphi = c^{n,k}_\varepsilon$ is an admissible test function. The concentration equation then shows that $\Phi\partial_t c^{n,k}_\varepsilon \in L^2(0,T;(H^1(\Omega))')$.

Definition 3.2 *(Solution-by-approximation to (3.6)).* Assume (2.1). A *solution-by-approximation to* (3.6) is a triple $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$ satisfying (3.6) and such that there exists $(\nu_n)_{n\in\mathbb{N}}\subset L^2(\Omega)$, $(a_n)_{n\in\mathbb{N}}\subset L^\infty(0,T;C(\overline{\Omega}))$ and $(p^n_\varepsilon, \mathbf{u}^n_\varepsilon, c^n_\varepsilon)_{n\in\mathbb{N}}$, with

- $\nu_n\geq 0$, $\nu_n\to\nu$ in $(C(\overline{\Omega}))'\cap(W^{1,\ell}(\Omega))'$ weak-$*$ as $n\to\infty$ (for all $\ell>2$), and for all $\eta>0$ there exists $N\in\mathbb{N}$ such that if $n\geq N$, $\mathrm{supp}(\nu_n)\subset\mathrm{supp}(\nu)+B(0,\eta)$,
- $a_n\geq 0$, $(a_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(0,T;C(\overline{\Omega}))$ and $a_n\to a$ a.e. on $\Omega\times(0,T)$ as $n\to\infty$,
- (ν_n, a_n, b) satisfy the compatibility condition (2.1j),
- $(p^n_\varepsilon, \mathbf{u}^n_\varepsilon, c^n_\varepsilon)$ is a solution-by-truncation to (3.5) with (ν, a) replaced by (ν_n, a_n),

and, along a sequence as $n\to\infty$,

$$\begin{aligned} &p^n_\varepsilon \to p_\varepsilon \text{ strongly in } L^2(0,T;W^{1,q}(\Omega)) \text{ for all } q<\tfrac{d}{d-1},\\ &\mathbf{u}^n_\varepsilon \to \mathbf{u}_\varepsilon \text{ strongly in } L^2(0,T;L^q(\Omega)^d) \text{ for all } q<\tfrac{d}{d-1}, \text{ and}\\ &c^n_\varepsilon \to c_\varepsilon \text{ a.e. on } \Omega\times(0,T) \text{ and weakly in } L^2(0,T;H^1(\Omega)). \end{aligned} \tag{3.10}$$

Table 1
Notions of solution to (1.1).

Data		Solution
$\mathbf{D}_\varepsilon^k$, $\varepsilon = d_m > 0$, $\nu_n \in L^2(\Omega)$		$(p_\varepsilon^{n,k}, \mathbf{u}_\varepsilon^{n,k}, c_\varepsilon^{n,k})$
	$\downarrow k \to \infty$	
$\varepsilon = d_m > 0$, $\nu_n \in L^2(\Omega)$		solution-by-truncation $(p_\varepsilon^n, \mathbf{u}_\varepsilon^n, c_\varepsilon^n)$
	$\downarrow n \to \infty$	
$\varepsilon = d_m > 0$, $\nu \in \mathcal{M}_+(\Omega)$		solution-by-approximation $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$
	$\downarrow \varepsilon \to 0$	
$d_m = 0$, $\nu \in \mathcal{M}_+(\Omega)$		solution $(p, \mathbf{u}, c)$ to (2.2)

Remark 3.3. The existence of a solution-by-approximation to (3.6) is known [9, Section 4.3]. Fabrie–Gallouët [11, Section 5] establish the existence of an approximation (ν_n, a_n) of (ν, a) that satisfies the requirements of Definition 3.2.

Table 1 helps to visualise the relationship between these notions of solution to (1.1). Access to the solution $(p_\varepsilon^{n,k}, \mathbf{u}_\varepsilon^{n,k}, c_\varepsilon^{n,k})$ of the truncated problem is only required for Lemma 4.2 and the first step of Lemma 4.3. The rest of the analysis is largely conducted on the solution-by-approximation $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$.

We now recall the estimates necessary for our subsequent analysis. Taking $c_\varepsilon^{n,k}$ as a test function in its own equation [9, Eq. (3.8)] (see Remark 3.2) gives a bound on $\mathbf{D}_\varepsilon^k(\cdot, \mathbf{u}_\varepsilon^{n,k})\nabla c_\varepsilon^{n,k} \cdot \nabla c_\varepsilon^{n,k}$ in $L^1(\Omega \times (0,T))$ that is independent of k, n and ε. Passing to the limit as $k \to \infty$ and then as $n \to \infty$ (in that order) gives

$$\left\| \mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon \right\|_{L^2(0,T;L^2(\Omega)^d)} = \left\| \mathbf{D}_\varepsilon(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon \cdot \nabla c_\varepsilon \right\|_{L^1(\Omega\times(0,T))} \le C_1, \tag{3.11}$$

where C_1 does not depend on ε. It is well-known [3,4,11] that for all $q \in [1, \frac{d}{d-1})$ there exists a constant C_2 not depending on ε such that

$$\left\| p_\varepsilon \right\|_{L^\infty(0,T;W^{1,q}(\Omega))} \le C_2 \text{ and } \left\| \mathbf{u}_\varepsilon \right\|_{L^\infty(0,T;L^q(\Omega)^d)} \le C_2. \tag{3.12}$$

Estimates (3.4) and (3.12) give a bound on $|\mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon)|$ in $L^\infty(0,T;L^s(\Omega))$ for all $s < 2d/(d-1)$. Combined with (3.11), the decomposition $\mathbf{D}_\varepsilon(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon = \mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon)\mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon$ and Hölder's inequality, this shows that for all $r < \frac{2d}{2d-1}$, there exists a constant C_3 not depending on ε such that

$$\left\| \mathbf{D}_\varepsilon(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon \right\|_{L^2(0,T;L^r(\Omega)^d)} \le C_3. \tag{3.13}$$

Applying the coercivity (3.2) to (3.11) gives

$$\left\| |\mathbf{u}_\varepsilon|^{1/2}\nabla c_\varepsilon \right\|_{L^2(0,T;L^2(\Omega)^d)} \le C_1^{1/2}\phi_*^{-1/2}\min(d_l, d_t)^{-1/2}. \tag{3.14}$$

As for (3.13), from estimates (3.12) and (3.14), for every $r \in [1, \frac{2d}{2d-1})$ we obtain the existence of a constant C_4 not depending on ε such that

$$\left\| |\mathbf{u}_\varepsilon|\nabla c_\varepsilon \right\|_{L^2(0,T;L^r(\Omega)^d)} \le C_4. \tag{3.15}$$

Finally, from (3.6f) and the previous estimates, for every $s > 2d$ there is a constant C_5 not depending on ε such that

$$\left\| \Phi\partial_t c_\varepsilon \right\|_{L^2(0,T;(W^{1,s}(\Omega))')} \le C_5. \tag{3.16}$$

Remark 3.4. Using the regularity result of Monier and Gallouët [13] and the fact that $\nu \in (W^{1,\ell}(\Omega))'$ for all $\ell > 2$, as in Fabrie–Gallouët [11] we see that (3.12) holds for any $q < 2$. In order to demonstrate that this additional regularity is required in only a few places, we retain (3.12) and all subsequent estimates with $q < d/(d-1)$.

By using the Stampacchia formulation of the solution to linear elliptic equations with measures [23], we previously analysed [9] the model (1.1) for $d_m > 0$. This Stampacchia formulation provides the uniqueness of the solution to linear elliptic equations with measure data, without the additional $(W^{1,\ell}(\Omega))'$ regularity assumption. However, it is unclear if our reasoning below could be adapted to this formulation, rather than the (more natural) (4.24).

4. Proof of Theorem 2.2

4.1. Improving local elliptic regularity

Multiplying the elliptic equation (1.1a) by an appropriately chosen cutoff function θ excises the singularities caused by the measure sources and localises the problem to regions where the absolute permeability $\mathbf{K}$ is regular. Our next lemma follows an analogous procedure to Amirat–Ziani by rewriting the pressure equation in a form that yields higher local regularity of the solution.

Lemma 4.1. *Assume (2.1). For $\varepsilon > 0$, let $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$ be a solution-by-approximation to (3.6). Let $\theta \in C_c^\infty(\Omega)$ be such that* $\mathrm{supp}\,(\theta) \cap (\mathfrak{D}_\mathbf{K} \cup \mathrm{supp}\,(\nu)) = \emptyset$ *and take $r < \frac{2d}{2d-1}$. Then there exists C_6, depending on θ and r but not on ε, such that*

$$\|\theta p_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \le C_6, \tag{4.1}$$

$$\|\theta p_\varepsilon\|_{L^2(0,T;W^{2,r}(\Omega))} \le C_6, \tag{4.2}$$

$$\|\theta \mathbf{u}_\varepsilon\|_{L^2(0,T;W^{1,r}(\Omega)^d)} \le C_6. \tag{4.3}$$

Proof. *Step 1: proof of (4.1).*

Consider (3.7) with a and ν replaced by a_n and ν_n. For almost every $t \in (0,T)$, the local elliptic estimates in [7, Theorem 2] show that p_ε^n satisfies (4.1) with a bound not depending on n or ε. Passing to the limit as $n \to \infty$ shows that (4.1) holds.

Step 2: derivation of localised equation.

Take θ satisfying the hypotheses of the lemma, and consider (1.1a) with p, $\mathbf{u}$ and c replaced by p_ε, $\mathbf{u}_\varepsilon$ and c_ε, respectively. Multiplying the first equation by θ gives, in the sense of distributions,

$$\theta \mathbf{u}_\varepsilon = -\theta \frac{\mathbf{K}}{\mu(c_\varepsilon)} \nabla p_\varepsilon = -\frac{\mathbf{K}}{\mu(c_\varepsilon)} \nabla(\theta p_\varepsilon) + p_\varepsilon \frac{\mathbf{K}}{\mu(c_\varepsilon)} \nabla\theta.$$

Multiplying the second equation by θ yields

$$\mathrm{div}(\theta \mathbf{u}_\varepsilon) - \mathbf{u}_\varepsilon \cdot \nabla\theta = \theta(a-b)\nu.$$

The property of $\mathrm{supp}\,(\nu)$ and the choice of θ show that $\theta\nu = 0$, so the right-hand side of the previous equality vanishes. Combining these expressions using standard computations that are justified (in the sense of distributions) by the regularity (4.1), then simplifying where appropriate using the definition of $\mathbf{u}_\varepsilon$ leads to

$$-\,\mathrm{div}\,(\mathbf{K}\nabla(\theta p_\varepsilon)) = -p_\varepsilon\,\mathrm{div}\,(\mathbf{K}\nabla\theta) + \theta\mu'(c_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla c_\varepsilon + 2\mu(c_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla\theta. \tag{4.4}$$

In order to apply Grisvard's estimates we require that the diffusion matrix belongs to the class $C^{0,1}(\overline{\Omega}; \mathcal{S}_d(\mathbb{R}))$. Note that each term in (4.4) contains θ, so that both sides vanish outside the support of θ. We may therefore replace $\mathbf{K}$ in (4.4) by a uniformly coercive Lipschitz tensor $\widetilde{\mathbf{K}}$ that agrees with $\mathbf{K}$ on $\mathrm{supp}\,(\theta)$ whilst retaining equality. Take $\rho \in C_c^\infty(\Omega)$ with $0 \le \rho \le 1$ and such that $\mathrm{supp}\,(\theta) \subset \omega \subset \mathrm{supp}\,(\rho) \subset \Omega \setminus \mathfrak{D}_\mathbf{K}$, where ω is an open set such that $\rho \equiv 1$ on ω. Define

$$\widetilde{\mathbf{K}} := \rho\mathbf{K} + (1-\rho)\mathbf{I}. \tag{4.5}$$

Then $\widetilde{\mathbf{K}} \in C^{0,1}(\overline{\Omega}; \mathcal{S}_d(\mathbb{R}))$ and satisfies $\widetilde{\mathbf{K}} = \mathbf{K}$ on $\mathrm{supp}\,(\theta)$, $\widetilde{\mathbf{K}} = \mathbf{I}$ outside $\mathrm{supp}\,(\rho)$. Furthermore, for almost every $x \in \Omega$ and for all $\xi \in \mathbb{R}^d$ we have $\widetilde{\mathbf{K}}(x)\xi \cdot \xi \ge \min(1, k_*)|\xi|^2$. Replacing $\mathbf{K}$ with $\widetilde{\mathbf{K}}$ in the first two terms of (4.4), we are lead to the following localised pressure equation:

$$-\,\mathrm{div}\left(\widetilde{\mathbf{K}}\nabla(\theta p_\varepsilon)\right) = -p_\varepsilon\,\mathrm{div}\left(\widetilde{\mathbf{K}}\nabla\theta\right) + \theta\mu'(c_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla c_\varepsilon + 2\mu(c_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla\theta. \tag{4.6}$$

Step 3: proof of local estimates.

The equation (4.6) is satisfied on $\Omega \times (0,T)$, but due to the compact support of θ, it also holds on $B \times (0,T)$, where B is ball containing Ω and all functions are extended by 0 outside Ω. Estimates (3.12) and (3.15) then show that for every $1 \le r < 2d/(2d-1)$ the right-hand side of (4.6) is bounded in $L^2(0,T;L^r(\Omega))$ uniformly in ε. Then (4.2) follows from Grisvard [14, Eq. (2.3.3.1)] and from the fact that $\theta p_\varepsilon(\cdot,t) \in H_0^1(B)$ for almost every $t \in (0,T)$.

For (4.3), write $\widetilde{\mathbf{K}} = (\tilde{k}_{ij})_{i,j=1,\dots,d}$. Observing the summation convention, the i-th component of $\theta \mathbf{u}_\varepsilon$ is

$$(\theta \mathbf{u}_\varepsilon)_i = -\frac{\theta}{\mu(c_\varepsilon)} \tilde{k}_{ij} \partial_{x_j} p_\varepsilon.$$

By the regularity properties (3.6a) and (3.6e) of p_ε and c_ε, we can write, in the sense of distributions,

$$\begin{aligned}\partial_{x_l}(\theta \mathbf{u}_\varepsilon)_i &= (\partial_{x_l}\theta)\left(-\frac{\tilde{k}_{ij}}{\mu(c_\varepsilon)}\partial_{x_j} p_\varepsilon\right) - \theta \frac{\mu'(c_\varepsilon)}{\mu(c_\varepsilon)}(\mathbf{u}_\varepsilon)_i \partial_{x_l} c_\varepsilon - \frac{\theta}{\mu(c_\varepsilon)}(\partial_{x_l}\tilde{k}_{ij})(\partial_{x_j} p_\varepsilon) - \frac{\theta}{\mu(c_\varepsilon)}\tilde{k}_{ij}\partial^2_{x_l x_j} p_\varepsilon \\ &= T_1 + T_2 + T_3 + T_4.\end{aligned}$$

Thanks to (3.12), both T_1 and T_3 are bounded in $L^\infty(0,T;L^q(\Omega))$ for every $q < \frac{d}{d-1}$. For every $r < \frac{2d}{2d-1}$, estimate term T_2 in $L^2(0,T;L^r(\Omega))$ using (3.15). For T_4, use (4.2). □

4.2. Extraction of converging sequences

From (3.6a) the sequence $(c_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(\Omega \times (0,T))$, so that up to a subsequence

$$c_\varepsilon \rightharpoonup c \quad \text{in } L^\infty(\Omega \times (0,T)) \text{ weak-}*, \quad 0 \le c \le 1 \text{ a.e. in } \Omega \times (0,T), \tag{4.7}$$

which proves the first part of (2.2a); the second part follows at the end of Section 4.5. Estimate (3.12) implies the existence of extracted subsequences such that

$$p_\varepsilon \rightharpoonup p \quad \text{in } L^\infty(0,T;W_\star^{1,q}(\Omega)) \text{ weak-}* \quad \forall 1 \le q < \frac{d}{d-1}, \text{ and} \tag{4.8}$$

$$\mathbf{u}_\varepsilon \rightharpoonup \mathbf{u} \quad \text{in } L^\infty(0,T;L^q(\Omega)^d) \text{ weak-}* \quad \forall 1 \le q < \frac{d}{d-1}, \tag{4.9}$$

which proves (2.2e). The porosity is independent of time, so for every $s > 2d$, (3.16) provides an estimate in $L^2(0,T;(W^{1,s}(\Omega))')$ of the sequence $(\partial_t(\Phi c_\varepsilon))_{\varepsilon>0}$, from which we conclude that

$$\Phi \partial_t c_\varepsilon \rightharpoonup \Phi \partial_t c \quad \text{weakly in } L^2(0,T;(W^{1,s}(\Omega))') \text{ for all } s > 2d, \tag{4.10}$$

thus proving (2.2b). Furthermore, $(\Phi c_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(\Omega \times (0,T))$, and $L^\infty(\Omega)$ is compactly embedded in $(W^{1,s}(\Omega))'$ (since $W^{1,s}(\Omega)$ is compactly and densely embedded in $L^1(\Omega)$). A classical compactness lemma due to Simon [22] therefore ensures that, up to a subsequence, $\Phi c_\varepsilon \to \Phi c$ in $C([0,T];(W^{1,s}(\Omega))')$ for all $s > 2d$, which proves (2.2c).

4.3. Passing to the limit in the pressure equation

The proof that $(p, \mathbf{u}, c)$ satisfies the elliptic equation (2.2g) will be complete by passing to the limit in (3.6g), provided that we identify $\mathbf{u}$. For this we follow the ideas of Amirat–Ziani [1, Lemma 2.4], who rely on a variant of the compensated compactness phenomenon due to Kazhikhov [15]. Our proof necessarily departs from that of Amirat–Ziani due to our use of the cutoff functions θ. We also correct an error in their estimate of the term corresponding to our $\partial_t \mu(c_\varepsilon)$. They claim this sequence is bounded L^2 in time, when in fact it is only L^1 (for both regular and measure source terms). This necessitates our use of the $BV(0,T)$ spaces and a compensated compactness result adapted to this regularity, Lemma B.1 in the appendix.

Lemma 4.2. *Assume (2.1) and for $\varepsilon > 0$, let $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$ be a solution-by-approximation to (3.6). Assume that (4.7)–(4.9) hold. Then for almost every $(x,t) \in \Omega \times (0,T)$,*

$$\mathbf{u}(x,t) = -\frac{\mathbf{K}(x)}{\mu(c(x,t))}\nabla p(x,t). \tag{4.11}$$

Proof. By the assumptions (2.1d) on μ, there exists $\overline{\mu}$, $\underline{\mu} \in L^\infty(\Omega \times (0,T))$ such that $\mu_* \le \underline{\mu}, \overline{\mu} \le \mu^*$ and, up to a subsequence,

$$\mu(c_\varepsilon) \rightharpoonup \overline{\mu} \quad \text{and} \quad \frac{1}{\mu(c_\varepsilon)} \rightharpoonup \frac{1}{\underline{\mu}} \quad \text{in } L^\infty(\Omega \times (0,T)) \text{ weak-}*. \tag{4.12}$$

Step 1: BV estimates.

Take $\psi \in C_c^\infty(\Omega)$. To apply Lemma B.1, we must estimate the sequences $\left(\int_\Omega \Phi\mu(c_\varepsilon(x,\cdot))\psi(x)\,\mathrm{d}x\right)_{\varepsilon>0}$ and $\left(\int_\Omega \frac{\Phi}{\mu(c_\varepsilon(x,\cdot))}\psi(x)\,\mathrm{d}x\right)_{\varepsilon>0}$ in the space $BV(0,T)$. We first obtain these estimates on the solution to the truncated problem from Definition 3.1, and then deduce the corresponding estimates on $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$.

Replace ν and a by ν_n and a_n from Definition 3.2. Let $(p_\varepsilon^{n,k}, \mathbf{u}_\varepsilon^{n,k}, c_\varepsilon^{n,k})$ be the solution to the corresponding truncated problem, that is with $\mathbf{D}_\varepsilon$ replaced by $\mathbf{D}_\varepsilon^k$, defined by (3.8).

Take $\gamma \in C^2([0,1])$, $\psi \in C^\infty(\overline{\Omega})$ and choose $\varphi = \gamma'(c_\varepsilon^{n,k}))\psi$ as a test function in [9, Eq. (3.8)]. Then for almost every $t \in (0,T)$ we have

$$\begin{aligned}
&\langle(\Phi\partial_t c_\varepsilon^{n,k})(\cdot,t), \gamma'(c_\varepsilon^{n,k})(\cdot,t)\psi\rangle_{(H^1)',H^1} + \int_\Omega \mathbf{D}_\varepsilon^k(x,\mathbf{u}_\varepsilon^{n,k}(x,t))\nabla c_\varepsilon^{n,k}(x,t)\cdot\nabla[\gamma'(c_\varepsilon^{n,k}(x,t))\psi(x)]\,\mathrm{d}x \\
&\quad - \int_\Omega c_\varepsilon^{n,k}(x,t)\mathbf{u}_\varepsilon^{n,k}(x,t)\cdot\nabla[\gamma'(c_\varepsilon^{n,k}(x,t))\psi(x)]\,\mathrm{d}x + \int_\Omega c_\varepsilon^{n,k}(x,t)\gamma'(c_\varepsilon^{n,k}(x,t))\psi(x)b(x,t)\nu_n(x)\,\mathrm{d}x \\
&= \int_\Omega \hat{c}(x,t)\gamma'(c_\varepsilon^{n,k}(x,t))\psi(x)a_n(x,t)\nu_n(x)\,\mathrm{d}x.
\end{aligned} \tag{4.13}$$

Since $\Phi\partial_t c_\varepsilon^{n,k} \in L^2(0,T;(H^1(\Omega))')$ and $\gamma'(c_\varepsilon^{n,k}) \in L^2(0,T;H^1(\Omega))$, the product $\Phi\partial_t c_\varepsilon^{n,k}\gamma'(c_\varepsilon^{n,k})$ is well-defined as an element of $L^1(0,T;(C^\infty(\overline{\Omega}))')$. Reasoning by density of smooth functions, we also see that

$$\partial_t(\Phi\gamma(c_\varepsilon^{n,k})) = \Phi\gamma'(c_\varepsilon^{n,k})\partial_t c_\varepsilon^{n,k} \quad \text{in } L^1(0,T;(C^\infty(\overline{\Omega}))'). \tag{4.14}$$

Introducing $\zeta(s) = \int_0^s q\gamma''(q)\,\mathrm{d}q$, write

$$\begin{aligned}
c_\varepsilon^{n,k}\mathbf{u}_\varepsilon^{n,k}\cdot\nabla\left[\gamma'(c_\varepsilon^{n,k})\psi\right] &= \left[\mathbf{u}_\varepsilon^{n,k}\cdot\nabla c_\varepsilon^{n,k}\right]c_\varepsilon^{n,k}\gamma''(c_\varepsilon^{n,k})\psi + \left[\mathbf{u}_\varepsilon^{n,k}\cdot\nabla\psi\right]c_\varepsilon^{n,k}\gamma'(c_\varepsilon^{n,k}) \\
&= \mathbf{u}_\varepsilon^{n,k}\cdot\nabla(\zeta(c_\varepsilon^{n,k})\psi) + \left[\mathbf{u}_\varepsilon^{n,k}\cdot\nabla\psi\right]\left(c_\varepsilon^{n,k}\gamma'(c_\varepsilon^{n,k}) - \zeta(c_\varepsilon^{n,k})\right).
\end{aligned}$$

The equation (3.7) on $\mathbf{u}_\varepsilon^{n,k}$ then shows that

$$\begin{aligned}
&-\int_\Omega c_\varepsilon^{n,k}(x,t)\mathbf{u}_\varepsilon^{n,k}(x,t)\cdot\nabla[\gamma'(c_\varepsilon^{n,k}(x,t))\psi(x)]\,\mathrm{d}x \\
&= \int_\Omega \zeta(c_\varepsilon^{n,k}(x,t))(a_n - b)(x,t)\psi(x)\nu_n(x)\,\mathrm{d}x \\
&\quad - \int_\Omega \left[\mathbf{u}_\varepsilon^{n,k}(x,t)\cdot\nabla\psi(x)\right]\left(c_\varepsilon^{n,k}(x,t)\gamma'(c_\varepsilon^{n,k}(x,t)) - \zeta(c_\varepsilon^{n,k}(x,t))\right)\mathrm{d}x.
\end{aligned}$$

Substituted alongside (4.14) in (4.13), this gives

$$\begin{aligned}
&\langle \partial_t(\Phi\gamma(c_\varepsilon^{n,k})(\cdot,t)),\psi\rangle_{(C^\infty(\overline{\Omega}))',C^\infty(\overline{\Omega})} \\
&\quad+\int_\Omega \gamma'(c_\varepsilon^{n,k}(x,t))\mathbf{D}_\varepsilon^k(x,\mathbf{u}_\varepsilon^{n,k}(x,t))\nabla c_\varepsilon^{n,k}(x,t)\cdot\nabla\psi(x)\,\mathrm{d}x \\
&\quad+\int_\Omega \Big[\mathbf{D}_\varepsilon^k(x,\mathbf{u}_\varepsilon^{n,k}(x,t))\nabla c_\varepsilon^{n,k}(x,t)\cdot\nabla c_\varepsilon^{n,k}(x,t)\Big]\gamma''(c_\varepsilon^{n,k}(x,t))\psi(x)\,\mathrm{d}x \\
&\quad+\int_\Omega \zeta(c_\varepsilon^{n,k}(x,t))(a_n-b)(x,t)\psi(x)\nu_n(x)\,\mathrm{d}x \\
&\quad-\int_\Omega \Big[\mathbf{u}_\varepsilon^{n,k}(x,t)\cdot\nabla\psi(x)\Big]\Big(c_\varepsilon^{n,k}(x,t)\gamma'(c_\varepsilon^{n,k}(x,t))-\zeta(c_\varepsilon^{n,k}(x,t))\Big)\,\mathrm{d}x \\
&\quad+\int_\Omega c_\varepsilon^{n,k}(x,t)\gamma'(c_\varepsilon^{n,k}(x,t))\psi(x)b(x,t)\nu_n(x)\,\mathrm{d}x \\
&=\int_\Omega \hat{c}(x,t)\gamma'(c_\varepsilon^{n,k}(x,t))\psi(x)a_n(x,t)\nu_n(x)\,\mathrm{d}x.
\end{aligned} \tag{4.15}$$

All the integral terms can be bounded in the $L^1(0,T)$ norm by using $0\le c_\varepsilon^{n,k}\le 1$ and the estimates (3.11), (3.12) and (3.13) for $(p_\varepsilon^{n,k},\mathbf{u}_\varepsilon^{n,k},c_\varepsilon^{n,k})$ (and $\mathbf{D}_\varepsilon$ replaced by $\mathbf{D}_\varepsilon^k$), with constants that do not depend on k, n or ε. This gives

$$\left|\int_\Omega (\Phi\gamma(c_\varepsilon^{n,k}))(x,\cdot)\psi(x)\,\mathrm{d}x\right|_{BV(0,T)} = \left\|\partial_t\int_\Omega (\Phi\gamma(c_\varepsilon^{n,k}))(x,\cdot)\psi(x)\,\mathrm{d}x\right\|_{L^1(0,T)} \le C_7,$$

where C_7 may depend on ψ and γ, but not on k, n or ε. Letting $k\to\infty$, $c_\varepsilon^{n,k}\to c_\varepsilon^n$ almost-everywhere and so $\left|\int_\Omega \Phi\gamma(c_\varepsilon^n)(x,\cdot)\psi(x)\,\mathrm{d}x\right|_{BV(0,T)}\le C_7$. By the convergence (3.10) of c_ε^n to c_ε, we infer a uniform-in-ε estimate in $BV(0,T)$ of $\int_\Omega \Phi\gamma(c_\varepsilon)(x,\cdot)\psi(x)\,\mathrm{d}x$. Finally, set $\gamma=\mu$ or $\frac{1}{\mu}$ to see that

$$\left(\int_\Omega \Phi(x)\mu(c_\varepsilon(x,\cdot))\psi(x)\,\mathrm{d}x\right)_{\varepsilon>0} \text{ and } \left(\int_\Omega \frac{\Phi(x)}{\mu(c_\varepsilon(x,\cdot))}\psi(x)\,\mathrm{d}x\right)_{\varepsilon>0} \text{ are bounded in } BV(0,T). \tag{4.16}$$

Step 2: passing to the limit on $\mu(c_\varepsilon)\mathbf{u}_\varepsilon$.

For $q\in[1,\frac{d}{d-1})$, the sequence $(\mu(c_\varepsilon)\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is bounded in $L^\infty(0,T;L^q(\Omega)^d)$, so there exists $\overline{\mu\mathbf{u}}\in L^\infty(0,T;L^q(\Omega)^d)$ such that, up to a subsequence,

$$\mu(c_\varepsilon)\mathbf{u}_\varepsilon \rightharpoonup \overline{\mu\mathbf{u}} \quad \text{in } L^\infty(0,T;L^q(\Omega)^d) \text{ weak-}* \text{ for all } q<\tfrac{d}{d-1}. \tag{4.17}$$

The estimates (4.3) and (4.16) and the weak convergences (4.9) and (4.12) enable us to apply Lemma B.1 with $p=2$, $a=r$ (for a fixed $r<2d/(2d-1)$), $\alpha_\varepsilon=$ components of $\theta\mathbf{u}_\varepsilon$ and $\beta_\varepsilon=\Phi\mu(c_\varepsilon)$, to see that

$$\theta\Phi\mu(c_\varepsilon)\mathbf{u}_\varepsilon \rightharpoonup \theta\Phi\overline{\mu}\mathbf{u} \quad \text{in } \mathcal{D}'(\Omega\times(0,T)).$$

Combined with (4.17) multiplied by $\theta\Phi$, this shows that $\theta\Phi\overline{\mu\mathbf{u}}=\theta\Phi\overline{\mu}\mathbf{u}$ almost-everywhere. This holds for any $\theta\in C_c^\infty(\Omega)$ with $\mathrm{supp}(\theta)\cap\mathfrak{D}_\mathbf{K}=\emptyset$. By the freedom of θ and since Φ is uniformly positive, so we conclude that $\overline{\mu\mathbf{u}}=\overline{\mu}\mathbf{u}$ almost-everywhere and hence

$$\mu(c_\varepsilon)\mathbf{u}_\varepsilon \rightharpoonup \overline{\mu}\mathbf{u} \quad \text{in } L^\infty(0,T;L^q(\Omega)^d) \text{ weak-}* \text{ for all } q<\tfrac{d}{d-1}.$$

Note that by (4.8), $\mu(c_\varepsilon)\mathbf{u}_\varepsilon = -\mathbf{K}\nabla p_\varepsilon \rightharpoonup -\mathbf{K}\nabla p$ in $L^\infty(0,T;L^q(\Omega)^d)$ weak-$*$ for all $q < d/(d-1)$. Thus for almost every $(x,t) \in \Omega \times (0,T)$,

$$\mathbf{u}(x,t) = -\frac{\mathbf{K}(x)}{\overline{\mu}(x,t)}\nabla p(x,t). \tag{4.18}$$

Step 3: identifying the limit of $\mathbf{u}_\varepsilon$.

We seek to identify the limit of

$$\Phi\theta\mathbf{u}_\varepsilon = -\Phi\theta\frac{\widetilde{\mathbf{K}}}{\mu(c_\varepsilon)}\nabla p_\varepsilon. \tag{4.19}$$

Apply Lemma B.1 to the right-hand side, with $p = 2$, $a = r$ (for a fixed $r < 2d/(2d-1)$), α_ε = components of $-\theta\widetilde{\mathbf{K}}\nabla p_\varepsilon$ and $\beta_\varepsilon = \frac{\Phi}{\mu(c_\varepsilon)}$. The estimates (4.2) and (4.16) and the convergences (4.8) and (4.12) once again show that the assumptions of Lemma B.1 are satisfied. We then pass to the limit on both sides of (4.19) to obtain

$$\Phi\theta\mathbf{u} = -\Phi\theta\frac{\widetilde{\mathbf{K}}}{\underline{\mu}}\nabla p = -\Phi\theta\frac{\mathbf{K}}{\underline{\mu}}\nabla p.$$

Again using the freedom of θ and the strict positivity of Φ, for almost every $(x,t) \in \Omega \times (0,T)$,

$$\mathbf{u}(x,t) = -\frac{\mathbf{K}(x)}{\underline{\mu}(x,t)}\nabla p(x,t). \tag{4.20}$$

Comparing (4.18) and (4.20), for almost every $(x,t) \in \Omega \times (0,T)$,

$$(\overline{\mu}\mathbf{u})(x,t) = (\underline{\mu}\mathbf{u})(x,t).$$

To conclude the proof of (4.11), argue exactly as in Amirat–Ziani [1, Lemma 2.4]. □

4.4. *Strong convergence of the Darcy velocity*

The strong convergence of the Darcy velocity is necessary to handle the convergence of the diffusion–dispersion term, detailed in Section 4.5. Strong convergence of $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$ begins with strong convergence of $(\nabla p_\varepsilon)_{\varepsilon>0}$. When the source terms belong to $L^\infty(0,T;L^2(\Omega))$, the key to proving the latter is to use $p_\varepsilon - p$ as a test function in its own equation (see [1, Lemma 2.5]). In the non-variational setting of measure source terms, this is no longer possible as $p_\varepsilon - p$ does not have the required regularity. We first need to excise the support of the measure using localisation functions θ. While doing so, we create lower order terms in the right-hand side whose convergence needs to be assessed. This is the purpose of the following lemma, which establishes the strong convergence of $(p_\varepsilon)_{\varepsilon>0}$. Due to the lack of estimates on the time derivative of $(p_\varepsilon)_{\varepsilon>0}$, this result is not straightforward and requires careful use of the Aubin–Simon compactness lemma, alongside a uniqueness result for elliptic equations with source terms in $\mathcal{M}_+(\Omega) \cap (W^{1,\ell}(\Omega))'$ for all $\ell > 2$.

Lemma 4.3. *Assume (2.1). For $\varepsilon > 0$, let $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$ be a solution-by-approximation to (3.6). Assume that (4.7)–(4.9) hold along a subsequence. Then along the same subsequence,*

$$p_\varepsilon \to p \text{ strongly in } L^a(0,T;L^q(\Omega)) \text{ for all } a < \infty \text{ and all } q < \tfrac{d}{d-1}, \tag{4.21}$$

and for any $\theta \in C_c^\infty(\Omega)$ such that $\mathrm{supp}(\theta) \cap (\mathfrak{D}_{\mathbf{K}} \cup \mathrm{supp}(\nu)) = \emptyset$,

$$\theta p_\varepsilon \to \theta p \text{ strongly in } L^a(0,T;L^2(\Omega)) \text{ for all } a < \infty. \tag{4.22}$$

Proof. *Step 1: almost-everywhere convergence of* $1/\mu(c_\varepsilon)$.

Our aim is to apply an Aubin–Simon lemma to $1/\mu(c_\varepsilon)$. We can only estimate the time derivative of this function when multiplied by the porosity Φ. To eliminate this factor, we use a similar trick as in our previous work [9, Section 3.3]. Let $\delta \in (1,\infty)$ and set $\frac{1}{\Phi}W^{1,\delta}(\Omega) = \{v \in L^\delta(\Omega) : \Phi v \in W^{1,\delta}(\Omega)\}$, with norm $\|v\|_{\frac{1}{\Phi}W^{1,\delta}(\Omega)} = \|\Phi v\|_{W^{1,\delta}(\Omega)}$. By the Rellich theorem, $\frac{1}{\Phi}W^{1,\delta}(\Omega)$ is compactly and densely embedded in $L^\delta(\Omega)$. It follows that $L^{\delta'}(\Omega)$ is compactly embedded in $(\frac{1}{\Phi}W^{1,\delta}(\Omega))'$.

Take $(p_\varepsilon^{n,k}, \mathbf{u}_\varepsilon^{n,k}, c_\varepsilon^{n,k})$ as in the proof of Lemma 4.2 and fix $s > 2d$. The family $(p_\varepsilon^{n,k}, \mathbf{u}_\varepsilon^{n,k}, c_\varepsilon^{n,k}))_{\varepsilon>0}^{n,k\in\mathbb{N}}$ satisfies estimates (3.11)–(3.13), with constants not depending on n, k or ε. Used in (4.15) applied to $\gamma = 1/\mu$, these estimates give a uniform bound on $\partial_t(\Phi/\mu(c_\varepsilon^{n,k}))$ in $L^1(0,T;(W^{1,s}(\Omega))')$ and therefore in $L^1(0,T;(W^{1,s}(\Omega))' + (\frac{1}{\Phi}W^{1,\delta}(\Omega))')$.

Now $(\Phi/\mu(c_\varepsilon^{n,k}))_{\varepsilon>0}^{n,k\in\mathbb{N}}$ is bounded in $L^\infty(\Omega\times(0,T))$ and therefore in $L^1(0,T;L^{\delta'}(\Omega))$, with $L^{\delta'}(\Omega)$ compactly embedded in $(\frac{1}{\Phi}W^{1,\delta}(\Omega))'$. Classical Aubin–Simon lemmas show that $\mathcal{A} = \{\Phi/\mu(c_\varepsilon^{n,k}) : \varepsilon > 0;\, n,k\in\mathbb{N}\}$ is relatively compact in the space $L^1(0,T;(\frac{1}{\Phi}W^{1,\delta}(\Omega))')$. Write $\overline{\mathcal{A}}$ for the (compact) closure of $\mathcal{A}$ in this space.

By compactness of $\overline{\mathcal{A}}$, the limit in $\mathcal{D}'(\Omega\times(0,T))$ of any sequence in $\mathcal{A}$ also belongs to $\overline{\mathcal{A}}$. As $k\to\infty$ and $n\to\infty$ (in that order), we know that $c_\varepsilon^{n,k}\to c_\varepsilon$ almost-everywhere on $\Omega\times(0,T)$. Hence $\Phi/\mu(c_\varepsilon^{n,k})\to\Phi/\mu(c_\varepsilon)$ almost-everywhere on $\Omega\times(0,T)$, and thus in $\mathcal{D}'(\Omega\times(0,T))$ since these functions are uniformly bounded in $L^\infty(\Omega\times(0,T))$. As a consequence, $(\Phi/\mu(c_\varepsilon))_{\varepsilon>0}$ is a sequence in $\overline{\mathcal{A}}$ and thus, up to a subsequence, converges strongly in $L^1(0,T;(\frac{1}{\Phi}W^{1,\delta}(\Omega))')$. By (4.12), the limit of this sequence must be $\Phi/\underline{\mu}$. Extracting another subsequence, we can therefore assert that, as $\varepsilon\to 0$, for almost every $t\in(0,T)$,

$$\frac{\Phi}{\mu(c_\varepsilon(\cdot,t))}\to\frac{\Phi}{\underline{\mu}(\cdot,t)}\text{ strongly in }\left(\tfrac{1}{\Phi}W^{1,\delta}(\Omega)\right)'.$$

The definition of $\frac{1}{\Phi}W^{1,\delta}(\Omega)$ shows that, for all $Z\in L^\infty(\Omega)$, $\|\Phi Z\|_{(\frac{1}{\Phi}W^{1,\delta}(\Omega))'} = \|Z\|_{(W^{1,\delta}(\Omega))'}$. Then, along a subsequence as $\varepsilon\to 0$, for almost every $t\in(0,T)$

$$\frac{1}{\mu(c_\varepsilon(\cdot,t))}\to\frac{1}{\underline{\mu}(\cdot,t)}\text{ strongly in }(W^{1,\delta}(\Omega))',\text{ for all }\delta\in(1,\infty). \tag{4.23}$$

Step 2: proof of (4.21).

From here on, we work with the subsequence along which (4.23) holds and explicitly denote any other extraction of a subsequence. Let A_1 be the set of $t\in(0,T)$ such that (4.23) holds, and A_2 be the set of $t\in(0,T)$ such that, for all $q<2$, $(p_\varepsilon(\cdot,t))_{\epsilon>0}$ is bounded in $W_\star^{1,q}(\Omega)$ (see Remark 3.4). Take functions $(\theta_j)_{j\ge 3}$ in $C_c^\infty(\Omega)$ such that $\mathrm{supp}\,(\theta_j)\cap(\mathfrak{D}_{\mathbf{K}}\cup\mathrm{supp}\,(\nu)) = \emptyset$, $0\le\theta_j\le 1$ and $\theta_j\to 1$ almost-everywhere on Ω as $j\to\infty$. Apply Lemma B.2 to $(\theta_j p_\varepsilon)_{\varepsilon>0}$ and $E = W^{2,r}(\Omega)$ (see (4.2)), and let A_j be the set of $t\in(0,T)$ that satisfy the conclusion of the lemma. The complement of $A = \cap_{j\in\mathbb{N}}A_j$ has a zero measure.

Fix $t\in A$. Owing to (2.1i), as in [11, Step 3, proof of Theorem 2.1] we see that $(a(\cdot,t)-b(\cdot,t))\nu\in(W^{1,\ell}(\Omega))'$ for all $\ell>2$. Hence by [11, Proposition 3.2], there is a unique solution to $-\mathrm{div}(\frac{\mathbf{K}}{\underline{\mu}}\nabla P(t)) = (a(\cdot,t)-b(\cdot,t))\nu$ with zero average and homogeneous Neumann conditions in the sense

$$\begin{aligned}&P(\cdot,t)\in\bigcap_{q<2}W_\star^{1,q}(\Omega)\text{ and }\forall\psi\in C^\infty(\overline{\Omega}),\\&\int_\Omega\frac{\mathbf{K}(x)}{\underline{\mu}(x,t)}\nabla P(x,t)\cdot\nabla\psi(x)\,\mathrm{d}x = \int_\Omega(a-b)(x,t)\psi(x)\,\mathrm{d}\nu(x).\end{aligned}\tag{4.24}$$

Note that the formulation in Fabrie–Gallouët [11] is written for $\psi\in\bigcup_{z>d}W^{1,z}(\Omega)$ which, by density, is equivalent to the formulation above.

We first prove that, up to a subsequence (depending on t), $p_\varepsilon(t)\to P(t)$ strongly in $L^q(\Omega)$ for all $q<2$. By choice of $t\in A$, there exists a subsequence $(p_{\varepsilon'}(t))_{\varepsilon'>0}$ that converges weakly $W_\star^{1,q}(\Omega)$ for all $q<2$ — and strongly in the corresponding $L^q(\Omega)$ spaces — toward some function $\mathcal{P}$. Recalling the conclusion of Lemma B.2, we can also assume that this subsequence satisfies

$$(\theta_j p_{\varepsilon'}(\cdot,t))_{\varepsilon'>0}\text{ is bounded in }W^{2,r}(\Omega)\text{ for all }r<\tfrac{2d}{2d-1}\text{ and all }j\in\mathbb{N},$$

which shows that, for every $j\in\mathbb{N}$, $\theta_j p_{\varepsilon'}(\cdot,t)\rightharpoonup\theta_j\mathcal{P}$ in $W^{2,r}(\Omega)$ for all $r<\frac{2d}{2d-1}$. Substitute $\psi\in C^\infty(\overline{\Omega})$ into (3.7). Defining $\widetilde{\mathbf{K}}_j$ by (4.5), with $\rho=\rho_j$ associated with θ_j, this gives (dropping the explicit mention of the x variable)

$$\int_\Omega(a-b)(t)\psi\,\mathrm{d}\nu = \int_\Omega\frac{\mathbf{K}}{\mu(c_{\varepsilon'}(t))}\nabla p_{\varepsilon'}(t)\cdot\nabla\psi\,\mathrm{d}x$$

$$= \int_\Omega \frac{1}{\mu(c_{\varepsilon'}(t))} \theta_j \widetilde{\mathbf{K}}_j \nabla p_{\varepsilon'}(t) \cdot \nabla\psi \, dx + \int_\Omega (1-\theta_j) \frac{\mathbf{K}}{\mu(c_{\varepsilon'}(t))} \nabla p_{\varepsilon'}(t) \cdot \nabla\psi \, dx$$
$$= I_{\varepsilon',j,1} + I_{\varepsilon',j,2}. \tag{4.25}$$

The tensor $\widetilde{\mathbf{K}}_j$ is Lipschitz continuous and, as $\varepsilon' \to 0$, $\theta_j \nabla p_{\varepsilon'}(t) \rightharpoonup \theta_j \nabla \mathcal{P}$ in $W^{1,r}(\Omega)$ for all $r < \frac{2d}{2d-1}$. Hence, the convergence (4.23) (which holds since $t \in A$) gives

$$\lim_{\varepsilon' \to 0} I_{\varepsilon',j,1} = \int_\Omega \frac{1}{\underline{\mu}(t)} \theta_j \widetilde{\mathbf{K}}_j \nabla\mathcal{P} \cdot \nabla\psi \, dx = \int_\Omega \frac{\mathbf{K}}{\underline{\mu}(t)} \nabla\mathcal{P} \cdot \nabla\psi \, dx - J_j, \tag{4.26}$$

where

$$J_j = \int_\Omega \frac{\mathbf{K}}{\underline{\mu}(t)} (1-\theta_j) \nabla\mathcal{P} \cdot \nabla\psi.$$

Fix $q_0 \in \left(1, \frac{d}{d-1}\right)$. Since $(p_{\varepsilon'}(t))_{\varepsilon'>0}$ is bounded in $W^{1,q_0}(\Omega)$ and $\mathcal{P} \in W^{1,q_0}(\Omega)$, we find C_8 not depending on j or ε such that

$$|I_{\varepsilon',j,2}| + |J_j| \le C_8 \left\| 1-\theta_j \right\|_{L^{q_0'}(\Omega)}.$$

Plugged into (4.25), this gives

$$\left| \int_\Omega (a(t)-b(t))\psi \, d\nu - \int_\Omega \frac{\mathbf{K}}{\underline{\mu}(t)} \nabla\mathcal{P} \cdot \nabla\psi \, dx \right|$$
$$\le \left| I_{\varepsilon',j,1} - \left(\int_\Omega \frac{\mathbf{K}}{\underline{\mu}(t)} \nabla\mathcal{P} \cdot \nabla\psi \, dx - J_j \right) \right| + C_8 \left\| 1-\theta_j \right\|_{L^{q_0'}(\Omega)}. \tag{4.27}$$

Since $q_0' < \infty$, the properties of θ_j show that $\left\| 1-\theta_j \right\|_{L^{q_0'}(\Omega)} \to 0$ as $j \to \infty$. Then thanks to (4.26), taking the superior limit as $\varepsilon' \to 0$ and then the limit as $j \to \infty$ of (4.27) shows that $\mathcal{P}$ satisfies (4.24).

We infer that $\mathcal{P} = P(t)$ and thus that the limit of $(p_{\varepsilon'}(t))_{\varepsilon'>0}$ does not depend on the chosen subsequence. In other words, the whole sequence $(p_\varepsilon(t))_{\varepsilon>0}$ converges in $L^q(\Omega)$ to $P(t)$, for almost every $t \in (0,T)$. By the bound in $L^\infty(0,T;L^q(\Omega))$ on $(p_\varepsilon)_{\varepsilon>0}$ given by (3.12), the dominated convergence theorem shows that $p_\varepsilon \to P$ strongly in $L^a(0,T;L^q(\Omega))$ for all $a < \infty$. The convergence (4.8) imposes $P = p$ and the proof of (4.21) is complete.

Step 3: proof of (4.22).

This follows from the previous convergence by a simple interpolation technique. Let $\tau \in (0,1)$ be such that $\frac{1}{2} = \frac{\tau}{1} + \frac{1-\tau}{2^*}$, where $2^* > 2$ is a Sobolev exponent (that is, such that $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$). Fix $a < \infty$ and take $A \in (a, \infty)$ such that $\frac{1}{a} = \frac{\tau}{A} + \frac{1-\tau}{\infty}$. Then

$$\left\| \theta p_\varepsilon - \theta p \right\|_{L^a(0,T;L^2(\Omega))} \le \left\| \theta p_\varepsilon - \theta p \right\|^\tau_{L^A(0,T;L^1(\Omega))} \left\| \theta p_\varepsilon - \theta p \right\|^{1-\tau}_{L^\infty(0,T;L^{2^*}(\Omega))}$$
$$\le \|\theta\|_\infty \left\| p_\varepsilon - p \right\|^\tau_{L^A(0,T;L^1(\Omega))} \left\| \theta p_\varepsilon - \theta p \right\|^{1-\tau}_{L^\infty(0,T;L^{2^*}(\Omega))}.$$

The second term in the right-hand side converges to 0 by (4.21), and the third term is bounded by (4.1), which, combined with (4.8), proves in particular that $\theta p \in L^\infty(0,T;H^1(\Omega))$. □

The next lemma highlights an almost-everywhere convergence property of $(c_\varepsilon)_{\varepsilon>0}$ that is critical for obtaining strong convergence of $(\nabla p_\varepsilon)_{\varepsilon>0}$.

Lemma 4.4. *Assume (2.1) and for $\varepsilon > 0$, let $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$ be a solution-by-approximation to (3.6). Assume that (4.7)–(4.9) hold along a subsequence. Then, up to another subsequence,*

$$c_\varepsilon \to c \quad \text{a.e. on } \{(x,t) \in \Omega \times (0,T) : |\mathbf{u}(x,t)| \ne 0\}. \tag{4.28}$$

Proof. By Assumption (2.1d) on μ, $m := 2\min_{[0,1]}\mu''$ is strictly positive. Lagrange's Remainder Theorem therefore gives $\mu(c_\varepsilon) - \mu(c) - (c_\varepsilon - c)\mu'(c) \geq m(c_\varepsilon - c)^2$. Multiplying by $|\mathbf{u}| \geq 0$ and integrating yields

$$\int_0^T \int_\Omega \big(\mu(c_\varepsilon(x,t))|\mathbf{u}(x,t)| - \mu(c(x,t))|\mathbf{u}(x,t)|\big)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega (c_\varepsilon - c)(x,t)\mu'(c(x,t))|\mathbf{u}(x,t)|\,\mathrm{d}x\,\mathrm{d}t$$
$$\geq m \int_0^T \int_\Omega (c_\varepsilon(x,t) - c(x,t))^2 |\mathbf{u}(x,t)|\,\mathrm{d}x\,\mathrm{d}t.$$

By (4.7) and (4.12), since $|\mathbf{u}| \in L^1(\Omega \times (0,T))$, we pass to the limit in the left-hand side to obtain

$$\int_0^T \int_\Omega (\overline{\mu}(x,t)|\mathbf{u}(x,t)| - \mu(c(x,t))|\mathbf{u}(x,t)|)\,\mathrm{d}x\,\mathrm{d}t \geq m \limsup_{\varepsilon\to 0} \int_0^T \int_\Omega (c_\varepsilon(x,t) - c(x,t))^2 |\mathbf{u}(x,t)|\,\mathrm{d}x\,\mathrm{d}t.$$

Thanks to (4.11) and (4.18) we have $\overline{\mu}\mathbf{u} = \mu(c)\mathbf{u}$. Taking the norms, we see that left-hand side vanishes. This shows that $(c_\varepsilon - c)^2|\mathbf{u}| \to 0$ in $L^1(\Omega \times (0,T))$, and therefore almost-everywhere on $\Omega \times (0,T)$ up to a subsequence. □

Remark 4.1. The main purpose of this almost-everywhere convergence of $(c_\varepsilon)_{\varepsilon>0}$ is to prove the convergence of $(\mathbf{u}_\varepsilon)_{\varepsilon>0}$.

Lemma 4.4 is no longer valid if μ is constant. However, in that case, the system is decoupled: the pressure does not depend on the concentration (and then does not even depend on ε), and there are no difficulties with the convergence of $\mathbf{u}_\varepsilon$ as it does not depend on ε.

Lemma 4.5. *Assume (2.1). For $\varepsilon > 0$, let $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$ be a solution-by-approximation to (3.6). Assume that (4.7)–(4.9) hold along a subsequence. Then along the same subsequence,*

$$\nabla p_\varepsilon \to \nabla p \quad \textit{strongly in } L^a(0,T;L^q(\Omega)^d) \textit{ for all } a < \infty \textit{ and all } q < \tfrac{d}{d-1}. \tag{4.29}$$

Proof. *Step 1: strong convergence of localised functions.*

Let $\rho \in C_c^\infty(\Omega)$ such that $\mathrm{supp}\,(\rho) \cap (\mathfrak{D}_{\mathbf{K}} \cup \mathrm{supp}\,(\nu)) = \emptyset$ and $\rho \geq 0$. We want to prove that as $\varepsilon \to 0$,

$$\sqrt{\rho}\nabla p_\varepsilon \to \sqrt{\rho}\nabla p \text{ strongly in } L^2(0,T;L^2(\Omega)^d). \tag{4.30}$$

Let $\psi \in L^1(0,T;W^{1,q}(\Omega))$ for some $q > d$, and take $\rho\psi$ as a test function in the equation (3.6g) satisfied by $\mathbf{u}_\varepsilon$. Since $\mathrm{supp}\,(\rho) \cap \mathrm{supp}\,(\nu) = 0$, the source terms disappear and we find that

$$\int_0^T \int_\Omega \mathbf{u}_\varepsilon(x,t) \cdot \nabla(\rho\psi)(x,t)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{4.31}$$

Let U be an open set in Ω such that $\mathrm{supp}\,(\rho) \subset U$ and $\overline{U} \cap (\mathfrak{D}_{\mathbf{K}} \cup \mathrm{supp}\,(\nu)) = \emptyset$. Let $\theta \in C_c^\infty(\Omega)$ be such that $\theta = 1$ on $\overline{U}$ and $\mathrm{supp}\,(\theta) \cap (\mathfrak{D}_{\mathbf{K}} \cup \mathrm{supp}\,(\nu)) = \emptyset$. Applying (4.1), we see that $\mathbf{u}_\varepsilon \in L^2(0,T;L^2(U))$ and $p_\varepsilon - p \in L^2(0,T;H^1(U))$. Taking a sequence $(\psi_j)_{j\in\mathbb{N}} \subset L^1(0,T;W^{1,q}(\Omega))$ for some $q > d$ and such that $\psi_j \to p_\varepsilon - p$ in $L^2(0,T;H^1(U))$, we pass to the limit in (4.31) to see that this relation still holds with $p_\varepsilon - p$ instead of ψ. Expanding, we obtain

$$-\int_0^T \int_\Omega \rho(x)\mathbf{u}_\varepsilon(x,t) \cdot \nabla(p_\varepsilon - p)(x,t)\,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_\Omega (p_\varepsilon(x,t) - p(x,t))\mathbf{u}_\varepsilon(x,t) \cdot \nabla\rho(x)\,\mathrm{d}x\,\mathrm{d}t.$$

By the choice of θ above, this can be written as

$$-\int_0^T\int_\Omega \rho(x)\mathbf{u}_\varepsilon(x,t)\cdot\nabla(p_\varepsilon-p)(x,t)\,\mathrm{d}x\,\mathrm{d}t=\int_0^T\int_\Omega(\theta(x)p_\varepsilon(x,t)-\theta(x)p(x,t))\theta(x)\mathbf{u}_\varepsilon(x,t)\cdot\nabla\rho(x)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C_9\left\|\theta p_\varepsilon-\theta p\right\|_{L^2(0,T;L^2(\Omega))},\tag{4.32}$$

where the existence of C_9 (not depending on ε) is ensured by (4.1), which shows that $(\theta\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0,T;L^2(\Omega)^d)$. Now use the definition of $\mathbf{u}_\varepsilon=-\frac{\mathbf{K}}{\mu(c_\varepsilon)}\nabla p_\varepsilon$, estimate (4.32) and the properties of θ to write

$$\begin{aligned}
&\frac{k_*}{\mu^*}\left\|\sqrt{\rho}\nabla(p_\varepsilon-p)\right\|^2_{L^2(0,T;L^2(\Omega)^d)}\\
&\le\int_0^T\int_\Omega\rho(x)\frac{\mathbf{K}(x)}{\mu(c_\varepsilon(x,t))}\nabla(p_\varepsilon-p)(x,t)\cdot\nabla(p_\varepsilon-p)(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&=-\int_0^T\int_\Omega\rho(x)\mathbf{u}_\varepsilon(x,t)\cdot\nabla(p_\varepsilon-p)(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad-\int_0^T\int_\Omega\rho(x)\frac{\mathbf{K}(x)}{\mu(c_\varepsilon(x,t))}\nabla p(x,t)\cdot\nabla(p_\varepsilon-p)(x,t)\,\mathrm{d}x\,\mathrm{d}t\\
&\le C_9\left\|\theta p_\varepsilon-\theta p\right\|_{L^2(0,T;L^2(\Omega))}\\
&\quad-\int_0^T\int_\Omega\rho(x)\frac{\mathbf{K}(x)}{\mu(c_\varepsilon(x,t))}\theta(x)\nabla p(x,t)\cdot\theta(x)\nabla(p_\varepsilon-p)(x,t)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{4.33}$$

By (4.28), $\mu(c_\varepsilon)\to\mu(c)$ almost-everywhere on $\{|\mathbf{u}|\neq 0\}=\{|\nabla p|\neq 0\}$. Hence, by the dominated convergence theorem and (4.1), $(\frac{\mathbf{K}}{\mu(c_\varepsilon)}\theta\nabla p)_{\varepsilon>0}$ converges strongly in $L^2(0,T;L^2(\Omega)^d)$. Using (4.1) and (4.8) we also have $\theta\nabla p_\varepsilon\to\theta\nabla p$ weakly in $L^2(0,T;L^2(\Omega)^d)$. Hence, the last term in (4.33) tends to 0 as $\varepsilon\to 0$. Taking the superior limit of this estimate and using (4.22) shows that (4.30) holds.

Step 2: conclusion.

Since (4.30) is satisfied for all nonnegative $\rho\in C_c^\infty(\Omega)$ whose support does not intersect the closed set $\mathfrak{D}_{\mathbf{K}}\cup\operatorname{supp}(\nu)$, and since this set has a zero Lebesgue measure, up to a subsequence we can assume that $\nabla p_\varepsilon\to\nabla p$ almost-everywhere on $\Omega\times(0,T)$. The convergence (4.29) then follows from the Vitali theorem and the bound (3.12) on $(\nabla p_\varepsilon)_{\varepsilon>0}$ in $L^\infty(0,T;L^q(\Omega)^d)$ for all $q<d/(d-1)$. □

The strong convergence of the Darcy velocity and of $(c_\varepsilon\mathbf{u}_\varepsilon)_{\varepsilon>0}$ is then straightforward.

Lemma 4.6. *Assume (2.1). For $\varepsilon>0$, let $(p_\varepsilon,\mathbf{u}_\varepsilon,c_\varepsilon)$ be a solution-by-approximation of (3.6). Assume that (4.7)–(4.9) hold along a subsequence. Then along the same subsequence,*

$$\mathbf{u}_\varepsilon\to\mathbf{u}\quad\text{strongly in } L^a(0,T;L^q(\Omega)^d)\text{ for all } a<\infty\text{ and all } q<\tfrac{d}{d-1}.\tag{4.34}$$

Proof. The almost-everywhere convergence (4.28) of c_ε gives $\mu(c_\varepsilon)\mathbf{u}\to\mu(c)\mathbf{u}$ almost-everywhere on $\Omega\times(0,T)$. Since $\mathbf{u}\in L^a(0,T;L^q(\Omega)^d)$ for all $a<\infty$ and $q<d/(d-1)$, this convergence also holds in $L^a(0,T;L^q(\Omega)^d)$ by dominated convergence. Thanks to (4.11) and (4.29), letting $\varepsilon\to 0$ gives

$$\begin{aligned}
\mu_*\left\|\mathbf{u}_\varepsilon-\mathbf{u}\right\|_{L^a(0,T;L^q(\Omega)^d)}&\le\left\|\mu(c_\varepsilon)\mathbf{u}_\varepsilon-\mu(c_\varepsilon)\mathbf{u}\right\|_{L^a(0,T;L^q(\Omega)^d)}\\
&=\left\|-\mathbf{K}\nabla p_\varepsilon-\mu(c_\varepsilon)\mathbf{u}\right\|_{L^a(0,T;L^q(\Omega)^d)}\\
&\to\left\|-\mathbf{K}\nabla p-\mu(c)\mathbf{u}\right\|_{L^a(0,T;L^q(\Omega)^d)}=0.\quad\square
\end{aligned}$$

Corollary 4.7. *Assume (2.1). For $\varepsilon > 0$, let $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$ be a solution-by-approximation of (3.6). Assume that (4.7)–(4.9) hold along a subsequence. Then along the same subsequence,*

$$c_\varepsilon \mathbf{u}_\varepsilon \to c\mathbf{u} \quad \text{strongly in } L^a(0,T;L^q(\Omega)^d) \text{ for all } a < \infty \text{ and all } q < \tfrac{d}{d-1}.$$

Proof. Write $c_\varepsilon \mathbf{u}_\varepsilon - c\mathbf{u} = c_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{u}) + (c_\varepsilon - c)\mathbf{u}$. Owing to (4.34) and $0 \le c_\varepsilon \le 1$, the first term tends to 0 in $L^a(0,T;L^q(\Omega)^d)$ as $\varepsilon \to 0$. For the second term, use (4.28) and the fact that $\mathbf{u} \in L^a(0,T;L^q(\Omega)^d)$. □

4.5. Passing to the limit in the concentration equation

The proof that $(p, \mathbf{u}, c)$ satisfies (2.2d) and (2.2f) follows from the next two lemmas, which address the regularity and convergence of the diffusion–dispersion term.

Lemma 4.8. *Assume (2.1) and for $\varepsilon > 0$, let $(p_\varepsilon, \mathbf{u}_\varepsilon, c_\varepsilon)$ be a solution-by-approximation to (3.6). Assume that (4.7)–(4.9) hold along a subsequence. Then the function c defined by (4.7) has a $\{|\mathbf{u}| > 0\}$-gradient and*

$$\mathbf{D}_\circ(\cdot, \mathbf{u})\nabla_{\{|\mathbf{u}|>0\}} c \in L^2(0,T;L^r(\Omega)^d) \text{ for all } r < \tfrac{2d}{2d-1}. \tag{4.35}$$

Proof. From (4.34) and the partial converse to the dominated convergence theorem, up to a subsequence $\mathbf{u}_\varepsilon \to \mathbf{u}$ almost everywhere on $\Omega \times (0,T)$. Let $(\eta_i)_{i\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ with $\eta_i \to 0^+$ as $i \to \infty$ and such that for every $i \in \mathbb{N}$, $\text{meas}(\{|\mathbf{u}| = \eta_i\}) = 0$ (existence of such a sequence is guaranteed by Lemma A.1). On the set $\{|\mathbf{u}_\varepsilon| > \eta_i\}$ we have

$$\mathbf{D}_\varepsilon(x, \mathbf{u}_\varepsilon)\nabla c_\varepsilon \cdot \nabla c_\varepsilon \ge \min(d_l, d_t)\phi_* \eta_i |\nabla c_\varepsilon|^2.$$

Since $(\mathbf{D}_\varepsilon(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon \cdot \nabla c_\varepsilon)_{\varepsilon>0}$ is bounded in $L^1(0,T;L^1(\Omega))$ (see (3.11)), it follows that

$$(\mathbf{1}_{\{|\mathbf{u}_\varepsilon|>\eta_i\}}\nabla c_\varepsilon)_{\varepsilon>0} \quad \text{is bounded in } L^2(0,T;L^2(\Omega)^d) \text{ for all } i \in \mathbb{N}.$$

After performing a diagonal extraction upon the index i, we infer the existence of $\chi_{\eta_i} \in L^2(0,T;L^2(\Omega)^d)$ such that, up to a subsequence not depending on i,

$$\mathbf{1}_{\{|\mathbf{u}_\varepsilon|>\eta_i\}}\nabla c_\varepsilon \rightharpoonup \chi_{\eta_i} \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d). \tag{4.36}$$

The hypotheses of Definition 2.1 are therefore satisfied and so c has a $\{|\mathbf{u}| > 0\}$-gradient.

To prove (4.35), we begin by using the same splitting trick as in our previous work [9, Section 4.3] by writing

$$\mathbf{1}_{\{|\mathbf{u}_\varepsilon|>\eta_i\}}\mathbf{D}_\varepsilon(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon = \mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon)\left(\mathbf{1}_{\{|\mathbf{u}_\varepsilon|>\eta_i\}}\mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon\right) \tag{4.37}$$

and applying Lemma B.4 once to each term in the right-hand side product. By (4.34) with $a = q = 1$ and the estimate (3.4) on $\mathbf{D}_\varepsilon^{1/2}$, Lemma B.3 yields

$$\mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon) \to \mathbf{D}_\circ^{1/2}(\cdot, \mathbf{u}) \quad \text{strongly in } L^2(0,T;L^2(\Omega)^{d\times d}). \tag{4.38}$$

Since $(\mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0,T;L^2(\Omega)^d)$ (see (3.11)), the weak convergence (4.36) enables us to apply Lemma B.4 with $r_1 = r_2 = s_2 = s_2 = 2$ and $a = b = 2$, to $w_\varepsilon =$ components of $\mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon)$ and $v_\varepsilon =$ components of $\mathbf{1}_{\{|\mathbf{u}_\varepsilon|>\eta_i\}}\nabla c_\varepsilon$. This gives

$$\mathbf{1}_{\{|\mathbf{u}_\varepsilon|>\eta_i\}}\mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon \rightharpoonup \mathbf{D}_\circ^{1/2}(\cdot, \mathbf{u})\chi_{\eta_i} = \mathbf{D}_\circ^{1/2}(\cdot, \mathbf{u})\nabla_{\{|\mathbf{u}|>\eta_i\}} c \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d).$$

This weak convergence and (4.38) enable us to re-use Lemma B.4 with $w_\varepsilon =$ components of $\mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon)$ and $v_\varepsilon =$ components of $\mathbf{1}_{\{|\mathbf{u}_\varepsilon|>\eta_i\}}\mathbf{D}_\varepsilon^{1/2}(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon$. Owing to the decomposition (4.37), the bound (3.13) then shows that, for any $r < \frac{2d}{2d-1}$,

$$\mathbf{1}_{\{|\mathbf{u}_\varepsilon|>\eta_i\}}\mathbf{D}_\varepsilon(\cdot, \mathbf{u}_\varepsilon)\nabla c_\varepsilon \rightharpoonup \mathbf{D}_\circ(\cdot, \mathbf{u})\nabla_{\{|\mathbf{u}|>\eta_i\}} c \quad \text{weakly in } L^2(0,T;L^r(\Omega)^d). \tag{4.39}$$

In particular, this shows that

$$\left\|\mathbf{D}_\circ(\cdot,\mathbf{u})\nabla_{\{|\mathbf{u}|>\eta_i\}}c\right\|_{L^2(0,T;L^r(\Omega)^d)} = \left\|\mathbf{1}_{\{|\mathbf{u}|>\eta_i\}}\mathbf{D}_\circ(\cdot,\mathbf{u})\nabla_{\{|\mathbf{u}|>0\}}c\right\|_{L^2(0,T;L^r(\Omega)^d)} \le C_3.$$

Now $\mathbf{1}_{\{|\mathbf{u}|>\eta_i\}} \to \mathbf{1}_{\{|\mathbf{u}|>0\}}$ almost-everywhere as $i\to\infty$, so by the Fatou lemma (applied twice),

$$\begin{aligned}\int_0^T \left\|\mathbf{D}_\circ(\cdot,\mathbf{u}(\cdot,t))\nabla_{\{|\mathbf{u}|>0\}}c(\cdot,t)\right\|^2_{L^r(\Omega)^d}\,\mathrm{d}t &\le \int_0^T \liminf_{i\to\infty}\left\|\mathbf{1}_{\{|\mathbf{u}|>\eta_i\}}\mathbf{D}_\circ(\cdot,\mathbf{u}(\cdot,t))\nabla_{\{|\mathbf{u}|>0\}}c(\cdot,t)\right\|^2_{L^r(\Omega)^d}\,\mathrm{d}t\\ &\le \liminf_{i\to\infty}\int_0^T \left\|\mathbf{1}_{\{|\mathbf{u}|>\eta_i\}}\mathbf{D}_\circ(\cdot,\mathbf{u}(\cdot,t))\nabla_{\{|\mathbf{u}|>0\}}c(\cdot,t)\right\|^2_{L^r(\Omega)^d}\,\mathrm{d}t \le C_3. \quad \square\end{aligned}$$

Lemma 4.9. *Assume (2.1) and for $\varepsilon>0$ let $(p_\varepsilon,\mathbf{u}_\varepsilon,c_\varepsilon)$ be a solution-by-approximation to (3.6). Assume that (4.7)–(4.9) hold along a subsequence. Then along the same subsequence,*

$$\mathbf{D}_\varepsilon(\cdot,\mathbf{u}_\varepsilon)\nabla c_\varepsilon \rightharpoonup \mathbf{D}_\circ(\cdot,\mathbf{u})\nabla_{\{|\mathbf{u}|>0\}}c \quad \textit{weakly in } L^2(0,T;L^r(\Omega)^d) \textit{ for all } r<\tfrac{2d}{2d-1}. \tag{4.40}$$

Proof. Let $\boldsymbol{\psi}\in L^2(0,T;L^{r'}(\Omega)^d)$ and $i\in\mathbb{N}$. Write

$$\begin{aligned}&\int_0^T\int_\Omega \mathbf{D}_\varepsilon(x,\mathbf{u}_\varepsilon(x,t))\nabla c_\varepsilon(x,t)\cdot\boldsymbol{\psi}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad=\int_0^T\int_\Omega \mathbf{1}_{\{|\mathbf{u}_\varepsilon|>\eta_i\}}\mathbf{D}_\varepsilon(x,\mathbf{u}_\varepsilon(x,t))\nabla c_\varepsilon(x,t)\cdot\boldsymbol{\psi}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+\int_0^T\int_\Omega \mathbf{1}_{\{|\mathbf{u}_\varepsilon|\le\eta_i\}}\mathbf{D}_\varepsilon(x,\mathbf{u}_\varepsilon(x,t))\nabla c_\varepsilon(x,t)\cdot\boldsymbol{\psi}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad=T_1(i,\varepsilon)+T_2(i,\varepsilon),\end{aligned} \tag{4.41}$$

and

$$\begin{aligned}&\int_0^T\int_\Omega \mathbf{D}_\circ(x,\mathbf{u}(x,t))\nabla_{\{|\mathbf{u}|>0\}}c(x,t)\cdot\boldsymbol{\psi}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad=\int_0^T\int_\Omega \mathbf{1}_{\{|\mathbf{u}|>\eta_i\}}\mathbf{D}_\circ(x,\mathbf{u}(x,t))\nabla_{\{|\mathbf{u}|>0\}}c(x,t)\cdot\boldsymbol{\psi}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad+\int_0^T\int_\Omega \mathbf{1}_{\{0<|\mathbf{u}|\le\eta_i\}}\mathbf{D}_\circ(x,\mathbf{u}(x,t))\nabla_{\{|\mathbf{u}|>0\}}c(x,t)\cdot\boldsymbol{\psi}(x,t)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad=L_1(i)+L_2(i).\end{aligned} \tag{4.42}$$

Using (4.39) we obtain $\lim_{\varepsilon\to0}T_1(i,\varepsilon)=L_1(i)$. For $T_2(i,\varepsilon)$, use the estimate (3.11) on $\mathbf{D}_\varepsilon^{1/2}(\cdot,\mathbf{u}_\varepsilon)\nabla c_\varepsilon$ and the estimate (3.4) on $\mathbf{D}_\varepsilon^{1/2}$ to obtain

$$\begin{aligned}|T_2(i,\varepsilon)| &\le \int_0^T\int_\Omega \mathbf{1}_{\{|\mathbf{u}_\varepsilon|\le\eta_i\}}\left|\mathbf{D}_\varepsilon^{1/2}(x,\mathbf{u}_\varepsilon(x,t))\nabla c_\varepsilon(x,t)\cdot\mathbf{D}_\varepsilon^{1/2}(x,\mathbf{u}_\varepsilon(x,t))\boldsymbol{\psi}(x,t)\right|\mathrm{d}x\,\mathrm{d}t\\ &\le \left\|\mathbf{D}_\varepsilon^{1/2}(\cdot,\mathbf{u}_\varepsilon)\nabla c_\varepsilon\right\|_{L^2(0,T;L^2(\Omega)^d)}\left\|\mathbf{1}_{\{|\mathbf{u}_\varepsilon|\le\eta_i\}}\mathbf{D}_\varepsilon^{1/2}(\cdot,\mathbf{u}_\varepsilon)\boldsymbol{\psi}\right\|_{L^2(0,T;L^2(\Omega)^d)}\end{aligned}$$

$$\leq C_1 \phi_*^{-1/2} (\varepsilon + \max(d_l, d_t)\eta_i)^{1/2} \|\boldsymbol{\psi}\|_{L^2(0,T;L^2(\Omega)^d)}.$$

This shows that

$$\lim_{i\to\infty} \limsup_{\varepsilon\to 0} T_2(i,\varepsilon) = 0.$$

For $L_2(i)$, use (4.35) and write

$$\begin{aligned}
|L_2(i)| &\leq \int_0^T \int_\Omega \mathbf{1}_{\{0<|\mathbf{u}|\leq\eta_i\}} |\mathbf{D}_\circ(x,\mathbf{u})\nabla_{\{|\mathbf{u}|>0\}} c \cdot \boldsymbol{\psi}| \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \left\|\mathbf{D}_\circ(\cdot,\mathbf{u})\nabla_{\{|\mathbf{u}|>0\}} c\right\|_{L^2(0,T;L^r(\Omega)^d)} \left\|\mathbf{1}_{\{0<|\mathbf{u}|\leq\eta_i\}}\boldsymbol{\psi}\right\|_{L^2(0,T;L^{r'}(\Omega)^d)} \\
&\leq C_3 \left\|\mathbf{1}_{\{0<|\mathbf{u}|\leq\eta_i\}}\boldsymbol{\psi}\right\|_{L^2(0,T;L^{r'}(\Omega)^d)} \to 0 \quad \text{as } i\to\infty.
\end{aligned}$$

Then

$$\begin{aligned}
&\left| \int_0^T \int_\Omega \mathbf{D}_\varepsilon(x,\mathbf{u}_\varepsilon(x,t))\nabla c_\varepsilon(x,t)\cdot\boldsymbol{\psi}(x,t)\,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_\Omega \mathbf{D}_\circ(x,\mathbf{u}(x,t))\nabla_{\{|\mathbf{u}|>0\}} c(x,t)\cdot\boldsymbol{\psi}(x,t)\,\mathrm{d}x\,\mathrm{d}t \right| \\
&\quad = |T_1(i,\varepsilon) + T_2(i,\varepsilon) - (L_1(i) + L_2(i))| \\
&\quad \leq |T_1(i,\varepsilon) - L_1(i)| + |T_2(i,\varepsilon)| + |L_2(i)|.
\end{aligned}$$

Then taking (in this order) the limit superior as $\varepsilon\to 0$ and the limit as $i\to\infty$, we conclude that as $\varepsilon\to 0$

$$\int_0^T \int_\Omega \mathbf{D}_\varepsilon(x,\mathbf{u}_\varepsilon(x,t))\nabla c_\varepsilon(x,t)\cdot\boldsymbol{\psi}(x,t)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \mathbf{D}_\circ(x,\mathbf{u}(x,t))\nabla_{\{|\mathbf{u}|>0\}} c(x,t)\cdot\boldsymbol{\psi}(x,t)\,\mathrm{d}x\,\mathrm{d}t. \quad \square$$

The proof of Theorem 2.2 is now easy to complete. Equation (4.10), Corollary 4.7 and Lemma 4.9 enable us to take the limit of (3.6f), thus proving (2.2f). To prove the last two parts of (2.2a), that is, $c \in L^\infty(0,T;L^1(\Omega,\nu))$ and $0 \leq c(x,t) \leq 1$ for ν-almost-every $x\in\Omega$ and for almost-every $t\in(0,T)$, follow exactly the same argument as that employed by Fabrie–Gallouët [11, Lemma 5.1].

Remark 4.2. Note that we can use exactly the same method as in Lemmas 4.8 and 4.9 to show that

$\mathbf{D}_\varepsilon^{1/2}(\cdot,\mathbf{u}_\varepsilon)\nabla c_\varepsilon \rightharpoonup \mathbf{D}_\circ^{1/2}(\cdot,\mathbf{u})\nabla_{\{|\mathbf{u}|>0\}} c$ weakly in $L^2(0,T;L^2(\Omega)^d)$, and

$\mathbf{u}_\varepsilon\cdot\nabla c_\varepsilon \rightharpoonup \mathbf{u}\cdot\nabla_{\{|\mathbf{u}|>0\}} c$ weakly in $L^2(0,T;L^r(\Omega))$ for all $r < \frac{2d}{2d-1}$.

The latter is particularly relevant in the nonconservative formulation of (1.1b), in which a term of that form appears.

Conflict of interest statement

No conflict of interest.

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Appendix A. Properties of the concentration gradient

The results in this appendix attest to the consistency of Definition 2.1. Lemmas A.1 and A.2 give the necessary background for Proposition A.3, which makes precise the dependence of the $\{v>0\}$-gradient (Definition 2.1) on the sequences necessary to define it.

Lemma A.1. *Let* (Ω, Σ, μ) *be a* σ*-finite measure space and* $f : \Omega \to \mathbb{R}$ *be measurable. For almost every* $k \in \mathbb{R}$,

$$\mu(\{f = k\}) = 0. \tag{A.1}$$

Proof. We use the Fubini–Tonelli theorem to measure the graph $G = \{(x, f(x)) : x \in \Omega\}$ of f in $\Omega \times \mathbb{R}$. Denote by λ the Lebesgue measure on $\mathbb{R}$. For a given $k \in \mathbb{R}$, the slice G_k of G at k in the first direction is $G_k = \{x \in \Omega : f(x) = k\}$. For a fixed $x \in \Omega$, the slice G^x of G at x in the second direction is $G^x = \{f(x)\}$. By Fubini–Tonelli, we therefore have

$$\int_{\mathbb{R}} \mu(\{x \in \Omega : f(x) = k\}) d\lambda(k) = \int_{\Omega} \lambda(\{f(x)\}) d\mu(x).$$

Since $\lambda(\{f(x)\}) = 0$ for all $x \in \Omega$, this shows that $\int_{\mathbb{R}} \mu(\{x \in \Omega : f(x) = k\}) d\lambda(k) = 0$ and the conclusion follows. $\square$

Lemma A.2. *Let* (Ω, Σ, μ) *be a* σ*-finite measure space and for every* $\varepsilon > 0$ *let* $f_\varepsilon : \Omega \to \mathbb{R}$ *be measurable. Suppose there is a measurable function* $f : \Omega \to \mathbb{R}$ *such that* $f_\varepsilon \to f$ *almost-everywhere as* $\varepsilon \to 0$. *Then for every* $k \in \mathbb{R}$ *satisfying* (A.1),

$$\mathbf{1}_{\{f_\varepsilon > k\}} \to \mathbf{1}_{\{f > k\}} \quad \text{a.e. as } \varepsilon \to 0.$$

Proof. Take k such that $A = \{x \in \Omega : f(x) = k\}$ is null, and let B be the null set $\{x \in \Omega : f_\varepsilon(x) \not\to f(x)\}$. If $x \notin A \cup B$ we have either $f(x) > k$ or $f(x) < k$. In each respective case, for ε sufficiently small, $f_\varepsilon(x) > k$ (respectively $f_\varepsilon(x) < k$) and thus $\mathbf{1}_{\{f_\varepsilon > k\}} = \mathbf{1}_{\{f > k\}}$. $\square$

Proposition A.3. *Let* $f, v \in L^2(0, T; L^2(\Omega))$ *be such that* f *has a* $\{v > 0\}$*-gradient in the sense of Definition 2.1. Then*

(i) *The* $\{v > 0\}$*-gradient is independent of the choice of sequence* $(\eta_i)_{i \in \mathbb{N}}$.
(ii) $\nabla_{\{v > \eta_i\}} f = 0$ *on* $\{v \le \eta_i\}$.
(iii) *The* $\{v > 0\}$*-gradient is independent of the choice of sequence* $(v_\varepsilon)_{\varepsilon > 0}$.

Proof. Fix the sequences $(f_\varepsilon)_{\varepsilon > 0}$, $(v_\varepsilon)_{\varepsilon > 0}$ in Definition 2.1 and let $(\eta_i)_{i \in \mathbb{N}}$ and $(\zeta_i)_{i \in \mathbb{N}}$ be two sequences of real numbers such that for every $i \in \mathbb{N}$,

$$\text{meas}(\{v = \eta_i\}) = \text{meas}(\{v = \zeta_i\}) = 0.$$

Let $\chi_{\eta_i}, \chi_{\zeta_i} \in L^2(0, T; L^2(\Omega)^d)$ be such that as $\varepsilon \to 0$,

$$\mathbf{1}_{\{v_\varepsilon > \eta_i\}} \nabla f_\varepsilon \rightharpoonup \chi_{\eta_i} \quad \text{weakly in } L^2(0, T; L^2(\Omega)^d), \text{ and}$$
$$\mathbf{1}_{\{v_\varepsilon > \zeta_i\}} \nabla f_\varepsilon \rightharpoonup \chi_{\zeta_i} \quad \text{weakly in } L^2(0, T; L^2(\Omega)^d).$$

It suffices to show that for any $i \in \mathbb{N}$, $\chi_{\eta_i} = \chi_{\zeta_i}$ on $\{v > \eta_i\} \cap \{v > \zeta_i\}$. Without loss of generality, assume that $\eta_i > \zeta_i$ so that $\{v > \eta_i\} \cap \{v > \zeta_i\} = \{v > \eta_i\}$. We have

$$\mathbf{1}_{\{v_\varepsilon > \eta_i\}} \nabla f_\varepsilon = \mathbf{1}_{\{v_\varepsilon > \eta_i\}} \mathbf{1}_{\{v_\varepsilon > \zeta_i\}} \nabla f_\varepsilon. \tag{A.2}$$

Thanks to Lemma A.2, as $\varepsilon \to 0$,

$$\begin{aligned} \mathbf{1}_{\{v_\varepsilon > \eta_i\}} \nabla f_\varepsilon &\rightharpoonup \chi_{\eta_i} && \text{weakly in } L^2(0, T; L^2(\Omega)^d), \\ \mathbf{1}_{\{v_\varepsilon > \eta_i\}} &\to \mathbf{1}_{\{v > \eta_i\}} && \text{a.e. on } \Omega \times (0, T), \text{ and} \\ \mathbf{1}_{\{v_\varepsilon > \zeta_i\}} \nabla f_\varepsilon &\rightharpoonup \chi_{\zeta_i} && \text{weakly in } L^2(0, T; L^2(\Omega)^d). \end{aligned} \tag{A.3}$$

Passing to the weak limit in $L^2(0, T; L^2(\Omega)^d)$ on (A.2) shows that on $\{v > \eta_i\}$, $\chi_{\eta_i} = \chi_{\zeta_i}$ in $L^2(0, T; L^2(\Omega)^d)$, which proves (i).

For (ii), we have

$$\mathbf{1}_{\{v_\varepsilon>\eta_i\}}\nabla f_\varepsilon \rightharpoonup \nabla_{\{v>\eta_i\}} f \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d)\text{, and}$$
$$\mathbf{1}_{\{v_\varepsilon>\eta_i\}} \to \mathbf{1}_{\{v>\eta_i\}} \quad \text{in } L^2(0,T;L^2(\Omega)). \tag{A.4}$$

Then

$$\mathbf{1}_{\{v_\varepsilon>\eta_i\}}\nabla f_\varepsilon = \mathbf{1}_{\{v_\varepsilon>\eta_i\}}\mathbf{1}_{\{v_\varepsilon>\eta_i\}}\nabla f_\varepsilon \to \mathbf{1}_{\{v>\eta_i\}}\nabla_{\{v>\eta_i\}} f \text{ in } \mathcal{D}'(\Omega\times(0,T)). \tag{A.5}$$

Comparing (A.4) and (A.5), we see that $\nabla_{\{v>\eta_i\}} f = \mathbf{1}_{\{v>\eta_i\}}\nabla_{\{v>\eta_i\}} f$ in $\mathcal{D}'(\Omega\times(0,T))$, which shows that $\nabla_{\{v>\eta_i\}} f = 0$ on $\{v\le\eta_i\}$.

For (iii), fix the sequence $(f_\varepsilon)_{\varepsilon>0}$ and let $(v_\varepsilon)_{\varepsilon>0}$ and $(\bar{v}_\varepsilon)_{\varepsilon>0}$ be two sequences in $L^2(0,T;L^2(\Omega))$ such that as $\varepsilon\to 0$, $v_\varepsilon\to v$ and $\bar{v}_\varepsilon\to v$, both almost-everywhere $\Omega\times(0,T)$. Let $\eta_i>0$ be such that $\operatorname{meas}(\{v=\eta_i\})=0$ and suppose that there are functions $\chi_{\eta_i},\overline{\chi}_{\eta_i}\in L^2(0,T;L^2(\Omega)^d)$ such that, as $\varepsilon\to 0$,

$$\mathbf{1}_{\{v_\varepsilon>\eta_i\}}\nabla f_\varepsilon \rightharpoonup \chi_{\eta_i} \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d)\text{, and}$$
$$\mathbf{1}_{\{\bar{v}_\varepsilon>\eta_i\}}\nabla f_\varepsilon \rightharpoonup \overline{\chi}_{\eta_i} \quad \text{weakly in } L^2(0,T;L^2(\Omega)^d).$$

Observe that by Lemma A.2,

$$(\mathbf{1}_{\{v_\varepsilon>\eta_i\}}-\mathbf{1}_{\{\bar{v}_\varepsilon>\eta_i\}})\nabla f_\varepsilon = \mathbf{1}_{\{\bar{v}_\varepsilon\le\eta_i\}}(\mathbf{1}_{\{v_\varepsilon>\eta_i\}}\nabla f_\varepsilon) - \mathbf{1}_{\{v_\varepsilon\le\eta_i\}}(\mathbf{1}_{\{\bar{v}_\varepsilon>\eta_i\}}\nabla f_\varepsilon)$$
$$\rightharpoonup \mathbf{1}_{\{v\le\eta_i\}}\chi_{\eta_i} - \mathbf{1}_{\{v\le\eta_i\}}\overline{\chi}_{\eta_i} \text{ weakly in } L^2(0,T;L^2(\Omega)^d) \text{ as } \varepsilon\to 0.$$

By (ii), the last term vanishes, which shows that $\chi_{\eta_i}=\overline{\chi}_{\eta_i}$ in $L^2(0,T;L^2(\Omega)^d)$. □

Appendix B. Convergence lemmas

A similar result to the following appeared in Kazhikhov [15] with stronger assumptions. Here we give a proof of this “compensated compactness” lemma by following the ideas in the proof of Droniou–Eymard [8, Theorem 5.4].

Theorem B.1. *For $\psi\in C_c^\infty(\Omega)$ and $\gamma\in L^1(\Omega\times(0,T))$, define $F_\gamma^\psi\in L^1(0,T)$ by $F_\gamma^\psi(t)=\int_\Omega\gamma(x,t)\psi(x)\,\mathrm{d}x$. Let $a,p\in(1,\infty)$ and $(\alpha_\varepsilon)_{\varepsilon>0}$ and $(\beta_\varepsilon)_{\varepsilon>0}$ be sequences such that*

$$(\alpha_\varepsilon)_{\varepsilon>0} \text{ is bounded in } L^p(0,T;W^{1,a}(\Omega)),$$
$$\alpha_\varepsilon \rightharpoonup \alpha \text{ weakly in } L^p(0,T;L^a(\Omega)),$$
$$\beta_\varepsilon \rightharpoonup \beta \text{ weakly in } L^{p'}(0,T;L^{a'}(\Omega)) \text{ and}$$
$$\forall\psi\in C_c^\infty(\Omega)\,,\ (F_{\beta_\varepsilon}^\psi)_{\varepsilon>0} \text{ is bounded in } BV(0,T).$$

Then up to a subsequence,

$$\alpha_\varepsilon\beta_\varepsilon \rightharpoonup \alpha\beta \text{ in } \mathcal{D}'(\Omega\times(0,T)).$$

Proof. Let

$$\mathcal{A}(W,Z)=\int_0^T\int_\Omega W(x,t)Z(x,t)\,\mathrm{d}x\,\mathrm{d}t.$$

We prove that for every $\phi\in C_c^\infty(\Omega\times(0,T))$,

$$\mathcal{A}(\alpha_\varepsilon\phi,\beta_\varepsilon)\to\mathcal{A}(\alpha\phi,\beta). \tag{B.1}$$

In this proof, C denotes a generic constant that does not depend on ε.

Step 1: reduction to tensorial functions.

Consider a covering $(C_\ell^\delta)_{\ell=1,\dots,M}$ of Ω in cubes of length δ. For $g \in L^a(\mathbb{R}^d)$, define

$$R_\delta g = \sum_{\ell=1}^{M} \left(\frac{1}{\text{meas}(C_\ell^\delta)} \int_{C_\ell^\delta} g(x)\,\mathrm{d}x \right) \mathbf{1}_{C_\ell^\delta},$$

where $\mathbf{1}$ denotes the characteristic function. In what follows, take $g \in W_0^{1,a}(\Omega)$. Using Jensen's inequality, a linear change of variable, and a standard inequality for functions in $W_0^{1,a}(\Omega)$ (extended by 0 outside Ω),

$$\|R_\delta g - g\|_{L^a(\Omega)} \le C \sup_{z\in(-\delta,\delta)^d} \|g(\cdot + z) - g\|_{L^a(\mathbb{R}^d)} \le C\delta \|g\|_{W_0^{1,a}(\Omega)}.$$

The sequence of functions $(\alpha_\varepsilon\phi)_{\varepsilon>0}$ is bounded in $L^p(0,T;W_0^{1,a}(\Omega))$, the zero boundary condition coming from the support of ϕ. Hence

$$\|R_\delta(\alpha_\varepsilon\phi) - (\alpha_\varepsilon\phi)\|_{L^p(0,T;L^a(\Omega))} \le C\delta.$$

By the weak convergence of $(\alpha_\varepsilon)_{\varepsilon>0}$, this estimate also holds with α_ε replaced by α. Using the boundedness of $(\beta_\varepsilon)_{\varepsilon>0}$ in $L^{p'}(0,T;L^{a'}(\Omega))$ and the Hölder inequality,

$$|\mathcal{A}(\alpha_\varepsilon\phi,\beta_\varepsilon) - \mathcal{A}(\alpha\phi,\beta)| \le C\delta + |\mathcal{A}(R_\delta(\alpha_\varepsilon\phi),\beta_\varepsilon) - \mathcal{A}(R_\delta(\alpha\phi),\beta)|. \tag{B.2}$$

For a fixed δ, assume that we can prove that

$$\mathcal{A}(R_\delta(\alpha_\varepsilon\phi),\beta_\varepsilon) \to \mathcal{A}(R_\delta(\alpha\phi),\beta) \quad \text{as } \varepsilon \to 0. \tag{B.3}$$

Then taking the limit superior as $\varepsilon \to 0$ and then the limit as $\delta \to 0$ of (B.2) would show that (B.1) holds.

Step 2: reduction to smooth functions.

By construction of R_δ, we have

$$R_\delta(\alpha_\varepsilon\phi)(x,t) = \sum_{\ell=1}^{M} \xi_{\varepsilon,\ell}(t)\mathbf{1}_{C_\ell^\delta}(x), \quad \text{with} \quad \xi_{\varepsilon,l}(t) = \frac{1}{\text{meas}(C_\ell^\delta)} \int_{C_\ell^\delta} \alpha_\varepsilon(x,t)\phi(x,t)\,\mathrm{d}x.$$

Hence by the bilinearity of $\mathcal{A}$, (B.3) follows if we can establish that

$$\mathcal{A}(\xi_{\varepsilon,\ell}\otimes\mathbf{1}_{C_\ell^\delta},\beta_\varepsilon) \to \mathcal{A}(\xi_\ell\otimes\mathbf{1}_{C_\ell^\delta},\beta) \quad \text{as } \varepsilon \to 0, \tag{B.4}$$

where

$$\xi_\ell(t) = \frac{1}{\text{meas}(C_\ell^\delta)} \int_{C_\ell^\delta} \alpha(x,t)\phi(x,t)\,\mathrm{d}x.$$

Let $\psi \in C_c^\infty(\Omega)$. Using the bounds on $(\alpha_\varepsilon)_{\varepsilon>0}$ and $(\beta_\varepsilon)_{\varepsilon>0}$ we have

$$\left|\mathcal{A}(\xi_{\varepsilon,\ell}\otimes\mathbf{1}_{C_\ell^\delta},\beta_\varepsilon) - \mathcal{A}(\xi_{\varepsilon,\ell}\otimes\psi,\beta_\varepsilon)\right| \le C\left\|\mathbf{1}_{C_\ell^\delta} - \psi\right\|_{L^a(\Omega)}$$

where C may depend on ℓ and δ, but not on ε. A similar estimate holds with ξ_ℓ and β instead of $\xi_{\varepsilon,\ell}$ and β_ε. Since $\|\mathbf{1}_{C_\ell^\delta} - \psi\|_{L^a(\Omega)}$ can be made arbitrarily small by an appropriate choice of ψ, we see that (B.4) holds provided that, for any $\psi \in C_c^\infty(\Omega)$,

$$\mathcal{A}(\xi_{\varepsilon,\ell}\otimes\psi,\beta_\varepsilon) \to \mathcal{A}(\xi_\ell\otimes\psi,\beta) \quad \text{as } \varepsilon \to 0.$$

Step 3: conclusion.

We have

$$\mathcal{A}(\xi_{\varepsilon,\ell}\otimes\psi,\beta_\varepsilon) = \int_0^T \xi_{\varepsilon,\ell}(t)F_{\beta_\varepsilon}^{\psi}(t)\,\mathrm{d}t.$$

The weak convergence of $(\beta_\varepsilon)_{\varepsilon>0}$ ensures that $F^\psi_{\beta_\varepsilon} \to F^\psi_\beta$ in $\mathcal{D}'(0,T)$. Since $(F^\psi_{\beta_\varepsilon})_{\varepsilon>0}$ is bounded in $BV(0,T)$, this convergence also holds in $L^{p'}(0,T)$. On the other side, the weak convergence of $(\alpha_\varepsilon)_{\varepsilon>0}$ shows that $\xi_{\varepsilon,\ell} \to \xi_\ell$ weakly in $L^p(0,T)$. Hence as $\varepsilon \to 0$,

$$\mathcal{A}(\xi_{\varepsilon,\ell} \otimes \psi, \beta_\varepsilon) = \int_0^T \xi_{\varepsilon,\ell}(t) F^\psi_{\beta_\varepsilon}(t)\,\mathrm{d}t \to \int_0^T \xi_\ell(t) F^\psi_\beta(t)\,\mathrm{d}t = \mathcal{A}(\xi_\ell \otimes \psi, \beta). \quad \square$$

Lemma B.2. *Let E be a Banach space, $T > 0$, and $(f_m)_{m\in\mathbb{N}}$ be a bounded sequence in $L^1(0,T;E)$. Then for almost every $t \in (0,T)$, there exists a subsequence $(f_{m_k})_{k\in\mathbb{N}}$ such that $(f_{m_k}(t))_{k\in\mathbb{N}}$ is bounded in E.*

Proof. Let Z be the set of $t \in (0,T)$ such that no subsequence exists along which $\|f_m(t)\|_E$ is bounded. Then for every $t \in Z$, $\|f_m(t)\|_E \to \infty$. Using Fatou's lemma and denoting by M a bound of $(f_m)_{m\in\mathbb{N}}$ in $L^1(0,T;E)$ gives

$$\begin{aligned}\mathrm{meas}(Z) \times (\infty) &= \int_Z \liminf_{m\to\infty} \|f_m(t)\|_E \,\mathrm{d}t \\ &\le \int_0^T \liminf_{m\to\infty} \|f_m(t)\|_E \,\mathrm{d}t \le \liminf_{m\to\infty} \int_0^T \|f_m(t)\|_E \,\mathrm{d}t \le M.\end{aligned}$$

This shows that $\mathrm{meas}(Z) = 0$. $\square$

The following two lemmas are proved in [9].

Lemma B.3. *Let Ω be a bounded subset of $\mathbb{R}^d$, $d \in \mathbb{N}$, and for $\varepsilon > 0$ let $H_\varepsilon : \Omega \times \mathbb{R}^d \to \mathbb{R}$ be a Carathéodory function such that*

- *there exist positive constants C_{10}, γ such that for a.e. $x \in \Omega$,*

  $$|H_\varepsilon(x,\xi)| \le C_{10}(1 + |\xi|^\gamma) \quad \forall \xi \in \mathbb{R}^d,\ \forall \varepsilon > 0;$$

- *there is a Carathéodory function $H : \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that for a.e. $x \in \Omega$,*

  $$H_\varepsilon(x,\cdot) \to H(x,\cdot) \quad \text{uniformly on compact sets as } \varepsilon \to 0.$$

If $p, q \in [\max(1,\gamma), \infty)$ and $(u_\varepsilon)_{\varepsilon>0} \subset L^p(0,T;L^q(\Omega)^d)$ is a sequence with $u_\varepsilon \to u$ in $L^p(0,T;L^q(\Omega)^d)$ as $\varepsilon \to 0$, then $H_\varepsilon(\cdot,u_\varepsilon) \to H(\cdot,u)$ in $L^{p/\gamma}(0,T;L^{q/\gamma}(\Omega))$ as $\varepsilon \to 0$.

Lemma B.4. *Let Ω be a bounded, open subset of $\mathbb{R}^d$ and for $\varepsilon > 0$, let $w_\varepsilon : \Omega \times (0,T) \to \mathbb{R}$ and $v_\varepsilon : \Omega \times (0,T) \to \mathbb{R}$ be such that as $\varepsilon \to 0$,*

$$\begin{aligned}&w_\varepsilon \to w \quad \text{strongly in } L^{r_1}(0,T;L^{s_1}(\Omega)), \text{ and}\\ &v_\varepsilon \to v \quad \text{weakly in } L^{r_2}(0,T;L^{s_2}(\Omega)),\end{aligned}$$

where $r_1, r_2, s_1, s_2 \ge 1$ are such that $1/r_1 + 1/r_2 \le 1$ and $1/s_1 + 1/s_2 \le 1$. Suppose also that the sequence $(w_\varepsilon v_\varepsilon)_{\varepsilon>0}$ is bounded in $L^a(0,T;L^b(\Omega))$, where $a, b \in (1,\infty)$. Then $w_\varepsilon v_\varepsilon \to wv$ weakly in $L^a(0,T;L^b(\Omega))$.

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