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Quasistatic crack growth in 2d-linearized elasticity

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Abstract

In this paper we prove a two-dimensional existence result for a variational model of crack growth for brittle materials in the realm of linearized elasticity. Starting with a time-discretized version of the evolution driven by a prescribed boundary load, we derive a time-continuous quasistatic crack growth in the framework of generalized special functions of bounded deformation ($GSBD$). As the time-discretization step tends to zero, the major difficulty lies in showing the stability of the static equilibrium condition, which is achieved by means of a Jump Transfer Lemma generalizing the result of [19] to the $GSBD$ setting. Moreover, we present a general compactness theorem for this framework and prove existence of the evolution without imposing a-priori bounds on the displacements or applied body forces.

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1. Introduction

The mathematical foundations of the theory of brittle fracture were laid by the work of A. Griffith [26] in the 1920s. The fundamental idea is that the formation of cracks may be seen as the result of the competition between the elastic bulk energy of the body and the work needed to produce a new crack. This latter is modeled as a surface energy, which, in its simplest form, is proportional to the surface measure of the crack via a material constant, called the *toughness* of the material. The rigorous mathematical formulation of the problem, introduced in [23], requires that the function $t \to (u(t), \Gamma(t))$, associating to each time t a deformation $u(t)$ of the reference configuration and a crack set $\Gamma(t)$, is a quasistatic evolution satisfying the following conditions:

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- (a) static equilibrium: for every t the pair $(u(t), \Gamma(t))$ minimizes the energy at time t among all admissible competitors;
- (b) irreversibility: $\Gamma(s)$ is contained in $\Gamma(t)$ for $0 \le s < t$;
- (c) nondissipativity: the derivative of the internal energy equals the power of the applied forces.

Remarkable features of this approach are that it is able to show crack initiation, as well as a discontinuous evolution of the crack path, which *needs not to be a priori prescribed*. On the other hand, establishing a rigorous mathematical framework for the existence of a continuous-time evolution has proved to be quite a hard task.

1.1. Existence results for continuous-time evolution

The first breakthrough results in this direction are the ones in [15] and [9] tackling in a *planar setting* the case of anti-plane shear and linearized elasticity, respectively. The evolution is driven by a given prescribed load $g(t)$ on a Dirichlet part $\partial_D\Omega$ of the boundary of the reference configuration Ω. Namely, in the case considered in [9], the energy associated to a displacement u and a crack Γ is given by

$$\mathcal{E}(u,\Gamma) := \int_{\Omega\setminus\Gamma} Q(e(u))\,\mathrm{d}x + \mathcal{H}^1(\Gamma)\,, \tag{1}$$

where Q is a quadratic form acting on the symmetrized gradient $e(u)$. At each time t, the deformation $u(t)$, which fulfills the boundary condition $u(t) = g(t)$ on $\partial_D\Omega \setminus \Gamma(t)$ has to satisfy the minimality property

$$\mathcal{E}(u(t), \Gamma(t)) \le \mathcal{E}(v, \Gamma) \tag{2}$$

for all $\Gamma \supset \bigcup_{s<t} \Gamma(s)$ and all $v \in LD(\Omega \setminus \Gamma)$ with $v = g(t)$ on $\partial_D\Omega \setminus \Gamma$. Here LD is the space of displacements with square-integrable strains. The existence of an evolution is proved by following the natural idea, in the context of quasistatic brittle fracture, of starting with a time-discretized evolution, and then letting the time-step go to 0. Namely, for a given time step $\delta > 0$ and $n \in \mathbb{N}$, the pair $(u(n\delta), \Gamma(n\delta))$ is inductively defined as a solution for the problem

$$\arg\min \left\{ \int_{\Omega\setminus\Gamma} Q(e(u))\,\mathrm{d}x + \mathcal{H}^1(\Gamma) \right\} \tag{3}$$

among all cracks $\Gamma \supset \Gamma((n-1)\delta)$ and displacements $u \in LD(\Omega \setminus \Gamma)$ with $u = g(n\delta)$ on $\partial_D\Omega \setminus \Gamma$. Notice that the existence for the above minimum problems can be proved under the *additional restriction* that the admissible cracks have *at most a fixed number* of connected components. Indeed, in this case the direct method proves successful: crack sets are compact and lower semicontinuous with respect to the Hausdorff topology of sets via Gołab's Theorem (see [25]), while compactness of the displacements is recovered via the Poincaré–Korn inequality, upon noticing that the energy stays invariant under subtraction of rigid movements in the connected components of $\Omega \setminus \Gamma$ whose boundary has no intersection with $\partial_D\Omega \setminus \Gamma$.

The aforementioned important restriction plays furthermore a fundamental role in overcoming a stability issue, which arises when taking the limit for a time step δ going to 0. Indeed, if this hypothesis is dropped, the convergence in the Hausdorff metric of the approximating cracks $\Gamma_\delta(t)$ (obtained as piecewise constant interpolations of $\Gamma(n\delta)$, $n \in \mathbb{N}$) to a set $\Gamma(t)$ does not imply that piecewise constant interpolations of the time-discretized displacements $u_\delta(t)$ converge to a solution of the minimum problem (3). This issue, which is due to a *Neumann-sieve-type* phenomenon (see [30]), can be overcome in a planar setting imposing an a-priori bound on the connected components of the cracks and using some results from the analysis of Neumann problems in varying domains, contained in [7,11].

To avoid this restriction, a different and more powerful approach has been proposed in [19], and successfully applied to the case of *anti-plane shear* in arbitrary dimension N. In this case, the reference configuration is an infinite cylinder $\Omega \times \mathbb{R}$ with $\Omega \subset \mathbb{R}^N$ open and bounded, and admissible displacements are of the form $(0, \dots, 0, u(x))$ where x varies in Ω and the only nonzero component $u(x)$ is scalar-valued. In this case, the linear elastic energy reduces to the Dirichlet energy $\int_{\Omega\setminus\Gamma} |\nabla u|^2\,\mathrm{d}x$ and the incremental minimum problems become very similar to the strong formulation of the Mumford–Shah functional in image segmentation proposed in [29]. Inspired by De Giorgi's weak formulation in the space of special functions of bounded variation $SBV(\Omega)$ (see [16,17]), the authors model crack sets as (union of) jump sets of admissible displacements. The minimum problems to be solved at every time step essentially reduce (up to some modifications in order to allow for cracks running alongside the boundary) to

$$\arg\min\left\{\int_\Omega |\nabla u|^2\,\mathrm{d}x + \mathcal{H}^{N-1}(J_u \setminus \Gamma((n-1)\delta))\right\}, \tag{4}$$

with J_u denoting the jump set of u, among all displacement satisfying $u = g(n\delta)$ on $\partial_D\Omega$. Provided one assumes an L^∞ bound on the boundary datum, the maximum principle and Ambrosio's compactness theorem in SBV (see [1] and [2]) ensure well-posedness for the above problem. If u is a solution thereof, the crack set is then updated by setting $\Gamma(n\delta) := J_u \cup \Gamma((n-1)\delta)$.

A key tool introduced in the paper [19] in order to deal with the above mentioned stability issues, when the time step δ tends to 0, is the so-called *Jump Transfer Lemma*. It allows to transfer most of the jump set of any function in SBV that lies inside of the jump set of a function u onto that of u_n, if u_n is a sequence in SBV weakly converging to u. As a consequence of this lemma, the authors are able to recover a weak form of (2) in the limit. The existence result has been later generalized to finite hyperelastic energies and vector-valued deformations in [14], whereas the existence of a weak quasistatic evolution for the fully linear elastic model (1) has remained an open issue, due to at least two major difficulties.

1.2. Challenges for linear elastic models

As a first point, even in the static setting the existence of minimizers for the weak formulation is not clear. A natural attempt of generalizing (4) consists indeed in considering problems of the type

$$\arg\min\left\{\int_\Omega Q(e(u))\,\mathrm{d}x + \mathcal{H}^{N-1}(J_u \setminus \Gamma)\right\}, \tag{5}$$

under some prescribed boundary condition g, in the space SBD of special functions of bounded deformation (see [3,6]), for which a symmetrized gradient and an $\mathcal{H}^{N-1}$-rectifiable jump set are well defined. However, weak sequential compactness in SBD requires (see [6, Theorem 1.1]) a uniform bound on the L^∞ norm of the sequence, similarly to the SBV-case, which in this setting is not guaranteed along a minimizing sequence, due to the lack of a maximum principle. The addition of lower order terms, related for instance to the action of bulk forces, can at least provide some uniform bound on the L^p norm of the minimizing sequences, so that, mimicking a successful approach to similar problems in spaces of functions of bounded variation, one can recover an existence result in the space $GSBD$ of *generalized special functions of bounded deformation*. A correct definition of this space and the investigation of the related compactness and lower semicontinuity properties have proved to be a quite delicate issue, which has been overcome only recently in the paper [13]. On the other hand, it would be highly desirable to have an existence result also for the model (5) without the addition of lower order terms. This requires a suitable Korn-type inequality in $GSBD$ to be available, allowing in some sense to reproduce the steps of the existence proof for (3) in a weak setting.

The other major issue to be faced in order to give an existence proof of a quasistatic evolution with values in $GSBD$ is the generalization of the Jump Transfer Lemma to this setting. Actually, the proof strategy devised in [19] cannot be straightforwardly reproduced in this context. Indeed, there the jump set J_u is written as a countable union of pairwise intersections of level sets of u. The parts of the corresponding level sets for u_n lying outside J_{u_n} are then shown to have small length. With this, one can transfer onto pieces of these sets the jump $J_\phi \cap J_u$ for a given competitor ϕ. In this procedure, the *coarea formula* and the equiintegrability of ∇u_n play a crucial role. In the framework of linearized elasticity, however, only an a-priori control on the symmetrized gradient is available. Again, being able to estimate gradients in terms of their symmetrized part via a Korn-type inequality would remove parts of these obstacles and be a good starting point for proving an analog of the lemma in the $GSBD$ setting.

1.3. The present paper

This preliminary discussion leads us to the purpose of the present paper. Our goal is to provide an existence result, in dimension $N = 2$, for quasistatic crack growth in the sense of Griffith in a linearly elastic material. In Theorem 3.1

we show the existence of a pair $(u(t), \Gamma(t))$, with $u(t) \in GSBD^2(\Omega)$, $J_{u(t)} \subset \Gamma(t)$, and $\Gamma(t)$ nondecreasing in time, such that $u(t)$ minimizes

$$\int_\Omega Q(e(v))\,\mathrm{d}x + \mathcal{H}^1(J_v \setminus \Gamma(t))$$

among all $v \in GSBD^2(\Omega)$ satisfying the prescribed time-dependent Dirichlet condition $g(t)$, and the total energy satisfies the energy-dissipation balance

$$\mathcal{E}(u(t), \Gamma(t)) = \mathcal{E}(u(0), \Gamma(0)) + \int_0^t \int_\Omega \mathbb{C}e(u(s,x)) \cdot e(\dot{g}(s,x))\,\mathrm{d}s\,\mathrm{d}x\,.$$

In the above equality $\mathbb{C}$ is the elastic tensor generating the quadratic form Q, so that the integral term can be interpreted as the virtual work of the applied boundary load. We also mention that, as it is typical of variational problems in spaces of functions of bounded deformation, the boundary condition has to be understood in a relaxed sense (see Section 3 for details).

A starting point for our proof strategy is the use of a piecewise Korn inequality for $GSBD$ functions, proved in the planar setting in [22], extending other recent results in the literature ([20, Theorem 1.1] and [12, Theorem 1.2]). For every $1 \le p < 2$ it allows to control the L^p-norm of a displacement and its gradient in terms of the square norm of the symmetrized gradient, provided a suitable piecewise infinitesimal rigid motion is subtracted. With this construction the jump set is enlarged, but still controlled by the length of the original jump set.

A major ingredient is then a sharp version of the piecewise Korn inequality proved in Theorem 4.1. We show that the jump set can even only be enlarged by a small length at the prize of having only an L^1-control on the gradient. This control, however, involves constants which behave well with respect to scaling and particularly are small on small squares (see Remark 4.2).

Equipped with this result, we can prove Theorem 5.5, where, up to an arbitrarily small error θ, the jump set of a weakly compact sequence $(u_n)_n$ in $GSBD$ is shown to coincide with the one of a sequence $(v_n)_n$ of SBV functions, still L^1-converging to u up to some small exceptional sets. Furthermore, the L^1-norm of ∇v_n is uniformly small in a tubular neighborhood of the jump set J_u. Notice that the construction of v_n is quite involved and depends on the given covering of J_u (see Section 5 for details).

This allows to prove a Jump Transfer Lemma also in this setting (Theorem 5.1), adapting the arguments of [19, Theorem 2.1]. The reflection procedure that the authors use there in order to define the sequence $(\phi_n)_n$ corresponding to the competitor ϕ, which is not compatible with a control only on the symmetrized gradient $e(\phi)$, is here replaced by a suitable generalization introduced in [31] and adjusted to our purposes in Lemma 5.2.

The existence proof for the minimum problem (5) requires an additional step, namely a version of the sharp piecewise Korn inequality proved in Theorem 4.1 which also takes into account the relaxed boundary conditions. This is proved in Theorem 4.5. With this, we can derive a general compactness result for minimizing sequences of the energy (5) drawing some ideas from [20]: while typically sequences are not compact, it is always possible to pass to modifications by subtracting suitable piecewise infinitesimal rigid motions (which do not change the elastic part of the energy) at the expense of arbitrarily small additional fracture energy. This allows us to construct a minimizing sequence $(y_n)_n$ which satisfies the uniform bound

$$\int_\Omega \psi(|y_n|)\mathrm{d}x + \int_\Omega |e(y_n)|^2 \mathrm{d}x + \mathcal{H}^1(J_{y_n}) \le M$$

for an increasing, continuous functions $\psi : [0,\infty) \to [0,\infty)$ with $\psi(s) \to \infty$ as $s \to \infty$. This bound, in general weaker than any L^p-bound, is enough to apply the compactness result in [13, Theorem 11.3] deducing the existence of a minimizer (see Theorem 6.1 and Theorem 6.2 below). An additional delicate point of the proof is showing that the function ψ is only depending on the reference configuration Ω and the H^1 norm of the boundary displacement $g(t)$, so that, under the usual regularity assumptions on the boundary load, it is independent from the time t along an evolution. This is crucial in the proof of Theorem 7.5 where the global stability property is derived.

Once this two major hurdles have been fixed, the by now well-known machinery successfully exploited in [19] and in [14], in the linear antiplane and in the finite elastic context, respectively, can be adapted to our setting with minor modifications, which we however detail to some extent in Section 7. This leads to the proof of our main result stated in Theorem 3.1.

As already mentioned, we establish the result only in two dimensions as we make a heavy use of the piecewise Korn inequality of [22] which has been only derived in a planar setting due to technical difficulties, concerning the topological structure of crack geometries in higher dimensions. Additionally, also its generalization to the sharp version (Theorem 4.1) and the case of prescribed boundary conditions (Theorem 4.5) makes use of estimates holding in a planar setting (see Lemma 2.3, Lemma 4.4, and Lemma 4.6). Without these restrictions, the methods we use actually hold in any dimension. We therefore believe that our results can be extended to the N-dimensional case and that the proof provides the principal techniques being necessary to establish the result in arbitrary space dimension.

2. Preliminaries

In this section we introduce basic definitions and the function spaces which we will use in the paper. Moreover, we recall a piecewise Korn inequality for $GSBD$ functions proved in [22].

2.1. Basic definitions

For a bounded, measurable set $E \subset \mathbb{R}^N$ we define

$$\mathrm{diam}(E) = \operatorname{ess\,sup}\{|x-y| : x, y \in E\}.$$

The above definition is independent of the particular Lebesgue representative. If U is an open set in $\mathbb{R}^N$, and $u : U \to \mathbb{R}^m$ is a $\mathcal{L}^N$-measurable function, u is said to have an approximate limit $a \in \mathbb{R}^m$ at a point $x \in U$ if and only if

$$\lim_{\varrho \to 0^+} \frac{\mathcal{L}^N\left(\{|u-a| \geq \varepsilon\} \cap B_\varrho(x)\right)}{\varrho^N} = 0 \text{ for every } \varepsilon > 0\,,$$

where $B_\varrho(x)$ is the ball of radius ϱ centered at x. In this case, one writes $\mathrm{ap}\lim_{y\to x} u(y) = a$. The approximate jump set J_u is defined as the set of points $x \in U$ such that there exist $a \neq b \in \mathbb{R}^m$ and $\nu \in S^{N-1} := \{\xi \in \mathbb{R}^N : |\xi| = 1\}$ with

$$\mathrm{ap} \lim_{\substack{y\to x \\ (y-x)\cdot\nu>0}} u(y) = a\,, \quad \mathrm{ap} \lim_{\substack{y\to x \\ (y-x)\cdot\nu<0}} u(y) = b\,.$$

The triplet (a, b, ν) is uniquely determined up to a permutation of (a, b) and a change of sign of ν, and is denoted by $(u^+(x), u^-(x), \nu_u(x))$. The jump of u is the function $[u] : J_u \to \mathbb{R}^m$ defined by $[u](x) := u^+(x) - u^-(x)$ for every $x \in J_u$. It follows from Lusin's Theorem that u has $u(x)$ as approximate limit at $\mathcal{L}^N$-a.e. $x \in U$, in which case one says that u is approximately continuous at x, and therefore J_u is a $\mathcal{L}^N$-null set. Given $x \in U$ such that u is approximately continuous at x, an $m \times N$ matrix $\nabla u(x)$ is said to be an approximate gradient of u at x if and only if

$$\mathrm{ap}\lim_{y\to x} \frac{u(y) - u(x) - \nabla u(x)(y-x)}{|y-x|} = 0\,.$$

We say that u has an approximate symmetric differential $e(u)(x) \in \mathbb{R}^{N\times N}_{\mathrm{sym}}$ at x if

$$\mathrm{ap}\lim_{y\to x} \frac{(u(y) - u(x) - e(u)(x)(y-x)) \cdot (y-x)}{|y-x|^2} = 0\,.$$

We will make use of the following measure-theoretical result from [20]. A short proof is reported for the reader's convenience.

Lemma 2.1. *Let $F \subset \mathbb{R}^N$ with $\mathcal{L}^N(F) < +\infty$ and let $(s_n)_n$, $(t_n)_n$ be nonnegative, monotone sequences with $s_n \to \infty$ and $t_n \to 0$ as $n \to \infty$. Then there is a nonnegative, increasing, concave function ψ with*

$$\lim_{s\to+\infty} \psi(s) = +\infty \tag{6}$$

only depending on F, $(s_n)_n$, $(t_n)_n$ such that for every sequence $(u_n)_n \subset L^1(F;\mathbb{R}^m)$ with

$$\|u_n\|_{L^1(F)} \leq s_n\,, \quad \mathcal{L}^N\left(\bigcap\nolimits_{m\geq n} \{|u_m - u_n| \geq 1\}\right) \leq t_n$$

for all $n \in \mathbb{N}$ there is a not relabeled subsequence such that

$$\sup_{n\geq 1}\int_F \psi(|u_n|)\mathrm{d}x \leq 1\,.$$

Proof. Let $A_n = \bigcap_{m\geq n}\{|u_n - u_m| \leq 1\}$ and set $B_1 = A_1$ as well as $B_n = A_n \setminus \bigcup_{m=1}^{n-1} B_m$ for all $n \in \mathbb{N}$. The sets $(B_n)_n$ are pairwise disjoint with $\sum_n \mathcal{L}^N(B_n) = \mathcal{L}^N(F)$. We choose $0 = n_1 < n_2 < \ldots$ such that $\sum_{1\leq n\leq n_i} \frac{\mathcal{L}^N(B_n)}{\mathcal{L}^N(F)} \geq 1 - 4^{-i}$. We let $B^i = \bigcup_{n=n_i+1}^{n_{i+1}} B_n$ and observe $\mathcal{L}^N(B^i) \leq 4^{-i}\mathcal{L}^N(F)$.

From now on we consider the subsequence $(n_i)_{i\in\mathbb{N}}$ and observe that the choice of $(n_i)_{i\in\mathbb{N}}$ only depends on the sequence $(t_n)_n$. Choose $E^i \supset B^i$ such that $\mathcal{L}^N(E^i) = 4^{-i}\mathcal{L}^N(F)$. Let $b_i = \frac{s_{n_{i+1}}}{\mathcal{L}^N(E^i)} + 2 = 4^i \frac{s_{n_{i+1}}}{\mathcal{L}^N(F)} + 2$ for $i \in \mathbb{N}$ and note that $(b_i)_i$ is increasing with $b_i \to \infty$. By an elementary construction (see [20, Lemma 4.1]) we find an increasing concave function $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{s\to\infty}\psi(s) = \infty$ and $\psi(b_i) \leq \frac{2^i}{\mathcal{L}^N(F)}$ for all $i \in \mathbb{N}$.

For $\hat{B}^i := \Omega \setminus \bigcup_{n=1}^{n_i} B_n$ we have $\mathcal{L}^N(\hat{B}^i) \leq 4^{-i}\mathcal{L}^N(F)$ and choose $\hat{E}^i \supset \hat{B}^i$ with $\mathcal{L}^N(\hat{E}^i) = 4^{-i}\mathcal{L}^N(F)$. We then obtain $\frac{s_{n_i}}{\mathcal{L}^N(\hat{E}^i)} = 4^i \frac{s_{n_i}}{\mathcal{L}^N(F)} \leq b_i$. Now let $l = n_i$. Using Jensen's inequality, the definition of the sets B^i, $\|u_l\|_1 \leq s_l$ and the monotonicity of ψ we compute

$$\begin{aligned}
\int_F \psi(|u_l|) &= \sum\nolimits_{1\leq j\leq i-1} \int_{B^j} \psi(|u_l|)\mathrm{d}x + \int_{\hat{B}^i} \psi(|u_l|)\mathrm{d}x \\
&\leq \sum\nolimits_{1\leq j\leq i-1} \int_{B^j} \psi(|u_{n_{j+1}}| + 2)\mathrm{d}x + \int_{\hat{B}^i} \psi(|u_l|)\mathrm{d}x \\
&\leq \sum\nolimits_{1\leq j\leq i-1} \mathcal{L}^N(E^j)\psi\Big(\fint_{E^j} |u_{n_{j+1}}| + 2\Big) + \mathcal{L}^N(\hat{E}^i)\psi\Big(\fint_{\hat{E}^i} |u_l|\Big) \\
&\leq \sum\nolimits_{1\leq j\leq i-1} \mathcal{L}^N(F)4^{-j}\frac{2^j}{\mathcal{L}^N(F)} + \mathcal{L}^N(F)4^{-i}\frac{2^i}{\mathcal{L}^N(F)} \leq \sum\nolimits_{j\in\mathbb{N}} 2^{-j} = 1.
\end{aligned}$$

As the estimate is independent of $l \in (n_i)_i$, this yields $\int_F \psi(|u_l|)\mathrm{d}x \leq 1$ uniformly in l, as desired. □

Remark 2.2. Let u be a measurable function and $(u_n)_n \subset L^1(F;\mathbb{R}^m)$ a sequence such that $u_n \to u$ in measure. Then it follows from the previous lemma that there exist a subsequence $(u_{n_k})_k$ of $(u_n)_n$ and a nonnegative, increasing, concave function ψ satisfying (6), such that

$$\sup_{k\geq 1}\int_F \psi(|u_{n_k}|)\mathrm{d}x \leq 1\,.$$

Indeed, by definition of convergence in measure we can always find a subsequence $(u_{n_k})_k$ with the property that, setting $E_k := \{|u_{n_k} - u| \geq \frac{1}{2^k}\}$, one has $\mathcal{L}^N(E_k) \leq \frac{1}{2^k}$. Now, for all $k \in \mathbb{N}$ we have by the triangle inequality that

$$\bigcup_{m\geq k}\{|u_{n_m} - u_{n_k}| \geq 1\} \subseteq \bigcup_{m\geq k} E_m$$

and therefore

$$\mathcal{L}^N\Big(\bigcup\nolimits_{m\geq k}\{|u_{n_m} - u_{n_k}| \geq 1\}\Big) \leq \sum\nolimits_{m=k}^{+\infty}\frac{1}{2^m} = \frac{1}{2^{k-1}}\,.$$

Now it suffices to apply the previous lemma with $s_k := \max\{\max_{1\leq i\leq k}\|u_{n_i}\|_{L^1(F)}, k\}$ and $t_k := \frac{1}{2^{k-1}}$.

In a two-dimensional setting, we will often make use of the following simple lemma.

Lemma 2.3. *Let $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$, $b \in \mathbb{R}^2$.*

(a) There is a universal constant $c > 0$ independent of A and b such that for all measurable $E \subset \mathbb{R}^2$ we have $(\mathcal{L}^2(E))^{\frac{1}{2}}|A| \leq c\|A\cdot + b\|_{L^\infty(E;\mathbb{R}^2)}$.

(b) Let F be a bounded measurable subset of $\mathbb{R}^2$, $\delta > 0$ and let a continuous nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying (6) be given. Consider a measurable subset $E \subset F$ with $\mathcal{L}^2(E) \geq \delta$. Then, if

$$M \geq \int_E \psi(|Ax + b|)\,\mathrm{d}x\,,$$

there exists a constant C only depending on M, δ, ψ, and F such that

$$|A| + |b| \leq C\,. \tag{7}$$

If $\psi(s) = s^p$ for $p \in [1, \infty)$ we get $|A| + |b| \leq \tilde{C} M^{\frac{1}{p}}$ for a constant $\tilde{C}$ only depending on δ, p and F.

Proof. (a) It suffices to consider the case $A \neq 0$. If $A \neq 0$, the assumption $A \in \mathbb{R}^{2\times 2}_{\text{skew}}$ implies that A is invertible and that $|Ay| = \frac{\sqrt{2}}{2}|A||y|$ for all $y \in \mathbb{R}^2$. We notice that for all $z \in \mathbb{R}^2$ there exists $x \in E$ with $|x - z| \geq \frac{1}{4}\mathrm{diam}(E)$. For the special choice $z = -A^{-1}b$ we obtain $|A\,x + b| = |A\,(x - z)| = \frac{\sqrt{2}}{2}|A||x - z| \geq \frac{\sqrt{2}}{8}|A|\mathrm{diam}(E)$ which implies the result due to the isodiametric inequality.

(b) If $A = 0$, we have

$$\frac{M}{\delta} \geq \psi(|b|)$$

and the result follows from (6). If $A \neq 0$, we set $z := -A^{-1}b$ and $\lambda := \sqrt{\frac{\delta}{2\pi}}$. Then we have that $\mathcal{L}^2(E \setminus B_\lambda(z)) \geq \frac{\delta}{2}$. Since ψ is nonnegative and increasing, we get

$$\begin{aligned} M &\geq \int_E \psi\left(\frac{\sqrt{2}}{2}|A||x - z|\right)\mathrm{d}x \\ &\geq \int_{E\setminus B(z,\lambda)} \psi\left(\frac{\sqrt{2}}{2}|A||x - z|\right)\mathrm{d}x \geq \frac{\delta}{2}\psi\left(\frac{\sqrt{2}}{2}|A|\lambda\right). \end{aligned}$$

By this and (6) it exists a constant $\hat{C}$ only depending on M, δ, and ψ such that

$$|A| \leq \hat{C}. \tag{8}$$

It also follows that $|Ax| \leq C'$ for all $x \in F$, where C' is allowed to depend on F, too. If now $|b| \leq C'$ we are done, otherwise it holds $|Ax + b| \geq |b| - C' > 0$ for all $x \in F$. The monotonicty of ψ yields then

$$\frac{M}{\delta} \geq \psi(|b| - C')$$

and again (6) implies the conclusion. The case $\psi(s) = s^p$ may be proved along similar lines taking into account that (8) can be replaced by $|A| \leq \tilde{C} M^{\frac{1}{p}}$ for $\tilde{C}$ independent of M. □

2.2. Function spaces

In the whole paper we use standard notations for the spaces SBV and SBD. We refer the reader to [4] and [3,6,32], respectively, for definitions and basic properties. In this section we only give the definition and some properties of *generalized functions of bounded deformation* introduced in [13], being the setting of our existence result. For fixed $\xi \in S^{N-1}$, we set

$$\Pi^\xi := \{y \in \mathbb{R}^N : y \cdot \xi = 0\}\,, \qquad U^\xi_y := \{t \in \mathbb{R} : y + t\xi \in U\} \text{ for } y \in \Pi^\xi.$$

Definition 2.4. An $\mathcal{L}^N$-measurable function $u : U \to \mathbb{R}^N$ belongs to $GBD(U)$ if there exists a positive bounded Radon measure λ_u such that, for all $\tau \in C^1(\mathbb{R}^N)$ with $-\frac{1}{2} \leq \tau \leq \frac{1}{2}$ and $0 \leq \tau' \leq 1$, and all $\xi \in S^{N-1}$, the distributional derivative $D_\xi(\tau(u \cdot \xi))$ is a bounded Radon measure on U whose total variation satisfies

$$\left|D_\xi(\tau(u\cdot\xi))\right|(B) \le \lambda_u(B)$$

for every Borel subset B of U. A function $u \in GBD(U)$ belongs to the subset $GSBD(U)$ of special functions of bounded deformation if in addition for every $\xi \in S^{N-1}$ and $\mathcal{H}^{N-1}$-a.e. $y \in \Pi^\xi$, the function $u^\xi_y(t) := u(y+t\xi)$ belongs to $SBV_{\rm loc}(U^\xi_y)$.

By [13, Remark 4.5] one has the inclusions $BD(U) \subset GBD(U)$ and $SBD(U) \subset GSBD(U)$, which are in general strict. Some relevant properties of functions with bounded deformation can be generalized to this weak setting: in particular, in [13, Theorem 6.2 and Theorem 9.1] it is shown that the jump set J_u of a GBD-function is $\mathcal{H}^{N-1}$-rectifiable and that GBD-functions have an approximate symmetric differential $e(u)(x)$ at $\mathcal{L}^N$-a.e. $x \in U$, respectively. The space $GSBD^2(U)$ is defined through:

$$GSBD^2(U) := \{u \in GSBD(U) : e(u) \in L^2(U;\mathbb{R}^{N\times N}_{\rm sym}),\ \mathcal{H}^{N-1}(J_u) < +\infty\}.$$

Furthermore, the following compactness theorem has been proved in [13], which we slightly adapt for our purposes.

Theorem 2.5. *Let Γ be a measurable set with $\mathcal{H}^{N-1}(\Gamma) < +\infty$. Let $(y_k)_k$ be a sequence in $GSBD^2(U)$. Suppose that there exist a constant $M > 0$ and an increasing continuous functions $\psi : [0,\infty) \to [0,\infty)$ with $\lim_{s\to\infty}\psi(s) = +\infty$ such that*

$$\int_U \psi(|y_k|)\mathrm{d}x + \int_U |e(y_k)|^2\mathrm{d}x + \mathcal{H}^{N-1}(J_{y_k}) \le M$$

for every $k \in \mathbb{N}$. Then there exist a subsequence, still denoted by $(y_k)_k$, and a function $y \in GSBD^2(U)$ such that

$$\begin{aligned} &y_k \to y \quad \text{in measure in} \quad U,\\ &e(y_k) \rightharpoonup e(y) \quad \text{weakly in } L^2(U;\mathbb{R}^{N\times N}_{\rm sym}),\\ &\mathcal{H}^{N-1}(J_y\setminus\Gamma) \le \liminf_{k\to\infty}\mathcal{H}^{N-1}(J_{y_k}\setminus\Gamma). \end{aligned} \tag{9}$$

Proof. In [13] the assertion has been proved in the case $\Gamma = \emptyset$. We briefly indicate the necessary adaption for the derivation of (9)(iii) following the argumentation in [14, Theorem 2.8]. If Γ is compact, it suffices to replace Ω by $\Omega\setminus\Gamma$. In the general case let $K \subset \Gamma$ compact with $\mathcal{H}^1(\Gamma\setminus K) \le \varepsilon$. Since $J_y\setminus\Gamma \subset J_y\setminus K$ and $J_{y_k}\setminus K \subset (J_{y_k}\setminus\Gamma)\cup(\Gamma\setminus K)$ we have

$$\begin{aligned} \mathcal{H}^1(J_y\setminus\Gamma) &\le \mathcal{H}^1(J_y\setminus K) \le \liminf_{k\to\infty}\mathcal{H}^1(J_{y_k}\setminus K)\\ &\le \liminf_{k\to\infty}\mathcal{H}^1(J_{y_k}\setminus\Gamma) + \mathcal{H}^1(\Gamma\setminus K) \le \liminf_{k\to\infty}\mathcal{H}^1(J_{y_k}\setminus\Gamma) + \varepsilon. \end{aligned}$$

We conclude by letting $\varepsilon \to 0$. □

We now define a class of displacements with regular jump set. We say that $u \in L^1(U;\mathbb{R}^N)$ is a displacements with regular jump set if the following properties are satisfied

$$\begin{aligned} &(i)\quad u \in SBV^2(U;\mathbb{R}^N),\\ &(ii)\quad J_u = \bigcup_{k=1}^m \Sigma_k, \quad \Sigma_k \text{ closed connected pieces of } C^1\text{-hypersurfaces},\\ &(iii)\quad u \in H^1(U\setminus J_u;\mathbb{R}^N). \end{aligned} \tag{10}$$

Displacements with regular jump set are dense in $GSBD^2(U)\cap L^2(U;\mathbb{R}^N)$ in the sense given by the following statement, proved in [27] (cf. also [10, Theorem 3, Remark 5.3])).

Theorem 2.6. *Let $U \subset \mathbb{R}^N$ open, bounded with Lipschitz boundary. Let $u \in GSBD^2(U)\cap L^2(U;\mathbb{R}^N)$. Then there exists a sequence $(u_k)_k$ of displacements with regular jump set so that*

$(i)\quad \|u_k - u\|_{L^2(\Omega;\mathbb{R}^N)} \to 0$

$(ii)\quad \|e(u_k) - e(u)\|_{L^2(\Omega;\mathbb{R}^{N\times N}_{\mathrm{sym}})} \to 0,$

$(iii)\quad \mathcal{H}^{N-1}(J_{u_k} \triangle J_u) \to 0.$

2.3. Caccioppoli partitions

We say that a partition $\mathcal{P} = (P_j)_j$ of an open set $U \subset \mathbb{R}^N$ is a *Caccioppoli partition* of U if

$$\sum_j \mathcal{H}^1(\partial^* P_j) < +\infty,$$

where $\partial^* P_j$ denotes the *essential boundary* of P_j (see [4, Definition 3.60]). We say a partition is *ordered* if $\mathcal{L}^N(P_i) \geq \mathcal{L}^N(P_j)$ for $i \leq j$. In the whole article, when dealing with infinite partitions, we will always tacitly assume that they are ordered. Moreover, we say that a set of finite perimeter P_j is *indecomposable* if it cannot be written as $P^1 \cup P^2$ with $P^1 \cap P^2 = \emptyset$, $\mathcal{L}^N(P^1), \mathcal{L}^N(P^2) > 0$ and $\mathcal{H}^{N-1}(\partial^* P_j) = \mathcal{H}^{N-1}(\partial^* P^1) + \mathcal{H}^{N-1}(\partial^* P^2)$. The local structure of Caccioppoli partitions can be characterized as follows (see [4, Theorem 4.17]).

Theorem 2.7. *Let $(P_j)_j$ be a Caccioppoli partition of U. Then*

$$\bigcup_j (P_j)^1 \cup \bigcup_{i\neq j} (\partial^* P_i \cap \partial^* P_j)$$

contains $\mathcal{H}^{N-1}$-almost all of U.

Here $(P)^1$ denote the points where P has density one (see again [4, Definition 3.60]). Essentially, the theorem states that $\mathcal{H}^{N-1}$-a.e. point of U either belongs to exactly one element of the partition or to the intersection of exactly two sets $\partial^* P_i$, $\partial^* P_j$. We now state a compactness result for ordered Caccioppoli partitions (see [4, Theorem 4.19, Remark 4.20]) slightly adapted for our purposes.

Theorem 2.8. *Let $U \subset \mathbb{R}^N$ open, bounded with Lipschitz boundary. Let $\mathcal{P}_i = (P_{j,i})_j$, $i \in \mathbb{N}$, be a sequence of ordered Caccioppoli partitions of U with*

$$\sup_{i\geq 1} \sum_{j\geq 1} \mathcal{H}^{N-1}(\partial^* P_{j,i}) < +\infty.$$

Then there exists a Caccioppoli partition $\mathcal{P} = (P_j)_j$ and a not relabeled subsequence such that $\sum_{j\geq 1} \mathcal{L}^N(P_{j,i} \triangle P_j) \to 0$ as $i \to \infty$.

Proof. In [4] it was proved that $P_{j,i} \to P_j$ in measure for all $j \in \mathbb{N}$ as $i \to \infty$. We briefly show that this already implies $\sum_j \mathcal{L}^N(P_{j,i} \triangle P_j) \to 0$ as $i \to \infty$. Let $\varepsilon > 0$ and choose $j_0 \in \mathbb{N}$ sufficiently large such that $\sum_{j<j_0} \mathcal{L}^N(P_j) \geq \mathcal{L}^N(U) - \varepsilon$. Then the convergence in measure implies that for i_0 large enough depending on j_0 we have $\sum_{j<j_0} \mathcal{L}^N(P_{j,i} \triangle P_j) \leq \varepsilon$ for all $i \geq i_0$. Moreover, this overlapping property and the choice j_0 imply $\sum_{j\geq j_0} \mathcal{L}^N(P_{j,i}) \leq 2\varepsilon$ for $i \geq i_0$. Consequently, we find $\sum_j \mathcal{L}^N(P_{j,i} \triangle P_j) \leq 4\varepsilon$ for $i \geq i_0$. As $\varepsilon > 0$ was arbitrary, the assertion follows. □

2.4. Piecewise Korn inequality in GSBD

In this section we recall a piecewise Korn inequality for $GSBD$ functions, proved in the planar setting in [22] (cf. also [12] and [21] for previous results) and being one of the major ingredients of our proofs. It implies in particular a density result Theorem 2.10 which in the planar case improves upon Theorem 2.6. Here and henceforth we will call an affine mapping of the form $a_{A,b}(x) := Ax + b$ with $A \in \mathbb{R}^{2\times 2}_{\mathrm{skew}}$ and $b \in \mathbb{R}^2$ an *infinitesimal rigid motion*.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Let $p \in [1, 2)$. Then there is a constant $c = c(p) > 0$ and $C_{\mathrm{korn}} = C_{\mathrm{korn}}(p, \Omega) > 0$ such that for each $u \in GSBD^2(\Omega)$ there is a Caccioppoli partition $\Omega = \bigcup_{j=1}^{\infty} P_j$ and corresponding infinitesimal rigid motions $(a_j)_j = (a_{A_j,b_j})_j$ such that*

$$v := u - \sum_{j=1}^{\infty} a_j \chi_{P_j} \in SBV^p(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$$

and

$$\begin{aligned} &(i) \quad \sum_{j=1}^{\infty} \mathcal{H}^1(\partial^* P_j) \le c(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega)), \\ &(ii) \quad \|v\|_{L^\infty(\Omega;\mathbb{R}^2)} \le C_{\rm korn}\|e(u)\|_{L^2(\Omega;\mathbb{R}^{2\times2}_{\rm sym})}, \\ &(iii) \quad \|\nabla v\|_{L^p(\Omega;\mathbb{R}^{2\times2})} \le C_{\rm korn}\|e(u)\|_{L^2(\Omega;\mathbb{R}^{2\times2}_{\rm sym})}. \end{aligned} \tag{11}$$

Below in Section 4 we prove a refined version of Theorem 2.9 which (a) provides a sharp estimate for the boundary of the partition in (11)(i) and (b) takes into account boundary data. This refined result will then be fundamental in proving the jump transfer lemma and the existence theorem for the time-incremental minimum problems.

Applying the above result, approximating u by the sequence $v_n := u - \sum_{j=n+1}^{\infty} a_j \chi_{P_j} \in GSBD^2(\Omega) \cap L^\infty(\Omega;\mathbb{R}^2)$, and using Theorem 2.6, we obtain the following density result for $GSBD$ functions (see again [22]).

Theorem 2.10. *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Let $u \in GSBD^2(\Omega)$. Then there exists a sequence $(u_k)_k$ of displacements with regular jump set such that*

$$\begin{aligned} &(i) \quad u_k \to u \ \text{ in measure}, \\ &(ii) \quad \|e(u_k) - e(u)\|_{L^2(\Omega;\mathbb{R}^{2\times2}_{\rm sym})} \to 0, \\ &(iii) \quad \mathcal{H}^1(J_{u_k} \triangle J_u) \to 0. \end{aligned}$$

Note that in contrast to the original density result reported in Theorem 2.6 the assumption that $u \in L^2(\Omega)$ is not needed in the planar setting.

3. The model and statement of the main result

In this section we introduce the model we study and we fix the related notations. This preliminary discussion is still conducted in a general N-dimensional setting, while our main result, given at the end of the section, is stated and proved only in the planar case $N = 2$.

We analyze the evolution of a brittle material in the sense of Griffith [26] whose total energy consists of a linear elastic bulk term and a surface term proportional to the $(N-1)$-dimensional measure of the crack. The body is under the action of a time-dependent *prescribed boundary displacement* $g(t)$ on a relatively open part $\partial_D\Omega$ of the boundary (*Dirichlet part*) of the reference configuration $\Omega \subset \mathbb{R}^N$, which is supposed to be open, bounded with Lipschitz boundary. The rest of the boundary will be instead assumed to be force-free for simplicity. The variables of the model are a $GSBD$-valued displacement u and a (not a priori prescribed) crack Γ with finite $\mathcal{H}^{N-1}$ measure. The uncracked part of the body has a linear elastic stored energy of the form

$$\int_{\Omega\setminus\Gamma} Q(e(u))\,\mathrm{d}x.$$

In the above expression $e(u)$ is the approximate symmetrized gradient of u and $Q : \mathbb{R}^{N\times N}_{\rm sym} \to \mathbb{R}$ is the quadratic form associated to a symmetric bounded and positive definite stiffness tensor $\mathbb{C} : \mathbb{R}^{N\times N}_{\rm sym} \to \mathbb{R}^{N\times N}_{\rm sym}$, that is

$$Q(e) := \frac{1}{2}\mathbb{C}e : e, \tag{12}$$

with the colon denoting the Euclidean product between matrices.

The prescribed boundary displacement g is a time dependent function $g \in W^{1,1}_{\rm loc}([0,+\infty); H^1(\mathbb{R}^N;\mathbb{R}^N))$. As it is typical for the weak formulation of evolutionary problems in spaces of functions of bounded deformation, the boundary condition will be relaxed as follows. We will assume that it exists an open, bounded Lipschitz set $\Omega' \supset \Omega$ such that

$$\Omega' \cap \overline{\Omega} = \partial_D \Omega \qquad \Omega' \setminus \overline{\Omega} \text{ has Lipschitz boundary} \tag{13}$$

and impose, for every time t, that an admissible displacement $u(t)$ satisfies $u(t) = g(t)$ a.e. in $\Omega' \setminus \overline{\Omega}$. A competing crack may choose indeed to run alongside $\partial_D \Omega$, in which case the boundary condition is not attained in the sense of traces, at the expense of a crack energy.

The energy of a crack $\Gamma \subset \overline{\Omega}$ will be proportional to its $(N-1)$-dimensional Hausdorff measure, namely of the form

$$\kappa \mathcal{H}^{N-1}(\Gamma \cap \Omega'),$$

where the material parameter κ represents the toughness of the material. Within this choice, and because of (13), formation of cracks along $\partial_D \Omega$ is penalized, while no energy is spent for a crack sitting on the load-free part of the boundary $\partial \Omega \setminus \partial_D \Omega$. In the following we will set $\kappa = 1$ without loss of generality.

The quasistatic evolution problem associated to the model with the prescribed boundary displacement $g(t)$ consists in finding a displacement and crack path $(u(t), \Gamma(t))$ with $J_{u(t)} \subset \Gamma(t) \subset \overline{\Omega}$ and $u(t) = g(t)$ a.e. in $\Omega' \setminus \overline{\Omega}$ such that $\Gamma(t)$ is irreversible, namely $\Gamma(t) \supset \Gamma(s)$ whenever $t > s$, and the following two conditions hold:

- *global stability*. For each t, $u(t)$ minimizes

$$\int_\Omega Q(e(v))\, \mathrm{d}x + \mathcal{H}^{N-1}(J_v \setminus \Gamma(t)) \tag{14}$$

  among all $v \in GSBD^2(\Omega')$ such that $v = g(t)$ on $\Omega' \setminus \overline{\Omega}$;
- *energy-dissipation balance*. The total energy

$$\mathcal{E}(t) := \int_\Omega Q(e(u(t)))\, \mathrm{d}x + \mathcal{H}^{N-1}(\Gamma(t) \cap \Omega') \tag{15}$$

  is absolutely continuous and satisfies for all $t > 0$

$$\mathcal{E}(t) = \mathcal{E}(0) + \int_0^t \langle \sigma(s), e(\dot{g}(s)) \rangle \, \mathrm{d}s, \tag{16}$$

  where $\sigma(s) = \mathbb{C}e(u(s))$, $\langle \cdot, \cdot \rangle$ is the duality pairing in $L^2(\Omega; \mathbb{R}^{N\times N}_{\mathrm{sym}})$, and $\dot{g}(s)$ denotes the Frèchet derivative of g with respect to s.

Notice that even for a given Γ, the existence of a minimizer for the problem considered in (14) is a nontrivial issue, which we are able to overcome for the moment only in the planar case $N = 2$ (Theorem 6.2). Indeed, in the planar case we are able to show the existence of a quasistatic evolution according to the following statement, which constitutes the main result of the paper.

Theorem 3.1. *Let $N = 2$. Let $\Omega \subset \Omega'$ be bounded domains in $\mathbb{R}^2$ with Lipschitz boundary satisfying (13), $g \in W^{1,1}_{\mathrm{loc}}([0, +\infty); H^1(\mathbb{R}^2; \mathbb{R}^2))$, and consider Q as in (12). Then, for all $t \geq 0$ it exists an $\mathcal{H}^1$-rectifiable crack $\Gamma(t) \subset \overline{\Omega}$ and a field $u(t) \in GSBD^2(\Omega')$ such that*

- *$\Gamma(t)$ is nondecreasing in t;*
- *$u(0)$ minimizes*

$$\int_\Omega Q(e(v))\, \mathrm{d}x + \mathcal{H}^1(J_v)$$

  among all $v \in GSBD^2(\Omega')$ such that $v = g(0)$ on $\Omega' \setminus \overline{\Omega}$;
- *for all $t > 0$, $u(t)$ satisfies the global stability (14) for $N = 2$;*
- *$J_{u(0)} = \Gamma(0)$ and $J_{u(t)} \subset \Gamma(t)$ up to a set of $\mathcal{H}^1$-measure 0.*

Furthermore, the total energy $\mathcal{E}(t)$ defined by (15) satisfies the energy dissipation balance (16). Finally, for any countable, dense subset $I \subset [0, +\infty)$ containing zero, we have

$$\Gamma(t) = \bigcup_{\tau \in I\,,\,\tau \le t} J_{u(\tau)}$$

for all $t > 0$.

4. A sharp piecewise Korn inequality in GSBD

In this section we derive a piecewise Korn inequality with a sharp estimate for the surface energy and also prove a version taking Dirichlet boundary conditions into account.

4.1. A refined piecewise Korn inequality

The goal of this section is to prove the following result.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary and $0 < \theta < 1$. Then there is a universal constant $c > 0$, some $C_\Omega = C_\Omega(\Omega) > 0$ and some $C_{\theta,\Omega} = C_{\theta,\Omega}(\theta, \Omega) > 0$ such that the following holds: For each $u \in GSBD^2(\Omega)$ we find $u^\theta \in SBV(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$ such that $\{u \neq u^\theta\}$ is a set of finite perimeter with*

$$\begin{aligned} &(i) \quad \mathcal{L}^2(\{u \neq u^\theta\}) \le c\theta(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega))^2, \\ &(ii) \quad \mathcal{H}^1((\partial^*\{u \neq u^\theta\} \cap \Omega) \setminus J_u) \le c\theta(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega)), \end{aligned} \tag{17}$$

a (finite) Caccioppoli partition $\Omega = \bigcup_{i=0}^I P_i$, and corresponding infinitesimal rigid motions $(a_i)_{i=0}^I$ such that $v := u^\theta - \sum_{i=0}^I a_i \chi_{P_i} \in SBV(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$ and

$$\begin{aligned} &(i) \quad \sum\nolimits_{i=0}^I \mathcal{H}^1((\partial^* P_i \cap \Omega) \setminus J_u) \le c\theta(\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega)), \\ &(ii) \quad \mathcal{L}^2(P_i) \ge C_\Omega \theta^2 \quad \text{for all} \quad 1 \le i \le I, \quad \mathcal{L}^2(\{u \neq u^\theta\} \triangle P_0) = 0, \\ &(iii) \quad \|v\|_{L^\infty(\Omega;\mathbb{R}^2)} + \|\nabla v\|_{L^1(\Omega;\mathbb{R}^{2\times2})} \le C_{\theta,\Omega}\|e(u)\|_{L^2(\Omega;\mathbb{R}^{2\times2}_{\mathrm{sym}})}. \end{aligned} \tag{18}$$

Note that the refined estimate (18)(i) comes at the expense of the fact that we have to pass to a slightly modified function (see (17)) and that in (17)(iii) only the L^1-norm of the derivative is controlled.

Remark 4.2. Let C_{Q_1} and C_{θ,Q_1} be the constants in Theorem 4.1 for the unit square $\Omega = Q_1 = (0, 1)^2$. Using a rescaling argument, (18)(ii),(iii) in Theorem 4.1 applied for any square $\Omega = Q \subset \mathbb{R}^2$ read as

$$\begin{aligned} &(i) \quad \mathcal{L}^2(P_i) \ge C_{Q_1}\mathcal{L}^2(Q)\theta^2 \quad \text{for} \quad 1 \le i \le I, \\ &(ii) \quad \|v\|_{L^\infty(Q;\mathbb{R}^2)} + (\mathrm{diam}(Q))^{-1}\|\nabla v\|_{L^1(Q;\mathbb{R}^{2\times2})} \le C_{\theta,Q_1}\|e(u)\|_{L^2(Q;\mathbb{R}^{2\times2}_{\mathrm{sym}})}. \end{aligned} \tag{19}$$

Below after the proof of Theorem 4.1 we briefly indicate how Remark 4.2 can be derived from (18) for convenience of the reader. As a preparation we formulate two lemmas. Recall the notion of decomposable sets in Section 2.3 and the definition of diam in Section 2.1.

Lemma 4.3. *Let $B \subset \mathbb{R}^2$ be an indecomposable, bounded set with finite perimeter. Then* $\mathrm{diam}(B) \le \mathcal{H}^1(\partial^* B)$.

The proof can be found in [28, Proposition 12.19, Remark 12.28]. The following lemma investigates some properties of the jump set of a piecewise-defined function on the interface of two sets of finite perimeter.

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^2$ open, bounded and $y \in SBV(\Omega; \mathbb{R}^2) \cap L^\infty(\Omega; \mathbb{R}^2)$. Let $P_1, P_2 \subset \Omega$ be sets of finite perimeter and $a_i = a_{A_i,b_i}$, $i = 1, 2$, infinitesimal rigid motions. Then there is a ball $B \subset \mathbb{R}^2$ with*

(i) $\operatorname{diam}(B) \le 4\operatorname{diam}(P_2)\, \|a_1 - a_2\|^{-1}_{L^\infty(P_2;\mathbb{R}^2)} \sum_{i=1,2} \|y - a_i\|_{L^\infty(P_i;\mathbb{R}^2)}$,

(ii) $\mathcal{H}^1\big((\partial^* P_1 \cap \partial^* P_2) \setminus (B \cup J_y)\big) = 0$.

Proof. We define $\gamma = \|a_1 - a_2\|_{L^\infty(P_2;\mathbb{R}^2)}$ and $\delta = \sum_{i=1,2} \|y - a_i\|_{L^\infty(P_i;\mathbb{R}^2)}$ for shorthand. First, if $\delta \ge \frac{1}{2}\gamma$, we can choose B as a ball containing P_2 with $\operatorname{diam}(B) \le 2\operatorname{diam}(P_2)$. Consequently, it suffices to assume $\delta < \frac{1}{2}\gamma$.

For $i = 1, 2$ we denote by $T_i y$ the trace of y on $\partial^* P_i$, which exists by [4, Theorem 3.77] and satisfies

$$|T_i y(x) - a_i(x)| \le \|y - a_i\|_{L^\infty(P_i;\mathbb{R}^2)} \quad \text{for } \mathcal{H}^1\text{-a.e. } x \in \partial^* P_i.$$

Assume the statement was wrong. Then we would find two points x_1, x_2 with $|x_1 - x_2| > 4\gamma^{-1}\delta\operatorname{diam}(P_2)$ such that $x_1, x_2 \in (\partial^* P_1 \cap \partial^* P_2) \setminus J_y$ and for $i, j = 1, 2$

$$|T_i y(x_j) - a_i(x_j)| \le \|y - a_i\|_{L^\infty(P_i;\mathbb{R}^2)}.$$

Since $x_1, x_2 \notin J_y$ and thus $T_1 y(x_1) = T_2 y(x_1)$, $T_1 y(x_2) = T_2 y(x_2)$ we compute

$$|a_1(x_j) - a_2(x_j)| \le |T_1 y(x_j) - a_1(x_j)| + |T_2 y(x_j) - a_2(x_j)| \le \delta$$

for $j = 1, 2$. Combining the estimates for $j = 1, 2$ we get

$$\begin{aligned}|x_1 - x_2||A_1 - A_2| &\le 2|(A_1 x_1 + b_1) - (A_2 x_1 + b_2) - (A_1 x_2 + b_1) + (A_2 x_2 + b_2)| \\ &\le 2(|a_1(x_1) - a_2(x_1)| + |a_1(x_2) - a_2(x_2)|) \le 2\delta\end{aligned}$$

and therefore $|A_1 - A_2| \le \frac{1}{2}(\operatorname{diam}(P_2))^{-1}\gamma$ as well as

$$\gamma = \|a_1 - a_2\|_{L^\infty(P_2;\mathbb{R}^2)} \le |a_1(x_1) - a_2(x_1)| + \operatorname{diam}(P_2)|A_1 - A_2| \le \delta + \tfrac{1}{2}\gamma,$$

which contradicts $\gamma > 2\delta$. □

Proof of Theorem 4.1. Let $u \in GSBD^2(\Omega)$ be given and set for shorthand $\mathcal{E} = \|e(u)\|_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})}$ and $J_u' = J_u \cup \partial\Omega$. Without restriction we can assume $\theta^{-1} \in \mathbb{N}$ and that Ω is connected as otherwise the following arguments are applied for each connected component of Ω. Moreover, we may suppose that $\mathcal{H}^1(J_u) \le (\theta^{-1}\mathcal{L}^2(\Omega))^{\frac{1}{2}}$ as otherwise the assertion trivially holds with $u^\theta = 0$.

In the following $c > 0$ stands for a universal constant and $C_\Omega = C_\Omega(\Omega) > 0$, $C_{\theta,\Omega} = C_{\theta,\Omega}(\theta, \Omega) > 0$ represent generic constants which may vary from line to line. We may further assume that θ is chosen (depending on Ω) such that $\theta \le \theta_0 := \frac{1}{16} C_{\mathrm{korn}}^{-1}$, where C_{korn} is the constant from (11).[1]

Step 0 (Overview of the proof). The general idea behind the proof is to modify suitably the infinitesimal rigid motions provided by Theorem 2.9 so that all the sets P_j of the Caccioppoli partition are *almost completely disconnected* by J_u: by this we mean that the interface between different components will be contained in the jump set of u up to a small (in area and perimeter) exceptional set. In doing this, we must anyway be able not to lose the estimate in (11)(iii). These are the main observations that allow us to pursue this strategy:

(O1) If the L^∞ distance between two infinitesimal rigid motions a_{j_1} and a_{j_2}, that are subtracted from u on two sets P_{j_1} and P_{j_2}, respectively, lies below a fixed threshold depending on the error parameter θ (see (21)(iii)), we can replace a_{j_2} with a_{j_1} on P_{j_2}. Indeed, by construction and using Lemma 2.3(a), (11)(iii) will still hold up to enlarging $C_{\theta,\Omega}$ suitably.

(O2) If the L^∞ distance between two infinitesimal rigid motions a_{j_1} and a_{j_2}, that are subtracted from u on two sets P_{j_1} and P_{j_2}, respectively, lies above an (even larger) fixed threshold depending on θ (see (21)(iv)), using Lemma 4.4 the interface between P_{j_1} and P_{j_2} not contained in J_u can be covered by a small ball. This will lead to neglecting a small exceptional set with small perimeter, provided this is not done 'too often'. Some combinatorial arguments will indeed be needed (cf., for instance, the derivation of (27) later in the proof).

[1] If $\theta > \theta_0$, the result holds for $u^\theta = u^{\theta_0}$, upon replacing C_Ω by $C_\Omega \theta_0^2$ in (18)(ii).

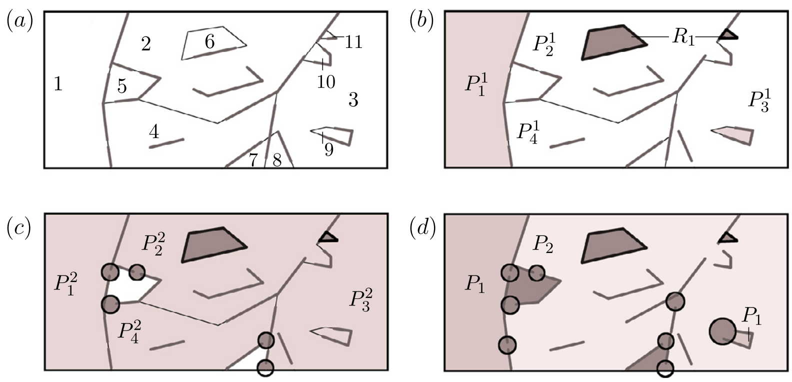

Fig. 1. Illustration of the constructions in the proof of Theorem 4.1. (a) The partition $(P'_j)_{j=1}^{11}$ is sketched (for convenience only the indices are given). Note that in general the jump set (depicted in light gray) is not a subset of $\bigcup_{j=1}^{11} \partial^* P'_j$. (b) The *large components* of $(P_j^1)_{j=1}^6$ are given by $P_1^1 = P'_1 \cup P'_9$, $P_2^1 = P'_2 \cup P'_{10}$, $P_3^1 = P'_3 \cup P'_8$, $P_4^1 = P'_4$ (i.e. $I' = 4$), the exceptional set is $R_1 = P'_6 \cup P'_{11}$ and the *small components* are P'_5, P'_7. Observe that P_1^1, depicted in light gray, is not connected. (c) The union of balls R_2 is illustrated and the set $\Omega_{\rm good} = \bigcup_{j=1}^4 P_j^1 \setminus R_2 = \bigcup_{j=1}^4 P_j^2$ is given in light gray. (d) In this example we have $R_4 = \emptyset$. The set $\Omega_{\rm bad}$ is depicted in dark gray and $\Omega \setminus \Omega_{\rm bad} = P_1^3 \cup P_2^3 = P_1 \cup P_2$ consists of two components, i.e. $I'' = 2$. We further have $\mathcal{Z}_1 = \emptyset$, $\mathcal{Z}_2 = \{(1,2),(1,3),(1,4),(3,4)\}$ and $\mathcal{Z}_3 = \{(2,3),(2,4)\}$.

(O3) On neighboring components P_{j_1} and P_{j_2}, whose size lies above a fixed threshold depending on θ, and that are *not* almost completely disconnected by J_u, the L^∞ estimate in (11)(iii), the continuity of u on part of the interface, together with Lemma 2.3(a), allow us to estimate the L^∞ distance between the corresponding infinitesimal rigid motions a_{j_1} and a_{j_2} basically only in terms of θ, and therefore we may apply (O1) to remove the artificially introduced boundaries.

Guided by these observations, the proof is organized as follows (the steps of the construction are depicted in Fig. 1).

In Step I we reorganize the partition given by Theorem 2.9 into *large sets*, of size at least $\theta^2\mathcal{L}^2(\Omega)$, *small sets*, covering only a small part of Ω and a *rest set*, denoted by R_1, which has small perimeter (see (20)). Using (O1) the partition has now the property that the infinitesimal rigid motions given on large and small components, respectively, differ very much (see (21)(iv)). This is the starting point for Step II, where, in the spirit of (O2), we show that the part of the interfaces between large and small components not contained in J_u can be covered by an exceptional set which is small in area and perimeter. In Step III we then investigate the difference of the infinitesimal rigid motions given on large components, again employing Lemma 4.4 to completely disconnect various components, and using (O3) on the others. In Step IV we collect all estimates and conclude the proof.
Step I (Identification of large components). The goal of this step is to define a set $R_1 \subset \Omega$ with

$$\mathcal{H}^1(\partial^* R_1) \le \theta \mathcal{H}^1(J'_u), \quad \mathcal{L}^2(R_1) \le c\theta^2(\mathcal{H}^1(J'_u))^2, \tag{20}$$

an (ordered) Caccioppoli partition $\Omega \setminus R_1 = \bigcup_{j=1}^\infty P_j^1$ and corresponding infinitesimal rigid motions $(a_j^1)_j$ such that $v_1 := u - \sum_{j\ge1} a_j^1 \chi_{P_j^1}$ satisfies for an index $I' \in \mathbb{N}$ with $I' \le \theta^{-2}$ and some $K_\theta \in \mathbb{N}$, $K_\theta \le \theta^{-1}$,

$$\begin{aligned}
&(i) \quad \mathcal{L}^2(P_j^1) \ge \theta^2\mathcal{L}^2(\Omega) \ \text{ for all } \ 1 \le j \le I', \ \ \mathcal{L}^2\Big(\Omega \setminus \bigcup\nolimits_{j=1}^{I'} P_j^1\Big) \le c\theta(\mathcal{H}^1(J'_u))^2,\\
&(ii) \quad \sum\nolimits_{j\ge1} \mathcal{H}^1(\partial^* P_j^1) \le c\mathcal{H}^1(J'_u),\\
&(iii) \quad \|v_1\|_{L^\infty(\Omega\setminus R_1;\mathbb{R}^2)} \le 2C_{\rm korn}\theta^{-4K_\theta}\mathcal{E}, \quad \|\nabla v_1\|_{L^1(\Omega\setminus R_1;\mathbb{R}^{2\times2})} \le C_{\theta,\Omega}\mathcal{E},\\
&(iv) \quad \min\nolimits_{1\le i\le I'} \|a_i^1 - a_j^1\|_{L^\infty(P_j^1;\mathbb{R}^2)} \ge \theta^{-4(K_\theta+1)}\mathcal{E} \ \text{ for all } j > I'.
\end{aligned} \tag{21}$$

Moreover, the sets $(P_j^1)_{j>I'}$ are indecomposable, while the sets $(P_j^1)_{j=1}^{I'}$ are possibly not indecomposable.

We first apply Theorem 2.9 to find an ordered Caccioppoli partition $(P'_j)_{j\geq 1}$ of Ω and corresponding infinitesimal rigid motions $(a'_j)_j = (a_{A'_j,b'_j})_j$ such that $v' := u - \sum_{j\geq 1} a'_j \chi_{P'_j} \in SBV(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$ satisfies (11), in particular

$$\|v'\|_{L^\infty(\Omega;\mathbb{R}^2)} + \|\nabla v'\|_{L^1(\Omega;\mathbb{R}^{2\times 2})} \leq C_{\mathrm{korn}}\mathcal{E}. \tag{22}$$

Without restriction we assume that the sets $(P'_j)_{j\geq 1}$ are indecomposable. Let $I' \in \mathbb{N}$ be the largest index such that $\mathcal{L}^2(P'_{I'}) \geq \theta^2\mathcal{L}^2(\Omega)$. (Recall that the partition is assumed to be ordered.) Then $I' \leq \theta^{-2}$ and by the isoperimetric inequality and (11)(i)

$$\begin{aligned}
&(i)\quad \sum\nolimits_{j\geq 1}(\mathcal{L}^2(P'_j))^{\frac{1}{2}} \leq c\sum\nolimits_{j\geq 1}\mathcal{H}^1(\partial^* P'_j) \leq c\mathcal{H}^1(J'_u) \leq C_{\theta,\Omega},\\
&(ii)\quad \sum\nolimits_{j>I'}\mathcal{L}^2(P'_j) \leq \theta(\mathcal{L}^2(\Omega))^{\frac{1}{2}}\sum\nolimits_{j>I'}(\mathcal{L}^2(P'_j))^{\frac{1}{2}}\\
&\qquad\qquad \leq c\theta\mathcal{H}^1(\partial\Omega)\sum\nolimits_{j>I'}\mathcal{H}^1(\partial^* P'_j) \leq c\theta(\mathcal{H}^1(J'_u))^2,
\end{aligned} \tag{23}$$

where in the last step of (i) we used the assumption $\mathcal{H}^1(J_u) \leq (\theta^{-1}\mathcal{L}^2(\Omega))^{\frac{1}{2}}$. We introduce a decomposition for the small components according to the difference of infinitesimal rigid motions as follows. For $k \in \mathbb{N}$ we introduce the set of indices

$$\begin{aligned}
&\mathcal{J}^0 = \{j > I' : \min\nolimits_{1\leq i\leq I'}\|a'_j - a'_i\|_{L^\infty(P'_j;\mathbb{R}^2)} \leq \mathcal{E}\theta^{-4}\},\\
&\mathcal{J}^k = \{j > I' : \mathcal{E}\theta^{-4k} < \min\nolimits_{1\leq i\leq I'}\|a'_j - a'_i\|_{L^\infty(P'_j;\mathbb{R}^2)} \leq \mathcal{E}\theta^{-4(k+1)}\}
\end{aligned} \tag{24}$$

and define $s_k = \sum_{j\in\mathcal{J}^k}\mathcal{H}^1(\partial^* P'_j)$ for $k \in \mathbb{N}_0$. In view of (23)(i) we find some $K_\theta \in \mathbb{N}$, $K_\theta \leq \theta^{-1}$, such that $s_{K_\theta} \leq c\theta\mathcal{H}^1(J'_u)$.

We let $R_1 := \bigcup_{j\in\mathcal{J}^{K_\theta}} P'_j$ and the choice of K_θ together with the isoperimetric inequality shows (20). We introduce the Caccioppoli partition $(P^1_j)_{j\geq 1}$ of $\Omega \setminus R_1$ by combining different components of $(P'_j)_{j\geq 1}$. We decompose the indices in $\bigcup_{k=0}^{K_\theta-1}\mathcal{J}^k$ into sets $\mathcal{J}'_i$ with $\bigcup_{i=1}^{I'}\mathcal{J}'_i = \bigcup_{k=0}^{K_\theta-1}\mathcal{J}^k$ according to the following rule: an index $j \in \mathcal{J}^k$ is assigned to $\mathcal{J}'_i$ whenever i is the smallest index such that the minimum in (24) is attained.

Define the *large components* $P^1_i = P'_i \cup \bigcup_{j\in\mathcal{J}'_i} P'_j$ for $1 \leq i \leq I'$ and by $(P^1_i)_{i>I'}$ we denote the *small components*

$$\left\{P'_j : j > I',\ j \in \bigcup\nolimits_{k=K_\theta+1}^{\infty}\mathcal{J}^k\right\}. \tag{25}$$

Then (21)(i) holds by (20), (23)(ii) and we see that the sets $(P^1_j)_{j>I'}$ are indecomposable. Likewise, (21)(ii) follows from (23)(i). Moreover, we define $a^1_j = a'_j$ for $1 \leq j \leq I'$ and let $a^1_j = a'_{k_j}$ for $j > I'$, where $k_j \in \mathbb{N}$ such that $P^1_j = P'_{k_j}$. We introduce $v_1 = u - \sum_{j\geq 1} a^1_j\chi_{P^1_j}$ and observe that by (22), (24) and the definition of P^1_j for $1 \leq j \leq I'$ we have

$$\begin{aligned}
\|v_1\|_{L^\infty(P^1_j;\mathbb{R}^2)} &\leq \|v'\|_{L^\infty(P^1_j;\mathbb{R}^2)} + \|v_1 - v'\|_{L^\infty(P^1_j;\mathbb{R}^2)} \leq C_{\mathrm{korn}}\mathcal{E} + \theta^{-4K_\theta}\mathcal{E}\\
&\leq 2C_{\mathrm{korn}}\theta^{-4K_\theta}\mathcal{E}.
\end{aligned}$$

Moreover, by Lemma 2.3, (22), (23)(i), (24) and the definition of $\mathcal{J}'_i$ we find

$$\begin{aligned}
\sum\nolimits_{i=1}^{I'}\|\nabla v_1\|_{L^1(P^1_i;\mathbb{R}^{2\times 2})} &\leq \sum\nolimits_{i=1}^{I'}\Big(\|\nabla v'\|_{L^1(P^1_i;\mathbb{R}^{2\times 2})} + \sum\nolimits_{j\in\mathcal{J}'_i}\mathcal{L}^2(P'_j)|A'_j - A^1_i|\Big)\\
&\leq \|\nabla v'\|_{L^1(\Omega;\mathbb{R}^{2\times 2})} + c\sum_{i=1}^{I'}\sum_{j\in\mathcal{J}'_i}(\mathcal{L}^2(P'_j))^{\frac{1}{2}}\|a'_j - a'_i\|_{L^\infty(P'_j;\mathbb{R}^2)}\\
&\leq C_{\mathrm{korn}}\mathcal{E} + c\theta^{-4K_\theta}\mathcal{E}\sum\nolimits_{j\geq 1}(\mathcal{L}^2(P'_j))^{\frac{1}{2}} \leq C_{\theta,\Omega}\mathcal{E}.
\end{aligned}$$

Note that the last constant $C_{\theta,\Omega}$ indeed only depends on θ and Ω since $K_\theta \le \theta^{-1}$ and C_{korn} only depends on Ω. The last two estimates together with (22) show (21)(iii). Finally, the definition of the small components in (25) together with (24) implies (21)(iv).

Step II (Interface between large and small components). We now show that there is a union of balls $R_2 \subset \Omega$ and a Caccioppoli partition $\bigcup_{j=1}^{I'} P_j^2$ of $\Omega_{\mathrm{good}} := \bigcup_{j=1}^{I'} (P_j^1 \setminus R_2)$ and corresponding infinitesimal rigid motions $(a_j^2)_{j=1}^{I'}$ such that with $v_2 := u - \sum_{j=1}^{I'} a_j^2 \chi_{P_j^2}$ we have

$$\begin{aligned} &(i) \quad \mathcal{L}^2(\Omega \setminus \Omega_{\mathrm{good}}) \le c\theta(\mathcal{H}^1(J_u'))^2, \\ &(ii) \quad \mathcal{H}^1(\partial^*\Omega_{\mathrm{good}} \setminus J_u') \le c\theta\mathcal{H}^1(J_u'), \\ &(iii) \quad \sum\nolimits_{j=1}^{I'} \mathcal{H}^1(\partial^* P_j^2) \le c\mathcal{H}^1(J_u'), \\ &(iv) \quad \|v_2\|_{L^\infty(\Omega_{\mathrm{good}};\mathbb{R}^2)} + \|\nabla v_2\|_{L^1(\Omega_{\mathrm{good}};\mathbb{R}^{2\times2})} \le C_{\theta,\Omega}\mathcal{E}. \end{aligned} \tag{26}$$

First, for each $1 \le i \le I'$ and $j > I'$ we apply Lemma 4.4 for $P_1 = P_i^1$ and $P_2 = P_j^1$ and obtain a ball $B_{i,j}$ with $\mathcal{H}^1((\partial^* P_i^1 \cap \partial^* P_j^1) \setminus (B_{i,j} \cup J_u)) = 0$ such that by (21)(iii),(iv)

$$\mathrm{diam}(B_{i,j}) \le 16C_{\mathrm{korn}}\,\mathrm{diam}(P_j^1) \cdot \theta^{-4K_\theta} \cdot (\theta^{-4(K_\theta+1)})^{-1} \le \theta^3\,\mathrm{diam}(P_j^1),$$

where the last step follows from the fact that $\theta \le \frac{1}{16}C_{\mathrm{korn}}^{-1}$. Then by Lemma 4.3 and the fact that P_j^1 is indecomposable (see below (21)) we get $\mathrm{diam}(B_{i,j}) \le \theta^3\mathcal{H}^1(\partial^* P_j^1)$.

Define $R_2 = \bigcup_{i \le I' < j} B_{i,j}$ and compute by (21)(ii) and $I' \le \theta^{-2}$ (cf. (21)(i))

$$\sum\nolimits_{i \le I' < j} \mathcal{H}^1(\partial B_{i,j}) \le \theta^3 I' \sum\nolimits_{j > I'} \mathcal{H}^1(\partial^* P_j^1) \le c\theta\mathcal{H}^1(J_u'). \tag{27}$$

Then the isoperimetric inequality yields $\mathcal{L}^2(R_2) \le c\theta^2(\mathcal{H}^1(J_u'))^2$ and this together with (21)(i) shows (26)(i). Let $P_j^2 = P_j^1 \setminus R_2$ and $a_j^2 = a_j^1$ for $1 \le j \le I'$. Then (26)(iii) follows from (21)(ii) and (27). To see (26)(ii), we calculate by Theorem 2.7, (20) and (27) recalling that $\Omega_{\mathrm{good}} \cup \bigcup_{j>I'}(P_j^1 \setminus R_2) \cup (R_1 \setminus R_2) \cup R_2$ is a partition of Ω

$$\begin{aligned} \mathcal{H}^1(\partial^*\Omega_{\mathrm{good}} \setminus (J_u \cup \partial\Omega)) \le &\sum_{i \le I' < j} \Big(\mathcal{H}^1\big((\partial^* P_i^1 \cap \partial^* P_j^1) \setminus (J_u \cup B_{i,j})\big) + \mathcal{H}^1(\partial B_{i,j})\Big) \\ &+ \mathcal{H}^1(\partial^* R_1) \le 0 + c\theta\mathcal{H}^1(J_u') = c\theta\mathcal{H}^1(J_u'). \end{aligned}$$

Finally, (26)(iv) follows from (21)(iii), the definition of v_2 and the fact that $K_\theta \le \theta^{-1}$.

Step III (Interface between large components). We now investigate the difference of the infinitesimal rigid motions $(a_j^2)_{j=1}^{I'}$. We show that there is a union of balls $R_3 \subset \Omega$ and a Caccioppoli partition $\Omega_{\mathrm{good}} \setminus R_3 = \bigcup_{i=1}^{I''} P_i^3$ with $I'' \le I'$ and corresponding infinitesimal rigid motions $(a_i^3)_{i=1}^{I''}$ such that with $v_3 := u - \sum_{i=1}^{I''} a_i^3 \chi_{P_i^3}$ we have

$$\begin{aligned} &(i) \quad \mathcal{H}^1(\partial^* R_3) \le c\theta\mathcal{H}^1(J_u'), \quad \mathcal{L}^2(R_3) \le c\theta^2(\mathcal{H}^1(J_u'))^2, \\ &(ii) \quad \sum\nolimits_{i=1}^{I''} \mathcal{H}^1(\partial^* P_i^3 \setminus J_u') \le c\theta\mathcal{H}^1(J_u'), \\ &(iii) \quad \|v_3\|_{L^\infty(\Omega_{\mathrm{good}} \setminus R_3;\mathbb{R}^2)} + \|\nabla v_3\|_{L^1(\Omega_{\mathrm{good}} \setminus R_3;\mathbb{R}^{2\times2})} \le C_{\theta,\Omega}\mathcal{E}. \end{aligned} \tag{28}$$

In the following we denote the constant given in (26)(iv) by $\bar{C} = \bar{C}(\theta, \Omega)$ to distinguish it from other generic constants $C_{\theta,\Omega}$. We introduce the set of indices $\mathcal{Z}_1 = \{1 \le j \le I' : \mathrm{diam}(P_j^2) \le \theta^3\mathcal{H}^1(\partial\Omega)\}$ [2] and let $\mathcal{Z}_2 = \{(i, j) : 1 \le i < j \le I',\ i, j \notin \mathcal{Z}_1\}$ be the collection of pairs with

$$\max\nolimits_{k=i,j} \|a_i^2 - a_j^2\|_{L^\infty(P_k^2;\mathbb{R}^2)} > \bar{C}\theta^{-5}\mathcal{E}. \tag{29}$$

Finally, let $\mathcal{Z}_3 = \{(i, j) : 1 \le i < j \le I',\ i, j \notin \mathcal{Z}_1, (i, j) \notin \mathcal{Z}_2\}$.

[2] The introduction of $\mathcal{Z}_1$ is only a technical point due to the fact that by the previous step some large components may have become small after cutting of R_2.

For each $j \in \mathcal{Z}_1$ we find a ball B^1_j with $\mathcal{H}^1(\partial B^1_j) \le c\theta^3\mathcal{H}^1(\partial\Omega)$ and $P^2_j \subset B^1_j$. Moreover, by Lemma 4.4 we find for each $(i,j) \in \mathcal{Z}_2$ a ball $B^2_{i,j}$ satisfying $\mathcal{H}^1\big((\partial^* P^2_i \cap \partial^* P^2_j) \setminus (B^2_{i,j} \cup J_u)\big) = 0$ and by (26)(iv)

$$\begin{aligned}\mathrm{diam}(B^2_{i,j}) &\le c \max_{k=i,j} \mathrm{diam}(P^2_k)\,\big(\max_{k=i,j} \|a^2_i - a^2_j\|_{L^\infty(P^2_k;\mathbb{R}^2)}\big)^{-1}\,\|v_2\|_{L^\infty(\Omega_{\mathrm{good}};\mathbb{R}^2)}\\ &\le c\mathrm{diam}(\Omega)(\bar C\theta^{-5}\mathcal{E})^{-1}\bar C\mathcal{E} \le c\theta^5\mathcal{H}^1(\partial\Omega),\end{aligned}$$

where in the last step $\mathrm{diam}(\Omega) \le \mathcal{H}^1(\partial\Omega)$ follows from the fact that Ω is assumed to be connected.

We define $R_3 = \bigcup_{j\in\mathcal{Z}_1} B^1_j \cup \bigcup_{(i,j)\in\mathcal{Z}_2} B^2_{i,j}$ and the fact that $\#\mathcal{Z}_1 \le \theta^{-2}$, $\#\mathcal{Z}_2 \le I'(I'-1) \le \theta^{-4}$ yields

$$\sum\nolimits_{j\in\mathcal{Z}_1} \mathcal{H}^1(\partial B^1_j) + \sum\nolimits_{(i,j)\in\mathcal{Z}_2} \mathcal{H}^1(\partial B^2_{i,j}) \le c\theta\mathcal{H}^1(\partial\Omega), \tag{30}$$

which together with the isoperimetric inequality gives (28)(i). We now combine different components $(P^2_j)^{I'}_{j=1}$: we can find a decomposition $\mathcal{I}_1\dot\cup\ldots\dot\cup\mathcal{I}_{I''}$ of the indices $\{1,\ldots,I'\}\setminus\mathcal{Z}_1$ with the property that for each pair $i_1, i_2 \in \mathcal{I}_j$, $i_1 < i_2$, we find a chain $i_1 = l_1 < l_2 < \ldots < l_n = i_2$ such that $(l_k, l_{k+1}) \in \mathcal{Z}_3$ for all $k = 1,\ldots,n-1$.

Then we introduce a partition of $\Omega_{\mathrm{good}} \setminus R_3$ consisting of the sets $P^3_i = \bigcup_{j\in\mathcal{I}_i}(P^2_j \setminus R_3)$, $1 \le i \le I''$. (Note that this is indeed a partition of $\Omega_{\mathrm{good}} \setminus R_3$ since, by construction, $P^2_j \subset \Omega_{\mathrm{good}}$ for $j \in \mathcal{I}_i$ and $P^2_j \subset R_3$ for $j \in \mathcal{Z}_1$.) To see (28)(ii), we now compute using the property of the balls B^1_j, $B^2_{i,j}$, as well as (26)(ii), (30) and Theorem 2.7

$$\begin{aligned}\sum\nolimits_{i=1}^{I''} \mathcal{H}^1(\partial^* P^3_i \setminus J'_u) &\le \mathcal{H}^1(\partial^*\Omega_{\mathrm{good}} \setminus J'_u) + \sum\nolimits_{j\in\mathcal{Z}_1} \mathcal{H}^1(\partial B^1_j)\\ &\quad + \sum\nolimits_{(i,j)\in\mathcal{Z}_2} \Big(\mathcal{H}^1\big((\partial^* P^2_i \cap \partial^* P^2_j) \setminus (B^2_{i,j} \cup J_u\big) + \mathcal{H}^1(\partial B^2_{i,j})\Big)\\ &\le c\theta\mathcal{H}^1(J'_u) + c\theta\mathcal{H}^1(\partial\Omega) + 0 \le c\theta\mathcal{H}^1(J'_u).\end{aligned}$$

It remains to define v_3 and to show (28)(iii). Fix $(i,j) \in \mathcal{Z}_3$. Then by the fact that (29) does not hold and $\min_{k=i,j}\mathrm{diam}(P^2_k) \ge \theta^3\mathcal{H}^1(\partial\Omega) \ge \theta^3\mathrm{diam}(\Omega)$ a short calculation implies $\|a^2_i - a^2_j\|_{L^\infty(\Omega;\mathbb{R}^2)} \le C\mathcal{E}$ for some $C = C(\Omega,\theta,\bar C)$. Then the triangle inequality together with $\#\mathcal{I}_j \le I' \le \theta^{-2}$ yields

$$\max\nolimits_{i_1,i_2\in\mathcal{I}_j} \|a^2_{i_1} - a^2_{i_2}\|_{L^\infty(\Omega;\mathbb{R}^2)} \le C_{\theta,\Omega}\mathcal{E}$$

for all $1 \le j \le I''$, which by Lemma 2.3 implies $\max_{i_1,i_2\in\mathcal{I}_j} |A^2_{i_1} - A^2_{i_2}| \le C_{\theta,\Omega}\mathcal{E}$. For each P^3_j, $1 \le j \le I''$, we choose an infinitesimal rigid motion a^3_j, which coincides with an arbitrary a^2_i, $i \in \mathcal{I}_j$. Then (28)(iii) follows from (26)(iv).

Step IV (Conclusion). We are now in a position to prove the assertion of the theorem. Suppose that the partition $(P^3_j)^{I''}_{j=1}$ is ordered and choose the smallest index I such that $\mathcal{L}^2(P^3_{I+1}) \le (\theta\mathcal{H}^1(J'_u))^2$. Define $R_4 = \bigcup_{j=I+1}^{I''} P^3_j$ and compute by the isoperimetric inequality and (28)(ii)

$$\mathcal{L}^2(R_4) \le \theta\mathcal{H}^1(J'_u)\sum_{j=I+1}^{I''}(\mathcal{L}^2(P^3_j))^{\frac12} \le c\theta\mathcal{H}^1(J'_u)\sum_{j=I+1}^{I''}\mathcal{H}^1(\partial^* P^3_j) \le c\theta(\mathcal{H}^1(J'_u))^2.$$

Then we define $\Omega_{\mathrm{bad}} := (\Omega \setminus \Omega_{\mathrm{good}}) \cup (R_3 \cup R_4)$ and by (26)(i),(ii), (28)(i),(ii) we get $\mathcal{H}^1(\partial^*\Omega_{\mathrm{bad}} \setminus J'_u) \le c\theta\mathcal{H}^1(J'_u)$ and $\mathcal{L}^2(\Omega_{\mathrm{bad}}) \le c\theta(\mathcal{H}^1(J'_u))^2$.

We define $u^\theta \in SBV(\Omega;\mathbb{R}^2) \cap L^\infty(\Omega;\mathbb{R}^2)$ by $u^\theta = u\chi_{\Omega\setminus\Omega_{\mathrm{bad}}} + t_0\chi_{\Omega_{\mathrm{bad}}}$ for some $t_0 \in \mathbb{R}^2$ such that $\mathcal{L}^2(\{u = t_0\}) = 0$, which is possible since u is measurable. With this, $\mathcal{L}^2(\{u^\theta \ne u\}\triangle\Omega_{\mathrm{bad}}) = 0$. Observe that the previous calculation yields (17). Let $(P_i)^I_{i=0}$ be the Caccioppoli partition consisting of the sets $P_0 = \Omega_{\mathrm{bad}}$ and $P_i = P^3_i$ for $1 \le i \le I$. Set $a_i = a^3_i$ for $1 \le i \le I$ and $a_0 = t_0$. Then (18)(iii) follows from (28)(iii) and (28)(ii) yields (18)(i). Finally, the choice of the index I together with the fact that $\mathcal{H}^1(J'_u) \ge \mathcal{H}^1(\partial\Omega)$ implies (18)(ii). □

Proof of Remark 4.2. Let $Q_\lambda = x + (0,\lambda)^2$ be given and $u \in GSBD^2(Q_\lambda)$. After translation we may assume $x = 0$. Define $\bar u \in GSBD^2(Q_1)$ by $\bar u(x) = u(\lambda x)$ and also note that $\nabla\bar u(x) = \lambda\nabla u(\lambda x)$ and $\mathcal{H}^1(J_{\bar u}) = \lambda^{-1}\mathcal{H}^1(J_u)$. Applying the above theorem for $\bar u$ on Q_1 we obtain $\bar u^\theta \in SBV(Q_1;\mathbb{R}^2) \cap L^\infty(Q_1;\mathbb{R}^2)$ such that

$$(i)\quad \mathcal{L}^2(\{\bar u \neq \bar u^\theta\}) \le c\theta(\mathcal{H}^1(J_{\bar u}) + \mathcal{H}^1(\partial Q_1))^2,$$
$$(ii)\quad \mathcal{H}^1((\partial^*\{\bar u \neq \bar u^\theta\} \cap Q_1) \setminus J_{\bar u}) \le c\theta(\mathcal{H}^1(J_{\bar u}) + \mathcal{H}^1(\partial Q_1)),$$

a (finite) Caccioppoli partition $Q_1 = \bigcup_{i=0}^I \bar P_i$, and corresponding infinitesimal rigid motions $(\bar a_i)_{i=0}^I$ such that $\bar v := \bar u^\theta - \sum_{i=0}^I \bar a_i \chi_{\bar P_i} \in SBV(Q_1;\mathbb{R}^2) \cap L^\infty(Q_1;\mathbb{R}^2)$ is constant on $\bar P_0$ and satisfies

$$(i)\quad \sum_{i=0}^I \mathcal{H}^1((\partial^* \bar P_i \cap Q_1) \setminus J_{\bar u}) \le c\theta(\mathcal{H}^1(J_{\bar u}) + \mathcal{H}^1(\partial Q_1)),$$
$$(ii)\quad \mathcal{L}^2(\bar P_i) \ge C_{Q_1}\theta^2 = C_{Q_1}\theta^2\mathcal{L}^2(Q_1), \quad 1 \le i \le I,$$
$$(iii)\quad \|\bar v\|_{L^\infty(Q_1;\mathbb{R}^2)} + \|\nabla \bar v\|_{L^1(Q_1;\mathbb{R}^{2\times 2})} \le C_{\theta,Q_1}\|e(\bar u)\|_{L^2(Q_1;\mathbb{R}^{2\times2}_{\rm sym})}.$$

Set $P_i = \lambda \bar P_i$, $u^\theta(x) = \bar u^\theta(\lambda^{-1}x)$ and $v(x) = \bar v(\lambda^{-1}x) \in SBV(Q_\lambda;\mathbb{R}^2)$. The estimates for the modification in (17) follow since the estimate in (i) is two homogeneous and the estimate in (ii) is one homogeneous. For the same reason (18)(i) and (19)(i) hold. We finally show (19)(ii).

By transformation formula and the fact that $\nabla \bar u(x) = \lambda \nabla u(\lambda x)$ we have $\|e(u)\|^2_{L^2(Q_\lambda)} = \|e(\bar u)\|^2_{L^2(Q_1)}$. Likewise, $\|\nabla v\|_{L^1(Q_\lambda)} = \lambda\|\nabla \bar v\|_{L^1(Q_1)}$ and finally we clearly have $\|v\|_\infty = \|\bar v\|_\infty$. Then (ii) follows as $\mathrm{diam}(Q_\lambda) = \sqrt{2}\lambda \ge \lambda$. □

4.2. A version with Dirichlet boundary conditions

We now state a version of the piecewise Korn inequality with Dirichlet boundary conditions, which will be needed for the general existence result in Section 6, but not for the jump transfer lemma in Section 5. The reader more interested in the derivation of the latter may therefore wish to skip the remainder of this section and to proceed directly with Section 5.

Theorem 4.5. *Let $\Omega \subset \Omega'$ be bounded domains in $\mathbb{R}^2$ with Lipschitz boundary such that (13) holds. Let $\theta > 0$. Then there is a constant $\bar c = \bar c(\Omega, \Omega') > 0$ and some $C_{\theta,\Omega'} = C_{\theta,\Omega'}(\theta, \Omega') > 0$ such that for each $w \in H^1(\Omega';\mathbb{R}^2)$ and $u \in GSBD^2(\Omega')$ with $u = w$ on $\Omega' \setminus \overline{\Omega}$ there is a modification $u^\theta \in SBV(\Omega';\mathbb{R}^2)$ satisfying*

$$(i)\quad \mathcal{L}^2(\{u \neq u^\theta\}) \le \bar c\theta(\mathcal{H}^1(J_u) + 1)^2, \quad \mathcal{H}^1(J_{u^\theta} \setminus J_u) \le \bar c\theta(\mathcal{H}^1(J_u) + 1)$$
$$(ii)\quad \|e(u^\theta)\|^2_{L^2(\Omega;\mathbb{R}^{2\times2}_{\rm sym})} \le \|e(u)\|^2_{L^2(\Omega;\mathbb{R}^{2\times2}_{\rm sym})} + \|\nabla w\|^2_{L^2(\{u\neq u^\theta\};\mathbb{R}^{2\times2})}, \tag{31}$$

a Caccioppoli partition $\Omega' = \bigcup_{j=1}^\infty P_j$ and corresponding infinitesimal rigid motions $(a_j)_j = (a_{A_j,b_j})_j$ such that $v := u^\theta - \sum_{j=1}^\infty a_j\chi_{P_j} \in SBV(\Omega';\mathbb{R}^2) \cap L^2(\Omega';\mathbb{R}^2)$, and

$$(i)\quad \sum_{j=1}^\infty \mathcal{H}^1((\partial^* P_j \cap \Omega') \setminus J_u) \le \bar c\theta(\mathcal{H}^1(J_u) + 1),$$
$$(ii)\quad v = w \ \text{ on } \Omega' \setminus \overline{\Omega}, \tag{32}$$
$$(iii)\quad \|v\|_{L^2(\Omega';\mathbb{R}^2)} + \|\nabla v\|_{L^1(\Omega';\mathbb{R}^{2\times2})} \le C_{\theta,\Omega'}\|e(u)\|_{L^2(\Omega';\mathbb{R}^{2\times2}_{\rm sym})} + C_{\theta,\Omega'}\|w\|_{H^1(\Omega';\mathbb{R}^2)}.$$

As a preparation, we need the following lemma.

Lemma 4.6. *Let $A \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Then there exists $\delta = \delta(A)$ such that for all indecomposable sets $E \subset A$ with finite perimeter satisfying $\mathcal{H}^1(\partial^* E \cap A) \le \delta(A)$ one has either*

$$(i)\ \mathcal{L}^2(E) > \tfrac12\mathcal{L}^2(A) \quad \text{or} \quad (ii)\ \mathrm{diam}(E) \le C_A\mathcal{H}^1(\partial^* E \cap A)$$

for some constant C_A only depending on A.

Proof. Fix $\varepsilon > 0$. By [8] (see also [5]) there is a constant $K = K(A)$ and a Borel set $B_\varepsilon \subset \mathbb{R}^2$ with $B_\varepsilon \cap A = E$ such that $\mathcal{H}^1(\partial^* B_\varepsilon) \le K\mathcal{H}^1(\partial^* E \cap A) + \varepsilon$. It is not restrictive to assume that B_ε is indecomposable as otherwise we simply take the component containing E. By the isoperimetric inequality we derive

$$\min\{\mathcal{L}^2(B_\varepsilon), \mathcal{L}^2(\mathbb{R}^2 \setminus B_\varepsilon)\} \le c(\mathcal{H}^1(\partial^* B_\varepsilon))^2 < \tfrac{1}{2}\mathcal{L}^2(A),$$

where the last inequality holds provided that $\delta = \delta(A, c)$ is small enough. If $\mathcal{L}^2(\mathbb{R}^2 \setminus B_\varepsilon) < \frac{1}{2}\mathcal{L}^2(A)$, we find

$$\mathcal{L}^2(E) = \mathcal{L}^2(B_\varepsilon \cap A) = \mathcal{L}^2(A) - \mathcal{L}^2(A \setminus B_\varepsilon) \ge \mathcal{L}^2(A) - \mathcal{L}^2(\mathbb{R}^2 \setminus B_\varepsilon) > \tfrac{1}{2}\mathcal{L}^2(A)$$

and (i) holds. Otherwise, we particularly obtain $\mathcal{L}^2(\mathbb{R}^2 \setminus B_\varepsilon) = +\infty$ and $\mathcal{L}^2(B_\varepsilon) < +\infty$. Since B_ε has finite perimeter, by an approximation argument we may assume that B_ε is bounded. As B_ε is also indecomposable, Lemma 4.3 yields

$$\operatorname{diam}(E) \le \operatorname{diam}(B_\varepsilon) \le \mathcal{H}^1(\partial^* B_\varepsilon) \le K\mathcal{H}^1(\partial^* E \cap A) + \varepsilon.$$

The claim follows with $\varepsilon \to 0$. □

Proof of Theorem 4.5. By Theorem 4.1 applied with Ω' in place of Ω we obtain a Caccioppoli partition $(P_i')_{i=0}^I$, corresponding $(a_i')_{i=0}^I$ as well as $\bar{u}^\theta \in SBV(\Omega'; \mathbb{R}^2)$ and $v' := \bar{u}^\theta - \sum_{i=0}^I a_i' \chi_{P_i'} \in SBV(\Omega'; \mathbb{R}^2) \cap L^\infty(\Omega'; \mathbb{R}^2)$ such that (17)–(18) hold. Define $u^\theta = u\chi_{\Omega \setminus P_0'} + w\chi_{P_0'}$. Then (31) follows directly from (17) and (18)(ii).

Let $\mathcal{P}' = (P_j')_{j=1}^I$. Let $\mathcal{P}_1 \subset \mathcal{P}'$ be the components completely contained in Ω and let $\mathcal{P}_2 \subset \mathcal{P}'$ be the components P_j' satisfying $\mathcal{L}^2(P_j' \cap (\Omega' \setminus \overline{\Omega})) \ge \theta$. Moreover, we set $\mathcal{P}_3 = \mathcal{P}' \setminus (\mathcal{P}_1 \cup \mathcal{P}_2)$. We now define the partition $\mathcal{P} = (P_j)_{j=1}^\infty$ consisting of the components

$$\{P_0'\} \cup \mathcal{P}_1 \cup \mathcal{P}_2 \cup \{P_j' \cap \Omega : P_j' \in \mathcal{P}_3\} \cup \{P_j' \setminus \overline{\Omega} : P_j' \in \mathcal{P}_3\}.$$

(Strictly speaking, the number of components is even finite.) For $P_1 := P_0'$ we let $a_1 = 0$. For $P_j = P_k' \in \mathcal{P}_1$ we set $a_j = a_k'$ and for $P_j = P_k' \in \mathcal{P}_2$ we set $a_j = 0$. If $P_j \in \mathcal{P}$ with $P_j = P_k' \cap \Omega$ for some $P_k' \in \mathcal{P}_3$, we let $a_j = a_k'$. Finally, if $P_j \in \mathcal{P}$ with $P_j = P_k' \setminus \overline{\Omega}$ for some $P_k' \in \mathcal{P}_3$, we set $a_j = 0$.

Now define $v = u^\theta - \sum_{j=1}^\infty a_j \chi_{P_j}$. By construction we get $v = w$ on $\Omega' \setminus \overline{\Omega}$. It remains to confirm (32)(i),(iii). To see (iii), we first note that, since $u^\theta = v = w$ on the open Lipschitz set $\Omega' \setminus \overline{\Omega}$, by (18)(iii) with v' in place of v and [4, Corollary 3.89], it suffices to show that the restriction of v to Ω belongs to $SBV(\Omega; \mathbb{R}^2) \cap L^2(\Omega; \mathbb{R}^2)$ and that

$$\|v - v'\|_{L^2(\Omega;\mathbb{R}^2)} + \|\nabla v - \nabla v'\|_{L^1(\Omega;\mathbb{R}^{2\times 2})} \le C_{\theta,\Omega'}\|e(u)\|_{L^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + C_{\theta,\Omega'}\|w\|_{H^1(\Omega';\mathbb{R}^2)}. \tag{33}$$

By construction we have that $\{v \neq v'\} \cap \Omega \subset (P_0' \cap \Omega) \cup \bigcup_{P_j \in \mathcal{P}_2} P_j$ (up to a set of negligible measure). First, (33) with $P_0' \cap \Omega$ in place of Ω follows directly from (18)(iii) and the fact that $v = w$ on P_0'. Fix $P_j \in \mathcal{P}_2$. We first observe that $u = \bar{u}^\theta$ by (18)(ii) and thus $(v - v')\chi_{P_j} = (u - v')\chi_{P_j} = a_k'\chi_{P_j}$ with k such that $P_j = P_k'$. Since $u = w$ on $\Omega' \setminus \overline{\Omega}$, we then deduce

$$a_k'\chi_{(\Omega'\setminus\overline{\Omega})\cap P_j} = (w - v')\chi_{(\Omega'\setminus\overline{\Omega})\cap P_j}$$

and therefore

$$\|a_k'\|_{L^2((\Omega'\setminus\overline{\Omega})\cap P_j;\mathbb{R}^2)} \le \|w\|_{L^2(\Omega'\setminus\overline{\Omega};\, \mathbb{R}^2)} + \|v'\|_{L^2(\Omega'\setminus\overline{\Omega};\, \mathbb{R}^2)}.$$

Consequently, using $\mathcal{L}^2(P_j \cap (\Omega' \setminus \overline{\Omega})) \ge \theta$, (18)(iii) and Lemma 2.3 for $\psi(s) = s^2$ we find

$$|A_k'| + |b_k'| \le C_{\theta,\Omega'}\|e(u)\|_{L^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + C_{\theta,\Omega'}\|w\|_{L^2(\Omega';\mathbb{R}^2)}.$$

Since $\#\mathcal{P}_2 \le \theta^{-1}\mathcal{L}^2(\Omega') = C(\Omega', \theta)$, this yields

$$\sum\nolimits_{P_j \in \mathcal{P}_2} \|a_k'\|_{L^2(P_j;\mathbb{R}^2)} + \|A_k'\|_{L^1(P_j;\mathbb{R}^{2\times 2}_{\mathrm{skew}})} \le C_{\theta,\Omega'}(\|e(u)\|_{L^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \|w\|_{L^2(\Omega';\mathbb{R}^2)}),$$

where for each j the index k is chosen such that $P_j = P_k'$. This implies $v \in SBV(\Omega; \mathbb{R}^2) \cap L^2(\Omega; \mathbb{R}^2)$, as well as (33), and establishes (32)(iii).

We now show (32)(i). To this end, we fix $\theta_0 = \theta_0(\Omega, \Omega') > 0$ to be specified below and we first observe that it suffices to treat the case where $\mathcal{H}^1(J_u) + \mathcal{H}^1(\partial\Omega') \le \theta_0\theta^{-1}$. In fact, otherwise (32)(i) follows directly from (18)(i) for $\bar{c} = \bar{c}(\Omega, \Omega')$ large enough.

Without restriction we suppose that each $P'_j \cap (\Omega' \setminus \overline{\Omega})$, $P'_j \in \mathcal{P}_3$, is indecomposable as otherwise we consider the indecomposable components. We show that each $P'_j \cap (\Omega' \setminus \overline{\Omega})$ is contained in some ball of diameter $\bar{C}\mathcal{H}^1(\partial^* P'_j \cap (\Omega' \setminus \overline{\Omega}))$ for $\bar{C} = \bar{C}(\Omega, \Omega')$ large enough. To see this, we first observe that due to the fact that $J_u \subset \overline{\Omega}$ we have

$$\mathcal{H}^1(\partial^* P'_j \cap (\Omega' \setminus \overline{\Omega})) \le c\theta\mathcal{H}^1(J_u) + c\theta\mathcal{H}^1(\partial\Omega') \le c\theta_0$$

by (18)(i). Choose θ_0 so small that $c\theta_0 \le \delta(\Omega' \setminus \overline{\Omega})$ with $\delta(\Omega' \setminus \overline{\Omega})$ as in Lemma 4.6. Then Lemma 4.6 and the fact that $\mathcal{L}^2(P'_j \cap (\Omega' \setminus \overline{\Omega})) \le \theta$ imply for θ small

$$\mathrm{diam}(\partial^* P'_j \cap (\Omega' \setminus \overline{\Omega})) \le \bar{C}\mathcal{H}^1(\partial^* P'_j \cap (\Omega' \setminus \overline{\Omega})) \le c\bar{C}\theta_0. \tag{34}$$

We cover $\Theta := \partial(\Omega' \setminus \Omega)$ with sets $U_1, \dots, U_n$ such that $U_i \cap \Theta$ is the graph of a Lipschitz function for $i = 1, \dots, n$ and the sets pairwise overlap such that each ball with radius $c\bar{C}\theta_0$ and center in Θ is contained in one U_i provided that θ_0 is chosen sufficiently small. Consequently, recalling (34), each $P'_j \cap (\Omega' \setminus \overline{\Omega})$ is contained in some U_i. Since $U_i \cap \Theta$ is the graph of a Lipschitz function f_i and $P'_j \cap (\Omega' \setminus \overline{\Omega}) \subset\subset U_i$, it follows that

$$\mathcal{H}^1(\partial\Omega \cap P'_j) \le \mathrm{Lip}_{f_i} \mathrm{diam}(\partial^* P'_j \cap (\Omega' \setminus \overline{\Omega})) \le \hat{C}\bar{C}\mathcal{H}^1(\partial^* P'_j \cap (\Omega' \setminus \overline{\Omega})),$$

where $\hat{C} = \max_i \mathrm{Lip}_{f_i}$. For the last inequality we again used (34). Finally, noting that $\bigcup_{j=1}^\infty \partial^* P_j \setminus \bigcup_{j=0}^I \partial^* P'_j \subset \bigcup_{P'_j \in \mathcal{P}_3} (\partial\Omega \cap P'_j)$ we find using (18)(i)

$$\sum\nolimits_{j=1}^\infty \mathcal{H}^1((\partial^* P_j \cap \Omega') \setminus J_u) \le (1 + \hat{C}\bar{C}) \sum\nolimits_{j=0}^I \mathcal{H}^1((\partial^* P'_j \cap \Omega') \setminus J_u) \le \bar{c}\theta(\mathcal{H}^1(J_u) + 1)$$

for $\bar{c} = \bar{c}(\Omega, \Omega')$ large enough. □

5. Jump transfer lemma in GSBD

In this section we prove a jump transfer lemma which will be essential for the stability of the static equilibrium condition in the derivation of the existence result (Theorem 3.1).

Theorem 5.1. *Let $\Omega \subset \Omega'$ be bounded domains in $\mathbb{R}^2$ with Lipschitz boundary such that (13) holds. Let $\ell \in \mathbb{N}$ and let $(w_n^l)_n \subset H^1(\Omega'; \mathbb{R}^2)$ be bounded sequences for $l = 1, \dots, \ell$. Let $(u_n^l)_n$ be sequences in $GSBD^2(\Omega')$ and $u^l \in GSBD^2(\Omega')$ such that*

$$\begin{aligned} &(i) \quad \|e(u_n^l)\|_{L^2(\Omega'; \mathbb{R}^{2\times2}_{\mathrm{sym}})} + \mathcal{H}^1(J_{u_n}) \le M \quad \text{for all } n \in \mathbb{N}, \\ &(ii) \quad u_n^l \to u^l \text{ in measure in } \Omega', \quad u_n^l = w_n^l \text{ on } \Omega' \setminus \overline{\Omega}, \end{aligned} \tag{35}$$

for $l = 1, \dots, \ell$. Then it exists a (not relabeled) subsequence of $n \in \mathbb{N}$ with the following property: For each $\phi \in GSBD^2(\Omega')$ there is a sequence $(\phi_n)_n \subset GSBD^2(\Omega')$ with $\phi_n = \phi$ on $\Omega' \setminus \overline{\Omega}$ such that for $n \to \infty$

$$\begin{aligned} &(i) \quad \phi_n \to \phi \text{ in measure in } \Omega, \\ &(ii) \quad e(\phi_n) \to e(\phi) \text{ strongly in } L^2(\Omega; \mathbb{R}^{2\times2}_{\mathrm{sym}}), \\ &(iii) \quad \mathcal{H}^1\big((J_{\phi_n} \setminus \bigcup\nolimits_{l=1}^\ell J_{u_n^l}) \setminus (J_\phi \setminus \bigcup\nolimits_{l=1}^\ell J_{u^l})\big) \to 0. \end{aligned} \tag{36}$$

5.1. *Proof of the jump transfer lemma*

The general strategy is to follow the proof of the SBV jump transfer (see [19, Theorem 2.1]) with the essential difference that (a) in the definition of ϕ_n we transfer the jump not by a reflection but by a suitable extension and (b) the control of the derivatives, which is needed for the application of the coarea formula, is recovered from (35)(i) by means of the piecewise Korn inequality in Theorem 4.1. The auxiliary results, allowing us to overcome such difficulties, are the following Lemmas 5.2 and 5.7, as well as Theorem 5.5. We postpone their proofs to the next subsection and first show that with these additional techniques Theorem 5.1 can be derived following the lines of [19].

For problem (a) we need the following extension lemma, based on an argument of [31].

Lemma 5.2. *Let $R \subset \mathbb{R}^2$ be an open rectangle, let R^- be the reflection of R with respect to one of its sides, and let $\hat{R}$ be the open rectangle obtained by joining R, R^- and their common side. Let $\phi \in GSBD^2(R)$. Then it exists an extension $\hat{\phi} \in GSBD^2(\hat{R})$ of ϕ satisfying*

$$(i) \quad \mathcal{H}^1(J_{\hat{\phi}}) \leq c\mathcal{H}^1(J_\phi)$$

$$(ii) \quad \|e(\hat{\phi})\|_{L^2(\hat{R};\mathbb{R}^{2\times 2}_{\mathrm{sym}})} \leq c\|e(\phi)\|_{L^2(R;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} \tag{37}$$

for some universal constant c independent of R and ϕ.

We concern ourselves with problem (b). A key point in the proof of the jump transfer lemma is to write the jump set of a limiting function u^l as a countable union of pairwise intersections of boundaries of super-level sets by the BV coarea formula. Thus, as a first ingredient we state that the jump set of an $GSBD^2$ function can be approximated suitably by the jump set of an SBV function.

Lemma 5.3. *Let $\Omega' \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary and $\epsilon > 0$. For each $u \in GSBD^2(\Omega')$ there is $v \in SBV(\Omega';\mathbb{R}^2) \cap GSBD^2(\Omega')$ with $\mathcal{H}^1(J_{u-v}) \leq \epsilon$ and $\mathcal{H}^1(J_u \triangle J_v) \leq \epsilon$. If in addition there exist an open subset $\Omega \subset \Omega'$ and $w \in H^1(\Omega';\mathbb{R}^2)$ with $u = w$ on $\Omega' \setminus \overline{\Omega}$, the function v can be also taken with $v = w$ on $\Omega' \setminus \overline{\Omega}$.*

Recall that the main assumption in the SBV jump transfer lemma (see [19, Theorem 2.1]) was that the derivatives $|\nabla u^l_n|$, $n \in \mathbb{N}$, are equiintegrable. Although Theorem 4.1 allows us to reduce the problem to the SBV setting, we need further arguments since Theorem 4.1 only provides an L^1-bound for the derivatives and the bound is not given in terms of the displacement field, but holds only after subtraction of a piecewise infinitesimal rigid motion.

To overcome this difficulty, given a fine covering of the jump set of u^l, we have to construct explicitly modifications of the functions u^l_n on the given covering which have almost the same jump set and whose gradients are small. Notice that this differs substantially from the proof strategy devised in [19] where, due to equiintegrability, one could ensure a priori that gradients do not concentrate on small neighborhoods of the jump set of u^l.

In the following, for $u \in GSBD^2$ and $x \in J_u$ with unit normal $\nu(x)$ we denote by $Q_r(x)$ the square with sidelength $2r$, center x and two faces perpendicular to $\nu(x)$.

Definition 5.4. Let $u \in GSBD^2(\Omega)$, $x \in J_u$ and $\eta > 0$. We say r is an *η-fine radius* and $Q_r(x)$ an *η-fine square* of u at x if there are two sets $K^r_+, K^r_- \subset Q_r(x)$ such that

$$\mathcal{L}^2(K^r_\pm) \geq \frac{1}{2}(1-\eta)\mathcal{L}^2(Q_r(x)), \qquad \|u - u^\pm(x)\|_{L^\infty(K^r_\pm;\mathbb{R}^2)} \leq \frac{1}{2}\eta.$$

For given x, u, and η we set $r(u,x,\eta)$ as the maximal radius such that r is an η-fine radius of u at x for all $r < r(u,x,\eta)$. Observe that both notions are well defined for almost every jump point and that $r(u,x,\eta) > 0$ for $\mathcal{H}^1$-a.e. $x \in J_u$.

We have the following approximation result.

Theorem 5.5. *Let $\Omega' \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary. Let $M > 0$, $0 < \theta, \delta < 1$ with $\delta \leq \frac{1}{4}C_{Q_1}\theta^8$ with the constant C_{Q_1} from Remark 4.2. Consider a sequence $(u_n)_n$ in $GSBD^2(\Omega')$ and $u \in GSBD^2(\Omega')$ with*

$$(i) \quad \|e(u_n)\|^2_{L^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \mathcal{H}^1(J_{u_n}) \leq M \ \text{ for all } n \in \mathbb{N},$$

$$(ii) \quad u_n \to u \text{ in measure in } \Omega'. \tag{38}$$

Let $Q_* = \bigcup_{i=1}^m Q_{r_i}(x_i)$ be a union of pairwise disjoint δ-fine squares for u at $x_i \in J_u$ with $\sum_{i=1}^m r_i \leq M$ and $r_i \leq \delta^2$. Then there exist a sequence $(v^{\delta,\theta}_n)_n \subset SBV(Q_*;\mathbb{R}^2)$, a universal constant $c > 0$, and $C_\theta = C_\theta(\theta) > 0$ independent of the sequence $(u_n)_n$ and δ, such that

$$\begin{aligned}
&(i) \quad \limsup_{n\to\infty} \mathcal{H}^1(J_{v_n^{\delta,\theta}} \setminus J_{u_n}) \le cM\theta,\\
&(ii) \quad \limsup_{n\to\infty} \|\nabla v_n^{\delta,\theta}\|_{L^1(Q_*;\mathbb{R}^{2\times2})} \le C_\theta M\delta,\\
&(iii) \quad \liminf_{n\to\infty} \|v_n^{\delta,\theta} - u\|_{L^1(Q_{r_i}(x_i)\setminus F_i;\mathbb{R}^2)} = 0 \quad \text{for } i = 1,\dots,m,
\end{aligned} \tag{39}$$

where F_i, $i = 1,\dots,m$, are Borel sets with

$$\mathcal{L}^2(F_i) \le c\theta^2(r_i^2 + \theta^2 \liminf_{n\to\infty}(\mathcal{H}^1(J_{u_n} \cap Q_{r_i}(x_i)))^2). \tag{40}$$

For the proof of the jump transfer lemma we will need the following extension to the case with boundary conditions.

Corollary 5.6. *Under the assumptions of Theorem 5.5, suppose in addition that there is an open subset Ω of Ω' so that $u_n = w_n$ in $\Omega' \setminus \overline{\Omega}$, for a bounded sequence $(w_n)_n \subset H^1(\Omega';\mathbb{R}^2)$. Then the sequence $(v_n^{\delta,\theta})_n$ can be taken such that $J_{v_n^{\delta,\theta}} \subset Q_* \cap \overline{\Omega}$, provided the constant C_θ is allowed to additionally depend on $\sup_n \|w_n\|_{H^1(\Omega';\mathbb{R}^2)}$.*

We now proceed with the proof of Theorem 5.1.

Proof of Theorem 5.1. We first consider the case $\ell = 1$ and drop the superscript. Let $(u_n)_n$, u and ϕ be given as in the hypothesis.

Step 0. We first show that it is not restrictive to assume that the limiting function u additionally satisfies $u \in SBV(\Omega';\mathbb{R}^2)$. Indeed, assume the theorem has been proved in this case. Then, in the general case of $u \in GSBD^2(\Omega')$, for a fixed $\epsilon > 0$ we choose $v_\epsilon \in SBV(\Omega';\mathbb{R}^2) \cap GSBD^2(\Omega')$ with $v_\epsilon = u$ on $\Omega' \setminus \overline{\Omega}$ such that $\mathcal{H}^1(J_{u-v_\epsilon}) \le \epsilon$ and $\mathcal{H}^1(J_u \triangle J_{v_\epsilon}) \le \epsilon$, thanks to Lemma 5.3. We notice that the sequence $v_{n,\epsilon} := u_n + (v_\epsilon - u)$ converges in measure to v_ϵ and satisfies the assumptions (35). Furthermore,

$$\mathcal{H}^1(J_{v_{n,\epsilon}} \triangle J_{u_n}) \le \mathcal{H}^1(J_{v_\epsilon - u}) \le \epsilon. \tag{41}$$

If we apply the theorem to the function v_ϵ and the sequence $(v_{n,\epsilon})_n$, we find a sequence $\phi_{n,\epsilon}$ satisfying (i) and (ii) in (36) as well as

$$\limsup_{n\to+\infty} \mathcal{H}^1\big((J_{\phi_{n,\epsilon}} \setminus J_{v_{n,\epsilon}}) \setminus (J_\phi \setminus J_{v_\epsilon})\big) = 0\,.$$

From (41) we then get

$$\limsup_{n\to+\infty} \mathcal{H}^1\big((J_{\phi_{n,\epsilon}} \setminus J_{u_n}) \setminus (J_\phi \setminus J_u)\big) \le 2\epsilon\,,$$

whence the conclusion follows, by arbitrariness of ϵ, through a diagonal argument.

Step 1. We now prove the theorem for $\ell = 1$ and $u \in SBV(\Omega';\mathbb{R}^2)$, mainly following the proof of the SBV jump transfer (see [19, Theorem 2.1]) employing additionally our auxiliary results.

Let $\theta > 0$. In the following all appearing generic constants c are always independent of θ. As a shortcut, for $\lambda \in \mathbb{R}$ we introduce the scalar, auxiliary function $u^\lambda := u^1 + \lambda u^2$, where u^1 and u^2 denote the two components of the function u, respectively. We may fix $\lambda \in (0,1)$ such that

$$\mathcal{H}^1(J_{u^\lambda} \triangle J_u) = 0. \tag{42}$$

This follows from the fact that $A_\lambda := \{x \in J_u : [u^1(x)] + \lambda[u^2(x)] = 0\}$ satisfies $\mathcal{H}^1(A_\lambda) = 0$ except for a countable number of λ's. We further denote by E_t the set of all Lebesgue-density 1 points for $\{x \in \Omega' : u^\lambda(x) > t\}$. Let $L = \{t \in \mathbb{R} : \mathcal{L}^2(\{x \in \Omega' : u^\lambda(x) = t\}) = 0\}$. Then there is a countable, dense subset $D \subset L$ such that J_{u^λ} (and thus, J_u) coincides up to a set of negligible $\mathcal{H}^1$-measure with

$$G := \bigcup_{t_1,t_2 \in D, t_1 < t_2} (\partial^* E_{t_1} \cap \partial^* E_{t_2} \cap \Omega').$$

For $x \in G$ we can choose $t_1(x) < t_2(x)$ in D such that $x \in \partial^* E_{t_1(x)} \cap \partial^* E_{t_2(x)}$ and $t_2(x) - t_1(x) \ge \frac{1}{2}|[u^\lambda(x)]|$. It can be shown that $\partial^* E_{t_1(x)}, \partial^* E_{t_2(x)}$ have a common outer unit normal $\nu(x)$. Let N be the set of points, where $\partial\Omega$ is not differentiable. We define

$$G_j = \Big\{x \in G \setminus N : |[u^\lambda(x)]| \geq \tfrac{1}{j},\ \ \lim_{r\to 0} \frac{\mathcal{H}^1((J_u \setminus \partial^* E_{t_1(x)}) \cap Q_r(x))}{2r} = 0\Big\},$$

where $Q_r(x)$ is a square with sidelength $2r$ and faces perpendicular to the normal $\nu(x)$. As in the proof of [19, Theorem 2.1], and recalling (42) we have that for fixed $\theta > 0$ and $j = j(\theta)$ large enough

$$\mathcal{H}^1(J_u \setminus G_j) = \mathcal{H}^1(J_{u^\lambda} \setminus G_j) \leq \theta. \tag{43}$$

We also fix the half squares

$$Q_r^+(x) := \{y \in Q_r(x) : (y - x) \cdot \nu(x) > 0\}, \quad Q_r^-(x) := Q_r(x) \setminus Q_r^+(x)$$

and the one-dimensional faces

$$H_r(x, s) = \{y \in Q_r(x) : (y - x) \cdot \nu(x) = s\}, \quad H_r(x) := H_r(x, 0).$$

Let $\delta = \theta(2\sqrt{2}MjC_\theta)^{-1} \wedge \frac{1}{4}C_{Q_1}\theta^8$, with the constant C_θ from (39)(ii) and C_{Q_1} from Remark 4.2. Following [19, (2.3),(2.5)–(2.6)] and covering G_j using the Morse–Besicovitch Theorem (see e.g. [18]) we find a finite number of pairwise disjoint squares $Q_i := Q_{r_i}(x_i)$, $i = 1, \dots, m$, with $x_i \in J_u$, $r_i < r(u, x_i, \delta) \wedge \delta^2$ (cf. Definition 5.4) such that

$$\begin{aligned}
&(i) \quad \mathcal{L}^2\big(\textstyle\bigcup_{i=1}^m Q_i\big) < \theta, \quad \mathcal{H}^1(G_j \setminus \textstyle\bigcup_{i=1}^m Q_i) < \theta,\\
&(ii) \quad \mathcal{H}^1((J_\phi \setminus J_u) \cap Q_i) \leq \theta r_i,\\
&(iii) \quad r_i \leq \mathcal{H}^1(J_u \cap Q_i) \leq 3r_i,\\
&(iv) \quad \mathcal{H}^1\big((J_u \setminus \partial^* E_{t_1(x_i)}) \cap Q_i\big) \leq \theta r_i,\\
&(v) \quad \mathcal{H}^1\big(\{y \in \partial^* E_{t_1(x_i)} \cap Q_i : \mathrm{dist}(y, H_{r_i}(x_i)) \geq \tfrac{\theta}{2} r_i\}\big) \leq \theta r_i,\\
&(vi) \quad \mathcal{L}^2\big((E_{t_k(x_i)} \cap Q_i) \triangle Q_i^-)\big) \leq \theta^2 r_i^2, \ k = 1, 2,\\
&(vii) \quad Q_i \subset \Omega \text{ if } x_i \in \Omega, \quad \mathcal{H}^1(\partial\Omega \cap Q_i) \leq c r_i \text{ if } x_i \in \partial\Omega,
\end{aligned} \tag{44}$$

where $Q_i^- := Q_{r_i}^-(x_i)$. Recall that r_i is a δ-fine radius of u at x_i in the sense of Definition 5.4 since $r_i < r(u, x_i, \delta)$. Thus, each Q_i is a δ-fine square. By (44)(iii) we can now apply Theorem 5.5 and Corollary 5.6 to obtain a sequence $(v_n^{\delta,\theta})_n \subset SBV(Q_*; \mathbb{R}^2)$ with $Q_* = \bigcup_{i=1}^m Q_i$ satisfying (39), in particular we have

$$\mathcal{H}^1(J_{v_n^{\delta,\theta}} \setminus J_{u_n}) \leq cM\theta, \quad J_{v_n^{\delta,\theta}} \subset \overline{\Omega}. \tag{45}$$

For brevity we write v_n instead of $v_n^{\delta,\theta}$ in the following. For the same λ that we fixed in (42) we analogously define the scalar-valued auxiliary functions v_n^λ and we denote by E_t^n the set of all Lebesgue-density 1 points for $\{x \in \Omega' : v_n^\lambda(x) > t\}$. By construction, and applying (39)(ii) we obtain, recalling $\delta \leq \theta(2\sqrt{2}MjC_\theta)^{-1}$

$$\|\nabla v_n^\lambda\|_{L^1(Q_*;\mathbb{R}^2)} \leq \sqrt{2}\|\nabla v_n\|_{L^1(Q_*;\mathbb{R}^{2\times 2})} \leq \sqrt{2}C_\theta M\delta \leq \tfrac{\theta}{2j}.$$

In view of the coarea formula in BV this implies that there are $t_i \in [t_1(x_i), t_2(x_i)]$ with (see [19, (2.7)])

$$\textstyle\sum_{i=1}^m \mathcal{H}^1\big((\partial^* E_{t_i}^n \cap Q_i) \setminus J_{v_n^\lambda}\big) \leq \theta. \tag{46}$$

By construction it holds that $J_{v_n^\lambda} \subset J_{v_n}$: combining with (45), we deduce

$$\textstyle\sum_{i=1}^m \mathcal{H}^1\big((\partial^* E_{t_i}^n \cap Q_i) \setminus J_{u_n}\big) \leq (1 + cM)\theta. \tag{47}$$

We now denote with $\mathcal{I} \subset \{1, \dots, m\}$ the subset of *good squares* such that

$$\liminf_{n\to\infty} \mathcal{H}^1(J_{u_n} \cap Q_i) \leq \theta^{-1} r_i \tag{48}$$

if and only if $i \in \mathcal{I}$. For $t \in L$, (39)(iii), (40), and (48) imply that

$$\liminf_{n\to\infty} \mathcal{L}^2((E_t^n \triangle E_t) \cap Q_i) \leq c\theta^2 r_i^2$$

for $i \in \mathcal{I}$. Then taking (44)(vi) into account and following [19, (2.8)–(2.9)] we find $N(\theta)$ such that for $n \geq N(\theta)$

$$\mathcal{L}^2((E^n_{t_i} \cap Q_i)\triangle Q_i^-) + \mathcal{L}^2((E_{t_i} \cap Q_i)\triangle Q_i^-) \le c\theta^2 r_i^2.$$

Following [19, (2.10)–(2.14)] and using (44)(i),(iii)–(v) we get $s_i^+, s_i^- \in [\frac{\theta}{2} r_i, \theta r_i]$ for $i \in \mathcal{I}$ such that for $n \ge N(\theta)$

$$\begin{aligned} &(i) \ \ \mathcal{H}^1(H_i^- \setminus E^n_{t_i}) \le c\theta r_i, \quad \mathcal{H}^1(H_i^+ \cap E^n_{t_i}) \le c\theta r_i, \ \ i \in \mathcal{I}, \\ &(ii) \ \ \mathcal{H}^1\big(G_j \setminus \big(\bigcup\nolimits_{i\in\mathcal{I}} R_i \cup \bigcup\nolimits_{i\notin\mathcal{I}} Q_i\big)\big) \le c\theta, \end{aligned} \tag{49}$$

where $H_i^+ = H_{r_i}(x_i, s_i^+)$, $H_i^- = H_{r_i}(x_i, s_i^-)$ and R_i the open rectangle between H_i^+ and H_i^-.

On the *bad squares* Q_i with $i \notin \mathcal{I}$ the above estimates are not available. On the other hand, there is not much jump of u in those squares. Indeed, since $\mathcal{H}^1(J_{u_n} \cap Q_i) > \theta^{-1} r_i$ for all $i \notin \mathcal{I}$ and $n \in \mathbb{N}$ large enough, we derive, using (44)(iii)

$$\sum\nolimits_{i\notin\mathcal{I}} \mathcal{H}^1(J_u \cap Q_i) \le \sum\nolimits_{i\notin\mathcal{I}} 3r_i \le 3\theta \sum\nolimits_{i\notin\mathcal{I}} \mathcal{H}^1(J_{u_n} \cap Q_i) \le 3\theta M. \tag{50}$$

Therefore, our aim is now to transfer the jump set J_ϕ in $G_j \cap \bigcup_{i\in\mathcal{I}} Q_i$ to $\bigcup_{i\in\mathcal{I}}(\partial^* E^n_{t_i} \cap Q_i)$. Assume first $x_i \notin \partial\Omega$. We set $\phi_- = \phi\chi_{Q_i^-\setminus R_i}$ extended to R_i according to Lemma 5.2. (This is possible when θ is small enough since R_i is a small neighborhood of $H_{r_i}(x_i)$.) In a similar way we define ϕ_+ on $(Q_i \setminus Q_i^-) \cup R_i$.

Now we let

$$\phi_n = \begin{cases} \phi_- & \text{on } Q_i^- \setminus R_i, \\ \phi_+ & \text{on } Q_i \setminus (Q_i^- \cup R_i), \\ \phi_- & \text{on } R_i \cap E^n_{t_i}, \\ \phi_+ & \text{on } R_i \setminus E^n_{t_i}. \end{cases}$$

If $x_i \in \partial\Omega$, we proceed similarly using (44)(vii) and modifying ϕ to ϕ_n only in the part contained in Ω.[3] We repeat the modification for all Q_i, $i \in \mathcal{I}$, so ϕ_n is defined on $\bigcup_{i\in\mathcal{I}} Q_i$. Outside this union we let $\phi_n = \phi$. By the construction and (37) we observe

$$\begin{aligned} &(i) \ \ \{\phi_n \neq \phi\} \subset \big(\bigcup\nolimits_{i\in\mathcal{I}} R_i\big) \cap \overline{\Omega}, \\ &(ii) \ \ \mathcal{H}^1\big(J_{\phi_n} \cap \bigcup\nolimits_{i\in\mathcal{I}} (R_i \setminus \partial^* E^n_{t_i})\big) \le c\mathcal{H}^1\big(J_\phi \cap \bigcup\nolimits_{i\in\mathcal{I}} (Q_i \setminus R_i)\big), \\ &(iii) \ \ \|e(\phi_n)\|_{L^2(\bigcup_{i\in\mathcal{I}} Q_i;\mathbb{R}^{2\times 2}_{\rm sym})} \le c\|e(\phi)\|_{L^2(\bigcup_{i\in\mathcal{I}} Q_i;\mathbb{R}^{2\times 2}_{\rm sym})}. \end{aligned} \tag{51}$$

Taking a sequence $\theta_k \to 0$ generates a sequence ϕ_n by choosing ϕ_n as above using θ_k for $n \in [N(\theta_k), N(\theta_{k+1}))$. With (44)(i) and (51) we immediately deduce (36)(i),(ii).

Finally, to see (36)(iii) we again follow the argumentation in [19] and refer therein for details. By (43), (49)(ii), (50), and (51)(i) we find

$$\mathcal{H}^1\Big(\big((J_{\phi_n} \setminus J_{u_n}) \setminus (J_\phi \setminus J_u)\big) \setminus \big(\bigcup\nolimits_{i\in\mathcal{I}} \overline{R_i}\big)\Big) \le O(\theta).$$

Consequently, to conclude it suffices to show

$$\mathcal{H}^1\Big((J_{\phi_n} \setminus J_{u_n}) \cap \bigcup\nolimits_{i\in\mathcal{I}} \overline{R_i}\Big) \le O(\theta). \tag{52}$$

To this end, we consider $(J_{\phi_n} \setminus J_{u_n}) \cap \overline{R_i}$ for a fixed $i \in \mathcal{I}$ and assume $x_i \in \Omega$ (the case $x_i \in \partial\Omega$ is similar). We break $\overline{R_i}$ into the parts

$$\begin{aligned} \overline{R_i} = \bigcup\nolimits_{k=1}^4 P_i^k :=& (\overline{R_i} \cap \partial^* E^n_{t_i}) \cup (R_i \setminus \partial^* E^n_{t_i}) \cup ((H_i^+ \cup H_i^-) \setminus \partial^* E^n_{t_i}) \\ &\cup (\partial R_i \setminus (H_i^+ \cup H_i^- \cup \partial^* E^n_{t_i})). \end{aligned}$$

First, by (47) we have

[3] See again the proof of [19, Theorem 2.1] for details. Let us just mention that in this context it is crucial that the jump sets of J_u, J_{v_n} are contained in $\overline{\Omega}$, cf. (45), as hereby the function has to be indeed only modified in $\overline{\Omega}$.

$$\sum_{i\in\mathcal{I}}\mathcal{H}^1(P_i^1\setminus J_{u_n})\le O(\theta).$$

Moreover, by (43), (44)(ii),(iii), (49)(ii), and (51)(ii) we derive

$$\begin{aligned}\sum_{i\in\mathcal{I}}\mathcal{H}^1(P_i^2\cap J_{\phi_n}) &\le c\mathcal{H}^1\Big(J_\phi\cap\bigcup_{i\in\mathcal{I}}(Q_i\setminus R_i)\Big)\\ &\le c\mathcal{H}^1\Big((J_\phi\cap J_u)\cap\bigcup_{i\in\mathcal{I}}(Q_i\setminus R_i)\Big)+c\mathcal{H}^1\Big((J_\phi\setminus J_u)\cap\bigcup_i Q_i\Big)\\ &\le O(\theta).\end{aligned}$$

By our construction the only possible jumps of ϕ_n in $H_i^+\cup H_i^-$ are $H_i^+\cap E_{t_i}^n$ and $H_i^-\setminus E_{t_i}^n$ so that

$$\sum_{i\in\mathcal{I}}\mathcal{H}^1(J_{\phi_n}\cap P_i^3)=\sum_{i\in\mathcal{I}}\big(\mathcal{H}^1(H_i^+\cap E_{t_i}^n)+\mathcal{H}^1(H_i^-\setminus E_{t_i}^n)\big)\le O(\theta),$$

where the last inequality follows from (44)(iii) and (49)(i). Finally, the estimate $\sum_{i\in\mathcal{I}}\mathcal{H}^1(P_i^4)\le O(\theta)$ is a consequence of (44)(iii) and $|s_i^+|,|s_i^-|\le\theta r_i$. Collecting the previous estimates we obtain (52). This concludes the proof for $\ell=1$.

Step 2. In the general case $\ell>1$ it suffices to observe that the same trick in (42) can be inductively applied also to a finite number of $GSBD$ functions. If there are sequences $(u_n^l)_n$ in $GSBD^2(\Omega')$ and $u^l\in GSBD^2(\Omega')$, one can find a single sequence $(\bar u_n)_n\subset GSBD^2(\Omega')$ converging to some $\bar u$ in measure with

$$\mathcal{H}^1\Big(J_{\bar u_n}\triangle\bigcup_{l=1}^{\ell}J_{u_n^l}\Big)=\mathcal{H}^1\Big(J_{\bar u}\triangle\bigcup_{l=1}^{\ell}J_{u^l}\Big)=0.$$

With this, we reduce the problem to a single sequence $(\bar u_n)_n$ for which the hypotheses of the theorem are satisfied for a suitable bounded sequence $(\bar w_n)_n$ of boundary data. □

5.2. *Proof of the auxiliary results*

We begin with the Proof of Lemma 5.2.

Proof of Lemma 5.2. We can assume $R=(-l,l)\times(0,h)$ with $l,h>0$ and $R^-=(-l,l)\times(-h,0)$. For a given parameter $0<\xi<1$ and a distribution T on $(-l,l)\times(0,\xi h)$ the symbol T^ξ denotes the distribution on R^- obtained by composition of T with the diffeomorphism $(x,y)\to(x,-\frac{1}{\xi}y)$. We first assume $\phi:=(\phi_1,\phi_2)$ is a regular displacement in the sense of (10). Given $0<\lambda<\mu<1$ and $p>0$ we set for all $(x,y)\in R^-$

$$\begin{aligned}\hat\phi_1(x,y)&=p\phi_1(x,-\lambda y)+(1-p)\phi_1(x,-\mu y)\\ \hat\phi_2(x,y)&=-\lambda p\phi_2(x,-\lambda y)+(1+\lambda p)\phi_2(x,-\mu y)\,.\end{aligned}\tag{53}$$

Furthermore, ϕ and $\hat\phi$ have by construction the same trace on the common boundary $(-l,l)\times\{0\}$ so that no jump occurs there. With this, (37)(i) follows. In order to show (ii), we calculate the component $(E\hat\phi)_{12}$ of the symmetrized distributional gradient of $\hat\phi$. A direct computation gives

$$2(E\hat\phi)_{12}=-\lambda p(\partial_1\phi_2+\partial_2\phi_1)^\lambda+(1+\lambda p)(\partial_1\phi_2)^\mu-\mu(1-p)(\partial_2\phi_1)^\mu\,.$$

Choosing $p=\frac{1+\mu}{\mu-\lambda}$ we get

$$2(E\hat\phi)_{12}=-\lambda p(\partial_1\phi_2+\partial_2\phi_1)^\lambda+(1+\lambda p)(\partial_1\phi_2+\partial_2\phi_1)^\mu\,.$$

Taking the absolutely continuous parts with respect to the Lebesgue measure we derive that the L^2 norm of $(e(\hat\phi))_{12}$ can be controlled with the L^2 norm of $(e(\phi))_{12}$ independently of R and ϕ, which was the only thing to be shown to get (ii).

Before coming to the general case, we notice that the function $\hat\phi$ has the following property: If $\psi:[0,+\infty)\to[0,+\infty)$ is an increasing continuous subadditive function satisfying (6) and $\int_R\psi(|\phi|)\,\mathrm{d}x\le 1$, then

$$\int_{\hat R}\psi(|\hat\phi|)\,\mathrm{d}x\le c\tag{54}$$

again for an absolute constant c independent of R and ϕ. Indeed, it follows from the construction and the properties of ψ that (54) holds for a constant c only depending on λ, μ, and p.

In the general case $\phi \in GSBD^2(R)$ we consider an approximating sequence of displacements with regular jump set $(\phi_k)_k$ in the sense of Theorem 2.10. (Again the reader willing to assume an L^2-bound may replace Theorem 2.10 by Theorem 2.6.) It follows by Remark 2.2 that there exists a nonnegative concave (thus, continuous and subadditive) increasing function ψ satisfying (6) and such that

$$\int_R \psi(|\phi_k|)\,\mathrm{d}x \leq 1$$

for all $k \in \mathbb{N}$. To the functions ϕ_k we associate extensions $\hat{\phi}_k \in GSBD^2(\hat{R})$ satisfying (37) and (54). In particular, there is a constant C independent of k such that

$$\int_{\hat{R}} \psi(|\hat{\phi}_k|) + |e(\hat{\phi}_k)|^2\,\mathrm{d}x + \mathcal{H}^1(J_{\hat{\phi}_k}) \leq C,$$

so that (9) implies the existence of $\hat{\phi} \in GSBD^2(\hat{R})$ such that $\hat{\phi}_k \to \hat{\phi}$ in measure in $\hat{R}$ and

$$\mathcal{H}^1(J_{\hat{\phi}}) \leq \liminf_{k\to+\infty} \mathcal{H}^1(J_{\hat{\phi}_k}) \qquad \|e(\hat{\phi})\|_{L^2(\hat{R};\mathbb{R}^{2\times 2}_{\mathrm{sym}})} \leq \liminf_{k\to+\infty} \|e(\hat{\phi}_k)\|_{L^2(\hat{R};\mathbb{R}^{2\times 2}_{\mathrm{sym}})}. \tag{55}$$

Passing to the limit and using (55), Theorem 2.10, and the corresponding inequalities for ϕ_k we get (37).[4] □

We go further by proving Lemma 5.3.

Proof of Lemma 5.3. We apply Theorem 2.9 to u and find a Caccioppoli partition $\Omega' = \bigcup_{j=1}^\infty P_j$ and infinitesimal rigid motions $(a_j)_{j=1}^\infty$ such that $u - \sum_{j\geq 1} a_j \chi_{P_j}$ lies in $SBV(\Omega';\mathbb{R}^2) \cap L^\infty(\Omega';\mathbb{R}^2)$. Using $\sum_{j=1}^{+\infty} \mathcal{H}^1\left(\partial^* P_j\right) < +\infty$ and Theorem 2.7, we choose j_0 as the smallest index j such that

$$\mathcal{H}^1\big(\bigcup\nolimits_{j\geq j_0} \partial^* P_j\big) + \mathcal{H}^1\big(J_u \cap \bigcup\nolimits_{j\geq j_0} (P_j)^1\big) \leq \epsilon. \tag{56}$$

Then we define $\Omega_{\mathrm{good}} = \bigcup_{j=1}^{j_0-1} P_j$ and see that the function $v := u\chi_{\Omega_{\mathrm{good}}}$ lies in $SBV(\Omega';\mathbb{R}^2) \cap GSBD^2(\Omega') \cap L^\infty(\Omega';\mathbb{R}^2)$. Since by construction we have

$$J_{u-v} \subseteq \bigcup\nolimits_{j\geq j_0} \partial^* P_j \,\cup\, \Big(J_u \cap \bigcup\nolimits_{j\geq j_0} (P_j)^1\Big),$$

we get $\mathcal{H}^1(J_{u-v}) \leq \epsilon$. As $J_u \triangle J_v \subseteq J_{u-v}$, the first part of the statement follows.

For the second part, observe that, since $v = u\chi_{\Omega_{\mathrm{good}}}$, it holds $v = u$ on the set $\{u = 0\}$. Therefore, if $w = 0$, the proof is concluded. In the general case we set $\hat{u} = u - w$ and apply the above procedure to $\hat{u}$ to construct a function $\hat{v}$ with $\mathcal{H}^1(J_{\hat{u}-\hat{v}}) \leq \epsilon$ and $\hat{v} = 0$ in $\Omega' \setminus \overline{\Omega}$, since $\hat{u} = 0$ there. We then conclude setting $v = \hat{v} + w$. □

For the proof of Theorem 5.5 we need the following preliminary lemma, which is a consequence of the piecewise Korn inequality in Theorem 4.1.

Lemma 5.7. *Let $\theta, \delta > 0$ with $\delta\theta^{-2} \leq \frac{1}{4}C_{Q_1}$ with the constant C_{Q_1} from Remark 4.2. Consider a square $Q \subset \mathbb{R}^2$ and $u \in GSBD^2(Q)$, and assume that there are two sets $K_1, K_2 \subset Q$ and $t_1, t_2 \in \mathbb{R}^2$ with*

$$\mathcal{L}^2(K_m) \geq (\tfrac{1}{2} - \delta)\mathcal{L}^2(Q), \qquad \|u - t_m\|_{L^\infty(K_m;\mathbb{R}^2)} \leq \delta, \qquad m = 1, 2. \tag{57}$$

Then there is a universal constant $c > 0$, some $C_\theta = C_\theta(\theta) > 0$, both independent of Q and u, and a modification $u^\theta \in SBV(Q;\mathbb{R}^2) \cap L^\infty(Q;\mathbb{R}^2)$ such that u^θ is constant on $\{u \neq u^\theta\}$ and

[4] Notice that by the explicit construction of $\hat{\phi}_k$ in (53) and the convergence in measure of ϕ_k one can also show that ϕ and $\hat{\phi}$ have the same trace on the common boundary $(-l, l) \times \{0\}$.

$$\begin{aligned} &(i) \quad \mathcal{L}^2(\{u \neq u^\theta\}) \le c\theta(\mathcal{H}^1(J_u \cap Q) + \mathrm{diam}(Q))^2, \\ &(ii) \quad \mathcal{H}^1(\partial^*\{u \neq u^\theta\} \setminus J_u) \le c\theta(\mathcal{H}^1(J_u \cap Q) + \mathrm{diam}(Q)), \\ &(iii) \quad \|\nabla u^\theta\|_{L^1(Q;\mathbb{R}^{2\times 2})} \le C_\theta \,\mathrm{diam}(Q)\big(\|e(u)\|_{L^2(Q;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \delta\big), \\ &(iv) \quad \|u^\theta\|_{L^\infty(Q;\mathbb{R}^2)} \le C_\theta\big(\|e(u)\|_{L^2(Q;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \delta\big) + c(t_1 + t_2). \end{aligned} \tag{58}$$

Remark 5.8. The essential point is that (58)(iii), differently from (18)(iii), holds now for the modification u^θ, coinciding with u outside a small set. Moreover, the estimate for ∇u^θ scales with the diameter of the square which is fundamental for the proof of (39)(ii).

Proof of Lemma 5.7. We apply Theorem 4.1 and obtain $u^\theta \in SBV(Q;\mathbb{R}^2) \cap L^\infty(Q;\mathbb{R}^2)$ as well as $v = u^\theta - \sum_{j=0}^I a_j \chi_{P_j} \in SBV(Q;\mathbb{R}^2)$ for a partition $(P_j)_{j=0}^I$ and infinitesimal rigid motions $(a_j)_{j=0}^I$ such that (17), (18)(i) hold. Recall that $P_0 = \{u \neq u^\theta\}$ (see (18)(ii)) and that u^θ can be defined constantly on P_0. Now (58)(i),(ii) follow from (17). From Remark 4.2 we get

$$\begin{aligned} &(i) \quad \mathcal{L}^2(P_j) \ge C_{Q_1}\mathcal{L}^2(Q)\theta^2 \quad \text{for} \quad 1 \le j \le I, \\ &(ii) \quad \|v\|_{L^\infty(Q;\mathbb{R}^2)} + (\mathrm{diam}(Q))^{-1}\|\nabla v\|_{L^1(Q;\mathbb{R}^{2\times 2})} \le C_{\theta,Q_1}\|e(u)\|_{L^2(Q;\mathbb{R}^{2\times 2}_{\mathrm{sym}})}. \end{aligned} \tag{59}$$

By (57), (59)(i) and the assumption that $\delta\theta^{-2} \le \frac{1}{4}C_{Q_1}$, we find

$$\begin{aligned} \mathcal{L}^2((K_1 \cup K_2) \cap P_j) &\ge \mathcal{L}^2(P_j) - \mathcal{L}^2(Q \setminus (K_1 \cup K_2)) \\ &\ge (C_{Q_1}\theta^2 - 2\delta)\mathcal{L}^2(Q) \ge \frac{1}{2}C_{Q_1}\theta^2\mathcal{L}^2(Q) \end{aligned}$$

for each P_j, $1 \le j \le I$. Fix now $1 \le j \le I$. By the above argument it holds that $\max_{m=1,2}\mathcal{L}^2(K_m \cap P_j) \ge \frac{1}{4}C_{Q_1}\theta^2\mathcal{L}^2(Q)$. Assuming without loss of generality that the maximum is achieved by K_1, we then have by the previous and the isodiametric inequality, Lemma 2.3, (57), and (59)(ii), that

$$\begin{aligned} \theta\mathrm{diam}(Q)|A_j| &\le c\|a_j - t_1\|_{L^\infty(P_j\cap K_1;\mathbb{R}^2)} \le c\|v\|_{L^\infty(P_j\cap K_1;\mathbb{R}^2)} + c\|u - t_1\|_{L^\infty(P_j \cap K_1;\mathbb{R}^2)} \\ &\le cC_{\theta,Q_1}\|e(u)\|_{L^2(Q;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + c\delta \end{aligned}$$

for $c = c(C_{Q_1})$ universal, where we used that $u = u^\theta$ on P_j. Since $\mathcal{L}^2(P_j) \le (\mathrm{diam}(Q))^2$ and $\# I \le c\theta^{-2}$ by (59)(i), we calculate by (59)(ii)

$$\begin{aligned} \|\nabla u^\theta\|_{L^1(Q;\mathbb{R}^{2\times 2})} &= \|\nabla u\|_{L^1(Q\setminus P_0;\mathbb{R}^{2\times 2})} \le \|\nabla v\|_{L^1(Q;\mathbb{R}^{2\times 2})} + \sum\nolimits_{j=1}^I \mathcal{L}^2(P_j)|A_j| \\ &\le C_\theta\mathrm{diam}(Q)\big(\|e(u)\|_{L^2(Q;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \delta\big). \end{aligned}$$

This gives (58)(iii) and finally (58)(iv) can be seen along similar lines using that, for a fixed $1 \le j \le I$,

$$\min_{m=1,2}\|a_j - t_m\|_{L^\infty(Q;\mathbb{R}^2)} \le cC_{\theta,Q_1}\|e(u)\|_{L^2(Q;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + c\delta. \qquad \square$$

We now proceed with the proof of Theorem 5.5.

Proof of Theorem 5.5. Let $Q_* := \bigcup_{i=1}^m Q_i$ be given as in the statement. By Definition 5.4, for each i we find two subsets $K^+_{r_i}(x_i)$ and $K^-_{r_i}(x_i)$ of Q_i such that

$$\mathcal{L}^2(K^+_{r_i}(x_i)) \ge \frac{1}{2}(1-\delta)\mathcal{L}^2(Q_i), \qquad \mathcal{L}^2(K^-_{r_i}(x_i)) \ge \frac{1}{2}(1-\delta)\mathcal{L}^2(Q_i), \tag{60}$$

where $\|u - u^+(x_i)\|_{L^\infty(K^+_{r_i}(x_i);\mathbb{R}^2)} \le \frac{1}{2}\delta$ and $\|u - u^-(x_i)\|_{L^\infty(K^-_{r_i}(x_i);\mathbb{R}^2)} \le \frac{1}{2}\delta$, respectively. For each i and n we set $K^\pm_{i,n} = \{y \in Q_i : |u_n(y) - u^\pm(x)| \le \delta\}$ and observe that due to (60) and the fact that $u_n \to u$ in measure, we obtain for n large enough

$$\mathcal{L}^2(K^+_{i,n}) \geq \Big(\frac{1}{2} - \delta\Big)\mathcal{L}^2(Q_i), \quad \mathcal{L}^2(K^-_{i,n}) \geq \Big(\frac{1}{2} - \delta\Big)\mathcal{L}^2(Q_i).$$

Since $\delta\theta^{-8} \leq \frac{1}{4}C_{Q_1}$, we can now apply Lemma 5.7 on the sequence $(u_n)_n$ and on each Q_i with θ^4 in place of θ, with K_1 and K_2 being given by $K^+_{i,n}$, and $K^-_{i,n}$, respectively, $t_1 = u^+(x_i)$, and $t_2 = u^-(x_i)$. Therefore, we obtain a sequence of functions $v_n^{\delta,\theta,i} \in SBV(Q_i;\mathbb{R}^2) \cap L^\infty(Q_i;\mathbb{R}^2)$ such that (58) holds (with θ^4 in place of θ). Recall that the constants in (58) are independent of i and n. The functions $v_n^{\delta,\theta}$ are then defined as being given by $v_n^{\delta,\theta,i}$ on each of the disjoint squares Q_i.

By (38)(i) and (58)(ii) for each i the perimeter of the sets $\{v_n^{\delta,\theta,i} \neq u_n\}$ is uniformly bounded in n and by a compactness theorem for sets of finite perimeter together with (58)(i) we thus obtain a set $F_i \subset Q_i$ such that $\chi_{\{v_n^{\delta,\theta,i} \neq u_n\}} \to \chi_{F_i}$ in measure as $n \to \infty$, after passing to a suitable (not relabeled) subsequence. Therefore, we obtain by (58)(i) (again with θ^4 in place of θ)

$$\mathcal{L}^2(F_i) \leq \liminf_{n\to\infty} \mathcal{L}^2(\{v_n^{\delta,\theta,i} \neq u_n\}) \leq c \liminf_{n\to\infty} \theta^4(\mathcal{H}^1(J_{u_n} \cap Q_i) + \mathrm{diam}(Q_i))^2,$$

which implies (40). Notice that by construction, the sequence $(v_n^{\delta,\theta,i})_n$ converges to u in measure on $Q_i \setminus F_i$. By (58)(iv), $v_n^{\delta,\theta,i}$ is bounded uniformly in L^∞ so that we deduce

$$\liminf_{n\to\infty} \|v_n^{\delta,\theta,i} - u\|_{L^1(Q_i \setminus F_i;\mathbb{R}^2)} = 0.$$

Recall that, by Lemma 5.7, $v_n^{\delta,\theta,i}$ are constant on the sets $\{v_n^{\delta,\theta,i} \neq u_n\}$, which therefore contain no jump of them. With this, (58)(ii) together with $\sum_{i=1}^m r_i \leq M$ yields (39)(i). Finally, to confirm (39)(ii), we use (58)(iii) to compute by Hölder's inequality and the fact that $r_i \leq \delta^2$, $\sum_{i=1}^m r_i \leq M$

$$\begin{aligned}\|\nabla v_n^{\delta,\theta}\|_{L^1(Q_*;\mathbb{R}^{2\times 2})} &= \sum_{i=1}^m \|\nabla v_n^{\delta,\theta,i}\|_{L^1(Q_i;\mathbb{R}^{2\times 2})} \leq C_\theta \sum_{i=1}^m r_i \Big(\|e(u_n)\|_{L^2(Q_i;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \delta\Big) \\ &\leq C_\theta\Big(\sum\nolimits_{i=1}^m r_i^2\Big)^{\frac{1}{2}} \|e(u_n)\|_{L^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + C_\theta \delta M \leq C_\theta \delta M. \qquad \square\end{aligned}$$

We conclude with the proof of Corollary 5.6.

Proof of Corollary 5.6. Consider the functions $v_n^{\delta,\theta}$ constructed above. By (58)(ii) and the assumption $\sum_{i=1}^m r_i \leq M$ it holds

$$\mathcal{H}^1(\partial^*\{v_n^{\delta,\theta} \neq u_n\} \setminus J_{u_n}) \leq c\theta \sum\nolimits_{i=1}^m (\mathcal{H}^1(J_{u_n} \cap Q_i) + \mathrm{diam}(Q_i)) \leq 2Mc\theta. \tag{61}$$

We now set

$$\hat{v}_n^{\delta,\theta} := (w_n - v_n^{\delta,\theta})\chi_{\{v_n^{\delta,\theta} \neq u_n\}} + v_n^{\delta,\theta}.$$

These functions satisfy $\hat{v}_n^{\delta,\theta} = w_n$ on $Q_* \setminus \overline{\Omega}$ and thus $J_{v_n^{\delta,\theta}} \subset Q_* \cap \overline{\Omega}$. Since the sets F_i were constructed as limits of $\{v_n^{\delta,\theta} \neq u_n\} \cap Q_i$, the new sequence still satisfies (39)(iii). From $J_{\hat{v}_n^{\delta,\theta}} \setminus J_{v_n^{\delta,\theta}} \subset \partial^*\{v_n^{\delta,\theta} \neq u_n\}$ and (61) we get (39)(i).

Finally, the assumptions $\sum_{i=1}^m r_i \leq M$ and $r_i \leq \delta^2$ imply that $\mathcal{L}^2(Q_*) \leq 4\delta^2 M$, so that by Hölder's inequality $\|\nabla w_n\|_{L^1(Q_*;\mathbb{R}^{2\times 2})} \leq 2\delta\sqrt{M}\|\nabla w_n\|_{L^2(Q_*;\mathbb{R}^{2\times 2})}$. With this and the trivial inequality

$$\|\nabla \hat{v}_n^{\delta,\theta}\|_{L^1(Q_*;\mathbb{R}^{2\times 2})} \leq \|\nabla v_n^{\delta,\theta}\|_{L^1(Q_*;\mathbb{R}^{2\times 2})} + \|\nabla w_n\|_{L^1(Q_*;\mathbb{R}^{2\times 2})}$$

we get (39)(ii) for a constant also depending on $\sup_n \|w_n\|_{H^1(\Omega';\mathbb{R}^2)}$. $\square$

6. A general compactness and existence result

Notice that for the compactness theorem in $GSBD$ (see Theorem 2.5) it is necessary that the integral for some integrand ψ with $\lim_{s\to\infty}\psi(s) = \infty$ is uniformly bounded. However, in many application, e.g. in our model presented below, such an a priori bound is not available. Partially following ideas in [20] we now show that by means of Theorem 4.5 it is possible to establish a compactness and existence result for suitably modified functions.

We first prove the following general compactness result.

Theorem 6.1. *Let $\Omega \subset \Omega' \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary such that (13) holds. Let $M > 0$, $w \in H^1(\Omega', \mathbb{R}^2)$ and Γ be a rectifiable set with $\mathcal{H}^1(\Gamma) \le M$. Define*

$$E(u) = \int_{\Omega'} Q(e(u))\,\mathrm{d}x + \mathcal{H}^1(J_u \setminus \Gamma) \tag{62}$$

for $u \in GSBD^2(\Omega')$, where Q is a positive definite quadratic form on $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$.

Then there is an increasing concave function $\psi : [0,\infty) \to [0,\infty)$ satisfying (6) only depending on Ω, Ω', M such that for every sequence $(u_k)_k \subset GSBD^2(\Omega')$ with $\sup_{k\ge 1} E(u_k) \le M$ and $u_k = w$ on $\Omega' \setminus \overline{\Omega}$ we find a (not relabeled) subsequence and modifications $(y_k)_k \subset GSBD^2(\Omega')$ with $y_k = w$ on $\Omega' \setminus \overline{\Omega}$ and

$$E(y_k) \le E(u_k) + \tfrac{1}{k}, \qquad \sup_{k\ge 1} \int_{\Omega'} \psi(|y_k|)\,\mathrm{d}x \le 1. \tag{63}$$

Moreover, there is a function $y \in GSBD^2(\Omega')$ with $y = w$ on $\Omega' \setminus \overline{\Omega}$ such that $\int_{\Omega'} \psi(|y|)\,\mathrm{d}x \le 1$ and for $k \to \infty$

$$\begin{aligned} &(i) \quad y_k \to y \quad \text{in measure on } \Omega',\\ &(ii) \quad e(y_k) \rightharpoonup e(y) \quad \text{weakly in } L^2(\Omega', \mathbb{R}^{2\times 2}_{\mathrm{sym}}),\\ &(iii) \quad \mathcal{H}^1(J_y \setminus \Gamma) \le \liminf_{k\to\infty} \mathcal{H}^1(J_{y_k} \setminus \Gamma). \end{aligned} \tag{64}$$

Note that properties (64)(ii),(iii) also hold with u_k in place of y_k. Moreover, observe that in general a passage to modifications is indispensable since the behavior on components completely detached from the rest of the body cannot be controlled.

Proof. Let be given a sequence $(u_k)_k$ with $E(u_k) \le M$ and $u_k = w$ on $\Omega' \setminus \overline{\Omega}$. This implies $\|e(u_k)\|^2_{L^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \mathcal{H}^1(J_{u_k}) \le cM$ for all $k \in \mathbb{N}$. Let $\theta_l = 2^{-2l}$ for all $l \in \mathbb{N}$. By Theorem 4.5 we find functions $(v_k^l)_k \subset SBV(\Omega';\mathbb{R}^2) \cap L^2(\Omega';\mathbb{R}^2)$ of the form

$$v_k^l = u_k^l - \sum\nolimits_{j=1}^{\infty} a_j^{k,l} \chi_{P_j^{k,l}}, \tag{65}$$

where u_k^l are modifications, $(P_j^{k,l})_j$ are partitions of Ω' and $(a_j^{k,l})_j$ infinitesimal rigid motions. In particular, for all $l \in \mathbb{N}$, $k \in \mathbb{N}$ we have $v_k^l = w$ on $\Omega' \setminus \overline{\Omega}$ and by (31) the modifications satisfy

$$\begin{aligned} &(i) \quad \mathcal{L}^2(E_k^l) \le \bar{c}\theta_l, \quad \mathcal{H}^1(J_{u_k^l} \setminus J_{u_k}) \le \bar{c}\theta_l,\\ &(ii) \quad \|e(u_k^l)\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} \le \|e(u_k)\|^2_{L^2(\Omega;\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \varepsilon_l \end{aligned} \tag{66}$$

for some $\bar{c} = \bar{c}(M, \Omega, \Omega') > 0$, where $E_k^l := \{u \neq u_k^l\}$ and $(\varepsilon_l)_l$ is a null sequence only depending on w. Moreover, by (32) we get

$$(i)\ \|v_k^l\|_{L^2(\Omega';\mathbb{R}^2)} \le \hat{C}_l, \ \ (ii)\ \|e(v_k^l)\|^2_{L^2(\Omega',\mathbb{R}^{2\times 2}_{\mathrm{sym}})} \le cM, \ \ (iii)\ \mathcal{H}^1(J_{v_k^l} \setminus J_{u_k}) \le \bar{c}\theta_l \tag{67}$$

for $\hat{C}_l = \hat{C}_l(\theta_l, M, \Omega, \Omega')$. Here we used, possibly passing to a larger M, that $\varepsilon_l \le M$ for all $l \in \mathbb{N}$. Without restriction we assume that $\hat{C}_l$ is increasing in l.

Using a diagonal argument we get a (not relabeled) subsequence of $(k)_{k\in\mathbb{N}}$ such that by Theorem 2.5 for every $l \in \mathbb{N}$ we find a function $v^l \in GSBD^2(\Omega')$ with $v_k^l \to v^l$ in $L^1(\Omega';\mathbb{R}^2)$ for $k \to \infty$ and

$$e(u_k^l) = e(v_k^l) \rightharpoonup e(v^l) \text{ weakly in } L^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}}), \ \ \mathcal{H}^1(J_{v^l} \setminus \Gamma) \le \liminf_{k\to\infty} \mathcal{H}^1(J_{v_k^l} \setminus \Gamma).$$

In particular, by (67) we have

$$\|v^l\|_{L^1(\Omega';\mathbb{R}^2)} \le \hat{C}_l, \quad \|e(v^l)\|^2_{L^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})} + \mathcal{H}^1(J_{v^l}) \le cM + \bar{c}. \tag{68}$$

Likewise, we can establish a compactness result for the Caccioppoli partitions. By construction (see (65)) and (67)(iii) we have

$$\sum_j \mathcal{H}^1(\partial^* P_j^{k,l} \cap \Omega') \le 2\mathcal{H}^1(J_{u_k} \cup J_{v_k^l}) \le 2cM + 2\bar{c} \tag{69}$$

for all $k, l \in \mathbb{N}$. Thus, by Theorem 2.8 we find for all $l \in \mathbb{N}$ an (ordered) partition $(P_j^l)_j$ with $\sum_j \mathcal{H}^1(\partial^* P_j^l \cap \Omega') \le 2cM + 2\bar{c}$ such that for a suitable subsequence one has $P_j^{k,l} \to P_j^l$ in measure for all $j \in \mathbb{N}$ as $k \to \infty$ and $\sum_j \mathcal{L}^2\left(P_j^{k,l} \triangle P_j^l\right) \to 0$ for $k \to \infty$. As $\sum_j \mathcal{H}^1(\partial^* P_j^l \cap \Omega') \le 2cM + 2\bar{c}$ for all $l \in \mathbb{N}$, we can repeat the arguments and obtain a partition $(P_j)_j$ such that $\sum_j \mathcal{L}^2\left(P_j^l \triangle P_j\right) \to 0$ for $l \to \infty$ after extracting a suitable subsequence. Consequently, using a diagonal argument we can choose a (not relabeled) subsequence of $(l)_{l\in\mathbb{N}}$ and afterwards of $(k)_{k\in\mathbb{N}}$ such that

$$\sum_j \mathcal{L}^2\left(P_j^l \triangle P_j\right) \le 2^{-l}, \qquad \sum_j \mathcal{L}^2\left(P_j^{k,l} \triangle P_j^l\right) \le 2^{-l} \quad \text{for all } k \ge l. \tag{70}$$

We now want to pass to the limit $l \to \infty$ for the sequence $(v^l)_l$. However, we see that the compactness result in $GSBD$ cannot be applied directly as the L^1 bound depends on θ_l (cf. (68)). We show that by choosing the infinitesimal rigid motions on the elements of the partitions appropriately (see (65)) we can construct the sequence $(v^l)_l$ such that

$$\mathcal{L}^2\left(\bigcap\nolimits_{m\ge n}\{|v^n - v^m| \ge 1\}\right) \le \hat{c}2^{-n} \quad \text{for all } n \in \mathbb{N} \tag{71}$$

for a constant $\hat{c} = \hat{c}(M, \Omega, \Omega') > 0$, whence Lemma 2.1 is applicable.

We fix $k \in \mathbb{N}$ and describe an iterative procedure to redefine $a_j^{k,l} = a_{A_j^{k,l}, b_j^{k,l}}$ for all $l, j \in \mathbb{N}$. Let $\hat{v}_k^1 = v_k^1$ as defined in (65) and assume $\hat{v}_k^l$ as well as $(\hat{a}_j^{k,l})_j$ have been chosen (which may differ from $(a_j^{k,l})_j$) such that (67)(i) still holds possibly passing to a larger constant $\hat{C}_l$. Fix some $P_j^{k,l+1}$, $j \in \mathbb{N}$. If $\mathcal{L}^2(P_j^{k,l} \cap P_j^{k,l+1}) > 3\bar{c}\theta_l$, we define $\hat{a}_j^{k,l+1} = \hat{a}_j^{k,l}$ on $P_j^{k,l+1}$. Otherwise, we set $\hat{a}_j^{k,l+1} = a_j^{k,l+1}$. In the first case we then obtain by the triangle inequality and the fact that $u_k^l = u$ on $\Omega' \setminus E_k^l$

$$\begin{aligned}
&\|\hat{a}_j^{k,l+1} - a_j^{k,l+1}\|_{L^2((P_j^{k,l} \cap P_j^{k,l+1}) \setminus (E_k^l \cup E_k^{l+1});\mathbb{R}^2)} \\
&\quad \le \|u - \hat{a}_j^{k,l}\|_{L^2(\Omega' \setminus E_k^l;\mathbb{R}^2)} + \|u - a_j^{k,l+1}\|_{L^2(\Omega' \setminus E_k^{l+1};\mathbb{R}^2)} \\
&\quad \le \|\hat{v}_k^l\|_{L^2(\Omega';\mathbb{R}^2)} + \|v_k^{l+1}\|_{L^2(\Omega';\mathbb{R}^2)} \le \hat{C}_l + \hat{C}_{l+1} \le 2\hat{C}_{l+1}.
\end{aligned}$$

In the penultimate step we have used that (67)(i) holds for $\hat{v}_k^l$ and v_k^{l+1}. By (66)(i) we get $\mathcal{L}^2((P_j^{k,l} \cap P_j^{k,l+1}) \setminus (E_k^l \cup E_k^{l+1})) \ge \bar{c}\theta_l$. Consequently, by Lemma 2.3 for $\psi(s) = s^2$ we find $|\hat{A}_j^{k,l+1} - A_j^{k,l+1}| + |\hat{b}_j^{k,l+1} - b_j^{k,l+1}| \le C_*^{l+1}$ for a constant C_*^{l+1} only depending on Ω, Ω', $\hat{C}_{l+1}$, θ_l and M. We define $\hat{v}_k^{l+1}$ as in (65) replacing $a_j^{k,l+1}$ by $\hat{a}_j^{k,l+1}$ and summing over all components we derive

$$\|\hat{v}_k^{l+1}\|_{L^2(\Omega';\mathbb{R}^2)} \le \|v_k^{l+1}\|_{L^2(\Omega';\mathbb{R}^2)} + C_{\Omega'}C_*^{l+1} \le \hat{C}_{l+1} + C_{\Omega'}C_*^{l+1}$$

for a constant $C_{\Omega'}$ depending only on Ω'. I.e., (67)(i) is also satisfied for $\hat{v}_k^{l+1}$ after possibly passing to a larger constant $\hat{C}_{l+1} = \hat{C}_{l+1}(\theta_{l+1}, M, \Omega, \Omega')$.

For simplicity the modified functions and the infinitesimal rigid motions will still be denoted by v_k^l and $a_j^{k,l}$ in the following. We now show that (71) holds. To this end, we define $A_{k,l}^n = \bigcap_{n\le m\le l}\{v_k^m = v_k^n\}$ for all $n \in \mathbb{N}$ and $n \le l \le k$. If we show

$$\mathcal{L}^2\left(\Omega' \setminus A_{k,l}^n\right) \le \hat{c}2^{-n}, \tag{72}$$

then (71) follows. Indeed, for given $l \ge n$ we can choose $K = K(l) \ge l$ so large that $\mathcal{L}^2\left(\{|v_K^m - v^m| > \frac{1}{2}\}\right) \le 2^{-m}$ for all $n \le m \le l$ since $v_k^m \to v^m$ in $L^1(\Omega';\mathbb{R}^2)$ for $k \to \infty$. This implies

$$\mathcal{L}^2\left(\bigcap\nolimits_{n\le m\le l}\{|v^m - v^n|\ge 1\}\right)$$
$$\le \mathcal{L}^2\left(\Omega'\setminus A^n_{K,l}\right) + \sum\nolimits_{n\le m\le l}\mathcal{L}^2\left(\{|v^m_K - v^m| > \tfrac{1}{2}\}\right) \le \hat{c}2^{-n}.$$

Passing to the limit $l\to\infty$ we then derive $\mathcal{L}^2\left(\bigcap_{m\ge n}\{|v^m - v^n|\ge 1\}\right)\le \hat{c}2^{-n}$, as desired.

We now confirm (72). To this end, fix $k\ge l$ and first observe that by (65) and (66)(i)

$$\mathcal{L}^2\left(\bigcap\nolimits_{n\le m\le l}\{T^n_k = T^m_k\}\setminus A^n_{k,l}\right) \le \sum\nolimits_{n\le m\le l}\mathcal{L}^2(E^m_k) \le 2\bar{c}\theta_n \le \bar{c}2^{-n}, \tag{73}$$

where $T^n_k = \sum_j a^{k,n}_j \chi_{P^{k,n}_j}$. We consider $\{T^m_k = T^{m+1}_k\}$ for $n\le m\le l-1$ and from (70) we deduce

$$\sum_j \mathcal{L}^2\left(P^{k,m+1}_j \triangle P^{k,m}_j\right) \le 3\cdot 2^{-m}.$$

Define $J_1\subset\mathbb{N}$ such that $\mathcal{L}^2\left(P^{k,m+1}_j\right)\le 6\bar{c}\theta_m$ for $j\in J_1$. Then let $J_2\subset\mathbb{N}\setminus J_1$ such that $\mathcal{L}^2\left(P^{k,m+1}_j\cap P^{k,m}_j\right) > \frac{1}{2}\mathcal{L}^2\left(P^{k,m+1}_j\right)$ for all $j\in J_2$. Finally, we observe that $\mathcal{L}^2\left(P^{k,m+1}_j\right)\le 2\mathcal{L}^2\left(P^{k,m+1}_j\setminus P^{k,m}_j\right)$ for $j\in J_3 := \mathbb{N}\setminus(J_1\cup J_2)$. Using the isoperimetric inequality and (69) we derive

$$\sum\nolimits_{j\in J_1}\mathcal{L}^2\left(P^{k,m+1}_j\right) \le \sqrt{6\bar{c}\theta_m}\sum\nolimits_{j\in J_1}\mathcal{L}^2\left(P^{k,m+1}_j\right)^{\frac{1}{2}}$$
$$\le c2^{-m}\sum\nolimits_{j\in J_1}\mathcal{H}^1(\partial^* P^{k,m+1}_j) \le c(M+\bar{c})2^{-m}.$$

Due to the above construction of the infinitesimal rigid motions we obtain $\{T^m_k = T^{m+1}_k\}\supset\bigcup_{j\in J_2}(P^{k,m+1}_j\cap P^{k,m}_j)$ and therefore

$$\mathcal{L}^2\left(\Omega'\setminus\{T^m_k = T^{m+1}_k\}\right) \le \sum_{j\in J_2}\mathcal{L}^2\left(P^{k,m+1}_j\setminus P^{k,m}_j\right) + \sum_{j\in J_1\cup J_3}\mathcal{L}^2\left(P^{k,m+1}_j\right)$$
$$\le \sum_{j\in J_2}\mathcal{L}^2\left(P^{k,m+1}_j\setminus P^{k,m}_j\right) + \sum_{j\in J_3}2\mathcal{L}^2\left(P^{k,m+1}_j\setminus P^{k,m}_j\right) + c(M+\bar{c})2^{-m} \le c2^{-m}$$

for c only depending on M,Ω,Ω'. Summing over $n\le m\le l-1$ and recalling (73), we establish (72) and consequently (71).

In view of (68) and (71) we can apply Lemma 2.1 on the sequences $s_l = \hat{C}_l$ and $t_l = \hat{c}2^{-l}$ to obtain an increasing, concave function $\tilde{\psi}$ with (6) such that $\sup_{l\ge 1}\int_{\Omega'}\tilde{\psi}(|v^l|)\,\mathrm{d}x\le 1$. Define $\psi(s) = \frac{1}{2}\min\{\tilde{\psi}(s), s\}$ and observe that ψ has the desired properties. In particular, the choice of ψ only depends on Ω,Ω' and M. Recalling $v^l_k\to v^l$ in $L^1(\Omega';\mathbb{R}^2)$, (66) and (67)(iii) we can now select a subsequence of $(u_k)_k$ and a diagonal sequence $(y_k)\subset(v^l_k)_{k,l}$ such that $\|y_k - v^l\|_{L^1(\Omega';\mathbb{R}^2)}\le 1$ for some v^l and $E(y_k)\le E(u_k)+\frac{1}{k}$. Then we get that (63) holds.

The existence of a function $y\in GSBD^2(\Omega')$ with $y = w$ on $\Omega'\setminus\overline{\Omega}$ and $\int_{\Omega'}\psi(|y|)\,\mathrm{d}x\le 1$ as well as the convergence (64) now directly follow from Theorem 2.5. □

As a consequence we now obtain the following existence result.

Theorem 6.2. *Let $\Omega\subset\Omega'\subset\mathbb{R}^2$ open, bounded with Lipschitz boundary such that (13) holds. Let $w\in H^1(\Omega',\mathbb{R}^2)$ with $\|w\|_{H^1(\Omega';\mathbb{R}^2)}\le M$ and E as given in (62). Then the following holds:*

(i) There is a minimizer of $E(u)$ among all functions $u\in GSBD^2(\Omega')$ with $u = w$ on $\Omega'\setminus\overline{\Omega}$.
(ii) There is an increasing concave function $\psi:[0,\infty)\to[0,\infty)$ with (6) only depending on Ω,Ω',M such that $\int_{\Omega'}\psi(|u|)\,\mathrm{d}x\le 1$ for at least a minimizer u of the minimization problem in (i).

Proof. Let $\mathcal{A} := \{u\in GSBD(\Omega') : u = w \text{ on } \Omega'\setminus\overline{\Omega}\}$ and $(u_k)_k\subset\mathcal{A}$ with $E(u_k)\to\inf_{u\in\mathcal{A}}E(u)$. We employ Theorem 6.1 and let $(y_k)_k$ be a (sub-)sequence of modifications converging to $u\in\mathcal{A}$ in the sense of (64). Then we find by (63), (64)

$$E(u) \leq \liminf_{k\to\infty} E(y_k) = \liminf_{k\to\infty} E(u_k) = \inf_{u\in\mathcal{A}} E(u).$$

Consequently, u is a minimizer for the problem (i). Moreover, by Theorem 6.1 we find a function ψ with the desired properties such that $\int_{\Omega'} \psi(|u|)\,\mathrm{d}x \leq 1$. □

Remark 6.3. By inspection of the proof, the above compactness and existence result also holds for more general energies in $GSBD^2$ of the form

$$\int_{\Omega'} f(x, e(u)(x))\,\mathrm{d}x + \int_{J_u\setminus\Gamma} g(x,\nu)\mathrm{d}\mathcal{H}^1$$

which are lower semicontinuous with respect to the convergence in measure. Here, it is crucial that the surface density g, while possibly depending on the material point and the orientation of the jump, is insensitive to the jump height. Likewise, the existence result stated in Section 3 may be generalized in this direction.

We also mention that, in the same spirit, a derivation of an existence result in the realm of finite elasticity (see [14]) without a-priori bounds on the deformations or applied body forces is possible. We defer a more thorough analysis of these issues to a subsequent work.

We later will use property (ii) to derive compactness in $GSBD^2$ of the minimizers of our incremental problems. Concerning the stability of minimizers with respect to converging sequences of boundary data we have the following corollary being a consequence of the jump transfer lemma. As before Q is a strictly positive quadratic form on $\mathbb{R}^{2\times 2}_{\mathrm{sym}}$.

Corollary 6.4. *Let $\Omega \subset \Omega' \subset \mathbb{R}^2$ open, bounded with Lipschitz boundary such that (13) holds. Let $\Gamma \subset \mathbb{R}^2$ be a measurable set with $\mathcal{H}^1(\Gamma) < \infty$, let $(u_n)_n, u \in GSBD^2(\Omega')$ and $u_n = w_n$ in $\Omega' \setminus \overline{\Omega}$ for $(w_n)_n \subset H^1(\Omega';\mathbb{R}^2)$ such that $u_n \to u$ in measure, $e(u_n) \rightharpoonup e(u)$ weakly in $L^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})$. If u_n minimize*

$$\int_{\Omega} Q(e(v))\,\mathrm{d}x + \mathcal{H}^1(J_v \setminus (J_{u_n} \cup \Gamma))$$

among all functions with the same Dirichlet data, then u minimizes

$$\int_{\Omega} Q(e(v))\,\mathrm{d}x + \mathcal{H}^1(J_v \setminus (J_{u} \cup \Gamma))$$

among all functions v such that $v = u$ on $\Omega' \setminus \overline{\Omega}$. If furthermore $(w_n)_n$ is a constant sequence, we have $e(u_n) \to e(u)$ strongly in $L^2(\Omega';\mathbb{R}^{2\times 2}_{\mathrm{sym}})$.

The proof is omitted as it is completely analogous to Corollary 2.10 in [19] provided one substitutes the Dirichlet energy with the linearized elastic energy and the gradient by the symmetrized gradient.

7. Proof of the existence result

Equipped with the theoretical results in the previous section, we can obtain the announced existence result Theorem 3.1 by passing to the limit in the usual scheme of time-incremental minimization. The discussion in this section will closely follow the analogous one in [19, Section 3] and therefore not all the proofs will be detailed. For the reader's convenience we will only focus on some points where our $GSBD^2$ setting involves some modifications of the arguments developed there. Through all this section we will write $\mathcal{H}^1(\Gamma)$ in place of $\mathcal{H}^1(\Gamma \cap \Omega')$, since all the cracks we consider in the proof will have by construction no intersection with $\partial\Omega \setminus \partial_D\Omega$.

We fix a time interval $[0, T]$ and consider a countable dense subset I_∞ thereof. We can assume that 0 and T belong to I_∞. For each $n \in \mathbb{N}$ we choose a subset $I_n := \{0 = t_0^n < t_1^n < \cdots < t_n^n = T\}$ such that $(I_n)_n$ form an increasing sequence of nested sets whose union is I_∞. Setting $\Delta_n := \sup_{1\leq k\leq n} (t_k^n - t_{k-1}^n)$, we have that $\Delta_n \to 0$ when $n \to +\infty$. As discussed in Section 3, we consider a boundary datum $g \in W^{1,1}([0,T]; H^1(\mathbb{R}^2;\mathbb{R}^2))$ and the corresponding left-continuous piecewise constant interpolation

$$g^n(t) := g(t^n_k) \text{ for all } t \in [t^n_k, t^n_{k+1})$$

which satisfies $g(t) = g^n(t)$ for all $t \in I_\infty$, when n is large enough. Moreover, $g^n(t) \to g(t)$ strongly in H^1 for all $t \in [0, T]$. We set $u^n(0) = u(0)$, the given initial datum, while for all $k = 1, \dots, n$ we recursively define u^n_k as a minimizer of the problem

$$\int_\Omega Q(e(v))\,\mathrm{d}x + \mathcal{H}^1\Bigg(J_v \setminus \bigcup_{0\le j\le k-1} J_{u^n_j}\Bigg) \tag{74}$$

among the functions $v \in GSBD^2(\Omega')$ satisfying $v = g(t^n_k)$ in $\Omega' \setminus \overline{\Omega}$. The existence of such a minimizer follows from Theorem 6.2. We then construct left-continuous piecewise constant interpolation

$$u^n(t) := u^n_k \text{ for all } t \in [t^n_k, t^n_{k+1})\,.$$

The following a-priori estimates on the interpolations can be then derived combining similar arguments as those developed in [15] and [19] with the additional property (ii) of Theorem 6.2.

Lemma 7.1. *There exists an increasing concave function $\psi : [0, \infty) \to [0, \infty)$, satisfying (6), which only depends on Ω, Ω' and $\sup_{t\in[0,T]} \|g(t)\|_{H^1}$, such that the interpolations $u^n(t)$ satisfy*

$$\int_{\Omega'} \psi(|u^n(t)|)\,\mathrm{d}x + \|e(u^n(t))\|_{L^2(\Omega';\mathbb{R}^{2\times2}_{\mathrm{sym}})} + \mathcal{H}^1\Bigg(\bigcup_{\tau\in I_\infty,\,\tau\le t} J_{u^n(\tau)}\Bigg) \le M \tag{75}$$

for a constant M independent of $t \in [0, T]$. Furthermore, setting $\sigma^n(t) := \mathbb{C}e(u^n(t))$ with $\mathbb{C}$ as in (12), it exists a modulus of continuity ω such that the following energy inequality holds at every $t \in [0, T]$:

$$\int_\Omega Q(e(u^n(t)))\,\mathrm{d}x + \mathcal{H}^1\Bigg(\bigcup_{\tau\in I_\infty,\,\tau\le t} J_{u^n(\tau)}\Bigg)$$
$$\le \int_\Omega Q(e(u(0)))\,\mathrm{d}x + \mathcal{H}^1\left(J_{u(0)}\right) + \int_0^t \langle\sigma^n(s), e(\dot g(s))\rangle\,\mathrm{d}s + \omega(\Delta_n)\,. \tag{76}$$

Proof. The bound on $\|e(u^n(t))\|_{L^2(\Omega',\mathbb{R}^{2\times2}_{\mathrm{sym}})}$ is simply obtained by comparing the minimizer $u^n(t)$ with the admissible competitor $g^n(t)$, while the existence of a ψ as in (75) follows from (ii) in Theorem 6.2 and the assumptions on g. Fix now $t \in [0, T]$, and for fixed n, let k be such that $t \in [t^n_k, t^n_{k+1})$. By construction, since $I_n \subset I_\infty$, one has

$$\bigcup_{\tau\in I_\infty,\tau\le t} J_{u^n(\tau)} = \bigcup_{j=0}^k J_{u^n_j}.$$

Testing for every $1 \le j \le k$ the minimality of $u^n(t^n_j)$ with the admissible competitor $u^n(t^n_{j-1}) + g(t^n_j) - g(t^n_{j-1})$, summing up all steps until step k and using the above equality, we obtain (76) (for the details, use the same arguments leading to [19, (3.4)], upon replacing the Dirichlet energy with the linearized elastic energy). Once (76) is proved, the uniform a-priori bound on $\mathcal{H}^1\left(\bigcup_{\tau\in I_\infty,\tau\le t} J_{u^n(\tau)}\right)$ simply follows by the Cauchy–Schwarz inequality and the already proven bound on $\sigma^n(t)$. □

The following lower semicontinuity result will be needed in order to pass to the limit in the previous bounds. We do not report the proof, which is *verbatim* the same as in [19, Lemma 3.1], provided one uses the $GSBD$ compactness and lower semicontinuity theorem in place of the one in SBV.

Lemma 7.2. *Let $A \subset \mathbb{R}^2$ be open, bounded. For all $\ell \in \mathbb{N}$, let $(v^n_\ell)_n$ be a sequence of functions in $GSBD^2(A)$ satisfying the assumptions of Theorem 2.5, and let $v_\ell \in GSBD^2(A)$ be such that $v^n_\ell \to v_\ell$ in measure when $n \to +\infty$. Then*

$$\mathcal{H}^1\left(\bigcup_{\ell=0}^{+\infty} J_{v_\ell}\right) \le \liminf_{n\to+\infty} \mathcal{H}^1\left(\bigcup_{\ell=0}^{+\infty} J_{v_\ell^n}\right).$$

Using the bounds in (75), we will initially define $u(t)$ only for $t \in I_\infty$. This will already allow us to define a crack set $\Gamma(t)$ for all $t \in [0,T]$ with $J_{u(t)} \subset \Gamma(t)$ for $t \in I_\infty$. The function $u(t)$ will be later extended to all t in a way that the inclusion $J_{u(t)} \subset \Gamma(t)$ still holds.

Theorem 7.3. *There exists a (not relabeled) subsequence $(u^n(t))_n$ independently of $t \in I_\infty$ and a function $u\colon I_\infty \to GSBD^2(\Omega')$ such that $u^n(t) \to u(t)$ in measure for all $t \in I_\infty$ and, setting*

$$\Gamma(t) := \bigcup_{\tau\in I_\infty,\, \tau\le t} J_{u(\tau)} \text{ for all } t \in [0,T], \tag{77}$$

the following properties are satisfied:

$$\begin{aligned} &(i)\quad u(t) = g(t) \quad \text{in } \Omega'\setminus\overline{\Omega} \text{ for all } t\in I_\infty,\\ &(ii)\quad e(u^n(t)) \to e(u(t)) \quad \text{strongly in } L^2(\Omega', \mathbb{R}^{2\times2}_{\text{sym}}) \text{ for all } t \in I_\infty,\\ &(iii)\quad \mathcal{H}^1(\Gamma(t)) \le \liminf_{n\to\infty} \mathcal{H}^1\left(\bigcup_{\tau\in I_\infty,\, \tau\le t} J_{u^n(\tau)}\right) \text{ for all } t\in[0,T]. \end{aligned} \tag{78}$$

Furthermore, for all $t \in I_\infty$, $u(t)$ minimizes

$$\int_\Omega Q(e(v))\,\mathrm{d}x + \mathcal{H}^1(J_v \setminus \Gamma(t)) \tag{79}$$

among all functions v such that $v = g(t)$ on $\Omega'\setminus\overline{\Omega}$.

Proof. By (75) the sequence $(u^n(t))_n$ satisfies the assumptions of Theorem 2.5 for every $t \in I_\infty$. With this, up to extracting a diagonal sequence, there exists $u\colon I_\infty \to GSBD^2(\Omega')$ such that $u^n(t) \to u(t)$ in measure and $e(u^n(t)) \rightharpoonup e(u(t))$ weakly in $L^2(\Omega', \mathbb{R}^{2\times2}_{\text{sym}})$ for all $t \in I_\infty$. Since $u^n(t) = g^n(t)$ in $\Omega'\setminus\overline{\Omega}$ and $g^n(t) = g(t)$ for n large enough, (78)(i) follows. At the expense of a numbering of I_∞, (78)(iii) follows from Lemma 7.2.

From the definition of $u^n(t)$ and $g^n(t)$ (cf. (74)), for all $t \in [0,T]$ we have that $u^n(t)$ is minimizing

$$\int_\Omega Q(e(v))\,\mathrm{d}x + \mathcal{H}^1\left(J_v \setminus \bigcup_{\tau\in I_n,\, \tau\le t} J_{u^n(\tau)}\right) \tag{80}$$

among the functions $v \in GSBD^2(\Omega')$ satisfying $v = g^n(t)$ in $\Omega'\setminus\overline{\Omega}$. *A fortiori*, we deduce that $u^n(t)$ is a minimizer with respect to its own jump set, that is with $J_{u^n(t)}$ in place of $\bigcup_{\substack{\tau\in I_n\\ \tau\le t}} J_{u^n(\tau)}$ in the above problem. If additionally $t \in I_\infty$, we can choose n so large that $t \in I_n \cap I_\infty$, and thus $g^n(t) = g(t)$. With this, Corollary 6.4 gives (78)(ii).

We now fix $\delta > 0$ and $t \in I_\infty$. Since $\mathcal{H}^1(\Gamma(t))$ is finite, we can find $\ell \in \mathbb{N}$ so that $t \in I_\ell$ and the subset $\Gamma_\ell(t)$ of $\Gamma(t)$ defined by

$$\Gamma_\ell(t) = \bigcup_{\tau\in I_\ell,\, \tau\le t} J_{u(\tau)}$$

satisfies $\mathcal{H}^1(\Gamma(t)\setminus\Gamma_\ell(t)) < \delta$. For all $n \ge \ell$, we similarly define $\Gamma_\ell^n(t)$ with $u^n(\tau)$ in place of $u(\tau)$. Notice that $J_{u(t)} \subset \Gamma_\ell(t)$ and $J_{u^n(t)} \subset \Gamma^n_\ell(t)$ since $t \in I_\ell$. With this and using (80) we have that $u^n(t)$ is minimizing $\int_\Omega Q(e(v))\,\mathrm{d}x + \mathcal{H}^1\left(J_v \setminus \Gamma_\ell^n(t)\right)$ among the functions $v \in GSBD^2(\Omega')$ which satisfy $v = g(t)$ in $\Omega'\setminus\overline{\Omega}$.

We observe that by Lemma 7.1 the sequences $(u^n(\tau))_n$ with $\tau \in I_\ell$, $\tau \le t$, and the corresponding limiting functions $u(\tau)$ defined above satisfy (35). Consequently, for any v with $v = g(t)$ in $\Omega'\setminus\overline{\Omega}$ we can apply Theorem 5.1 to $\phi = v - u(t)$ and to the finite unions of jump sets $\Gamma_\ell^n(t)$ and $\Gamma_\ell(t)$. Therefore, we get the existence of a sequence $(\phi_n)_n$ such that $\phi_n = v - u(t) = 0$ in $\Omega'\setminus\overline{\Omega}$ satisfying, by (36) and (78)(ii),

$$\|e(u^n(t)+\phi_n)-e(v)\|_{L^2(\Omega',\mathbb{R}^{2\times 2}_{\rm sym})}\to 0, \qquad \limsup_{n\to+\infty}\mathcal{H}^1(J_{\phi_n}\setminus\Gamma^n_\ell(t))\le\mathcal{H}^1(J_\phi\setminus\Gamma_\ell(t)) \tag{81}$$

as $n\to+\infty$. Furthermore, since $t\in I_\infty$, when n is so big that $g^n(t)=g(t)$ in $\Omega'\setminus\overline{\Omega}$ we have that $u^n(t)+\phi_n=g(t)$ in $\Omega'\setminus\overline{\Omega}$. The minimality of $u^n(t)$, (78)(ii), and (81) then imply that

$$\begin{aligned}&\int_\Omega Q(e(u(t)))\,\mathrm{d}x=\lim_{n+\infty}\int_\Omega Q(e(u^n(t)))\,\mathrm{d}x\\&\le\limsup_{n\to+\infty}\int_\Omega Q(e(u^n(t)+\phi_n))\,\mathrm{d}x+\mathcal{H}^1\left(J_{u^n(t)+\phi_n}\setminus\Gamma^n_\ell(t)\right)\\&\le\int_\Omega Q(e(v))\,\mathrm{d}x+\mathcal{H}^1\left(J_v\setminus\Gamma_\ell(t)\right)\le\int_\Omega Q(e(v))\,\mathrm{d}x+\mathcal{H}^1\left(J_v\setminus\Gamma(t)\right)+\delta\,,\end{aligned}$$

where in the third step we used that $J_{u(t)}\subset\Gamma_\ell(t)$ and $J_{u^n(t)}\subset\Gamma^n_\ell(t)$. This concludes the proof of (79) since δ is arbitrary. □

Remark 7.4. Let $t\notin I_\infty$ and let $w\in GSBD^2(\Omega')$ be such that $(u^n(t))_n$ has a subsequence, possibly depending on t, which converges to w in the sense of (9). Fix $\delta>0$ and $\Gamma_\ell(t)$ and $\Gamma^n_\ell(t)$ as in the previous proof, without the request $t\in I_\ell$. We can apply Theorem 5.1 for the finite number of sequences $(u^n(t))_n$ and $(u^n(\tau))_n$ with $\tau\in I_\ell$, $\tau\le t$, and thus for any v with $v=g(t)$ in $\Omega'\setminus\overline{\Omega}$, we can apply (36) to $\phi=v$ to obtain a corresponding sequence $(\phi_n)_n$. It follows now from (80) that (with $v_n:=\phi_n+g^n(t)-g(t)$)

$$\int_\Omega Q(e(u^n(t)))\,\mathrm{d}x\le\int_\Omega Q(e(v_n))\,\mathrm{d}x+\mathcal{H}^1(J_{v_n}\setminus\left(\Gamma^n_\ell(t)\cup J_{u^n(t)}\right)).$$

By (9)(ii), the strong convergence of $g^n(t)$ to $g(t)$ in H^1, (36) and the arbitrariness of δ we deduce the minimality property

$$\int_\Omega Q(e(w))\,\mathrm{d}x\le\int_\Omega Q(e(v))\,\mathrm{d}x+\mathcal{H}^1(J_v\setminus(\Gamma(t)\cup J_w)). \tag{82}$$

For $v=w$ one also gets $\lim_{n\to+\infty}\int_\Omega Q(e(u^n(t)))\,\mathrm{d}x=\int_\Omega Q(e(w))\,\mathrm{d}x$, which implies

$$\|e(u^n(t))-e(w)\|_{L^2(\Omega',\mathbb{R}^{2\times 2}_{\rm sym})}\to 0 \tag{83}$$

by the strict convexity of Q.

In the next theorem we extend u from I_∞ to a function defined on all of $[0,T]$. We prove that this extension satisfies the inclusion $J_{u(t)}\subset\Gamma(t)$ for all $t\in[0,T]$ (notice that, at this stage of the proof, the crack set $\Gamma(t)$ is already defined on the whole interval $[0,T]$), the global minimality condition, as well as the "$\le$"-inequality in the energy balance of Theorem 3.1. The proof follows very closely in the footsteps of [19, Lemma 3.8]: A sketch is reported for the reader's convenience.

Theorem 7.5. *There exists a function $u\colon[0,T]\to GSBD^2(\Omega')$ with $u(t)=g(t)$ in $\Omega'\setminus\overline{\Omega}$ and an $\mathcal{H}^1$-rectifiable crack $\Gamma(t)\subset\overline{\Omega}$, nondecreasing in t, such that $J_{u(t)}\subset\Gamma(t)$ up to an $\mathcal{H}^1$-negligible set for all $t\in[0,T]$ and:*

- *(global stability) for all $t\in[0,T]$, $u(t)$ minimizes*

$$\int_\Omega Q(e(v))\,\mathrm{d}x+\mathcal{H}^1\left(J_v\setminus\Gamma(t)\right)$$

among the functions $v\in GSBD^2(\Omega')$ which satisfy $v=g(t)$ in $\Omega'\setminus\overline{\Omega}$.

- *(energy inequality) defining the stress $\sigma(t)$ and the total energy $\mathcal{E}(t)$ as in Theorem 3.1, it holds*

$$\mathcal{E}(t) \le \mathcal{E}(0) + \int_0^t \langle \sigma(s), e(\dot{g}(s)) \rangle \, \mathrm{d}s \,.$$

Proof. We consider $u\colon I_\infty \to GSBD^2(\Omega')$ as in Theorem 7.3. Accordingly, we define $\Gamma(t)$ as in (77) for all $t \in [0,T]$. Thus, we simply have to define u when $t \notin I_\infty$. We fix $t \notin I_\infty$ and an increasing sequence $(t_k)_k \subset I_\infty$ converging to t. Notice that for the interpolants $u^n(t)$ the inequality (75) holds with a constant M and a function ψ which are not depending on k. Since, for all k, $u^n(t_k) \to u(t_k)$ in measure when $n \to +\infty$ and thus also $u^n(t_k) \to u(t_k)$ a.e. for a not relabeled subsequence, by Fatou's lemma and (9), also the sequence $(u(t_k))_k$ satisfies (75). Then, it exists a limit point $u(t) \in GSBD^2(\Omega')$ with $u(t_k) \to u(t)$ in measure and $e(u(t_k)) \rightharpoonup e(u(t))$ weakly in L^2 as $k \to \infty$. It is obvious that, $u(t) = g(t)$ in $\Omega' \setminus \overline{\Omega}$ while an application of (79) together with the arguments leading to [19, (3.24)], again simply using $GSBD$ in place of SBV compactness, shows that the inclusion $J_{u(t)} \subset \Gamma(t)$ holds up to an $\mathcal{H}^1$-negligible set.

We now prove the global stability property. Notice that for all k one has by definition $\Gamma(t_k) \subset \Gamma(t)$ and, since the sequence of cracks $\Gamma(t_k)$ is nondecreasing, it holds that $\mathcal{H}^1(\Gamma(t) \setminus \Gamma(t_k)) \to 0$ when $k \to +\infty$. For each $v \in GSBD^2(\Omega')$ with $v = g(t)$ in $\Omega' \setminus \overline{\Omega}$, the sequence $v_k = v + g(t_k) - g(t)$ has the same jump set as v and clearly satisfies $e(v_k) \to e(v)$ in $L^2(\Omega', \mathbb{R}^{2\times 2}_{\mathrm{sym}})$. By Theorem 7.3 we have

$$\int_\Omega Q(e(u(t_k)))\,\mathrm{d}x \le \int_\Omega Q(e(v_k))\,\mathrm{d}x + \mathcal{H}^1(J_v \setminus \Gamma(t_k)) \,.$$

Taking the limit we get the global stability because of the inclusion $J_{u(t)} \subset \Gamma(t)$. We also get, for $v = u(t)$ in the above argument, that

$$\lim_{n\to+\infty} \int_\Omega Q(e(u^n(t)))\,\mathrm{d}x = \int_\Omega Q(e(u(t)))\,\mathrm{d}x,$$

which implies the strong convergence of $e(u(t_k))$ to $e(u(t))$. Furthermore, due to the strict convexity of Q, the function $e(u(t))$ is uniquely determined by the global stability and the condition $J_{u(t)} \subset \Gamma(t)$. Thus, $e(u(t))$ is uniquely determined once I_∞ is fixed. This implies the strong convergence of $e(u(t_k))$ to $e(u(t))$ on the whole sequence $(t_k)_k$ and not only along a subsequence, and that the mapping $t \to e(u(t))$ is strongly left continuous in L^2 at any $t \in [0,T] \setminus I_\infty$.

The energy inequality immediately follows from (76) and (78)(iii), once the following claim is proved:

$$e(u^n(t)) \to e(u(t)) \text{ strongly in } L^2(\Omega', \mathbb{R}^{2\times 2}_{\mathrm{sym}})$$

for a.e. $t \in [0,T]$. In fact, one can then pass to the limit in (76) also in the term associated to the work of the external loads. Because of (78), it suffices to show the claim for $t \in [0,T] \setminus I_\infty$. Notice that because of (75) the L^2-norm of the sequence $e(u^n(t))$ is bounded. Furthermore, again (75), together with Theorem 2.5 imply that any weak accumulation point of $e(u^n(t))$ must be of the form $e(w)$, where w is a $GSBD^2$ function such that a subsequence, possibly depending on t, of $u^n(t)$ converges to w in the sense of (9). Therefore, to prove the claim it suffices to show that $e(w) = e(u(t))$ for a.e. t, so that the limit is independent of the chosen subsequence and the strong convergence holds because of (83).

Let us a consider a weak accumulation point w. Notice that $u(t)$ is an admissible competitor for the problem (82), which as shown above additionally satisfies $J_{u(t)} \subset \Gamma(t)$. Therefore, if we prove

$$\int_\Omega Q(e(u(t)))\,\mathrm{d}x \le \int_\Omega Q(e(w))\,\mathrm{d}x \tag{84}$$

for a.e. t, we will get $e(w) = e(u(t))$ as requested, otherwise, using the strict convexity of Q, we would contradict (82) with $v = \frac{1}{2}(w + u(t))$. Now, using (83) and the left continuity of $t \to e(u(t))$ at $t \notin I_\infty$, the inequality (84) follows from the minimality of $u^n(t)$ arguing exactly as in the proof of part (d) in [24, Lemma 4.3], again upon substituting the Dirichlet with the linear elastic energy. We omit the details. □

We are finally in a position to give the proof of Theorem 3.1.

Proof of Theorem 3.1. Defining $u(t)$ as in Theorems 7.3 and 7.5 and $\Gamma(t)$ as in (77), the only thing left to show is the "$\geq$"-inequality in the energy balance. This follows from global stability by a well-known argument (see [24, Lemma 4.6]) that we sketch for the reader's convenience. We first notice that the map $t \mapsto \mathcal{H}^1(\Gamma(t))$ is bounded monotone increasing, so that it is continuous at each $t \in [0,T] \setminus \mathcal{N}$, where $\mathcal{N}$ has 0-Lebesgue measure. At each $t \in [0,T] \setminus (I_\infty \cup \mathcal{N})$ we already now that $e(u(\cdot))$ is left continuous with respect to the L^2-norm. We can show that it is indeed continuous, arguing as follows. Fixing a decreasing sequence $t_k \to t$, any weak-L^2 accumulation point $e(w)$ of $e(u(t_k))$ satisfies, because of (9), the inclusion $\Gamma(t) \subset \Gamma(t_k)$, and the continuity of $\mathcal{H}^1(\Gamma(\cdot))$ at time t, that

$$\mathcal{H}^1(J_w \setminus \Gamma(t)) \leq \liminf_{k\to+\infty} \mathcal{H}^1\left(J_{u(t_k)} \setminus \Gamma(t_k)\right) = 0\,.$$

Consequently, $J_w \subset \Gamma(t)$ up to an $\mathcal{H}^1$-negligible set. With this, testing the global stability of $u(t_k)$ with $v + g(t_k) - g(t)$ for any v with $v = g(t)$ in $\Omega' \setminus \overline{\Omega}$ and arguing as in the proof of Theorem 7.5, we obtain $e(w) = e(u(t))$ as well as the claimed strong convergence.

Fix now $t \in [0,T]$. Setting for every $k \in \mathbb{N}$ and every $i = 0, \dots, k$, $s_k^i = \frac{i}{k} t$ and $u_k(s) = u(s_k^{i+1})$ whenever $t \in (s_k^i, s_k^{i+1}]$, we have that $\|e(u_k(s))\|_{L^2(\Omega',\mathbb{R}^{2\times2}_{\rm sym})}$ is uniformly bounded because of the energy inequality (see Theorem 7.5), and

$$\|e(u_k(s)) - e(u(s))\|_{L^2(\Omega',\mathbb{R}^{2\times2}_{\rm sym})} \to 0 \text{ for all } s \in [0,t] \setminus (I_\infty \cup \mathcal{N})\,, \tag{85}$$

that is a.e. in $[0,t]$. Testing the global stability of $u(s_k^i)$ with $u(s_k^{i+1}) - g(s_k^{i+1}) + g(s_k^i)$, summing up on i and exploiting the absolute continuity of $t \mapsto g(t)$, one obtains

$$\mathcal{E}(t) \geq \mathcal{E}(0) + \int_0^t \langle \sigma_k(s), e(\dot g(s)) \rangle \,\mathrm{d}s + \eta_k\,,$$

where $\sigma_k(s) := \mathbb{C}e(u_k(s))$ and η_k is an infinitesimal remainder. The thesis now follows by dominated convergence and (85) when taking the limit $k \to +\infty$. $\square$

Conflict of interest statement

The authors state that there is no conflict of interests.

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